UNAVOIDABLE TOPOLOGICAL MINORS OF INFINITE GRAPHS

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ABSTRACT. A graph G is *loosely-c*-connected, or ℓ-c-connected, if there exists a number d depending on G such that the deletion of fewer than c vertices from G leaves precisely one infinite component and a graph containing at most d vertices. In this paper, we give the structure of a set of ℓ-c-connected infinite graphs that form an unavoidable set among the topological minors of ℓ-c-connected infinite graphs. Corresponding results for minors and parallel minors are also obtained.

1. INTRODUCTION

In this paper, we explore unavoidable topological minors in ℓ-c-connected infinite graphs, building on König's Infinity Lemma for connected infinite graphs, which is stated as follows.

Lemma 1.1. *If G is a connected infinite graph, then G contains a vertex of infinite degree or a oneway infinite path.*

The purpose of this paper is to extend this result by identifying unavoidable structures in better connected infinite graphs. We prove a stronger form of an infinite graph result by Oporowski, Oxley, and Thomas from 1993 found in [2], which we state later as Theorem 1.2(b).

Since we only consider vertex connectivity in this paper, we restrict our attention to simple graphs. We say that a graph is *connected* if every pair of vertices is contained in a path in the graph. As stated in the abstract, an infinite graph G is *loosely-c-connected*, or *ℓ-c-connected* if there exists a number d depending on G such that the deletion of fewer than c vertices from G leaves precisely one infinite component and a graph containing at most d vertices. (We learned after the first draft of this paper that ℓ-c-connected graphs are called *essentially c-connected* in [2]. We continue to use our abbreviation since e-c-connectivity could be misunderstood as an edge connectivity.)

We now define some more terms and notation for use throughout this paper. All other graph terminology and notation are defined in [1]. For an edge e in a graph G, we may *contract* e in G, written G/e, by replacing the two ends of e with a single vertex adjacent to every vertex that is adjacent to either end of e in G. A *subdivision* of a graph M is any graph obtained from M by replacing some edges of M with finite paths. We say that a graph M is a *topological minor*, or

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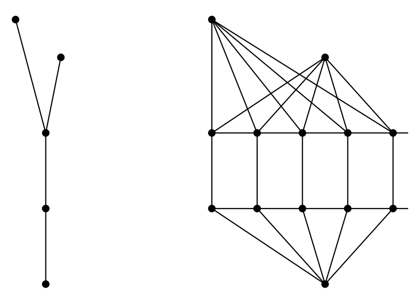
series minor, of a graph G, written $M \preceq_t G$, if G contains a subdivision of M as a subgraph. A graph N is a *minor* of a graph G, written $N \preceq G$, if N can be obtained by contracting a set Y of edges in a subgraph H of G, where N can be written H/Y. A graph P is a *parallel minor* of a graph G, written $P \preceq_{\|} G$, if P can be obtained from G by contracting edges. We note that parallel minor is the matroid dual operation of series minor. Parallel minor is related to *induced minor*, which is obtained from a graph by deleting vertices and contracting edges. Observe that a parallel minor is an induced minor, and an induced minor is a minor.

A *ray* is a oneway infinite path and a *star* is a vertex u and an infinite vertex set V together with edge set $\{uv : v \in V\}$. A *fan* is the graph of a vertex adjacent to each vertex in a ray. A *ladder* on two rays Y and Z is the graph consisting of the disjoint rays $Y = y_1y_2y_3\ldots$ and $Z = z_1z_2z_3\ldots$, and edges $y_1z_1, y_2z_2, y_3z_3, \ldots$. If the edges $y_2z_1, y_3z_2, \ldots$ are added to this ladder, we get a *zigzag ladder* on rays Y and Z. Note that in this zigzag ladder rays Y and Z are not symmetric, since Y contains a vertex of degree two and Z does not, but observe that after the contraction of the edge y_1y_2, ray Z contains a vertex of degree two and Y does not.

Next, we define the *expansion* of a finite tree T. A *leaf* is a vertex with degree one. If T has one vertex then the expansion of T is a ray. If T has two vertices then the expansion is a fan. These are the two special cases of expansion. If T has three or more vertices, then let $t_1, t_2, \ldots, t_m$ be its leaves and $t_{m+1}, t_{m+2}, \ldots, t_n$ be its internal vertices. Then the expansion of T is the graph consisting of vertices $s_1, s_2, \ldots, s_m$ and rays $R_{m+1}, R_{m+2}, \ldots, R_n$, with a ladder on rays R_i and R_j exactly when $t_it_j \in E(T)$, and a fan on vertex s_k and ray R_l exactly when $t_kt_l \in E(T)$. We say that $s_1, s_2, \ldots, s_m$ are the stars of the expansion and $R_{m+1}, R_{m+2}, \ldots, R_n$ are the rays of the expansion. Though there are other rays in the expansion, when we refer to the rays of the expansion, we mean these particular rays. An example of expansion is given in Figure 1, where tree T in Figure 1a is expanded in Figure 1b.



(a) (b)

FIGURE 1. (a) Tree T. (b) The expansion of T.

The graph $K_{c,\infty}$ is the infinite bipartite graph containing an independent set A with c vertices and an infinite independent set B, such that $A \cup B = V(K_{c,\infty})$ and each vertex in A is adjacent to every vertex in B. Note that $K_{1,\infty}$ is a star. We add an edge between each pair of vertices in A to $K_{c,\infty}$ to obtain the graph $K'_{c,\infty}$.

The countable version of part (b) of the following theorem is proved in [2]; part (a) is mentioned without proof.

Theorem 1.2. *For each positive integer c, let $\mathcal{M}_c$ be the set of graphs that consists of $K'_{c,\infty}$ and expansions of c-vertex trees. Then the following hold.*

(a) *Every graph in $\mathcal{M}_c$ is ℓ-c-connected.*
(b) *Every ℓ-c-connected graph has a minor that is isomorphic to a graph in $\mathcal{M}_c$.*
(c) *No graph in $\mathcal{M}_c$ contains another graph in $\mathcal{M}_c$ as a minor.*

In the definition of expansion, we could use zigzag ladders instead of ladders. Since zigzag ladders are not symmetric with respect to their two poles, such an expansion would not be unique for a given tree. Parts (a) and (b) in the above theorem would still be true, but we would have to modify part (c). Let us call two graphs *minor-equivalent* if each one contains the other as a minor. It is not difficult to show that all such expansions of a single tree are minor-equivalent. If we use this modified definition, statement (c) would be "if a graph in $\mathcal{M}_c$ contains another graph in $\mathcal{M}_c$ as a minor then the two graphs are minor-equivalent," which would not be as clean as the current formulation. Thus we refrain from using the zigzag ladder in our definition of expansion.

Note that Theorem 1.2 completely characterizes all unavoidable (or minimal) minors of ℓ-c-connected graphs, and it generalizes König's Infinity Lemma. In this paper, we actually prove two stronger results of which Theorem 1.2(b) is a corollary.

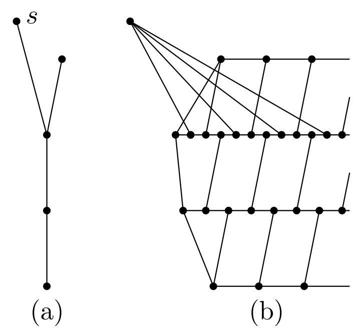


FIGURE 2. (a) Tree T. (b) The series expansion of T.

To state our next result we first define the *series expansion* of (T, S), where T is a finite tree and S is a set of leaves of T and $S \neq V(T)$. Note that S may be empty. The series expansion is basically a subgraph of the expansion of T, except that leaves not in S correspond to rays. The reader may choose to skip the following detailed definition since the idea is clearly illustrated in Figure 2.

For the purpose of avoiding notation clutter, we first describe an intermediate graph G. Let $V(T) = \{t_1, t_2, \ldots, t_n\}$ with $S = \{t_1, t_2, \ldots, t_m\}$. Let $R_i = r^i_1 r^i_2 \ldots$ be disjoint rays for $i = m+1, m+2, \ldots, n$. Then G is constructed from vertices $s_1, s_2, ..., s_m$, and rays R_{m+1}, R_{m+2}, ..., R_n by adding edges $s_i r^j_i, s_i r^j_{i+n}, s_i r^j_{i+2n}, \ldots,$ for each $t_i t_j \in E(T)$ such that $i \le m < j$, and edges $r^i_j r^j_i, r^i_{j+n} r^j_{i+n}, r^i_{j+2n} r^j_{i+2n}, \ldots,$ for each $t_i t_j \in E(T)$ such that $i, j > m$. Notice that G may have many vertices of degree at most two, all of which are incident only with edges of the rays. The graph obtained from G by contracting, one by one, the edges incident with a vertex of degree at most two is the *cosimplification* of G, which we call the *series expansion* of (T, S). Note that the resulting series expansion depends not only on T and S, but also on how vertices of T are labelled. It is straightforward to verify that all series expansions of the pair (T, S) are *series-equivalent*, meaning that any one contains the other as a topological minor. We will refer to vertices in S and $V(T) - S$ as *star vertices* and *ray vertices*, respectively. In our figures, star vertices are labelled with s and ray vertices are unlabelled.

In addition to series expansions of trees, we also need to define different versions of $K_{c,\infty}$. A tree is *branching* if it has no vertices of degree two. Let T be a finite branching tree with exactly $c \geq 3$ leaves, which are labeled $1, 2, ..., c$. The *duplication* of T is obtained by taking infinitely many disjoint copies of T and identifying the leaves that have the same label. Note that the duplication of $K_{1,c}$ is exactly $K_{c,\infty}$. For $c = 1, 2$, we will also consider $K_{1,c}$ a branching tree with c leaves, and we define its duplication to be $K_{c,\infty}$. Each duplication of a branching tree with c leaves is a *version of* $K_{c,\infty}$.

For each positive integer c, let $\mathcal{T}_c$ be the set of graphs that consists of duplications of branching trees with c leaves and series expansions of (T, S) with $|T| = c$. The following is the main result in this paper, which characterizes a complete set of unavoidable topological minors of ℓ-c-connected graphs.

Theorem 1.3. *The following hold for every positive integer c.*

- (a) *Every graph in $\mathcal{T}_c$ is ℓ-c-connected.*
- (b) *Every ℓ-c-connected graph has a topological minor that is isomorphic to a graph in $\mathcal{T}_c$.*
- (c) *If $M, N \in \mathcal{T}_c$ and $N \preceq_t M$, then M and N are series-equivalent and are both congruent to a version of $K_{c,\infty}$ or are series expansions of a pair (T, S).*

Note that 1.3(c) states that nonequivalent graphs in $\mathcal{T}_c$ are not comparable, which means that, up to equivalence, there is no redundancy in $\mathcal{T}_c$. We could define $\mathcal{T}_c$ by taking one representative from each equivalence class, which would give rise to a formulation similar to 1.2(c). Since no natural representatives are available, we leave the formulation as it is.

The following figure illustrates all pairs (T, S) for $c \leq 4$. These are finite descriptions of the unavoidable topological minors other than duplications of branching trees.

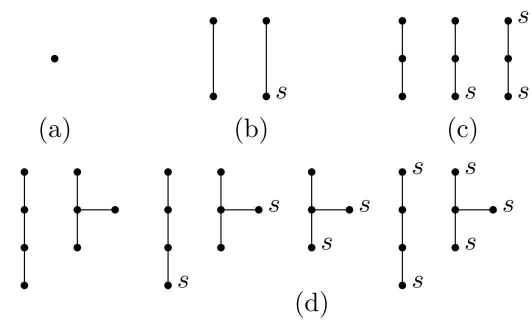


FIGURE 3. All possible pairs (T, S) for (a) $c = 1$, (b) $c = 2$, (c) $c = 3$, and (d) $c = 4$.

Our final result is a similar theorem on parallel minors. Since no vertex or edge deletions are allowed, the unavoidable structures will be expansions of general graphs, instead of trees. A spanning tree T of a finite graph is called *leaf-maximal* if the graph has no spanning tree such that its set of leaves properly contains the set of leaves of T.

We consider pairs (H, S), where H is a connected finite graph and $S \subset V(H)$. If H has one or two vertices, we require that $|S| = |H| - 1$, and we define the *expansion* of (H, S) to be a ray or a fan, respectively. If H has three or more vertices, we require that $H - S$ is a tree, $H[S]$ is a clique, and H has a leaf-maximal spanning tree with S as its set of leaves. Let $S = \{t_1, t_2, \ldots, t_m\}$ and $V(H) - S = \{t_{m+1}, t_{m+2}, \ldots, t_n\}$. The *expansion of* (H, S) is the graph consisting of vertices $s_0, s_1, s_2, \ldots, s_m$ and rays $R_{m+1}, R_{m+2}, \ldots, R_n$, with a zigzag ladder on rays R_i and R_j exactly when $t_it_j \in E(H)$, a fan on vertex s_k and ray R_l exactly when $t_kt_l \in E(H)$, an edge between any two vertices of $\{s_0, s_1, ..., s_n\}$, and an edge between s_0 and the first vertex of each ray R_i. Note that there are two ways to put a zigzag ladder onto a pair of rays, therefore there may be several different graphs that are expansions of a single pair. For any pair of graphs G and G' in such a set, $G \cong G'/Y$, where Y consists of initial segments of the rays, so we call the two graphs G and G' *parallel-equivalent*.

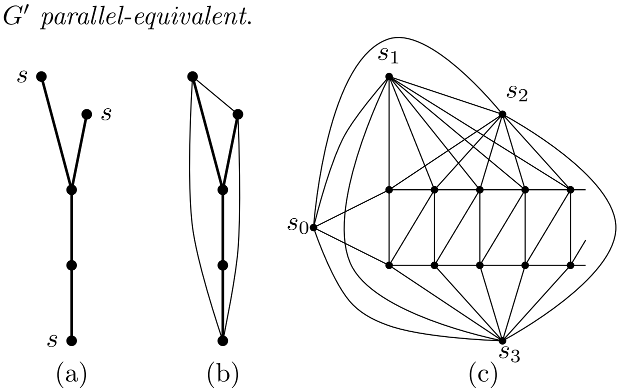


FIGURE 4. (a) Tree T with leaves S. (b) Graph $H \supseteq T$. (c) The expansion of (H, S).

For each positive integer c, let $\mathcal{P}_c$ be the set of graphs that consists of K_∞, $K'_{c,\infty}$, and expansions of (H, S), over all pairs as described in the last paragraph, such that $|H| = c$. The following is our final result, a characterization of unavoidable parallel minors of ℓ-c-connected graphs.

Theorem 1.4. *The following hold for every positive integer c.*

(a) *Every graph in $\mathcal{P}_c$ is ℓ-c-connected.*
(b) *Every ℓ-c-connected connected graph has a parallel minor that is isomorphic to a graph in $\mathcal{P}_c$.*
(c) *If $M, N \in \mathcal{P}_c$ and $N \preceq_{\|} M$, then M and N are parallel-equivalent and are congruent to $K'_{c,\infty}$, congruent to K_∞, or expansions of a pair (H, S).*

We point out that this result gives a characterization of the set of unavoidable induced minors of ℓ-c-connected graphs: besides K_∞ and $K'_{c,\infty}$, this set consists of members of $\mathcal{P}_c - \{K_\infty, K'_{c,\infty}\}$ with s_0 being deleted.

Figure 5 contains all possible graphs H for $c = 3$ and $c = 4$. Vertices in S are labelled by s. The darker edges indicate edges in a leaf-maximal spanning tree of H.

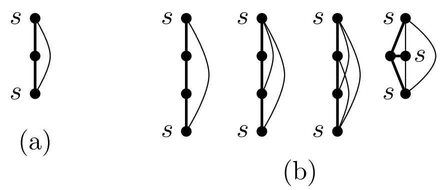


FIGURE 5. All possible pairs (H, S) for (a) $c = 3$ and (b) $c = 4$.

The rest of the paper is organized as follows. In Section 2, we prove parts (a) and (c) of our three theorems. In Section 3, we prove a result on augmenting path, which will be used in later analysis. In Section 4 and Section 5 we prove Theorem 1.3(b) and Theorem 1.4(b), respectively.

2. The qualification of unavoidable sets

We first prove that all the unavoidable graphs are ℓ-c-connected. We then address nonredundancy.

Lemma 2.1. *The series expansion of $(T, \emptyset)$ is ℓ-c-connected if T is a tree containing c vertices.*

Proof. Let T be a tree with c vertices, let G be the series expansion of $(T, \emptyset)$, and let Δ be the maximum degree of the vertices of T. Suppose that G is not ℓ-c-connected. Then, for every integer d, there is a set of fewer than c vertices that divides G into a component and a graph with more than d vertices. We prove that G is ℓ-c-connected by showing that $d = c(\Delta c)^c$ satisfies the requirements. Take vertex set V' of order at most $(c-1)$ such that $G \backslash V' = X \cup H$, where X is a component and $|H| \geq d$.

Let R_1, R_2,..., R_c be the rays of the series expansion G. An average of $\frac{d}{c}$ vertices of H are in each ray. Therefore at least one ray, say R_1, contains at least $\frac{d}{c} = (\Delta c)^c$ vertices of H. Each component of $R_1 \cap H$ is adjacent with one or two vertices in R_1, and each of these vertices is in V', thus the number of components of $R_1 \cap H$ is at most c. Ray R_1 therefore contains a path P_1 with order at least $\frac{(\Delta c)^c}{c} = \Delta^c c^{(c-1)}$. Fewer than Δ rays in G have neighbors in R_1, and each such ray neighboring R_1 contains a path with over $\frac{\Delta^c c^{(c-1)}}{\Delta} = (\Delta c)^{(c-1)}$ vertices adjacent with P_1. These neighbors are in $V' \cup H$, and since $|V'| < c$, there is a path in each ray neighboring R_1 of length at least $\frac{(\Delta c)^{(c-1)}}{c} = \Delta^{(c-1)} c^{(c-2)}$ in H.

Ray R_1 contains a path in H with length at least $\Delta^c c^{(c-1)}$. Each ray neighboring R_1 in G contains a path in H with length at least $\Delta^{(c-1)} c^{(c-2)}$. We apply the same argument to conclude that each ray adjacent to a ray neighboring R_1 contains a path in H of length at least $\Delta^{(c-2)} c^{(c-3)}$. Continuing in this fashion, a ray in G that is a distance i from R_1 contains a path in H with length at least $\Delta^{(c-i)} c^{(c-1-i)}$. Since G contains c rays, the greatest distance between R_1 and any other ray in G is $(c-1)$, therefore every ray in G will contain a path in H with length at least Δ. The graph H therefore contains vertices in each of the c rays.

Since $|X| \geq d$, we may also conclude that X meets each ray in G. Between a vertex of X and a vertex of H in a ray, there must be a vertex of V', so we conclude that V' meets every ray in G. This contradicts the fact that $|V'| < c$. □

Lemma 2.2. *Every graph in $\mathcal{M}_c \cup \mathcal{T}_c \cup \mathcal{P}_c$ is ℓ-c-connected.*

Proof. Clearly K_∞ and every version of $K_{c,\infty}$ is ℓ-c-connected. Since graphs in $\mathcal{M}_c \cup \mathcal{P}_c$ are obtained from graphs in $\mathcal{T}_c$ by adding edges, it suffices to show that every graph in $\mathcal{T}_c$ is ℓ-c-connected. Take a tree T with c vertices. Let G be a series expansion of $(T, \emptyset)$. We apply Lemma 2.1 to conclude that G is ℓ-c-connected, so there is an integer d such that any cut set of G with fewer than c vertices separates the graph into a component and a graph with at most d vertices. Let R be a ray of the series expansion G that labels a leaf of T. The vertices $V(R)$ are adjacent with the vertex set of only one other ray of G. We will show that G/R is ℓ-c-connected. Since contracting such a ray will not decrease the connectivity of the graph, we will conclude that every member of $\mathcal{T}_c$ is ℓ-c-connected, which will complete our proof.

Contract R to a vertex r and let $G' = G/R$. Take $V' \subset V(G')$, a cut set of G' with fewer than c vertices. Let X be the infinite component of $G' \backslash V'$ and let $H = G' \backslash (V' \cup X)$.

If $r \notin V'$, then $r \in V(X)$, since r is adjacent with infinitely many vertices. The cut set V' is a cut set of G, and $G \backslash V'$ consists of graph H and the infinite component with vertex set $V(X - r) \cup V(R)$. Therefore, $|H| \leq d$. Suppose $r \in V'$. By Lemma 2.1, $G' - r$ is ℓ-$(c-1)$-connected, so any vertex cut set in G' with fewer than c vertices that contains vertex r will separate G' into a component and a graph with fewer than d' vertices for some integer d' depending on G'.

The deletion of any set of fewer than c vertices from G' results in a component and a graph with fewer than $max\{d, d'\}$ vertices, and we conclude that G' is ℓ-c-connected. □

We say that a graph G is *k-disconnected*, for a positive integer k, if there is a set of finite graphs G_1, G_2,... such that G is obtained by identifying V_i, a set of $a_i \leq k$ vertices of G_i, with a_i vertices of G_{i+1} for all positive integers i. Graph $G[H]$ is the graph that G *induces* on subgraph H, that is, $G[H]$ contains the edges and vertices in H and also every edge of G with both ends in the vertices of H. We assume that the edges in $G_i[V_i]$ are identical to the edges in $G_{i+1}[V_i]$. Then G is the *k-path-sum* of $\{G_i\}_{i=1,2,...}$. Since V_i is a cut set for $i = 1, 2, \ldots$, graph G is not ℓ-$(k+1)$-connected. Observe that any minor G' of G is the k-path-sum of some sequence $\{G'_i\}_{i=1,2,...}$ such that $G'_i \preceq G_i$ for $i = 1, 2, \ldots$. We make the following observation.

Lemma 2.3. *Every minor of a k-disconnected graph is k-disconnected.*

If a graph contains $k+1$ pairwise disjoint rays in one end, then there is some V_j that meets all of them, which contradicts our assumption that $|V_j| \leq k$. We conclude with the following observation.

Lemma 2.4. *If a graph is k-disconnected, then it does not have $(k+1)$ pairwise disjoint rays.*

Let S be the set of vertices in G that are in infinitely many graphs G_i in the k-path-sum. Let $m = k - |S|$. We make the following observation.

Lemma 2.5. *Graph $G \backslash S$ is m-disconnected.*

We say that two rays R and R' are *equivalent* if $R \backslash P = R' \backslash P'$ for some finite paths P and P'. Two sets of rays $\{R_1, \ldots, R_m\}$ and $\{R'_1, \ldots, R'_m\}$ are equivalent if there is a permutation σ such that R_i is equivalent with $R'_{\sigma(i)}$ for all i. The following observation is another consequence of our structure.

Lemma 2.6. *If $|V_i| = k$ for all positive integers i and each graph G_i contains a unique set of pairwise disjoint paths from the vertices in V_i to the vertices in V_{i+1}, then let R_1, R_2,..., R_m be a set of m pairwise disjoint rays in G. If R'_1, R'_2,..., R'_m are pairwise disjoint rays of M, then $\{R'_1, R'_2, \ldots, R'_m\}$ and $\{R_1, R_2, \ldots, R_m\}$ are equivalent.*

Suppose that G has k unique pairwise disjoint rays and consider $G \backslash X$. Suppose $X \cap R$ is infinite for a ray R. Then $G \backslash X$ has at most $k-1$ pairwise disjoint rays, hence $G \backslash X$ is $(k-1)$-disconnected. We conclude the following.

Lemma 2.7. *If G has k unique pairwise disjoint rays, then the deletion of infinitely many edges from any of the k rays results in a $(k-1)$-disconnected graph.*

Take a set of m pairwise disjoint rays: R_1, R_2,..., R_m, and let S be the set of vertices in infinitely many different graphs G_i. Let Q be the set of edges in $G[V(R_1) \cup V(R_2) \cup \cdots \cup V(R_m) \cup S]$ that are not in $E(R_1) \cup E(R_2) \cup \cdots \cup E(R_m)$.

Lemma 2.8. *If set $Y \cap Q$ is infinite then G/Y is $(c-1)$-disconnected.*

Suppose not. If Y contains infinitely many edges between R_1 and R_2, then R_1 is not disjoint from R_2 in G/Y. If instead Y contains infinitely many edges between R_1 and a vertex s in S, then R_1 is not a ray in G/Y. Since any other set of m rays in G is equivalent to $\{R_1, R_2, \ldots, R_m\}$, we conclude that G/Y contains only $m-1$ pairwise disjoint rays, hence it is $(k-1)$-disconnected. We apply Lemma 2.3 and conclude that N is not ℓ-c-connected, a contradiction.

Let G_Y be the subgraph of G with no isolated vertices and with edge set exactly equal to Y. If any component of G_Y contains two or more vertices in S, then G/Y

contains fewer than $|S|$ vertices that are in infinitely many graphs G_i, hence G/Y is $(k-1)$-disconnected. We make the following observation.

Lemma 2.9. *If any component of G_Y contains two or more star vertices, then G/Y is $(k-1)$-disconnected.*

The following proof shows nonredundancy among the members of $\mathcal{M}_c$.

Proof of Theorem 1.2(c). Observe that since $K_{c,\infty}$ does not contain any ray, none of its minors contains a ray. Therefore, $K_{c,\infty}$ does not contain any other graph in $\mathcal{M}_c$ as a minor.

Take M in $\mathcal{M}_c - \{K_{c,\infty}\}$ and tree T such that M is the expansion of T. Let k be the minimal number such that M is k-disconnected and M is the k-path-sum of $\{M_i\}_{i=1,2,\ldots}$. Note that M contains infinitely many copies of T, each of which has a vertex set that is a separating set of M and which we may order into sets V_1, V_2,..., thus $k \leq c$. Since M is ℓ-c-connected, we conclude that $k = c$. Let G be the c-path-sum of G_1, G_2,... over V_1, V_2,.... Observe that each graph G_i contains a unique set of pairwise disjoint paths from the c vertices in V_i to the c vertices in V_{i+1}. Let R be the set of edges contained in the rays of M labelling the internal vertices of T. Let $Q = E(M\backslash R)$. For every edge $e = t_it_j$ of T, let Q_e be the set of edges of M that are between R_i or s_i and R_j or s_j, where R_i, R_j, s_i, s_j, t_i, and t_j are as specified in the definition of expansion. Let S be the set of star vertices of the expansion M. We apply Lemma 2.6 to G_1, G_2,... and conclude that every set of $c - |S|$ pairwise disjoint rays in G are equivalent to the rays of R.

Let $N = M\backslash X/Y$ for some N in $\mathcal{M}_c$. We apply part (a) of Theorem 1.2 to conclude that N is ℓ-c-connected. We apply Lemma 2.7 to conclude that $X \cap E(R)$ is finite, or else $M\backslash X$ is ℓ-$(c-1)$-disconnected, hence N is not ℓ-c-connected by Lemma 2.3, a contradiction.

Suppose, for some ray R_i, the set $E(R_i)\backslash Y$ is finite. If t_i is adjacent to a leaf of T, then $M/\{Y \cap E(R_i)\}$ is $(c-1)$-disconnected, and by Lemma 2.3 N is not ℓ-c-connected. If e is not adjacent to a leaf of T, then $M/\{Y \cap E(R_i)\}$ contains two ends, each with at least one ray, so $M/\{Y \cap E(R_i)\}$ is at most $(c-1)$-disconnected, and by Lemma 2.3 N is not ℓ-c-connected. In either case, we contradict our assumption and make the following observation.

Lemma 2.10. *For each ray R_i, the set $E(R_i)\backslash Y$ is infinite.*

This together with Lemma 2.7 implies that N is not isomorphic to $K_{c,\infty}$. Lemma 2.10 and Lemma 2.7 also imply that, for each ray R of the expansion M, there is a ray R' of the expansion N such that a subray of R' consists entirely of edges in R. That is, R' contains a subray of R except that some of the edges in R are in Y, hence they are contracted in N.

Suppose $Q_e\backslash X$ is finite for some edge $e \in E(T)$. If e is incident with a leaf of T, then $M\backslash X$ is $(c-1)$-disconnected, and by Lemma 2.3 N is not ℓ-c-connected. If e is not incident with a leaf of T, then $M\backslash\{Q_e \cap X\}$ contains two ends, each with at least one ray, so $M\backslash\{Q_e \cap X\}$ is at most $(c-1)$-disconnected, and by Lemma 2.3 N is not ℓ-c-connected. In either case, we contradict our assumption and make the following observation.

Lemma 2.11. *The set $Q_e\backslash X$ is infinite for all edges $e \in E(T)$.*

Lemma 2.7, Lemma 2.11, Lemma 2.10, and Lemma 2.8 together imply that N has m pairwise disjoint rays. Furthermore, Lemma 2.10, Lemma 2.8, and Lemma 2.9 together imply that every component of $G[Y]$ is finite, though $G[Y]$ may contain infinitely many components. Thus, N has precisely $|S|$ vertices of infinite degree.

If we contract all of the edges in the m pairwise disjoint rays of N then the result is a graph with finitely many vertices. Let Z be its subgraph formed by edges from infinite parallel families. The simplification of Z must be isomorphic to T. Graph N is therefore not the expansion of any tree other than T. □

We now prove part (c) of Theorem 1.3.

Proof of Theorem 1.3(c). Take positive integer c, and take $M, N \in \mathcal{T}_c$ such that $N \preceq_t M$.

Observe that no version of $K_{c,\infty}$ contains a ray, so if M is a version of $K_{c,\infty}$, then so is N. Also, observe that the number of vertices of infinite degree in a graph does not increase under the operation of topological minor, therefore if N is a version of $K_{c,\infty}$, then so is M. Thus we assume that M and N are the expansions of (T_M, S_M) and (T_N, S_N), respectively. Since $N \prec_t M$, $N = M\backslash X/Y$, where each edge $e \in Y$ is a *series edge*, that is e is incident with a vertex of degree two, in $M\backslash X/\{Y - e\}$. Since N is cosimple, it is exactly the cosimplification of $M\backslash X$.

Observe that M contains infinitely many copies of tree T_M, and the vertex set of each copy of T_M is a cut set of M. Let T_1 be a copy of T_M such that $M\backslash T_1$ is connected. Let T_i be a copy of T_M such that T_i is in the finite component of $M\backslash T_j$ for all $j > i$ and $E(T_i) \cap E(T_j) = \emptyset$ if $i \neq j$. Let G be the c-path-sum of G_1, $G_2, \ldots$, over $V(T_1)$, $V(T_2), \ldots$. Since M is ℓ-c-connected, we conclude that $k = c$. Observe that each graph G_i contains a unique set of pairwise disjoint paths from the c vertices in V_i to the c vertices in V_{i+1}. Let R be the set of edges contained in the rays of M labelling the internal vertices of T_M. Let $Q = E(M\backslash R)$. For every edge $e = t_it_j$ of T_M, let Q_e be the set of edges of M that are between R_i or s_i and R_j or s_j, where R_i, R_j, s_i, s_j, t_i, and t_j are as specified in the definition of series expansion. Let S be the set of star vertices of the expansion M and let $m = c - |S|$. We apply Lemma 2.6 to G_1, $G_2, \ldots$ and conclude that every set of m pairwise disjoint rays in G are equivalent to the rays of M labelling the vertices in $V(T)\backslash S$.

We apply Lemma 2.7 to G and conclude that $X \cap R$ must be finite. Suppose $Q_e\backslash X$ is finite for some edge $e \in E(T)$. If e is incident with a vertex in S, then $M\backslash X$ is $(c-1)$-disconnected, and by Lemma 2.3 N is not ℓ-c-connected. If e is not incident with a vertex in S, then $M\backslash\{Q_e \cap X\}$ contains two ends, each with at least one ray, so $M\backslash\{Q_e \cap X\}$ is at most $(c-1)$-disconnected, and by Lemma 2.3 N is not ℓ-c-connected. In either case, we contradict our assumption and make the following observation.

Lemma 2.12. *The set $Q_e\backslash X$ is infinite for all edges $e \in E(T)$.*

We apply Lemma 2.7 to $M\backslash X$ and conclude that $M\backslash X$ contains a subray R'_i of each ray R_i of M, and we apply Lemma 2.6 to conclude that the set $\{R_1, R_2, \ldots, R_m\}$ and $\{R'_1, R'_2, \ldots, R'_m\}$ are equivalent. We apply Lemma 2.12 to $M\backslash X$ and conclude that there is a permutation σ such that R_i has vertices adjacent with R_j if and only if $R'_{\sigma(i)}$ has vertices adjacent with $R'_{\sigma(j)}$ and each vertex that is a star of M has infinitely many neighbors in N. Since N is the cosimplification of $M\backslash X$, no edge between a star and a ray R'_i is in Y and no edge between two rays R'_i and R'_j is in Y. Furthermore, if we contract all of the edges in the m pairwise disjoint rays of N then the result is a graph with finitely many vertices. Let Z be its subgraph formed by edges from infinite parallel families. The simplification of Z must be isomorphic to T_M and the vertices labelling rays of N must be the set S_M. Therefore M and N are both expansions of (T_M, S_M). □

In the remainder of this section, we prove part (c) of Theorem 1.4.

Proof of Theorem 1.4(c). Take positive integer c. Take M and N in $\mathcal{P}_c$ that are expansions of (H_M, S_M) and (H_N, S_N), respectively, such that $N \preceq_{\|} M$. Let T_M and T_N be leaf-maximal spanning trees of H_M and H_N with leaf sets S_M and S_N, respectively. Take Y such that $N = M/Y$. Observe that M contains infinitely many copies of H_M such that the vertex set of each copy is a cut set of M. Furthermore, these cut sets may be ordered V_1, V_2,..., such that M is the c-path-sum of an infinite sequence of graphs G_1, G_2,... and $V_i = V(G_i) \cap V(G_{i+1})$. Vertex s_0 occurs in some graph, say G_1, and each graph G_i contains a copy of H_M plus some edges and vertices from the zigzag ladders in M. Graph M is c-disconnected. Let $m = c - |S|$. Since G_i contains m unique pairwise disjoint paths from V_i to V_{i+1} for each positive integer i, we apply Lemma 2.6 and conclude that any set of m rays is equivalent to $\{R_1, R_2, \ldots, R_m\}$. We will show that $H_M \cong H_N$ by showing that they have exactly the same edges.

We apply Lemma 2.5 to $M\backslash S_M$ and conclude that M has exactly m pairwise disjoint rays. Let R be the set of edges in the rays of M. Suppose $E(R_i)\backslash Y$ is finite for some ray R_i. Since N is infinite, it must be the case that M has a ray other than R_i. By Lemma 2.3, M/Y is not $(c-1)$-disconnected, so each vertex in S_M with neighbors in $V(R_i)$ must also have neighbors in another ray of M. Clearly R_i is not adjacent with two other rays of M. Since the stars of M adjacent with $V(R_i)$ are also adjacent with other rays, and R_i is adjacent with at most one, hence exactly one, other ray R_j of M, we may delete all of the edges in T_M incident with t_i except t_it_j to obtain a spanning tree of T'_M of H_M with more leaves than T_M. This contradicts the fact that T_M is leaf-maximal, and we conclude with the following observation.

Lemma 2.13. *For each ray R_i, the set $E(R_i)\backslash Y$ is infinite.*

We apply Lemma 2.10 to conclude that $E(R_i)\backslash Y$ is infinite for each ray and apply Lemma 2.8 to conclude that $Y\backslash R$ is finite. Thus the m rays in N are contractions of rays contained in M and we apply Lemma 2.6 to conclude that these sets of m rays are equivalent.

We then apply Lemma 2.9 and conclude that if star vertex $s_j \in S_M$ has infinitely many neighbors in R_i, then star vertex $s'_j \in S_N$ has infinitely many neighbors in R'_i. Furthermore, for every star $s_k \in S_M$ nonadjacent with all of the vertices of a subray of R_i, star $s'_k \in S_N$ is nonadjacent with all the vertices of a subray of R'_i. Thus, $R_is_k \in E(H_M)$ if and only if $R'_is'_k \in E(H_N)$. Another consequence of Lemma 2.9 is that $s_js_k \in E(H_M)$ if and only if $s'_js'_k \in E(H_N)$.

Let R_i' be the ray of N that contains a subray of R_i in M. We see that if M contains a zigzag ladder on R_iR_j, then N contains a zigzag ladder on $R_i'R_j'$, thus $R_iR_j \in E(H_M)$, implies that $R_i'R_j' \in E(H_N)$. On the other hand, if $R_i'R_j' \in E(H_N)$, then there is a zigzag ladder on subrays of R_i and R_j in M, thus $R_iR_j \in E(H_M)$. We conclude that $E(H_M) \cong E(H_N)$, thus $H_M \cong H_N$. □

3. Unavoidable End Behavior in Locally Finite Infinite Graphs

In this section we prove a result for augmenting paths, which will be essential for finding the unavoidable topological minors in locally finite ℓ-c-connected graphs. We begin with a stronger form of König's Infinity Lemma.

Lemma 3.1. *If G is a connected, locally finite infinite graph, then G contains an induced ray.*

Proof. Let G be a connected, locally finite infinite graph. Since G is locally finite, we apply Lemma 1.1 and conclude that G has a ray $v_1v_2....$ In addition, for each positive integer i, there exists the largest integer $n(i) > i$ such that v_i is adjacent to $v_{n(i)}$. It follows that $v_1v_{n(1)}v_{n(n(1))}...$ is an induced ray of G. □

A *comb* is a ray, called the *spine* of the comb, combined with an infinite set of pairwise disjoint, finite paths, each containing exactly one vertex in the spine, as shown in Figure 6. These finite paths are called *teeth.* Note that a path is a comb, and all its vertices are teeth. The following theorem is proved in [1].

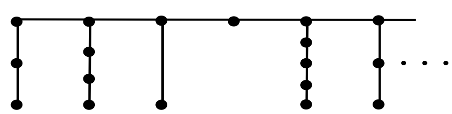


FIGURE 6. Example of a comb graph.

Theorem 3.2. *If X_1, X_2, ... are pairwises disjoint non-empty sets of vertices in a connected graph G, then G has either a comb with a tooth in each of infinitely many of these sets or a subdivided star with a leaf in infinitely many of these sets.*

We define an *end* in a graph, not to be confused with the endpoints of an edge, as it is defined in [1]: An *end* of a graph G is an equivalence class of rays in G, where two rays are considered equivalent if, for every finite set $S \subset V(G)$, both have an infinite subray in the same component of $G \backslash S$. Note that two rays are joined by infinitely many disjoint paths if and only if they are equivalent.

We now state and prove the following small lemma, which we use in the proof of the theorem later in this section.

Lemma 3.3. *If P and Q are disjoint rays in graph G joined by an infinite set Π of pairwise disjoint paths, then G contains a subdivision of a ladder with poles contained in $P \cup Q$, with an infinite subset of Π forming the rungs.*

Proof. In graph G, let P and Q be disjoint rays $p_1p_2\dots$ and $q_1q_2\dots$, respectively. Let P and Q be joined by an infinite set Π of pairwise disjoint paths, $\{P_1, P_2, \dots\}$, where P_i has ends p_{m_i} and q_{n_i}. The sequence $n_1, n_2, \ldots$ takes infinitely many values, so it contains an infinite subsequence that is strictly increasing. Take such a subsequence, $n_\alpha, n_\beta, \ldots$. The sequence $m_\alpha, m_\beta, \ldots$ takes on infinitely many values, hence it contains a strictly increasing sequence: let S be the set of the indices in this sequence. Let $\Pi' = \{P_i : i \in S\}$. The set $\Pi' \subseteq \Pi$ contains the rungs of a subdivision of a ladder with poles contained in $P \cup Q$. □

We now state and prove the main result of this section, an essential theorem concerning the locally finite case of ℓ-c-connected infinite graphs. We will use this theorem in the proof of our main result.

Theorem 3.4. *Suppose G is a locally finite, ℓ-c-connected graph, for some positive integer c. If G contains an end with $c-1$ pairwise disjoint rays, then G contains c pairwise disjoint rays in that end such that infinitely many vertices from each original ray are contained in the set of c rays.*

Proof. Observe that Lemma 1.1 implies the result when $c = 1$.

Let c be an integer greater than one. Let G be a locally finite, ℓ-c-connected infinite graph with an end containing $c-1$ pairwise disjoint rays, R_1, R_2,..., R_{c-1}, where $R_i = r_1^i r_2^i \ldots$, for $i = 1, 2, \ldots, c-1$. Take integer d such that any separating set of order $(c-1)$ divides G into an infinite component and a graph containing at most d vertices. Let $H = R_1 \cup R_2 \cup \cdots \cup R_{c-1}$. We say that vertex v *precedes* vertex w in H if the two vertices are in the same path of H and vertex v has index less than that of w.

Now consider the parts of G that are not among the $c-1$ paths. We will call each component of $G \backslash V(H)$, together with all edges incident with it in G, a *bridge*. Also, we will call each edge in G that is not in H but has both vertices in H a *bridge*. For a bridge B, we will let the *neighborhood* $N(B)$, also called the *attachments of* B, be the set of vertices in H incident with B.

Suppose there is a bridge B that contains infinitely many neighbors in H. Then, B has infinitely many neighbors in some ray. Without loss of generality we suppose it is R_1. Let S be the set of vertices in $B \backslash N(B)$ adjacent to vertices in R_1. Since G contains no vertices of infinite degree, $B - N(B)$ is connected, and we apply Theorem 3.2 to obtain a comb, C, with each tooth containing one vertex in S. Let the spine of the comb be $R_c = r_1^c r_2^c \ldots$. The teeth of the comb are an infinite set of pairwise disjoint paths between R_1 and R_c, so R_1 and R_c are in the same end of G. Thus, G meets the criteria of the lemma.

Therefore, assume that there is no bridge with infinitely many neighbors in H. A vertex pair $\{y, z\}$ *crosses* a vertex pair $\{w, x\}$ if y or z, say y, is in a finite component of $H \backslash \{w, x\}$, and z is in an infinite component, unless y precedes w, x, and z. We say that vertex set V_1 *crosses* vertex set V_2 if V_1 has a vertex pair that crosses a vertex pair in V_2. We say that bridge B_1 *crosses* bridge B_2 if vertex set $N(B_1)$ crosses $N(B_2)$. Observe that bridge B_1 may cross bridge B_2 such that B_2 does not cross B_1. We define the *crossing graph* of H in G, a simple graph, written $\chi_G(H)$, to have vertex set equal to the set of bridges, with directed edge set $\{(B_k, B_l) : B_l \text{ crosses } B_k\}$.

We will now show that $\chi_G(H)$ contains an infinite directed induced path.

If S is a set of vertices in H, then $X(S)$ is the set of vertices of highest index from each of the $c-1$ rays that are in S. The following observation can be easily verified, and the proof is omitted.

3.4.1. *If y and z are in an infinite component and a finite component of $H\backslash X(S)$, respectively, then vertex set $\{y, z\}$ crosses S unless z precedes every vertex of $S-\{z\}$.*

We will now prove the following.

3.4.2. *There exists a sequence of bridges B_1, B_2,... such that $N(B_i)$ crosses $\{N(B_1) \cup N(B_2) \cup \cdots \cup N(B_{i-1})\}$ for each positive integer i.*

We may assume that r_1^1 is not a cut vertex since, if it is, we may reassign the indices such that r_{d+2}^1 is the first vertex in the ray and path $r_1^1 r_2^1 \ldots r_{d+1}^1$ is in a bridge. The new initial vertex will not be a cut vertex, since it would divide G into a component and a graph with $d+1$ vertices, a contradiction.

If $c = 2$, then take vertex v in R_1 that is the neighbor of a bridge of R_1 and precedes every other vertex in R_1 that is the neighbor of a bridge. Take vertex w in the neighborhood of a bridge that has v as a neighbor, such that every other vertex in the neighborhood of a bridge with neighbor v precedes w. Let B_1 be the bridge with neighbors v and w.

Since w is not a cut vertex of G, there is some bridge B_2 with neighbors in both components of $R_1 - w$. By our selection of B_1, no neighbor of B_2 precedes v, so B_2 crosses B_1, by 3.4.1. Take vertex $z \in N(B_2)$ with highest index in R_1. Since z is not a cut vertex of G, there is a bridge B_3 with neighbors in both components of $R_1 - z$. Observe that the vertices in $N(B_3)$ cross $\{N(B_1) \cup N(B_2)\}$. We may continue in this way to obtain a set of bridges $\{B_1, B_2, \ldots\}$ where each set $N(B_i)$ crosses $\{N(B_1) \cup N(B_2) \cup \cdots \cup N(B_{i-1})\}$. The case $c = 2$ for 3.4.2 is complete. We now consider $c > 2$.

Since our rays are in the same end of G, if $c > 2$, then there is a bridge, B_1, with neighbors in rays R_1 and R_2 Let $S_1 = X(N(B_1))$. Note that $|S_1| \leq c-1$. Since S_1 is not a cut set of G, there is a bridge B_2 that has a neighbor in a finite component of $H\backslash S_1$, and a neighbor in an infinite component of $H\backslash S_1$. Observe that B_2 crosses B_1. Let S_2 be the set of vertices in $N(B_1) \cup N(B_2)$ with highest index in each of the $c-1$ rays of H. There is a bridge B_3 that meets a finite component and an infinite component of $H\backslash S_2$. Bridge B_3 must cross either B_1 or B_2. Let $S_i = X(N(B_1) \cup N(B_2) \cup \cdots \cup N(B_i))$. Choose B_{i+1}, a bridge with neighbors in a finite component and an infinite component of $H\backslash S_i$. This completes the proof of 3.4.2. We claim the following.

3.4.3. *Bridge B_{i+1} crosses B_1, B_2, ..., or B_i.*

By the choice of B_i for $c \geq 2$, there are y and z in $N(B_i)$ that belong to an infinite component and a finite component of $H\backslash X(N(B_1) \cup N(B_2) \cup \cdots \cup N(B_{i-1}))$. Let j be the smallest index such that z belongs to a finite component of $H\backslash X(N(B_1) \cup N(B_2) \cup \cdots \cup N(B_j))$. Clearly, $j < i$. We claim that $\{y, z\}$ crosses $N(B_j)$. By the minimality of j, vertex z belongs to a finite component of $H\backslash X(N(B_j))$. If our claim is false, then, by 3.4.1, z precedes all vertices in $N(B_j) - \{z\}$. Let P be the minimal path in H that contains all of the vertices in $N(B_j)$. By our choice of B_1, we conclude that $j \neq 1$. By induction, B_j crosses some B_k with $k < j$. It follows that some vertex v in $N(B_k)$ belongs to the interior of P, which implies that z precedes v, and thus z belongs to a finite component of $H\backslash X(N(B_k))$, contradicting the minimality of j. This completes our proof of 3.4.3.

3.4.4. *Each vertex of $\chi_G(H)$ has finitely many outflowing edges.*

Suppose 3.4.4 is not true, and vertex $B \in V(\chi_G(H))$ has infinitely many outflowing edges. Then bridge B in G is crossed by infinitely many bridges. These bridges each have an attachment in a finite component of $H\backslash N(B)$, thus a vertex in a finite component of $H\backslash N(B)$ has infinite degree in G. This contradicts our assumption that G is locally finite.

A *dipath* is a directed path. We now prove the following statement, which states that $\chi_G(H)$ has an infinite *dipath*.

3.4.5. *The sequence $B_1, B_2, \dots$ contains a subsequence $B_{n_1}, B_{n_2}, \dots$ such that, for each $i > 1$, the set $N(B_{n_i})$ has two vertices y_i and z_i such that $N(B_{n_{i+1}})$ crosses $\{y_i, z_i\}$, and $\{y_i, z_i\}$ crosses $N(B_{n_{i-1}})$.*

There are outflowing edges from B_1, such as the edge (B_1, B_2). Consider the subgraph χ' of $\chi_G(H)$ that consists of vertices $\{B_i\}$ and, for each $i > 1$, all edges (B_i, B_j) in $E(\chi_G(H))$ such that $j > i$. Note that χ' is a tree with all edges directed away from B_1. We apply 3.4.4 and conclude that the tree is locally finite. We now apply Lemma 1.1 to conclude that χ' contains the dipath $B_{n_1}, B_{n_2}, \dots$ we are looking for.

By the choice of the bridges, $(B_{n_{i+1}})$ has a vertex z_{i+1} that belongs to an infinite component of $H\backslash\{X(N(B_{n_1})) \cup \dots \cup X(N(B_{n_i}))\}$. Clearly, z_{i+1} also belongs to an infinite component of $H\backslash X(N(B_{n_i}))$. Since $B_{n_{i+1}}$ crosses B_{n_i}, there is a vertex y_{i+1} of $N(B_{n_{i+1}})$ that belongs to a finite component of $H\backslash X(N(B_{n_i}))$. Take $z_i \in X(N(B_{n_i}))$ such that y_{i+1} precedes z_i. Since $B_{n_{i+1}}$ has no vertex v that precedes all vertices in $X(N(B_{n_1}) \cup \dots \cup N(B_{n_{i-2}}))$, vertex z_i must belong to an infinite component of $H\backslash X(N(B_{n_1}) \cup \dots \cup N(B_{n_{i-1}}))$. Repeating this argument, we can find $y_i \in N(B_{n_i})$ that precedes a vertex $z_{i-1} \in X(N(B_{n_{i-1}}))$. This completes the proof of 3.4.5.

Statement 3.4.5 implies that we may assume that each B_{n_i} is a path, although since obtaining the paths may require some deletions, we sacrifice our assumption that G is ℓ-c-connected as we will not need it for the rest of the proof. For the rest of the proof we assume each bridge to be a path, and relabel the vertices of R to be $P_1 P_2 \dots$. Let y_j be the neighbor of P_j in a finite component of $H\backslash N(P_{j-1})$, and let z_j be the remaining neighbor of P_j. We show that this sequence of crossing paths and the rays in H together contain c pairwise disjoint rays. The explanation is quite technical, and the reader may see Figure 7 below for the general idea when $c = 3$.

Let k be the number of rays in H that are adjacent to vertices in the set of bridges $\{P_1, P_2, \dots\}$ in G. Without loss of generality, assume these rays to be R_1, R_2, …, R_k, and assume that the sequence of bridges P_1, P_2, … meets them in order, that is, if bridge P_i meets ray R_j, then bridges with indices at most i meet rays R_1, R_2,..., R_{j-1}. Let ϕ be a function such that $P_{\phi(l)}$ is the bridge with lowest index that has a neighbor in R_l.

We will now show that there are c pairwise disjoint rays, Q_1, Q_2, ..., and Q_c, and that these rays are in the same end of H. Let q_1^i be the vertex $r_{\phi(i)}^i$ for $i = 1, 2, \dots, k$, and let Q_i be ray R_i for $i = k+1, k+2, \dots, c-1$. Let q_1^c be $y_{\phi(k+1)}$. Observe that $z_{\phi(k+1)}$ is in an infinite component of $H \backslash \{r_{\phi(1)}^1, r_{\phi(2)}^2, \dots, r_{\phi(k)}^k, r_1^{k+1}, r_1^{k+2}, \dots, r_1^{c-1}\}$. Vertex $y_{\phi(k+2)}$ is in the same ray of H as $y_{\phi(k+1)}$ or $z_{\phi(k+1)}$. If $y_{\phi(k+2)}$ is not in the ray of H with $z_{\phi(k+1)}$, then it is in ray R_k with $y_{\phi(k+1)}$, so $P_{\phi(k+2)}$ crosses a bridge with index lower than that of $P_{\phi(k+1)}$, which contradicts our assumption. Vertex $y_{\phi(k+2)}$ is therefore in the finite component of $H - z_{\phi(k+1)}$, thus $y_{\phi(k+2)}$ precedes $z_{\phi(k+1)}$. Vertex $y_{\phi(k+2)}$ precedes $z_{\phi(k+1)}$, and is proceded by q_1^m for some $i \in \{1, 2, \dots, k\}$. For the same reason, for integer $i > \phi(k+1)$, vertex y_{i+1} will precede z_i, and y_{i+1} will not precede y_i. Furthermore, y_{i+1} will precede no vertex in $\{z_{i-1}, z_{i-2}, \dots, z_{\phi(k)}\}$. Let $Q_i = R_i$ for $i = k+1, k+2, \dots, c-1$. Path Q_i will obey the following rules for $i = 1, 2, \dots, k, c$. Vertex q_1^i has degree one in Q_i. For any vertex q_m^i, the vertex it immediately precedes is q_{m+1}^i unless $q_m^i = y_j$ for some integer $j > \phi(k)$, in which case the entire path $q_{m+1}^i q_{m+2}^i \dots q_n^i$ in P_j follows q_m^i, and $q_{n+1}^i = z_j$. Rays $Q_1, Q_2, \dots, Q_c$ in G are pairwise disjoint and this set of rays contains infinitely many vertices from each ray in $R_1, R_2, \dots, R_{c-1}$. This completes the inductive argument of our proof.

To sum up, of the original rays in H, at least $(c-1) - k$ are contained in H. A very rough sketch of the remaining $k+1$ rays is as follows. Ray Q_c includes bridge $P_{\phi(k+1)}$ and vertex $z_{\phi(k+1)}$, which is in a ray R_a of H, but the ray containing first vertex q_1^a includes the bridge that crosses $P_{\phi(k+1)}$, namely $P_{\phi(k+2)}$, and $z_{\phi(k+2)}$ in ray R_b of H. The new ray Q_b that was traveling along R_b includes the bridge $P_{\phi(k+3)}$, so it does not meet Q_a, and so on. This situation may resemble the diagram in Figure 7 if $c = 3$, in which one ray is dotted, one dashed, and the third dashed and dotted.

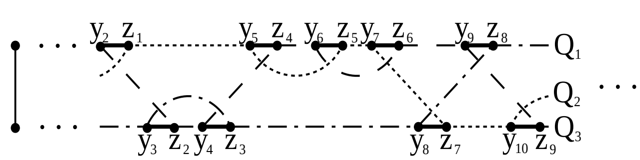


FIGURE 7. Continuation of three pairwise disjoint rays in G.

For $c = 2$, we give a rough illustration in Figure 8, in which one ray is dashed and one ray is dotted.

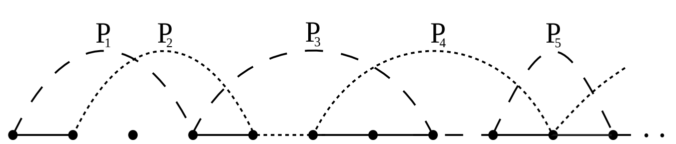


FIGURE 8. Example showing $L_\infty \preceq_t R_1 \cup \{P_1, P_2, \ldots\}$.

This completes our proof. □

4. Unavoidable Topological Minors of c-connected Infinite Graphs

For a graph G that is a subdivision of a member H of $\mathcal{T}_c$, we will say that a graph is a *direct augmentation* of G, written $G^{\oplus}$, if it contains a subdivision of a subgraph of H that is isomorphic to a subdivision of G and $G^{\oplus}$ is a subdivision of a member of $\mathcal{T}_{c+1}$.

We now prove the following theorem, which implies Theorem 1.3(b).

Theorem 4.1. *For integer c at least two, let G be a ℓ-c-connected infinite graph, and D a subdivision of a graph in $\mathcal{T}_{c-1}$ with the maximal number of star vertices among the subgraphs of G. One of the following occurs:*

1. *D contains a star vertex and G contains a graph $D^{\oplus}$; or*
2. *D is locally finite and G contains a graph Y that is a subdivision of a member of $\mathcal{T}_c$, such that Y contains infinitely many vertices from each ray of D.*

Proof. We will prove this theorem by induction on c.

Let $c = 2$, and let G be a ℓ-c-connected infinite graph. Suppose G contains a vertex v adjacent to an infinite set S of vertices. Let D be the graph with vertex set $S \cup \{v\}$ and edge set $\{vw\}_{w \in S}$. If $G - v$ contains a subdivision of a star with all of its leaves in S, then observe that G contains a subdivision of $K_{2,\infty}$, which itself contains an infinite subgraph of D and is a direct augmentation of D. Suppose not. We apply Theorem 3.2 to $N(v)$ in $G - v$ to obtain a comb C with infinitely many teeth that meet S. Observe that $D \cup C$ contains a subdivision of a fan, which is a direct augmentation of D. If G has no vertex of infinite degree, then G is locally finite. We apply Lemma 1.1 to obtain D, a ray. We then apply Theorem 3.4 to D in G to obtain R_1 and R_2, vertex disjoint rays in the same end of G that contain infinitely many vertices in $V(D)$. We apply Lemma 3.3 to R_1 and R_2 and the set of paths between them to obtain a subdivision of a ladder with poles contained in $R_1 \cup R_2$. We conclude that the theorem is true if $c = 2$. This completes the initial step of the proof by induction.

We now assume the theorem holds if $c = n$ for some integer n at least two. Let $c = n + 1$, and let G be a ℓ-c-connected infinite graph. Take D, a subdivision of a member of $\mathcal{T}_{c-1}$ with the maximal number of star vertices such that $D \subseteq G$. As an example, observe that any member of T_c that contains $k < c$ star vertices contains a subdivision of a member of T_{c-1} with k stars. We will now consider two cases.

1. D contains a vertex of infinite degree.
2. D is locally finite.

We introduce a bit of notation before addressing these cases. For any subdivision of a member of $\mathcal{T}_i$, the *bag graphs* are the components of the graph after the deletion of the star vertices and the edges in each ray. If the member contains a ray, then the bag graphs are ordered by the indices of that ray. If it contains no ray, then the bag graphs are ordered arbitrarily. The *bags* are the vertex sets of the bag graphs.

Suppose case (1) occurs. Graph D contains a star vertex v. We will show that we may augment a subgraph of $D-v$ that will form part of a direct augmentation of D. Let G_v be vertex v together with the paths from v to the rest of D. That is, let G_v be the subdivided star in G containing v such that each leaf has degree at least three in G and each interior vertex of G_v has degree two in G. Let D' be D after the deletion of the interior vertices of G_v. Observe that D' is a subdivision of a member of $\mathcal{T}_{c-2}$ and D' has the maximal number of star vertices of all such subgraphs of $G-v$. Since graph $G-v$ is ℓ-$(c-1)$-connected, we apply the induction assumption and conclude that $G-v$ contains a graph $D'^{\oplus}$ or $G-v$ contains Y, a subdivision of a member of $\mathcal{T}_{c-1}$ such that Y contains infinitely many vertices from each ray of D'. Thus $G-v$ contains a graph Y such that Y is a subdivision of a member of $\mathcal{T}_{c-1}$ and Y contains vertices from infinitely many bags of D'. We may delete the edge sets of each bag graph that contains no vertex of Y, so without loss of generality, we assume that each bag meets Y.

We will now show that G contains a graph $Y^{\oplus}$ in $Y \cup G_v$.

Observe that $\{V(G_v) \cap V(D')\}$ is infinite, therefore G_v meets infinitely many bags of D'. Since we may delete some paths in G_v and the edge sets of some bag graphs in D', we assume without loss of generality that each leaf of G_v is contained in exactly one bag of D'. Let G_{vY} be the extension of the subdivided star G_v through the bag graphs such that $G_{vY} \cap Y$ is exactly the set of leaves of G_{vY}. If G_{vY} contains infinitely many leaves in a ray R_i of Y, then observe that $G_{vY} \cup Y$ contains a direct augmentation of Y that is also a direct augmentation of D, as desired. Suppose not. Let $Q_{t_i t_j}$ be the set of paths between star s_i or ray R_i and star s_j or ray R_j. Graph G_{vY} must contain infinitely many leaves in $Q_{t_i t_j}$ for some integers i and j. Observe that $G_{vY} \cup Y$ contains a direct augmentation of Y that is also a direct augmentation of D, as desired.

By the preceding argument, we have shown that the theorem holds if D contains a vertex of infinite degree. Suppose this is not the case. Then case (2) occurs and D is locally finite.

It follows that G is locally finite, and we apply Lemma 3.4 to obtain c rays, R_1, $R_2, \ldots, R_c$, in G, which contain infinitely many vertices from each ray of D. We conclude this proof with the following lemma.

Lemma 4.2. *The series expansion of $(T, \emptyset)$, for some c-vertex tree T, is contained in G and has rays contained in $\{R_1 \cup R_2 \cup \cdots \cup R_c\}$.*

Proof. Between each pair of rays are infinitely many pairwise disjoint paths, since they are in the same end. Observe that some pair of rays, say R_1 and R_2, is joined by infinitely many pairwise disjoint paths that meet none of the other rays. Let H_1 be the subgraph of G containing R_1, R_2, and an infinite set Π_1 of pairwise disjoint paths that join them but meet none of the other rays. There is a ray, say R_3, such that G contains infinitely many pairwise disjoint paths between R_3 and H_1 that meet none of the remaining rays. Let H_2 be the union of R_3, H_1, and an infinite set Π_2 of pairwise disjoint paths that join them but meet none of the other rays. We may continue in this way all the way through, finally adding R_c to H_{c-1} with an infinite set Π_{c-1} of pairwise disjoint paths that join them.

We apply Lemma 3.3 to R_1, R_2, and Π_1 to obtain a subdivided ladder L_1 in their union, with poles contained in $R_1 \cup R_2$ and rungs contained in Π_1. For simplicity, we will assume Π_1 to be the set of rungs of L_1, and let the paths be labelled $\{P_1^1, P_2^1, \dots\}$ such that, for positive integers i and j, the vertex in $R_1 \cap P_i^1$ precedes $R_1 \cap P_j^1$ in R_1 if $i < j$. Observe that an infinite subset of the paths in Π_2 from R_3 to H_1 either meet L_1 or may be extended through the members of Π_1 that are not in L_1 to meet L_1 in R_1 or R_2. We therefore assume for simplicity that each member of Π_2 meets L_1. If infinitely many members of Π_2 meet L_1 in a pole R_{i_1}, then we apply Lemma 3.3 to R_3, R_{i_1}, and Π_2 to obtain a ladder L_2 with poles in $R_3 \cup R_{i_1}$ and rungs in Π_2. We again assume, for simplicity, that each member of $\Pi_2 = \{P_1^2, P_2^2, \dots\}$ is a rung in L_2. Since we may delete some of the rungs to ensure that the rungs that meet R_{i_1} alternate from L_1 to L_2, that is, $P_1^1 \cap R_{i_1}$ precedes $P_1^2 \cap R_{i_1}$, which precedes $P_2^1 \cap R_{i_1}$, which precedes $P_2^2 \cap R_{i_1}$, and so on; we assume that the rungs of L_1 and L_2 alternate in this way.

If, instead, infinitely many members of Π_2 meet L_1 in the paths in Π_1, then we may assume that the members of Π_2 meet each path in Π_1 exactly once. We may remove P_{2i}^1 from L_1 for $i \in \mathbb{N}$ and extend the members of Π_2 that meet them along the paths P_{2i}^1 to R_1. We may also remove each member of Π_2 that meets a path P_{2j-1}^1 for $j \in \mathbb{N}$, and obtain an infinite set of pairwise disjoint paths from R_3 to R_1 in an infinite subladder of L_1. In this case, we may apply Lemma 3.3 as before.

By repeating this argument $c - 3$ more times, we can attach each ray R_k onto the growing infinite graph to ultimately obtain an infinite graph H with an ∞-representation that is a tree with c ray vertices. Furthermore, the rays of H are contained in $\{R_1 \cup R_2 \cup \cdots \cup R_c\}$, which contain infinitely many vertices from each ray of D, so H contains infinitely many vertices from each ray of D. □

This concludes our proof. □

5. Unavoidable Parallel Minors of ℓ-c-connected Infinite Graphs

For the proof in this section, we will need the following lemma, which is one application of "Ramsey's Theorem A" from Reference [3], stated and proved therein.

Lemma 5.1. *If G is an infinite graph, then G has an induced subgraph isomorphic to K_∞ or $\overline{K_\infty}$.*

In the remainder of this paper, we prove Theorem 1.4(b).

Proof of Theorem 1.4. Take positive integer c. Let G be a ℓ-c-connected infinite graph that contains no minor isomorphic to K_∞. Graph G contains an infinite component, so we may ignore the finite components of G and assume that G is connected. We apply Theorem 1.2(b) to obtain a minor of G in $\mathcal{M}_c$. Let M be the minor of G in $\mathcal{M}_c$ containing the most star vertices and let $M = G\backslash X/Y$, where M spans G/Y.

If $M \cong K_{c,\infty}$, then we may add some edges to Y to obtain Y' such that $G\backslash X/Y' = M' \cong K'_{c,\infty}$. Since K_∞ is not a minor of G, K_∞ is not a subgraph of G/Y', thus we apply Lemma 5.1 to obtain an infinite independent set $A \subset V(G/Y')$. Let S be the set of star vertices in M'. Take $s \in S$. We contract the edges in G/Y' between s and each vertex in $V(M')\backslash\{S\cup A\}$ to obtain a parallel minor of G isomorphic to $K'_{c,\infty}$.

Suppose then that M is not isomorphic to $K_{c,\infty}$. Then M is the expansion of some tree T. Let S be the set of leaves of T. It is simple to add edges to Y to obtain Y' such that M/Y' is the expansion of (T,S). That is, $G\backslash X/Y'$ is isomorphic to M with a complete graph on the star vertices, a vertex s_0 that is adjacent with each star and the first vertex of each ray, and a zigzag ladder between each pair of ladder poles in M. Now, let $M' = G\backslash X/Y'$. Take H, S, and T such that M' is the expansion of (H,S) and T is a leaf-maximal spanning tree of H with leaf set S. Consider the edges X in G/Y'.

For each vertex pair $\{t_i,t_j\}$ of $V(T)$, let $Q_{t_it_j}$ be the set of edges in G/Y' between R_i or s_i and R_j or s_j. We say that each edge in $Q_{t_it_j}$ is *between the vertex pair* t_i *and* t_j. Let n be the number of vertex pairs of $V(H)$ that are not edges of H such that X contains edges between the vertex pair. We prove the theorem by induction on n. If $n=0$, then $X=\emptyset$ and the expansion of (H,S) is a parallel minor of G and the theorem holds. Suppose the theorem holds for $(n-1)$.

Suppose that G/Y' contains edges between n vertex pairs of $V(H)$ that are not edges of H. Take one such vertex pair $\{t_i,t_j\}$.

If $Q_{t_it_j}$ is finite, then take a vertex r_l^k from a ray R_k incident with an edge in $Q_{t_it_j}$ such that no edge in $Q_{t_it_j}$ is incident with a vertex r_b^a such that $b>l$. Take star vertex s of M'. For each ray R_a, we contract the path $sr_1^ar_2^a\dots r_l^a$ to vertex s to eliminate the edges in $Q_{t_it_j}$ and obtain a graph that contains M' and has edges between at most $(n-1)$ vertex pairs of $V(H)$ that are not edges of H. We apply the inductive hypothesis and conclude that the theorem holds.

Suppose then that $Q_{t_it_j}$ is infinite. The following three cases are exhaustive:
(1) $t_i=R_i=t_j$;
(2) $t_i=R_i$ and $t_j=s_j$; or
(3) $t_i=R_i\neq t_j=R_j$.

For the rest of the proof, it will be convenient to let $E(r_lr_{l+1})$ denote the edge set $\{r_l^kr_{l+1}^k : R_k \text{ is a ray of } M'\}$.

Suppose Case (1) occurs. Let R' be the graph that $Q_{t_it_j}$ induces on $V(R_i)$. If R' contains a vertex r of infinite degree, then we contract the edge sets $E(r_lr_{l+1})$ if and only if $r_l^i\notin N(r)$, where $N(r)$ is the neighborhood of vertex r. Observe that r is a star of the resulting graph, thus G contains a minor in $\mathcal{M}_c$ with more star vertices than M, a contradiction. We make the following observation, where S is the set of stars of M'.

5.1.1. *The graph that edge set $Q_{t_it_j}$ induces in $M'\backslash S$ is locally finite.*

If R' is locally finite, then let $r_1^i=r_{n_1}$. Let r_{n_2} be the vertex with highest index among the neighbors of r_{n_1} in R'. Let r_{n_i} be the vertex with highest index that is a neighbor of a vertex in the path $r_{n_{i-2}}r_{n_{i-2}+1}\dots r_{n_{i-1}}$. We contract the edge set $E(r_lr_{l+1})$ if and only if $l\notin\{n_1,n_2,\dots\}$. Observe that by these contractions in R', we contract each edge of $Q_{t_it_j}$ to a single vertex. In this way, we obtain a parallel minor of G that contains a copy of M' and has edges between at most $(n-1)$ vertex pairs of $V(H)$ that are not edges of H. We apply the inductive hypothesis and conclude that the theorem holds. We therefore assume that Case (1) does not occur.

Suppose Case (2) occurs: $t_i = R_i$ and $t_j = s_j$. We contract the edge set $E(r_l r_{l+1})$ if and only if $l \notin N(s_j)$ to obtain the expansion of $(H \cup t_i t_j, S)$. Tree T is a leaf maximal spanning tree, and we obtain a parallel minor of G that contains a copy of M' and has edges between at most $(n-1)$ vertex pairs of $V(H)$ not in $E(H \cup \{t_i t_j\})$. We apply the inductive hypothesis and conclude that the theorem holds. We also make the following observation.

5.1.2. *If a star s is adjacent with infinitely many vertices in a ray R_i in Z, then we may assume s to be adjacent with every vertex in R_i.*

We therefore assume that Case (2) does not occur.

Suppose Case (3) occurs: $t_i = R_i \neq t_j = R_j$. We apply 5.1.1 and conclude that $Q_{t_i t_j}$ contains no infinite set of edges adjacent with a single vertex, thus $Q_{t_i t_j}$ contains an infinite set Π of pairwise non-adjacent edges.

The following argument is technical and amounts to obtaining a zigzag ladder on R_i and R_j. We break up the edge set $E(r_l r_{l+1})$ into two sets. Edge $t_i t_j$ is a cut edge of tree T and divides the graph into a component containing t_i and a component containing t_j. Let $E_i(r_l r_{l+1})$ be the set of edges corresponding to the edges in $E(r_l r_{l+1})$ that are in the rays labelling vertices in the component of $T \backslash t_i t_j$ containing t_i. Let $E_j(r_l r_{l+1})$ be the set of edges $E(r_l r_{l+1}) \backslash E_i(r_l r_{l+1})$. We apply Lemma 3.3 to obtain L, a subdivided ladder with poles in R_i and R_j and with rung set ρ in Π. This allows us to assume that, for every integer $k > 0$, we may find a rung in ρ with ends in the infinite components of $R_i - r^i_k$ and $R_j - r^j_k$. Let $i_1 = 1$. Let j_1 be the lowest index such that $r^j_1 r^j_2 \dots r^j_{j_1}$ has a neighbor in $R_i - r^i_1$ and $j_1 \geq m$ for each vertex r^j_m adjacent with $r^i_{i_1}$. For $n = 2, 3, \dots$, let i_n be the lowest index such that $i_n > m$ for each vertex r^i_m adjacent with a vertex in $r^j_1 r^j_2 \dots r^j_{j_{n-1}}$ and $r^i_{i_{n-1}+1} r^i_{i_{n-1}+2} \dots r^i_{i_n}$ has a neighbor in the infinite component of $R_j - r^j_{n-1}$; and let j_n be the lowest index such that $j_n \geq m$ for each vertex r^j_m adjacent with a vertex in $r^i_1 r^i_2 \dots r^i_{i_n}$ and $r^j_{j_{n-1}+1} r^j_{j_{n-1}+2} \dots r^j_{j_n}$ has a neighbor in the infinite component of $R_i - r^i_n$. Contract edge set $E_i(r_l r_{l+1})$ if and only if $l \notin \{i_1, i_2, \dots\}$ and contract edge set $E_j(r_l r_{l+1})$ if and only if $l \notin \{j_1, j_2, \dots\}$ to obtain a zigzag ladder on R_i and R_j. Let Z be the resulting graph. Observe that the graph that Z induces on rays R_i and R_j is a zigzag ladder.

If $t_i t_j \in E(T)$, then Z is the expansion of (H, S), and the theorem holds.

If $t_i t_j \notin E(T)$, then $T \cup R_i R_j$ contains a cycle $C = R_{k_1} R_{k_2} \dots R_{k_l}$ of interior vertices, where $k_1 = i$ and $k_2 = j$. Observe that T is not leaf-maximal in $H \cup t_i t_j$. We will show that G contains a member of $\mathcal{M}_c$ with more star vertices than M and obtain a contradiction. We begin by identifying a set of l rays in Z each of which contains infinitely many vertices of each ray in this cycle. Since there are two different ways of expressing a zigzag ladder between two rays, we will have to be careful with this construction. Let $\phi(a)$ be equal to one if $r^{k_a}_1 r^{k_{a+1}}_2 \in E(Z)$, where we say that $l+1=1$, otherwise $\phi(a)=0$. Let $\Sigma(a)=1+\sum_{m=1}^{a}\phi(m)$. Let ray R'_1 be $r_1^{k_1}r_{\Sigma(1)}^{k_2}r_{\Sigma(2)}^{k_3}r_{\Sigma(3)}^{k_4}\ldots$. For $m=2,3,\ldots,l$, let

$$R'_m = r_1^{k_m} r_{\Sigma(m)}^{k_{m+1}} r_{\Sigma(m+1)}^{k_{m+2}} r_{\Sigma(m+2)}^{k_{m+3}} \cdots$$

Observe that these l rays are pairwise disjoint and each contains infinitely many vertices of each of the l original rays of Z. The graph that Z induces on each pair of rays R'_m and R'_{m+1}, where $l+1=1$, is a zigzag ladder. We also conclude the following.

5.1.3. *Every ray and star labelling a vertex of H with infinitely many neighbors in R'_1 contains infinitely many neighbors in R'_m for $m=2,3,\ldots,l$.*

We will now show that Z/R'_1 is ℓ-c-connected. We will deduce that G contains a minor in $\mathcal{M}_c$ with more star vertices than M, a contradiction that will conclude our proof.

Let S_Z be the star set of Z. We will show that R'_1 is not a cut set of $Z\backslash S_Z$ and that no star has infinitely many neighbors only in R'_1 and conclude that we may contract R'_1 without losing ℓ-c-connectivity. Let R be a ray containing infinitely many vertices adjacent with R'_1. Apply 5.1.3 and conclude that R has infinitely many neighbors in R'_2. We apply Lemma 3.3 and conclude that the graph that Z induces on $R\cup R'_1$ contains a subdivision of a ladder. Let s be a star with infinitely many neighbors in R'_1. We apply 5.1.3 and conclude that s is adjacent to an infinite subset of vertices in R'_2, and we may apply 5.1.2 to this pair and assume that s is adjacent to each vertex in R'_1. We contract ray R'_1 in Z to obtain an ℓ-c-connected graph that contains a member of $\mathcal{M}_c$ with more star vertices than M, a contradiction. We may assume that Case (3) does not occur. This concludes our proof. □

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