UNAVOIDABLE PARALLEL MINORS AND SERIES MINORS OF REGULAR MATROIDS

CAROLYN CHUN AND JAMES OXLEY

To the memory of Tom Brylawski, who contributed so much to matroid theory.

ABSTRACT. We prove that, for each positive integer k, every sufficiently large 3-connected regular matroid has a parallel minor isomorphic to $M^*(K_{3,k})$, $M(\mathcal{W}_k)$, $M(K_k)$, the cycle matroid of the graph obtained from $K_{2,k}$ by adding paths through the vertices of each vertex class, or the cycle matroid of the graph obtained from $K_{3,k}$ by adding a complete graph on the vertex class with three vertices.

1. Introduction

For 3-connected graphs, the collections of unavoidable parallel and unavoidable series minors were determined by Chun, Ding, Oporowski, and Vertigan [3] and by Oporowski, Oxley, and Thomas [8]. In this paper, we combine these results with Seymour's decomposition theorem for regular matroids [12] to determine the collection of unavoidable parallel minors for the class of 3-connected regular matroids. In particular, we prove that the last collection is precisely the union of the collections of unavoidable parallel minors for the classes of 3-connected graphic and 3-connected cographic matroids. The collections of unavoidable minors for binary 3-connected matroids and for all 3-connected matroids were determined in [6, 7]. We would like to extend our main theorem to find the unavoidable parallel minors for the class of binary 3-connected matroids, but this will require some new ideas.

Our terminology for matroids and graphs generally follows [9] and [4]. If M and N are both matroids or are both graphs, N is a *parallel minor* of M if N can be obtained from M by a sequence of moves each consisting of contracting an element (in the graph case, an edge) or deleting an element that is in a 2-element circuit. When M and N are both matroids, N is a *series minor* of M if N^* is a parallel minor of M^*. If G and H are graphs and H is a parallel minor of G, then $M(H)$ is a parallel minor of $M(G)$. Conversely, when G and H are loopless 3-connected graphs, if $M(H)$ is a parallel minor of $M(G)$, then H is a parallel minor of G.

Let M be a matroid with ground set E and rank function r. The simplification of M will be denoted by $\mathrm{si}(M)$. The *connectivity function* λ_M of M is defined for all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. Thus $\lambda_M(X) = \lambda_{M^*}(X)$. For a positive integer m, when $\lambda_M(X) < m$, a partition (X, Y) of E is an *m-separation* if $\min\{|X|, |Y|\} \geq m$ and is a *vertical m-separation* if $\min\{r(X), r(Y)\} \geq m$. A matroid is *$n$-connected* if, for all $m < n$, it has no m-separations [13]. A 3-connected matroid is *internally*

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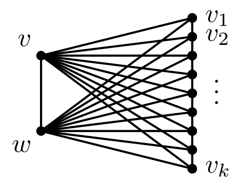

FIGURE 1. A double fan graph DF_k.

4-*connected* if it has no 3-separation (X, Y) with $\min\{|X|, |Y|\} \geq 4$. A matroid M is *vertically* 3-*connected* if it is loopless and has no vertical 1-separations and no vertical 2-separations. Note that this adds the requirement that M be loopless to the usual definition of vertical 3-connectedness. Thus M is vertically 3-connected if and only if $\mathrm{si}(M)$ is 3-connected and M is loopless.

In the following theorem, the main result of the paper, $\mathcal{W}_k$ denotes the k-spoked wheel, $K'_{i,j}$ is the bipartite graph $K_{i,j}$ together with a complete graph on the vertex class of i vertices, and DF_k is a double fan, as shown in Figure 1.

Theorem 1.1. *There is a function $f_{1.1}$ such that, for each integer k exceeding three, every* 3*-connected regular matroid with at least $f_{1.1}(k)$ elements has a parallel minor isomorphic to $M(K'_{3,k})$, $M^*(K_{3,k})$, $M(\mathcal{W}_k)$, $M(DF_k)$, or $M(K_k)$.*

By using duality, we immediately obtain the set of unavoidable series minors of 3-connected regular matroids. We denote the dual of the double fan DF_k by V_k. It can be obtained from two cycles $v_1v_2v_3 \dots v_k$ and $v_1u_2u_3 \dots u_k$ that share a single vertex by adding the edges $\{v_iu_i : i \in \{2, 3, \dots, k\}\}$.

Corollary 1.2. *There is a function $f_{1.2}$ such that, for each integer k exceeding three, every* 3*-connected regular matroid with at least $f_{1.2}(k)$ elements has a series minor isomorphic to $M^*(K'_{3,k})$, $M(K_{3,k})$, $M(\mathcal{W}_k)$, $M(V_k)$, or $M^*(K_k)$.*

By a result of Seymour, stated below as Theorem 2.1, an internally 4-connected regular matroid with at least eleven elements is graphic or cographic. This means that the sets of unavoidable parallel minors and unavoidable series minors of internally 4-connected regular matroids can be immediately determined by combining results in [3] and [8] that determine the sets of unavoidable parallel minors and unavoidable series minors, respectively, of internally 4-connected graphs.

2. PRELIMINARIES

In this section, we introduce some more terminology and prove some lemmas that will be used in the proof of the main theorem, which appears in the next section. Of particular importance here is the operation of generalized parallel connection of matroids, which was introduced and examined in detail by Tom Brylawski [2]. We shall only use one special case of this operation.

For binary matroids M_1 and M_2 with ground sets E_1 and E_2 such that $E_1 \cap E_2 = \Delta$ and $M_1|\Delta$ and $M_2|\Delta$ are triangles, the *generalized parallel connection* of M_1 and M_2 with respect to Δ, written $P_\Delta(M_1, M_2)$, is the matroid with ground set $E_1 \cup E_2$ in which F is a flat if and only if $F \cap E_i$ is a flat of M_i for each i. Then $P_\Delta(M_2, M_1) = P_\Delta(M_1, M_2)$. Moreover, one can show that if cl, cl_1, and cl_2 are the closure operators of $P_\Delta(M_1, M_2)$, M_1, and M_2, then, for every subset X of $E_1 \cup E_2$,

$$\mathrm{cl}(X) = \mathrm{cl}_1([X \cup \mathrm{cl}_2(X \cap E_2)] \cap E_1) \cup \mathrm{cl}_2([X \cup \mathrm{cl}_1(X \cap E_1)] \cap E_2). \qquad (1)$$

This correction to [9, Exercise 12.4.5] appears in the errata to that book available at the second author's website and in the second edition of the book [10].

When M_1 and M_2 both have at least seven elements and Δ does not contain a cocircuit of M_1 or M_2, Seymour [12] defined the 3-*sum*, $M_1 \oplus_\Delta M_2$, of M_1 and M_2 to be the matroid $P_\Delta(M_1, M_2)\backslash\Delta$. In much of what we do, it will be convenient to work with generalized parallel connections rather than 3-sums because of the additional constraints that must be satisfied in order for the latter to be defined. The generalized parallel connection across a triangle of two graphic matroids is easily seen to be graphic. Hence so is their 3-sum. Note, however, that the 3-sum of two cographic matroids need not be cographic. For example, the non-cographic matroid R_{12} can be written as a 3-sum of $M(K_5\backslash e)$ and $M^*(K_{3,3})$ (see, for example, [9, Exercise 1(ii), p. 440]). When G_1 and G_2 are graphs and both have Δ as a vertex bond, $P_\Delta(M^*(G_1), M^*(G_2))$ and $P_\Delta(M^*(G_1), M^*(G_2))\backslash\Delta$ are easily shown to be cographic. Hence so is $M^*(G_1) \oplus_\Delta M^*(G_2)$ when it is defined.

The next theorem was proved by Seymour [12]. The matroid R_{10} is the 10-element matroid that arises as a graft matroid from $K_{3,3}$ by taking the graft hyperedge to contain all the vertices (see [9, p. 518]).

Theorem 2.1. *Let M be a 3-connected regular matroid. Then*

(i) *M is graphic;*
(ii) *M is cographic;*
(iii) *$M \cong R_{10}$; or*
(iv) *there are regular matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \Delta$, where Δ is a triangle of both M_1 and M_2, and $M = M_1 \oplus_\Delta M_2$; and, for each i in $\{1, 2\}$,*
  (a) *M_i is 2-connected and, for every 2-separation (X, Y) of it, either X or Y has exactly two elements and meets Δ, so $\mathrm{si}(M_i)$ is 3-connected;*
  (b) *M_i is isomorphic to a minor of M; and*
  (c) *$|E(M_i) - \mathrm{cl}_{M_i}(\Delta)| \geq 6$ and $|E(\mathrm{si}(M_i))| \geq 9$.*

Let M_1 and M_2 be binary matroids with $E(M_1) \cap E(M_2) = \Delta_2$, where Δ_2 is a triangle of both M_1 and M_2. Let $P(M_1, M_2)$ and (M_1, Δ_2, M_2) be $P_{\Delta_2}(M_1, M_2)$ and $P_{\Delta_2}(M_1, M_2)\backslash\Delta_2$, respectively. Now assume, for some $n \geq 3$, that $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$ and $P(M_1, M_2, \dots, M_{n-1})$ have been defined, that

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}) = P(M_1, M_2, \dots, M_{n-1})\backslash(\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_{n-1}),$$

and that the flats of $P(M_1, M_2, \dots, M_{n-1})$ are those subsets F of its ground set such that $F \cap E(M_i)$ is a flat of M_i for all i in $\{1, 2, \dots, n-1\}$. Let M_n be a binary matroid whose ground set meets that of $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$ in a set Δ_n that is a triangle of both $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$ and M_n. Define

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n) = P_{\Delta_n}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}), M_n)\backslash\Delta_n$$

and $P(M_1, M_2, \dots, M_n) = P_{\Delta_n}(P(M_1, M_2, \dots, M_{n-1}), M_n)$. Then one easily checks that $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n) = P(M_1, M_2, \dots, M_n)\backslash(\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_n)$ and that the flats of $P(M_1, M_2, \dots, M_n)$ are those subsets F of its ground set such that $F \cap E(M_i)$ is a flat of M_i for all i in $\{1, 2, \dots, n\}$. It will be convenient to abbreviate $P(M_1, M_2, \dots, M_n)$ as $M^P_{[n]}$. Observe that the construction guarantees that $\Delta_2, \Delta_3, \dots, \Delta_n$ are disjoint.

Lemma 2.2. *If $\mathrm{si}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n))$ is 3-connected, then $\mathrm{si}(M_i)$ is 3-connected for all i.*

Proof. By definition, $\mathrm{si}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n))$ is

$$\mathrm{si}(P_{\Delta_n}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}), M_n)\backslash \Delta_n).$$

Assume that $\mathrm{si}(P_{\Delta_2}(M_1, M_2)\backslash \Delta_2)$ is 3-connected. If we can show that both $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ are 3-connected, then the result will follow by induction. For some k in $\{1, 2\}$, suppose that (X, Y) is a vertical k-separation of M_1. Without loss of generality, we may assume that $|X \cap \Delta_2| \geq 2$. Then

$$r(X \cup \Delta_2) + r(Y - \Delta_2) - r(M_1) \leq r(X) + r(Y) - r(M_1) \leq k - 1.$$

Now, by [9, Lemma 8.2.10],

$$\begin{aligned}
&r((X \cup E(M_2)) - \Delta_2) + r(Y - \Delta_2) - r(P_{\Delta_2}(M_1, M_2)\backslash \Delta_2) \\
&\leq r(X \cup E(M_2) \cup \Delta_2) + r(Y - \Delta_2) - r(P_{\Delta_2}(M_1, M_2)) \\
&\leq [r(X \cup \Delta_2) + r(M_2) - r(\Delta_2)] + r(Y - \Delta_2) - [r(M_1) + r(M_2) - r(\Delta_2)] \\
&= r(X \cup \Delta_2) + r(Y - \Delta_2) - r(M_1) \leq k - 1.
\end{aligned}$$

Thus $P_{\Delta_2}(M_1, M_2)\backslash \Delta_2$ has a vertical k-separation; a contradiction. Therefore M_1 is vertically 3-connected and, by symmetry, so is M_2. □

The next lemma will be helpful in the proof of Lemma 2.4, where we use Seymour's theorem to obtain a sequential decomposition of a regular matroid.

Lemma 2.3. *Let M_1 and M_2 be binary matroids whose ground sets meet in a set Δ_2 that is a triangle of both matroids. If Δ_3 is a triangle of $P_{\Delta_2}(M_1, M_2)\backslash \Delta_2$, then, for some $\{i, j\} = \{1, 2\}$, either*

- (i) *$\Delta_3 \subseteq E(M_i)$; or*
- (ii) *$|\Delta_3 \cap E(M_i)| = 2$ and $|\Delta_3 \cap E(M_j)| = 1$, and the element c of $\Delta_3 \cap E(M_j)$ is parallel to some element g of M_i. Moreover, if M'_j and M'_i are obtained by deleting c from M_j, and adding c in parallel to g in M_i, then $P_{\Delta_2}(M'_1, M'_2) = P_{\Delta_2}(M_1, M_2)$, while $\mathrm{si}(M'_1) = \mathrm{si}(M_1)$ and $\mathrm{si}(M'_2) = \mathrm{si}(M_2)$.*

Proof. Let $E_1 = E(M_1)$ and $E_2 = E(M_2)$. We may assume that $|\Delta_3 \cap E_1| = 2$ and $|\Delta_3 \cap E_2| = 1$. Then, in $P_{\Delta_2}(M_1, M_2)$, the intersection of $\mathrm{cl}(E_1)$ and $\mathrm{cl}(E_2)$ is $\mathrm{cl}(\Delta_2)$. Thus the element c of $\Delta_3 \cap E(M_2)$ is parallel to some element of $\mathrm{cl}(\Delta_2)$, and the lemma follows. □

Lemma 2.4. *Let M be a vertically 3-connected regular matroid such that $\mathrm{si}(M)$ has at least six elements and is not isomorphic to R_{10}. Then either M is graphic or cographic, or, for some $n \geq 2$, there is a sequence $M_1, M_2, \dots, M_n$ of graphic and cographic matroids such that $M = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ where, for all i with $2 \leq i \leq n$, the triangle $\Delta_i \subseteq E(M_j)$ for some $j < i$, and all of $\mathrm{si}(M_1), \mathrm{si}(M_2), \dots, \mathrm{si}(M_n)$ are 3-connected having at least nine elements.*

Proof. We shall assume that M is simple since it suffices to prove the lemma in that case. We proceed by induction on $|E(M)|$. Since M is regular, if $|E(M)| \leq 9$, then either M is graphic, or M is isomorphic to $M^*(K_{3,3})$ and so is cographic. In both cases, the lemma holds. Now suppose that the lemma holds for matroids with fewer than k elements and let $|E(M)| = k \geq 10$.

Assume that M is neither graphic nor cographic. Then, by Theorem 2.1, M is the 3-sum of some matroids N_1 and N_2, where both $\mathrm{si}(N_1)$ and $\mathrm{si}(N_2)$ are 3-connected having at least nine elements. Choose such a 3-sum decomposition in which $|E(N_2)|$ is minimized. Let Δ be the common triangle of N_1 and N_2. We may assume that $\Delta \subseteq E(\mathrm{si}(N_i))$ for each i.

Since N_2 has a triangle, it is not isomorphic to R_{10}. Suppose $\mathrm{si}(N_2)$ is not graphic or cographic. Then, by Theorem 2.1, N_2 is the 3-sum of matroids N_2' and N_2'' across a common triangle Δ' where each of $\mathrm{si}(N_2')$ and $\mathrm{si}(N_2'')$ is 3-connected and contains at least nine elements. As Δ is a triangle of $P_{\Delta'}(N_2', N_2'')\backslash\Delta'$, Lemma 2.3 implies that, without altering $\mathrm{si}(N_2')$ or $\mathrm{si}(N_2'')$, we can assume that $\Delta \subseteq E(N_2')$. Then, by comparing flats, we can show that $P_\Delta(N_1, P_{\Delta'}(N_2', N_2'')) = P_{\Delta'}(P_\Delta(N_1, N_2'), N_2'')$, so $M = (N_1 \oplus_\Delta N_2') \oplus_{\Delta'} N_2''$. By Lemma 2.2, $\mathrm{si}(N_1 \oplus_\Delta N_2')$ is 3-connected; a contradiction, since $|E(N_2)|$ was chosen to be minimal.

We may now assume that $\mathrm{si}(N_2)$ is graphic or cographic. Hence so is N_2. By the inductive hypothesis, $N_1 = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ and the desired conditions hold. Now Δ is a triangle of N_1. Pick the smallest integer k such that $\Delta \subseteq E((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_k, M_k))$. Then Δ meets $E(M_k)$.

Suppose that $|\Delta \cap E(M_k)| \geq 2$. Then, by moving at most one element of Δ from being parallel to an element of Δ_k in $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{k-1}, M_{k-1})$ to being parallel to that element of Δ_k in M_k, we ensure that $\Delta \subseteq E(M_k)$, as desired.

It remains to consider when $\Delta \cap E(M_k)$ contains a single element, say c. Then, by Lemma 2.3 again, we move c from being parallel to an element of Δ_k in M_k to being parallel with that element in $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{k-1}, M_{k-1})$. We now have $\Delta \subseteq E((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{k-1}, M_{k-1}))$ and we can repeat the above process until we eventually obtain $\Delta \subseteq E(M_i)$ for some i. Thus the lemma holds. $\square$

Let M be a vertically 3-connected regular matroid having at least six elements. If $M = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ for some $n \geq 2$, we call $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ a *good decomposition* of M if, for all i with $2 \leq i \leq n$, the triangle $\Delta_i \subseteq E(M_j)$ for some $j < i$. Also, we view (M) as a good decomposition of M.

Two disjoint triangles X_1 and X_2 in a binary matroid are *parallel* if $r(X_1 \cup X_2) = 2$. Recall that a regular matroid M is vertically 3-connected if $\mathrm{si}(M)$ is 3-connected and M is loopless. For a good decomposition $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ of a vertically 3-connected regular matroid, define the *associated tree* T to have vertex set $\{M_1, M_2, \dots, M_n\}$ and edge set $\{\Delta_2, \Delta_3, \dots, \Delta_n\}$ where Δ_i joins M_i to the vertex M_j with $j < i$ such that $\Delta_i \subseteq E(M_j)$. We shall sometimes write M_T for M. Note that this labelling means that, for every path $M_{i_1} M_{i_2} \dots M_{i_k}$ in T, there is a j in $\{1, 2, \dots, k\}$ such that $i_1 > i_2 > \cdots > i_j$ and $i_j < i_{j+1} < \cdots < i_k$. The reader may find some features of the tree disconcerting. For example, the matroids labelling two non-adjacent vertices may contain triangles that are parallel in $M^P_{[n]}$. In spite of this apparent shortcoming, this tree will be adequate for our needs.

Lemma 2.5. *Let M be a vertically 3-connected regular matroid for which $|E(\mathrm{si}(M))| \geq 9$ and $\mathrm{si}(M) \not\cong R_{10}$. Let $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ be a good decomposition of M and $M_i M_j$ be an edge of the associated tree with $j < i$. Then*

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_j, (M_j, \Delta_i, M_i), \Delta_{j+1}, \dots, M_{i-1}, \Delta_{i+1}, M_{i+1}, \dots, \Delta_n, M_n)$$

is a good decomposition of M. Moreover, $\mathrm{si}((M_j, \Delta_i, M_i))$ is 3-connected.

Proof. We shall show first that

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_j, (M_j, \Delta_i, M_i), \Delta_{j+1}, \dots, \Delta_{i-1}, M_{i-1}) \\ = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_i, M_i). \quad (2)$$

Now $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_i, M_i)$ is obtained from $P(M_1, M_2, \dots, M_i)$ by deleting $\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_i$. Moreover, $P(M_1, M_2, \dots, M_i)$ has, as its flats, those sets F such that $F \cap E(M_s)$ is a flat of M_s for all s with $1 \leq s \leq i$. The matroid on the left-hand side of (2) is obtained from $P(M_1, M_2, \dots, M_{j-1}, P_{\Delta_i}(M_j, M_i) \backslash \Delta_i, M_{j+1}, \dots, M_{i-1})$ by deleting $\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_{i-1}$. Thus it is obtained from $P(M_1, M_2, \dots, M_{j-1}, P_{\Delta_i}(M_j, M_i), M_{j+1}, \dots, M_{i-1})$ by deleting $\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_i$. The flats of the last matroid coincide with the flats of $P(M_1, M_2, \dots, M_i)$. Hence (2) holds. It follows that M has the decomposition specified in the lemma, and one easily checks that this decomposition is good. Finally, $\mathrm{si}((M_j, \Delta_i, M_i))$ is 3-connected by Lemma 2.2. $\square$

We shall repeatedly use the following routine consequence of the last lemma.

Corollary 2.6. *Let T be a tree associated with a vertically 3-connected matroid M. Delete an edge M_aM_b of T and let T_a be the component of the resulting forest that contains M_a. A new tree associated with M can be obtained from T by contracting the edges of T_a, one by one, each time labelling the composite vertex that results from contracting the edge Δ joining M_i and M_j by (M_j, Δ, M_i).*

When we have a good decomposition of a regular matroid M, the next two lemmas will be useful in obtaining good decompositions of certain minors of M.

Lemma 2.7. *Let $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ be a good decomposition of a regular matroid M. For e in $E(M_i) - (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_n)$, if $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_j)$ for some j, then $e \in \mathrm{cl}_{M_i}(\Delta_k)$ for some k in $\{2, 3, \dots, n\}$ where $\Delta_k \subseteq E(M_i)$.*

Proof. Choose j to be the smallest integer t for which $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_t)$. If $\Delta_j \subseteq E(M_i)$, then the result holds with $j = k$. Thus we may assume that $\Delta_j \not\subseteq E(M_i)$ so $\Delta_j \cap E(M_i) = \emptyset$ and $j \neq i$. Now e is parallel in $M^P_{[n]}$ to some element of Δ_j.

Assume $j < i$. Then $e \in \mathrm{cl}_{M^P_{[i]}}(\Delta_j)$ so, in $M^P_{[i]}$, the element e is in the intersection of $\mathrm{cl}(E(M_i))$ and $\mathrm{cl}(E(P(M_1, M_2, \dots, M_{i-1}))$. Hence $e \in \mathrm{cl}_{M^P_{[i]}}(\Delta_i)$. Thus $e \in \mathrm{cl}_{M_i}(\Delta_i)$ and the result holds with $k = i$.

We may now assume that $j > i$ so $j \geq 2$. We know that $\Delta_j \subseteq E(M_j)$ and $\Delta_j \subseteq E(M_s)$ for some $s < j$. If $s < i$, then, it follows, as above, that $e \in \mathrm{cl}_{M_i}(\Delta_i)$. Hence we may assume that $s > i$. Then $e \in \mathrm{cl}_{M^P_{[s]}}(\Delta_j)$ so $e \in \mathrm{cl}_{M^P_{[s]}}(\Delta_s)$ and hence $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_s)$. But $s < j$; a contradiction. $\square$

Lemma 2.8. *Let $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ be a good decomposition of a regular matroid M. For e in $E(M_i) - (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_n)$, if $e \in \mathrm{cl}_{M^P_{[n]}}(E(M_j))$ for some $j \neq i$, then $e \in \mathrm{cl}_{M_i}(\Delta_k)$ for some k in $\{2, 3, \dots, n\}$ where $\Delta_k \subseteq E(M_i)$.*

Proof. First we show the following.

2.8.1. *The lemma holds if $e \in \mathrm{cl}_{M^P_{[q+1]}}(E(M_j)) - \mathrm{cl}_{M^P_{[q]}}(E(M_j))$ for some q with $j \leq q < n$.*

By definition, $M^P_{[q+1]} = P_{\Delta_{q+1}}(M^P_{[q]}, M_{q+1})$. Suppose $E(M_j) \cap E(M_{q+1}) \neq \emptyset$. Then the construction of M means that $E(M_j) \cap E(M_{q+1}) = \Delta_{q+1}$. Thus, by (1), $\mathrm{cl}_{M^P_{[q+1]}}(E(M_j)) = \mathrm{cl}_{M^P_{[q]}}(E(M_j)) \cup \mathrm{cl}_{M_{q+1}}(\Delta_{q+1})$, so $e \in \mathrm{cl}_{M_{q+1}}(\Delta_{q+1})$. Hence $e \in \mathrm{cl}_{M^P_{[q+1]}}(\Delta_{q+1})$, so $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_{q+1})$ and the lemma follows by Lemma 2.7. Hence 2.8.1 holds.

Now assume that $j > i$. If $e \notin \mathrm{cl}_{M^P_{[j]}}(E(M_j))$, then, since $e \in \mathrm{cl}_{M^P_{[n]}}(E(M_j))$, the lemma follows by 2.8.1. Hence we may assume that $e \in \mathrm{cl}_{M^P_{[j]}}(E(M_j))$. Then $e \in E(M_i) \cap \mathrm{cl}_{M^P_{[j]}}(E(M_j))$. Hence $e \in \mathrm{cl}_{M^P_{[j]}}(\Delta_j)$, so $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_j)$ and again the lemma follows by Lemma 2.7.

Finally, assume that $j < i$. By 2.8.1, we may assume that $e \in \mathrm{cl}_{M^P_{[i]}}(E(M_j))$. But $e \in E(M_i)$, so $e \in \mathrm{cl}_{M^P_{[i]}}(E(M_j)) \cap \mathrm{cl}_{M^P_{[i]}}(E(M_i)) \subseteq \mathrm{cl}_{M^P_{[i]}}(\Delta_i)$. Thus $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_i)$ and the lemma follows by Lemma 2.7. □

Corollary 2.9. *Let $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ be a good decomposition of a regular matroid M. For some i in $\{1, 2, \dots, n\}$, let N_i be a minor of M_i such that if $\Delta_j \subseteq E(M_i)$ for some j in $\{2, 3, \dots, n\}$, then Δ_j is a triangle of N_i. Then*

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, M_{i-1}, \Delta_i, N_i, \Delta_{i+1}, M_{i+1}, \dots, \Delta_n, M_n)$$

is a good decomposition of a minor of M.

Proof. It suffices to prove this when N_i is $M_i \backslash e$ or M_i/e for some element e. In this case, the result follows without difficulty using the last lemma and properties of the generalized parallel connection [2] summarized in [9, Proposition 12.4.16]. □

Let A and B be parallel triangles in a loopless binary matroid N. Then $N|(A \cup B)$ is a *double triangle*. We call N a *multi-K_4 with respect to A and B* if $\mathrm{si}(N) = M(K_4)$; and we call N a *multi-triangle with respect to A and B* if $r(N) = 2$ and N contains at least one element not in $A \cup B$.

The following result is an immediate consequence of the Scum Theorem.

Lemma 2.10. *If a binary matroid M has as a minor a multi-triangle or a multi-K_4 with respect to two parallel triangles A and B, then $E(M)$ has a subset Y such that M/Y is, respectively, a multi-triangle or a multi-K_4 with respect to A and B.*

The next lemma [10] was proved by Jim Geelen and is useful for finding a double triangle as a parallel minor of a 3-connected graphic or cographic matroid. If X and Y are disjoint subsets of the ground set of a matroid M, we define $\kappa_M(X, Y)$ to be $\min\{\lambda_M(Z) : X \subseteq Z \subseteq E(M) - Y\}$.

Lemma 2.11. *Let C and X be disjoint sets in a matroid M such that C is a circuit and $\kappa_M(C, X) = 2$. Then there are elements $a, b,$ and c of C and a minor N of M that has $\{a, b, c\}$ as a circuit and $X \cup \{a, b, c\}$ as its ground set such that $\kappa_N(\{a, b, c\}, X) = 2$.*

Lemma 2.12. *Let M be a vertically 3-connected regular matroid for which $|E(\mathrm{si}(M))| \geq 9$ and $\mathrm{si}(M) \not\cong R_{10}$. Let $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ be a good decomposition of M such that each $\mathrm{si}(M_i)$ has at least nine elements. Let T be the tree associated with this decomposition. Let T' be a connected subgraph of T. Then T' is a tree associated with the matroid M' that labels the one vertex that results after all the edges of T' are contracted. Moreover, $\mathrm{si}(M')$ is a 3-connected matroid that is isomorphic to a parallel minor of M.*

Proof. It suffices to prove the lemma in the case that $T' = T - M_i$ for some vertex M_i of degree one. Let M_j be the neighbor of M_i in T and let Δ_k be the triangle common to M_i and M_j. By Corollary 2.6, $M = P_{\Delta_k}(M_i, M'_j) \backslash \Delta_k$ where M'_j labels the vertex other than M_i in the graph that is obtained by contracting every edge of

T other than M_iM_j. By Lemma 2.2, $\mathrm{si}(M_j')$ is 3-connected. We may assume that the only 2-circuits of M_i meet $\mathrm{cl}_{M_i}(\Delta_k)$.

Because the vertex M_i has degree one in T, in $M^P_{[n]}$, the intersection of the closures of $E(M_i)$ and $E(M_1)\cup\cdots\cup E(M_{i-1})\cup E(M_{i+1})\cup\cdots\cup E(M_n)$ is the closure of Δ_k. Let $Y_i = E(M_i) - \mathrm{cl}_{M_i}(\Delta_k)$. Then $|Y_i| \geq 6$ so, as M_i is regular and cosimple, $r^*(Y_i) \geq 3$. Now $2 = \lambda_{M_i}(\Delta_k) = \lambda_{M_i}(Y_i) = r(Y_i) + r^*(Y_i) - |Y_i|$. Thus $r(Y_i) < |Y_i|$ so Y_i contains a circuit C. By Lemma 2.11, there are elements a, b, and c of C and a minor N_i of M_i that has $\{a, b, c\}$ as a circuit and $\Delta_k \cup \{a, b, c\}$ as its ground set such that $\kappa_{N_i}(\{a, b, c\}, \Delta_k) = 2$. Thus $2 = \lambda_{N_i}(\{a, b, c\}) = r(\{a, b, c\}) + r(\Delta_k) - r(N_i) \leq r(\Delta_k) \leq 2$, so equality holds throughout and $r(\Delta_k) = r(N_i) = 2$. Therefore N_i is a double triangle that is a minor of M_i. Hence, by the Scum Theorem, since M_i is binary, N_i is a parallel minor of M_i. Then (N_i, Δ_k, M_j') is isomorphic to M_j' and the latter is a parallel minor of M. The lemma now follows using Corollary 2.9. □

The next lemma is from an unpublished paper [5] of Ding and Oporowski. The proof is given here for completeness.

Lemma 2.13. *Let G be a 3-connected simple graph containing distinct 3-element bonds S_1 and S_2. Then one of the following occurs.*

(i) *S_1 and S_2 are both vertex bonds.*
(ii) *G has a subgraph H that is a subdivision of K_4 such that H has a degree-three vertex v so that $S_1 \cup S_2$ is contained in the union of the minimal paths in H from v to the other degree-three vertices of H.*

Proof. Let $S_1 = \{e_1, f_1, g_1\}$ and $S_2 = \{e_2, f_2, g_2\}$. Either $S_1 \cap S_2 = \emptyset$ or $|S_1 \cap S_2| = 1$. In each case, since G is 3-connected, $S_2 - S_1$ is a bond of $G \backslash S_1$, and $S_1 - S_2$ is a bond of $G \backslash S_2$. Let A be the component of $G \backslash S_1$ that does not contain $S_2 - S_1$, and let C be the component of $G \backslash S_2$ that does not contain $S_1 - S_2$. Then A and C are vertex disjoint.

Suppose A contains no cycles. Then A is a tree and, since G is 3-connected, all the leaves of A must meet edges of S_1. Assume that A contains an edge. Then A has at least two vertices of degree one, so G has a vertex of degree at most two; a contradiction. Hence A contains no edges, and S_1 is a vertex bond. Likewise, if C contains no cycles, then S_2 is a vertex bond.

We may now assume that A or C, say A, contains a cycle D, otherwise (i) holds. Take a vertex v in $V(C)$. By Menger's Theorem, G contains three paths from v to $V(D)$ that have no internal vertices in $V(D)$ and that are disjoint except that all contain v. Each such path contains exactly one edge of S_1 and exactly one edge of S_2. The union of these three paths with D is a subdivision of K_4 satisfying (ii). □

3. The proof of the main theorem

The following theorem is well-known (see, for example, [4]).

Theorem 3.1. *There is an integer-valued function $f_{3.1}$ such that, for each positive integer d, every tree with at least $f_{3.1}(d)$ vertices has an induced subgraph isomorphic to $K_{1,d}$ or a path with d vertices.*

The next two theorems [3, 8] will be crucial in the proof of Theorem 1.1.

Theorem 3.2. *There is an integer-valued function $f_{3.2}$ such that, for each integer k exceeding three, every 3-connected graph with at least $f_{3.2}(k)$ vertices has a parallel minor isomorphic to $K_{3,k}'$, $\mathcal{W}_k$, DF_k, or K_k.*

Theorem 3.3. *There is an integer-valued function $f_{3.3}$ such that, for each integer k exceeding two, every 3-connected graph with at least $f_{3.3}(k)$ vertices has a subgraph that is isomorphic to a subdivision of V_k, $\mathcal{W}_k$, or $K_{3,k}$.*

We will also use the following result of Oxley [11].

Lemma 3.4. *Let N be a 3-connected binary matroid having rank and corank at least three and suppose $\{x, y, z\} \subseteq E(N)$. Then N has a minor isomorphic to $M(K_4)$ whose ground set contains $\{x, y, z\}$.*

The proof of our main result will occupy the rest of the paper.

Proof of Theorem 1.1. Let k be an integer exceeding three. Let $f_{3.2}$ and $f_{3.3}$ be the functions described in Theorems 3.2 and 3.3, respectively. Let $s = f_{3.2}(k) + f_{3.3}(k) + 11$. Let $m = \lceil (k+2)\frac{1}{3} f_{3.3}(k) \rceil + 2$ and $l = \max\{\binom{s}{3}(k+2), 2(2m+1)\}$. Let $t = (s-1)f_{3.1}(l)$. Set $f_{1.1}(k) = t$. Let M be a 3-connected regular matroid with at least t elements. Then $t \geq 11$.

By Lemma 2.4, M has a good decomposition into matroids each of which is graphic or cographic and has a 3-connected simplification with at least nine elements. By Lemma 2.5, we retain a good decomposition satisfying these additional conditions if we contract, one by one, the edges between vertices labelling graphic matroids. Let the resulting good decomposition be $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$, and let T be the tree associated with this decomposition.

By Lemma 2.2, for each i, the matroid $\mathrm{si}(M_i)$ is 3-connected. Suppose that some such $\mathrm{si}(M_i)$ has at least s elements. By Lemma 2.12, $\mathrm{si}(M_i)$ is isomorphic to a parallel minor N of M. If N is graphic, then, by Theorem 3.2, M has a parallel minor isomorphic to $M(K'_{3,k})$, $M(\mathcal{W}_k)$, $M(DF_k)$, or $M(K_k)$, and the theorem holds. If, instead, N is cographic, then, by Theorem 3.3, N^* has a series minor isomorphic to $M(K_{3,k})$, $M(V_k)$, or $M(\mathcal{W}_k)$. Thus N, and hence M, has a parallel minor isomorphic to $M^*(K_{3,k})$, $M(DF_k)$, or $M(\mathcal{W}_k)$, and again the theorem holds.

We may now assume that no vertex of T labels a matroid whose simplification has at least s elements. As $|E(M)| \leq \sum_{i=1}^{n} |E(\mathrm{si}(M_i))|$, we have $n > \frac{t}{s-1} = f_{3.1}(l)$.

Suppose next that T contains a vertex M_i of degree at least l. We will show that M has a parallel minor isomorphic to $M(K'_{3,k})$. Since $\mathrm{si}(M_i)$ has fewer than s elements, $\mathrm{si}(M_i)$ has fewer than $\binom{s}{3}$ triangles. As M_i has degree at least l, for some triangle S in $\mathrm{si}(M_i)$, at least $l/\binom{s}{3}$ of the matroids labelling vertices adjacent with the vertex M_i have a triangle whose union with S has rank 2 in $M^P_{[n]}$. We may assume that $\mathrm{si}(M_i)$ is labelled so that $S = \Delta_h$ for some h. Clearly $j > i$ for all but at most one neighbor M_j of M_i in T; and Δ_h is contained in the ground set of a unique neighbor of M_i in T. By definition, $l/\binom{s}{3} \geq k+2$. Take a subgraph T' of T induced by M_i and k of its higher-indexed neighbors, $M_{i_1}, M_{i_2}, \dots, M_{i_k}$, that contain triangles that are parallel to and so disjoint from Δ_h. By Lemma 2.12, the simplification of the matroid M' associated with T' is isomorphic to a parallel minor Q of M. We relabel M_i, M_{i_j}, Δ_{i_j}, and Δ_h as M_0, M_j, Δ_j, and Δ_0. Then $V(T') = \{M_0, M_1, \dots, M_k\}$ and M_0 has $\Delta_0, \Delta_1, \dots, \Delta_k$ as parallel triangles.

By Lemma 3.4, for all j in $\{1, 2, \dots, k\}$, the matroid M_j has an $M(K_4)$-minor M'_j having Δ_j as a triangle. Because M_j has no Fano minor, by the Scum Theorem, M'_j is a parallel minor of M_j. Take two distinct elements d_1 and d_2 in Δ_0 and extend $\{d_1, d_2\}$ to a basis B of M_0. Let $M'_0 = M_0/(\mathrm{cl}_{M_0}(B - \{d_1, d_2\}))$. Then $\Delta_0 \subseteq E(M'_0)$. Therefore, if $i \geq 1$, for every parallel deletion that is done in M_i to produce M'_i, there is a corresponding parallel deletion in Q. It follows by Corollary 2.9 that $(M'_0, \Delta_1, M'_1, \Delta_2, \dots, \Delta_k, M'_k)$ is a parallel minor N of Q. Moreover, $\mathrm{si}(N)$ can be obtained by identifying a triangle in each of k matroids isomorphic to $M(K_4)$, so $\mathrm{si}(N) \cong M(K'_{3,k})$. Hence M has a parallel minor isomorphic to $M(K'_{3,k})$.

We may now suppose that every vertex of T has degree at most $l-1$. By Theorem 3.1, T contains a path $M_{i_1}M_{i_2}\dots M_{i_l}$ with l vertices. By construction, there is some index j such that $i_1 > i_2 > \dots > i_j$ and $i_j < i_{j+1} < \dots < i_l$. Now $\frac{l}{2} \geq 2m+1$, so T contains a path T' with at least $2m+1$ vertices such that the indices on the vertices are increasing. As no two adjacent vertices of this path label graphic matroids, by removing vertices from the ends of the path, we can get a path T' with $2m$ vertices such that the first vertex of T' labels a non-graphic matroid. We relabel the vertices of T' so that $T' = M_1M_2\dots M_{2m}$ and relabel each edge M_iM_{i+1} as Δ_{i+1}. Let $M' = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{2m}, M_{2m})$ and $\bar{M} = \mathrm{si}(M')$. By Lemma 2.12, $\bar{M}$ is 3-connected and is isomorphic to a parallel minor of M. We can modify the decomposition we have for M' to obtain a good decomposition for $\bar{M}$ by *deleting superfluous parallel elements*. Specifically, we replace each M_i by its restriction to the set $(E(\bar{M}) \cap E(M_i)) \cup (\Delta_i \cup \Delta_{i+1})$. Note that Δ_1 and Δ_{2m+1} do not exist so we take these sets to be empty. This process gives us a good decomposition of $\bar{M}$ for which we shall retain the labelling $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{2m}, M_{2m})$.

Next we prove two lemmas to deal with this kind of situation. Let N be a 3-connected regular matroid having $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$ as a good decomposition such that the associated tree is a path $N_1N_2\dots, N_d$; each $\mathrm{si}(N_i)$ has at least nine elements and is graphic or cographic, with no two consecutive matroids being graphic; and N_1 is not graphic. We call such a good decomposition a *fine decomposition* of N. Note that, in a fine decomposition, every non-trivial parallel class of each N_i meets Δ_i or Δ_{i+1}. When $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$ is a fine decomposition of N, if $1 < i < d$, we denote $(N_1, \Delta_2, N_2, \dots, \Delta_{i-i}, N_{i-1})$ and $(N_{i+1}, \Delta_{i+2}, N_{i+2}, \dots, \Delta_d, N_d)$ by $\hat{N}_{i-1}$ and $\check{N}_{i+1}$. As a graph, the *triangular prism* consists of the vertices and edges of the eponymous polyhedron. This graph is the planar dual of the graph $K_5\backslash e$.

Lemma 3.5. *Let $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$ be a fine decomposition of a 3-connected regular matroid. For all i with $1 < i < d$, one of the following occurs:*

- (i) *N_i is graphic and $E(N_i)$ has a subset Y_i such that N_i/Y_i is a multi-triangle with respect to Δ_i and Δ_{i+1};*
- (ii) *N_i is the cycle matroid of a triangular prism, and N_{i-1} and $\hat{N}_{i-1}$ have no triads meeting Δ_i, while N_{i+1} and $\check{N}_{i+1}$ have no triads meeting Δ_{i+1};*
- (iii) *N_i is not graphic and $N_i = M^*(G_i)$ for some graph G_i where Δ_i and Δ_{i+1} are vertex bonds of G_i; or*
- (iv) *N_i is cographic but not graphic and $E(N_i)$ has a subset Y_i such that N_i/Y_i is a multi-K_4 with respect to Δ_i and Δ_{i+1}.*

Proof. If Δ_i and Δ_{i+1} are parallel in N_i, then Lemma 3.4 implies that $E(N_i)$ has a subset Y_i such that N_i/Y_i is a multi-K_4 with respect to Δ_i and Δ_{i+1}. In the first case, (i) holds; in the second, (i) or (iv) holds depending on whether N_i is graphic or not. We may now assume that Δ_i and Δ_{i+1} are not parallel in N_i.

Suppose that N_i is graphic and let G_i be the 3-connected graph such that $M(G_i) = N_i$. By Menger's Theorem, G_i has three vertex-disjoint paths, P_1, P_2, and P_3, from $V(\Delta_i)$ to $V(\Delta_{i+1})$.

We assume first that $G_i\backslash(E(\Delta_i) \cup E(\Delta_{i+1}))$ has a component C that contains at least two of the chosen paths. Then $G_i\backslash(E(\Delta_i) \cup E(\Delta_{i+1}))$ contains a path R with ends in two different chosen paths and no other vertices in any chosen path. Evidently, G_i has a multi-triangle as a minor whose restriction to each of $E(\Delta_i)$ and $E(\Delta_{i+1})$ is a triangle. By Lemma 2.10, $E(N_i)$ contains a set Y_i such that N_i/Y_i is a multi-triangle with respect to Δ_i and Δ_{i+1}, and (i) holds.

We may now assume that $G_i\backslash(E(\Delta_i) \cup E(\Delta_{i+1}))$ has three disjoint components each containing one chosen path. Since G_i is 3-connected, no P_i has an internal vertex since its ends do not form a vertex cut. Thus $V(G_i) = V(P_1) \cup V(P_2) \cup V(P_3)$. If G_i has a non-trivial parallel class, then this class meets Δ_i or Δ_{i+1}, and (i) holds with $Y_i = P_1 \cup P_2 \cup P_3$. Thus we may assume that G_i is simple. Then $|E(G_i)| = |E(\mathrm{si}(N_i))| \geq 9$, and it follows that G_i is a triangular prism.

Let $\{x_1, x_2, x_3\} = E(N_i) - (\Delta_i \cup \Delta_{i+1})$. By Lemma 2.5, $N_{i-1} \oplus_{\Delta_i} N_i$ and $\hat{N}_{i-1} \oplus_{\Delta_i} N_i$ have no series pairs. Thus N_{i-1} and $\hat{N}_{i-1}$ have no triads meeting Δ_i. Similarly, N_{i+1} and $\check{N}_{i+1}$ have no triads meeting Δ_{i+1}, and (ii) holds.

We may now assume that N_i is not graphic. Then N_i is cographic and so too is $\mathrm{si}(N_i)$. Hence $\mathrm{si}(N_i) = M^*(H_i)$ for some 3-connected simple graph H_i. Now Δ_i and Δ_{i+1} are not parallel in N_i. Thus $r(\Delta_i \cup \Delta_{i+1})$ is 3 or 4. Hence we can choose H_i so that either both Δ_i and Δ_{i+1} label bonds of it, or so that Δ_i and $(\Delta_{i+1} - e_{i+1}) \cup e_i$ label bonds of it where $\{e_i, e_{i+1}\}$ is a circuit of N_i with each e_j in Δ_j. Consider the bonds Δ_i and Δ'_{i+1} of H_i where Δ'_{i+1} is Δ_{i+1} or $(\Delta_{i+1} - e_{i+1}) \cup e_i$. Suppose first that both Δ_i and Δ'_{i+1} are vertex bonds. Then, by replacing edges of H_i by paths if necessary, we can get a graph G_i such that $N_i = M^*(G_i)$ and Δ_i and Δ_{i+1} are both vertex bonds of G_i. Thus (iii) holds.

It remains to consider when Δ_i or Δ'_{i+1} is not a vertex bond of H_i. By Lemma 2.13, H_i has a subgraph J that is a subdivision of K_4 such that J has a degree-three vertex v so that $\Delta_i \cup \Delta'_{i+1}$ is contained in the union of the minimal paths in J from v to the other degree-three vertices of J. If $\Delta'_{i+1} \neq \Delta_{i+1}$, form J' from J by replacing e_i by a 2-edge path $\{e_i, e_{i+1}\}$; otherwise let J' be J. Then $M^*(J')$ is a minor of N_i. By Lemma 2.10, $E(N_i)$ has a subset Y_i such that N_i/Y_i is a multi-K_4 with respect to Δ_i and Δ_{i+1}, and (iv) holds. □

We will say that N_i is *type (i)* if it meets the conditions of (i) in the preceding lemma. Likewise, we will say that N_i is *type (ii)*, *type (iii)*, or *type (iv)* if it meets the conditions of (ii), (iii), or (iv), respectively. The goal of the next lemma is to eliminate the graphic matroids in a fine decomposition.

Lemma 3.6. *Let $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$ be a fine decomposition of a 3-connected regular matroid N. For some i with $2 \leq i \leq d-1$, suppose $N_1, N_2, \dots, N_{i-1}$ are not graphic. When N_i is type (i), let N'_i be a contraction of N_i that is a multi-triangle with respect to Δ_i and Δ_{i+1}. When N_i is type (ii), let N'_i be the double triangle obtained by contracting each element not in a triangle of N_i. Then either $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_i, N'_i, \Delta_{i+1}, \dots, \Delta_d, N_d)$ is vertically 3-connected, or there is an element a of $E(N_j) - (\mathrm{cl}_{N_j}(\Delta_j) \cup \mathrm{cl}_{N_j}(\Delta_{j+1}))$ for some $j \leq i-1$ such that*

$$(N_1, \Delta_2, N_2, \dots, \Delta_j, N_j/a, \Delta_{j+1}, \dots, N_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \dots, \Delta_d, N_d)$$

is vertically 3-connected, and N_j/a is not graphic.

Proof. By Lemma 2.12, both $\hat{N}_{i-1}$ and $\check{N}_{i+1}$ are vertically 3-connected. We show first that:

3.6.1. *Either $(\hat{N}_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \check{N}_{i+1})$ is vertically 3-connected, or that there is an element a of $E(\hat{N}_{i-1}) - \Delta_i$ such that $(\hat{N}_{i-1}/a, \Delta_i, N'_i, \Delta_{i+1}, \check{N}_{i+1})$ is vertically 3-connected.*

Now N'_i is either a double triangle with ground set $\Delta_i \cup \Delta_{i+1}$, or it is obtained from this matroid by adding some elements in parallel with elements of Δ_{i+1}. In both cases, we let $\hat{N}'_{i-1} = \hat{N}_{i-1} \oplus_{\Delta_i} N'_i$. Then $\hat{N}'_{i-1}$ may be obtained from $\hat{N}_{i-1}$ by relabelling the elements of Δ_i by the appropriate elements in Δ_{i+1} and, when N'_i is type (i), adding some non-empty set of elements in parallel with those of Δ_{i+1}. Let $\bar{N}$ be the matroid $P_{\Delta_{i+1}}(\hat{N}'_{i-1}, \check{N}_{i+1})$. Then every non-trivial parallel class of $\bar{N}$ meets Δ_{i+1}. Let $\Delta_{i+1} = \{x, y, z\}$. We shall distinguish the following two cases:

(a) no element of Δ_{i+1} is in a non-trivial parallel class of $\bar{N}$; and
(b) some element, say z, of Δ_{i+1} is in a non-trivial parallel class of $\bar{N}$.

Observe that if N_i is type (i), then (b) holds.

Assume first that (a) holds. Then N_i is type (ii), so $\hat{N}'_{i-1}$ has no triad meeting Δ_{i+1} because $\hat{N}_{i-1}$ has no triad meeting Δ_i. Moreover, $\bar{N}$ is simple and, since it is the generalized parallel connection across a triangle of two 3-connected matroids, it too is 3-connected. Let C^* be a cocircuit of $\bar{N}$ meeting Δ_{i+1}. Then $|C^* \cap \Delta_{i+1}| = 2$. Furthermore, as $C^* \cap E(\hat{N}'_{i-1})$ and $C^* \cap E(\check{N}_{i+1})$ contain cocircuits of $\hat{N}'_{i-1}$ and $\check{N}_{i+1}$, it follows that both $|C^* \cap E(\hat{N}'_{i-1})|$ and $|C^* \cap E(\check{N}_{i+1})|$ exceed 3, so $|C^*| \geq 6$. Thus, if $Z \subseteq \Delta_{i+1}$, then $\bar{N} \backslash Z$ has no 2-cocircuits. Since $\bar{N}/x$ has a non-minimal 2-separation, it follows, by a well-known result of Bixby [1] (see also [9, Proposition 8.4.6]), that $\bar{N} \backslash x$ is 3-connected. Similarly, $\bar{N} \backslash x/y$ and $\bar{N} \backslash x, y/z$ have non-minimal 2-separations, so $\bar{N} \backslash x, y$ is 3-connected and then so is $\bar{N} \backslash x, y, z$. Hence, in case (a), $(\hat{N}_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \check{N}_{i+1})$ is vertically 3-connected.

Now assume that (b) holds. Then $\bar{N}$ has $\{e, z\}$ as a 2-circuit for some element e, so $\mathrm{si}(\bar{N} \backslash z)$ is 3-connected. We shall show next that $\mathrm{si}(\bar{N} \backslash z, y)$ is 3-connected. Suppose not. Then y is not in a 2-circuit of $\bar{N}$. Clearly $\mathrm{si}(\bar{N} \backslash z)/y$ has a non-minimal 2-separation. Thus, by Bixby's Lemma, $\mathrm{co}(\mathrm{si}(\bar{N} \backslash z) \backslash y)$ is 3-connected, that is, $\mathrm{co}(\mathrm{si}(\bar{N} \backslash z, y))$ is 3-connected. As $\mathrm{si}(\bar{N} \backslash z, y)$ is not 3-connected, $\mathrm{si}(\bar{N} \backslash z) \backslash y$ has a 2-cocircuit. Thus $\mathrm{si}(\bar{N} \backslash z)$ has a triad C^* containing y. As each of $\mathrm{si}(\hat{N}'_{i-1})$ and $\mathrm{si}(\check{N}_{i+1})$ is a restriction of $\mathrm{si}(\bar{N} \backslash z)$, and either $C^* \cap E(\mathrm{si}(\hat{N}'_{i-1}))$ or $C^* \cap E(\mathrm{si}(\check{N}_{i+1}))$ has exactly two elements, we deduce that $\mathrm{si}(\hat{N}'_{i-1})$ or $\mathrm{si}(\check{N}_{i+1})$ has a cocircuit with at most two elements; a contradiction. Thus $\mathrm{si}(\bar{N} \backslash z, y)$ is indeed 3-connected.

Now $\mathrm{si}(\bar{N} \backslash z, y)/x$ has a non-minimal 2-separation. Thus, by Bixby's Lemma again, $\mathrm{co}(\mathrm{si}(\bar{N} \backslash z, y) \backslash x)$ is 3-connected. As $\mathrm{si}(\bar{N} \backslash z, y, x) \cong \mathrm{si}(P(\hat{N}'_{i-1}, \check{N}_{i+1}) \backslash \Delta_{i+1})$, we assume that $\mathrm{si}(\bar{N} \backslash z, y, x)$ is not 3-connected, otherwise the lemma holds. Then

3.6.2. *$\bar{N}$ has no 2-circuit containing x or y.*

As $\mathrm{si}(\bar{N} \backslash z, y)$ is 3-connected, $\bar{N}$ has no 2-circuit containing x. By symmetry, $\bar{N}$ has no 2-circuit containing y.

Now $\mathrm{si}(\bar{N} \backslash z, y)$ must have a triad containing x. Assume that $\{a, b, x\}$ and $\{c, d, x\}$ are such triads. Then their symmetric difference is a disjoint union of cocircuits of $\mathrm{si}(\bar{N} \backslash z, y)$. Thus $\{a, b\} \cap \{c, d\} = \emptyset$. Now $\mathrm{si}(\bar{N} \backslash z) \backslash y$ is 3-connected. Therefore $\{a, b, x, y\}$ and $\{c, d, x, y\}$ contain cocircuits of $\mathrm{si}(\bar{N} \backslash z)$ containing $\{a, b, x\}$ and $\{c, d, x\}$. By considering the intersections of these cocircuits with $E(\mathrm{si}(\hat{N}'_{i-1}))$ and $E(\mathrm{si}(\check{N}_{i+1}))$, we see that each such cocircuit has four elements. Moreover, we may assume that the first contains $\{a, c\}$ and the second contains $\{b, d\}$. Thus $\{a, x, y\}$ and $\{c, x, y\}$ are cocircuits of $\mathrm{si}(\hat{N}'_{i-1})$. Hence $\mathrm{si}(\hat{N}'_{i-1})$ has a cocircuit contained in $\{a, c\}$; a contradiction. We deduce that $\mathrm{si}(\bar{N}\backslash z, y)$ has exactly one triad, say $\{a, b, x\}$, containing x. Moreover, we may assume that $\{a, x, y\}$ and $\{b, x, y\}$ are triads of $\mathrm{si}(\hat{N}'_{i-1})$ and $\mathrm{si}(\check{N}_{i+1})$, respectively.

3.6.3. *$\hat{N}'_{i-1}$ has no 2-circuit containing a.*

If a is in a 2-circuit of $\hat{N}'_{i-1}$, then, by 3.6.2, a is parallel to z. Thus $\{a, x, y\}$ is both a triangle and a triad of $\mathrm{si}(\hat{N}'_{i-1})$; a contradiction.

By 3.6.2 and 3.6.3, $\{a, x, y\}$ is a triad of $\hat{N}'_{i-1}$. Since $\{a, b\}$ is the only 2-cocircuit of $\mathrm{si}(\hat{N}'_{i-1} \oplus_{\Delta_{i+1}} \check{N}_{i+1})$, the matroid $\mathrm{si}(\hat{N}'_{i-1} \oplus_{\Delta_{i+1}} \check{N}_{i+1})/a$ is 3-connected, so $\mathrm{si}((\hat{N}'_{i-1}/a) \oplus_{\Delta_{i+1}} \check{N}_{i+1})$ is 3-connected. This completes the proof of 3.6.1.

Observe that the construction of $\hat{N}'_{i-1}$ means that we can label the triangle Δ_i of N_{i-1} by $\{x_i, y_i, z_i\}$ where $\{x, x_i\}$, $\{y, y_i\}$, and $\{z, z_i\}$ are circuits of N'_i. Clearly $\hat{N}_{i-1}$ can be obtained from $\hat{N}'_{i-1}$ by first relabelling the elements x, y, and z of the latter as x_i, y_i, and z_i and then deleting some elements that are parallel to x_i, y_i, or z_i. By 3.6.2 and 3.6.3, none of a, x, or y is in a 2-circuit of $\hat{N}'_{i-1}$. Hence none of a, x_i, or y_i is in a 2-circuit of $\hat{N}_{i-1}$. Moreover, as $\{a, x, y\}$ is a triad of $\hat{N}'_{i-1}$, and $\mathrm{si}(\hat{N}_{i-1})$ is 3-connected, $\{a, x_i, y_i\}$ is a triad of $\hat{N}_{i-1}$.

For all p with $2 \leq p \leq i-1$, let $\Delta_p = \{x_p, y_p, z_p\}$. Now $\hat{N}_{i-1} = P_{\Delta_{i-1}}(\hat{N}_{i-2}, N_{i-1})\backslash\Delta_{i-1}$. Since $\{a, x_i, y_i\}$ is a triad of $\hat{N}_{i-1}$, either $\{a, x_i, y_i\}$ is a triad of N_{i-1}; or $\{a, x_i, y_i\} \cup Z$ is a cocircuit of $P_{\Delta_{i-1}}(\hat{N}_{i-2}, N_{i-1})$ for some 2-element subset Z of Δ_{i-1}. In the latter case, we may assume that $Z = \{x_{i-1}, y_{i-1}\}$. Then $\{a, x_{i-1}, y_{i-1}\}$ contains and so is a cocircuit of $\hat{N}_{i-2}$. By repeating this argument, we deduce that, for some j with $1 \leq j \leq i-1$, after possibly relabelling the elements of Δ_{j+1}, we have $\{a, x_{j+1}, y_{j+1}\}$ as a triad of N_j.

Next we shall show that a is not in the closure of Δ_j or Δ_{j+1} in N_j. Note that, when $j = 1$, the set Δ_j does not exist. We have $\{a, x_{j+1}, y_{j+1}\}$ as a triad of N_j. If N_j has a circuit containing a and contained in $a \cup \Delta_j$, then we contradict orthogonality. If N_j has a circuit containing a and contained in $a \cup \Delta_{j+1}$, then a is parallel to some element of Δ_{j+1}. Thus $\mathrm{si}(N_j)$ has a 2-cocircuit, a contradiction since $\mathrm{si}(N_j)$ is 3-connected having at least nine elements.

We now show that N_j/a is not graphic. Assume it is and let G be a graph such that $M(G) = N_j^*$. Since $\{a, x_{j+1}, y_{j+1}\}$ is a triad of N_j, it is a triangle of G. As $\{x_{j+1}, y_{j+1}, z_{j+1}\}$ is a triad of G, the vertex v common to x_{j+1} and y_{j+1} has degree 3. Since N_j is not graphic, G has a minor isomorphic to K_5 or $K_{3,3}$. Assume first that G has a $K_{3,3}$-minor. Since $K_{3,3}$ is cubic, G contains a subgraph H that is a subdivision of $K_{3,3}$. As $M^*(G\backslash a)$ is graphic, $G\backslash a$ has no $K_{3,3}$-minor. Thus a is in H. Since H has no triangles, at most one of x_{j+1} and y_{j+1} is in H. Either v has degree two in H, or v is not in $V(H)$. In each case, by interchanging x_{j+1} and y_{j+1} if necessary, we get that G/x_{j+1} has a $K_{3,3}$-minor. But $\{a, y_{j+1}\}$ is a cycle of G/x_{j+1}, so $G/x_{j+1}\backslash a$ has a $K_{3,3}$-minor. Hence so does $G\backslash a$; a contradiction.

We may now assume that G has a K_5-minor. Then G has five disjoint connected subgraphs G_1, G_2, G_3, G_4, and G_5 that together contain all of the vertices in G and such that G has at least one edge between every pair of these subgraphs. Suppose first that a is in G_1. Then two of the three neighbors of v are in G_1, and we may assume that v is in G_1. Hence x_{j+1} and y_{j+1} are in G_1. Then $G_1\backslash a$ is connected, since $\{a, x_{j+1}, y_{j+1}\}$ is a triangle, and $G\backslash a$ contains a minor isomorphic to K_5; a contradiction. Finally, assume that a is a G_1-G_2-edge. If x_{j+1} or y_{j+1} is a G_1-G_2-edge, then $G\backslash a$ has a minor isomorphic to K_5. In the exceptional case, without loss of generality, we may assume that x_{j+1} is a G_2-G_3-edge and y_{j+1} is a G_3-G_1-edge. Then v is in G_3. Since v has degree three in G, it has degree one in the graph G_3. Hence $G_3 - v$ is a connected graph and, for each i in $\{4, 5\}$, there is an edge of G with one end in $G_3 - v$ and the other in G_i. We contract the subgraphs G_1, G_2, $G_3 - v$, G_4, and G_5 to vertices v_1, v_2, v_3, v_4, and v_5, respectively, and delete the edge a. The resulting 6-vertex graph has $K_{3,3}$ as a subgraph, where the vertex classes are $\{v_1, v_2, v_3\}$ and $\{v, v_4, v_5\}$. Thus $G\backslash a$ has a $K_{3,3}$-minor; a contradiction. We conclude that N_j/a is not graphic and the lemma is proved. □

Now returning to the proof of the main theorem, recall that, immediately before Lemma 3.5, we showed that we could obtain a fine decomposition $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{2m}, M_{2m})$ of a 3-connected matroid $\bar{M}$ that is isomorphic to a parallel minor of M. Each M_i with $1 < i < 2m$ satisfies one of (i)-(iv) of Lemma 3.5.

Suppose that some matroid in the set $\{M_1, M_2, \dots, M_{2m-1}\}$ is graphic. In that case, let M_i be the lowest-indexed such matroid. Then $i > 1$, so M_i labels a type (i) or type (ii) matroid. By Lemma 3.6, we may contract elements from M_i to obtain a matroid M_i' that is a double triangle or a multi-triangle containing Δ_i and Δ_{i+1}, and we may contract at most one element of some M_j with $j \leq i-1$ to obtain a non-graphic matroid M_j'' such that

$$(M_1, \Delta_2, M_2, \dots, \Delta_j, M_j'', \Delta_{j+1}, \dots, M_{i-1}, \Delta_i, M_i', \Delta_{i+1}, \dots, \Delta_{2m}, M_{2m}) \qquad (3)$$

is vertically 3-connected. Now let M_{i-1}'' be M_j'' when $j = i-1$ and let $M_{i-1}'' = M_{i-1}$ when $j < i-1$. Then M_{i-1}'' is cographic but not graphic, hence so is $(M_{i-1}'', \Delta_i, M_i')$. Thus, in (3), when we remove Δ_i and M_i', and replace M_{i-1}'' by $(M_{i-1}'', \Delta_i, M_i')$, we get a good decomposition of a vertically 3-connected matroid whose simplification is a parallel minor $\bar{M}'$ of M. We can convert this good decomposition into a fine decomposition for $\bar{M}'$ by deleting superfluous parallel elements. This means that we can repeat the above process. Thus, from our original fine decomposition, we eliminate graphic matroids one by one, beginning with the lowest-indexed such matroid. After each such move, we recover a fine decomposition of a 3-connected parallel minor of M. Since no two consecutive matroids in $M_1, M_2, \dots, M_{2m}$ are graphic and M_1 is non-graphic, we eventually obtain a fine decomposition for which the corresponding path has at least $m+1$ vertices, where each vertex except possibly the last labels a cographic matroid that is not graphic. If this path ends in a graphic matroid, that matroid has been unaltered in the above process and so its simplification has at least nine elements. Hence we can apply Lemma 2.12 and remove at least one vertex from the end of this path to obtain a path Q with m vertices each of which is labelled by a cographic matroid that is not graphic. Again by deleting superfluous parallel elements, we may assume that M_Q, which is a parallel minor of M, is simple. Relabel Q as $N_1 N_2 \dots N_m$. By Lemma 3.5, each N_i with $1 < i < m$ is type (iii) or type (iv).

Recall that $m = \lceil (k+2)\frac{1}{3}f_{3.3}(k) \rceil + 2$. Suppose that the interior vertices of Q contain a subpath Q' of at least $\lfloor \frac{1}{3}f_{3.3}(k) \rfloor$ vertices each of which is labelled by a type (iii) matroid. Then it is not difficult to check that the associated matroid $M_{Q'}$ is cographic. Because each $\mathrm{si}(N_i)$ has at least nine elements, $\mathrm{si}(M_{Q'})$ has at least $f_{3.3}(k)$ elements and, by Lemma 2.12, $M_{Q'}$ is vertically 3-connected. Since DF_k is the dual of V_k, we deduce by Theorem 3.3, that M has a parallel minor isomorphic to $M(DF_k)$, $M(\mathcal{W}_k)$, or $M^*(K_{3,k})$. Hence, in this case, Theorem 1.1 holds.

We may now assume that every interior subpath of Q with at least $\frac{1}{3}f_{3.3}(k)$ vertices contains a vertex labelled by a type (iv) matroid. Thus Q has at least $\lfloor (m-2)/(\frac{1}{3}f_{3.3}(k)) \rfloor$ vertices that are labelled by type (iv) matroids, so Q has at least $k+2$ such vertices.

We now modify each N_i with $1 < i < m$ to produce N'_i as follows. If N_i is type (iv), we let $N'_i = N_i/Y_i$, where N_i/Y_i is a multi-K_4 with respect to Δ_i and Δ_{i+1}. Now suppose N_i is type (iii). Then $N_i = M^*(G_i)$ for some graph G_i that has Δ_i and Δ_{i+1} as vertex bonds. By Menger's Theorem, G_i has a subgraph H_i that is a subdivision of $K_{2,3}$ where Δ_i and Δ_{i+1} are vertex bonds of H_i. Thus N_i has, as a minor, a double triangle with ground set $\Delta_i \cup \Delta_{i+1}$. Hence, by the Scum Theorem, for some subset Y_i of $E(N_i)$, the matroid N_i/Y_i is either this double triangle or a multi-triangle with respect to Δ_i and Δ_{i+1}. In this case, we let $N'_i = N_i/Y_i$.

Let $R = N'_2 N'_3 \dots N'_{m-1}$. Using Corollary 2.9 and Lemma 2.12, we can show that $\mathrm{si}(M_R)$ is a parallel minor of $\mathrm{si}(M_Q)$. Furthermore, M_R may be obtained by identifying at least $k+2$ copies of $M(K_4)$ across a triangle and either deleting elements from the common triangle or adding elements parallel with the elements in the common triangle. Evidently M_R, and hence M, has a parallel minor isomorphic to $M(K'_{3,k})$, and this completes the proof of the theorem. $\square$

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Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
E-mail address: chchchun@math.lsu.edu

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
E-mail address: oxley@math.lsu.edu