UNAVOIDABLE PARALLEL MINORS OF 4-CONNECTED GRAPHS

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ABSTRACT. A *parallel minor* is obtained from a graph by any sequence of edge contractions and parallel edge deletions. We prove that, for any positive integer k, every internally 4-connected graph of sufficiently high order contains a parallel minor isomorphic to a variation of $K_{4,k}$ with a complete graph on the vertices of degree k, the k-partition triple fan with a complete graph on the vertices of degree k, the k-spoke double wheel, the k-spoke double wheel with axle, the $(2k+1)$-rung Möbius zigzag ladder, the $(2k)$-rung zigzag ladder, or K_k. We also find the unavoidable parallel minors of 1-, 2-, and 3-connected graphs.

1. INTRODUCTION

In this paper, we will explore unavoidable parallel minors in 1-, 2-, 3-, and internally 4-connected graphs of large order, building on the results of [1] in the last two cases. A graph is *internally c-connected* if it is $(c-1)$-connected and every separating set of order $(c-1)$ divides the graph into exactly one component and one single vertex. Since we only consider vertex connectivity in this paper, we may, without loss of generality, restrict our attention to simple graphs, which are graphs containing no loops or parallel edges.

We will begin by defining some terms and establishing a convenient notation for use throughout this paper. All other graph terminology and notation is defined in [2]. In particular, we denote a subgraph H of graph G as in [2] by the notation $H \subseteq G$. We say that a graph M is a *parallel minor* of a graph G, written $M \preceq_{\parallel} G$, if M is obtained by contracting some edges of G, then contracting all loops and deleting multiple edges to yield a simple graph. A graph N is a *minor* of a graph G, written $N \preceq G$, if N is a subgraph of a parallel minor of G. We use the notation $\Phi(G, N)$ to refer to the set $\{M \preceq_{\parallel} G : N \subseteq M \text{ and } |N| = |M|\}$, where $|N|$ is the order of N. Throughout this paper, note that, in order to ensure that $\Phi(G, N)$ is nonempty, N must contain exactly one component in each component of G. Since we will only use this notation in the context of a connected minor of a connected graph, we will not need to worry about this qualification. Observe that N can be obtained from any member of $\Phi(G, N)$ by deleting edges. Conversely, a member of $\Phi(G, N)$ is the graph N with extra edges.

The following statement of a Ramsey theorem will be used several times in this paper. This theorem tells us what induced subgraphs to expect in large graphs.

Theorem 1.1. *There is a function $f_{1.1}$ such that, for any natural number k, every graph with order at least equal to the integer $f_{1.1}(k)$ contains an induced subgraph isomorphic to K_k or $\overline{K_k}$.*

We will also use the following two theorems concerning 3- and internally 4-connected graphs. In these theorems, and throughout this paper, W_k is a wheel with k spokes, D_k is a double-wheel with k spokes at each hub, M_k is a Möbius zigzag ladder with $2k+1$ rungs, and Z_k is a zigzag ladder will $2k$ rungs (see Figure 3). These theorems are the main results of [1], and are restated for our purposes as follows.

Theorem 1.2. *There is a function $f_{1.2}$ such that, for any integer k exceeding two, every 3-connected graph with order at least equal to the integer $f_{1.2}(k)$ contains a minor isomorphic to W_k, or $K_{3,k}$.*

Theorem 1.3. *There is a function $f_{1.3}$ such that, for any integers q and r exceeding three, every internally 4-connected graph with order at least equal to the integer $f_{1.3}(q,r)$ contains a minor isomorphic to $K_{4,q}$, D_q, M_r, or Z_r.*

Note that Theorems 1.2 and 1.3 give the sets of unavoidable minors of large 3- and 4-connected graphs, respectively. These theorems are an integral step in our determination of the unavoidable parallel minors of 3 and 4-connected graphs.

The four main results of this paper give the sets of unavoidable parallel minors of 1-, 2-, 3-, and 4-connected graphs. The families of graphs that we introduce in the figures for use in Theorems 1.5, 1.6, and 1.7 are discussed in Section 2, and referred to throughout this paper. Our results build on one another, and may be stated as follows.

Theorem 1.4. *There is a function $f_{1.4}$ such that, for any positive integer k, every connected graph with order at least equal to the integer $f_{1.4}(k)$ contains a parallel minor isomorphic to $K_{1,k}$, C_k, P_k, or K_k.*
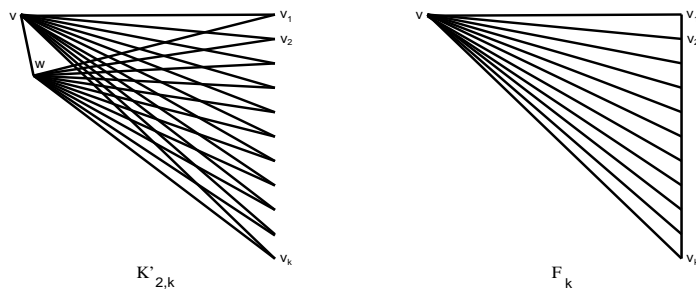

FIGURE 1. Families of 2-connected graphs.

Theorem 1.5. *There is a function $f_{1.5}$ such that, for any integer k exceeding two, every 2-connected graph with order at least equal to the integer $f_{1.5}(k)$ contains a parallel minor isomorphic to $K'_{2,k}$, C_k, F_k, or K_k.*
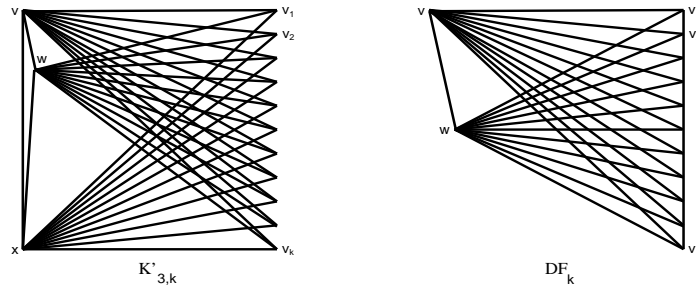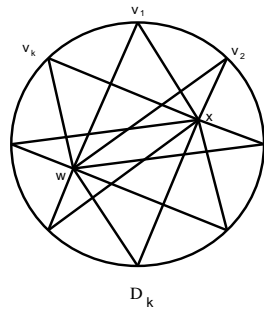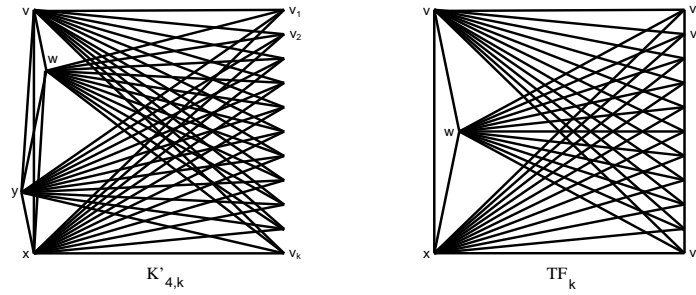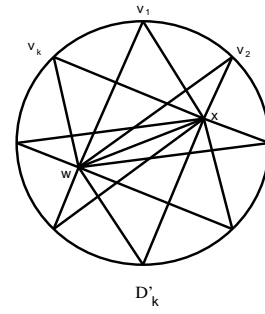
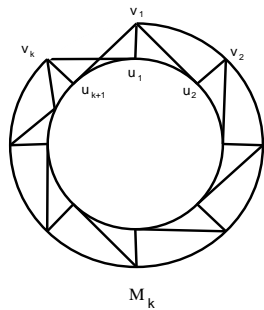FIGURE 2. Families of 3-connected graphs.

Theorem 1.6. *There is a function $f_{1.6}$ such that, for any integer k exceeding three, every 3-connected graph with order at least equal to the integer $f_{1.6}(k)$ contains a parallel minor isomorphic to $K'_{3,k}$, W_k, DF_k, or K_k.*
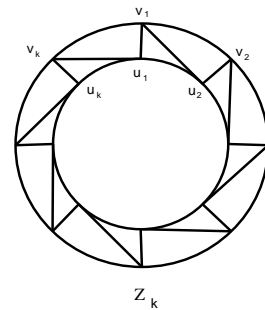

FIGURE 3. Families of 4-connected graphs.

Theorem 1.7. *There is a function $f_{1.7}$ such that, for any integer k exceeding four, every internally 4-connected graph with order at least equal to the integer $f_{1.7}(k)$ contains a parallel minor isomorphic to $K'_{4,k}$, D_k, D'_k, TF_k, M_k, Z_k, or K_k.*

Observe that the minors listed in Theorem 1.2 almost form a subset of the minors listed in Theorem 1.6, and likewise for Theorem 1.3 and Theorem 1.7.

2. Families of 1-, 2-, 3-, and Internally 4-Connected Graphs

We will not prove any theorems in this section: instead, we will provide a motivation for the specific families of graphs chosen to comprise our sets of unavoidable parallel minors in our variously connected graphs. The reader may feel free to turn directly to the internally 4-connected result, proved in Section 6, which assumes the 3-connected result, with the understanding that vertex labeling throughout this paper will follow Figures 1, 2, and 3.

We may have chosen to include the families of graphs from Theorem 1.4 in the list for Theorem 1.5, since every 2-connected graph is 1-connected. Observe, however, that each family in the unavoidable set stated in Theorem 1.5 is 2-connected. Likewise, Theorem 1.6 gives a list of families of 3-connected graphs and Theorem 1.7 gives a list of families of internally 4-connected graphs. We will show each family to be necessary among the unavoidable c-connected parallel minors of c-connected graphs.

Consider the 2-connected family of graphs, $\{F_k\}_{k>2}$. No large parallel minor of a member of this family is 2-connected, unless it is another member of this same family. This is true of each family of graphs listed in Theorem 1.5. Therefore, no family listed contains another in the list. The same statement can be made with respect to the 1-connected graphs listed in Theorem 1.4, the 3-connected graphs listed in Theorem 1.6, and the internally 4-connected graphs listed in Theorem 1.7. We leave it to the reader to convince himself or herself of this fact.

We therefore conclude that any set of 1-connected graphs that comprise an unavoidable set of parallel minors of large, 1-connected graphs must contain the four families $K_{1,k}$, C_k, P_k, and K_k. Similarly, each family in each of the three other sets is necessary. The rest of this paper will prove that each set stated in Theorems 1.4, 1.5, 1.6, and 1.7 is sufficient.

We make a special remark on Theorem 1.7. Notice that all the listed parallel minors are 4-connected. Since 4-connectivity implies internal 4-connectivity, Theorem 1.7 still holds if we replace internal 4-connectivity with 4-connectivity. In other words, the listed graphs are not only unavoidable in large internally 4-connected graphs, they are also unavoidable in large 4-connected graphs.

The unavoidable parallel minors of large, variously connected graphs are significant both because parallel minors are interesting, and because this work complements work done on unavoidable topological minors, the dual operation of parallel minor. Unavoidable, topological minors of large, 1- and 2-connected graphs are discussed in [2]. For 3- and internally 4-connected graphs, the unavoidable topological minors are described in [1].

3. Some Parallel Minors in Connected Graphs

In this section, we prove a weak result for 1-connected graphs, as a step towards proving Theorem 1.4. We begin this section by stating a simple proposition, which provides a set of minors we cannot avoid in a large, connected graph. The proof is left to the reader, who may note that the proposition still holds when the word "minor" is replaced with the word "subgraph".

Proposition 3.1. *There is a function $f_{3.1}$ such that, for positive integers r and q, every connected graph of order at least equal to integer $f_{3.1}(r, q)$ contains $K_{1,r}$, or P_q as a minor.*

We begin the proof of our connected result by proving the following lemma. Theorem 1.4 will be proved in the next section.

Lemma 3.2. *There is a function $f_{3.2}$ such that, for positive integers k and l, a connected graph G with order at least equal to integer $f_{3.2}(k, l)$ contains a parallel minor isomorphic to $K_{1,k}$, P_k, or K_k; or, G has a 2-connected graph of order at least l as a parallel minor and has no minor isomorphic to $K_{1,r}$, where $r = f_{1.1}(k)$.*

Proof. Let k and l be positive integers. We will now select our variables in a particular way to ease the later steps in the proof. Let $f_{1.1}$ and $f_{3.1}$ be the functions described in Theorem 1.1 and Proposition 3.1 respectively. Let $r = f_{1.1}(k)$, let $q = l(k + 1)$, and let $s = f_{3.1}(r, q)$. Set $f_{3.2}(k, l) = s$. Let G be a connected graph with order at least s.

Apply Proposition 3.1 to divide our proof into the following two cases, which are exhaustive. Case 1: Graph G contains a minor isomorphic to $K_{1,r}$. Case 2: Graph G contains no minor isomorphic to $K_{1,r}$, and G contains a minor isomorphic to P_q.

If G meets the conditions of Case 1, then take $M \preceq G$ such that M is isomorphic to $K_{1,r}$. Fix $H \in \Phi(G, M)$. Take vertex $v \in V(H)$ with degree r. By Theorem 1.1, the graph $H - v$ has an induced subgraph isomorphic to K_k or $\overline{K_k}$.

If $H - v$ has an induced subgraph isomorphic to K_k, then H has a parallel minor isomorphic to K_k. Assume, therefore, that $H - v$ has an induced subgraph S isomorphic to $\overline{K_k}$. In H, vertex v is adjacent to every other vertex. Contract each edge vu, where $u \notin V(S)$, deleting the multiple edges and loops, to obtain a parallel minor isomorphic to $K_{1,k}$, which completes Case 1.

If G meets the conditions of Case 2, then G has no minor isomorphic to $K_{1,r}$, and we take $M \preceq G$ such that M is isomorphic to P_q. Fix $H \in \Phi(G, M)$. Let V_c be the set of cut vertices of H.

If $|V_c| \geq k + 1$, then let H' be obtained recursively from H by contracting, one by one, each edge that is incident with a vertex not in V_c and deleting loops and multiple edges. The parallel minor H' is isomorphic to a path of length at least k, hence G has a parallel minor isomorphic to P_k. We are not finished with Case 2, however, since H may have fewer than $k + 1$ cut vertices.

If $|V_c| < k + 1$, then there is a large piece of H between cut vertices. Let N be a 2-connected subgraph of H of highest order. Subgraph N is an end of H or a piece of H between two vertices of V_c, so there are at most $k + 1$ places in H to find N. The order $|N|$ is therefore at least $\frac{q}{k+1} = l$. Let H' be the parallel minor of G obtained from H by contracting, one by one, each edge not in N. The graph $H' \preceq_\| G$ is 2-connected and has order at least l. This completes the proof of Lemma 3.2. □

4. Unavoidable Parallel Minors of 1- and 2-Connected Graphs

We will prove two lemmas before proving the main lemma, Lemma 4.4, of this section. We begin by establishing some notation.

Let M be a minor of G, where $M = G/X \backslash Y$. Take an edge e in M. Let S be the set of edges in the multigraph $M \cup Y$ that are parallel with e. We say that S is the set of edges in G that are in a parallel class with e in M.

For a Hamiltonian parallel minor M of G and a Hamilton cycle C of M, the following statements describe an H-set. Let P be a path in M along C such that each vertex of P has degree two in M except for one endpoint of P, which may have degree exceeding two. Let e be an edge of P. Let S be a set of edges in G that belong to the same parallel class as edge e in M. The quintuple (M, C, S, P, e) is an H-set. Furthermore, we say that the H-set (M', C', S', P', e) is an H-minor of the H-set (M, C, S, P, e), which we write as $(M', C', S', P', e) \preceq_H (M, C, S, P, e)$, if the following conditions hold.
  (1) $E(C') \subseteq E(C)$.
  (2) The graph M' is obtained from M by contracting all edges in $E(C - E(C'))$.
Observe that C' is a Hamilton cycle of M', and that the H-minor relation is transitive.

We say that the *weight* of an H-set is the pair $(|S|, |P|)$. We say that weight $(|S|, |P|)$ is greater than weight $(|S'|, |P'|)$ if $|S| > |S'|$, or if $|S| = |S'|$ and $|P| > |P'|$.

We will now prove a helpful lemma, which will give us the conditions for finding a longer induced path or a larger parallel set in a Hamiltonian graph by using the H-set construction.

Lemma 4.1. *For positive integers d and k and for a graph G, if (M, C, S, P, e) is an H-set such that $|M| > dk$ and $\Delta(M) < d$, then $|P| \geq k$, or H-set (M, C, S, P, e) has an H-minor (M', C', S', P', e) of greater weight such that $|M'| > \frac{|M|}{d}$.*

Proof. Let d and k be positive integers. Let (M, C, S, P, e) be an H-set of weight $(|S|, |P|) = (\sigma, \pi)$ such that $|M| = n > dk$, $\Delta(M) < d$, and $\pi < k$. By hypothesis, C is the Hamilton cycle of M, e is an edge in P, which is contained in C, and S is a set of σ edges in G that are in a parallel class with e in M. Order the vertices of $C = v_1 v_2 \ldots v_n$ such that the path $P = v_1 v_2 \ldots v_\pi$, where $d_M(v_i) = 2$ for $i = 1, 2, \ldots, (\pi - 1)$. Let $e = v_a v_{a+1}$.

Consider the neighbors of v_π in M. The vertices in $\{v_\pi\} \cup N(v_\pi)$ divide the

cycle C into at most d path segments, since v_π has fewer than d neighbors. There must be a path $v_l v_{l+1} \ldots v_{m-1} v_m$ of length at least $\frac{n}{d} > k$ along C, with ends in $\{v_\pi\} \cup N(v_\pi)$ and no other vertices in that set. With the following vertex indices, addition is computed modulo n.

In the case where the long path segment contains $P - v_\pi$, index m is equal to $\pi - 1$, and we do the following operations. Let M' be obtained from M by the contraction of the path $v_{\pi+1} v_{\pi+2} \ldots v_{l-1} v_l$ to the vertex v_l; let C' be the cycle C after this contraction; let $S' = S$; and let $P' = v_1 v_2 \ldots v_\pi v_l$. The H-set (M', C', S', P', e) has weight $(\sigma, \pi + 1)$ and $(M', C', S', P', e) \preceq_H (M, C, S, P, e)$, which is what we wanted to show.

We can therefore assume that the long path segment does not meet path P. In this case, take $f \in E(G)$ such that f is represented by the edge $v_\pi v_m$. Let $S' = S \cup \{f\}$. We obtain M' from M by performing the following contractions.

(1) Contract the path $v_m v_{(m+1)} \ldots v_{a-1} v_a$ to vertex v_a.
(2) Contract the path $v_{a+1} v_{a+2} \ldots v_\pi$ to v_{a+1}.
(3) Contract the path $v_{\pi+1} v_{\pi+2} \ldots v_l$ to vertex v_l. Note that l is not equal to π, by construction.

Note that vertex v_{a+1} has degree two. Let C' be obtained from C by these same contractions, and let $P' = v_a v_{a+1}$. The H-set (M', C', S', P', e) has weight $(\sigma + 1, \pi')$ and $(M', C', S', P', e) \preceq_H (M, C, S, P, e)$, which is what we wanted to show. This concludes the proof of Lemma 4.1. $\square$

With the use of this lemma, we will now prove a second lemma.

Lemma 4.2. *There is a function $f_{4.2}$ such that, for integers k and d exceeding two, any Hamilton cycle of a graph with order at least $f_{4.2}(k, d)$ contains edges that may be contracted to obtain either a vertex with d neighbors or a parallel minor isomorphic to C_k.*

Proof. Let k and d be integers greater than two. Let $r_H = d^{(k-1)(d^2-1)+2}$. Set $f_{4.2}(k, d) = r_H$. Any Hamiltonian graph with at least r_H vertices has a Hamiltonian minor of order r_H, so it suffices for our lemma to prove that an arbitrary Hamiltonian graph with order equal to r_H will have our desired structure. Let G_H be a Hamiltonian graph of order r_H such that edges of a Hamilton cycle may not be contracted to obtain either a vertex of degree d or a parallel minor isomorphic to C_k.

Let C_H be a Hamilton cycle of G_H. Take vertex v of C_H. Vertex v has degree less than d, so the vertices of $\{v\} \cup N(v)$ divide C_H into at most d path segments. There is some path segment of length at least $\frac{|C_H|}{d} = \frac{d^{(k-1)(d^2-1)+2}}{d} = d^{(k-1)(d^2-1)+1}$. Choose such a path segment, and let C be the cycle obtained from C_H by contracting all edges of C_H that are not in this path segment and that are not incident with v. Let G be the graph obtained from G_H by the same contractions. Observe that C is a Hamilton cycle of G, and $|G| \geq d^{(k-1)(d^2-1)+1}$. Without loss of generality, suppose Hamilton graph G to have order exactly $d^{(k-1)(d^2-1)+1}$. Let $r = d^{(k-1)(d^2-1)+1}$.

Let e and f be the two edges in C incident with v. If $G = C$, then observe

that G contains a parallel minor isomorphic to C_k. We assume not. Let $S = \{e\}$ and let P be the path with endpoint v containing the edge e such that each internal vertex of P has degree two and P has an endpoint with degree exceeding two. If $|P| \geq k$, then we may contract edges in $C - E(P) - \{e\}$ to obtain a parallel minor isomorphic to C_k. This is forbidden by our assumptions. If we find an H-set that is an H-minor $(M', C', S', P', e) \preceq_H (G, C, S, P, e)$ such that $|S'| \geq d^2$, then we may contract a path along C in G that contains exactly one end of each edge in S' to obtain a vertex of degree at least d. This is also forbidden by our assumptions. Our restrictions also require that P' have fewer than k vertices, for the same reason that path P does.

The H-set (G, C, S, P, e) has weight at least $(1, 1)$, and $|P| < k$. By applying Lemma 4.1, we may find an H-set $(M', C', S', P', e) \preceq_H (G, C, S, P, e)$ of greater weight, where $|M'| > \frac{r}{d} = d^{(k-1)(d^2-1)}$. We may do this another $(k-1)(d^2-1)$ times to find a sequence of H-sets of strictly increasing weight, each of which is an H-minor of the preceding one. By our assumptions, for each H-set (M'', C'', S'', P'', e) in this sequence, $|P''| < k$. Since this sequence must include at least $(k-1)(d^2-1)+1$ weights greater than $(1, 1)$, none of which may repeat, we may apply the pigeonhole principle to conclude that there must be one H-set $(M''', C''', S''', P''', e)$ among this sequence with weight greater than $(d^2 - 1, k - 1)$, so that $|S'''| > d^2 - 1$.

By transitivity, this H-set is an H-minor of (G, C, S, P, e). This concludes the proof. $\square$

For the final lemma of this section, we will require the following result concerning 2-connected graphs from [2], the proof of which is available in the reference. The following proposition names two minors which cannot both be absent from a large, 2-connected graph. This will provide a natural way of dividing into two cases the 2-connected graphs of high order that we will study in this section.

Proposition 4.3. *There is a function $f_{4.3}$ such that, for any integer r exceeding two, every 2-connected graph of order at least equal to integer $f_{4.3}(r)$ contains a minor isomorphic to C_r or $K_{2,r}$.*

As a next step toward proving our 1- and 2-connected results, Theorems 1.4 and 1.5, we will now prove a lemma concerning 2-connected graphs that is analogous to Lemma 3.2 in the preceding section for connected graphs.

Lemma 4.4. *There is a function $f_{4.4}$ such that, for integers k and q exceeding two, every 2-connected graph with order at least equal to the integer $f_{4.4}(k, q)$ has a parallel minor isomorphic to $K'_{2,k}$, C_k, F_k, K_k, or a 3-connected graph of order at least q.*

Proof. Let k and q be integers exceeding two. Let $f_{3.2}$, $f_{1.1}$, $f_{4.2}$, and $f_{4.3}$ be the functions described in Lemma 3.2, Theorem 1.1, Lemma 4.2, and Proposition 4.3 respectively. Let $s = f_{3.2}(k, q)$, $r = f_{1.1}(k + 1) + f_{4.2}(k, s)$, and $l = f_{4.3}(r)$. Set $f_{4.4}(k, q) = l$. Let G be a 2-connected graph of order at least l.

Proposition 4.3 implies that the following two cases are exhaustive. Case 1: Graph G has a minor isomorphic to $K_{2,r}$. Case 2: Graph G has no minor isomorphic to $K_{2,r}$, but G has a minor isomorphic to C_r.

If G meets the conditions of Case 1, then let M be a minor of G that is isomorphic to $K_{2,r}$. Fix $H \in \Phi(G, M)$. Take v and w in $V(H)$ with degree at least r in M. By Theorem 1.1, the graph $H - \{v, w\}$ has an induced subgraph isomorphic to K_{k+1} or $\overline{K_{k+1}}$.

If $H - \{v, w\}$ has an induced subgraph isomorphic to K_{k+1}, then H has a parallel minor isomorphic to K_k. Assume, therefore, that $H - \{v, w\}$ has an independent set X of order $(k + 1)$. In H, vertices v and w are adjacent to all other vertices. Contract, one by one, each edge that does not have both ends in $X \cup \{v\}$, deleting the multiple edges and loops, to obtain a parallel minor isomorphic to $K_{2,k+1}$ or $K'_{2,k+1}$. Contract any edge to obtain a parallel minor isomorphic to $K'_{2,k}$. This completes Case 1.

If G meets the conditions of Case 2, then let M be a minor of G that is isomorphic to C_r. Fix $H \in \Phi(G, M)$. The graph H is Hamiltonian.

Let C be a Hamilton cycle of H. We may contract edges of C to obtain a parallel minor isomorphic to C_k or a vertex of degree s by Lemma 4.2. If the former, then we are done, since C_k is among our list of parallel minors. If the latter, then contract edges of C to find a vertex of degree s. This vertex is contained in a Hamiltonian graph, so we can find a minor N of H isomorphic to F_s. Choose $H' \in \Phi(H, N)$.

Take vertex v of degree s in H'. The graph $H' - v$ is connected, so we may apply Lemma 3.2 with the following result. The graph $H' - v$ has a parallel minor isomorphic to $K_{1,k}$, P_k, K_k, or a 2-connected graph of order at least q. Therefore, H' has a parallel minor isomorphic to $K'_{2,k}$, F_k, K_k, or a 3-connected graph of order at least q, respectively. This completes Case 2, and the proof of Lemma 4.4. $\square$

Using Lemma 4.4 with Lemma 3.2, we will now prove our first major result of this paper, Theorem 1.4, concerning connected graphs.

Proof of Theorem 1.4. Let k be a positive integer. Let $f_{1.1}$, $f_{1.2}$, $f_{4.4}$, and $f_{3.2}$ be the functions described in Theorem 1.1, Theorem 1.2, Lemma 4.4, and Lemma 3.2 respectively. Let $r = f_{1.1}(k)$, $q = f_{1.2}(r)$, $l = f_{4.4}(2k, q)$, and $s = f_{3.2}(k, l)$. Set $f_{1.4}(k) = s$. Let G be a connected graph of order at least s.

By Lemma 3.2, graph G has a parallel minor isomorphic to $K_{1,k}$, P_k, or K_k; or G has a 2 connected parallel minor of order at least l that has no minor isomorphic to $K_{1,r}$. If G has a parallel minor isomorphic to $K_{1,k}$, P_k, or K_k, then the theorem holds. Suppose that G has a 2-connected parallel minor H of order at least l, and H has no minor isomorphic to $K_{1,r}$.

We apply Lemma 4.4 to H to obtain a 3-connected parallel minor of H with order q, or a parallel minor isomorphic to $K'_{2,2k}$, C_{2k}, F_{2k}, or K_{2k}. If $K'_{2,2k}$ is isomorphic to a parallel minor of H, then $K_{1,k}$ is isomorphic to a parallel minor of G. If C_{2k} is isomorphic to a parallel minor of H, then C_k is isomorphic to a parallel minor of G. If F_{2k} is isomorphic to a parallel minor of H, then we contract every

other spoke of the fan to obtain a parallel minor of G isomorphic to $K_{1,k}$. If K_{2k} is isomorphic to a parallel minor of H, then K_k is isomorphic to a parallel minor of G. Therefore, suppose that none of these four parallel minors occur in G.

Let H' be a 3-connected parallel minor of H with order q. From Theorem 1.2, we know that H must have a minor isomorphic to W_r or $K_{3,r}$, so H has a minor isomorphic to $K_{1,r}$, which contradicts our assumption. This completes our proof. $\square$

With the connected result in hand, we continue on to the 2-connected result.

Proof of Theorem 1.5. Let k be an integer exceeding two. Let $f_{1.2}$, $f_{1.4}$ and $f_{4.4}$ be the functions described in Theorem 1.2, Theorem 1.4, and Lemma 4.4, respectively. Let $r = f_{1.4}(k+2)$, let $q = f_{1.2}(r)$ and let $l = f_{4.4}(k,q)$. Set $f_{1.5}(k) = l$. Let G be a 2-connected graph of order at least l.

By Lemma 4.4, G has a parallel minor isomorphic to $K'_{2,k}$, C_k, F_k, K_k, or a 3-connected graph of order at least q. It remains only to investigate the last possibility. Let G contain a 3-connected graph, G', of order at least q as a parallel minor. Graph G' has a minor isomorphic to W_r or $K_{3,r}$, by Theorem 1.2.

Let M be a minor in G' isomorphic to W_r or $K_{3,r}$. Take $H \in \Phi(G, M)$, and take $v \in V(H)$ of degree at least r. The graph H is 3-connected, hence $H - v$ is 2-connected. Since $H - v$ is connected and $H - v$ has order $f_{1.4}(k+2)$, the graph $H - v$ has a parallel minor H' isomorphic to $K_{1,k+2}$, C_{k+2}, P_{k+2}, or K_{k+2}, by Theorem 1.4. Since v is non-adjacent to at most two other vertices in H', the graph H must have a parallel minor isomorphic to $K'_{2,k}$, F_k, or K_k, as required. $\square$

5. Unavoidable Parallel Minors of 3-Connected Graphs

We will now prove our third result, Theorem 1.6, using our second result, Theorem 1.5. Recall that Theorem 1.6 states that, for an appropriate integer k, every 3-connected graph of high enough order contains $K'_{3,k}$, W_k, DF_k, or K_k as a parallel minor.

Proof of Theorem 1.6. Let k be an integer exceeding three. Let $f_{1.5}$ and $f_{1.2}$ be the functions described in Theorem 1.5 and Theorem 1.2, respectively. Let $r = f_{1.5}(k+2)$ and $q = f_{1.2}(r)$. Set $f_{1.6}(k) = q$. Let G be a 3-connected graph of order at least q. By Theorem 1.2, the graph G contains a minor M isomorphic to W_r or $K_{3,r}$. Choose $H \in \Phi(G, M)$.

Take $v \in V(H)$ of highest degree. Graph $H - v$ is 2-connected, and has order at least r, so $H - v$ contains a parallel minor H' isomorphic to $K'_{2,k+2}$, C_{k+2}, F_{k+2}, or K_{k+2}, by Theorem 1.5. Vertex v is adjacent to all but at most two other vertices in H', hence G has a parallel minor isomorphic to $K'_{3,k}$, W_k, DF_k, or K_k, respectively. This completes our proof. $\square$

6. Unavoidable Parallel Minors of Internally 4-Connected Graphs

Recall that Theorem 1.3 states the set of unavoidable minors in large, internally 4-connected graphs, $\{K_{4,q}, D_q, M_r, Z_r\}$, which will provide the basis for this proof.

In this section, we will prove our main result, Theorem 1.7, which states that an internally 4-connected graph of sufficiently high order will contain as a parallel minor $K'_{4,k}$, D_k, D'_k, TF_k, M_k, Z_k, or K_k.

Proof of Theorem 1.7. Let k be an integer exceeding four. Let $f_{1.6}$, $f_{4.2}$, and $f_{1.3}$ be the functions described in Theorem 1.6, Lemma 4.2, and Theorem 1.3, respectively. Let $q = f_{1.6}(k+3)$, $r = f_{4.2}(2k+1, 4q)$, and $n = f_{1.3}(q, r)$. Set $f_{1.7}(k) = n$. Let G be an internally 4-connected graph of order at least n. The graph G has a minor isomorphic to $K_{4,q}$, D_q, M_r, or Z_r, by Theorem 1.3.

If G has a minor, M, isomorphic to $K_{4,q}$ or D_q, then choose $H \in \Phi(G, M)$. Take v of highest degree in H. Graph H is 4-connected. Graph $H - v$ is 3-connected, so it has a parallel minor H' isomorphic to $K'_{3,k+3}$, W_{k+3}, DF_{k+3}, or K_{k+3}, by Theorem 1.6. Since v is adjacent to all but at most 3 vertices of H', graph H has a parallel minor isomorphic to $K'_{4,k}$, D_k or D'_k, TF_k, or K_k, respectively.

Therefore, suppose G has no minor isomorphic to $K_{4,q}$ or D_q. Then, G has a minor M isomorphic to M_r or Z_r. To deal with these two cases simultaneously, we consider a minor M' of M, which is defined as follows. Let vertices of M be labeled as in Figure 3. If $M \cong Z_r$, let $M' = M \backslash \{v_1 u_2, v_2 u_3, \ldots, v_{r-1} u_r, v_r u_1\}$; if $M \cong M_r$, let $M' = M \backslash \{v_1 u_2, v_2 u_3, \ldots, v_r u_{r+1}, u_{r+1} u_1\} / \{u_r u_{r+1}\}$. Notice that M' is a circular ladder in the first case and a Möbius ladder in the second case. Take $H \in \Phi(G, M')$.

We will work with a collapsed form of H. Let H' be $H/\{v_1 u_1, v_2 u_2, \ldots, v_r u_r\}$ after deleting multiple edges, and let C be the cycle representing the collapsed ladder. We apply Lemma 4.2 to conclude that edges of C, a Hamilton cycle, may be contracted to obtain a vertex of degree $4q$ or a parallel minor isomorphic to C_{2k+1}.

Suppose we can obtain a vertex of degree $4q$ from H' by only contracting edges in C. Then we may obtain a graph D from H by the contractions of the corresponding pairs of edges in H. In this case, D contains a vertex x_1 of degree at least $2q$. Observe that D is a parallel minor of H with the same order as the ladder subgraph contained inside it, so it maintains the ladder structure, which we may label the same way we label M', but with x and y vertices instead of u and v vertices, respectively. Let $|V(D)| = 2s$.

The vertex x_1 must be adjacent with at least q vertices in either the x-vertices or the y-vertices of D. If x_1 has q neighbors among the x-vertices, we may contract the path, $y_3 y_4 \ldots y_{s-2} y_{s-1}$, in D to a vertex y of degree at least q. The vertices y and x_1 are then the two hubs of a minor isomorphic to D_q. If x_1 has more than q neighbors among the y-vertices, we may contract the path $x_2 x_3 \ldots x_{s-2} x_{s-1}$ in D and then obtain a minor isomorphic to D_q. In both cases we conclude that $D_q \preceq G$, which contradicts our assumptions.

Suppose we can obtain no vertex of degree $4q$ from H' by contracting edges in the representative Hamilton cycle. Then, by Lemma 4.2, we must be able to find in H' a parallel minor N' isomorphic to C_{2k+1} by contracting edges in the Hamilton cycle. For every edge that we contract in the Hamilton cycle of H' to obtain N', we contract corresponding pair of edges in H to obtain a parallel minor N. Observe

that N is simply a Möbius or circular ladder, possibly with extra edges between consecutive rungs. Let us label the vertices of N the same way we label M', but with x and y vertices instead of u and v vertices, respectively. If N is a Möbius ladder, then contracting edges $x_1x_2, x_3x_4, \dots , x_{2k-1}x_{2k}, x_{2k+1}y_1, y_2y_3, y_4y_5, \dots ,$ $y_{2k}y_{2k+1}$ results in a parallel minor M_k. If N is a circular ladder, then contracting edges $x_1x_2, x_3x_4, \dots , x_{2k-1}x_{2k}, y_2y_3, y_4y_5, \dots , y_{2k}y_{2k+1}$, and also $x_{2k+1}x_1$ and $y_{2k+1}y_1$ results in a parallel minor Z_k. This completes the proof. $\qquad\square$

7. Related Conclusions and Further Applications

Since a parallel minor is an induced minor, the reader should note that the set of unavoidable parallel minors in a c-connected graph contains the set of unavoidable induced minors. With the exception of C_k in the 1-connected graph case, the families of unavoidable parallel minors and unavoidable c-connected induced minors in c-connected graphs are identical.

Parallel minor is the dual matroid operation of series minor. The sets of unavoidable series minors in k-connected graphs are known for $k \in \{1, 2, 3, 4\}$. Since any regular matroid can be constructed from graphic and cographic submatroids, together with submatroids isomorphic to R_{10}, the results of this paper may contribute to finding the sets of unavoidable minors of variously connected matroids.

Other avenues of investigation related to the result of this paper include obtaining a set of minors unavoidable in large, 5-connected graphs. Also, it is natural to consider the dual operation of the induced minor, and the unavoidable minors under this dual operation.

Acknowledgements

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