Numerical Modelling of Some Systems in the Biomedical Sciences

A thesis submitted for the degree of

Doctor of Philosophy

by

Faisal Nasser Mohammed Al-Showaikh

Department of Mathematics and Statistics, Brunel University,

Uxbridge, Middlesex, England. UB8 3PH

January 1998

Dedicated to my wife, Bushra,

my brother, Mohammed

and in

memory of my parents

Abstract

Finite-difference numerical methods are developed for the solution of some systems in the biomedical sciences; namely, a predator-prey model and the SEIR (Susceptible/Exposed/ Infectious/Recovered) measles model. First-order methods are developed to solve the predator-prey model and one second-order method is developed to solve the SEIR measles model.

The predator-prey model is extended to one-space dimension to incorporate diffusion. The SEIR measles model is extended to one-space dimension to incorporate (i) diffusion, (ii) convection and (iii) diffusion-convection. The SEIR measles model is extended further to model diffusion in two-space dimensions.

The reaction terms in these systems of partial differntial equations contain non-linear expressions. Nevetheless, it is seen that the numerical solutions are obtained by solving a *linear* algebraic system at each time step, as opposed to solving a non-linear algebraic systems, which is often required when integrating non-linear partial differential equations. The development of each numerical method is made in the light of experience gained in solving the system of ordinary differential equations for each system.

The numerical methods proposed for the solution of the initial-value problem for the predator-prey and measles models are characterized to be implicit. However, in each case it is seen that the numerical solutions are obtained *explicitly*. In a series of numerical experiments, in which the ordinary differential equations are solved first of all, it is seen that the proposed methods have superior stability properties to those of the well-known, first-order, Euler method to which they are compared. Incorporating the proposed methods into the numerical solution of partial differential equations is seen to lead to economical and reliable methods for solving the systems.

Acknowledgements

First of all I would like to express my gratitudes to my supervisor Professor E.H. Twizell for his helpful guidance, invaluable suggestions and continuous encouragement during both the period of research and the writing of the thesis. He has always given patiently of his time and his endeavours have extended well beyond the bounds of mere supervision.

I am also indebted to Dr. S.A. Matar, my second advisor, for his help with Fortran programming and his continuous encouragement.

I express my sincere gratitude to Dr. M.K. Warby for helping me with the computer packages LaTeX and Matlab and Dr. J-E. Furter for his invaluable discussions about dynamical systems.

Also, I wish to express my gratitude to the University of Bahrain for my scholarship and the support provided throughout the duration of this thesis.

Finally, I would like to gratefully acknowledge the help and encouragement of my family, especially, my wife and my brother Mohammed for their understanding throughout this research.

Contents

List of Figures

List of Tables

Chapter 1

Preliminaries

1.1 Introduction

Chaos can be observed in many areas of the chemical, physical and biological sciences and in many areas of engineering. Some examples can be found in such areas as catalysis, turbulent fluid flows, predator-prey interactions, epidemiology, flame propagation, electronic circuits, meteorology, neurophysiological reactions, and ship dynamics. A profusion of examples of chaos are given in the popular book by Gleick[19], which gives those interested a worthy introduction to the phenomenon.

The examples given above are of dynamic behaviour. In recent times, the phenomenon of chaos has brought about useful collaboration between biologists and mathematicians. Such collaboration has usually resulted in the mathematical modelling of a biological system by a non-linear, time-dependent differential equation, ordinary or partial, or by a system of such equations. Often, the aim of such a model is to successfully reproduce laboratory findings before using the model to make predictions.

Careful analysis must, however, be carried out to ensure that the mathematical model does not predict chaos in the system under investigation, when chaos is not a feature of that system. Further care must be taken to ensure that a numerical method chosen to solve the model equations does not predict chaos. Such chaos, when it is a feature of neither the system nor the theoretical solution of the associated model equations, was described as *contrived chaos* in the article by Twizell et al.[69].

It is the purpose of the present thesis to develop simple and inexpensive computational techniques that may be employed to produce numerical results of mathematical models in the biomedical sciences in which chaotic behaviour is not inherent. The model equations that will feature in the illustrations will be a system of ordinary differential equations, herein denoted as ODEs, and systems of partial differential equations (hereafter abbreviated PDEs).

In the case of the system of ODEs, the Euler forward-difference method, arguably the best-known and most widely used, low-order, explicit, numerical method, which is also inexpensive to implement, can induce chaos whenever parameter(s) exceed critical value(s). Explicit methods are also well known to be inexpensive to implement when used to compute the solutions of non-linear ODEs. However, for the ODEs and PDEs, explicit methods have poor numerical-stability properties (see, for instance, Lambert[31] and Twizell et al.[69]) and the user may be forced to use an implicit numerical method.

Implicit methods are more expensive to implement and are usually used as the corrector formula in a predictor-corrector combination, which employs a low-order explicit formula as predictor. Such combinations have enjoyed considerable success with systems of ODEs, for which they were originally intended, but they have also been used to great effect in solving non-linear PDEs.

The restricted stability intervals of predictor-corrector combinations in PECE mode (see Lambert[31, p.117]) often lead the user to employ implicit methods directly, so that their good stability properties may be exploited fully. Solving systems of non-linear ODEs or PDEs requires the solution of a non-linear algebraic system using, for example, the well known Newton-Raphson method for a system.

In order to obviate the need to use a relatively expensive, non-linear algebraic solver such as the Newton-Raphson method for a system, while continuing to benefit from the superior stability properties of implicit methods, Twizell et al.[69] proposed numerical methods for the solution of differential equations of the forms

$$\dot{x} = dx/dt = f(x) \qquad (1.1.1)$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \phi\ (u)\ ; \qquad (1.1.2)$$

in (1.1.1) $x = x(t)$ and in (1.1.2) $u = u(x,t)$ are real-valued functions, $x \in I\!R$ is a space variable and $t \in I\!R^+$ represents time. In their paper Twizell et al.[69] approximated the non-linear functions $f(x)$ and $\phi(u)$ by splitting them and evaluating terms in the splittings at different time levels. This idea will be employed in subsequent chapters of this thesis in ways which permit the solutions of ODEs to be determined *explicitly* from what appear to be *implicit* numerical methods and the solutions of non-linear PDEs to be obtained by solving a *linear* algebraic system at each time step.

The subsequent sections of Chapter 1 outline the various preliminary definitions and theorems needed in the development and analyses of the numerical methods in later chapters.

In Chapter 2, three numerical methods are proposed for the solution of the system of partial differential equations of reaction-diffusion type, which model the behaviour of a continuous predator-prey system on a spatial gradient that affects the intrinsic growth rate of the prey. The numerical methods which are first-order accurate in time and second-order accurate in space, are seen to require the solution of only a *linear* algebraic system at each time step, even though the PDEs in the model are non-linear. This analogue is modelled mathematically by a system of two non-linear ODEs, which are solved by the Euler forward-difference method, and by two alternative methods of the same order. The system of ODEs is solved in Section 2.4, while the system of

PDEs is solved in Section 2.5.

The SEIR (Susceptibles/Exposed/Infectious/Recovered) measles model will be discussed in Chapter 3 where the stationary (equilibrium) points are found and analysed. Two numerical methods are developed, analysed and tested to solve the SEIR measles model. The first method is the well-known first-order explicit Euler method and the other is a second-order method. Numerical results will be given and compared for the two methods using two experiments.

In Chapter 4, the SEIR measles model is extended to one-space dimension. The reaction-diffusion SEIR measles model is analysed and solved numerically for different values of the time step, ℓ.

Two-space dimensions of the SEIR measles model is analysed and solved numerically in Chapter 5. Different numerical results are found for different values of the diffusion rates.

In Chapter 6, a one-dimensional measles model of convection type is analysed and solved. The maximum principle analysis is used to examine convergence for the developed method and numerical results are given for susceptibles, exposed and infectives.

A mixed initial/boundary-value problem for measles dynamics of diffusion-convection type is given in Chapter 7. As in Chapter 6, the maximum principle analysis is used to prove convergence of the developed methods. Four sets of numerical results for different values of diffusion and convection rates are obtained and discussed.

1.2 Some Definitions and Theorems

Given an independent variable t and a function $f(t,x)$, a typical (single) first-order initial-value problem associated with the biomedical sciences takes the general form

$$\dot{x}(t) = f(t,x)\,, \quad t > t_0\,; \quad x(t_0) = g \tag{1.2.1}$$

The following theorem outlined in Lambert[31], with proof contained in Henrici[24], states conditions on $f(t,x)$ which guarantee the existence of a unique solution of the initial-value problem (1.2.1).

Theorem 1.2.1 *Let $f(t,x)$, where $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, be defined and continuous for all points (t,x) in the region D defined by $t_0 \le t \le t_1$, $-\infty < x < \infty$, t_0 and t_1 finite, and let there exist a constant L such that, for every t, x, x^* such that (t,x) and (t,x^*) are both in D,*

$$|\,f(t,x) - f(t,x^*)\,| \;\le\; L\;|\,x - x^*\,|\,. \tag{1.2.2}$$

Then, if t_0 is any given number, there exists a unique solution $x(t)$ of the initial-value problem (1.2.1), where $x(t)$ is continuous and differentiable for all (t,x) in D.

The requirement (1.2.2) is known as the Lipschitz condition, and the constant L as a Lipschitz constant. This condition may be thought of as being intermediate between differentiability and continuity (see Khaliq[27]) in the sense that

$f(t,x)$ contiuously differentiable with respect to x for all (t,x) in D

$\Longrightarrow$ $f(t,x)$ satisfies a Lipschitz condition with respect to x for all (t,x) in D

$\Longrightarrow$ $f(t,x)$ continuous with respect to x for all (t,x) in D.

In particular, if $f(t,x)$ possesses a continuous derivative with respect to x for all

(t, x) in D, then, by the Mean Value Theorem,

$$f(t,x) - f(t,x^*) = \frac{\partial f(t,\bar{x})}{\partial x}\,(x - x^*)\ ,$$

where $\bar{x}$ is a point in the interior of the interval having endpoints x and x^*, and (t, x) and (t, x^*) are both in D. Clearly, (1.2.2) is then satisfied if L is chosen to be

$$L = \sup_{(t,x)\in D} \left| \frac{\partial f(t,x)}{\partial x} \right| . \qquad (\ 1.2.3\ )$$

The next theorem is of fundamental importance in deriving methods for error estimation. The proof of this theorem may be found in any standard calculus text (see, for example, Faires & Faires[17]).

Theorem 1.2.2 (Mean Value Theorem) *If $f \in C[a, b]$ and f is differentiable on (a, b), then a number c, $a < c < b$, exists such that*

$$f(b) - f(a) = f'(c)\,(b - a)\,.$$

The following theorem is a generalization of the Mean Value Theorem, Theorem 1.2.2, in two variables which will be used frequently in the stability analyses in Chapters 4–7. This theorem may be found in Lynch et al.[39] or Sandefur[56]).

Theorem 1.2.3 (Mean Value Theorem in two variables) *If $f(x, y)$ is differentiable, then there exists a point (x_0, y_0) on the line connecting the points (x_1, y_1) and (x_2, y_2) such that*

$$f(x_2, y_2) - f(x_1, y_1) = f_x(x_0, y_0)\,(x_2 - x_1) + f_y(x_0, y_0)\,(y_2 - y_1)\,.$$

One of the most important theorems in numerical analysis is due to Taylor (Brook Taylor 1685–1731, English mathematician) which will be stated here and may be found in Burden & Faires[9, p.240].

Theorem 1.2.4 (Taylor, one-dimensional) *Suppose $f \in C^n[a,b]$ and $f^{(n+1)}$ exists on $[a,b]$. Let $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists $\xi(x)$ between x_0 and x with*

$$f(x) = P_n(x) + R_n(x)\,, \qquad (1.2.4)$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\,,$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}\,.$$

Here $P_n(x)$ is called the nth Taylor polynomial for f about x_0 and $R_n(x)$ is called the remainder term (or truncation error) associated with $P_n(x)$. The infinite series obtained by taking the limit of $P_n(x)$ as $n \longrightarrow \infty$ is called the Taylor series for f about x_0. In the case $x_0 = 0$, the Taylor polynomial is often called a Maclaurin polynomial and the Taylor series is called a Maclaurin series. Usually Taylor's formula is used to express a function in a power series.

Theorem 1.2.5 (Taylor, two-dimensional) *Suppose $f(x,y)$ and all of its first partial derivatives of order less than or equal to n+1 are continuous on $D = \{(x,y)\,|\, a \le x \le b,\ c \le y \le d\}$. Let $(x_0,y_0) \in D$. For every $(x,y) \in D$, there exist ξ between x and x_0 and η between y and y_0 with*

$$f(x,y) = P_n(x,y) + R_n(x,y) \qquad (1.2.5)$$

where

$$\begin{aligned} P_n(x,y) &= f(x_0,y_0) + \left[(x-x_0)\tfrac{\partial f(x_0,y_0)}{\partial x} + (y-y_0)\tfrac{\partial f(x_0,y_0)}{\partial y}\right] \\ &+ \tfrac{1}{2!}\left[(x-x_0)^2\tfrac{\partial^2 f(x_0,y_0)}{\partial x^2} + 2\,(x-x_0)(y-y_0)\tfrac{\partial^2 f(x_0,y_0)}{\partial x\,\partial y} + (y-y_0)^2\tfrac{\partial^2 f(x_0,y_0)}{\partial y^2}\right] + \cdots \\ &+ \tfrac{1}{n!}\textstyle\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (x-x_0)^{n-k}\,(y-y_0)^k\,\tfrac{\partial^n f(x_0,y_0)}{\partial x^{n-k}\,\partial y^k} \end{aligned}$$

and

$$R_n(x,y) = \frac{1}{(n+1)!}\sum_{k=0}^{n+1} \begin{pmatrix} n+1 \\ k \end{pmatrix} (x-x_0)^{n+1-k}\,(y-y_0)^k\,\frac{\partial^{n+1} f(\xi,\eta)}{\partial x^{n+1-k}\,\partial y^k}\,.$$

The function $P_n(x, y)$ is called the nth Taylor polynomial in two variables for the function f about (x_0, y_0) and $R_n(x, y)$ is the remainder term associated with $P_n(x, y)$.

The truncation error generally refers to the error involved in using a truncated or finite summation to approximate the sum of an infinite series.

In connection with Taylor's expansion (1.2.4) it is useful to introduce the rate of convergence of a function limit, say $\lim_{x\to\infty} f(x)$. This is done by comparison with a reference function, say $g(x)$, see Wiktor[71], using the classical Landau order symbols O and o:

Definition 1.2.1 (Rate of convergence) *Let $f(x)$ and $g(x)$ be real continuous functions of $x \in (0, a]$.*
(i) $f(x) = O\left(g(x)\right)$ for $x \longrightarrow 0$ if there exist positive constants k and c such that

$$|\, f(x)\,| \leq k\ |\, g(x)\,| \quad for \quad 0 < x < c \tag{1.2.6}$$

$$(ii) \quad f(x) = o\left(g(x)\right) \quad if \quad \lim_{x\to 0} \left\{\frac{f(x)}{g(x)}\right\} \quad exists \quad and \quad equals \quad zero. \tag{1.2.7}$$

The idea of finite-difference methods is to replace the differential of a function, say $f(x)$, which is defined in terms of a limit

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{1.2.8}$$

by a finite-difference approximation. Not taking the limit in (1.2.8) (which a computer couldn't do anyway) but stopping at a sufficiently small value of h leads to a discretization or truncation error. The Taylor series gives

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!}\, f''(x) + \cdots \tag{1.2.9}$$

and

$$f(x-h) = f(x) - h\, f'(x) + \frac{h^2}{2!}\, f''(x) - \cdots . \tag{1.2.10}$$

It is easy then, using the "O"-notation defined in (1.2.6) to derive the following formulae with which to approximate $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$.

First-order backward derivative replacement, $f \in C^2[x-h, x]$

$$\frac{df}{dx} = \frac{f(x) - f(x-h)}{h} + O(h) \quad as \quad h \longrightarrow 0. \qquad (1.2.11)$$

First-order forward derivative replacement, $f \in C^2[x, x+h]$

$$\frac{df}{dx} = \frac{f(x+h) - f(x)}{h} + O(h) \quad as \quad h \longrightarrow 0. \qquad (1.2.12)$$

Second-order centred derivative replacement, $f \in C^3[x-h, x+h]$

$$\frac{df}{dx} = \frac{f(x+h) - f(x-h)}{2\,h} + O(h^2) \quad as \quad h \longrightarrow 0. \qquad (1.2.13)$$

Second-order centred second derivative replacement, $f \in C^4[x-h, x+h]$

$$\frac{d^2f}{dx^2} = \frac{f(x-h) - 2\,f(x) + f(x-h)}{h^2} + O(h^2) \quad as \quad h \longrightarrow 0. \qquad (1.2.14)$$

1.3 General Theory

In many areas such as chemical kinetics and biology, the dynamic behaviour is modelled with a system of n simultaneous first-order equations in n dependent variables $x_1, x_2, \ldots, x_n$. If each of these variables satisfies a given condition at the same value t_0 of t, then the initial-value problem for a first-order system may be written as

$$\left.\begin{array}{lcllll} \dot{x}_1 & = & f_1(t, x_1, x_2, \ldots, x_n)\,, & t > t_0\,, & x_1(t_0) = g_1 \\ \dot{x}_2 & = & f_2(t, x_1, x_2, \ldots, x_n)\,, & t > t_0\,, & x_2(t_0) = g_2 \\ \vdots & & \vdots & \vdots & \vdots \\ \dot{x}_n & = & f_n(t, x_1, x_2, \ldots, x_n)\,, & t > t_0\,, & x_n(t_0) = g_n \end{array}\right\}, \qquad (1.3.1)$$

where each $f_i : D \subseteq I\!\!R \times I\!\!R^n \longrightarrow I\!\!R^n \; (i = 1, 2, \ldots, n)$.

Introducing the vector notations $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T = \mathbf{f}(t, \mathbf{x})$, $\mathbf{g} = (g_1, g_2, \ldots, g_n)^T$, where T denotes transpose, the initial-value problem (1.3.1) may be written as

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad t > t_0, \quad \mathbf{x}(t_0) = \mathbf{g}. \qquad (1.3.2)$$

Equations where time does not appear explicitly on the right-hand side are called *autonomous* or *time-invariant* differential equations, whilst if time does appear explicitly they will be called *non-autonomous* or *time-variant.*

Theorem 1.2.1 generalizes immediately to give necessary conditions for the existence of a unique solution to (1.3.2); all that is required is that the region D now defined by $a \leq t \leq b, -\infty < x_i < \infty, i = 1, 2, \ldots, n$, and (1.2.2) be replaced by the condition

$$\| \mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{x}^*) \| < L \, \|\mathbf{x} - \mathbf{x}^*\| \, , \qquad (1.3.3)$$

where $(t, \mathbf{x})$ and $(t, \mathbf{x}^*)$ are in D, and $\|.\|$ denotes a vector norm. For the properties of vector and matrix norms see, for example, Twizell[66]. In the case when each of the $f_i(x_1, x_2, \ldots, x_n), i = 1, 2, \ldots, n$, possesses a continuous derivative with respect to each of the $x_i \, (i = 1, 2, \ldots, n)$ then

$$L = \sup_{(t,\mathbf{x}) \in D} \left\| \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right\| \qquad (1.3.4)$$

may be chosen analogously to (1.2.3), where $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ is the Jacobian of $\mathbf{f}$ with respect to $\mathbf{x}$ and is given by

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & \vdots & & \vdots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{pmatrix}. \qquad (1.3.5)$$

The first-order initial-value problem (1.3.2) is said to be *non-linear* if $\mathbf{f}$ depends non-linearly on $\mathbf{x}$. If $\mathbf{f}$ depends linearly on $\mathbf{x}$ the system is said to be *linear*.

In many situations the natural rest points of a dynamic system are as much of interest as the mechanism of change. These points are called *equilibrium points.* The general definition, applied to both discrete- and continuous-time systems and to non-linear as well as linear systems, is given by

Definition 1.3.1 (Equilibrium point) *A vector $\bar{\mathbf{x}}$ is an equlibrium point for a dynamic system if, once the state vector is equal to $\bar{\mathbf{x}}$, it remaines equal to $\bar{\mathbf{x}}$ for all future time.*

In particular, if a system is described by a set of differential equations (continuous-time system) as in (1.3.2), an equilibrium point is a state $\bar{\mathbf{x}}$ satisfying

$$\mathbf{f}(t, \bar{\mathbf{x}}) = \mathbf{0}$$

for all t. The case for a discrete-time system will be given below when a one-step discrete system is discussed.

In most situations of practical interest the system is time-invariant, in which case the equilibrium points $\bar{\mathbf{x}}$ are solutions of an n-dimensional system of algebraic equations. Specifically, in a continuous-time system,

$$\mathbf{f}(\bar{\mathbf{x}}) = \mathbf{0}.$$

Stability properties characterize how a system behaves if its state is initiated close to, but not precisely at, a given equilibrium point. If a system is initiated with the state exactly equal to an equilibrium point, then it will never move. When initiated close by, however, the state may remain close by, or it may move away. Roughly speaking, an equilibrium point is *stable* if, whenever the system state is initiated near that point, the state remains near it, perhaps even tending toward the equilibrium point as time increases.

In the following the notion of (asymptotically) stable and unstable equilibrium points of a continuous-time system is introduced (see Luenberger[38]).

Definition 1.3.2 (Stable point) *An equilibrium point* $\bar{\mathbf{x}}$ *is called (asymptotically) stable if there exists a number* $\varepsilon > 0$ *such that* $\forall \mathbf{x_0} \in I\!R^n$ *with* $\| \bar{\mathbf{x}} - \mathbf{x}_0 \| < \varepsilon$

$$\lim_{t \to \infty} \mathbf{f}(\bar{\mathbf{x}}) = \bar{\mathbf{x}} \,.$$

Definition 1.3.3 (Unstable point) *An equilibrium point* $\bar{\mathbf{x}}$ *is called unstable if it is not stable. Equivalently,* $\bar{\mathbf{x}}$ *is unstable if there exists a number* $r > 0$ *such that for all* $\varepsilon > 0$ *with* $0 < \| \bar{\mathbf{x}} - \mathbf{x}_0 \| < \varepsilon$

$$\| \bar{\mathbf{x}} - \mathbf{f}(\bar{\mathbf{x}}) \| > r \quad for \quad some \quad t \,.$$

The following theorem states a necessary and sufficient condition for an equilibrium point of system (1.3.2) to be asymptotically stable by assuming that the function **f** in Definitions 1.3.2 and 1.3.3 is sufficiently smooth.

Theorem 1.3.1 *Let* $J = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ *be the Jacobian of* $\mathbf{f}$ *at* $\bar{\mathbf{x}}$ *as in (1.3.5) with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. *A necessary and sufficient condition for an equilibrium point* $\bar{\mathbf{x}}$ *of the continuous-time system (1.3.2) to be asymptotically stable is that the eigenvalues of* J *all have negative real part (that is, the eigenvalues must lie in the left-half of the complex plane). If at least one eigenvalue has positive real part, the equilibrium point* $\bar{\mathbf{x}}$ *is unstable.*

Definition 1.3.4 *Suppose* $\mathbf{f}$ *is a function defined from* $D \subset I\!R^n$ *into* $I\!R^n$ *with* $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T$. *The function* $\mathbf{f}$ *is said to be continuous at* $\mathbf{x_0} \in D$ *provided* $\lim_{\mathbf{x} \to \mathbf{x_0}} \mathbf{f}(\mathbf{x})$ *exists and* $\lim_{\mathbf{x} \to \mathbf{x_0}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x_0})$. *In addition,* $\mathbf{f}$ *is said to be continuous on the set* D *if* $\mathbf{f}$ *is continuous at each point in* D. *This concept is expressed by writing* $\mathbf{f} \in C\,(D)$.

The following theorem relates the continuity of the function f of n variables at a point to the partial derivatives of the function at the given point.

Theorem 1.3.2 *Let f be a function from $D \subset I\!R^n$ into $I\!R$ and suppose that $\mathbf{x_0} \in D$. Suppose that δ and $k > 0$ are constants such that*

$$\left|\frac{\partial f(\mathbf{x})}{\partial x_i}\right| \leq k\,, \quad i = 1, 2, \ldots, n\,,$$

wherever $\|\mathbf{x} - \mathbf{x_0}\| < \delta$ and $\mathbf{x} \in D$. Then f is continuous at $\mathbf{x_0}$.

Depending on which approximations for the first and second derivatives equations (1.2.11)–(1.2.14) are chosen, a differential equation can be transformed into an m-step discrete dynamical system of the form (Herges[25])

$$\mathbf{x}_{k+m} = F\left(\mathbf{x}_{k+m-1}, \mathbf{x}_{k+m-2}, \ldots, \mathbf{x}_k\right), \quad k = 0, 1, 2, \ldots .$$

with $F : I\!R^{m.n} \longrightarrow I\!R^n$ and initial values $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1} \in I\!R^n$.

When solving numerically an initial-value problem of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})\,, \quad a \leq t \leq b, \quad \mathbf{x}(a) = \mathbf{x}_a \qquad (1.3.6)$$

with $\mathbf{x} : [a, b] \longrightarrow I\!R^n$ and $\mathbf{f} : [a, b] \times I\!R^n \longrightarrow I\!R^n$, the approximating discrete dynamical system is often only a one-step difference equation of the form

$$\mathbf{x}_{k+1} - \mathbf{x}_k = h\,G\,(t, \mathbf{x}_k; h) \qquad (1.3.7)$$

with $h > 0$ the steplength of equidistant grid points $t_i = a + i\,h\,(i = 1, 2, \ldots, M)$, $M = (b - a)/h$. The solution $\mathbf{x}(t)$ of equation (1.3.6) at $t_i = a + i\,h\,(i = 1, 2, \ldots, M)$ is approximated by the numerical solution $\mathbf{x}_i \in I\!R^n$ $(i = 1, 2, \ldots, M)$ obtained by iterating the difference equation (1.3.7), where $G : [a, b] \times I\!R^n \longrightarrow I\!R^n$ and $\mathbf{x}_0 = \mathbf{x}_a \in I\!R^n$ is the initial value. Scheme (1.3.7) is a special case of the general one-step discrete dynamical system.

The local truncation error at a specified grid point measures the amount by which the exact solution of the differential equation (1.3.6) fails to satisfy the difference equation. Hence, the local truncation error $\mathcal{L}[x(t_i); h]$ at $t_i = a + i\,h$, $i = 1, 2, \ldots, M$ for scheme (1.3.7) is defined by

$$\mathcal{L}[x(t_i); h] = x(t_i) - x(t_{i-1}) - h\,G\,(t, x(t_{i-1}); h)\,, \quad i = 1, 2, \ldots, M$$

and gives the accuracy of the numerical method at grid point t_i, $i = 1, 2, \ldots, M$ assuming the method was exact in the previous step.

Definition 1.3.5 (Order of a one-step difference method) *Let the solution* $\mathbf{x}(t)$ *of equation (1.3.6) be* $(p+1)$*-times continuously differentiable,* $p \in I\!N$*, then the local truncation errors* $\mathcal{L}[x(t_i); h], i = 1, 2, \ldots, M$ *can be expressed in terms of a finite Taylor series of the form*

$$\mathcal{L}[x(t_i); h] = \sum_{k=0}^{p+1} c_k\, h^k\, \frac{d^k x(t_{i-1})}{dt^k}, \quad i = 1, 2, \ldots, M.$$

The local truncation errors and with them the associated one-step difference method are said to be of order p *if* $c_0 = c_1 = \cdots = c_p = 0$ *and* $c_{p+1} \neq 0$.

Definition 1.3.6 (Consistency of a one-step difference method) *A one-step difference method with local truncation errors* $\mathcal{L}[x(t_i); h], i = 1, 2, \ldots, M$ *is said to be consistent with the differential equation it approximates if*

$$\lim_{h \to 0} \max_{1 \leq i \leq M} \frac{\|\mathcal{L}[x(t_i); h]\|}{h} = 0\,.$$

A one-step difference method is consistent precisely when the function $G\,(t, x_k; h)$ in equation (1.3.7) approaches $\mathbf{f}(t, \mathbf{x})$, the right-hand side of the differential equation (1.3.6), as the step size h goes to zero. Clearly, a one-step difference method is consistent if it is of order $p \geq 1$ and $d^{p+1}x/dt^{p+1}$ is bounded on $[a, b]$.

Definition 1.3.7 (Convergence of a one-step difference method) *A one-step difference method is said to be convergent with respect to the differential equation it approximates if*

$$\lim_{h\to 0}\max_{1\le i\le M}\|x_i - x(t_i)\| = 0\,,$$

where $x(t_i)$ is the value of the solution of the differential equation at $t_i = a + i\,h$ and x_i is the approximation obtained from the difference method at the ith step.

A one-step method is convergent precisely when the solution to the difference equation approaches the solution of the differential equation as the step size goes to zero.

Another type of error, known as round-off error, is introduced to the solution obtained when implementing a numerical scheme on a computer; neither the initial condition nor the arithmetic performed subsequently is represented exactly. Consequently, a numerical method must be used that is stable in the sense that small perturbations in the initial conditions cause only small perturbations in the subsequent approximations; that is, a stable method is one that depends continuously on the initial data. The following theorem (see Burden & Faires[9]) connects the notions of consistency, convergence and stability of a one-step difference method and states an error bound of the numerical solution.

Theorem 1.3.3 *Suppose the initial-value problem (1.3.6) is approximated by the one-step difference method (1.3.7). Suppose also there exist numbers $c > 0$ and $h_0 > 0$ such that $G(t, \mathbf{x}; h)$ is continuous and satisfies a Lipschitz condition with respect to $\mathbf{x} \in I\!R^n$ with Lipschitz constant L on the set $D = \{(t, \mathbf{x}, h)\,|\,a \le t \le b, \|\mathbf{x} - \mathbf{x}_0\| < c, 0 \le h \le h_0\}$. Then*

1. *the one-step difference method depends continuously on the initial value;*
2. *the one-step difference method is convergent if and only if it is consistent; that*

is, if and only if

$$G(t, \mathbf{x}; 0) = f(t, \mathbf{x}), \quad \forall\, t \in [a, b];$$

3. if for each $i = 1, 2, \ldots, M$ *the local truncation error* $\mathcal{L}[x(t_i); h]$ *satisfies*

$$\| \mathcal{L}[x(t_i); h]\| \leq g(h) \quad for \quad 0 \leq h \leq h_0 \quad and \quad g : [0, h_0] \longrightarrow I\!R, \quad then$$

$$\| x(t_i) - x_i\| \leq \frac{g(h)}{L} exp\left(L\,(t_i - a)\right) .$$

The notion of stable and unstable fixed points of a one-step discrete dynamical system is similar to the continuous systems. Here the definition of a fixed point of a discrete dynamical system and the criteria to test that a fixed point of a discrete dynamical system is stable or not will be given

Definition 1.3.8 (Fixed point) *Let* $\mathbf{x}_{k+1} = F(\mathbf{x}_k), k = 0, 1, 2, \ldots,$ *be a one-step discrete dynamical system with* $F : I\!R^n \longrightarrow I\!R^n$. *Then* $\bar{\mathbf{x}} \in I\!R^n$ *is called a fixed point of the dynamical system if* $F(\bar{\mathbf{x}}) = \bar{\mathbf{x}}$.

Theorem 1.3.4 *Let* $\bar{\mathbf{x}}$ *be a fixed point of the one-step dicrete dynamical system* $\mathbf{x}_{k+1} = F(\mathbf{x}_k), k = 0, 1, 2, \ldots, F : I\!R^n \longrightarrow I\!R^n$ *continuously differentiable. Let* $J = \frac{\partial F}{\partial \mathbf{x}}$ *be the Jacobian of* F *at* $\bar{\mathbf{x}}$ *with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. *The spectral radius* ρ *of* J *is defined by* $\rho(J) = \max_{1 \leq i \leq n} |\lambda_i|$. *Then a necessary and sufficient condition for* $\bar{\mathbf{x}}$ *to be asymptotically stable is that* $\rho(J) < 1$. *If* $\rho(J) > 1$ *then* $\bar{\mathbf{x}}$ *is unstable.*

1.4 Partial Differential Equations

Consider a partial differential equation (PDE) in which the independent variables are denoted by $x, y, z, \ldots$ and the dependent variables by $u, v, w, \ldots$.

$$u = u(x, y, z),$$

which, in this particular case, designates u as a function of the independent variables x, y, and z. Partial derivatives are often denoted as follows

$$u_x = \frac{\partial u}{\partial x}; \quad u_y = \frac{\partial u}{\partial y}; \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}; \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}; \cdots.$$

Employing the above notations, a PDE can be represented in the general form

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, \ldots\right) = 0, \qquad (1.4.1)$$

where F is a function of the indicated quantities and at least one partial derivative exists.

1.4.1 Second-Order PDE

Second-order partial differential equations arise in many mathematical models of biomedical systems; for example, in the study of heat flow in the human body, in neurophysiological problems, in genetics, and in drug absorption problems.

Consider the following second-order PDE written in two independent variables

$$a(\cdot)\, u_{xx} + 2\, b(\cdot)\, u_{xy} + c(\cdot)\, u_{yy} + d(\cdot)\, u_x + e(\cdot)\, u_y + f(\cdot)\, u + g(\cdot) = 0 \qquad (1.4.2)$$

The linearity of this equation is established by the functionality of the coefficients $a(\cdot), b(\cdot), \cdots$, and $g(\cdot)$. In the case of (1.4.2), if the coefficients are constants or functions of the independent variables only, the PDE is linear; quasilinear if the coefficients are functions of x, y, u, u_x and u_y; and non-linear otherwise. Typical examples of second-order PDEs are the following well-known equations

$$u_{xx} + u_{yy} = 0 \qquad \textit{Laplace's equation}$$

$$u_{xx} + u_{yy} = f(x, y) \qquad \textit{Poisson's equation}$$

$$u_x = u_{yy} \qquad \textit{heat flow or diffusion equation}$$

$$u_x = u_{yy} + u_{zz} \qquad \textit{heat flow or diffusion equation}$$

$$u_{xx} = u_{yy} \qquad \textit{wave equation.}$$

There exists an extensive body of knowledge regarding linear PDEs, see, for example, Lapidus & Pinder[32]. This information is generally catalogued according to the form of the PDE. Every linear second-order PDE in two independent variables can be converted into one of three standard or *canonical* forms which is identified as hyperbolic, parabolic, or elliptic. In this canonical form at least one of the second-order terms in (1.4.2) is not present.

The classification takes the forms that if

$$b^2 - a\,c > 0 \qquad \textit{the} \quad \textit{PDE} \quad \textit{is} \quad \textit{hyperbolic}$$

$$b^2 - a\,c = 0 \qquad \textit{the} \quad \textit{PDE} \quad \textit{is} \quad \textit{parabolic}$$

$$b^2 - a\,c < 0 \qquad \textit{the} \quad \textit{PDE} \quad \textit{is} \quad \textit{elliptic.}$$

The analytical solution of a PDE, which may be written

$$u = u(x, y),$$

denotes a function that, when substituted back into the PDE, generates an identity. Of course, when the solution of a PDE is discussed, it is necessary to consider appropriate auxiliary initial and boundary conditions.

The concept of *stability* is concerned with the boundedness of the solution of the finite difference equations that approximate the differential equations. Perhaps the most widely used procedure for determining stability (or instability) of a finite difference approximation is called the *von Neumann* stability analysis. In essence, it introduces an initial line of errors as represented by a finite Fourier series and considers the growth (or decay) of these errors as time increases. The method applies, in a theoretical sense, only to pure initial value problems with periodic initial data; as such it neglects completely the influence of boundary conditions. Further, it applies only to linear, constant coefficient, finite difference approximations. If the linearization condition is not met, some form of local linearization is necessary. Because of the linearity, each Fourier component can be treated separately and superposition used to add all other components. It is of practical interest that the von Neumann approach always yields a necessary condition for stability and in many cases this is also a sufficient condition. In Chapter 2, some other methods to discuss the stability are introduced along with the von Neumann method which is applied to prove stability.

One of the most useful and best known tools employed in the study of partial differential equations is the *maximum principle.* This principle is a generalization of the elementary fact of calculus that any function $f(x)$ which satisfies the inequality $f'' > 0$ on an interval $[a, b]$ achieves its maximum value at one of the endpoints of the interval. The solutions of the inequality $f'' > 0$ are said to satisfy a maximum principle. More generally, functions which satisfy a differential inequality in a domain D and, because of it, achieve their maxima on the boundary of D are said to be possess a maximum principle.

The maximum principle enables information about solutions of differential equations to be obtained without any explicit knowledge of the solutions themselves. In particular, the maximum principle is a useful tool in the approximation of solutions, a subject of great interest to many scientists.

In the following, convergence of solutions of mixed initial/boundary-value problems for a certain class of non-linear parabolic equations will be estimated using a maximum principle. Similar estimations will be found in Chapters 4, 5 and 7.

Consider the non-linear parabolic differential equation

$$\frac{\partial^2 u}{\partial x^2} = F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \quad u = u(x,t), \qquad (1.4.3)$$

in the strip $0 < t \leq T$, $0 < x < X$, with the initial condition

$$-\beta_0(x)\, u(x, 0) = f_0(x) \qquad (1.4.4)$$

and boundary conditions

$$\begin{aligned} \alpha_1(t)\, u_x(0,t) &- \beta_1(t)\, u(0,t) = f_1(t), \\ \alpha_2(t)\, u_x(X,t) &- \beta_2(t)\, u(X,t) = f_2(t), \end{aligned} \qquad (1.4.5)$$

assuming that the solution $u(x,t)$ is unique and exists with suitable regularity properties in the strip.

Consider the non-linear parabolic operator

$$L[u] \equiv \frac{\partial^2 u}{\partial x^2} - F\left(x,t,u,u_x,u_t\right). \qquad (1.4.6)$$

In (1.4.6), $F\left(x,t,u,u_x,u_t\right)$ denotes a fixed continuous function of its variables (x,t) in a region Ω in the (x,t)-plane and for all u, u_x and u_t. Assume that the partial derivatives F_u, F_{u_x} and F_{u_t} exist, are continuous and satisfy

$$\begin{aligned} 0 < a_0 < \left|F_{u_t}\right| \le a_1 < \infty, \\ \left|F_{u_x}\right| \le b_1 < \infty, \\ 0 \le c_0 \le |F_u| \le c_1 < \infty, \end{aligned} \qquad (1.4.7)$$

where a_0, a_1, b_1, c_0 and c_1 are fixed constants.

Let $\Omega\,[0 < x < X, 0 < t \le T]$ be bounded by the coordinate lines $x = 0$, $t = 0$ and the lines $x = X$ and $t = T$; the closure of Ω will be denoted by $\bar{\Omega}$. The set composed of the segments $\partial\Omega_0(0 < x < X, t = 0)$, $\partial\Omega_1(x = 0, 0 < t \le T)$ and $\partial\Omega_2(x = X, 0 < t \le T)$ will be denoted by $\partial\Omega$ and called the boundary of Ω.

The boundary operators are Λ_0, Λ_1 and Λ_2 and are defined by

$$\begin{aligned} \Lambda_0[u] &= -\beta_0(x)\, u(x,0) \qquad on \quad \partial\Omega_0 \\ \Lambda_1[u] &= \alpha_1(t)\, u_x(0,t) - \beta_1(t)\, u(0,t) \qquad on \quad \partial\Omega_1 \\ \Lambda_2[u] &= \alpha_2(t)\, u_x(X,t) - \beta_2(t)\, u(X,t) \qquad on \quad \partial\Omega_2. \end{aligned} \qquad (1.4.8)$$

Here β_0, β_1 and β_2 are continuous positive functions, and α_1 and α_2 are continuous non-negative functions on $\partial\Omega_0$, $\partial\Omega_1$ and $\partial\Omega_2$. Let f_0, f_1 and f_2 be fixed functions defined on $\partial\Omega_0$, $\partial\Omega_1$ and $\partial\Omega_2$, respectively.

A mixed initial/boundary-value problem $\mathcal{P}$ may be formulated as follows: for fixed T, determine a function $u(x,t)$ defined in Ω with certain regularity properties satisfying the equation

$$L[u] = 0 \qquad in \quad \Omega$$

and the initial and boundary conditions

$$\Lambda_i[u] = f_i \qquad on \quad \partial\Omega_i \quad (i = 0,1,2).$$

It is assumed that this problem has at most one solution which exists with suitable regularity properties under appropriate regularity conditions on the operators L, Λ_i $(i = 0,1,2)$ and on the initial and boundary data; specifically, it is assumed that u_{xxxx}, u_{xxt}, u_{tt} and lower-order mixed partial derivatives exists and are continuous in Ω.

Next, let $\mathcal{P}_h$ be the approximate solution of the problem $\mathcal{P}$ which consists of finding a function u^h defined on Ω and satisfying the equation

$$L_h\left[u^h\right] = 0 \qquad in \quad \Omega$$

and the initial and boundary conditions

$$\Lambda_i^h\left[u^h\right] = f_i^h \qquad on \quad \partial\Omega_i \quad (i = 0,1,2),$$

where f_i^h is a given function on $\partial\Omega_i$, and L_h and Λ_i^h will be defined in (7.4.3)–(7.4.4).

Rose[53] approximated the differential equation (1.4.6) by a family of implicit-difference equations

$$\begin{aligned}\phi\,\nabla^2\Psi(x,t) \;&+\; (1-\phi)\,\nabla^2\Psi(x,t-k) = F\left(x,t,\Psi(x,t),\phi\,\nabla_x\Psi(x,t)\right.\\ &+\; \left.(1-\phi)\,\nabla_x\Psi(x,t),\nabla_t\Psi(x,t)\right), \qquad (1.4.9)\end{aligned}$$

with $0 \le \phi \le 1$ and ∇^2, ∇_x and ∇_t as defined in (7.4.4). Rose[53] showed that

$$\|u - \Psi\|_\infty = O\left(h^2 + \ell\right)$$

for any value of the mesh ratio $p = \ell/h^2$ provided that

$$0 \leq 2\,(1-\phi)\,p \leq a_0 .$$

The following theorem which may be found in Rose[53] and Lees[33] will be used frequently in the analysis in Chapters 4, 5 and 7.

Theorem 1.4.1 *Let problem $\mathcal{P}$ be approximated by $\mathcal{P}_h$ in the sense that*

$$\max_i \max_{\partial\Omega_i} \left| f_i - f_i^h \right| = \begin{cases} \alpha^* \, O(h) & if \quad \alpha^* \neq 0, \\ O(h^2) & if \quad \alpha^* = 0. \end{cases}$$

Then, if $h, \ell \longrightarrow 0$ in such a way that

$$p = \frac{\ell}{h^2} \leq \frac{a_0}{2\,\phi},$$

the solution u^h of $\mathcal{P}_h$ approximates the solution u of problem $\mathcal{P}$ uniformly in $\bar{\Omega}$; that is,

$$\left\| u - u^h \right\|_\infty = \alpha^* \, O(h) + O(h^2 + \ell) \, . \qquad (\ 1.4.10\ )$$

Here $\alpha^* = \max_i \, \max_{\partial\Omega_i} \, |\alpha_i| \quad (i = 1, 2)$.

Chapter 2

Predator-Prey Model

2.1 Introduction

When species interact the population dynamics of each species is affected. In general there is a web of interacting species, called a *trophic web* (see May[41] and Murray[48]), leading to structurally complicated communities. There are three main types of interaction:

1. If the growth rate of one population is decreased and the other is increased the populations are in a predator-prey situation; examples include sharks-food fish, foxes-rabbits and ladybird-cottony cushion.

2. If the growth rate of each population is decreased then it is competition, example is *paramecium aurelia* and *paramecium caudatum.*

3. If each population's growth rate is enhanced then it is called *mutualism* or *symbiosis.* Plant and seed is an example.

A system of the first type is discussed and analysed in detail and some numerical results are given in this chapter.

The classical model for the predator-prey system (see, for instance, Comins & Blatt[11]; May[41]; Murray[48]; Sandefur[56]) with continuous growth is that of Lotka[37] and Volterra[70] (L-V)

$$\frac{dP}{dt} = P(a - \alpha H), \qquad (2.1.1)$$

$$\frac{dH}{dt} = H(-M + \beta P). \qquad (2.1.2)$$

Here $P = P(t)$ and $H = H(t)$ are the populations (densities) of prey and predators, respectively, at time t. The parameter a relates to the birth rate of the prey, M to the death rate of the predator and α, β to the interaction between the species: all are positive numbers. These equations constitute the simplest representation of the essentials of a non-linear predator-prey interaction.

Some assumptions have been considered for the L-V model, (2.1.1) and (2.1.2). These assumptions (see Murray[48]; Sandefur[56]) are:

- The prey in the absence of any predation grows unboundedly; this is modelled by the term aP in (2.1.1).
- The prey's contribution to the predator's growth rate is βPH; that is, it is proportional to the available prey as well as to the size of the predator population. This means that the number of species P that are eaten by species H during time period t depends on both P and H. Specifically, since each individual animal of species H is hunting for prey, the larger H, the more of species P that are eaten. Also, the larger P, the easier it is to find animals of species P, and thus each predator will eat more prey. In short, the number of species P that is eaten is proportional to both P and H.
- The effect of the predation is to reduce the prey's *per capita* growth rate by a term proportional to the prey and predator populations so that the prey do not

inhibit their own growth; this is modelled by the term $-\alpha P H$ in (2.1.1).

- In the absence of any prey for sustenance the predator's death rate results in exponential decay: this is modelled by the term $-MH$ in (2.1.2).

The $P H$ terms can be thought of as representing the conversion of energy from one source to another: $\alpha P H$ is taken from the prey and $\beta P H$ accrues to the predators.

The L-V model exhibits neutrally stable dynamics (Britton[7]; May[42]): predator and prey populations oscillate with constant amplitudes that are determined by their initial distances from equilibrium. Because of its neutral stability, the L-V model has been criticized as a poor description of persisting systems (May[42]).

Dispersal of predators and prey has also been well studied, both as a potentially stabilizing mechanism and as a means to generate or maintain spatial heterogeneity in species distributions. Most models of dispersing populations can be divided into two classes according to the way that they treat space. Discrete space or patch models partition the environment into two or more compartments the dynamics of which are coupled by migration (examples are given by Chewing[10]; Hastings[22]; McMurtie[45]). The second class of models treat space as an explicit and continuous variable (Comins & Blatt[11]; McLaughlin & Roughgarden[44]; Pascual[52]).

2.2 A More Realistic Predator-Prey Model

The Lotka-Volterra (L-V) model, (2.1.1) and (2.1.2), unrealistic though it is, does show that simple predator-prey interaction can result in oscillatory behaviour of the densities. Reasoning heuristically (Murray[48]) this is not unexpected since, if a prey density increases, it encourages growth of its predator. More predators, however, consume more prey, the population of which starts to decline. With less food available the predator population declines and when it is low enough, this allows the prey population to increase and the whole cycle starts over again. Depending on the detailed system such oscillations can grow or decay or go into a limit cycle or even exhibit chaotic behaviour.

One of the unrealistic assumptions in the L-V model (2.1.1) and (2.1.2) is that the prey growth is unbounded in the absence of predation. In the form that equations (2.1.1) and (2.1.2) have been written, the bracketed terms on the right-hand sides are the density dependent *per capita* growth rates. To be more realistic, Kolmogorov[29] (see May[41], p.86) has written these growth rates as

$$\frac{dP}{dt} = P\,F(P,H)\,, \qquad (\ 2.2.1\ )$$

$$\frac{dH}{dt} = H\,G(P,H)\,, \qquad (\ 2.2.2\ )$$

where the forms of F and G depend on the interaction, the species and so on, and then set out, in general terms, conditions which necessarily lead to the systems having *either* a stable point *or* a stable limit cycle.

Theorem 2.2.1 (Kolmogorov) *Predator-prey systems of the form (2.2.1) and (2.2.2) have either a stable equilibrium point or a stable limit cycle, provided that F and G are continuous functions of H and P, with continuous first derivatives, throughout the domain $H \geq 0$, $P \geq 0$ and that*

(i) $\partial F/\partial H < 0$

(ii) $H\,(\partial F/\partial H) + P\,(\partial F/\partial P) < 0$

(iii) $\partial G/\partial H < 0$

(iv) $H\,(\partial G/\partial H) + P\,(\partial G/\partial P) > 0$

(v) $F(0,0) > 0$

It is also required that there exist quantities A, B, and C such that

(vi) $F(0,A) = 0$, *with* $A > 0$

(vii) $F(B,0) = 0$, *with* $B > 0$

(viii) $G(C,0) = 0$, *with* $C > 0$

(ix) $B > C$.

The proof follows straight forwardly from the Poincaré-Bendixon theorem, one of the key theorems of non-linear stability analysis. The proof of the Kolmogorov theorem will not be given here, because the necessary theory is covered well in Minorsky[46]. The theorem also usually holds when certain of the above conditions are equalities rather than inequalities.

In more biological terms, Kolmogorov's conditions are that (i) for any given population size (as measured by numbers, etc.), the *per capita* rate of increase of the prey species is a decreasing function of the number of predators. For any given ratio between the two species, (ii), the rate of increase of the prey is a decreasing function of population size. Condition (iii) states that the the rate of increase of predators decreases with their population size. For any given ratio between the two species, (iv), the rate of increase of the predators is an increasing function. It also requires, (v), that when both populations are small the prey have a positive rate of increase, and that (vi) there can be a predator population size sufficiently large to stop further prey increase, even when the prey are rare. Condition (vii) requires a critical prey population size B, beyond which they cannot increase even in the absence of predators (a resource or other self limitation), and (viii) requires a critical prey size C that stops further increase in predators, even if they are rare; unless (ix) $B > C$, the system will collapse.

As a reasonable step (see Murray[48]; p.71), the prey might be expected to satisfy a logistic growth rate, say, in the absence of any predators. So, for example, a more realistic prey population equation might take the form

$$\frac{dP}{dt} = P\,F(P,H), \qquad F(P,H) = r\,(1 - \frac{P}{K}) - H\,E(P)\,, \tag{2.2.3}$$

where K is the constant carrying capacity for the prey when $H = 0$ and $E(P)$ is one of the predation terms, which are the functional responses of the predator to change in the prey density. Some examples are (May[41], Ch.4; Murray[48], Ch.3):

$$E(P) = \frac{A\,C_1}{C_2 + P}, \qquad (2.2.4)$$

$$E(P) = \frac{A\,P}{C_2^2 + P^2}, \qquad (2.2.5)$$

$$E(P) = \frac{A\,[1 - e^{-aP}]}{P}, \qquad (2.2.6)$$

in which A, C_1 and C_2 are positive constants.

The predator population equation, (2.1.2), could also be made more realistic by taking, for example,

$$G(P, H) = d\,(1 - \frac{e\,H}{P}), \qquad (2.2.7)$$

$$G(P, H) = -f + g\,E(P), \qquad (2.2.8)$$

in which d, e, f and g are positive constants and $E(P)$ has one of the forms given in (2.2.4)-(2.2.6).

Many models of prey-predator interactions have been suggested; some with very complicated terms depending, for example, on resource limitations and random fluctuations in time (see May[41]). In most of these the spatial environments have been completely homogeneous, with populations assumed uniform in space, although the case of migration between a number of these spatially nonvarying environments has also been considered (Levins[35]; Maynard Smith[43]).

Some models which exhibit chaotic dynamics because of non-linearities in population growth and interspecific interaction are being reported in the literature, see, for example, Hastings & Powell[23]. The models have, for the most part, ignored space. Explicit consideration of space, however, can alter the dynamics of non-linear interactions fundamentally (Levin & Segel[34]; Segel & Jackson[60]).

The effect of environmental heterogeneity on models of prey-predator systems was investigated by Comins & Blatt[11], McLaughlin & Roughgarden[44] and Pascual[52]. It was found by Comins & Blatt[11] that environmental heterogeneity can have important effects on prey-predator population dynamics. It is not necessary to have inaccessible regions or discontinuities in the environment in order to observe refuge features. In addition, Comins & Blatt[11] found that the heterogeneity has an important stabilizing effect, not produced by simple diffusion. In contrast, Pascual[52] found that local oscillations gave rise to complex temporal dynamics. This was because of the scaled diffusion coefficient. Chaos and quasiperiodicity occurred for the diffusion coefficient in the order of 10^{-4} to 10^{-3}.

The few existing ecological studies of chaos in spatial systems consider models in discrete time and space (Hassell et al.[21]; Solé & Valls[61]) or in discrete time and continuous space (Kot[30]). In all these models, the diffusive dispersal of organisms drives a predator-prey or a host-parasitoid system into chaotic dynamics.

The results of discrete models cannot be applied to non-linear interactions and dispersal in continuous time and space. It is well known that discrete models exhibit chaos more readily than their continuous counterparts. For instance, chaotic dynamics is possible for discrete time models of even a single species, but requires at least three variables in continuous time.

2.3 The Model

The mathematical model, in its simplest form, which will be studied in the following sections is the predator-prey model which was given by Pascual[52] in which a single spatial dimension is considered along which both species diffuse at the same rate D. At any point X and time T, the dynamics of the prey $(P(X,T))$ and predator $(H(X,T))$ populations are given by a reaction-diffusion model with logistic growth of the prey, as described by Pascual[52], and a "type II functional response" of the predator:

$$\frac{\partial P}{\partial T} = R\,P(1-\frac{P}{K}) - \frac{A\,C_1\,P}{C_2+P}H + D\frac{\partial^2 P}{\partial X^2} \qquad (2.3.1)$$

$$\frac{\partial H}{\partial T} = \frac{C_1\,P}{C_2+P}H - M\,H + D\frac{\partial^2 H}{\partial X^2} \qquad (2.3.2)$$

The parameters R, K, M and $1/A$ denote the intrinsic growth rate and carrying capacity of the prey, the death rate of the predator, and the yield coefficient of prey to predator, respectively. The constants C_1 and C_2 parametrize the saturating functional response. All parameters and constants are positive.

To describe an environment surrounded by dispersal barriers, Pascual[52] assumed zero flux at the boundaries. Hence, at the boundaries $X = 0$ and $X = L$, for all T,

$$\frac{\partial P}{\partial X} = \frac{\partial H}{\partial X} = 0. \qquad (2.3.3)$$

A simple form of environmental heterogeneity can be introduced by allowing the parameters in (2.3.1) and (2.3.2) to vary with X. Thus, the case where the prey rate of increase R is a linear function of X is considered.

In order to decrease the number of parameters, the model described by (2.3.1) and (2.3.2) can be simplified by introducing the dimensionless variables $p = \frac{P}{K}$ and $h = \frac{AH}{K}$. Space is scaled by the total length of the gradient L, and time is scaled by a characteristic value of the prey growth rate $\overline{R}$. Thus, $x = \frac{X}{L}$ and $t = \overline{R}T$ where $\overline{R} = R(X_0)$ for some X_0 in $(0, L)$. System (2.3.1) and (2.3.2) thus becomes

$$\frac{\partial p}{\partial t} = r\,p\,(1-p) - \frac{a\,p}{1+b\,p}\,h + d\,\frac{\partial^2 p}{\partial x^2}\,, \qquad (2.3.4)$$

$$\frac{\partial h}{\partial t} = \frac{a\,p}{1+b\,p}\,h - m\,h + d\,\frac{\partial^2 h}{\partial x^2}\,, \qquad (2.3.5)$$

in which the new parameters are $r = \frac{R}{\overline{R}} = e + f\,x$, $a = \frac{C_1 K}{C_2 \overline{R}}$, $b = \frac{K}{C_2}$, $m = \frac{M}{\overline{R}}$ and $d = \frac{D}{L^2 \overline{R}}$. At the boundaries, given now by $x = 0$ and $x = 1$, and for all t,

$$\frac{\partial p}{\partial x} = \frac{\partial h}{\partial x} = 0\,. \qquad (2.3.6)$$

Following Pascual[52], the set of parameters considered in numerical experiments will be

$$\begin{aligned} a &= 5.0 \\ b &= 5.0 \\ m &= 0.6 \\ e &= 2.0 \\ f &= -1.4 \\ x &= 0.85\,. \end{aligned} \qquad (2.3.7)$$

The initial conditions (see May[41], p.43), $0 \le x \le 1$, are

$$\begin{aligned} p(x,0) &= 0.5\,, \\ h(x,0) &= 1.0\,. \end{aligned} \qquad (2.3.8)$$

2.4 The Initial-Value Problem

In the absence of diffusion, the system (2.3.1) and (2.3.2) is a standard predator-prey system, which exhibits, using the set of parameters given in (2.3.7), stable equilibria or limit cycles (May[41]), see Figure 2.1[1]. Therefore the resulting system is the initial-value problem (IVP)

$$\frac{dp}{dt} = r\,p\,(1-p) - \frac{a\,p}{1+b\,p}\,h\,, \qquad (2.4.1)$$

[1] Figure 2.1 was drawn by the software package "mathematica", version 2.2, ©1988-93 by Wolfram Research, Inc. The other graphs were drawn using the software package "matlab", version 4.0a, ©1984-92 The Math Works, Inc.

$$\frac{dh}{dt} = \frac{a\,p}{1+b\,p}\,h - m\,h\,, \qquad (2.4.2)$$

with the intial conditions

$$\begin{aligned} p(0) &= 0.5, \quad 0 \leq x \leq 1\,, \\ h(0) &= 1.0, \quad 0 \leq x \leq 1\,. \end{aligned} \qquad (2.4.3)$$

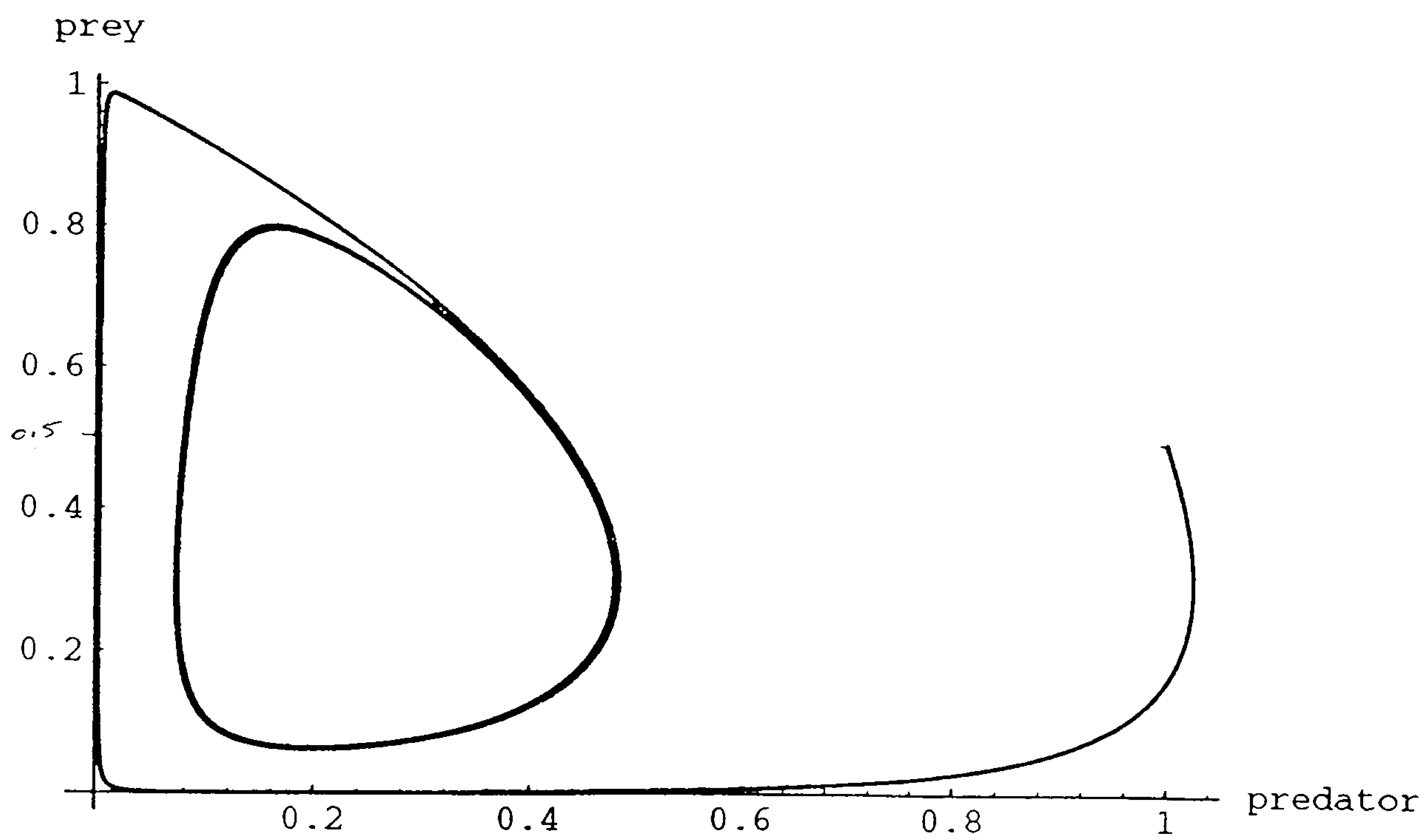

Figure 2.1: *Predator-prey dynamics in the absence of diffusion*

2.4.1 Stationary Points

The first thing to investigate when analysing a dynamical system is the existence of stationary or equilibrium or critical points. For the predator-prey system {(2.4.1), (2.4.2)}, the stationary points are determined by equating the right-hand sides of (2.4.1) and (2.4.2) to zero. Clearly there are three distinct stationary points, namely,

$$s_1 = (0,0) \quad (\textit{trivial stationary point}), \quad s_2 = (1,0), \quad \textit{and} \quad s_3 = (p_s, h_s) \quad . \qquad (2.4.4)$$

where $p_s = \frac{m}{a-mb}$ and $h_s = \frac{(a-m-bm)\,r}{(a-bm)^2}$. The first two are not of interest since then there would not be two species remaining.

Using the set of parameters in (2.3.7), the stationary point s_3 becomes

$$s_3 = (0.3, 0.2835). \qquad (2.4.5)$$

The Jacobian of the system is given by

$$J(p,h) = \begin{pmatrix} \frac{abhp}{(1+bp)^2} - \frac{ah}{1+bp} + (1-p)\,r - pr & -\frac{ap}{1+bp} \\ -\frac{abhp}{(1+bp)^2} + \frac{ah}{1+bp} & -m + \frac{ap}{1+bp} \end{pmatrix}. \qquad (2.4.6)$$

Using the equilibrium points s_1, s_2 and s_3 and the set of parameters given in (2.3.7), it is easy to show that for s_1 and s_2, the Jacobian has one positive eigenvalue and one negative eigenvalue showing that these are unstable stationary points (these are called *saddle points*) while for s_3, the Jacobian has a pair of complex eigenvalues with positive real parts, namely $0.0486 \pm 0.3657\,i$. Thus, s_3 is an unstable equilibrium point. However, Kolmogorov's theorem shows that for this particular model there is either a stable equilibrium point or a stable limit cycle. Thus, the system has a stable limit cycle, as shown in Figure 2.1.

2.4.2 Numerical Methods

The solution of the predator-prey model given by (2.4.1)-(2.4.3) is sought for $t > 0$ and to obtain a numerical solution, the time interval $t \geq 0$ is discretized at the points $t_n = nl \quad (n = 0, 1, 2, \ldots)$; ℓ is called the *time step.* The theoretical solutions of the system at the typical point $t = t_n$ are given, respectively, by $p(t_n)$ and $h(t_n)$, while the corresponding solutions of a numerical method will be denoted by P^n and H^n, respectively.

The numerical methods are based on the replacement of $\frac{dp}{dt}$ and $\frac{dh}{dt}$ by the first-order approximations

$$\frac{dp(t)}{dt} = [\,p(t+\ell) - p(t)\,]/\ell + O\,(\ell)\ , \qquad (2.4.7)$$

and

$$\frac{dh(t)}{dt} = [\,h(t+\ell) - h(t)\,]/\ell + O\,(\ell)\ , \qquad (\ 2.4.8\ )$$

as $\ell \longrightarrow 0$.

Method 1 (Euler's Method):

Using (2.4.7) in (2.4.1) and (2.4.8) in (2.4.2) with $t = t_n$ and evaluating p and h on the right-hand side of (2.4.1) and (2.4.2) at $t = t_n$ leads to

$$P^{n+1} = P^n + \ell\, r\, P^n\,(1 - P^n) - \frac{\ell\, a\, P^n\, H^n}{1 + b\, P^n}\,; \quad n = 0, 1, 2, \ldots\ , \qquad (\ 2.4.9\ )$$

$$H^{n+1} = H^n + \frac{\ell\, a\, P^n\, H^n}{1 + b\, P^n} - \ell\, m\, H^n\,; \quad n = 0, 1, 2, \ldots\ . \qquad (\ 2.4.10\ )$$

which is the familiar Euler explicit method.

Method 2:

Here the linear factor $1 - p$ on the right-hand side of (2.4.1) is evaluated at time $t = t_n$ while the other factor, p, is evaluated at time $t = t_{n+1}$. Replacing $\frac{dp}{dt}$ in (2.4.1) by (2.4.7) and $\frac{dh}{dt}$ in (2.4.2) by (2.4.8) then gives the implicit formulae

$$P^{n+1} = P^n + \ell\, r\, P^{n+1}\,(1 - P^n) - \frac{\ell\, a\, P^{n+1}\, H^n}{1 + b\, P^n}\,; \quad n = 0, 1, 2, \ldots\ , \qquad (\ 2.4.11\ )$$

$$H^{n+1} = H^n + \frac{\ell\, a\, P^{n+1}\, H^{n+1}}{1 + b\, P^n} - \ell\, m\, H^{n+1}\,; \quad n = 0, 1, 2, \ldots\ . \qquad (\ 2.4.12\ )$$

which, provided that both $\ell r(1 - P^n) - \frac{\ell a H^n}{1+bP^n} \neq 1$ and $\frac{\ell a P^{n+1}}{1+bP^n} - \ell m \neq 1$, may be rearranged to give the explicit method

$$P^{n+1} = \frac{P^n}{E^n}\,; \quad n = 0, 1, 2, \ldots\ , \qquad (\ 2.4.13\ )$$

$$H^{n+1} = \frac{H^n}{1 + \ell m - \frac{\ell a P^n}{(1+bP^n)\,E^n}}\,; \quad n = 0, 1, 2, \ldots\ , \qquad (\ 2.4.14\ )$$

where $E^n = 1 - \ell\, r\,(1 - P^n) + \frac{\ell a H^n}{1+b\,P^n}$.

Method 3:

Now, replacing the derivatives in (2.4.1) and (2.4.2) once more by (2.4.7) and (2.4.8), respectively, evaluating the linear factor $1-p$ on the right- hand side of (2.4.1) at time $t = t_{n+1}$ and the linear factor p at time $t = t_n$ gives the second pair of implicit formulae

$$P^{n+1} = P^n + \ell\, r\, P^n\,(1 - P^{n+1}) - \frac{\ell\, a\, P^n\, H^n}{1 + b\, P^n}; \quad n = 0, 1, 2, \ldots, \qquad (2.4.15)$$

$$H^{n+1} = H^n + \frac{\ell\, a\, P^{n+1}\, H^{n+1}}{1 + b\, P^n} - \ell\, m\, H^{n+1}; \quad n = 0, 1, 2, \ldots, \qquad (2.4.16)$$

which, provided that both $\ell\, r\, P^n \neq -1$ and $\frac{\ell\, a\, P^{n+1}}{1+bP^n} - \ell\, m \neq 1$, may be rearranged to give the method

$$P^{n+1} = \frac{(1 + \ell\, r - \frac{\ell\, a\, H^n}{1+bP^n})\, P^n}{1 + \ell\, r\, P^n}; \quad n = 0, 1, 2, \ldots, \qquad (2.4.17)$$

$$H^{n+1} = H^n / \left[1 + \ell\, m - \frac{\ell\, a\, P^n\,(1 + \ell\, r - \frac{\ell\, a\, H^n}{1+bP^n})}{(1 + bP^n)(1 + \ell\, r\, P^n)} \right]; \quad n = 0, 1, 2, \ldots. \qquad (2.4.18)$$

Clearly, this method generates the solution explicitly.

2.4.3 Local Truncation Errors

Method 1:

The local truncation errors (l.t.e.'s) of this method at time $t = t_n$ may be determined (see, for example, Lambert[31], p.48) from (2.4.9) and (2.4.10) and are given by

$$\mathcal{L}_p[p(t), h(t); \ell] = p(t + \ell) - p(t) - \ell\, r\, p(t)\,[1 - p(t)] + \frac{\ell\, a\, p(t)\, h(t)}{1 + b\, p(t)},$$

$$\mathcal{L}_h[p(t), h(t); \ell] = h(t + \ell) - h(t) - \frac{\ell\, a\, p(t)\, h(t)}{1 + b\, p(t)} + \ell\, m\, h(t),$$

from which it follows that, after expanding the functions $p(t+\ell)$ and $h(t+\ell)$ using Taylor's expansion,

$$\mathcal{L}_p[p(t),h(t);\ell] = \frac{1}{2}\ell^2\, p'' + O\left(\ell^3\right) \qquad as \quad \ell \quad \longrightarrow 0 \quad . \tag{2.4.19}$$

and

$$\mathcal{L}_h[p(t),h(t);\ell] = \frac{1}{2}\ell^2\, h'' + O\left(\ell^3\right) \qquad as \quad \ell \longrightarrow 0 \quad , \tag{2.4.20}$$

at some point $t = t_n$, verifying that this familiar numerical method is first-order accurate.

Method 2:

The local truncation errors (l.t.e.'s) of this method at time $t = t_n$ are obtained from (2.4.11) and (2.4.12) and are given by

$$\mathcal{L}_p[p(t),h(t);\ell] = p(t+\ell) - p(t) - \ell\, r\, p(t+\ell)\,[1-p(t)] + \frac{\ell\, a\, p(t+\ell)\, h(t)}{1+b\,p(t)},$$

$$\mathcal{L}_h[p(t),h(t);\ell] = h(t+\ell) - h(t) - \frac{\ell\, a\, p(t+\ell)\, h(t+\ell)}{1+b\,p(t)} + \ell\, m\, h(t+\ell),$$

in which $t = t_n$, which, after simplifying, give

$$\mathcal{L}_p[p(t),h(t);\ell] = \left(\frac{1}{2}p'' - rp' + rpp' + \frac{ahp'}{1+bp}\right)\ell^2 + O\left(\ell^3\right) \qquad as \quad \ell \longrightarrow 0 \quad . \tag{2.4.21}$$

and

$$\mathcal{L}_h[p(t),h(t);\ell] = \left(\frac{1}{2}h'' - \frac{ahp' + ah'p}{1+bp} + mh'\right)\ell^2 + O\left(\ell^3\right) \qquad as \quad \ell \longrightarrow 0 \quad , \tag{2.4.22}$$

verifying that this method is also first-order accurate.

Method 3:

The local truncation errors of this method at time $t = t_n$ are obtained from (2.4.15) and (2.4.16) and are given by

$$\mathcal{L}_p[p(t), h(t); \ell] = p(t+\ell) - p(t) - \ell\, r\, p(t)\,[1 - p(t+\ell)] + \frac{\ell\, a\, p(t)\, h(t)}{1 + b\, p(t)},$$

$$\mathcal{L}_h[p(t), h(t); \ell] = h(t+\ell) - h(t) - \frac{\ell\, a\, p(t+\ell)\, h(t+\ell)}{1 + b\, p(t)} + \ell\, m\, h(t+\ell),$$

which, after simplifying, give

$$\mathcal{L}_p[p(t), h(t); \ell] = \left(\frac{1}{2} p'' + r p p'\right) \ell^2 + O\left(\ell^3\right) \qquad as \quad \ell \longrightarrow 0 \quad , \qquad (2.4.23)$$

and

$$\mathcal{L}_h[p(t), h(t); \ell] = \left(\frac{1}{2} h'' - \frac{a h' p + a h p'}{1 + b p} + m h'\right) \ell^2 + O\left(\ell^3\right) \qquad as \quad \ell \longrightarrow 0 \quad , \qquad (2.4.24)$$

at time $t = t_n$, revealing that this method is first-order accurate, too.

2.4.4 Analysis of the Fixed Points of the Methods

Finding the fixed points of the finite-difference method is equivalent to finding the stationary points of the initial-value problem (2.4.1)-(2.4.3). It can be shown that the fixed points of each finite-difference method, as $n \longrightarrow \infty$, are the same as the stationary points of the system.

Method 1 (Euler's Method):

This method is explicit and hence can be written in the form

$$P^{n+1} = g_1\left(P^n, H^n\right), \qquad (2.4.25)$$

$$H^{n+1} = g_2\left(P^n, H^n\right); \qquad (2.4.26)$$

$n = 0, 1, 2, \ldots$. Associate, now, with (2.4.25) and (2.4.26) the functions

$$P = g_1(P, H) \qquad (2.4.27)$$

and

$$H = g_2(P, H), \qquad (2.4.28)$$

then from (2.4.9) and (2.4.10) g_1 and g_2 have the forms

$$g_1(P, H) = P + \ell r P(1 - P) - \frac{\ell a H P}{1 + bP}, \qquad (2.4.29)$$

$$g_2(P, H) = H - \ell m H + \frac{\ell a H P}{1 + bP}, \qquad (2.4.30)$$

provided that $1 + bP \neq 0$.

To examine the stability of the fixed points s_i $(i = 1, 2, 3)$ in (2.4.4) of (2.4.9), (2.4.10) or (2.4.11), (2.4.12) or (2.4.15) and (2.4.16), the eigenvalues $\lambda_i \in \mathbb{C}\,(i = 1, 2)$ of the Jacobian

$$J(P_s, H_s) = \begin{pmatrix} \frac{\partial g_1(P_s,H_s)}{\partial P} & \frac{\partial g_1(P_s,H_s)}{\partial H} \\ \frac{\partial g_2(P_s,H_s)}{\partial P} & \frac{\partial g_2(P_s,H_s)}{\partial H} \end{pmatrix} \qquad (2.4.31)$$

must be calculated, where

$$\begin{aligned} \frac{\partial g_1}{\partial P} &= 1 - \ell P r + \ell r(1 - P) + \frac{abH\ell P}{(1+bP)^2} - \frac{aH\ell}{1+bP}, \\ \frac{\partial g_1}{\partial H} &= -\frac{a\ell P}{1+bP}, \\ \frac{\partial g_2}{\partial P} &= \frac{aH\ell}{1+bP} - \frac{abH\ell P}{(1+bP)^2}, \\ \frac{\partial g_2}{\partial H} &= 1 - \ell m + \frac{\ell a P}{1+bP}. \end{aligned} \qquad (2.4.32)$$

Evaluating the derivatives in (2.4.32) at the fixed points s_i $(i = 1, 2, 3)$ and substituting, using the set of parameters in (2.3.7), in the Jacobian matrix (2.4.31) gives the following results

$$J_1 = \begin{pmatrix} 1 + 0.81\,\ell & 0 \\ 0 & 1 - 0.6\,\ell \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 1 - 0.81\,\ell & -\frac{5}{6}\,\ell \\ 0 & 1 + \frac{7}{30}\,\ell \end{pmatrix},$$

and

$$J_3 = \begin{pmatrix} 1 + 0.0972\,\ell & -0.6\,\ell \\ 0.2268\,\ell & 1 \end{pmatrix}.$$

For the trivial fixed points, s_1 and s_2, the eigenvalues of J_1 and J_2 are equal to unity at $\ell = 0$, while for $\ell > 0$, the spectral radius is greater than unity.

Considering the nontrivial point s_3, the eigenvalues of the Jacobian are equal to unity for $\ell = 0$. When $\ell > 0$, the eigenvalues are complex with positive real part; the spectral radius is greater than unity. Thus all the fixed points of this method are unstable for all $\ell > 0$.

Method 2:

This method can be written explicitly in the form

$$P^{n+1} = g_1\left(P^n, H^n\right), \qquad (2.4.33)$$

$$H^{n+1} = g_2\left(P^n, H^n\right); \qquad (2.4.34)$$

$n = 0, 1, 2, \ldots$. Associate with (2.4.33) and (2.4.34) the functions

$$P = g_1\left(P, H\right) \qquad (2.4.35)$$

and

$$H = g_2\left(P, H\right), \qquad (2.4.36)$$

then (2.4.11) and (2.4.12) can be written in the forms

$$g_1(P,H) = \frac{P}{E}, \tag{2.4.37}$$

$$g_2(P,H) = \frac{H}{1+\ell m - \frac{\ell a P D}{E}}, \tag{2.4.38}$$

where

$$\begin{aligned} D &= \tfrac{1}{1+bP}, \\ E &= 1 - \ell r(1-P) + \tfrac{\ell a H}{1+bP}. \end{aligned} \tag{2.4.39}$$

Note that, $\frac{dD}{dP} = -bD^2$, $\frac{\partial E}{\partial P} = \ell r - \ell a b H D^2$ *and* $\frac{\partial E}{\partial H} = \ell a D.$

It may be shown that

$$\begin{aligned} \frac{\partial g_1}{\partial P} &= \frac{1-\ell r+\ell a H D(1+bDP)}{E^2}, \\ \frac{\partial g_1}{\partial H} &= -\frac{\ell a D P}{E^2}, \\ \frac{\partial g_2}{\partial P} &= \frac{\ell a H[-bD^2 EP + DE - \ell DPr + \ell a b D^3 HP]}{(1+\ell m - \frac{\ell a D P}{E})^2 E^2}, \\ \frac{\partial g_2}{\partial H} &= \frac{(1+\ell m)E^2 - \ell a D P(E+\ell a D H)}{(1+\ell m - \frac{\ell a D P}{E})^2 E^2}. \end{aligned} \tag{2.4.40}$$

Evaluating the derivatives in (2.4.40) at the fixed points s_i $(i = 1,2,3)$ and substituting, using the set of parameters in (2.3.7), in the Jacobian matrix given in (2.4.31) gives the following results:

For s_1, the eigenvalues are all equal to unity at $\ell = 0$. The spectral radius is greater than unity for $0 < \ell < 2.466$, while it is less than unity provided $\ell \geq 2.466$. The trivial fixed point s_1 is unstable for $0 < \ell < 2.466$ and stable for $\ell \geq 2.466$.

Considering the fixed point s_2, the spectral radius is greater than unity for all $\ell > 0$ and equals unity at $\ell = 0$.

For s_3, the spectral radius equals unity at $\ell = 0$. When $0 < \ell < 0.287$, the eigenvalues are complex with positive real part; the spectral radius of the Jacobian, $\rho(J)$, is greater than unity. For $0.287 \leq \ell \leq 2.895$, $\rho(J) < 1$ and $\rho(J) > 1$ provided $\ell > 2.895$.

Method 3:

This method can be written in the form

$$P^{n+1} = g_1\left(P^n, H^n\right), \quad (2.4.41)$$

$$H^{n+1} = g_2\left(P^n, H^n\right); \quad (2.4.42)$$

$n = 0, 1, 2, \ldots$. Associate with (2.4.41) and (2.4.42) the functions

$$P = g_1\left(P, H\right) \quad (2.4.43)$$

and

$$H = g_2\left(P, H\right) \quad (2.4.44)$$

then (2.4.17) and (2.4.18) can be written in the forms

$$g_1\left(P, H\right) = \frac{F\,P}{1 + \ell\,Pr}, \quad (2.4.45)$$

$$g_2\left(P, H\right) = \frac{H}{1 + \ell\,m - \frac{\ell aDFP}{1+\ell Pr}}, \quad (2.4.46)$$

where

$$\left.\begin{aligned} D &= \tfrac{1}{1+bP} \\ F &= 1 + \ell\,r - \ell\,a\,D\,H\,. \end{aligned}\right. \quad (2.4.47)$$

Note that, $\frac{dD}{dP} = -bD^2$, $\frac{\partial F}{\partial P} = \ell\,a\,b\,D^2\,H$ and $\frac{\partial F}{\partial H} = -\ell\,a\,D$.

It may be shown that

$$\begin{aligned} \frac{\partial g_1}{\partial P} &= \frac{F}{(1+\ell Pr)^2} + \frac{\ell abD^2\,HP}{1+\ell Pr}, \\ \frac{\partial g_1}{\partial H} &= -\frac{\ell aDP}{1+\ell Pr}, \\ \frac{\partial g_2}{\partial P} &= \frac{\ell aDH[F+\ell abD^2\,HP-bDFP-\frac{\ell FPr}{1+\ell Pr}]}{(1+\ell m-\frac{\ell aDFP}{1+\ell Pr})^2\,(1+\ell Pr)}, \\ \frac{\partial g_2}{\partial H} &= \frac{(1+\ell m)(1+\ell Pr)-\ell aDP(F+\ell aDH)}{(1+\ell m-\frac{\ell aDFP}{1+\ell Pr})^2\,(1+\ell Pr)}, \end{aligned} \quad (2.4.48)$$

Evaluating the derivatives in (2.4.48) at the fixed points s_i $(i = 1, 2, 3)$ and substituting, using the set of parameters in (2.3.7), in the Jacobian matrix given in (2.4.31). the following results are obtained:

For all s_i $(i = 1, 2, 3)$, all the eigenvalues are equal to unity at $\ell = 0$. For s_1, the spectral radius, $\rho(J)$, is greater than unity whenever $\ell > 0$.

Using s_2, $\rho(J)$ is greater than unity whenever $0 < \ell < 8.582$, while for $\ell \geq 8.582$, the spectral radius, $\rho(J)$, is less than unity.

Considering s_3, the eigenvalues are complex with real part equal to unity for $\ell = 0$; the spectral radius is greater than unity for $0 < \ell < 3.173$. When $3.173 \leq \ell < 3.986$, the spectral radius is less than unity and greater than unity for $\ell \geq 3.986$.

These findings are summarized in Table 2.1.

Table 2.1: *Stability properties of fixed points of Methods 1, 2 and 3, where unst. denotes unstable.*

fixed	**Method 1**		**Method 2**		**Method 3**	
point	**stable**	**unst.**	**stable**	**unst.**	**stable**	**unst.**
s_1	-	$\ell > 0$	$\ell \geq 2.466$	$0 < \ell < 2.466$	-	$\ell > 0$
s_2	-	$\ell > 0$	-	$\ell > 0$	$\ell \geq 8.582$	$0 < \ell < 8.582$
s_3	-	$\ell > 0$	$0.287 \leq \ell \leq 2.985$	$0 < \ell < 0.287$ $\wedge \ell > 2.895$	$3.173 \leq \ell < 3.986$	$0 < \ell < 3.173$ $\wedge \ell \geq 3.986$

2.4.5 Numerical Results

Table 2.1 summarizes the stability properties of the fixed points of Methods 1, 2 and 3. All methods have the same fixed points but, whereas for Method 1 these fixed points eventually become unstable as the time step ℓ is increased both Methods 2 and 3 show the opposite behaviour: as ℓ is increased the fixed point, s_3, always becomes attracting. The results represented here are based on the numerical methods for the set of parameters given in (2.3.7), chosen to obtain limit cycles in the absence of diffusion.

Method 1 (the Euler method) gives limit cycles for small values of the time step ℓ.

Using $\ell = 0.2$, the predator-prey phase-plane for this method is shown in Figure 2.2. As ℓ is increased (i.e. $\ell \geq 0.29$), Method 1 gives negative values of prey population (which are not a feature of the theoretical solution) and chaos was observed in the numerical solution.

Numerical solutions computed using Methods 2 and 3 converged to the stationary point $s_3 = (h_s, p_s) = (0.2835, 0.3)$ as the time step is increased. That is, the higher the value of ℓ, the quicker the stable stationary point s_3 is reached. All these findings are depicted in Figures 2.3–2.6

The profiles in Figure 2.3 illustrate that, in reaching the stationary point, the prey attains a lower value for a larger time step ($\ell = 1.0$) than it does for a smaller time step ($\ell = 0.05$). This suggests that Methods 2 and 3 always converge to the fixed point using an arbitrarily large time step, unlike Method 1 which permits the use of only a small time step if convergence is to be attained.

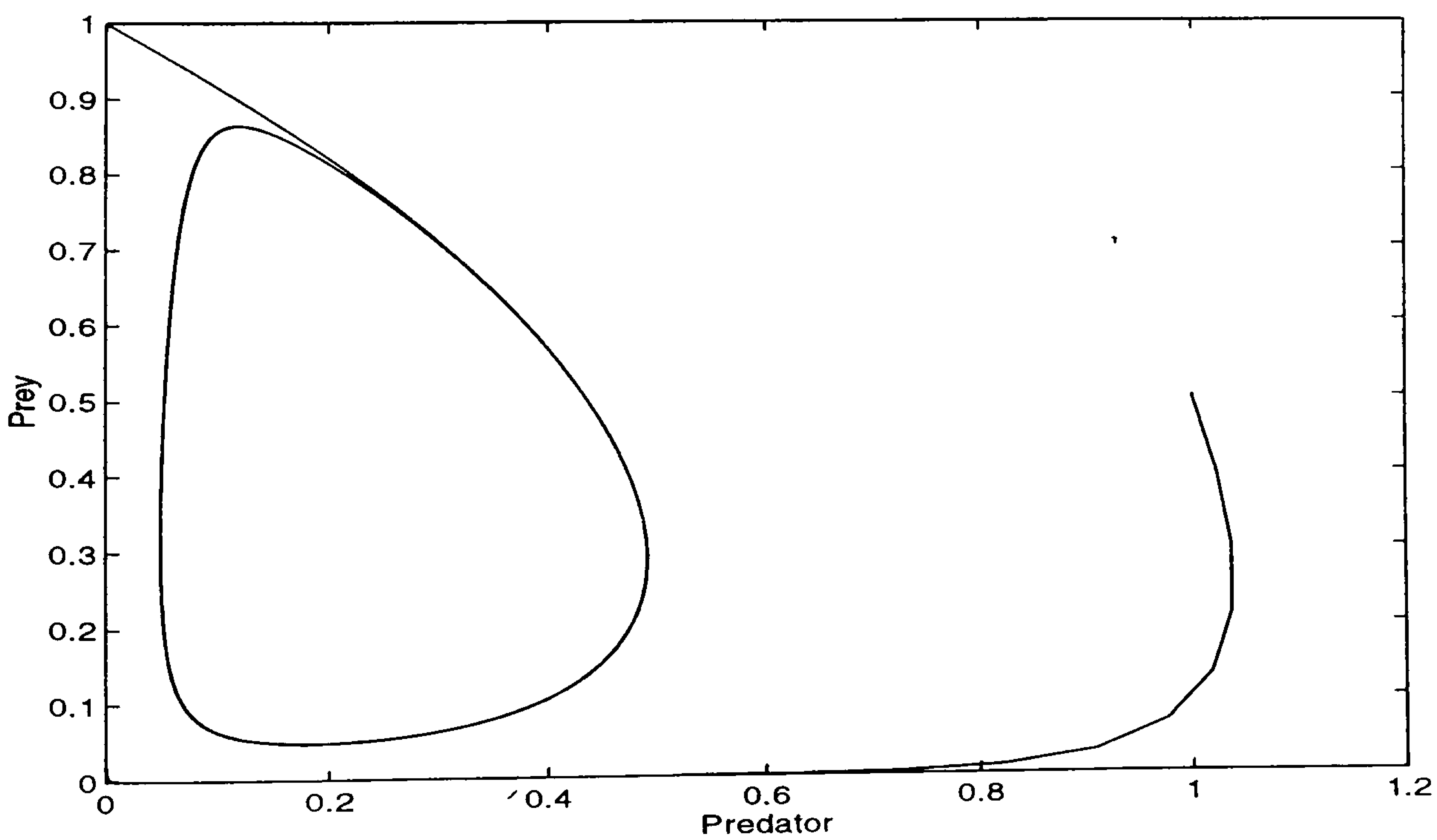

Figure 2.2: *Limit cycles obtained using Method 1 with* $\ell = 0.2$.

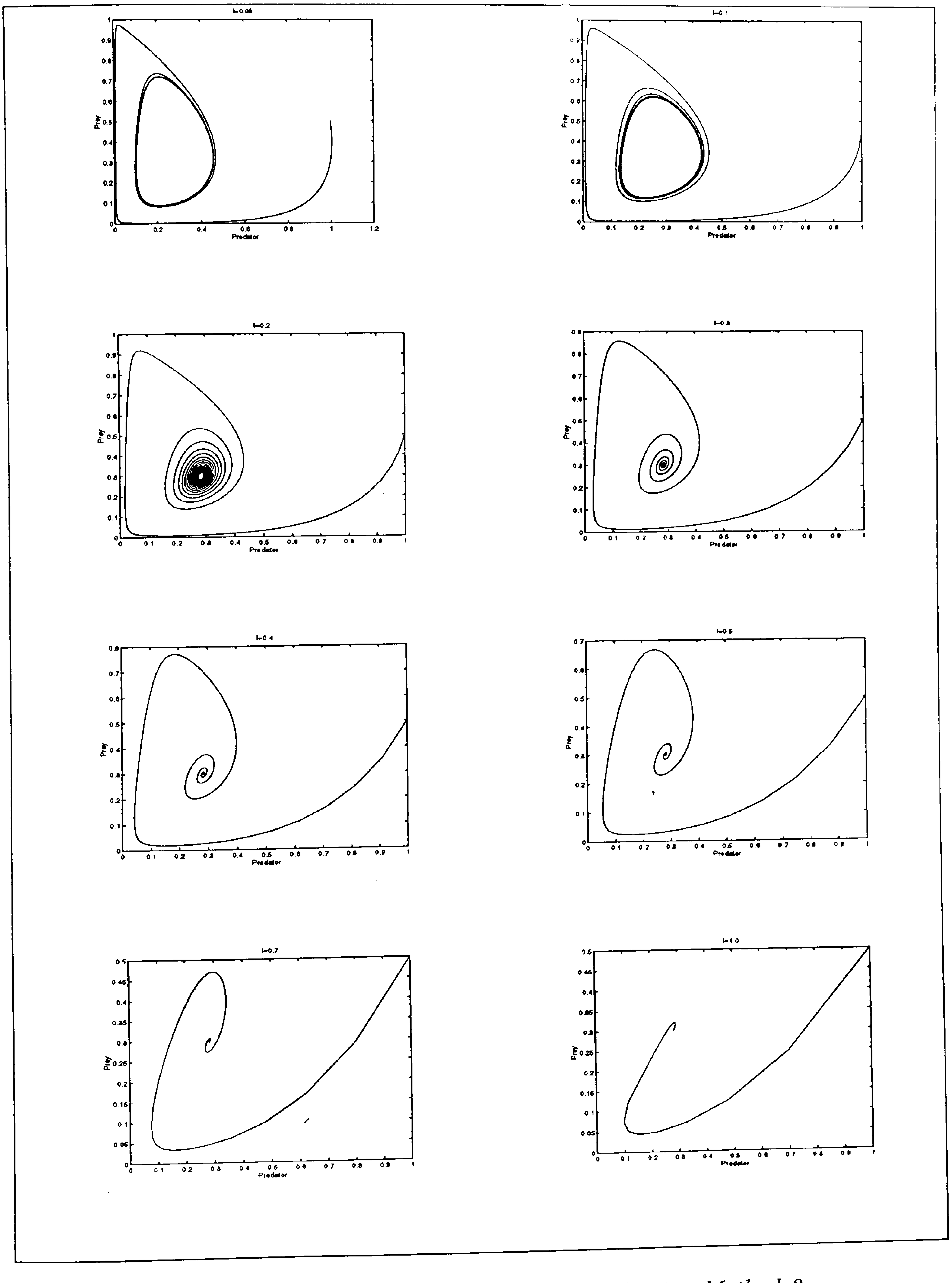

Figure 2.3: *Numerical solution computed using Method 2.*

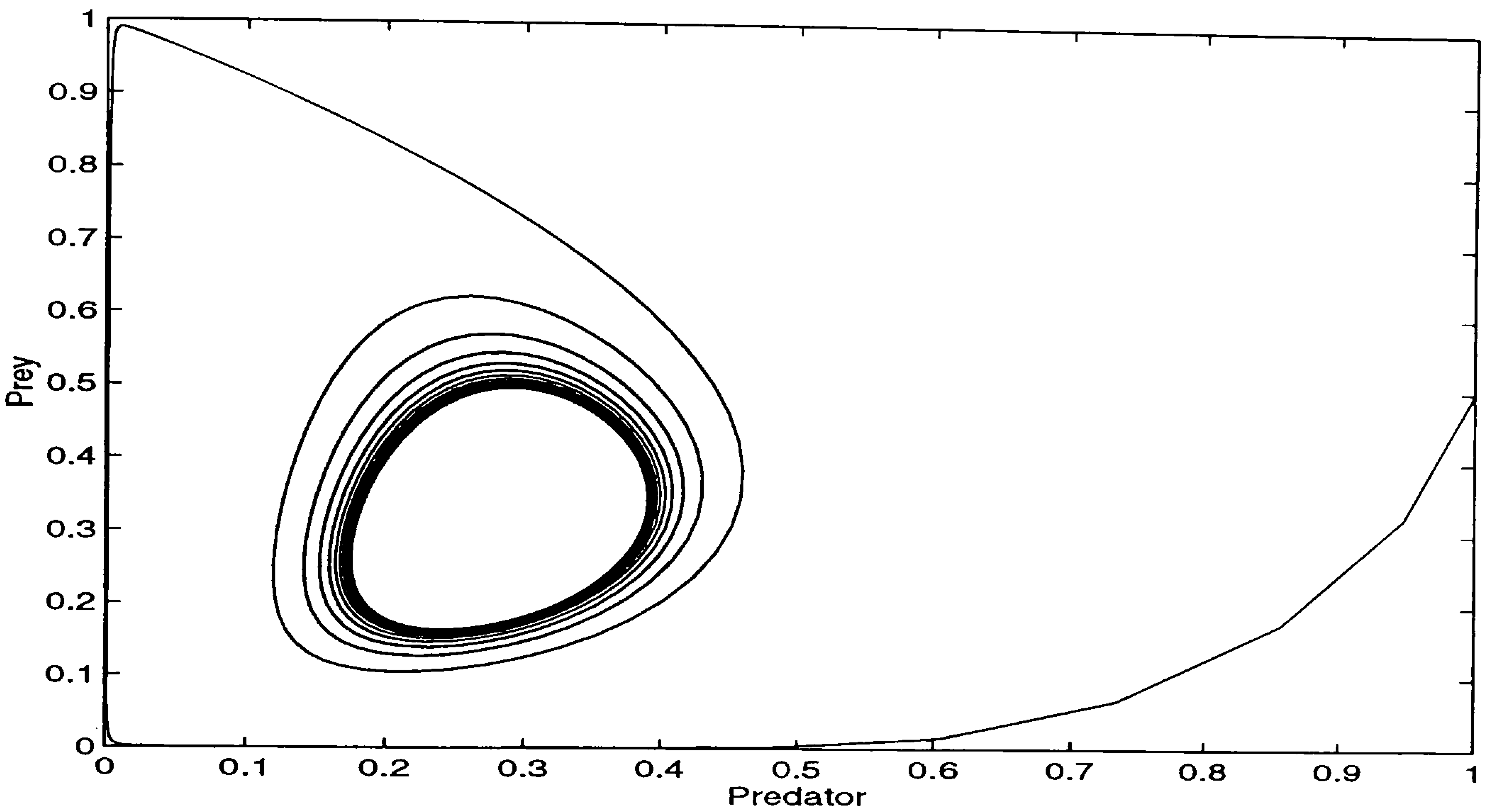

Figure 2.4: *Solution obtained using Method 3 with* $\ell = 0.4$.

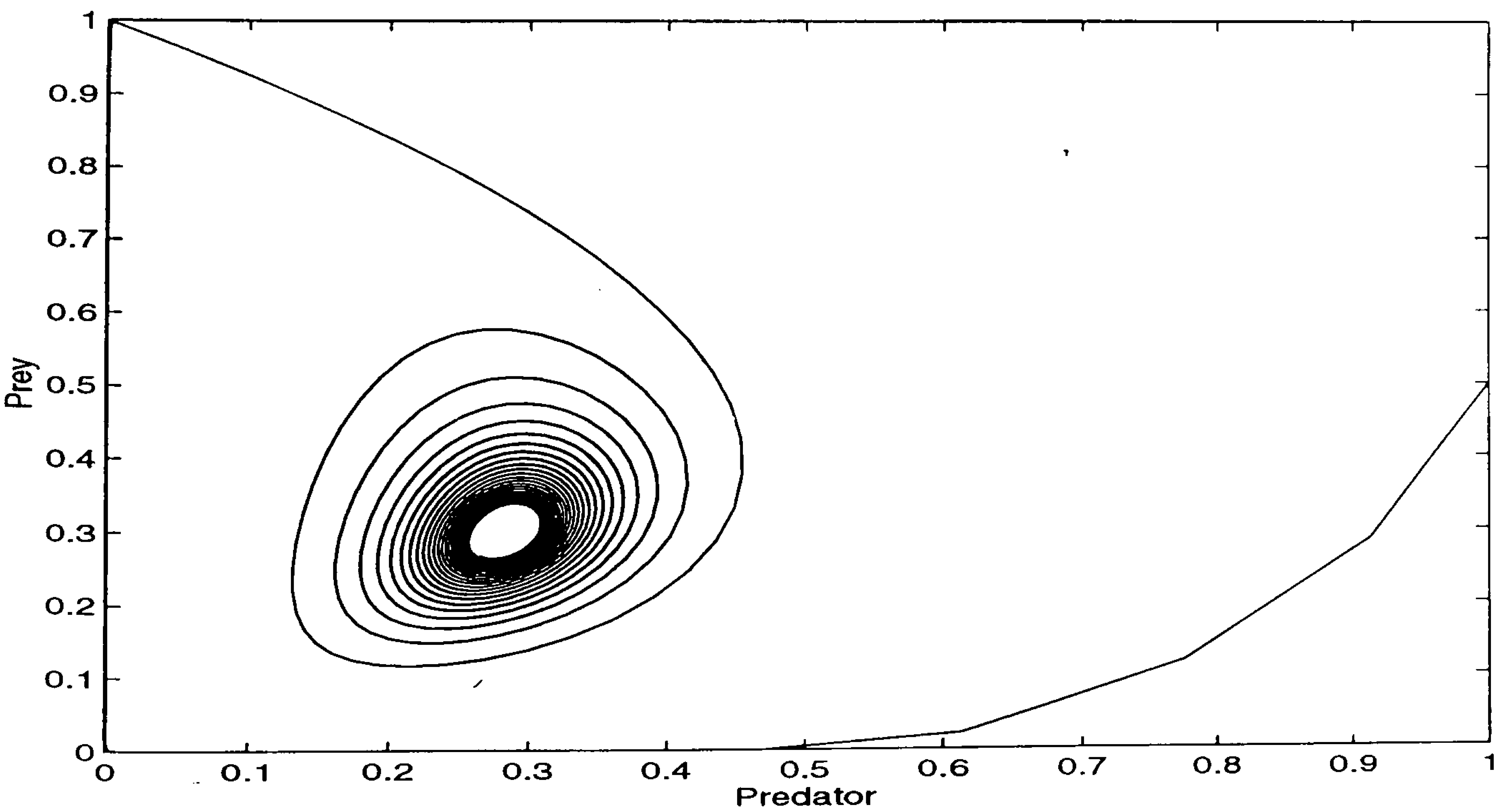

Figure 2.5: *Stable stationary point is reached using Method 3 with* $\ell = 0.5$.

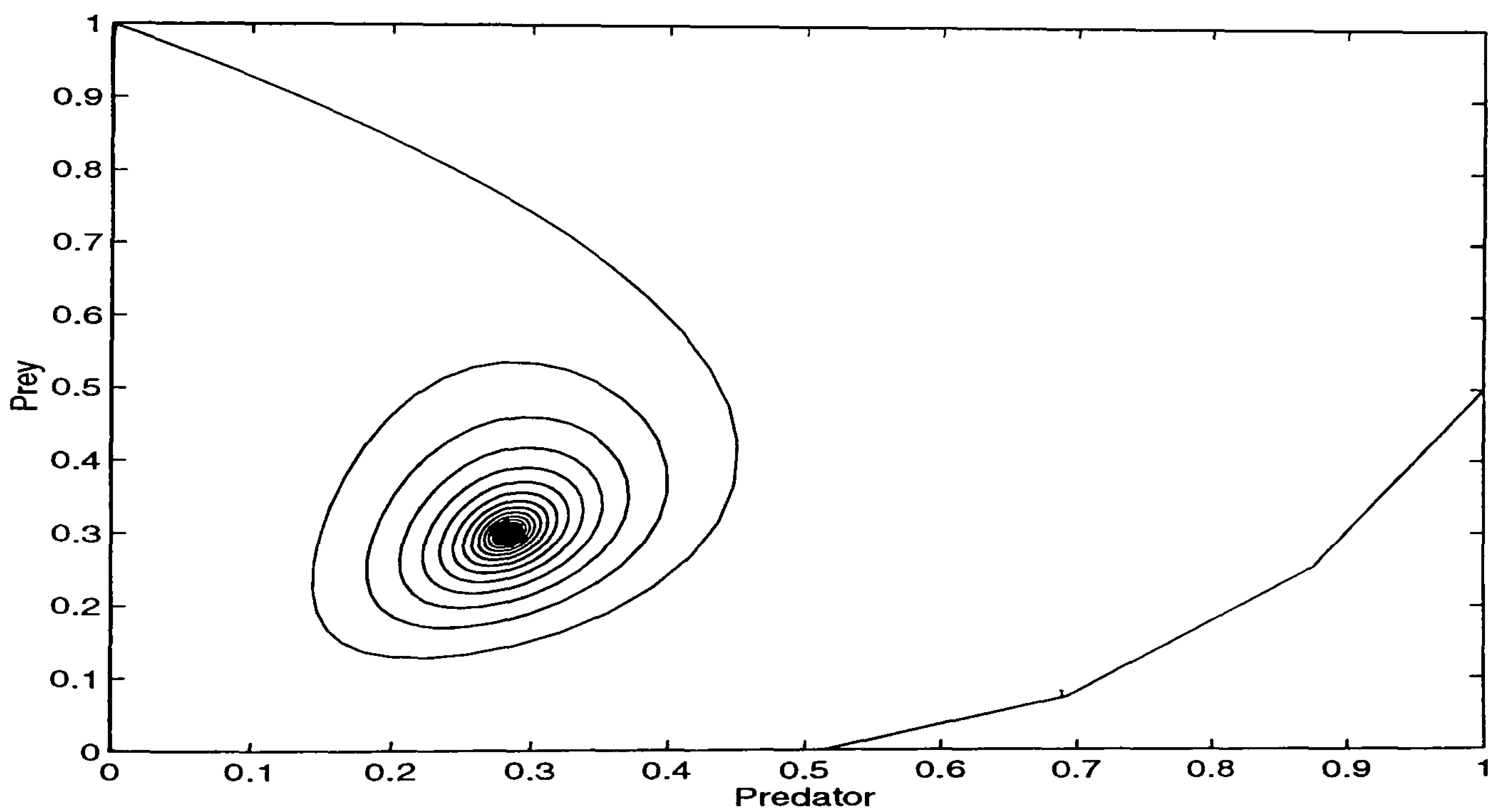

Figure 2.6: *Quicker convergence to the fixed point using Method 3 with* $\ell = 0.6$.

2.5 The Reaction-Diffusion Equations

The predator-prey system to be considered is the system given in (2.3.4) and (2.3.5) which involves the reaction-diffusion equations, namely,

$$\frac{\partial p}{\partial t} = r\,p(1-p) - \frac{a\,p}{1+b\,p}\,h + d\,\frac{\partial^2 p}{\partial x^2}\,; \quad 0 < x < 1, \quad t > 0 \tag{2.5.1}$$

$$\frac{\partial h}{\partial t} = \frac{a\,p}{1+b\,p}\,h - m\,h + d\,\frac{\partial^2 h}{\partial x^2}\,; \quad 0 < x < 1, \quad t > 0 \tag{2.5.2}$$

in which $p = p(x,t)$ and $h = h(x,t)$ are the densities in dimensionless forms of prey and predator, respectively; r, a, b and m are dimensionless parameters and $d > 0$ is the diffusion rate.

The boundary conditions (see Pascual[52]) have the form

$$\frac{\partial p(0,t)}{\partial x} = \frac{\partial p(1,t)}{\partial x} = 0\,; \quad t > 0\,, \tag{2.5.3}$$

$$\frac{\partial h(0,t)}{\partial x} = \frac{\partial h(1,t)}{\partial x} = 0\,; \quad t > 0\,, \tag{2.5.4}$$

and the initial conditions, $0 \leq x \leq 1$, are given by (2.4.3).

2.5.1 Numerical Methods

The problem {(2.5.1)-(2.5.4) and (2.4.3)} is solved by finite-difference methods by discretizing the space interval $0 \leq x \leq 1$ into $N+1$ subintervals each of width δ so that $(N+1)\,\delta = 1$, and by discretizing the time interval $t \geq 0$ into steps each of length ℓ as in Section 2.4.2. The open region $R = [0 < x < 1] \times [t > 0]$ and its boundary ∂R consisting of the lines $x = 0$, $x = 1$ and $t = 0$ are thus covered by a rectangular mesh, the mesh points having coordinates (x_k, t_n) where $x_k = k\,\delta \quad (k = 0, 1, 2, \ldots, N, N+1)$ and $t_n = n\,\ell \quad (n = 0, 1, 2, \ldots)$. The notations P_k^n and H_k^n will be used to distinguish the solution of an approximating finite-difference method from the theoretical solutions

$p(x_k, t_n)$ and $h(x_k, t_n)$ at the mesh point (x_k, t_n), while the values actually obtained, which may be subject to round-off errors, for example, will be denoted by $\widetilde{P_k^n}$ and $\widetilde{H_k^n}$.

Finite-difference methods are developed by approximating the time derivative in (2.5.1) by the first-order forward difference replacement

$$\frac{\partial p(x,t)}{\partial t} = [p(x,t+\ell) - p(x,t)]/\ell + O(\ell) \quad as \quad \ell \longrightarrow 0 \qquad (2.5.5)$$

and the space derivative by the weighted approximant

$$\begin{aligned}\frac{\partial^2 p(x,t)}{\partial x^2} &\approx \delta^{-2}\,[\phi\,\{p(x-\delta\,,t+\ell) - 2p(x,t+\ell) + p(x+\delta\,,t+\ell)\}\\ &+ (1-\phi)\,\{p(x-\delta\,,t) - 2p(x,t) + p(x+\delta\,,t)\}] \qquad (2.5.6)\end{aligned}$$

in which $x = x_k \quad (k = 0,1,2,\ldots,N,N+1)$, $t = t_n \quad (n = 0,1,2,\ldots)$ and $\phi\,(0 \leq \phi \leq 1)$ is a parameter. Similar replacements are used to approximate the derivatives in (2.5.2).

The terms in (2.5.1) and (2.5.2) may be replaced in the three ways used in Section 2.4.2, giving

$$(a) \qquad r\,P^n\,(1-P^n) - \frac{aP^n\,H^n}{1+b\,P^n} \quad in \quad (2.5.1) \quad and \quad \frac{aP^n\,H^n}{1+b\,P^n} - m\,H^n \quad in \quad (2.5.2) \qquad (2.5.7)$$

or

$$(b) \qquad r\,P^{n+1}\,(1-P^n) - \frac{aP^{n+1}\,H^n}{1+b\,P^n} \quad in \quad (2.5.1) \quad and \quad \frac{aP^{n+1}\,H^{n+1}}{1+b\,P^n} - m\,H^{n+1} \quad in \quad (2.5.2) \qquad (2.5.8)$$

or

$$(c) \qquad r\,P^n\,(1-P^{n+1}) - \frac{aP^n\,H^n}{1+b\,P^n} \quad in \quad (2.5.1) \quad and \quad \frac{aP^{n+1}\,H^{n+1}}{1+b\,P^n} - m\,H^{n+1} \quad in \quad (2.5.2). \qquad (2.5.9)$$

These approximations, together with the replacements for the derivatives of p and h give rise to three families of numerical methods, to be named A(ϕ), B (ϕ) and

$C(\phi)$ for the numerical solution of {(2.5.1)-(2.5.4) and (2.4.3)}. These methods are as follows:

Method A(ϕ), $0 \le \phi \le 1$:

$$\frac{P_k^{n+1} - P_k^n}{\ell} = r\,P_k^n\,(1 - P_k^n) \;-\; \frac{a\,P_k^n\,H_k^n}{1 + b\,P_k^n} + \frac{d}{\delta^2}\left\{\phi\left[P_{k-1}^{n+1} - 2P_k^{n+1} + P_{k+1}^{n+1}\right] + (1-\phi)\left[P_{k-1}^n - 2P_k^n + P_{k+1}^n\right]\right\}, \qquad (2.5.10)$$

which can be rearranged to give

$$\begin{aligned}-d\,q\,\phi\,P_{k-1}^{n+1} &+ (1 + 2d\,q\,\phi)\,P_k^{n+1} - d\,q\,\phi\,P_{k+1}^{n+1} = (1-\phi)\,d\,q\,P_{k-1}^n \\ &+ \left[1 - 2(1-\phi)\,d\,q + \ell\,r\,(1 - P_k^n) - \frac{\ell\,aH_k^n}{1 + b\,P_k^n}\right] P_k^n \\ &+ (1-\phi)\,d\,q\,P_{k+1}^n,\end{aligned} \qquad (2.5.11)$$

where $q = \ell/\delta^2$, and

$$\frac{H_k^{n+1} - H_k^n}{\ell} = \frac{aP_k^n\,H_k^n}{1 + b\,P_k^n} \;-\; m\,H_k^n + \frac{d}{\delta^2}\left\{\phi\left[H_{k-1}^{n+1} - 2H_k^{n+1} + H_{k+1}^{n+1}\right] + (1-\phi)\left[H_{k-1}^n - 2H_k^n + H_{k+1}^n\right]\right\}, \qquad (2.5.12)$$

which can be rearranged to give

$$\begin{aligned}-d\,q\,\phi\,H_{k-1}^{n+1} &+ (1 + 2\,d\,q\,\phi)\,H_k^{n+1} - d\,q\,\phi\,H_{k+1}^{n+1} = (1-\phi)\,d\,q\,H_{k-1}^n \\ &+ \left[1 - 2(1-\phi)\,d\,q - \ell\,m + \frac{\ell\,a\,P_k^n}{1 + b\,P_k^n}\right] H_k^n \\ &+ (1-\phi)\,d\,q\,H_{k+1}^n\,.\end{aligned} \qquad (2.5.13)$$

Method B(ϕ), $0 \le \phi \le 1$:

$$\frac{P_k^{n+1} - P_k^n}{\ell} = r\,P_k^{n+1}\,(1 - P_k^n) \;-\; \frac{a\,P_k^{n+1}\,H_k^n}{1 + b\,P_k^n} + \frac{d}{\delta^2}\left\{\phi\left[P_{k-1}^{n+1} - 2P_k^{n+1} + P_{k+1}^{n+1}\right] + (1-\phi)\left[P_{k-1}^n - 2p_k^n + P_{k+1}^n\right]\right\}, \qquad (2.5.14)$$

which can be rearranged to give

$$-d\,q\,\phi\,P_{k-1}^{n+1} \;+\; \left[1 - \ell\,r\,(1-P_k^n) + \frac{\ell\,a\,H_k^n}{1+b\,P_k^n} + 2d\,q\,\phi\right] P_k^{n+1} - d\,q\,\phi\,P_k^{n+1} \;=\; (1-\phi)\,d\,q\,P_{k-1}^n$$
$$+\;\; [1-2(1-\phi)\,d\,q]\;P_k^n + (1-\phi)\,d\,q\,P_{k+1}^n\,, \qquad (\ 2.5.15\ )$$

and

$$\frac{H_k^{n+1}-H_k^n}{\ell} = \frac{a\,P_k^{n+1}\,H_k^{n+1}}{1+b\,P_k^n} \;-\; m\,H_k^{n+1} + \frac{d}{\delta^2}\left\{\phi\left[H_{k-1}^{n+1} - 2H_k^{n+1} + H_{k+1}^{n+1}\right]\right.$$
$$\left. +\;\; (1-\phi)\left[H_{k-1}^n - 2H_k^n + H_{k+1}^n\right]\right\}, \qquad (\ 2.5.16\ )$$

which can be rearranged to give

$$-d\,q\,\phi\,H_{k-1}^{n+1} \;+\; \left[1 - \frac{\ell\,a\,P_k^{n+1}}{1+b\,P_k^n} + \ell\,m + 2d\,q\,\phi\right] H_k^{n+1} - d\,q\,\phi\,H_{k+1}^{n+1} \;=\; (1-\phi)\,d\,q\,H_{k-1}^n$$
$$+\;\; [1-2(1-\phi)\,d\,q]\;H_k^n + (1-\phi)\,d\,q\,H_{k+1}^n\,. \qquad (\ 2.5.17\ )$$

Method C(ϕ), $0 \le \phi \le 1$:

$$\frac{P_k^{n+1}-P_k^n}{\ell} = r\,P_k^n\left(1-P_k^{n+1}\right) \;-\; \frac{a\,P_k^n\,H_k^n}{1+b\,P_k^n} + \frac{d}{\delta^2}\left\{\phi\left[P_{k-1}^{n+1} - 2P_k^{n+1} + P_{k+1}^{n+1}\right]\right.$$
$$\left. +\;\; (1-\phi)\left[P_{k-1}^n - 2P_k^n + P_{k+1}^n\right]\right\}, \qquad (\ 2.5.18\ )$$

which can be rearranged to give

$$-d\,q\,\phi\,P_{k-1}^{n+1} \;+\; [1 + \ell\,r\,P_k^n + 2d\,q\,\phi]\;P_k^{n+1} - d\,q\,\phi\,P_{k+1}^{n+1} = (1-\phi)\,d\,q\,P_{k-1}^n$$
$$+\;\; \left[1 + \ell\,r - \frac{\ell\,a\,H_k^n}{1+b\,P_k^n} - 2(1-\phi)\,d\,q\right] P_k^n$$
$$+\;\; (1-\phi)\,d\,q\,P_{k+1}^n\,. \qquad (\ 2.5.19\ )$$

and

$$\frac{H_k^{n+1}-H_k^n}{\ell} = \frac{a\,P_k^{n+1}\,H_k^{n+1}}{1+b\,P_k^n} \;-\; m\,H_k^{n+1} + \frac{d}{\delta^2}\left\{\phi\left[H_{k-1}^{n+1} - 2H_k^{n+1} + H_{k+1}^{n+1}\right]\right.$$
$$\left. +\;\; (1-\phi)\left[H_{k-1}^n - 2H_k^n + H_{k+1}^n\right]\right\}, \qquad (\ 2.5.20\ )$$

which can be rearranged to give

$$-d\,q\,\phi\,H_{k-1}^{n+1} + \left[1 - \frac{\ell\,a\,P_k^{n+1}}{1+b\,P_k^n} + \ell\,m + 2d\,q\,\phi\right] H_k^{n+1} - d\,q\,\phi\,H_{k+1}^{n+1} = (1-\phi)\,d\,q\,H_{k-1}^n$$
$$+ \; [1 - 2(1-\phi)\,d\,q]\; H_k^n + (1-\phi)\,d\,q\,H_{k+1}^n\,. \qquad (\,2.5.21\,)$$

In (2.5.10) - (2.5.21), $k = 0, 1, 2, \ldots, N, N+1$ and $n = 0, 1, 2, \ldots$.

Computationally, methods A(ϕ), B(ϕ) and C(ϕ) are very economical in that they allow the solution of the non-linear partial differential equations (2.5.1) and (2.5.2) to be obtained by solving a linear algebraic systems at each time step (the solutions are obtained explicitly when $\phi = 0$).

2.5.2 Local Truncation Errors

Consider the use of (2.5.5) and (2.5.6) in (2.5.1) and analogous replacements for $\frac{\partial h}{\partial t}$ and $\frac{\partial^2 h}{\partial x^2}$ in (2.5.2). The local truncation errors (l.t.e.s) associated with (2.5.10), (2.5.14) and (2.5.18) are given by

$$\begin{aligned}
\mathcal{L}_p\,[p(x,t), h(x,t); \delta, \ell] \;=\;& [\,p(x, t+\ell) - p(x,t)\,]/\ell \\
-\;& d\,\phi\,\delta^{-2}\,[\,p(x-\delta, t+\ell) - 2\,p(x, t+\ell) + p(x+\delta, t+\ell)\,] \\
-\;& d\,(1-\phi)\,\delta^{-2}\,[\,p(x-\delta, t) - 2\,p(x,t) + p(x+\delta, t)\,] \\
-\;& r\,p(x, t+\alpha\,\ell)\,[1 - p(x, t+\beta\,\ell)\,] + \frac{a\,p(x, t+\alpha\,\ell)\,h(x,t)}{1+b\,p(x,t)} \\
-\;& \left\{ \frac{\partial p(x,t)}{\partial t} - r\,p(x,t)\;[1 - p(x,t)\,] \right. \\
+\;& \left. \frac{a\,p(x,t)\,h(x,t)}{1+b\,p(x,t)} - d\,\frac{\partial^2 p(x,t)}{\partial x^2} \right\}, \qquad (\,2.5.22\,)
\end{aligned}$$

where

(a) for family A(ϕ), $\alpha = \beta = 0$,

(b) for family B(ϕ), $\alpha = 1$ and $\beta = 0$,

(c) for family C(ϕ), $\alpha = 0$ and $\beta = 1$.

The local truncation errors associated with (2.5.12), (2.5.16) and (2.5.20) are given by

$$\begin{aligned}
\mathcal{L}_h\,[p(x,t), h(x,t); \delta, \ell] \;&=\; [\,h(x,t+\ell) - h(x,t)\,]/\ell \\
&- \; d\,\phi\,\delta^{-2}\,[\,h(x-\delta, t+\ell) - 2\,h(x,t+\ell) + h(x+\delta, t+\ell)\,] \\
&- \; d\,(1-\phi\,)\,\delta^{-2}\,[\,h(x-\delta,t) - 2\,h(x,t) + h(x+\delta,t)\,] \\
&- \; \frac{a\,p(x,t+\gamma\,\ell)\,h(x,t+\eta\,\ell)}{1+b\,p(x,t)} + m\,h(x,t+\eta\,\ell) \\
&- \; \left\{ \frac{\partial h(x,t)}{\partial t} - \frac{a\,p(x,t)\,h(x,t)}{1+b\,p(x,t)} \right. \\
&+ \; \left. m\,h(x,t) - d\,\frac{\partial^2 h(x,t)}{\partial x^2} \right\}, \qquad (\ 2.5.23\ )
\end{aligned}$$

where

(a) for family A(ϕ), $\gamma = \eta = 0$,

(b) for families B(ϕ) and C(ϕ), $\gamma = \eta = 1$.

It is then easy to verify that all three families are $O\,(\,\delta^2 + \ell\,)$ accurate as $\delta\,, \ell \longrightarrow 0$ with

(i) for family A(ϕ):

$$\mathcal{L}_p\,[p,h;\delta,\ell] = -\frac{1}{12}\,\delta^2\,\frac{\partial^4 p}{\partial x^4} + \left(\frac{1}{2}\,\frac{\partial^2 p}{\partial t^2} - d\,\phi\,\frac{\partial^3 p}{\partial x^2\,\partial t} \right)\,\ell + \cdots\,, \qquad (\ 2.5.24\ )$$

$$\mathcal{L}_h\,[p,h;\delta,\ell] = -\frac{1}{12}\,\delta^2\,\frac{\partial^4 h}{\partial x^4} + \left(\frac{1}{2}\,\frac{\partial^2 h}{\partial x^2} - d\,\phi\,\frac{\partial^3 h}{\partial x^2\,\partial t} \right)\,\ell + \cdots\,, \qquad (\ 2.5.25\ )$$

(ii) for family B(ϕ):

$$\mathcal{L}_p\,[p,h;\delta,\ell] = -\frac{1}{12}\,\delta^2\,\frac{\partial^4 p}{\partial x^4} + \left(\frac{1}{2}\,\frac{\partial^2 p}{\partial t^2} - d\,\phi\,\frac{\partial^3 p}{\partial x^2\,\partial t} - r\,(1-p)\,\frac{\partial p}{\partial t} + \frac{a\,h}{1+bp}\,\frac{\partial p}{\partial t} \right)\,\ell + \cdots\,, \qquad (\ 2.5.26\ )$$

$$\mathcal{L}_h\,[p,h;\delta,\ell] = -\frac{1}{12}\,\delta^2\,\frac{\partial^4 h}{\partial x^4} + \left(\frac{1}{2}\,\frac{\partial^2 h}{\partial t^2} - d\,\phi\,\frac{\partial^3 h}{\partial x^2\,\partial t} - \frac{a\left(p\,\frac{\partial h}{\partial t} + h\,\frac{\partial p}{\partial t}\right)}{1+bp} + m\,\frac{\partial h}{\partial t}\right)\ell + \cdots\,, \qquad (\ 2.5.27\ )$$

(iii) for family $C(\phi)$:

$$\mathcal{L}_p\,[p,h;\delta,\ell] = -\frac{1}{12}\,\delta^2\,\frac{\partial^4 p}{\partial x^4} + \left(\frac{1}{2}\,\frac{\partial^2 p}{\partial t^2} - d\,\phi\,\frac{\partial^3 p}{\partial x^2\,\partial t} + r\,p\,\frac{\partial p}{\partial t}\right)\ell + \cdots\,, \qquad (\ 2.5.28\ )$$

$$\mathcal{L}_h\,[p,h;\delta,\ell] = -\frac{1}{12}\,\delta^2\,\frac{\partial^4 h}{\partial x^4} + \left(\frac{1}{2}\,\frac{\partial^2 h}{\partial t^2} - d\,\phi\,\frac{\partial^3 h}{\partial x^2\,\partial t} - \frac{a\left(p\,\frac{\partial h}{\partial t} + h\,\frac{\partial p}{\partial t}\right)}{1+bp} + m\,\frac{\partial h}{\partial t}\right)\ell + \cdots\,. \qquad (\ 2.5.29\ )$$

2.5.3 Stability Analysis

The concept of *stability* (see Twizell[67], p.200) is concerned with the boundedness of the solution of the finite difference equations and this is examined by finding conditions under which $Z_k^n = U_k^n - \widetilde{U_k^n}$, where U_k^n represents P_h^n or H_k^n, remains bounded as n increases, for fixed δ and ℓ. A stability analysis considers the growth of perturbations in initial data or the growth of errors introduced at mesh points at a given time level. It will be convenient to define the vector $\mathbf{U}^n = [\,U_0^n, U_1^n, \cdots, U_N^n, U_{N+1}^n\,]^T$, T denoting transpose, with similar definitions of $\mathbf{Z}^n$ and $\widetilde{\mathbf{U}^n}$.

There are three common methods of investigating stability: the *energy method*, the *matrix method*, and the *von Neumann* or *Fourier method.* The energy method is powerful in dealing with boundary conditions, variable coefficient problems and non-linear problems. Its application can, unfortunately, become extremely complicated and its successful use will be due in no small part to the ingenuity of the user. Besides proving the stability of a finite-difference scheme, the energy method can indicate the correct choice of a method. However, the method provides only sufficient conditions for stability which may be far removed from what is necessary in certain initial/boundary-value problems.

The method calculates the sum of the squares of the errors Z_k^n at time level $t = n\ell$. This sum of squared errors is called the *energy*, from which the method gets its name, and the method determines criteria under which this energy remains bounded as the computation proceeds.

The matrix method of analysis is applicable to initial/boundary-value problems and is very popular among users. Unfortunately it can give erroneous results.

The correct stability interval may usually be found using the von Neumann or Fourier method developed by J.von Neumann in the early 1940s (see Twizell[67], p.201). A Fourier stability analysis determines the criterion governing the growth of a function which reduces to this Fourier series for $t = 0$.

The von Neumann condition is necessary only for three-level schemes; for two-level schemes the condition is sufficient as well as necessary. This also applies to problems with more than one space variable. Strictly speaking, the von Neumann or Fourier method applies only to pure initial-value problems with periodic initial data. In practice, however, it is used to analyse finite-difference schemes applied to initial/boundary-value problems also.

The von Neumann method of analysing stability will be used to give some insight into the stability of the families $A(\phi)$, $B(\phi)$ and $C(\phi)$. This method entails considering a small error Z_k^n of the form

$$Z_{p,k}^n = P_k^n - \widetilde{P_k^n} = e^{\alpha n \ell} e^{i \beta k \delta} \qquad (2.5.30)$$

and

$$Z_{h,k}^n = H_k^n - \widetilde{H_k^n} = e^{\mu n \ell} e^{i \nu k \delta}, \qquad (2.5.31)$$

where α and μ are complex, β and ν are real and $i = +\sqrt{-1}$. The errors in (2.5.30) and (2.5.31) will not grow if

$$\left|e^{\alpha \ell}\right| \leq 1 + M_p \ell \qquad and \qquad \left|e^{\mu \ell}\right| \leq 1 + M_h \ell, \qquad (2.5.32)$$

where M_p and M_h are non-negative constants independent of δ, ℓ. Inequalities (2.5.32) are the *von Neumann necessary conditions for stability*; they make no allowances for growing solutions if $M_p = 0$ and $M_h = 0$.

Method A(ϕ):

Substituting Z_p into (2.5.11) leads to the (local) stability equation

$$\left[1 + 4\, d\, q\, \phi\, \sin^2(\frac{1}{2}\beta\, \delta)\right] \xi_p = 1 - 4\,(1-\phi)\, d\, q\, \sin^2(\frac{1}{2}\beta\, \delta) + \ell\, r\,(1 - P_k^n) - \frac{\ell\, a\, H_k^n}{1 + b\, P_k^n}, \qquad (2.5.33)$$

in which P_k^n and H_k^n are treated as (local) constants and $\xi_p = e^{\alpha \ell}$. The von Neumann necessary condition for stability is $|\,\xi_p\,| \leq 1$. That is, the stability restrictions are

$$\begin{aligned}
&0 \leq \phi < \tfrac{1}{2}, && d\,q \leq \frac{2 + \ell\, r\left(1 - P_k^n\right) - \frac{\ell\, a\, H_k^n}{1 + b P_k^n}}{4\,(1 - 2\,\phi)}; \\
&\phi = \tfrac{1}{2}, && \frac{\ell\, a\, H_k^n}{1 + b\, P_k^n} \leq 2 + \ell\, r\,(1 - P_k^n); \\
&\tfrac{1}{2} < \phi \leq 1, && d\,q \geq \frac{2 + \ell\, r\left(1 - P_k^n\right) - \frac{\ell\, a\, H_k^n}{1 + b\, P_k^n}}{4\,(1 - 2\,\phi)}; \\
&0 \leq \phi \leq 1, && d\,q \geq \tfrac{1}{4}\left(\ell\, r\,(1 - P_k^n) - \frac{\ell\, a\, H_k^n}{1 + b\, P_k^n}\right).
\end{aligned} \qquad (2.5.34)$$

Now substituting Z_h into (2.5.13) gives the (local) stability equation

$$\left[1 + 4\, d\, q\, \phi\, \sin^2(\frac{1}{2}\nu\, \delta)\right] \xi_h = 1 - 4\,(1-\phi)\, d\, q\, \sin^2(\frac{1}{2}\nu\, \delta) - \ell\, m + \frac{\ell\, a\, P_k^n}{1 + b\, P_k^n}, \qquad (2.5.35)$$

in which P_k^n is treated as a (local) constant and $\xi_h = e^{\mu \ell}$, with the consequent stability restrictions

$$\begin{aligned} &0 \le \phi < \tfrac{1}{2}, && dq \le \frac{2-\ell m+\frac{\ell a P_k^n}{1+b P_k^n}}{4(1-2\phi)}; \\ &\phi = \tfrac{1}{2}, && \frac{\ell a P_k^n}{1+b P_k^n} \ge \ell m - 2; \\ &\tfrac{1}{2} < \phi \le 1, && dq \ge \frac{2-\ell m+\frac{\ell a P_k^n}{1+b P_k^n}}{4(1-2\phi)}; \\ &0 \le \phi \le 1, && dq \ge \tfrac{1}{4}\left(\frac{\ell a P_k^n}{1+b P_k^n} - \ell m\right). \end{aligned} \tag{2.5.36}$$

Method B(ϕ) :

Substituting Z_p into (2.5.15) gives the (local) stability equation

$$\left[1 + 4\,d\,q\,\phi\,\sin^2(\frac{1}{2}\,\beta\,\delta) - \ell\,r\,(1 - P_k^n) + \frac{\ell\,a\,H_k^n}{1 + b\,P_k^n}\right]\,\xi_p = 1 - 4\,(1-\phi)\,d\,q\,\sin^2(\frac{1}{2}\,\beta\,\delta), \tag{2.5.37}$$

in which P_k^n and H_k^n are treated as (local) constants and $\xi_p = e^{\alpha\,\ell}$, with the consequent stability restrictions

$$\begin{aligned} &0 \le \phi < \tfrac{1}{2}, && dq \le \frac{2-\ell r\left(1-P_k^n\right)+\frac{\ell a H_k^n}{1+b P_k^n}}{4(1-2\phi)}; \\ &\phi = \tfrac{1}{2}, && \frac{\ell a H_k^n}{1+b P_k^n} \ge \ell r\,(1 - P_k^n) - 2; \\ &\tfrac{1}{2} < \phi \le 1, && dq \ge \frac{2-\ell r\left(1-P_k^n\right)+\frac{\ell a H_k^n}{1+b P_k^n}}{4(1-2\phi)}; \\ &0 \le \phi \le 1, && dq \ge \tfrac{1}{4}\left(\ell r\,(1 - P_k^n) - \frac{\ell a H_k^n}{1+b P_k^n}\right), \end{aligned} \tag{2.5.38}$$

and substituting Z_h into (2.5.17) leads to the stability equation

$$\left[1 + 4\,d\,q\,\phi\,\sin^2(\frac{1}{2}\,\nu\,\delta) - \frac{\ell\,a\,P_k^{n+1}}{1 + b\,P_k^n} + \ell\,m\right]\,\xi_h = 1 - 4\,(1-\phi)\,d\,q\,\sin^2(\frac{1}{2}\,\nu\,\delta), \tag{2.5.39}$$

in which P_k^{n+1} and P_k^n are treated as (local) constants and $\xi_h = e^{\mu\,\ell}$. It may be shown that, for stability,

$$
\begin{aligned}
&0 \le \phi < \tfrac{1}{2}, && dq \le \frac{2+\ell m - \frac{\ell a P_k^{n+1}}{1+b P_k^n}}{4(1-2\phi)}; \\
&\phi = \tfrac{1}{2}, && \frac{\ell a P_k^{n+1}}{1+b P_k^n} \le 2 + \ell m; \\
&\tfrac{1}{2} < \phi \le 1, && dq \ge \frac{2+\ell m - \frac{\ell a P_k^{n+1}}{1+b P_k^n}}{4(1-2\phi)}; \\
&0 \le \phi \le 1, && dq \ge \tfrac{1}{4}\left(\frac{\ell a P_k^{n+1}}{1+b P_k^n} - \ell m\right).
\end{aligned}
\qquad (2.5.40)
$$

Method C(ϕ) :

Substituting Z_p into (2.5.19) gives the (local) stability equation, in which P_k^n and H_k^n are treated as (local) constants,

$$
\left[1 + 4\,d\,q\,\phi\,\sin^2(\frac{1}{2}\,\beta\,\delta) + \ell\,r\,P_k^n\right]\xi_p = 1 - 4\,(1-\phi)\,d\,q\,\sin^2(\frac{1}{2}\,\beta\,\delta) + \ell\,r - \frac{\ell\,a\,H_k^n}{1+b\,P_k^n}, \qquad (2.5.41)
$$

where $\xi_p = e^{\alpha\,\ell}$, with the consequent stability restrictions

$$
\begin{aligned}
&0 \le \phi < \tfrac{1}{2}, && dq \le \frac{2+\ell r P_k^n + \ell r - \frac{\ell a H_k^n}{1+b P_k^n}}{4(1-2\phi)}; \\
&\phi = \tfrac{1}{2}, && \frac{\ell a H_k^n}{1+b P_k^n} \le 2 + \ell r P_k^n + \ell r; \\
&\tfrac{1}{2} < \phi \le 1, && dq \ge \frac{2+\ell r P_k^n + \ell r - \frac{\ell a H_k^n}{1+b P_k^n}}{4(1-2\phi)}; \\
&0 \le \phi \le 1, && dq \ge \tfrac{1}{4}\left(\ell r - \ell r P_k^n - \frac{\ell a H_k^n}{1+b P_k^n}\right).
\end{aligned}
\qquad (2.5.42)
$$

Now substituting Z_h into (2.5.21) gives the (local) stability equation

$$
\left[1 + 4\,d\,q\,\phi\,\sin^2(\frac{1}{2}\,\nu\,\delta) - \frac{\ell\,a\,P_k^{n+1}}{1+b\,Pk^n} + \ell\,m\right]\xi_h = 1 - 4\,(1-\phi)\,d\,q\,\sin^2(\frac{1}{2}\,\nu\,\delta). \qquad (2.5.43)
$$

where $\xi_h = e^{\mu \ell}$. For stability, it may be shown that the restrictions are given by

$$\begin{aligned}
&0 \le \phi < \tfrac{1}{2}, \quad && dq \le \frac{2+\ell m - \frac{\ell a P_k^{n+1}}{1+b P_k^n}}{4(1-2\phi)}; \\
&\phi = \tfrac{1}{2}, && \frac{\ell a P_k^{n+1}}{1+b P_k^n} \le 2 + \ell m; \\
&\tfrac{1}{2} < \phi \le 1, && dq \ge \frac{2+\ell m - \frac{\ell a P_k^{n+1}}{1+b P_k^n}}{4(1-2\phi)}; \\
&0 \le \phi \le 1, && dq \ge \tfrac{1}{4}\left(\frac{\ell a P_k^{n+1}}{1+b P_k^n} - \ell m\right).
\end{aligned} \qquad (2.5.44)$$

Note, P_k^{n+1} is treated as a local constant when analysing H for stability, because P has already been analysed for stability.

2.5.4 Implementation

The derivative boundary conditions in (2.5.3) and (2.5.4), on the boundary $x = 0$, may be approximated by the second-order, central-difference replacements

$$\frac{\partial p(0,t)}{\partial x} = [\,p(x+\delta,t) - p(x-\delta,t)\,]/(2\,\delta) + O\left(\delta^2\right) \qquad (2.5.45)$$

and

$$\frac{\partial h(0,t)}{\partial x} = [\,h(x+\delta,t) - h(x-\delta,t)\,]/(2\,\delta) + O\left(\delta^2\right) \qquad (2.5.46)$$

as $\delta \longrightarrow 0$. These replacements reveal that, to second order,

$$P_{-1}^n = P_1^n \qquad \textit{and} \qquad H_{-1}^n = H_1^n \qquad (n = 0,1,2,\dots). \qquad (2.5.47)$$

Approximating the space derivatives in (2.5.3) and (2.5.4), once again, on the boundary $x = 1$, by its second-order, central-difference replacements, gives

$$P_{N+2}^n = P_N^n \qquad \textit{and} \qquad H_{N+2}^n = H_N^n \qquad (n = 0,1,2,\dots). \qquad (2.5.48)$$

The modifications to the formulae of the three families of numerical methods are as follows

Method A(ϕ) :

Taking $k = 0$ in (2.5.11) and (2.5.13) and using (2.5.47) and (2.5.48) gives

$$\begin{aligned}(1 + 2\,d\,q\,\phi)\;P_0^{n+1} - 2\,d\,q\,\phi\,P_1^{n+1} &= \left[1 - 2\,(1-\phi)\,d\,q + \ell\,r\;(1 - P_0^n\,) - \frac{\ell\,a\,H_0^n}{1 + b\,P_0^n}\right]\;P_0^n \\ &+\;\; 2\,(1-\phi)\,d\,q\,P_1^n\,, \end{aligned} \qquad (\ 2.5.49\ )$$

and

$$\begin{aligned}(1 + 2\,d\,q\,\phi)\;H_0^{n+1} - 2\,d\,q\,\phi\,H_1^{n+1} &= \left[1 - 2\,(1-\phi)\,d\,q - \ell\,m + \frac{\ell\,a\,P_0^n}{1 + b\,P_0^n}\right]\;H_0^n \\ &+\;\; 2\,(1-\phi)\,d\,q\,H_1^n\,. \end{aligned} \qquad (\ 2.5.50\ )$$

When $k = 1, 2, \ldots, N$, equations (2.5.11) and (2.5.13) are applied and when $k = N+1$ they become

$$\begin{aligned} -2\,d\,q\,\phi\,P_N^{n+1} + (1 + 2\,d\,q\,\phi)\;P_{N+1}^{n+1} = 2\,(1-\phi)\,d\,q\,\phi\,P_N^n \\ +\;\; \left[1 - 2\,(1-\phi)\,d\,q + \ell\,r\,\left(1 - P_{N+1}^n\right) - \frac{\ell\,a\,H_{N+1}^n}{1 + b\,P_{N+1}^n}\right]\;P_{N+1}^n \end{aligned} \qquad (\ 2.5.51\ )$$

and

$$\begin{aligned} -2\,d\,q\,\phi\,H_N^{n+1} &+ \;\;(1 + 2\,d\,q\,\phi)\;H_{N+1}^{n+1} = 2\,(1-\phi)\,d\,q\,H_N^n \\ &+ \;\;\left[1 - 2\,(1-\phi)\,d\,q - \ell\,m + \frac{\ell\,a\,P_{N+1}^n}{1 + b\,P_{N+1}^n}\right]\;H_{N+1}^n\,. \end{aligned} \qquad (\ 2.5.52\ )$$

The solution vectors $\mathbf{P}^{n+1}$ and $\mathbf{H}^{n+1}$ may be obtained using the following parallel algorithm

$$Processor\,1 : Solve \qquad E_1\,\mathbf{P}^{n+1} = F_1\,\mathbf{P}^n \qquad for \quad \mathbf{P}^{n+1}\,, \qquad (\ 2.5.53\ )$$

$$Processor\,2 : Solve \qquad E_1\,\mathbf{H}^{n+1} = G_1\,\mathbf{H}^n \qquad for \quad \mathbf{H}^{n+1}\,, \qquad (\ 2.5.54\ )$$

where E_1 is a constant, tridiagonal square matrix of order $N + 2$ and is of the form

$$E_1 = \begin{pmatrix} 1+2\,d\,q\,\phi & -2\,d\,q\,\phi & & & & \\ -d\,q\,\phi & 1+2\,d\,q\,\phi & -d\,q\,\phi & & & \\ & -d\,q\,\phi & 1+2\,d\,q\,\phi & -d\,q\,\phi & & \\ & \ddots & \ddots & \ddots & & \\ & & -d\,q\,\phi & 1+2\,d\,q\,\phi & -d\,q\,\phi & \\ & & & -2\,d\,q\,\phi & 1+2\,d\,q\,\phi & \end{pmatrix} . \qquad (\ 2.5.55\ )$$

The square matrices $F_1 = F_1\,(\mathbf{P}^n, \mathbf{H}^n)$ and $G_1 = G_1\,(\mathbf{P}^n)$ are also of order $(N+2)$ and are of the forms

$$F_1 = \begin{pmatrix} A_0 & 2\,(1-\phi)\,d\,q & & & & \\ (1-\phi)\,d\,q & A_1 & (1-\phi)\,d\,q & & & \\ & (1-\phi)\,d\,q & A_2 & (1-\phi)\,d\,q & & \\ & \ddots & \ddots & \ddots & & \\ & & (1-\phi)\,d\,q & A_N & (1-\phi)\,d\,q & \\ & & & 2\,(1-\phi)\,d\,q & A_{N+1} & \end{pmatrix} , \qquad (\ 2.5.56\ )$$

and

$$G_1 = \begin{pmatrix} B_0 & 2\,(1-\phi)\,d\,q & & & & \\ (1-\phi)\,d\,q & B_1 & (1-\phi)\,d\,q & & & \\ & (1-\phi)\,d\,q & B_2 & (1-\phi)\,d\,q & & \\ & \ddots & \ddots & \ddots & & \\ & & (1-\phi)\,d\,q & B_N & (1-\phi)\,d\,q & \\ & & & 2\,(1-\phi)\,d\,q & B_{N+1} & \end{pmatrix} , \qquad (\ 2.5.57\ )$$

where

$$\begin{aligned} A_k &= 1 - 2\,(1-\phi)\,d\,q + \ell\,r\,\left(1 - P_k^n\right) - \tfrac{\ell\,a\,H_k^n}{1+b\,P_k^n}\,; \\ B_k &= 1 - 2\,(1-\phi)\,d\,q - \ell\,m + \tfrac{\ell\,a\,P_k^n}{1+b\,P_k^n}\,; \\ k &= 0, 1, 2, \ldots, N, N+1\,. \end{aligned}$$

Method B(ϕ) :

Taking $k = 0$ in (2.5.15) and (2.5.17) and using (2.5.47) and (2.5.48) gives

$$\begin{aligned} \left[1 - \ell\,r\,\left(1 - P_0^n\right) + \frac{\ell\,a\,H_0^n}{1+b\,P_0^n} + 2\,d\,q\,\phi\right] P_0^{n+1} \;&-\; 2\,d\,q\,\phi\,P_1^{n+1} = \left[1 - 2\,(1-\phi)\,d\,q\right]\,P_0^n \\ &+\; 2\,(1-\phi)\,d\,q\,P_1^n\,, \end{aligned} \qquad (\ 2.5.58\ )$$

and

$$\left[1+\ell m+2\,d\,q\,\phi-\frac{\ell\,a\,P_0^{n+1}}{1+b\,P_0^n}\right]\,H_0^{n+1}\;-\;2\,d\,q\,\phi\,H_1^{n+1}=[1-2\,(1-\phi)\,d\,q]\;H_0^n$$
$$+\;2\,(1-\phi)\,d\,q\,H_1^n\,. \qquad (\,2.5.59\,)$$

Equations (2.5.15) and (2.5.17) are applied with $k=1,2,\ldots N$, and when $k=N+1$ they become

$$-2\,d\,q\,\phi\,P_N^{n+1}\;+\;\left[1-\ell\,r\,\left(1-P_{N+1}^n\right)+\frac{\ell\,a\,H_{N+1}^n}{1+b\,P_{N+1}^n}+2\,d\,q\,\phi\right]\,P_{N+1}^{n+1}$$
$$=\;2\,(1-\phi)\,d\,q\,P_N^n+[1-2\,(1-\phi)\,d\,q]\;P_{N+1}^n\,, \qquad (\,2.5.60\,)$$

and

$$-2\,d\,q\,\phi\,H_N^{n+1}\;+\;\left[1+\ell m+2\,d\,q\,\phi-\frac{\ell\,a\,P_{N+1}^{n+1}}{1+b\,P_{N+1}^n}\right]\,H_{N+1}^{n+1}=2\,(1-\phi)\,d\,q\,H_N^n$$
$$+\;[1-2\,(1-\phi)\,d\,q]\;H_{N+1}^n \qquad (\,2.5.61\,)$$

The solution vectors $\mathbf{P}^{n+1}$ and $\mathbf{H}^{n+1}$ may be obtained using the following parallel algorithm

$$Processor\,1:Solve \qquad E_2\,\mathbf{P}^{n+1}=F_2\,\mathbf{P}^n \qquad for \quad \mathbf{P}^{n+1}\,, \qquad (\,2.5.62\,)$$

$$Processor\,2:Solve \qquad J_2\,\mathbf{H}^{n+1}=F_2\,\mathbf{H}^n \qquad for \quad \mathbf{H}^{n+1}\,, \qquad (\,2.5.63\,)$$

where F_2 is a constant, tridiagonal square matrix of the form

$$F_2=\begin{pmatrix} X & 2\,(1-\phi)\,d\,q & & & & \\ (1-\phi)\,d\,q & X & (1-\phi)\,d\,q & & & \\ & (1-\phi)\,d\,q & X & (1-\phi)\,d\,q & & \\ & \ddots & \ddots & \ddots & & \\ & & (1-\phi)\,d\,q & X & (1-\phi)\,d\,q \\ & & & 2\,(1-\phi)\,d\,q & X \end{pmatrix}, \qquad (\,2.5.64\,)$$

and E_2 and J_2 are square matrices of order $(N+2)$ and are of the forms

$$E_2 = \begin{pmatrix} Y_0 & -2\,d\,q\,\phi & & & & \\ -d\,q\,\phi & Y_1 & -d\,q\,\phi & & & \\ & -d\,q\,\phi & Y_2 & -d\,q\,\phi & & \\ & \ddots & \ddots & \ddots & & \\ & & -d\,q\,\phi & Y_N & -d\,q\,\phi \\ & & & 2\,(1-\phi)\,d\,q & Y_{N+1} \end{pmatrix}, \qquad (\ 2.5.65\ )$$

$$J_2 = \begin{pmatrix} Z_0 & -2\,d\,q\,\phi & & & & \\ -d\,q\,\phi & Z_1 & -d\,q\,\phi & & & \\ & -d\,q\,\phi & Z_2 & -d\,q\,\phi & & \\ & \ddots & \ddots & \ddots & & \\ & & -d\,q\,\phi & Z_N & -d\,q\,\phi \\ & & & -2\,d\,q\,\phi & Z_{N+1} \end{pmatrix}, \qquad (\ 2.5.66\ )$$

where $E_2 = E_2\,(\mathbf{P}^n, \mathbf{H}^n)$, $J_2 = J_2\,(\mathbf{P}^{n+1}, \mathbf{P}^n)$ and

$$\begin{aligned} X &= 1 - 2\,(1-\phi)\,d\,q\,, \\ Y_k &= 1 - \ell\,r\,\left(1 - P_k^n\right) + 2\,d\,q\,\phi + \tfrac{\ell\,a\,H_k}{1+b\,P_k^n}\,, \\ Z_k &= 1 + \ell\,m + 2\,d\,q\,\phi - \tfrac{\ell\,a\,P_k^{n+1}}{1+b\,P_k^n}\,; \\ k &= 0, 1, 2, \ldots, N+1\,. \end{aligned}$$

Method C(ϕ) :

Taking $k = 0$ in (2.5.19) and (2.5.21) and using (2.5.47) and (2.5.48) gives

$$\begin{aligned} \left[1 + \ell\,r\,P_0^n + 2\,d\,q\,\phi\right]\,P_0^{n+1} - 2\,d\,q\,\phi\,P_1^{n+1} &= \left[1 + \ell\,r - 2\,(1-\phi)\,d\,q - \frac{\ell\,a\,H_0^n}{1+b\,P_0^n}\right]\,P_0^n \\ &+ \ 2\,(1-\phi)\,d\,q\,P_1^n\,, \end{aligned} \qquad (\ 2.5.67\ )$$

and

$$\begin{aligned} \left[1 + \ell\,m + 2\,d\,q\,\phi - \frac{\ell\,a\,P_0^{n+1}}{1+b\,P_0^n}\right]\,H_0^{n+1} &- \ 2\,d\,q\,\phi\,H_1^{n+1} = \left[1 - 2\,(1-\phi)\,d\,q\right]\,H_0^n \\ &+ \ 2\,(1-\phi)\,d\,q\,H_1^n\,. \end{aligned} \qquad (\ 2.5.68\ )$$

Equations (2.5.19) and (2.5.21) are applied with $k = 1, 2, \ldots, N$, and when $k = N+1$ they become

$$\begin{aligned} -2\,d\,q\,\phi\,P_N^{n+1} &+ \ \left[1 + \ell\,r\,P_{N+1}^n + 2\,d\,q\,\phi\right]\,P_{N+1}^{n+1} = 2\,(1-\phi)\,d\,q\,P_N^n \\ &+ \ \left[1 + \ell\,r - 2\,(1-\phi)\,d\,q - \frac{\ell\,a\,H_{N+1}^n}{1+b\,P_{N+1}^n}\right]\,P_{N+1}^n \end{aligned} \qquad (\ 2.5.69\ )$$

and

$$-2\,dq\,\phi\,H_N^{n+1} + \left[1+\ell m + 2\,dq\,\phi - \frac{\ell\, a\, P_{N+1}^{n+1}}{1+b\,P_{N+1}^{n}}\right] H_{N+1}^{n+1} = 2\,(1-\phi)\,dq\,H_N^n + [1-2\,(1-\phi)\,dq]\,H_{N+1}^n \qquad (2.5.70)$$

The solution vectors $\mathbf{P}^{n+1}$ and $\mathbf{H}^{n+1}$ may be obtained using the following parallel algorithm

$$Processor\,1 : Solve \quad E_3\,\mathbf{P}^{n+1} = F_3\,\mathbf{P}^n \quad for \quad \mathbf{P}^{n+1}, \qquad (2.5.71)$$

$$Processor\,2 : Solve \quad J_3\,\mathbf{H}^{n+1} = L_3\,\mathbf{H}^n \quad for \quad \mathbf{H}^{n+1}, \qquad (2.5.72)$$

where L_3, is a constant, tridiagonal square matrix of order $(N+2)$ and is of the form

$$L_3 = \begin{pmatrix} X & 2(1-\phi)\,dq & & & \\ (1-\phi)\,dq & X & (1-\phi)\,dq & & \\ & (1-\phi)\,dq & X & (1-\phi)\,dq & \\ & \ddots & \ddots & \ddots & \\ & & (1-\phi)\,dq & X & (1-\phi)\,dq \\ & & & 2(1-\phi)\,dq & X \end{pmatrix}, \qquad (2.5.73)$$

and $E_3 = E_3\,(\mathbf{P}^n)$, $F_3 = F_3\,(\mathbf{P}^n, \mathbf{H}^n)$ and $J_3 = J_3\,(\mathbf{P}^{n+1}, \mathbf{P}^n)$ are square matrices of order $(N+2)$ and are of the forms

$$E_3 = \begin{pmatrix} U_0 & -2\,dq\,\phi & & & \\ -dq\,\phi & U_1 & -dq\,\phi & & \\ & -dq\,\phi & U_2 & -dq\,\phi & \\ & \ddots & \ddots & \ddots & \\ & & -dq\,\phi & U_N & -dq\,\phi \\ & & & -2\,dq\,\phi & U_{N+1} \end{pmatrix}, \qquad (2.5.74)$$

$$F_3 = \begin{pmatrix} V_0 & 2(1-\phi)\,dq & & & \\ (1-\phi)\,dq & V_1 & (1-\phi)\,dq & & \\ & (1-\phi)\,dq & V_2 & (1-\phi)\,dq & \\ & \ddots & \ddots & \ddots & \\ & & (1-\phi)\,dq & V_N & (1-\phi)\,dq \\ & & & 2(1-\phi)\,dq & V_{N+1} \end{pmatrix}, \qquad (2.5.75)$$

$$J_3 = \begin{pmatrix} Z_0 & -2\,d\,q\,\phi & & & & \\ -d\,q\,\phi & Z_1 & -d\,q\,\phi & & & \\ & -d\,q\,\phi & Z_2 & -d\,q\,\phi & & \\ & \ddots & \ddots & \ddots & & \\ & & -d\,q\,\phi & Z_N & -d\,q\,\phi \\ & & & -2\,d\,q\,\phi & Z_{N+1} \end{pmatrix}, \qquad (\,2.5.76\,)$$

in which

$$\begin{aligned} X &= 1 - 2\,(1-\phi)\,d\,q\,; \\ U_k &= 1 + \ell\,r\,P_k^n + 2\,d\,q\,\phi\,; \\ V_k &= 1 + \ell\,r - 2\,(1-\phi)\,d\,q - \tfrac{\ell\,a\,H_k^n}{1+b\,P_k^n}\,; \\ Z_k &= 1 + \ell\,m + 2\,d\,q\,\phi - \tfrac{\ell\,a\,P_k^{n+1}}{1+b\,P_k^n}\,; \\ k &= 0,1,2,\ldots,N+1\,. \end{aligned}$$

2.5.5 Numerical Results and Discussion

Numerical results were obtained using Methods A(ϕ), B(ϕ) and C(ϕ) with $\phi = 1$, $d = 10^{-4}$ and the set of parameters given in (2.3.7). Methods A(ϕ) and B(ϕ) give constant populations for both prey and predators (see Figure 2.7). As ℓ is increased from zero using Method A(ϕ), the number of prey is decreased whereas that of the predator is increased till $\ell = 0.22$. At this value of ℓ, the number of prey is more than that of the predator. When $\ell > 0.22$, the prey population is less than the predator population. For $\ell \geq 0.29$, overflow occurs.

For small values of ℓ, Method B(ϕ) converges to the fixed point $(0.3, 0.2835)$ as shown in Figure 2.7. When $\ell \geq 2.5$ negative values of prey populations were obtained.

Method C(ϕ) converges to the fixed point (0.3,0.2835) for $\ell = 3.5$. For $\ell = 4.5$, the prey appear to exceed their maximum size, at $t \approx 210$, see Figure 2.8. This is because of the low order of the numerical method; the excess above unity may be accounted for by the $O\,(\ell^2)$ local truncation error. For this value of ℓ, i.e. $\ell = 4.5$, because predators become extinct at $t \approx 340$, see Figure 2.8, the prey quickly achieve their maximum density of unity. When $\ell = 4.65$, the behaviour in Figure 2.9 can be elucidated as follows: the predators ate so many of the prey that, at time $t = 45$ (approximately), there were fewer prey than predators. Because they could not get food some of the predators died. This gave the prey the chance to recover and at time $t = 60$ (approximately) the number of prey again exceeded the number of predators. This continued until $t \approx 240$ after which the pattern began to repeat itself. At $\ell = 4.7$, both prey and predator populations exceed their maximum values, unity, for $t > 60$ (approximately) as shown in Figure 2.10.

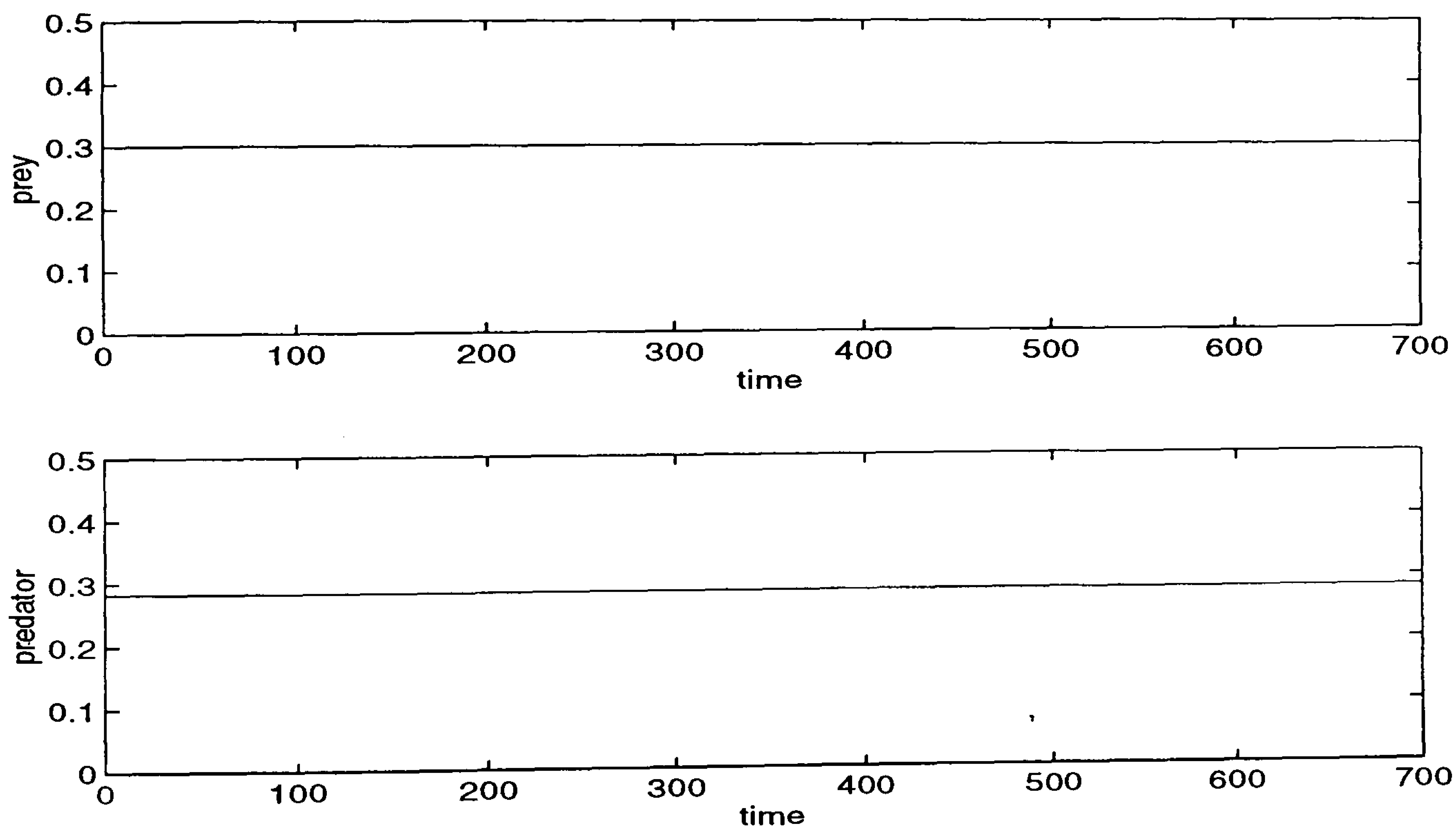

Figure 2.7: *Method B(ϕ) with* $\ell = 0.7$ *and* $x = 0.85$.

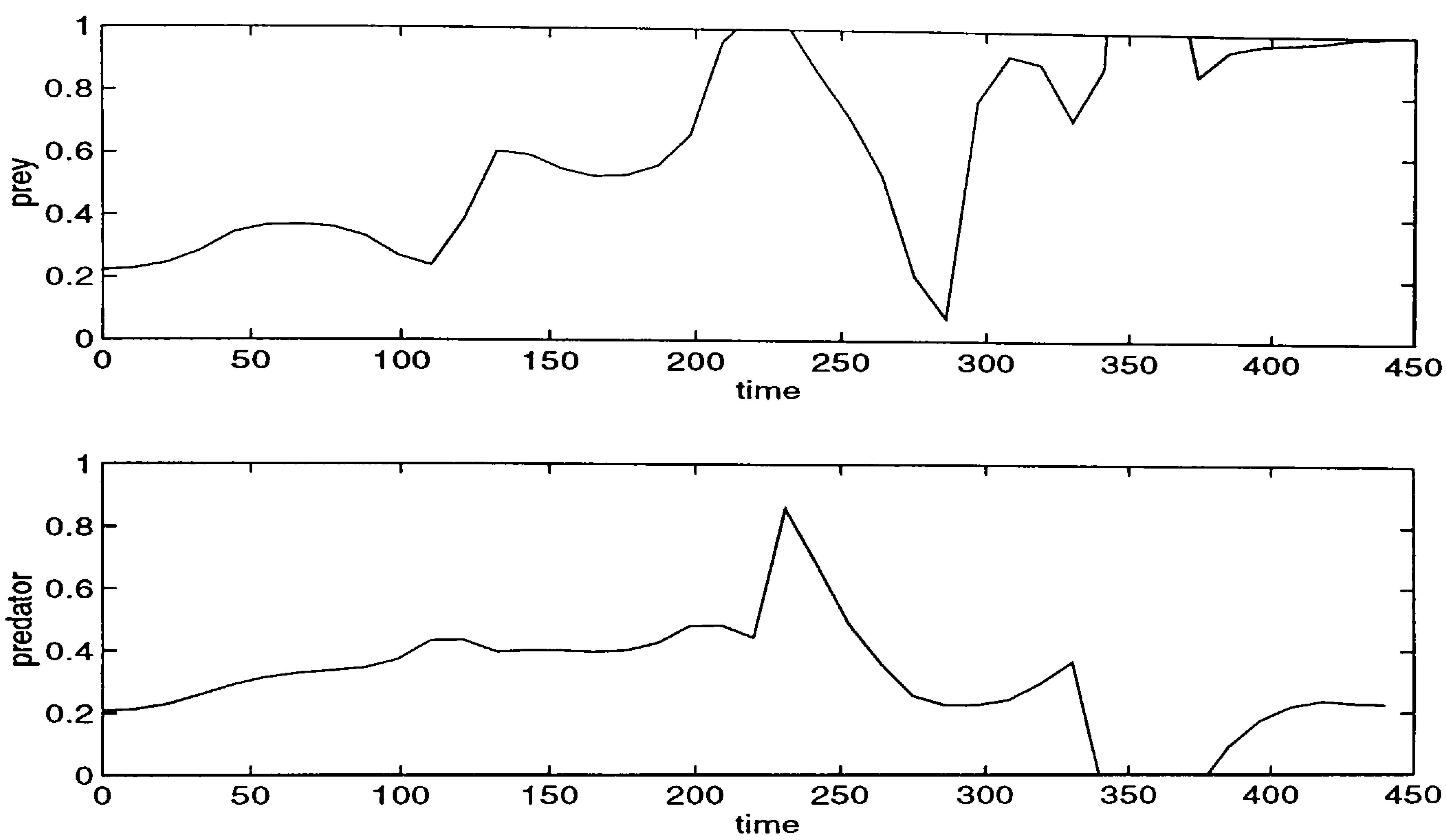

Figure 2.8: *Method $C(\phi)$ with $\ell = 4.5$ and $x = 0.85$.*

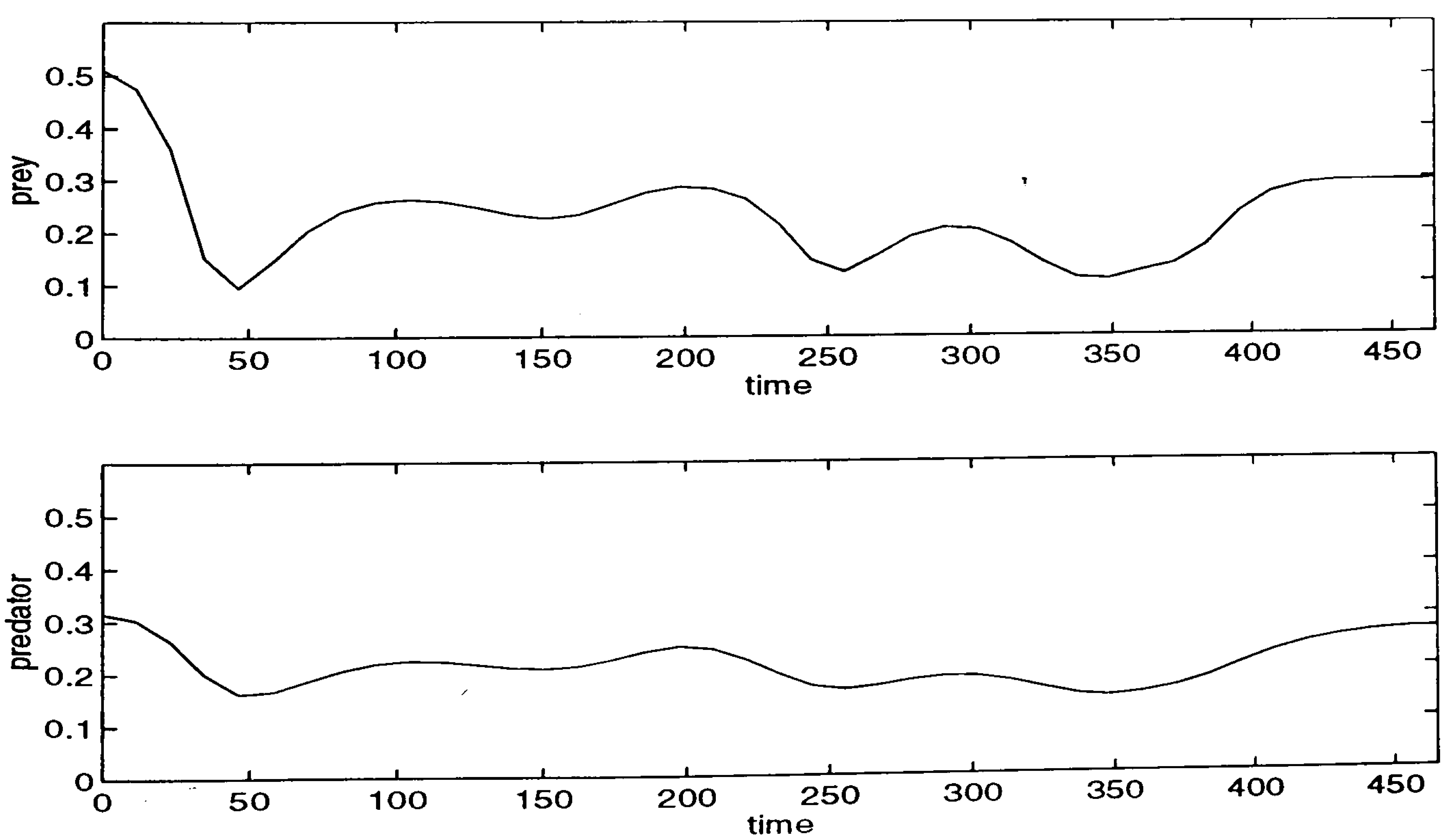

Figure 2.9: *Method $C(\phi)$ with $\ell = 4.65$ and $x = 0.85$.*

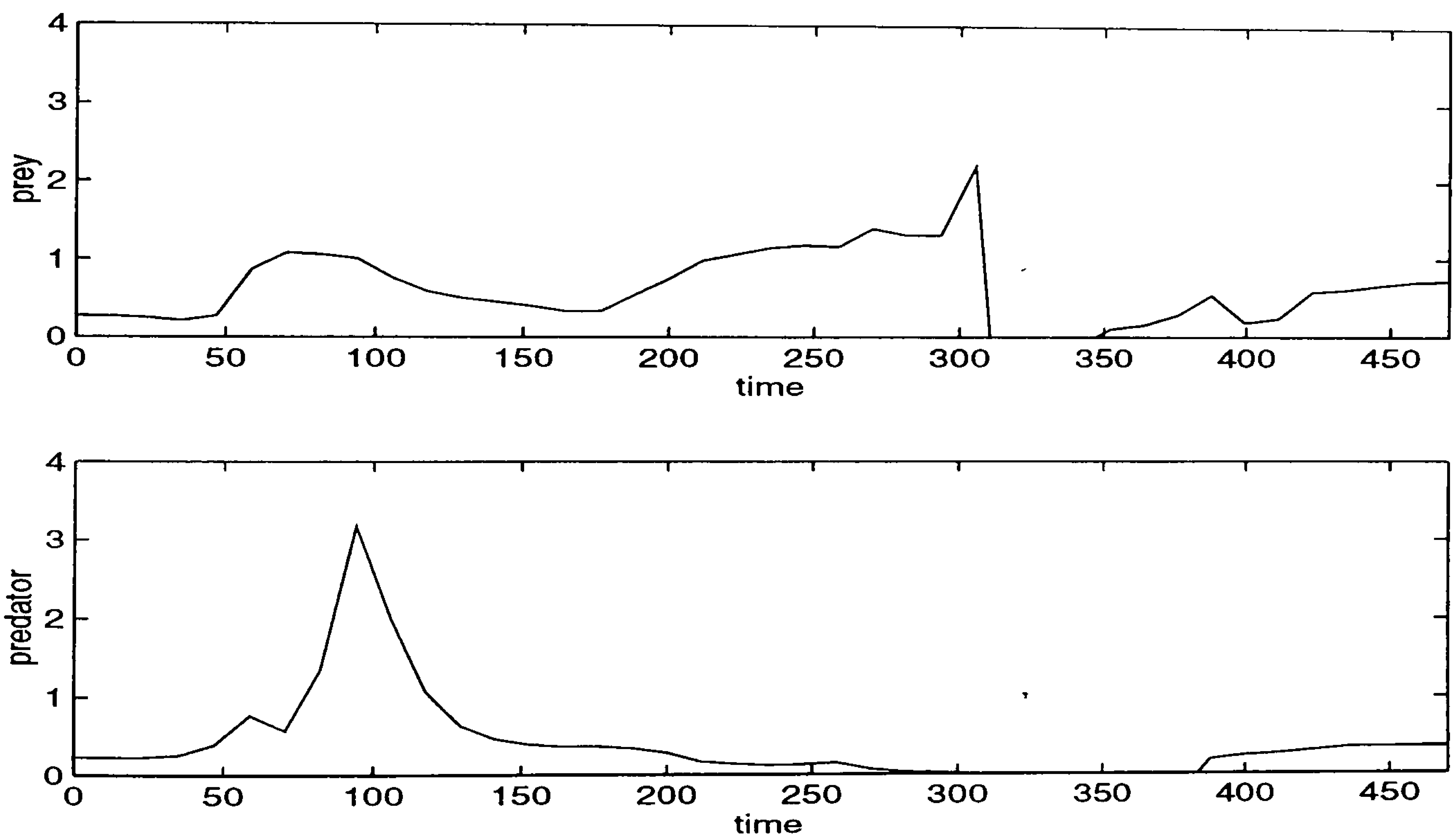

Figure 2.10: *Method C(ϕ) with* $\ell = 4.7$ *and* $x = 0.85$.

2.6 Conclusion

In the absence of diffusion, the predator-prey model, system (2.4.1)–(2.4.3), becomes an initial-value problem which was solved numerically using three methods. The first method is the Euler method and the other two methods were constructed by evaluating some factors at the time level $t = t_n$ and some at $t = t_{n+1}$. All the three methods are first-order accurate. It has been shown that Method 1 (Euler Method) gives limit cycles for small values of the time step ℓ and for larger values of ℓ, chaos has been observed.

Methods 2 and 3 converged to the stationary point $s_3 = (0.3, 0.2835)$ as the time step ℓ is increased; that is the higher the value of ℓ, the quicker the stable stationary point s_3 is reached.

For the reaction-diffusion predator-prey system which is given in equations (2.5.1)-(2.5.4), three numerical methods have been introduced, analysed, implemented and used to solve the system. Numerical results showed that, for the first and the second methods, Methods A(ϕ) and B(ϕ), a constant populations for both prey and predators has been found for small value of the time step ℓ. As $\ell \geq 0.29$, overflow occurred for Method A(ϕ).

Method B(ϕ), for small value of ℓ, converged to the fixed point $(0.3, 0.2835)$ and negative values of prey populations were obtained when $\ell \geq 2.5$).

Method C(ϕ) behaved much better than both Methods A(ϕ) and B(ϕ) as it converged to the fixed point $(0.3, 0.2835)$ for $\ell = 3.5$.

Chapter 3

Measles Dynamics

3.1 Introduction

Measles, a common, acute, contagious disease, chiefly of children, is characterized by fever, sore eyes, catarrh (inflammation and excessive mucus in nose and throat), and a spotty rash. Unlike the case in industrialized nations, measles is a common cause of death in children in underdeveloped countries, where they are likely to be already suffering from malnutrition and poor health. The disease is also highly contagious in monkeys, the only other known host in which measles develops spontaneously.

Measles is a disease of all climates and races, and susceptibility is universal. It must have been common in the ancient world, but no accurate account occurs in history until the classical description by Rhazes in A.D. 915. Thomas Sydenham, in the 17th century, clearly distinguished it from scarlet fever, with which it had been confused. P. L. Panum, in 1847, published the results of his brilliant study of a measles epidemic in the Faroe Islands, definitely establishing the incubation and infectivity periods of measles, see Mathews[40].

Measles is caused by an RNA virus of the paramyxovirus group, a group also containing the human mumps virus, the canine distemper virus, and the cattle rinderpest virus. The measles virus is closely related to the latter two, and, indeed, it is believed that the measles virus evolved from the canine distemper virus.

Measles is one of the most infectious diseases known. It is estimated that 95 percent of the world's urbanized population contract measles before the age of 21. The disease is transmitted largely by the inhalation of respiratory droplets from the nose, throat, and mouth of an infected individual and also by the inhalation of contaminated dust. The attack rate is very high: almost all those who are susceptible develop the disease when exposed. One attack confers lifelong immunity; second attacks are extremely rare. Infants whose mothers had measles will be immune up to the age of four or possibly six months because of the presence in their blood of protective antibodies derived from their mothers. The seasonal peak incidence of measles is late winter. For some obscure reason, in certain cities every second or third winter presents a much higher measles peak. Measles is commonly acquired in school, in one of the primary classes.

The incubation period, or the time from exposure to the virus to the onset of symptoms, is eight to 13 days (Anderson & May[2]): usually 10 days. The first symptom is fever; about 12 hours later the eyes become sore and bloodshot, and after a further 12 hours the catarrhal signs develop. Small, scattered, white spots (Koplik's spots) appear inside the cheeks approximately two to four days after the first symptoms. One or two days after the appearance of the Koplik's spots (three to five days after the first symptoms), the rash suddenly appears. It starts behind the ears and on the forehead, spreading down the body and limbs. When the rash is at its maximum, two to three days after it begins, body temperature may be very high, reaching $40.5°C$. The rash lasts about four to seven days, disappearing from parts of the body in the order of its appearance. It leaves behind a brownish discolouration, which fades in seven to 10 days.

Greatest communicability, or likelihood of transmitting the disease, is from 11 to

16 days after exposure to the virus: that is, approximately from the onset of fever until about the fourth day of the rash.

The mortality of uncomplicated measles is very low, but the complications are frequent and important. In the order of frequency of occurrence they are (Mathews[40]):

1) middle-ear infection, often with mastoiditis,

2) bronchopneumonia (inflammation of the lungs),

3) inflamed lymph nodes in the neck,

4) laryngitis (inflammation of the larynx), and

5) encephalitis (inflammation of the brain).

There is no specific drug directly useful in the treatment of measles. Isolation, bed rest, and a fluid diet are necessary. Darkening the room is not necessary. Antibiotics are often given to prevent bacterial complications.

Research to find a measles vaccine began in 1954 when a measles virus, the Edmonston strain, was successfully grown in tissue culture for the first time. From this strain the first line-virus vaccine was developed by Dr. John F. Enders (Mathews[40]). A field trial in 1961 proved the vaccine to be highly effective. In 1963 the U.S. Public Health Service licenced for manufacture two types of measles vaccine: a live, attenuated (weakened), virus vaccine and an inactivated ('killed') virus vaccine. The use of the short-term inactivated vaccine is not recommended, and none has been used in the United States since 1968. Passive immunization (with gamma globulin antibodies) is effective for prevention and attenuation of measles, but the immune serum globulin must be infected within five days – preferably as soon as possible – after exposure to the virus to prevent the development of the disease. Despite earlier predictions, however, that immunization would eliminate the disease, measles is actually increasing on a world-wide basis.

3.2 Mathematical Epidemiology

The application of mathematics to the study of infectious diseases appears to have been initiated by Daniel Bernoulli in 1760 (Anderson & May[2]). He used a mathematical method to evaluate the effectiveness of the techniques of variolation against smallpox, with the aim of influencing public health policy. There then followed a long gap until the middle of the nineteenth century when, in 1840, William Farr effectively fitted a normal curve to smoothed quarterly data on deaths from smallpox in England and Wales over the period 1837-1839. This descriptive approach was developed further by John Brownlee[8] who published a paper entitled "*Statistical studies in immunity; the theory of an epidemic*" in 1906, in which he fitted a Pearsonian frequency distribution curve to a large series of epidemics. The empirical approaches adopted by Farr and Brownlee (Anderson & May[2]) were in great contrast to the work of two other scientists of the same period, Hamer and Ross. Their contribution was to apply post-germ-theory-thinking towards the solution of two specific quantitative problems: the regular occurrence of measles epidemics and the relationship between numbers of mosquitoes and the incidence of malaria (Hamer[20]; Ross[54]; Moshkovskii[47]). They were the first to formulate specific theories about the transmission of infectious diseases in simple but precise mathematical statements and to investigate the properties of the resulting models. Their work, in conjunction with the studies of Ross & Hudson [55], Soper[62], and Kermack & McKendrick[28] began to provide a firm theoretical framework for the investigation of observed patterns.

Hamer[20] postulated that the course of an epidemic depends on the rate of contact between susceptible and infectious individuals. This notion has become one of the most important concepts in mathematical epidemiology; it is the so-called *mass action principle* in which the net rate of spread of infection is assumed to be proportional to the product of the density of susceptible people multiplied by the density of infectious individuals. The principle was originally formulated in a discrete-time model, but in

1908 Ronald Ross (celebrated as the discoverer of malaria transmission by mosquitoes) translated the problem into a continous-time framework in his pioneering work on the dynamics of malaria (Anderson & May[2]).

The ideas of Hamer and Ross were extended and explored in more detail by Soper[62] who deduced the underlying mechanisms responsible for the often-observed periodicity of epidemics, and by Kermack & McKendrick[28] who established the celebrated threshold theory. This theory, according to which the introduction of a few infectious individuals into a community of susceptibles will not give rise to an epidemic outbreak unless the density or number of susceptibles is above a certain critical value, is, in conjunction with the mass action principle, a cornerstone of modern theoretical epidemiology (Anderson & May[2]).

Since this early beginning, the growth in the literature concerned with mathematical epidemiology has been very rapid indeed. Recent reviews of the literature have been published by Bailey[4], Bolker & Grenfell[6], Dietz[14], Dietz & Schenzle[13], Schenzle[57] and Tidd et al.[64]. In particular, models incorporating seasonality (Aron & Schwartz[3]; Fine & Clarkson[18]; London & Yorke[36]; Olsen & Schaffer[50]) and age structure (Anderson & May[1]; Dietz & Schenzle[12]; Schenzle[57]) generate important predictions both about the likely performance of vaccination strategies and the observed dynamics of infection.

3.3 Compartmental Models

Measles epidemics may be described using compartmental models (see, for example, Anderson & May[2]; Jansen[26, p.5]). This approach divides the population into certain classes: susceptible, exposed, infectious and recovered individuals. In the course of an epidemic a person will then "flow" from the susceptible compartment into the exposed, the infectious and finally into the recovered compartment, as in the following diagram.

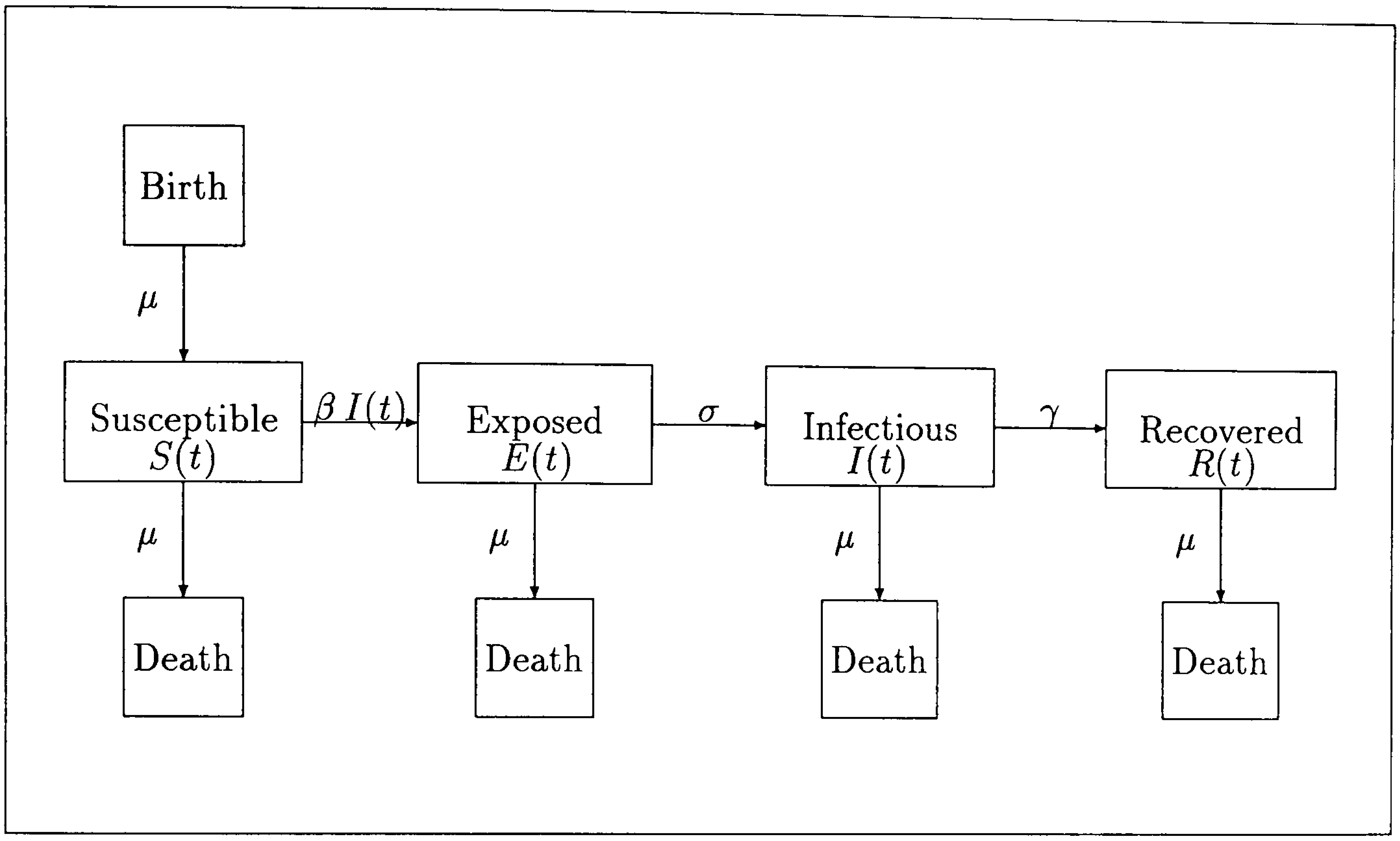

Figure 3.1: *Schematic representation of the flow of hosts through the compartments.*

Note that hosts both die and reproduce at the *per capita* rate, μ. It is assumed that nobody dies of measles, therefore the infected hosts do not experience a higher mortality rate. Recovered individuals do not flow back into the susceptibles compartment, as life-long immunity is supposed.

3.4 The SEIR Model

The SEIR (Susceptible/Exposed/Infectious/Recovered) model is expressed mathematically, from the compartmental model depicted in Figure 3.1, as a set of three non-linear ordinary differential equations given by

$$\begin{aligned}\frac{dS(t)}{dt} &= \mu N - (\mu + \beta I(t))\, S(t)\\ \frac{dE(t)}{dt} &= \beta I(t)\, S(t) - (\mu + \sigma)\, E(t)\\ \frac{dI(t)}{dt} &= \sigma E(t) - (\mu + \gamma)\, I(t)\end{aligned} \qquad (3.4.1)$$

with $t \geq 0$, subject to the initial conditions

$$S(0) = S^0, \quad E(0) = E^0, \quad I(0) = I^0 \qquad (3.4.2)$$

in the domain

$$\mathcal{D} = \{(S, E, I) \in I\!R^3_+ \,|\, S + E + I \leq N\}. \qquad (3.4.3)$$

Here, $S = S(t)$, $E = E(t)$, $I = I(t)$ and $R = R(t)$ represent, respectively, the density of susceptible, exposed, infectious and recovered individuals at time t in a constant population of size $N = S + E + I + R$. Thus, R is determined by S, E and I and the fourth differential equation derived from the compartmental model and describing the rate of change of R does not need to be considered. The terms $\frac{1}{\mu}$, $\frac{1}{\sigma}$, $\frac{1}{\gamma}$ are the average life expectancy, disease incubation and infectious period, respectively; β denotes the infection rate. It is assumed that the incubation period coincides with the latent period. All time-related parameters are measured in years. The incubation period is the period from the point of infection to the appearance of symptoms of disease, whereas the latent period is the period from the point of infection to the beginning of the state of infectiousness, see Anderson & May[2, p.14]. They estimated, for measles, the incubation period to be 8-13 days, the latent period to be 6-9 days and 6-7 days for the infectious period. Hence, in their view, the incubation time does not coincide with the latent period. However, they state that the consideration of latent times does not contribute much to the dynamics of the system (Anderson & May[2, p.59]). The infection rate β will be varied in the analysis and in essence will serve as a bifurcation parameter.

The SEIR model will be considered for the following set of parameter values

$$\begin{aligned} N &= 5 \times 10^7, \\ \mu &= 0.02\,years^{-1}, \\ \sigma &= 45.6\,years^{-1}, \\ \gamma &= 73.0\,years^{-1}. \end{aligned} \qquad (3.4.4)$$

Theses values represent a population size of 50 million, average life expectancy of 50 years and incubation and infectious periods of roughly eight and five days, respectively, as considered by Bolker & Grenfell[6].

3.5 Qualitative Analysis

3.5.1 Stationary Points

To study the stationary points of the SEIR model, the system (3.4.1) is written as

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u})$$

where

$$\mathbf{u}(t) = (S(t), E(t), I(t))^T \in I\!R^3,$$

$$\begin{aligned}\mathbf{f(u)} &= (f_1(\mathbf{u}), f_2(\mathbf{u}), f_3(\mathbf{u}))^T \\ &= (\mu N - (\mu + \beta I)S, \beta I S - (\mu + \sigma)E, \sigma E - (\mu + \gamma)I)^T, \qquad (3.5.1)\end{aligned}$$

and T denotes transpose.

Equating f_1, f_2 and f_3 to zero reveals the existence of two stationary points. The first stationary point, which is trivial, is given by

$$\mathbf{s}_1 = \left(\overline{S}_1, \overline{E}_1, \overline{I}_1\right)^T = (N, 0, 0)^T; \qquad (3.5.2)$$

the phrase "critical point" may also be used to describe $\mathbf{s}_1$.

This steady-state is trivial in the sense that it corresponds to the case of the existence of no exposed or infectious individuals in the population. Hence, all individuals are healthy and clearly stay healthy for all time.

The second, non-trivial, stationary point is

$$\mathbf{s}_2 = \left(\overline{S}_2, \overline{E}_2, \overline{I}_2\right)^T$$

where

$$\begin{aligned}\overline{S}_2 &= \frac{(\mu + \sigma)(\mu + \gamma)}{\sigma \beta}, \\ \overline{E}_2 &= \frac{\mu N}{\mu + \sigma} - \frac{\mu(\mu + \gamma)}{\sigma \beta}, \qquad (3.5.3) \\ \overline{I}_2 &= \frac{\mu \sigma N}{(\mu + \sigma)(\mu + \gamma)} - \frac{\mu}{\beta}.\end{aligned}$$

It is noted that, due to the magnitude of N, only $\overline{S}_2$ varies significantly with β.

Using the set of parameters in (3.4.4), this stationary point is

$$\mathbf{s}_2 = \left(\frac{73.052}{\beta}, 21920.21 - \frac{0.032}{\beta}, 13688.874 - \frac{0.02}{\beta}\right)^T. \qquad (3.5.4)$$

A graph of stationary points for $\beta \in [10^{-9}, 10^{-3}]$ is shown in Figure 3.2 (Jansen[26, p.14]).

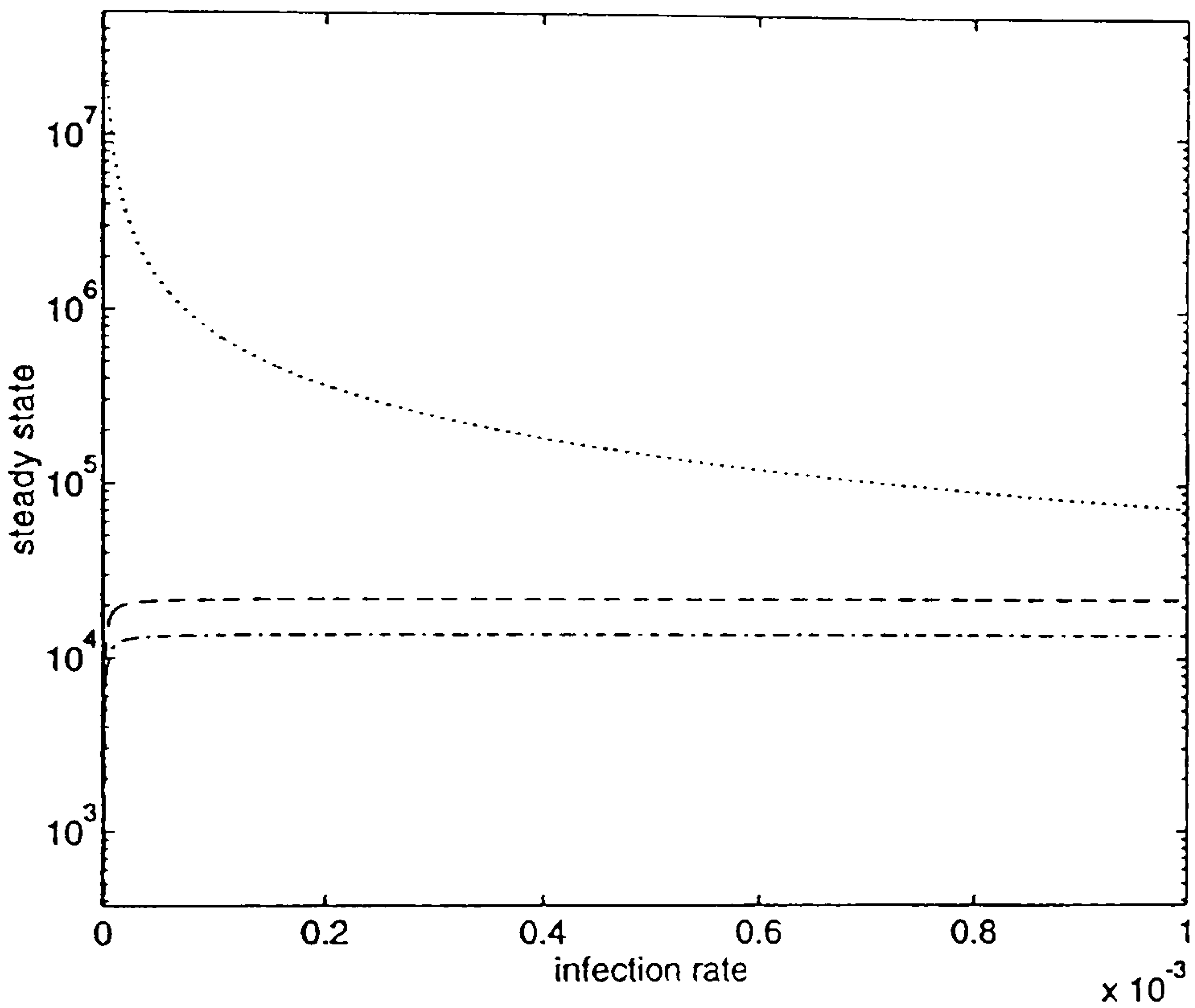

Figure 3.2: *Non-trivial steady-states for various infection rates on a logarithmic scale; susceptible[dots], exposed[dashes], infectious[dash-dot].*

3.5.2 Stability

The Jacobian of the system (3.4.1) is given by

$$J_f(\mathrm{u}) = \begin{pmatrix} -(\mu + \beta I) & 0 & -\beta S \\ \beta I & -(\mu + \sigma) & \beta S \\ 0 & \sigma & -(\mu + \gamma) \end{pmatrix}. \qquad (3.5.5)$$

Using the parameter set in (3.4.4), the Jacobian at the trivial critical point is given by

$$J_f(\mathrm{s}_1) = \begin{pmatrix} -0.02 & 0 & -5 \times 10^7 \beta \\ 0 & -45.62 & 5 \times 10^7 \beta \\ 0 & 45.6 & -73.02 \end{pmatrix}. \qquad (3.5.6)$$

The eigenvalues[1] of $J_f(\mathrm{s}_1)$ are all distinct and real for all $\beta \in [0, \infty)$. For $\beta < 1.46104 \times 10^{-6}$, all are negative, whereas for larger values of β, one eigenvalue is positive and two eigenvalues are negative.

Therefore, the trivial steady-state is asymptotically stable for $\beta \in [0, 1.46104 \times 10^{-6})$ and unstable for larger β. The value

$$\overline{\beta}_1 = 1.46104 \times 10^{-6} \qquad (3.5.7)$$

thus denotes a bifurcation point of the system.

Considering the non-trivial stationary point s_2 leads to

$$J_f(\mathrm{s}_2) = \begin{pmatrix} -13688.874\,\beta & 0 & -73.052 \\ 13688.874\,\beta - 0.02 & -45.62 & 73.052 \\ 0 & 45.60 & -73.02 \end{pmatrix}. \qquad (3.5.8)$$

Here, numerical computation of the eigenvalues revealed them to be distinct and real for $\beta < 1.45975 \times 10^{-6}$. In this range, one eigenvalue is positive while two are negative. For $1.45975 \times 10^{-6} \leq \beta \leq 1.46043 \times 10^{-6}$, the Jacobian has distinct, real and negative eigenvalues, while for larger values of β, it has a pair of complex eigenvalues with negative real part and one real, negative eigenvalue.

It follows that the non-trivial steady-state is unstable for $\beta \in [0, 1.45975 \times 10^{-6})$ and asymptotically stable for $\beta \in [1.45975 \times 10^{-6}, \infty)$.

[1] *All numerical computations were carried out by the software package "mathematica", version 2.2, ©1988-93 by Wolfram Research, Inc.*

The value

$$\overline{\beta}_2 = 1.45975 \times 10^{-6} \qquad (3.5.9)$$

therefore denotes a bifurcation point of the system.

The bifurcation value concerning the non-trivial steady-state $\overline{\beta}_2$, however, is likely to coincide with $\overline{\beta}_1$. The difference appears to be of a computational nature, as the matrices involved are very poorly conditioned (the condition number $K(J) = \|J\|.\|J^{-1}\| >> 1$) and the calculated values differ by only 1.29×10^{-9} (Jansen[26]).

As the relevant eigenvalues of the system's Jacobian at the trivial and the non-trivial stationary points change from real and negative to real and positive or *vice versa*, the bifurcation point may be determined to be the point at which the determinants of $J_f(\mathrm{s}_1)$ and $J_f(\mathrm{s}_2)$ vanish.

In this way, it is seen that the stationary points change their stability properties, simultaneously, so that

$$\beta^* = \frac{(\mu+\sigma)(\mu+\gamma)}{\sigma N} \qquad (3.5.10)$$

is the unique bifurcation point of the system.

Using the parameter set in (3.4.4), the bifurcation point is calculated as

$$\beta^* = 1.46104 \times 10^{-6}. \qquad (3.5.11)$$

For $\beta = \beta^*$, there is only one steady state since the coordinates of s_1 and s_2 coincide. This stationary point, s, is non-hyperbolic as the Jacobian at this point has eigenvalues {-118.64,-0.02,0}. Therefore, the stability cannot be determined using Theorem 1.3.1. It is estimated (Jansen[26]), however, that this steady-state is neutrally stable. The stability properties of the model are summarized in Table 3.1.

Table 3.1: *Stability properties of stationary points.*

β	s_1	s_2
$\beta < \beta^*$	asymptotically stable	unstable
$\beta > \beta^*$	unstable	asymptotically stable

3.6 Numerical Methods

The solution of the SEIR model given by (3.4.1) and (3.4.2) is required for $t > 0$ and to obtain a numerical solution, the time interval $t \geq 0$ is discretized at the points $t_n = n\,\ell \quad (n = 0, 1, 2, \ldots)$; ℓ is called the *time step.* The theoretical solution of the system at any typical point $t = t_n$ is given by $\mathbf{u}(t_n) = (S(t_n), E(t_n), I(t_n))^T$, while the solution of an approximating numerical method will be denoted by $\mathbf{u}^n = (S^n, E^n, I^n)^T$ at the same point t_n. The numerical methods are based on the replacement of the derivatives $\frac{dS}{dt}$, $\frac{dE}{dt}$ and $\frac{dI}{dt}$ in (3.4.1) by the first-order approximations

$$\begin{aligned} \frac{dS}{dt} &= [S(t+\ell) - S(t)]\,/\ell + O(\ell), \\ \frac{dE}{dt} &= [E(t+\ell) - E(t)]\,/\ell + O(\ell), \\ \frac{dI}{dt} &= [I(t+\ell) - I(t)]\,/\ell + O(\ell), \end{aligned} \qquad (3.6.1)$$

as $\ell \longrightarrow 0$.

3.6.1 Method 1 (Euler Method)

Using (3.6.1) in (3.4.1) with $t = t_n$ and evaluating S, E and I on the right-hand side of (3.4.1) at $t = t_n$ leads to

$$\begin{aligned} S^{n+1} &= S^n + \ell\,\mu\,N - \ell\,(\mu + \beta\,I^n)\,S^n\,; \\ E^{n+1} &= E^n + \ell\,\beta\,I^n\,S^n - \ell\,(\mu + \sigma)\,E^n\,; \\ I^{n+1} &= I^n + \ell\,\sigma\,E^n - \ell\,(\mu + \gamma)\,I^n\,; \end{aligned} \qquad (3.6.2)$$

$n = 0, 1, 2, \ldots$, which is the familiar Euler explicit method.

The local truncation error (l.t.e.) of S in (3.6.2) at $t = t_n$ is given by

$$\mathcal{L}_S\,[S(t), E(t), I(t); \ell] = S(t + \ell) - S(t) - \ell\,\mu\,N + \ell\,(\mu + \beta\,I(t))\,S(t)\,. \qquad (3.6.3)$$

Similar expressions for the local truncation errors of E and I are obtained at $t = t_n$. Using Taylor's expansion of $S(t + \ell)$, $E(t + \ell)$ and $I(t + \ell)$ about t, it follows that the l.t.e.'s are

$$\begin{aligned} \mathcal{L}_S\,[S(t), E(t), I(t); \ell] &= \frac{1}{2}\ell^2\,\ddot{S}(t) + O(\ell^3) \quad as \quad \ell \longrightarrow 0\,, \\ \mathcal{L}_E\,[S(t), E(t), I(t); \ell] &= \frac{1}{2}\ell^2\,\ddot{E}(t) + O(\ell^3) \quad as \quad \ell \longrightarrow 0\,, \\ \mathcal{L}_I\,[S(t), E(t), I(t); \ell] &= \frac{1}{2}\ell^2\,\ddot{I}(t) + O(\ell^3) \quad as \quad \ell \longrightarrow 0\,, \end{aligned} \qquad (3.6.4)$$

at some point $t = t_n$. This verifies that this familiar numerical method is first-order accurate.

Stability of Fixed Points of Method 1

Method 1 is explicit and hence the expressions for S^{n+1}, E^{n+1} and I^{n+1} in (3.6.2) can be written, respectively, in the forms

$$\begin{aligned} S^{n+1} &= g_1(S^n, E^n, I^n), \\ E^{n+1} &= g_2(S^n, E^n, I^n), \\ I^{n+1} &= g_3(S^n, E^n, I^n), \end{aligned} \tag{3.6.5}$$

$n = 0, 1, 2, \ldots$. The fixed points of (3.6.5) are the same as the stationary points of the system (3.4.1)-(3.4.2); that is, one is trivial $\overline{\mathbf{u}}_1 = (N, 0, 0)^T$ and the other is non-trivial $\overline{\mathbf{u}}_2 = (S_2, E_2, I_2)^T$, where

$$\begin{aligned} S_2 &= \frac{(\mu+\sigma)(\mu+\gamma)}{\sigma\,\beta}, \\ E_2 &= \frac{\mu\,N}{\mu+\sigma} - \frac{\mu\,(\mu+\gamma)}{\sigma\,\beta}, \\ I_2 &= \frac{\sigma\,\mu\,N}{(\mu+\sigma)(\mu+\gamma)} - \frac{\mu}{\beta}. \end{aligned} \tag{3.6.6}$$

Using the set of parameters given in (3.4.4) gives

$$\overline{\mathbf{u}}_1 = (5 \times 10^7, 0, 0)^T \tag{3.6.7}$$

$$\overline{\mathbf{u}}_2 = \left(\frac{73.052}{\beta}, 21920.21 - \frac{0.032}{\beta}, 13688.874 - \frac{0.02}{\beta}\right)^T. \tag{3.6.8}$$

To analyse the stability of this method, consider the associated functions

$$\begin{aligned} g_1(S,E,I) &= S + \ell\,\mu\,N - \ell\,(\mu + \beta\,I)\,S, \\ g_2(S,E,I) &= E + \ell\,\beta\,I\,S - \ell\,(\mu+\sigma)\,E, \\ g_3(S,E,I) &= I + \ell\,\sigma\,E - \ell\,(\mu+\gamma)\,I, \end{aligned} \tag{3.6.9}$$

so that the Jacobian of the Euler method is

$$J(\overline{\mathbf{u}}_s) = \begin{pmatrix} 1-\ell\,(\mu+\beta\, I_s) & 0 & -\ell\,\beta\, S_s \\ \ell\,\beta\, I_s & 1-\ell\,(\mu+\sigma) & \ell\,\beta\, S_s \\ 0 & \ell\,\sigma & 1-\ell\,(\mu+\gamma) \end{pmatrix}. \qquad (3.6.10)$$

Evaluating J at $\overline{\mathbf{u}}_1$, using the set of parameters in (3.4.4), gives

$$J(\overline{\mathbf{u}}_1) = \begin{pmatrix} 1-0.02\,\ell & 0 & -5\times 10^7\,\beta\,\ell \\ 0 & 1-45.62\,\ell & 5\times 10^7\,\beta\,\ell \\ 0 & 45.6\,\ell & 1-73.02\,\ell \end{pmatrix}. \qquad (3.6.11)$$

When $\ell = 0$ and for all β, all the eigenvalues are equal to unity. When $\beta = 0$, all the eigenvalues are positive provided $0 < \ell < 0.0136949$, two are positive and one is negative for $0.0136949 \leq \ell \leq 0.0219202$, while for larger ℓ, one eigenvalue is positive and two eigenvalues are negative. For this value of β, the spectral radius is less than unity provided $0 < \ell < 0.0273898$. When $\ell = 0.0273898$, one eigenvalue equals -1, one eigenvalue is positive and less than unity and the other eigenvalue is negative; the spectral radius is unity. The spectral radius is greater than unity for $\ell > 0.0273898$. If ℓ is increased further, one eigenvalue vanishes at $\ell = 50$ and again equals -1 at $\ell = 100$. For $\ell > 50$, all the eigenvalues are negative and still the spectral radius is greater than unity.

For $\beta = 1 \times 10^{-6} (< \beta^*)$, all the eigenvalues are real, positive and less than unity for $0 < \ell \leq 0.00917466$; two are positive and one is negative provided $0.00917466 < \ell < 0.018349324$. One eigenvalue equals -1 and two eigenvalues are positive and less than unity for $\ell = 0.018349324$. The spectral radius is less than unity provided $\ell < 0.018349324$ while it equals 1 for $\ell = 0.018349324$. For $0.018349324 < \ell < 0.10369$, the eigenvalues are distinct and real; one is negative and two are positive, while one is positive and two are negative provided $0.10369 \leq \ell < 50$. One eigenvalue equals zero at $\ell = 50$ and equals -1 at $\ell = 100$. For this value of β, the spectral radius passes through unity as ℓ exceeds 0.018349324.

For $\beta = 1.46104 \times 10^{-6} (= \beta^*)$, the eigenvalues are real, positive and distinct with one eigenvalue equal to unity and two less than unity provided $0 < \ell \leq 0.00842886$. For $0.00824886 < \ell < 0.016857723$, one eigenvalue equals unity, one eigenvalue is positive and less than unity and the other eigenvalue is negative. One eigenvalue equals -1, one eigenvalue equals unity and the other is positive and less than unity provided $\ell = 0.016857723$. For larger values of ℓ, two eigenvalues are positive and less than unity while the third eigenvalue is negative. For $\ell = 50$, one eigenvalue vanishes. The spectral radius is greater than unity provided $\ell > 0.016857723$.

For $\beta = 5 \times 10^{-4} (> \beta^*)$, all the eigenvalues are real and positive with one greater than unity provided $0 < \ell \leq 0.00088722$. For a larger time step, ℓ, one eigenvalue is negative and the other eigenvalues are positive. At $\ell = 50$, one eigenvalue equals zero, one eigenvalue is negative and the third is positive. For $\ell > 50$, one eigenvalue is positive and two eigenvalues are negative. For this value of β, the spectral radius is larger than unity for all positive time steps ($\ell > 0$).

Considering the non-trivial fixed point $\overline{\mathbf{u}}_2$ and the set of parameters given in (3.4.4), the Jacobian in (3.6.10) becomes

$$J(\overline{\mathbf{u}}_2) = \begin{pmatrix} 1 - 13688.874\,\beta\,\ell & 0 & -73.052\,\ell \\ (13688.874\,\beta - 0.02)\,\ell & 1 - 45.62\,\ell & 73.052\,\ell \\ 0 & 45.6\,\ell & 1 - 73.02\,\ell \end{pmatrix}. \qquad (3.6.12)$$

If $\ell = 0$, all the eigenvalues are equal to unity. For $0 < \ell < \infty$ and $0 < \beta < \beta^*$, all the eigenvalues are real and distinct. When $0 < \ell < 0.00842897$, the eigenvalues are all positive: one is greater than unity while two are less than unity. One eigenvalue is negative and two eigenvalues are positive provided $0.00842897 \leq \ell < 2.33306$, while two eigenvalues are negative and one eigenvalue is positive for $\ell \geq 2.33306$. The spectral radius is greater than unity for all $\ell > 0$.

For $\beta = 1.46104 \times 10^{-6} (= \beta^*)$, all the eigenvalues are real and distinct provided the time step, ℓ, is positive. When $0 < \ell < 0.00842887$, one eigenvalue equals unity and two eigenvalues are positive and less than unity, while one eigenvalue equals unity, one eigenvalue is positive and one eigenvalue is negative provided $0.00842887 \leq \ell < 0.016857723$. When $\ell = 0.016857723$, one eigenvalue equals -1, one eigenvalue equals 1 and the third eigenvalue is negative. One eigenvalue is negative, one eigenvalue is positive and less than unity and the other eigenvalue is positive and greater than unity provided $0.016857723 < \ell < 49.9735$. For $\ell \geq 49.9736$, one eigenvalue is positive and less than unity and two eigenvalues are negative. The spectral radius, for this value of β, equals unity provided $\ell \leq 0.016857723$ while it is greater than unity for $\ell > 0.016857723$.

For $\beta = 5 \times 10^{-4} (> \beta^*)$, the Jacobian, $J(\overline{\mathbf{u}}_2)$, has a pair of complex conjugate eigenvalues with positive real part and one real and positive eigenvalue provided $0 < \ell < 0.00831218$. When $0.00831218 \leq \ell < 0.01662435$, two of the eigenvalues are complex with positive real part and one is real and negative, while two are complex with positive real part and one is real and equal to -1 provided $\ell = 0.01662435$. When $0.01662435 < \ell < 0.38617601$, the Jacobian has a pair of complex conjugate eigenvalues with positive real part and one real and negative eigenvalue. Two of the eigenvalues are complex with negative real part and one is real and negative provided $\ell \geq 0.38617601$. The spectral radius of the Jacobian, $\rho(J(\overline{\mathbf{u}}_2))$, passes through unity as the time step, ℓ, becomes larger than 0.01662435; $\rho(J(\overline{\mathbf{u}}_2)) < 1$ when $\ell < 0.01662435$, $\rho(J(\overline{\mathbf{u}}_2)) = 1$ when $\ell = 0.01662435$ and $\rho(J(\overline{\mathbf{u}}_2)) > 1$ when $\ell > 0.01662435$.

Table 3.2 gives a summary of these observations.

Table 3.2: *Stability properties of fixed points of Method 1. Entries in different rows of the same type of stability correspond to alternative conditions.*

	trivial fixed point	non-trivial fixed point
neutral	$\ell = 0 \;\&\; \forall \beta$ $\ell = 0.0273898 \;\&\; \beta < \beta^*$ $\ell = 0.018349324 \;\&\; \beta < \beta^*$ $0 < \ell \leq 0.016857723 \;\&\; \beta = \beta^*$	$\ell = 0 \;\&\; \forall \beta$ $0 < \ell \leq 0.016857723 \;\&\; \beta = \beta^*$ $\ell = 0.01662435 \;\&\; \beta > \beta^*$
attracting (stable)	$0 < \ell < 0.0273898 \;\&\; \beta < \beta^*$ $0 < \ell < 0.018349324 \;\&\; \beta < \beta^*$	$0 < \ell < 0.01662435 \;\&\; \beta > \beta^*$
repelling (unstable)	$\ell > 0.0273898 \;\&\; \beta < \beta^*$ $\ell > 0.018349324 \;\&\; \beta < \beta^*$ $\ell > 0.016857723 \;\&\; \beta = \beta^*$ $\ell > 0 \;\&\; \beta > \beta^*$	$\ell > 0.016857723 \;\&\; \beta = \beta^*$ $\ell > 0.01662435 \;\&\; \beta > \beta^*$ $\ell > 0 \;\&\; \beta < \beta^*$

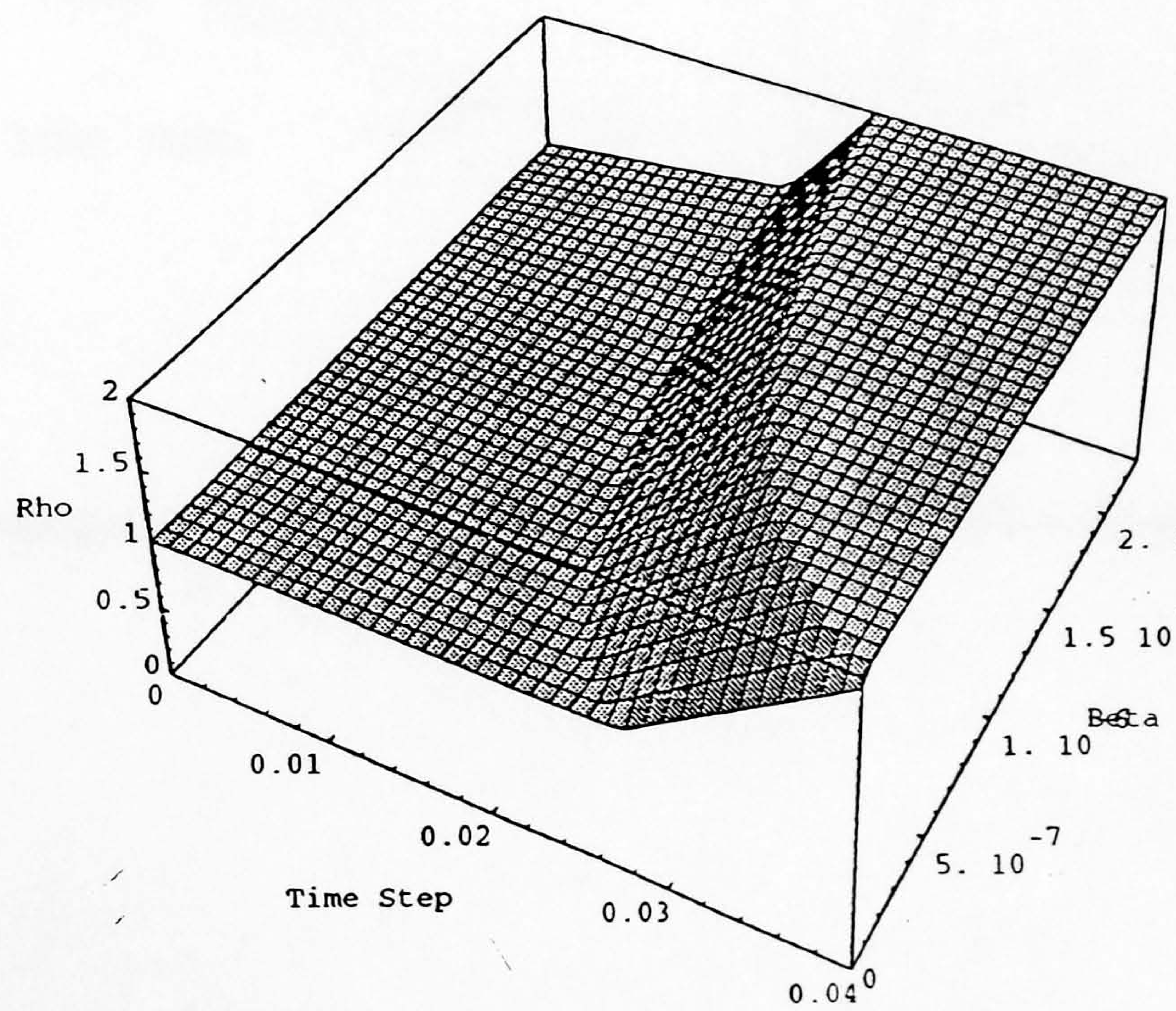

Figure 3.3: *Spectral radius of the Jacobian of Method 1 at the trivial fixed point.*

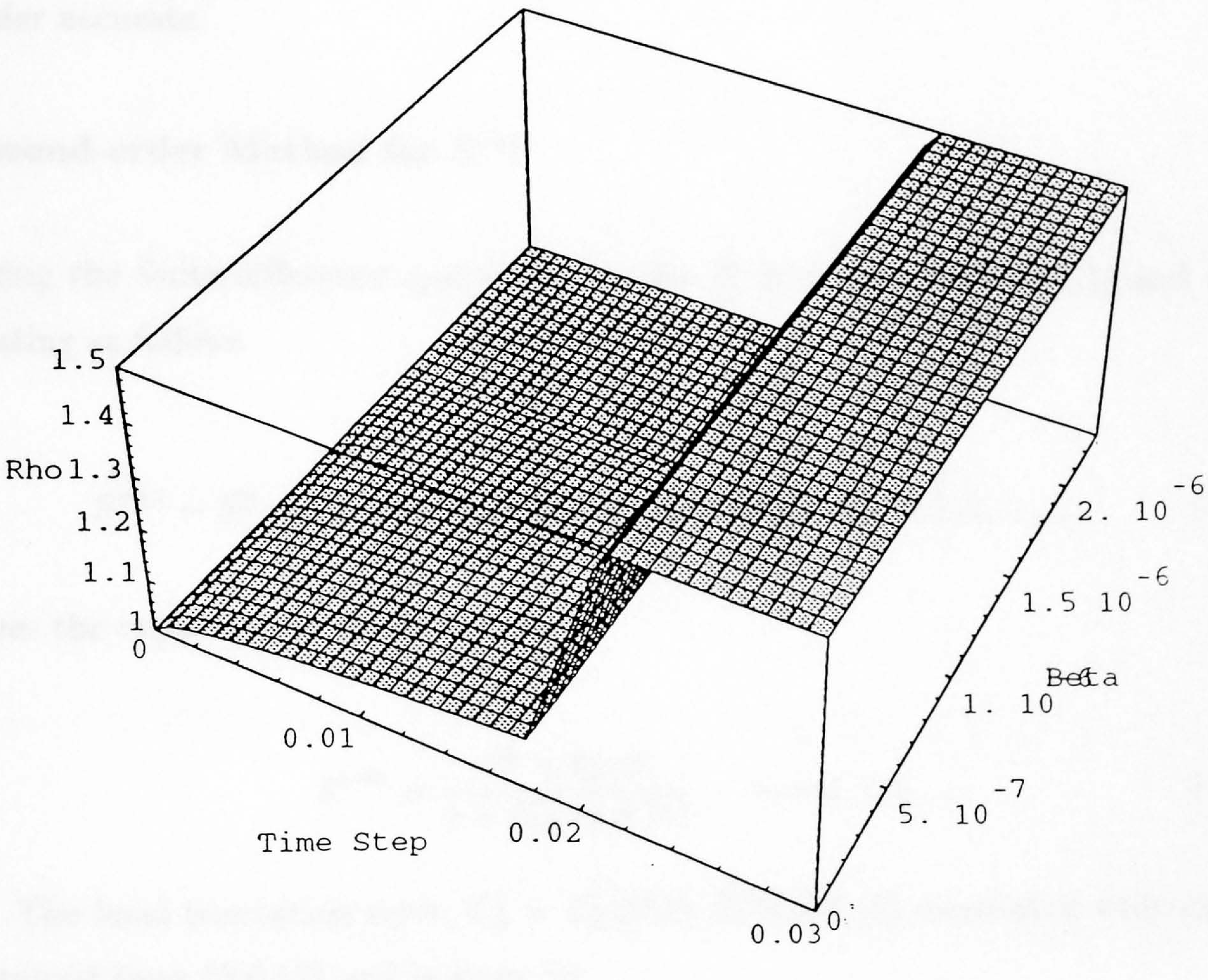

Figure 3.4: *Spectral radius of the Jacobian of Method 1 at the non-trivial fixed point.*

3.6.2 Method 2 (Second-order Method)

To achieve second-order accuracy when solving non-linear ODEs in (3.4.1), a linear combination of first-order methods is used so that the resulting method will be second-order accurate.

Second-order Method for S^{n+1}

Using the finite-difference approximation for $\frac{dS}{dt}$ from (3.6.1) in (3.4.1) and approximating as follows

$$S^{n+1} - S^n - \ell\,\mu\,N + \ell\,(\mu + \beta\,I^n)\,S^{n+1} = 0\,; \quad n = 0, 1, 2, \dots\,, \qquad (3.6.13)$$

gives the explicit formulation

$$S^{n+1} = \frac{S^n + \ell\,\mu\,N}{1 + \ell\,(\mu + \beta\,I^n)}\,; \quad n = 0, 1, 2, \dots\,. \qquad (3.6.14)$$

The local truncation error, $\mathcal{L}_S^1 = \mathcal{L}_S^1\,[S(t), E(t), I(t); \ell]$, associated with (3.6.14) is obtained from (3.6.13) and is given by

$$\begin{aligned} \mathcal{L}_S^1 &= S(t+\ell) - S(t) - \ell\,\mu\,N + \ell\,(\mu + \beta\,I(t))\,S(t+\ell) \\ &= \left(\frac{1}{2}\ddot{S} + (\mu + \beta\,I)\dot{S}\right)\ell^2 + O(\ell^3) \end{aligned} \qquad (3.6.15)$$

as $\ell \longrightarrow 0$, at some point $t = t_n$.

Now, using the difference approximation for $\frac{dS}{dt}$ from (3.6.1) in (3.4.1) again. S may be approximated alternatively as

$$S^{n+1} - S^n - \ell\,\mu\,N + \ell\,(\mu + \beta\,I^{n+1})\,S^n = 0\,; \qquad (3.6.16)$$

$n = 0, 1, 2, \ldots$, which, also, can be solved explicitly for S^{n+1} to get

$$S^{n+1} = \Big[1 - \ell\,(\mu + \beta\,I^{n+1})\Big]\;S^n + \ell\,\mu\,N\,; \quad n = 0, 1, 2, \ldots\,. \qquad (3.6.17)$$

The local truncation error, $\mathcal{L}_S^2 = \mathcal{L}_S^2\,[S(t), E(t), I(t); \ell]$, associated with (3.6.16) is given, at some point $t = t_n$, by

$$\begin{aligned} \mathcal{L}_S^2 &= S(t+\ell) - S(t) - \ell\,\mu\,N + \ell\,(\mu + \beta\,I(t+1))\,S(t) \\ &= \Big(\frac{1}{2}\,\ddot{S} + \beta\,\dot{I}\,S\Big)\;\ell^2 + O(\ell^3) \end{aligned} \qquad (3.6.18)$$

as $\ell \longrightarrow 0$. Differentiating the first equation in (3.4.1) with respect to time, t, gives

$$\ddot{S} + (\mu + \beta\,I)\,\dot{S} + \beta\,\dot{I}\,S = 0\,. \qquad (3.6.19)$$

By defining the quantity $\mathcal{L}_S^e = \mathcal{L}_S^e\,[S(t), E(t), I(t); \ell]$ by

$$\mathcal{L}_S^e = \mathcal{L}_S^1 + \mathcal{L}_S^2\,, \qquad (3.6.20)$$

it is easy to see that

$$\mathcal{L}_S^e = \Big(\ddot{S} + (\mu + \beta\,I)\,\dot{S} + \beta\,\dot{I}\,S\Big)\;\ell^2 + O(\ell^3) \qquad (3.6.21)$$

as $\ell \longrightarrow 0$, so that, using (3.6.19),

$$\mathcal{L}_S^e = O(\ell^3) \quad as \quad \ell \longrightarrow 0\,. \qquad (3.6.22)$$

The construction of the associated second-order method follows from (3.6.20). Thus, adding (3.6.13) and (3.6.16), the numerical method associated with (3.6.15) and (3.6.18) which are used in (3.6.20), gives

$$[2+\ell\,(\mu+\beta\,I^n)]\;S^{n+1}-2\,\ell\,\mu\,N+\ell\,(\mu+\beta\,I^{n+1})\,S^n=2\,S^n \qquad (3.6.23)$$

which can be written as

$$\left[1+\frac{1}{2}\,\ell\,(\mu+\beta\,I^n)\right]\;S^{n+1}-\ell\,\mu\,N+\frac{1}{2}\,\ell\,(\mu+\beta\,I^{n+1})\,S^n=S^n \qquad (3.6.24)$$

Second-order Method for E^{n+1}

Now using the difference approximation for $\frac{dE}{dt}$ from (3.6.1) in (3.4.1) and approximating as follows

$$E^{n+1}-E^n-\ell\,\beta\,I^n\,S^{n+1}+\ell\,(\mu+\sigma)\,E^{n+1}=0\,; \qquad (3.6.25)$$

$n=0,1,2,\ldots$, gives the explicit expression

$$E^{n+1}=\frac{E^n+\ell\,\beta\,I^n\,S^{n+1}}{1+\ell\,(\mu+\sigma)}\,;\quad n=0,1,2,\ldots \qquad (3.6.26)$$

for E^{n+1}. The local truncation error, $\mathcal{L}_E^1=\mathcal{L}_E^1\,[S(t),E(t),I(t);\ell]$, at some point $t=t_n$, associated with (3.6.25) is given by

$$\begin{aligned}\mathcal{L}_E^1 &= E(t+\ell)-E(t)-\ell\,\beta\,I(t)\,S(t+\ell)+\ell\,(\mu+\sigma)\,E(t+\ell)\\ &= \left(\frac{1}{2}\,\ddot{E}-\beta\,I\,\dot{S}+(\mu+\sigma)\,\dot{E}\right)\,\ell^2+O(\ell^3)\end{aligned} \qquad (3.6.27)$$

as $\ell \longrightarrow 0$. A second approximation to the differential equation involving $\frac{dE}{dt}$ gives the following

$$E^{n+1} - E^{n} - \ell\,\beta\, I^{n+1}\, S^{n} + \ell\,(\mu + \sigma)\, E^{n} = 0\,; \qquad (3.6.28)$$

$n = 0, 1, 2, \ldots$, which may be solved explicitly for E^{n+1}.

The local truncation error, $\mathcal{L}_E^2 = \mathcal{L}_E^2\,[S(t), E(t), I(t); \ell]$, at some point $t = t_n$, associated with (3.6.28) is given by

$$\begin{aligned} \mathcal{L}_E^2 &= E(t+\ell) - E(t) - \ell\,\beta\, S(t)\, I(t+\ell) + \ell\,(\mu+\sigma)\, E(t) \\ &= \left(\frac{1}{2}\,\ddot{E} - \beta\, S\, \dot{I}\right)\ell^2 + O(\ell^3)\,, \end{aligned} \qquad (3.6.29)$$

as $\ell \longrightarrow 0$. Differentiating the second equation in (3.4.1) with respect to time, t, gives

$$\ddot{E} - \beta\, I\, \dot{S} - \beta\, \dot{I}\, S + (\mu+\sigma)\, \dot{E} = 0\,. \qquad (3.6.30)$$

The expressions in (3.6.27) and (3.46.29) verify that both equations (3.6.26) and (3.6.28) are first-order approximations to E. However, it is easy to show that the quantity $\mathcal{L}_E^e = \mathcal{L}_E^2\,[S(t), E(t), I(t); \ell]$ defined by

$$\mathcal{L}_E^e = \mathcal{L}_E^1 + \mathcal{L}_E^2 = O(\ell^3) \quad as \quad \ell \longrightarrow 0 \qquad (3.6.31)$$

so that the numerical method associated with (3.6.31), which may be constructed simply by adding (3.6.25) and (3.6.28) to give

$$[2 + \ell\,(\mu+\sigma)]\; E^{n+1} - \ell\,\beta\, I^{n}\, S^{n+1} - \ell\,\beta\, I^{n+1}\, S^{n} + \ell\,(\mu+\sigma)\, E^{n} = 2\, E^{n}\,, \qquad (3.6.32)$$

is second-order accurate; equation (3.6.32) can be written in the form

$$\left[1+\frac{1}{2}\,\ell\,(\mu+\sigma)\right]\,E^{n+1}-\frac{1}{2}\,\ell\,\beta\,I^{n}\,S^{n+1}-\frac{1}{2}\,\ell\,\beta\,I^{n+1}\,S^{n}+\frac{1}{2}\,\ell\,(\mu+\sigma)\,E^{n}=E^{n}\,. \qquad (\ 3.6.33\ )$$

Second-order Method for I^{n+1}

One suitable approximation to the differential equation in (3.4.1) involving $\frac{dI}{dt}$ is given by

$$I^{n+1}-I^{n}-\ell\,\sigma\,E^{n+1}+\ell\,(\mu+\gamma)\,I^{n+1}=0\,;\quad n=0,1,2,\dots\,, \qquad (\ 3.6.34\ )$$

which may be solved explicitly for I^{n+1} to give

$$I^{n+1}=\frac{I^{n}+\ell\,\sigma\,E^{n+1}}{1+\ell\,(\mu+\gamma)}\,;\quad n=0,1,2,\dots\,. \qquad (\ 3.6.35\ )$$

The local truncation error, $\mathcal{L}_I^1=\mathcal{L}_I^1\,[S(t),E(t),I(t);\ell]$, associated with (3.6.34) is given, at some point $t=t_n$, by

$$\begin{aligned}\mathcal{L}_I^1 &= I(t+\ell)-I(t)-\ell\,\sigma\,E(t+\ell)+\ell\,(\mu+\gamma)\,I(t+\ell)\\ &= \left(\frac{1}{2}\,\ddot{I}-\sigma\,\dot{E}+(\mu+\gamma)\,\dot{I}\right)\,\ell^2+O(\ell^3)\end{aligned} \qquad (\ 3.6.36\ )$$

as $\ell\longrightarrow 0$. A second approximation to this differential equation is given by

$$I^{n+1}-I^{n}-\ell\,\sigma\,E^{n}+\ell\,(\mu+\gamma)\,I^{n}=0\,;\quad n=0,1,2,\dots\,, \qquad (\ 3.6.37\ )$$

which may also be written explicitly for I^{n+1}.

The local truncation error, $\mathcal{L}_I^2 = \mathcal{L}_I^2[S(t), E(t), I(t); \ell]$, at some point $t = t_n$, associated with (3.6.37) is given by

$$\begin{aligned} \mathcal{L}_I^2 &= I(t+\ell) - I(t) - \ell\,\sigma\,E(t) + \ell\,(\mu+\gamma)\,I(t) \\ &= \frac{1}{2}\ddot{I}\,\ell^2 + O(\ell^3) \end{aligned} \qquad (3.6.38)$$

as $\ell \longrightarrow 0$. Differentiating the third equation in (3.4.1) with respect to time, t, gives

$$\ddot{I} - \sigma\,\dot{E} + (\mu+\gamma)\,\dot{I} = 0\,. \qquad (3.6.39)$$

The expressions in (3.6.36) and (3.6.38) verify that both (3.6.35) and (3.6.37) are first-order approximations to I. However, it is easy to show that the quantity $\mathcal{L}_I^e = \mathcal{L}_I^e[S(t), E(t), I(t); \ell]$ defined by

$$\mathcal{L}_I^e = \mathcal{L}_I^1 + \mathcal{L}_I^2 = O(\ell^3) \quad as \quad \ell \longrightarrow 0\,, \qquad (3.6.40)$$

so that the numerical method associated with (3.6.40), which may be constructed simply by adding (3.6.34) and (3.6.37) to give

$$[2 + \ell\,(\mu+\gamma)]\; I^{n+1} - \ell\,\sigma\,E^{n+1} - \ell\,\sigma\,E^n + \ell\,(\mu+\gamma)\,I^n = 2\,I^n\,, \qquad (3.6.41)$$

is second-order accurate; equation (3.6.41) can be written in the form

$$\left[1 + \frac{1}{2}\ell\,(\mu+\gamma)\right]\; I^{n+1} - \frac{1}{2}\ell\,\sigma\,E^{n+1} - \frac{1}{2}\ell\,\sigma\,E^n + \frac{1}{2}\ell\,(\mu+\gamma)\,I^n = I^n\,. \qquad (3.6.42)$$

To solve the linear algebraic system (3.6.23), (3.6.32) and (3.6.41) for S^{n+1}, E^{n+1} and I^{n+1}, the system may be written for simplicity in the form

$$a\,S^{n+1} - 2\,\ell\,\mu\,N + \ell\,\mu\,S^n + \ell\,\beta\,I^{n+1}\,S^n = 2\,S^n\,, \qquad (3.6.43)$$

$$b\,E^{n+1} - \ell\,\beta\,I^n\,S^{n+1} - \ell\,\beta\,I^{n+1}\,S^n + \ell\,(\mu+\sigma)\,E^n = 2\,E^n\,, \qquad (3.6.44)$$

$$c\,I^{n+1} - \ell\,\sigma\,E^{n+1} - \ell\,\sigma\,E^n + \ell\,(\mu+\gamma)\,I^n = 2\,I^n\,, \qquad (3.6.45)$$

where

$$\begin{aligned} a &= 2+\ell\,(\mu+\beta\,I^n)\,, \\ b &= 2+\ell\,(\mu+\sigma)\,, \\ c &= 2+\ell\,(\mu+\gamma)\,. \end{aligned} \qquad (3.6.46)$$

Solving[2] the equations (3.6.43)-(3.6.46), gives

$$S^{n+1} = \frac{[(2-\ell\,\mu)\,S^n + 2\,\ell\,\mu\,N]\,[b\,c - \ell^2\,\beta\,\sigma\,S^n] - 4\,\ell^2\,\beta\,\sigma\,S^n\,E^n - \ell\,\beta\,b\,[2-\ell\,(\mu+\gamma)]\,I^n\,S^n}{\Delta}\,, \qquad (3.6.47)$$

$$E^{n+1} = \frac{c\,\ell\,\beta\,[(2-\ell\,\mu)\,S^n + 2\,\ell\,\mu\,N]\;I^n + [a\,c\,(2-\ell\,(\mu+\sigma)) + \ell^2\,\beta\,\sigma\,(2+\ell\,\mu)\,S^n]\;E^n}{\Delta} + \frac{\ell\,\beta\,(2+\ell\,\mu)\;[2-\ell\,(\mu+\gamma)]\;I^n\,S^n}{\Delta}\,, \qquad (3.6.48)$$

and

$$I^{n+1} = \frac{\ell^2\,\beta\,\sigma\;[(2-\ell\,\mu)\,S^n + 2\,\ell\,\mu\,N]\;I^n + 4\,a\,\ell\,\sigma\,E^n + a\,b\,[2-\ell\,(\mu+\gamma)]\;I^n}{\Delta}\,, \qquad (3.6.49)$$

where a, b and c are as in (3.6.46) and

$$\Delta = a\,b\,c - \ell^2\,\beta\,\sigma\,(2+\ell\,\mu)\,S^n\,. \qquad (3.6.50)$$

[2] *Solving the equations were carried out by the software package "mathematica", version 2.2, ©1988-93 by Walfram Research, Inc.*

Stability of Fixed Points of Method 2

The expressions for S^{n+1}, E^{n+1} and I^{n+1} in (3.6.47)-(3.6.49) may be written in the forms

$$\begin{aligned} S^{n+1} &= g_1(S^n, E^n, I^n), \\ E^{n+1} &= g_2(S^n, E^n, I^n), \\ I^{n+1} &= g_3(S^n, E^n, I^n); \end{aligned} \qquad (3.6.51)$$

$n = 0, 1, 2, \ldots$, respectively. The fixed points of (3.6.51) are given by solving the following equations

$$\begin{aligned} S &= g_1(S, E, I), \\ E &= g_2(S, E, I), \\ I &= g_3(S, E, I). \end{aligned} \qquad (3.6.52)$$

Computations[3], using the set of parameters in (3.4.4), revealed that the fixed points of this method, Method 2, are the same as the stationary points of the system (3.4.1)-(3.4.2); that is the fixed points are

$$\overline{\mathbf{u}}_1 = (N, 0, 0) \qquad (3.6.53)$$

and

$$\overline{\mathbf{u}}_2 = \left(\frac{73.052}{\beta}, 21920.21 - \frac{0.032}{\beta}, 13688.874 - \frac{0.02}{\beta}\right). \qquad (3.6.54)$$

To examine the stability of this method, consider the associated functions in (3.6.51), so that the Jacobian of this method at some fixed point, $\overline{\mathbf{u}}_s$, is

[3] *The computations were carried out by the software package "mathematica", version 2.2, ©1988-93 by Wolfram Research, Inc.*

$$J(\overline{\mathbf{u}}_s) = \begin{pmatrix} \frac{\partial g_1}{\partial S} & \frac{\partial g_1}{\partial E} & \frac{\partial g_1}{\partial I} \\ \frac{\partial g_2}{\partial S} & \frac{\partial g_2}{\partial E} & \frac{\partial g_2}{\partial I} \\ \frac{\partial g_3}{\partial S} & \frac{\partial g_3}{\partial E} & \frac{\partial g_3}{\partial I} \end{pmatrix}, \qquad (3.6.55)$$

where

$$\begin{aligned}
\frac{\partial g_1}{\partial S} &= \frac{[(2-\ell\,\mu)\,S + 2\,\ell\,\mu\,N]\;(-\ell^2\,\beta\,\sigma) + (2-\ell\,\mu)\,(b\,c - \ell^2\,\beta\,\sigma\,S) - 4\,\ell^2\,\beta\,\sigma\,E}{\Delta} \\
&- \frac{\ell\,\beta\,b\,[2-\ell\,(\mu+\gamma)]\;I}{\Delta} - \frac{[(2-\ell\,\mu)\,S + 2\,\ell\,\mu\,N]\;(b\,c - \ell^2\,\beta\,\sigma\,S)\,(-\ell^2\,\beta\,\sigma)\,(2+\ell\,\mu)}{\Delta^2} \\
&- \frac{4\,\ell^4\,\beta^2\,\sigma^2\,(2+\ell\,\mu)\,S\,E + \ell^3\,\beta^2\,\sigma\,b\,(2+\ell\,\mu)\;[2-\ell\,(\mu+\gamma)]\;S\,I}{\Delta^2}, \\
\frac{\partial g_1}{\partial E} &= -\frac{4\,\ell^2\,\beta\,\sigma\,S}{\Delta}, \\
\frac{\partial g_1}{\partial I} &= \frac{4\,\ell^3\,\beta^2\,\sigma\,b\,c\,S\,E - \ell\,\beta\,b\,c\;[(2-\ell\,\mu)\,S + 2\,\ell\,\mu\,N]\;(b\,c - \ell^2\,\beta\,\sigma\,S)}{\Delta^2} \\
&+ \frac{\ell^2\,\beta^2\,b^2\,c\;[2-\ell\,(\mu+\gamma)]\;S\,I}{\Delta^2} - \frac{\ell\,\beta.b\;[2-\ell\,(\mu+\gamma)]\;S}{\Delta}, \\
\frac{\partial g_2}{\partial S} &= \frac{\ell\,\beta\,c\,(2-\ell\,\mu)\,I + \ell^2\,\beta\,\sigma\,(2+\ell\,\mu)\,E + \ell\,\beta\,(2+\ell\,\mu)\;[2-\ell\,(\mu+\gamma)]\;I}{\Delta} \\
&+ \{\ell\,\beta\,c\;[(2-\ell\,\mu)\,S + 2\,\ell\,\mu\,N]\;I + \left[a\,c\,(2-\ell\,(\mu+\sigma)) + \ell^2\,\beta\,\sigma\,(2+\ell\,\mu)\,S\right]\;E \\
&+ \ell\,\beta\,(2+\ell\,\mu)\;[2-\ell\,(\mu+\gamma)]\;S\,I\} \times \frac{\ell^2\,\beta\,\sigma\,(2+\ell\,\mu)}{\Delta^2}, \\
\frac{\partial g_2}{\partial E} &= \frac{a\,c\;[2-\ell\,(\mu+\sigma)] + \ell^2\,\beta\,\sigma\,(2+\ell\,\mu)\,S}{\Delta}, \\
\frac{\partial g_2}{\partial I} &= \frac{\ell\,\beta\,c\;[(2-\ell\,\mu)\,S + 2\,\ell\,\mu\,N] + \ell\,\beta\,c\;[2-\ell\,(\mu+\sigma)]\;E + \ell\,\beta\,(2+\ell\,\mu)\;[2-\ell\,(\mu+\gamma)]\;S}{\Delta} \\
&- \{\ell\,\beta\,c\;[(2-\ell\,\mu)\,S + 2\,\ell\,\mu\,N]\;I + \left[a\,c\,(2-\ell\,(\mu+\sigma)) + \ell^2\,\beta\,\sigma\,(2+\ell\,\mu)\,S\right]\;E
\end{aligned}$$

$$
\begin{aligned}
& + \;\; \ell\,\beta\,(2+\ell\,\mu)\;[2-\ell\,(\mu+\gamma)]\;S\,I\,\}\times\frac{\ell\,\beta\,b\,c}{\Delta^2}\,,\\
\frac{\partial g_3}{\partial S} &= \frac{\ell^2\,\beta\,\sigma\,(2-\ell\,\mu)\,I}{\Delta}+\{\,\ell^2\,\beta\,\sigma\;[(2-\ell\,\mu)\,S+2\,\ell\,\mu\,N]\;I+4\,a\,\ell\,\sigma\,E\\
& + \;\; a\,b\,[2-\ell\,(\mu+\gamma)]\;I\,\}\times\frac{\ell^2\,\beta\,\sigma\,(2+\ell\,\mu)}{\Delta}\,,\\
\frac{\partial g_3}{\partial E} &= \frac{4\,a\,\ell\,\sigma}{\Delta}\,,\\
\frac{\partial g_3}{\partial I} &= \frac{\ell^2\,\beta\,\sigma\;[(2-\ell\,\mu)\,S+2\,\ell\,\mu\,N]+4\,\ell^2\,\beta\,\sigma\,E+b\;[2-\ell\,(\mu+\gamma)]\;(a+\ell\,\beta\,I)}{\Delta}\\
& - \;\; \{\,\ell^2\,\beta\,\sigma\;[(2-\ell\,\mu)\,S+2\,\ell\,\mu\,N]\;I+4\,a\,\ell\,\sigma\,E+a\,b\,[2-\ell\,(\mu+\gamma)]\,\}\times\frac{b\,c\,\ell\,\beta}{\Delta^2}\,.
\end{aligned}
\tag{3.6.56}
$$

Considering the trivial fixed point, $\overline{\mathrm{u}}_1$, in (3.6.53), the Jacobian in (3.6.55) becomes

$$
J(\overline{\mathrm{u}}_1)=\begin{pmatrix}
\frac{2-\ell\,\mu}{2+\ell\,\mu} & \frac{-4\,\ell^2\,\beta\,\sigma\,N}{(2+\ell\,\mu)\,X} & \frac{-4\,\ell\,\beta\,b\,N}{(2+\ell\,\mu)\,X}\\
0 & \frac{c\,(2-\ell\,(\mu+\sigma))+\ell^2\,\beta\,\sigma\,N}{X} & \frac{4\,\ell\,\beta\,N}{X}\\
0 & \frac{4\,\ell\,\sigma}{X} & \frac{\ell^2\,\beta\,\sigma\,N+b\,(2-\ell\,(\mu+\sigma))}{X}-\frac{\ell\,\beta\,b^2\,c\,(2-\ell\,(\mu+\gamma))}{(2+\ell\,\mu)\,X}
\end{pmatrix},
\tag{3.6.57}
$$

where

$$X = b\,c-\ell^2\,\beta\,\sigma\,N\,.$$

The eigenvalues of the Jacobian, (3.6.57), using the set of parameters in (3.4.4), are all real for all $\ell\,,\beta\in[0,1]$. When $\ell=0$, all the eigenvalues are equal to unity for all $\beta\in[0,1]$.

For $\beta = 1\times10^{-6}(<\beta^*)$, all the eigenvalues are real, positive, distinct and less than unity provided $0<\ell<0.018349323$, while, for $0.018349323\leq\ell<0.20737959$, one eigenvalue is negative and two eigenvalues are positive and less than unity. When

$0.20737959 \leq \ell \leq 100$, one eigenvalue is positive and less than unity and two are negative. All the eigenvalues are negative provided $\ell > 100$. For this value of β, the spectral radius of the Jacobian, $\rho(J(\overline{\mathbf{u}}_1))$, is less than unity for all positive time steps, ℓ.

For $\beta = 1.5 \times 10^{-6}(\approx \beta^*)$, all the eigenvalues are equal to unity provided $\ell = 0$. When $0 < \ell < 6.71 \times 10^{-6}$, two eigenvalues are equal to unity and one eigenvalue is less than unity. One eigenvalue is greater than unity, one eigenvalue equals unity and one eigenvalue is less than unity provided $6.71 \times 10^{-6} \leq \ell < 2.5 \times 10^{-5}$. When $2.5 \times 10^{-5} \leq \ell < 0.01675657$, two eigenvalues are less than unity and one eigenvalue is greater than unity. One eigenvalue is negative and less than unity in magnitude and two eigenvalues are positive; one is greater than unity and the other one is less than unity provided $0.016752657 \leq \ell < 2.688$. For $2.688 \leq \ell < 100$, one eigenvalue is positive and less than unity and two eigenvalues are negative; one is less than unity in magnitude and the other one is greater than unity in magnitude. All the eigenvalues are negative; one is less than unity in magnitude, one equals unity and the third is greater than unity in magnitude, provided $100 \leq \ell < 2 \times 10^6$. When $\ell \geq 2 \times 10^6$, all the eigenvalues are negative; two are equal to unity in magnitude and the third is less than unity in magnitude. For this value of β, the spectral radius equals unity for $0 \leq \ell < 6.71 \times 10^{-6}$ and $\ell \geq 2 \times 10^6$ and is greater than unity provided $6.71 \times 10^{-6} \leq \ell < 2 \times 10^6$.

Using $\beta = 5 \times 10^{-4}(> \beta^*)$, one eigenvalue is positive and greater than unity and two eigenvalues are positive and less than unity for $0 < \ell < 0.0017745$. For $0.0017745 \leq \ell < 0.019832$, one eigenvalue is positive and less than unity, one eigenvalue is negative and less than unity in magnitude and one eigenvalue is positive and greater than unity. Two eigenvalues are negative; one is greater than unity in magnitude and the other is less than unity in magnitude, and one eigenvalue is positive and less than unity provided $0.0019832 \leq \ell < 100$. At $\ell = 100$, one eigenvalue vanishes and equals -1 for $\ell \geq 794$. For $\ell > 100$, all the eigenvalues are negative with one greater than unity in magnitude. For this value of β, the spectral radius is greater than unity for $0 < \ell < 794$ and equal to unity whenever $\ell \geq 794$.

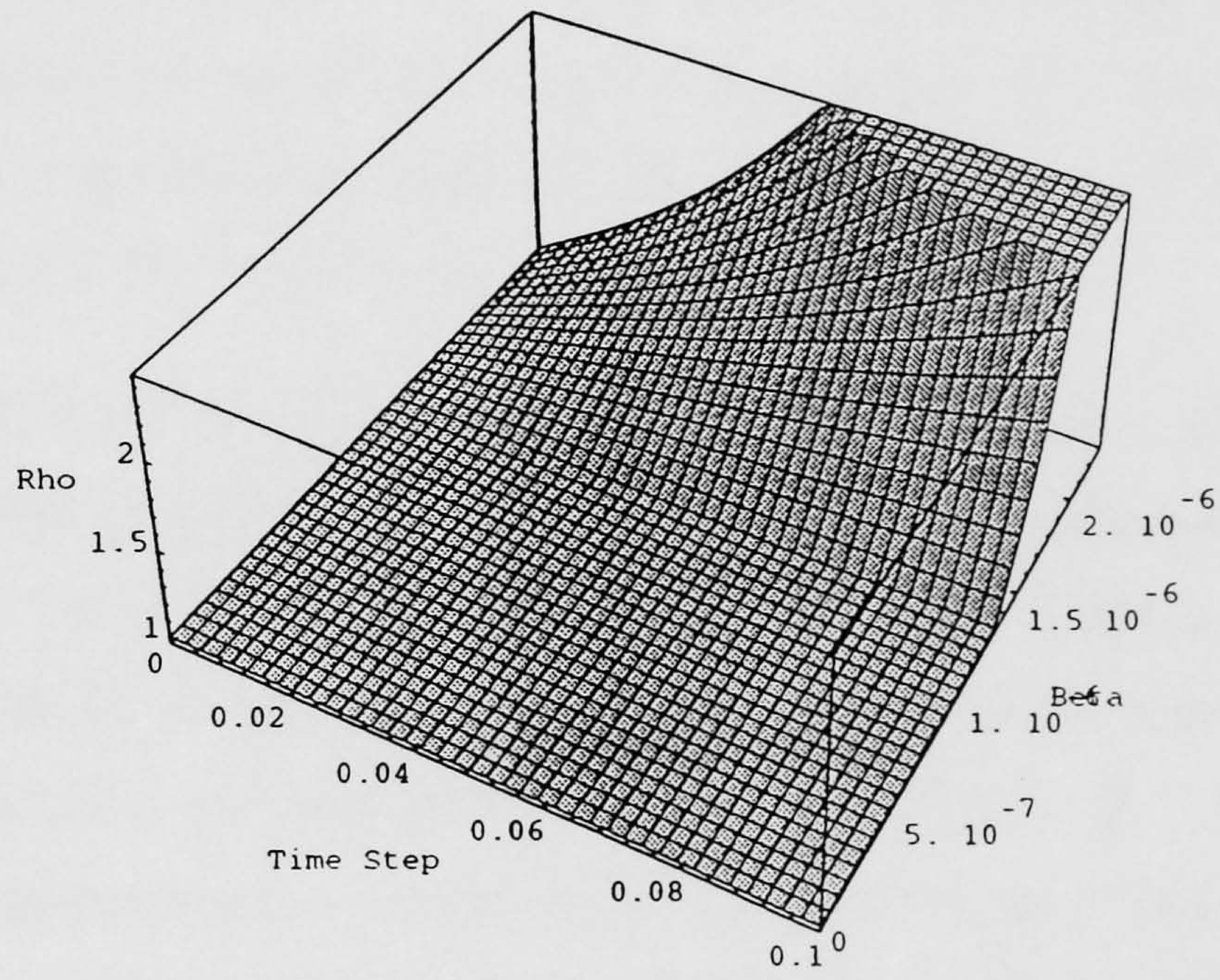

Figure 3.5: *Spectral radius of the Jacobian of Method 2 at the trivial fixed point.*

Now, evaluating the derivatives in (3.6.56) at the non-trivial fixed point, $\overline{\mathbf{u}}_2$, and substituting, using the set of parameters in (3.4.4), in the Jacobian (3.6.55) gives the following results:

The spectral radius of the Jacobian equals unity at $\ell = 0$ and for all $\beta \in [0, 1]$.

For $\beta = 1 \times 10^{-6} (< \beta^*)$, some of the eigenvalues are real and the others are complex. When $0 < \ell \leq 0.000012$, one eigenvalue equals unity and two eigenvalues are less than unity. One eigenvalue is greater than unity and two eigenvalues are less than unity provided $0.000012 < \ell < 0.01685794$. When $0.01685794 \leq \ell < 4.6730198$, one eigenvalue is negative and less than unity in magnitude and two eigenvalues are positive; one is less than unity and one is greater than unity, while, for $4.6730198 \leq \ell < 4.830043$, two eigenvalues are negative and one eigenvalue is positive and greater than unity.

For $4.830043 \leq \ell < 4.83345903$, one eigenvalue is negative and two eigenvalues are positive and greater than unity. For $4.83345903 \leq \ell < 4.8334823494$, the Jacobian has two complex eigenvalues with positive real part and one real and positive eigenvalue: it has two complex eigenvalues with negative real part and one real and negative eigenvalue provided $4.8334823494 \leq \ell < 4.83350769$. All the eigenvalues are negative for $\ell \geq 4.83350769$ and one eigenvalue equals -1 for $\ell \geq 2 \times 10^7$. The spectral radius, for this value of β, is greater than unity for $1.2 \times 10^{-5} < \ell < 2 \times 10^7$ and is equal to unity for $0 < \ell \leq 1.2 \times 10^{-5}$ and $\ell \geq 2 \times 10^7$.

For $\beta = 1.5 \times 10^{-6} (\approx \beta^*)$, all the eigenvalues are real and positive: each equals unity provided $\ell = 0$. Two eigenvalues are equal to unity and one eigenvalue is less than unity when $0 < \ell \leq 1 \times 10^{-11}$. When $1 \times 10^{-11} < \ell < 2.5 \times 10^{-6}$, all the eigenvalues are positive; one eigenvalue is less than unity, one equals unity and the other is greater than unity. All the eigenvalues are greater than unity provided $2.5 \times 10^{-6} \leq \ell < 7$. When $7 \leq \ell < 17$, two eigenvalues are complex with positive real part and one eigenvalue is real and negative. The Jacobian has a pair of complex-conjugate eigenvalues with negative real part and one real and negative eigenvalue provided $\ell \geq 17$. The spectral radius, for this value of β, equals unity provided $0 \leq \ell < 1 \times 10^{-11}$ and is greater than unity for $\ell > 1 \times 10^{-11}$, as shown in Table 3.3 and Figure 3.6.

For $\beta = 5 \times 10^{-4} (> \beta^*)$, the eigenvalues are a pair of complex conjugates and one real. When $0 < \ell < 0.0166244$, two eigenvalues are complex with positive real part and there is one real and positive eigenvalue, while, for $0.0166244 \leq \ell < 0.145925$, one eigenvalue is negative and two eigenvalues are complex with positive real part. The Jacobian has a pair of complex eigenvalues with negative real part and one real and negative eigenvalue. For this value of β, the spectral radius is less than unity for all positive time steps, ℓ.

Overall, the previous discussions can be summarized as in Table 3.3.

Table 3.3: *Stability properties of fixed points of Method 2. Entries in different rows of the same type of stability correspond to alternative conditions.*

	trivial fixed point	non-trivial fixed point
neutral	$\ell = 0 \,\&\, \forall \beta$ $0 \leq \ell < 6.71 \times 10^{-6} \,\&\, \beta = \beta^*$ $\ell \geq 794 \,\&\, \beta > \beta^*$ $\ell \geq 2 \times 10^6 \,\&\, \beta = \beta^*$	$\ell = 0 \,\&\, \forall \beta$ $0 \leq \ell \leq 1 \times 10^{-11} \,\&\, \beta = \beta^*$ $0 < \ell \leq 1.2 \times 10^{-5} \,\&\, \beta < \beta^*$ $\ell \geq 2 \times 10^7 \,\&\, \beta < \beta^*$
attracting (stable)	$\ell > 0 \,\&\, \beta < \beta^*$	$\ell > 0 \,\&\, \beta > \beta^*$
repelling (unstable)	$6.71 \times 10^{-6} \leq \ell < 2 \times 10^6 \,\&\, \beta = \beta^*$ $0 < \ell < 794 \,\&\, \beta > \beta^*$	$\ell > 1 \times 10^{-11} \,\&\, \beta = \beta^*$ $1.2 \times 10^{-5} < \ell < 2 \times 10^7 \,\&\, \beta < \beta^*$

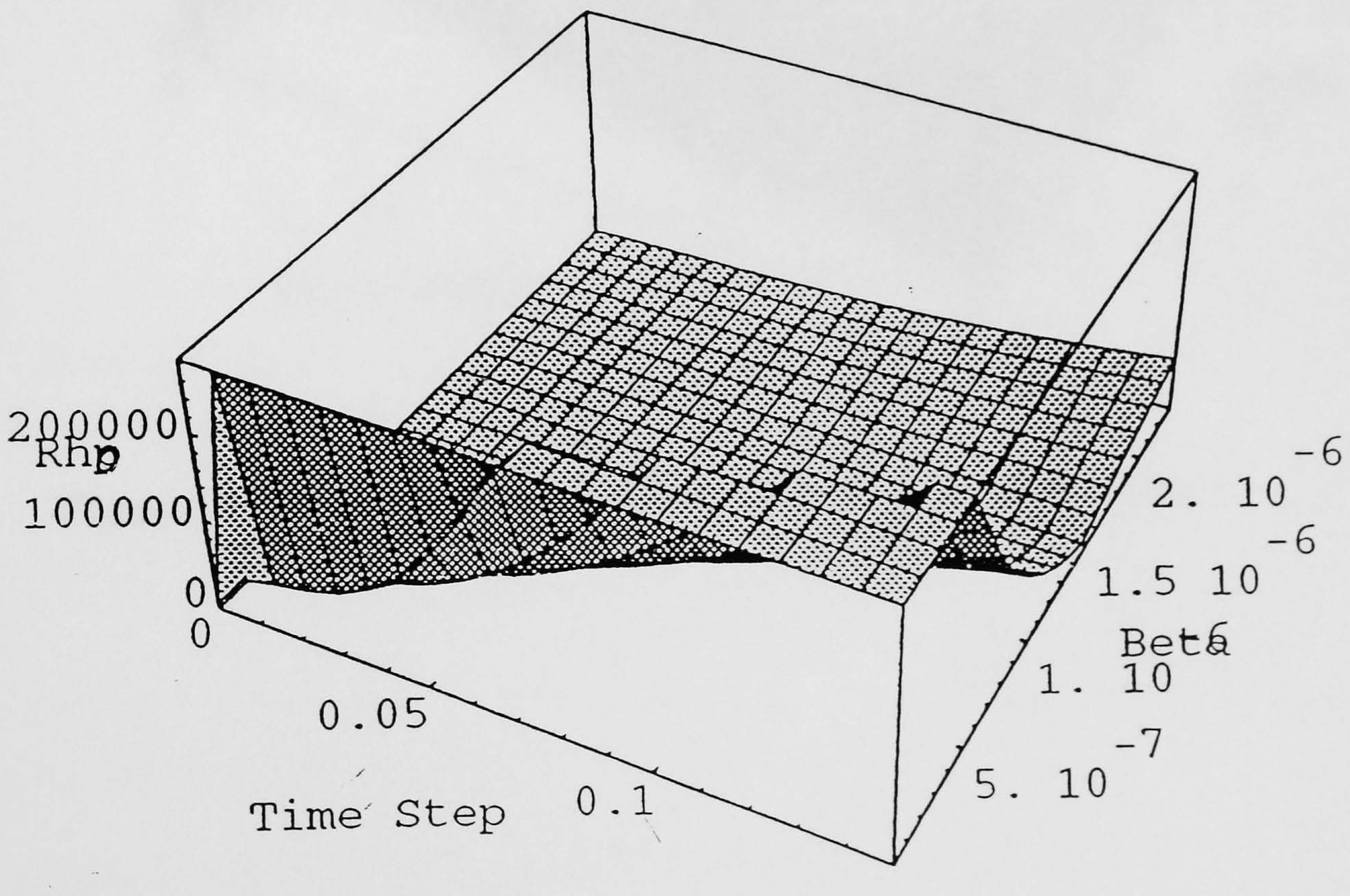

Figure 3.6: *Spectral radius of Method 2 at the non-trivial fixed point.*

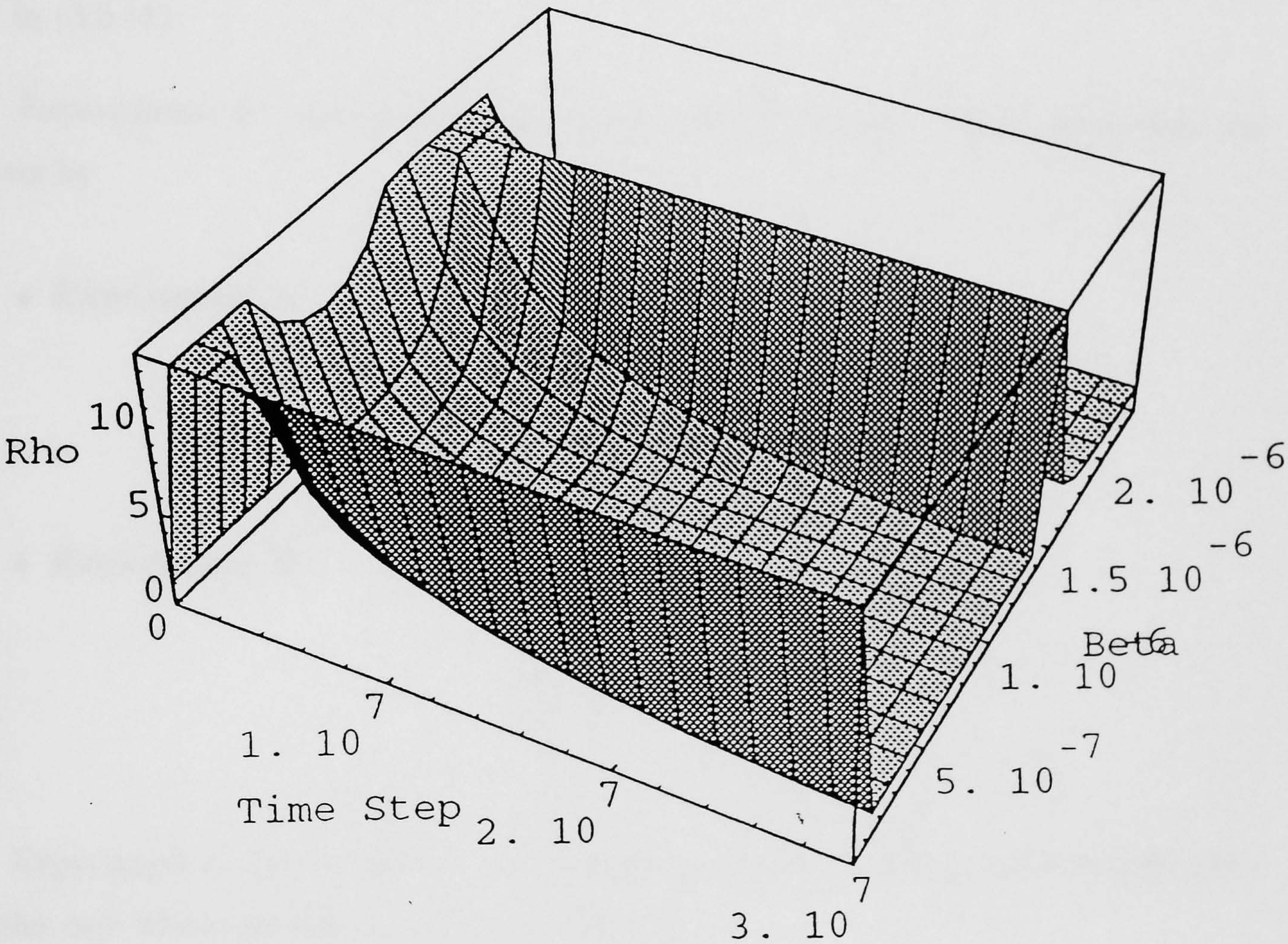

Figure 3.7: *Spectral radius of Method 2 at the non-trivial fixed point.*

3.7 Numerical Results

The numerical methods are tested, using the set of parameters in (3.4.4), to solve the SEIR model. The infection rate, β, was chosen to be $\beta = 1 \times 10^{-6}$ first and then $\beta = 5 \times 10^{-4}$, hence testing values of β smaller and larger than the bifurcation value β^* in (3.5.11).

Experiments are carried out with two sets of initial data. These initial data are given by

- **Experiment A**

$$\begin{aligned} S^0 &= 1.25 \times 10^7 &= 0.25\,N \\ E^0 &= 5 \times 10^4 &= 0.001\,N \\ I^0 &= 3 \times 10^4 &= 0.0006\,N \end{aligned}$$

- **Experiment B**

$$\begin{aligned} S^0 &= 1.25 \times 10^7 \\ E^0 &= 0 \\ I^0 &= 0 \end{aligned}$$

Experiment A corresponds to a mid-epidemic situation, while experiment B refers to the case where measles is not present at all.

Method 1

In experiment A, this method converges to the trivial fixed point for $\beta < \beta^*$ and to the non-trivial fixed point for $\beta > \beta^*$.

Taking $\beta = 1 \times 10^{-6}$ results in correct convergence for $\ell < 0.019$. For $\ell \in [0.017, 0.018]$, convergence takes place, but intermediate negative values are observed. For $\ell \in [0.019, 0.027]$, the method does not converge; it exhibits oscillations. It is seen in Figure 3.10 that up to time $t \approx 250$, the method seems to converge correctly. Note that, in Figure 3.11, the method settles down much more quickly than with $\ell = 0.019$ and the oscillations first have significantly larger amplitudes; the infectives range up to 5×10^7. The method produces overflow at $\ell = 0.028$. These findings are illustrated in Figures 3.8–3.11.

Now, with $\beta = 5\times10^{-4}$, the method converges to the non-trivial fixed point provided $\ell < 0.0015$ although intermediate oscillations are observed for susceptibles and exposed individuals when $\ell = 0.0014$ as shown in Figure 3.13. Overflow occurs for susceptibles, exposed and infectives when $\ell \geq 0.0015$.

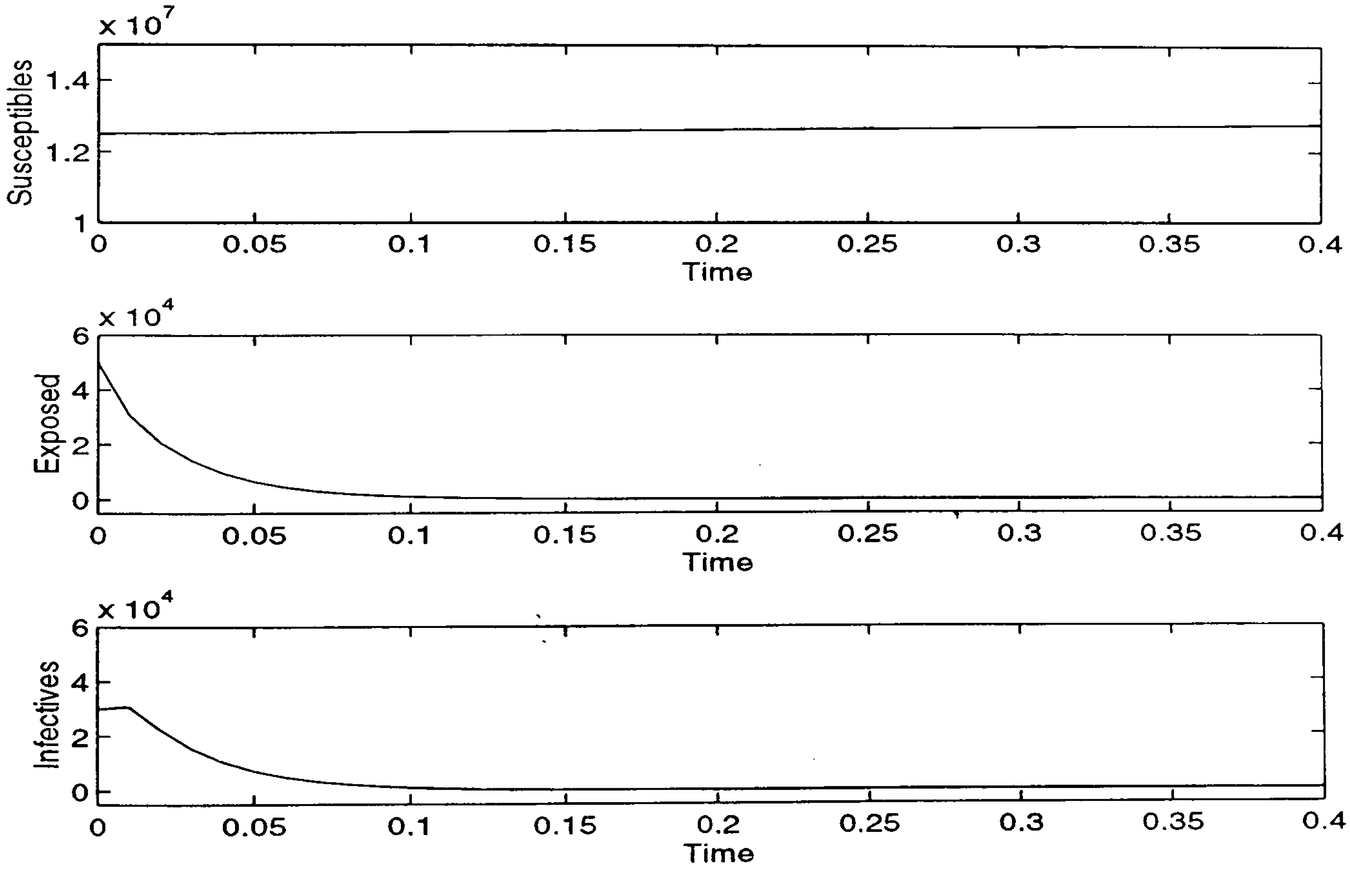

Figure 3.8: *Method 1, experiment A,* $\beta = 1 \times 10^{-6}, \ell = 0.01$. *Correct convergence to the trivial fixed point.*

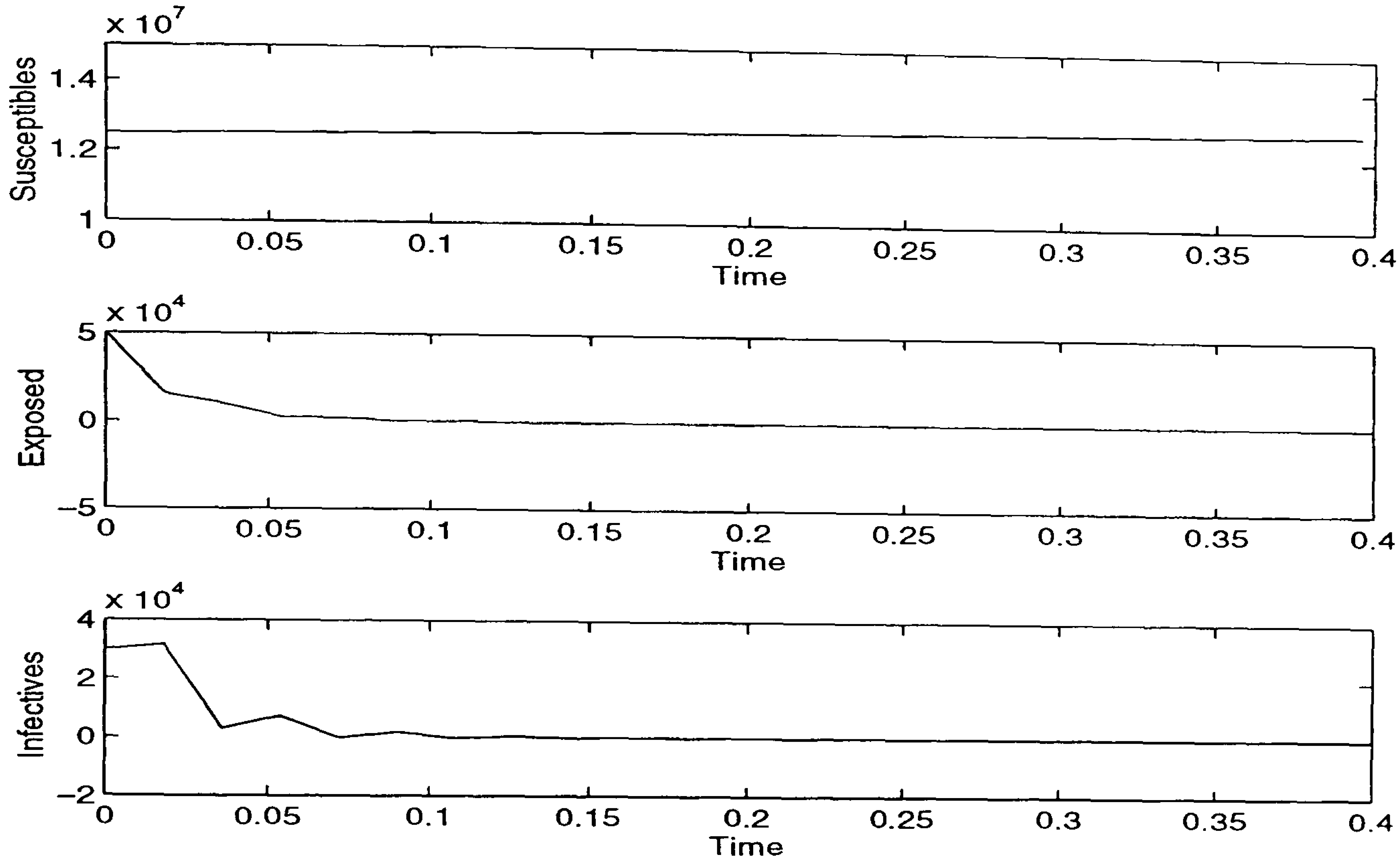

Figure 3.9: *Method 1, experiment A,* $\beta = 1 \times 10^{-6}, \ell = 0.018$. *Oscillatory convergence.*

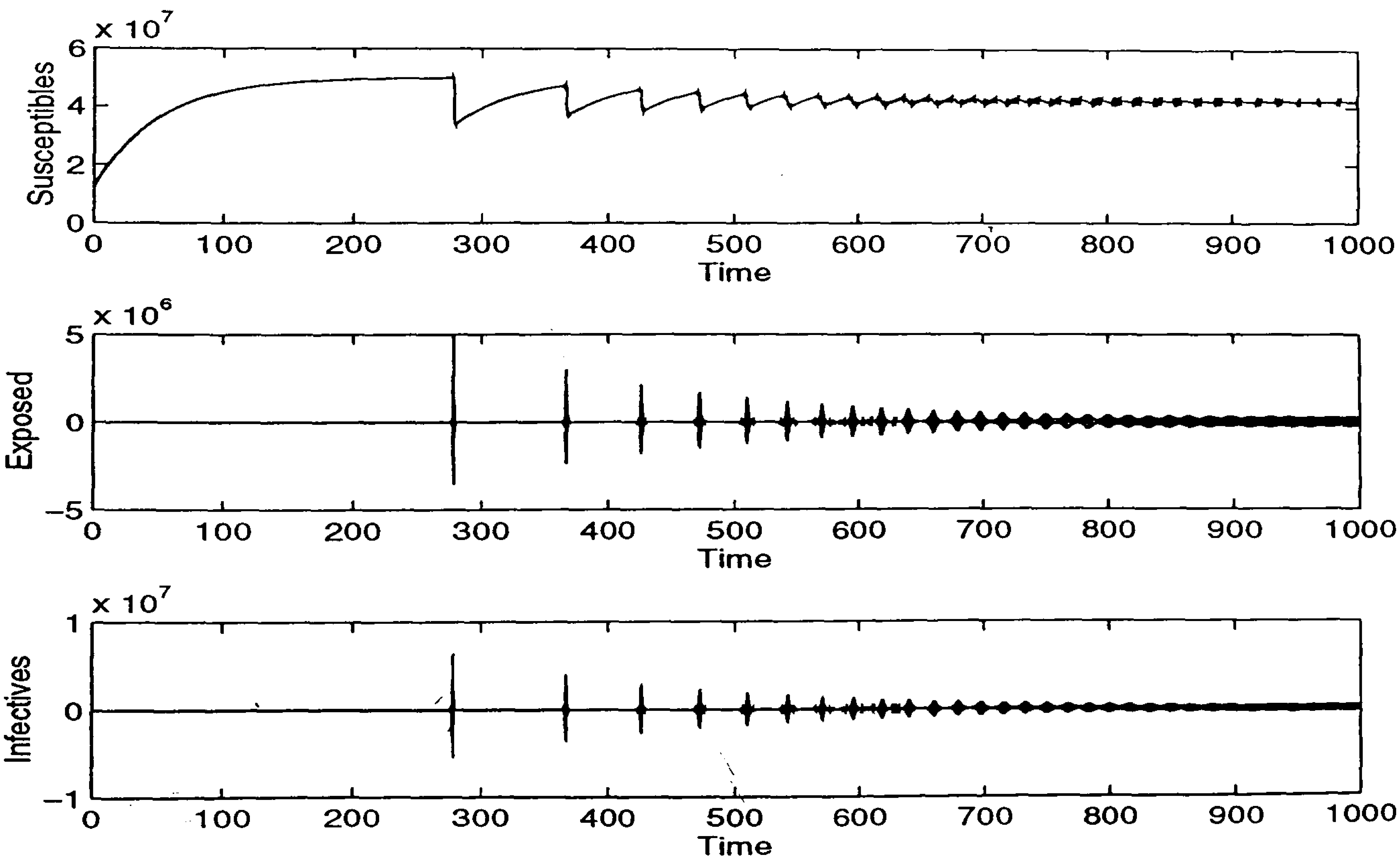

Figure 3.10: *Method 1, experiment A,* $\beta = 1 \times 10^{-6}, \ell = 0.019$. *The method swings into ocsillations.*

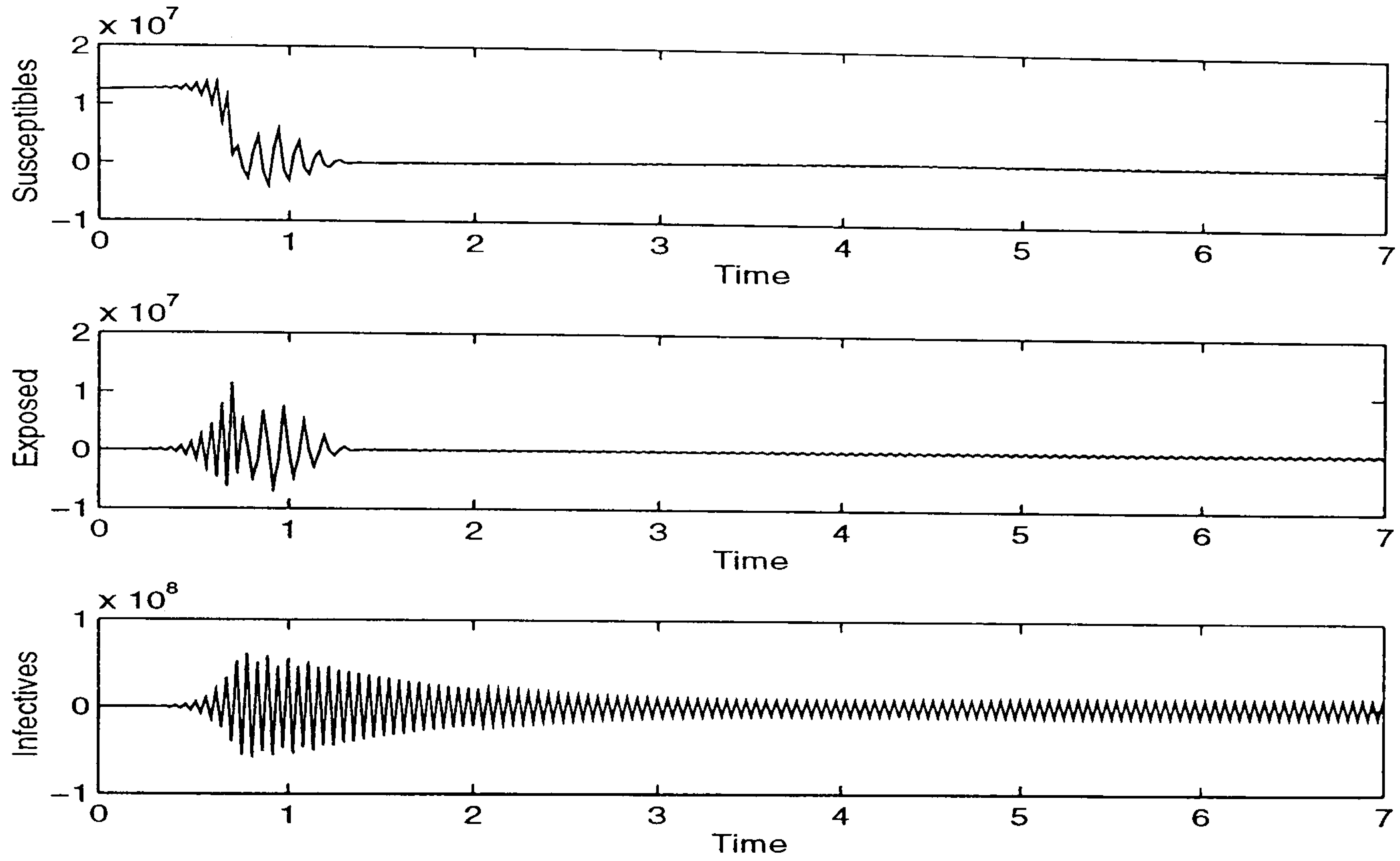

Figure 3.11: *Method 1, experiment A,* $\beta = 1 \times 10^{-6}, \ell = 0.027$.

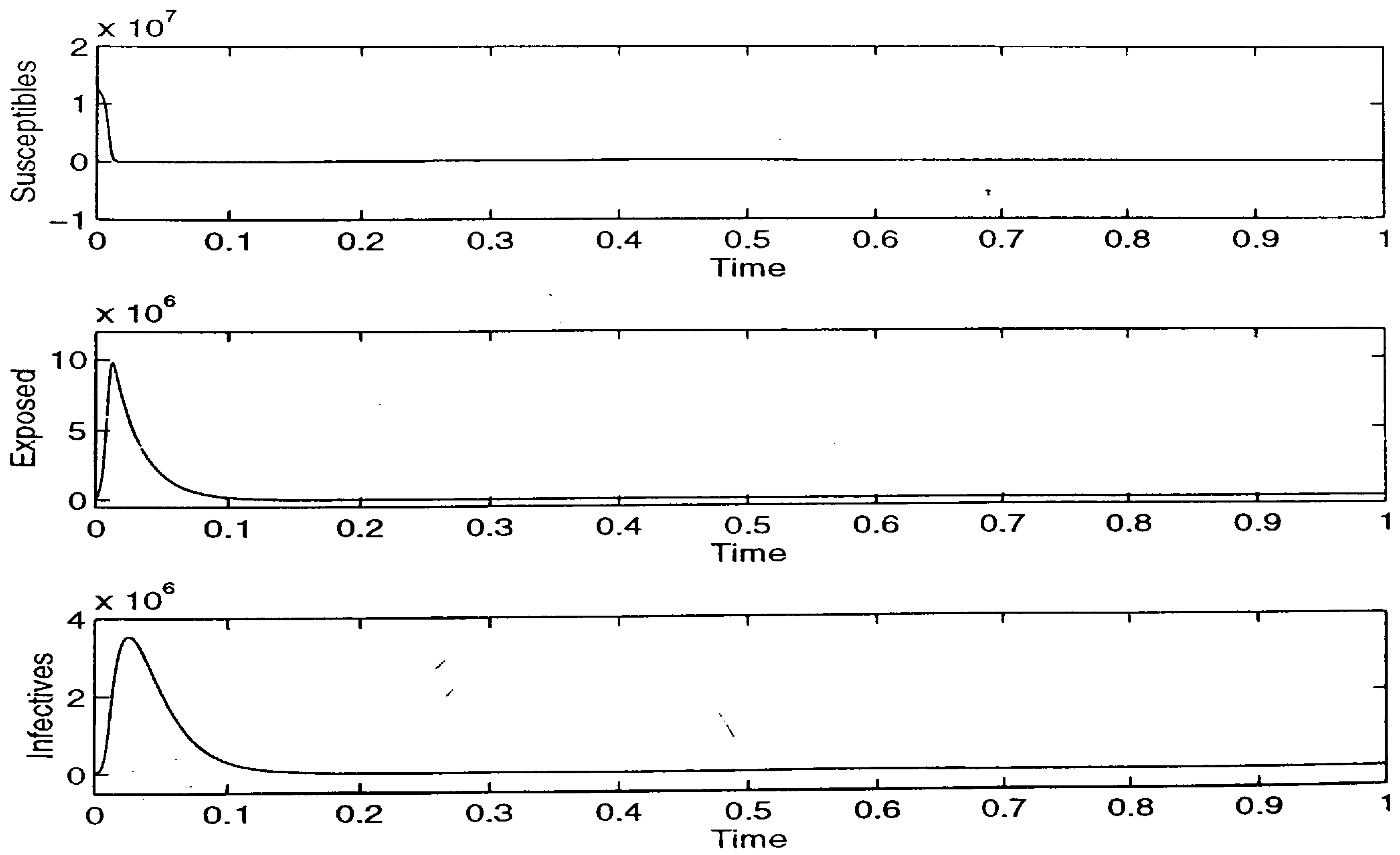

Figure 3.12: *Method 1, experiment A,* $\beta = 5 \times 10^{-4}, \ell = 0.0001$.

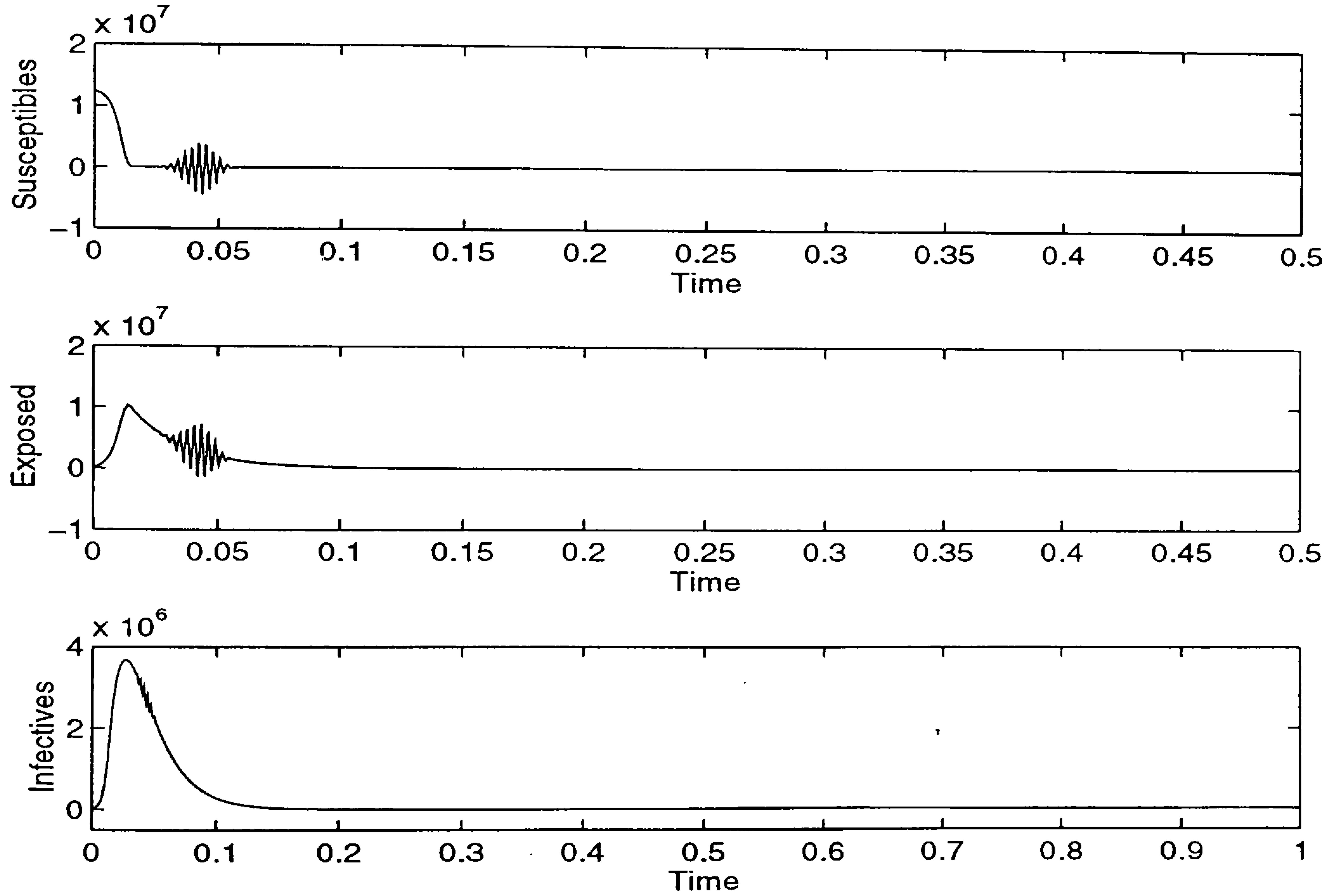

Figure 3.13: *Method 1, experiment A,* $\beta = 5 \times 10^{-4}, \ell = 0.0014$. *Intermediate oscillations are observed for susceptibles and exposed.*

In experiment B, the method approaches the trivial fixed point for both values of β. In fact, in this case, monotonic convergence takes place whenever $\ell \leq 50$. For $50 < \ell < 100$, convergence to the correct point takes place, but is oscillatory with susceptibles temporarily exceeding the size of the model, see Figures 3.14-3.16. When $\ell = 100$, the method does not converge but oscillates around the fixed point, thereby producing numbers of susceptibles exceeding the population size, as shown in Figure 3.17. Larger steplengths result in oscillations with increasing amplitude and eventually produce overflow, see Figure 3.18. They are connected to the fact that one eigenvalue of the Jacobian at the trivial fixed point is positive with magnitude smaller than unity for $0 < \ell < 50$, vanishes at $\ell = 50$, and equals -1 at $\ell = 100$, being of magnitude greater than unity for larger steplengths. Although the other two eigenvalues are of magnitude greater than unity, it is this eigenvalue that is responsible for the convergence to the fixed point. While the eigenvalue is positive, the method converges monotonically, then switches to oscillatory behaviour when the eigenvalue turns negative and finally diverges when the magnitude of the eigenvalue exceeds unity.

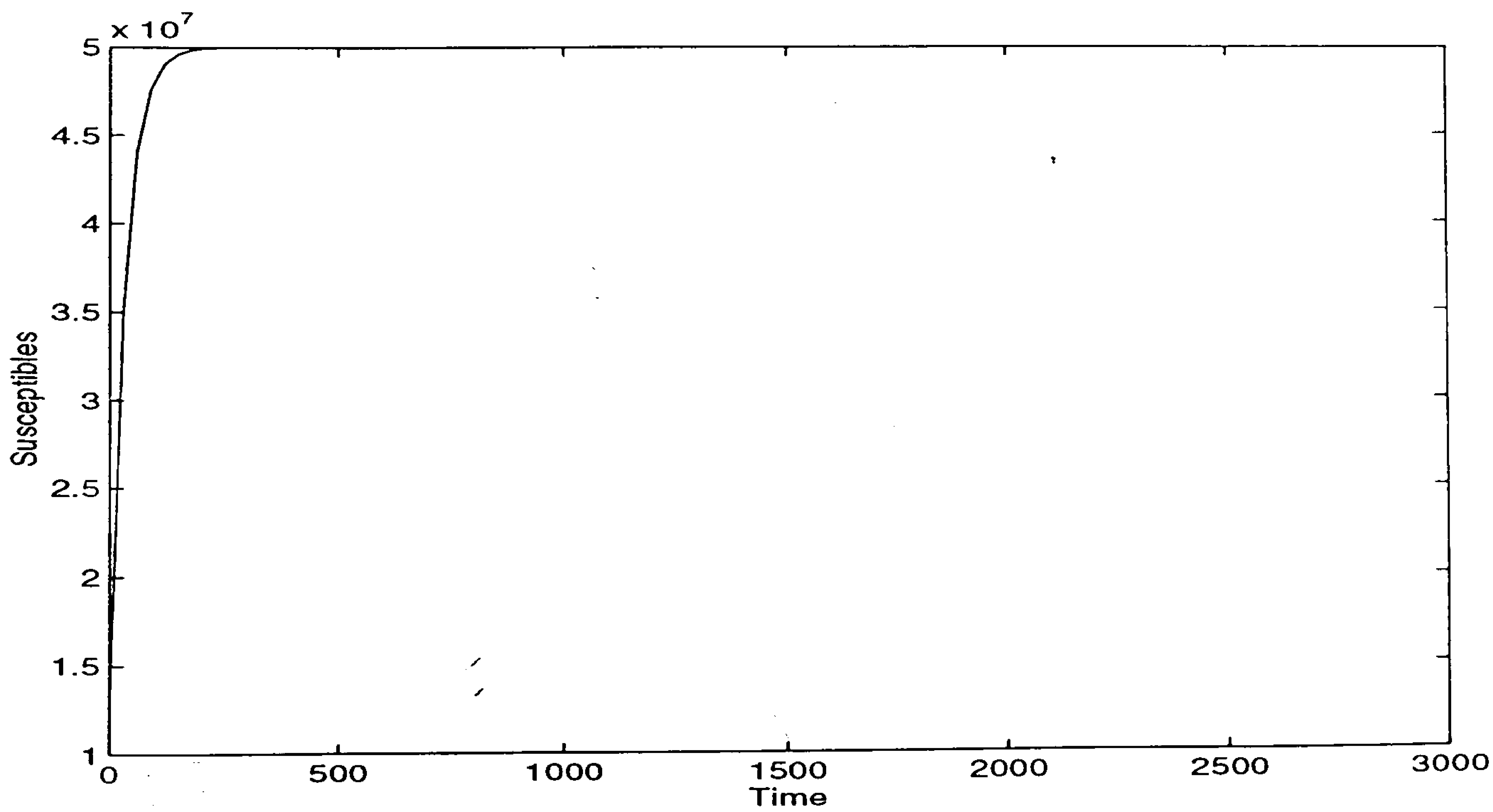

Figure 3.14: *Method 1, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 30$.

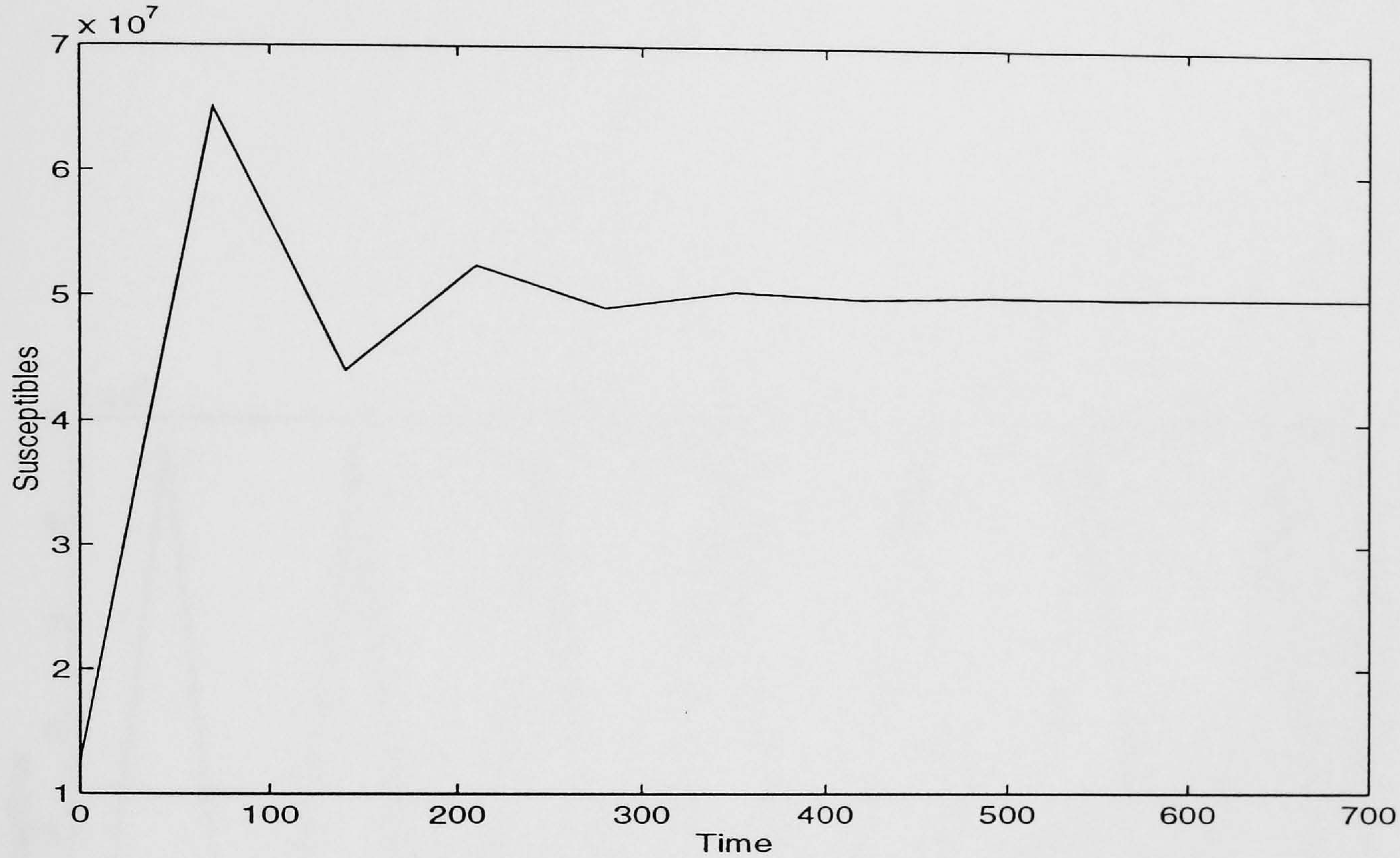

Figure 3.15: *Method 1, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 70$.

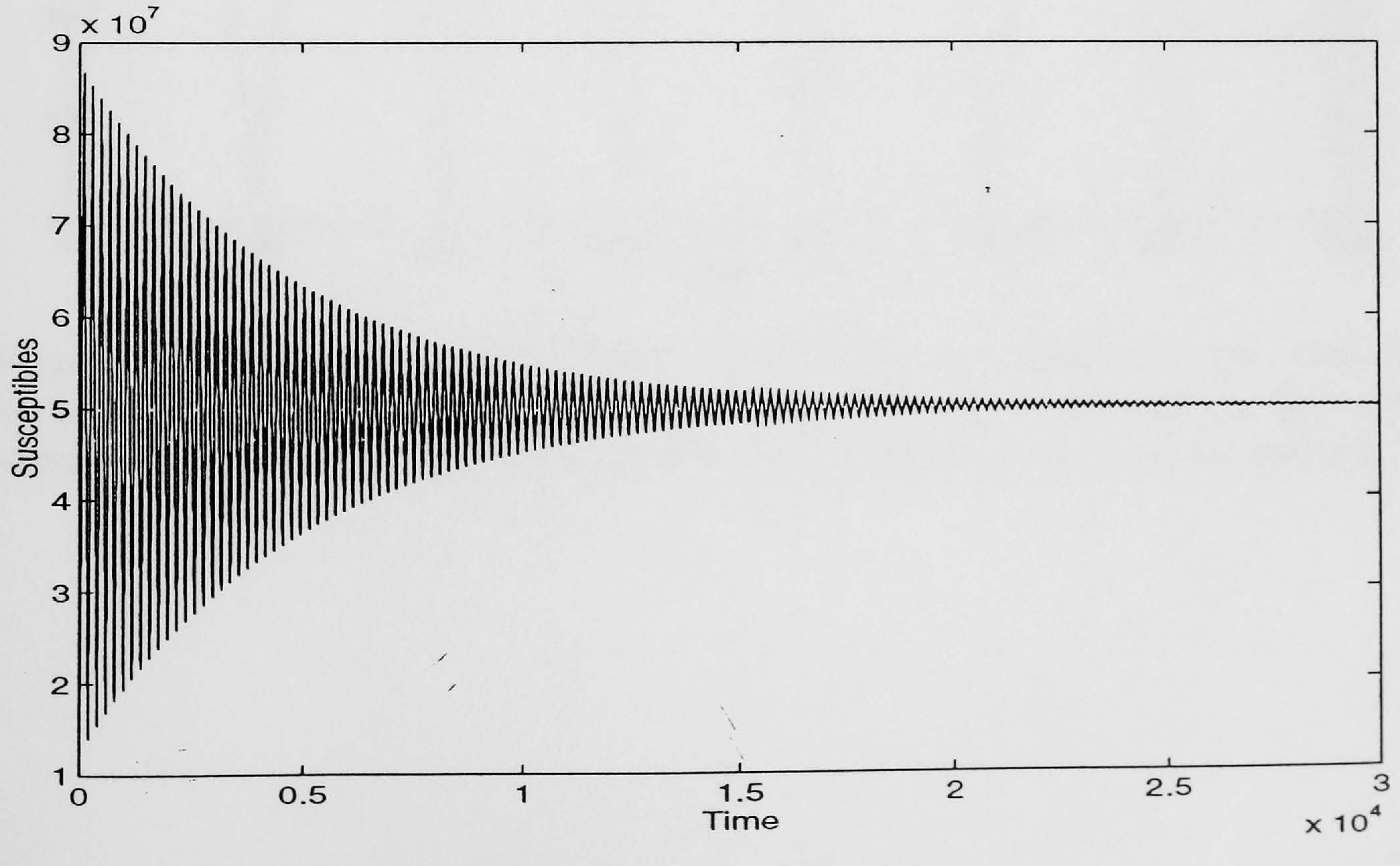

Figure 3.16: *Method 1, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 99$.

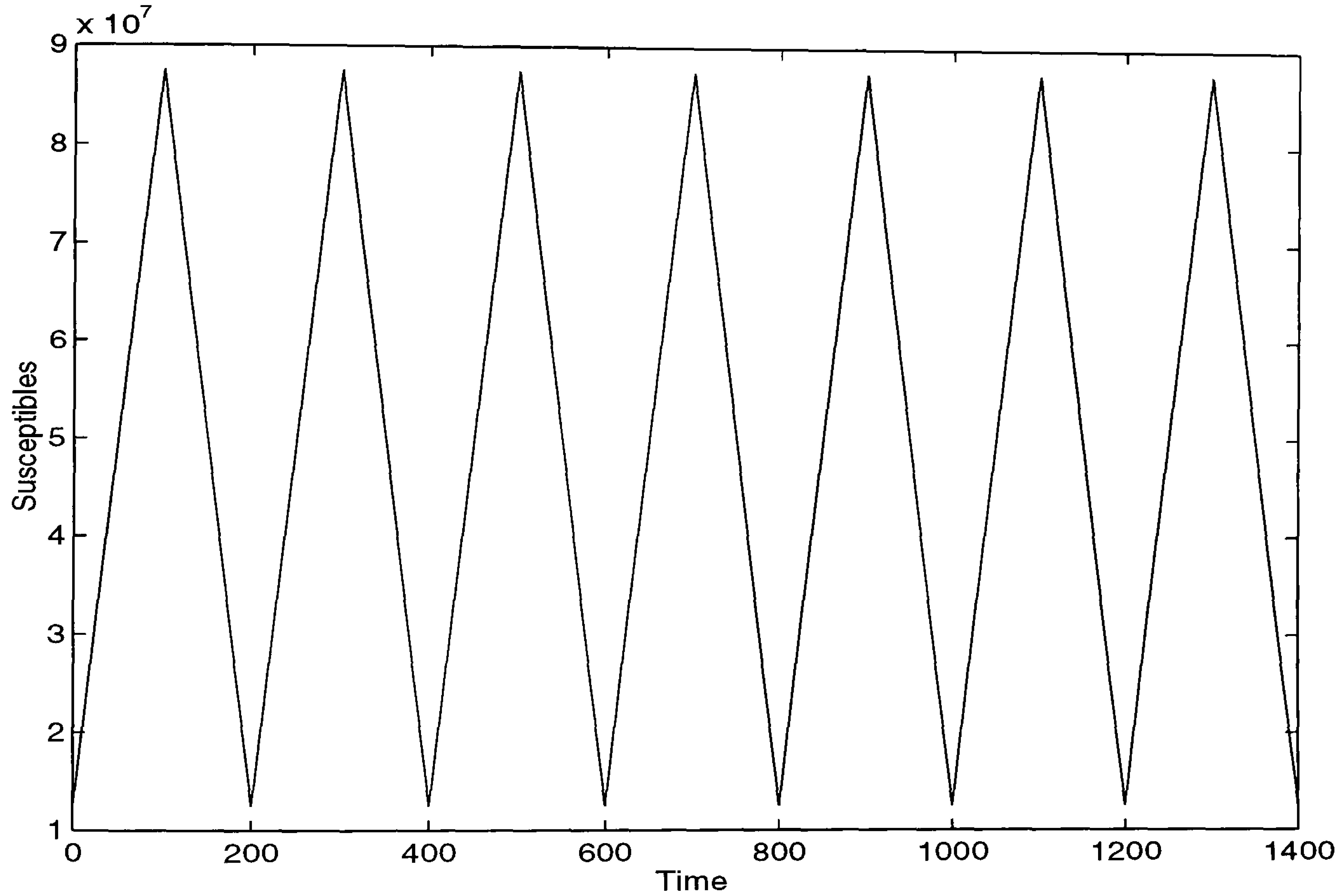

Figure 3.17: *Method 1, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 100$. *Note that the method oscillates between exactly two points. The higher value exceeds the correct number by* 3.75×10^7 *while the lower value underestimates it by the same amount.*

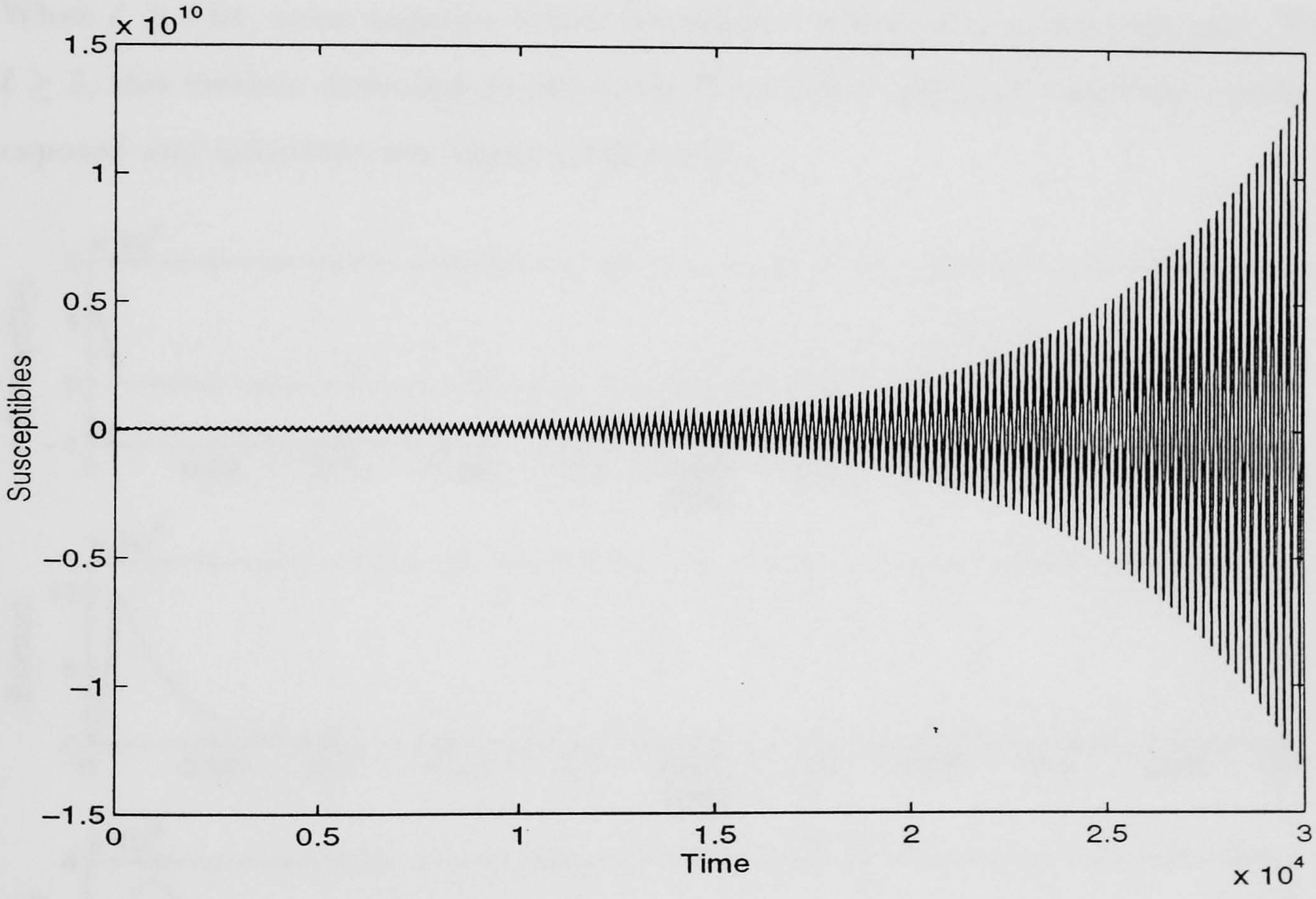

Figure 3.18: *Method 1, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 101$. *The method clearly diverges.*

Method 2

This method is seen to be very restrictive on stepsize, as for experiment A with $\beta = 5 \times 10^{-4}$ it produces overflow when $\ell \geq 0.01$. For smaller stepsizes, however, correct convergence to $(0,0,0)$ occurs, as shown in Figures 3.19–3.23.

When $\beta = 1 \times 10^{-6}$, this method converges to the trivial fixed point for $\ell < 0.04$. When $\ell \geq 0.04$, some negative values for exposed and infectives are observed. When $\ell \geq 3$, this method converges to the trivial fixed point, although oscillations occur for exposed and infectives, see Figures 3.24–3.27.

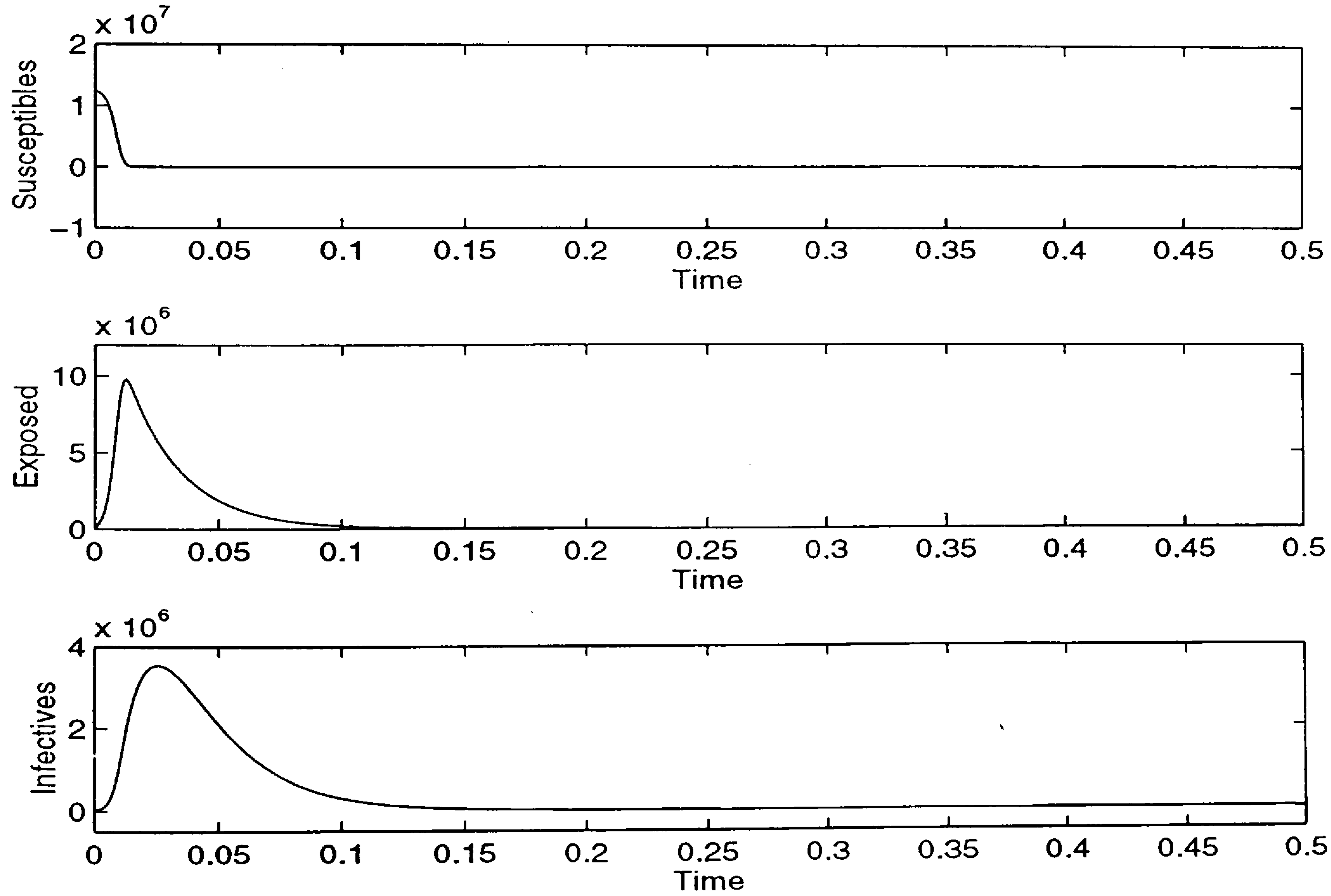

Figure 3.19: *Method 2, experiment A,* $\beta = 5 \times 10^{-4}, \ell = 0.0001$.

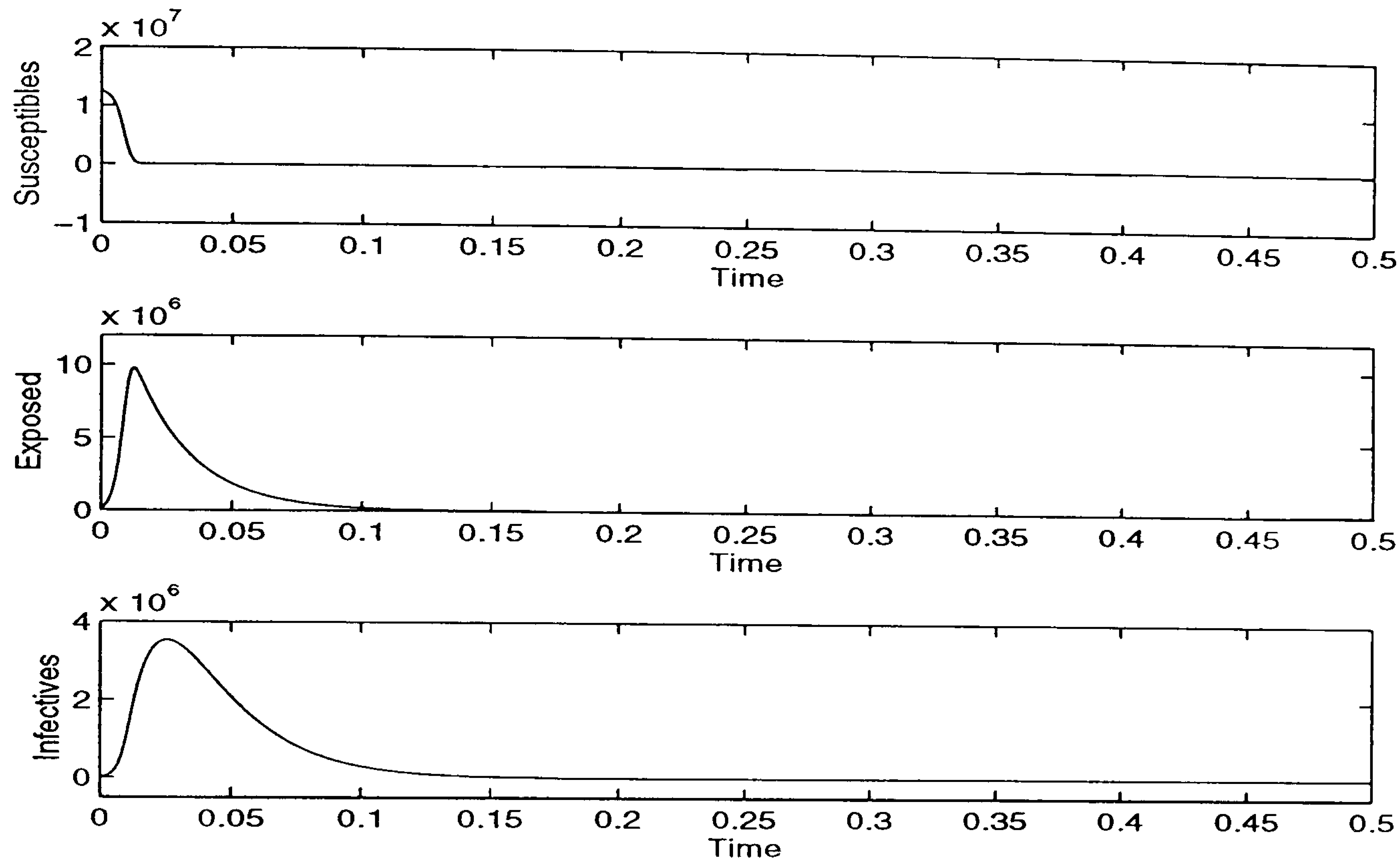

Figure 3.20: *Method 2, experiment A*, $\beta = 5 \times 10^{-4}, \ell = 0.0005$.

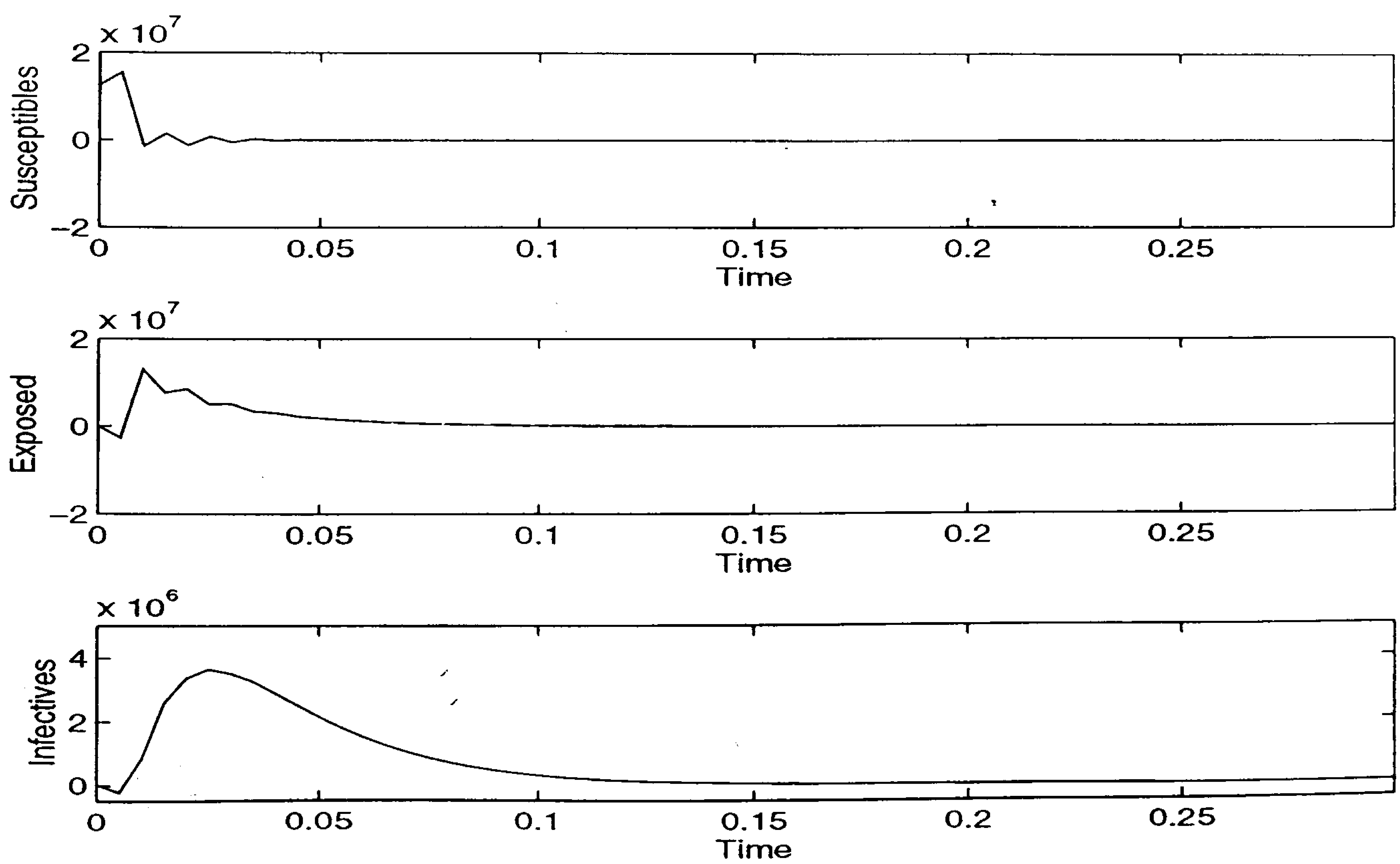

Figure 3.21: *Method 2, experiment A*, $\beta = 5 \times 10^{-4}, \ell = 0.005$.

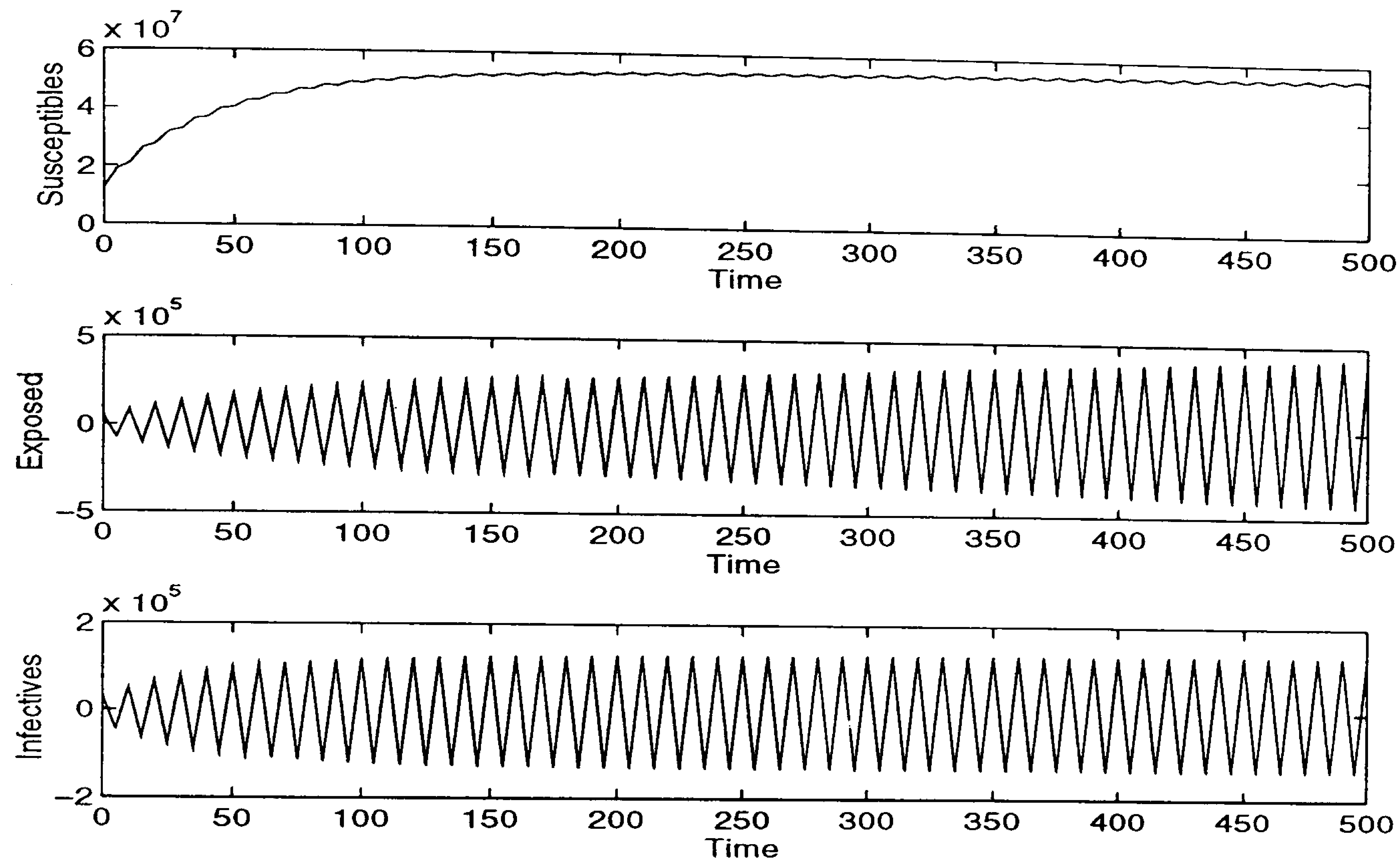

Figure 3.22: *Method 2, experiment A,* $\beta = 5 \times 10^{-4}, \ell = 5$. *Clearly this method diverges as the exposed and infectives diverge.*

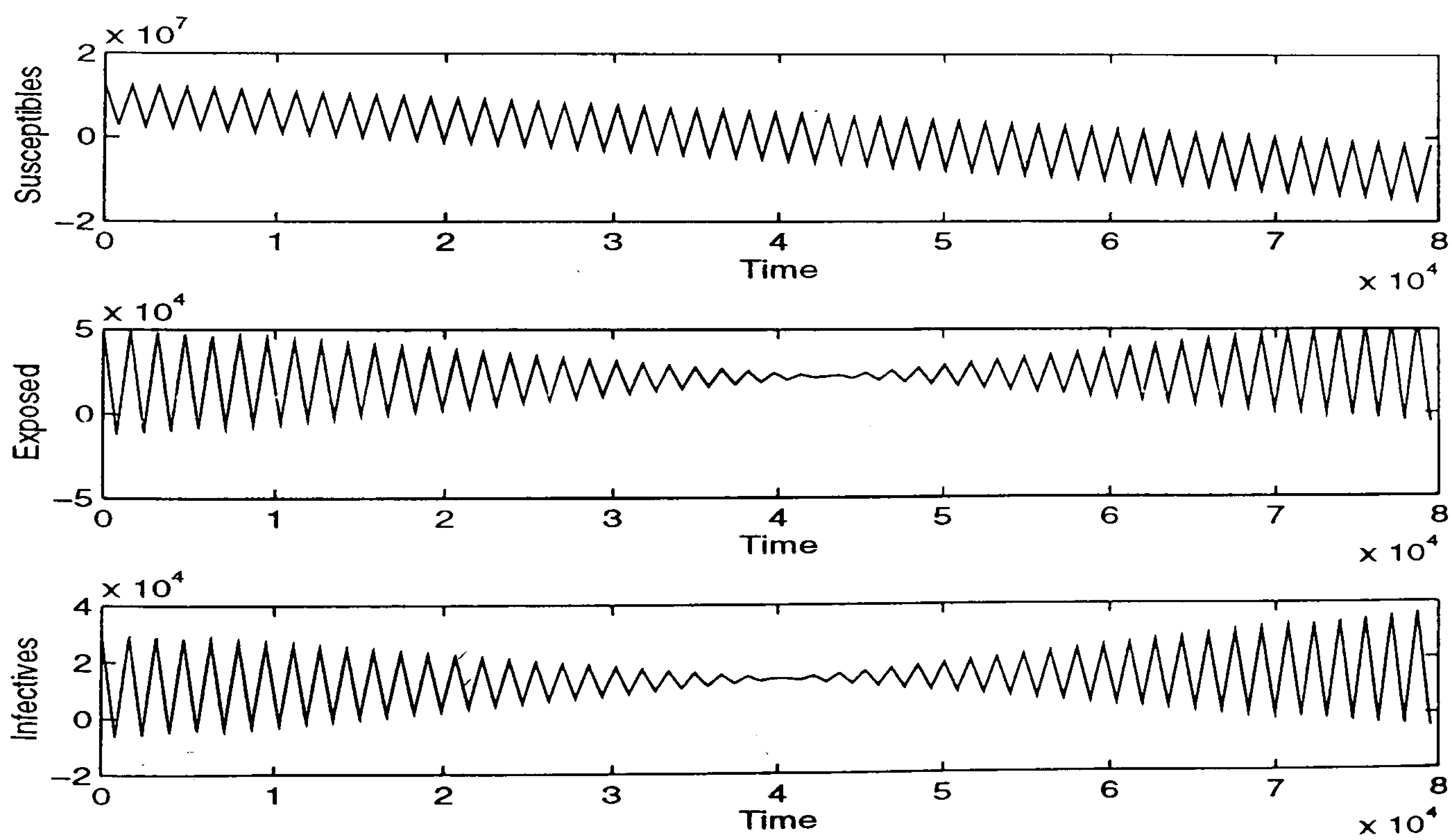

Figure 3.23: *Method 2, experiment A,* $\beta = 5 \times 10^{-4}, \ell = 795$.

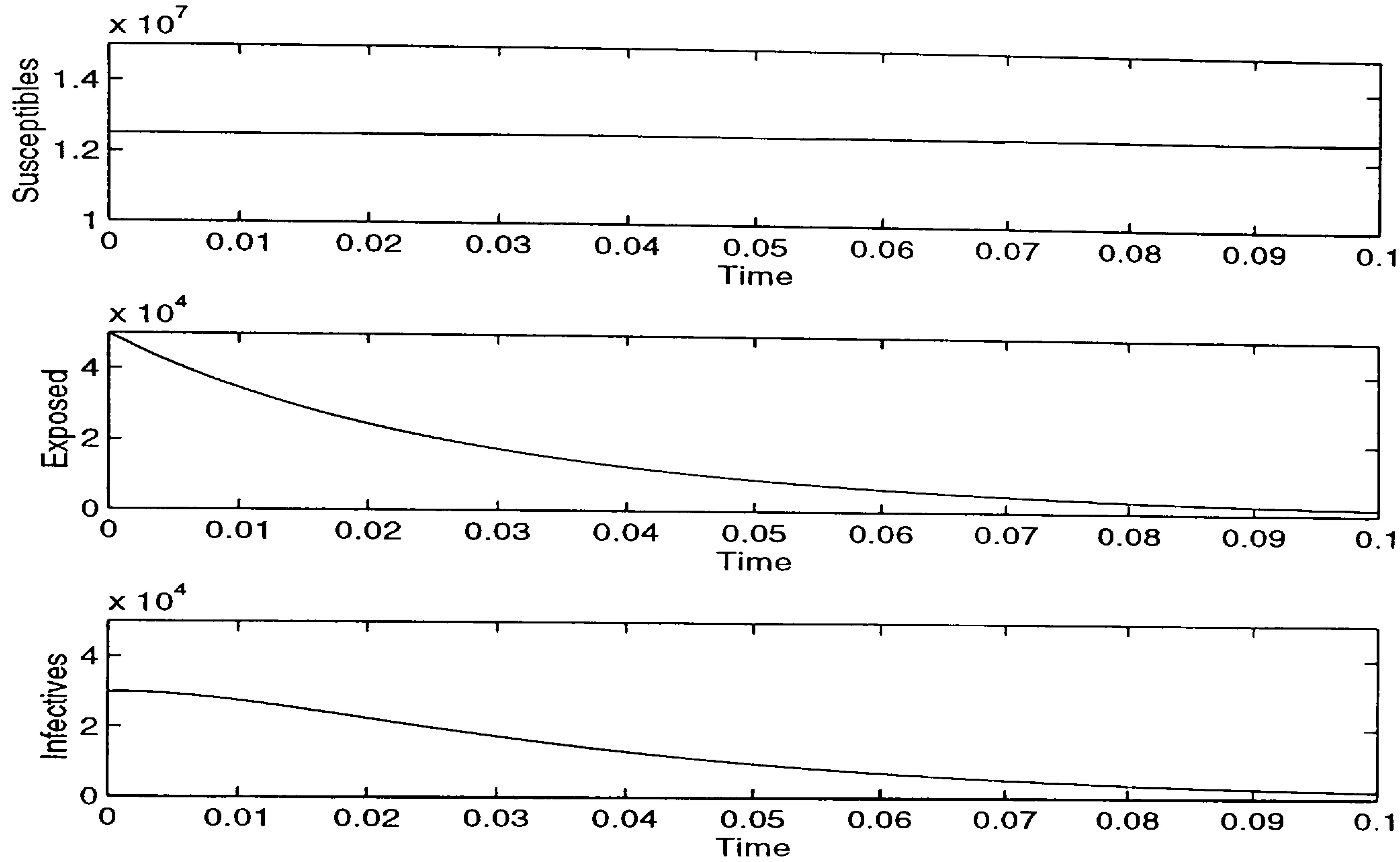

Figure 3.24: *Method 2, experiment A,* $\beta = 1 \times 10^{-6}, \ell = 0.0001$.

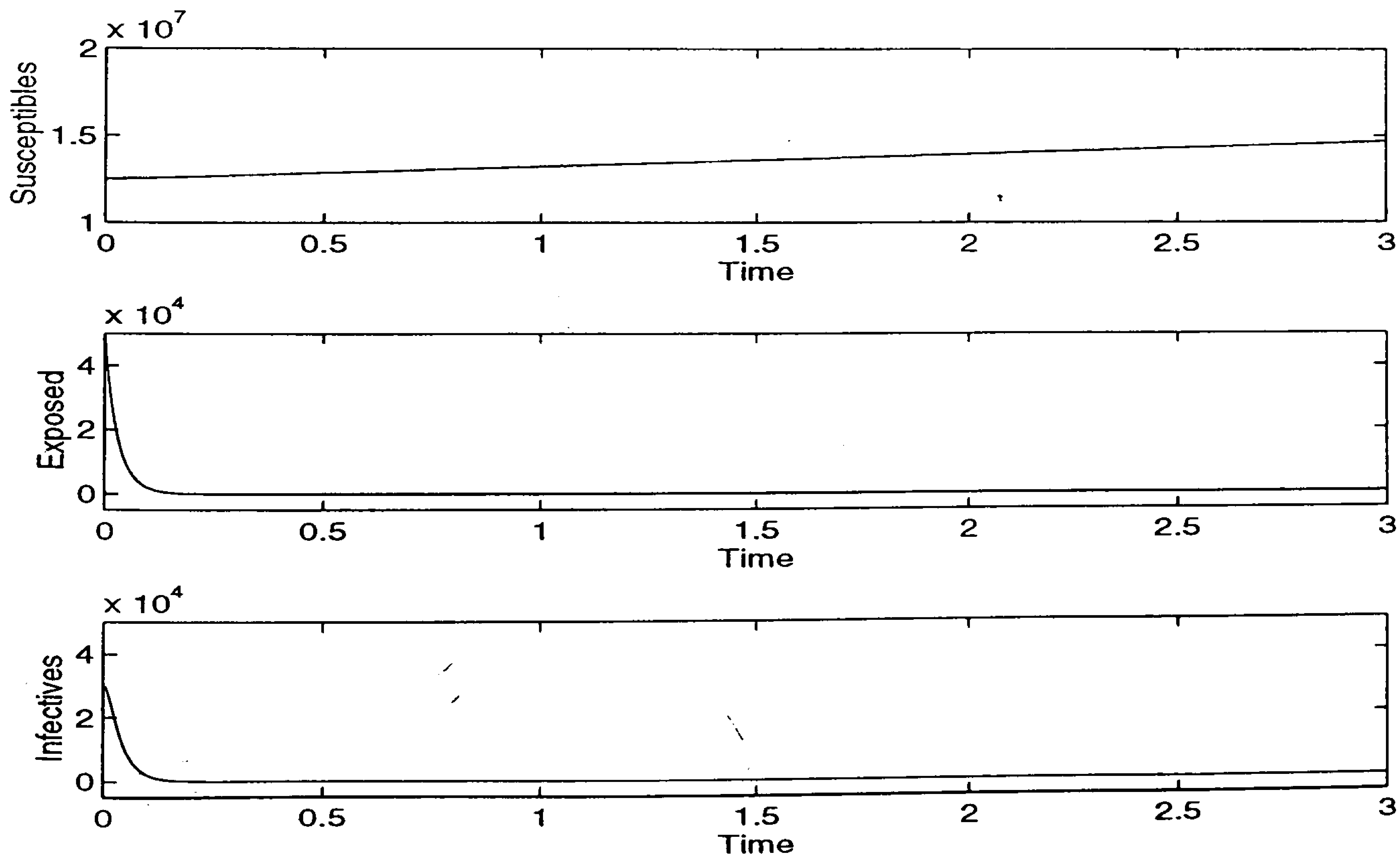

Figure 3.25: *Method 2, experiment A,* $\beta = 1 \times 10^{-6}, \ell = 0.005$.

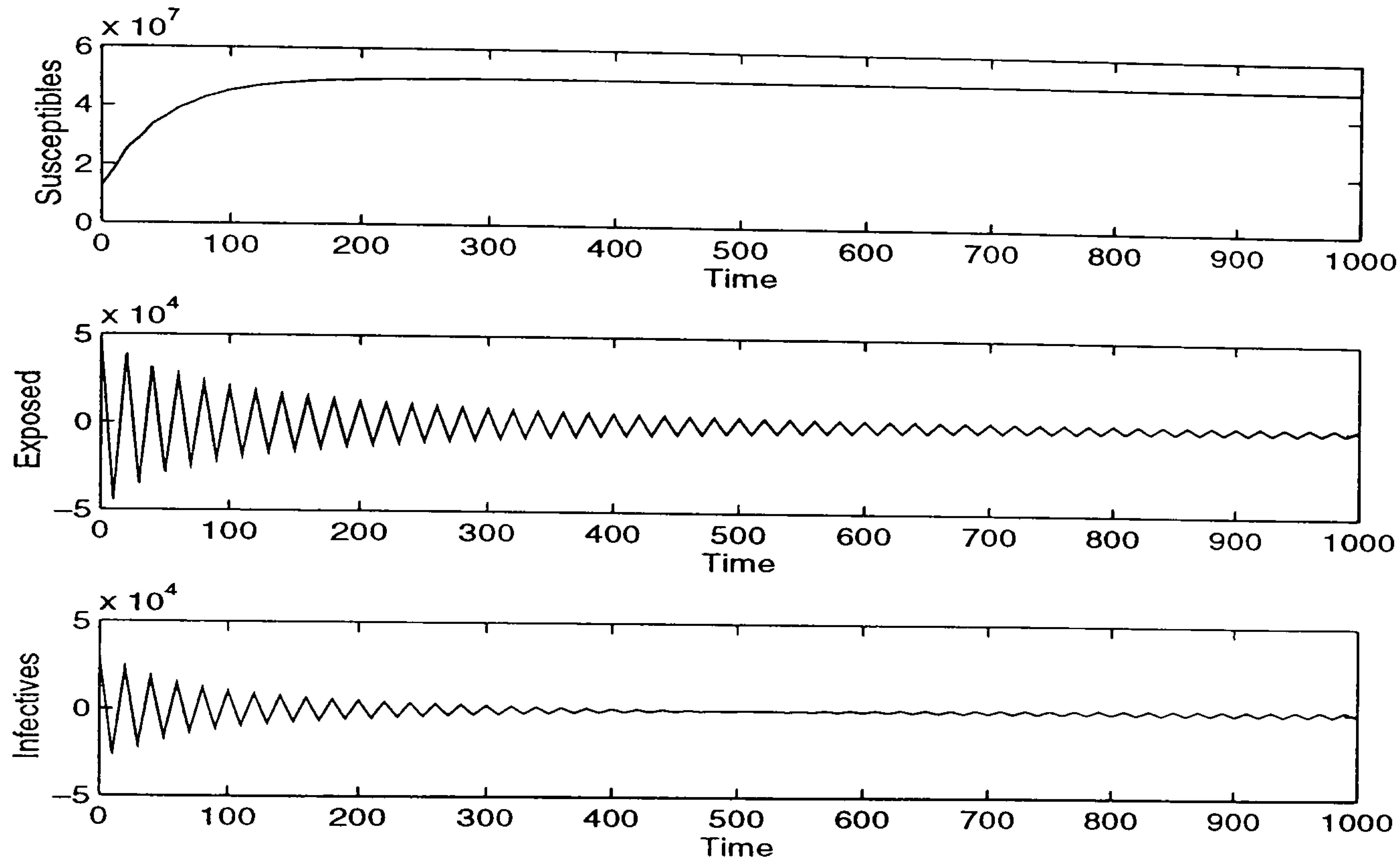

Figure 3.26: *Method 2, experiment A,* $\beta = 1 \times 10^{-6}, \ell = 10$.

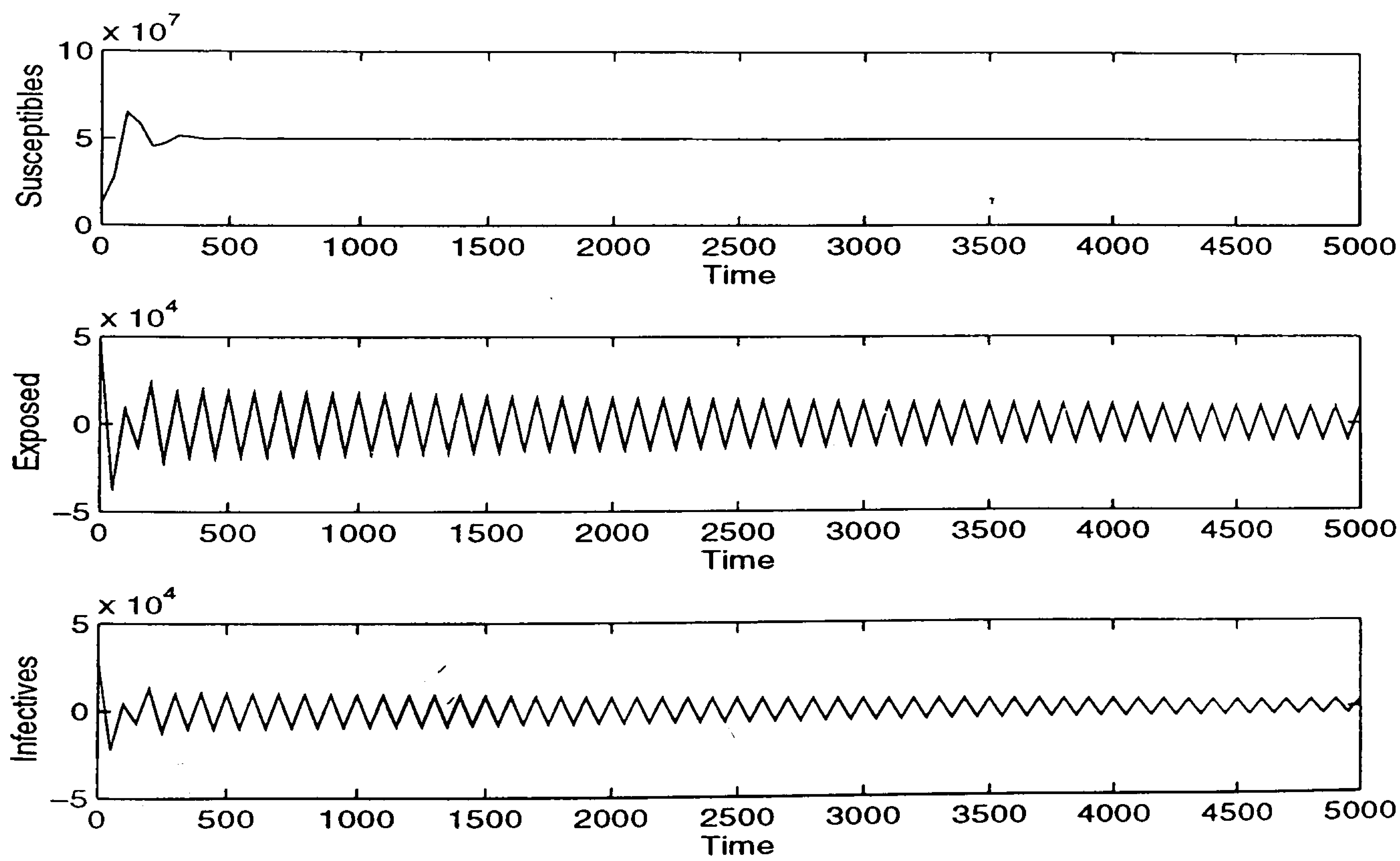

Figure 3.27: *Method 2, experiment A,* $\beta = 1 \times 10^{-6}, \ell = 50$.

In experiment B, the behaviour of this method is the same for both values of β. The method converges to the correct fixed point for $\ell < 1 \times 10^6$ with monotonic convergence provided $\ell < 100$. For $\ell = 100$, the method converges much faster to the trivial fixed point. For $100 < \ell < 1 \times 10^6$, convergence is oscillatory, thereby producing number of susceptibles exceeding the population size. When $\ell \geq 1 \times 10^6$, convergence does not take place; instead, this method produces periodic cycles around the fixed point. These findings are shown in Figures 3.28–3.34.

It is seen from the above results and discussions that the second method behaves much better than that of the well known Euler method, as the Euler method is first-order accurate, whereas the second method is second-order accurate. The maximum time step permitted with Method 2, especially in experiment B, is larger than that of the Euler method. Moreover, method 2 does not produce overflow, in experiment B, even with a very large steplength (see, again, Table 3.3).

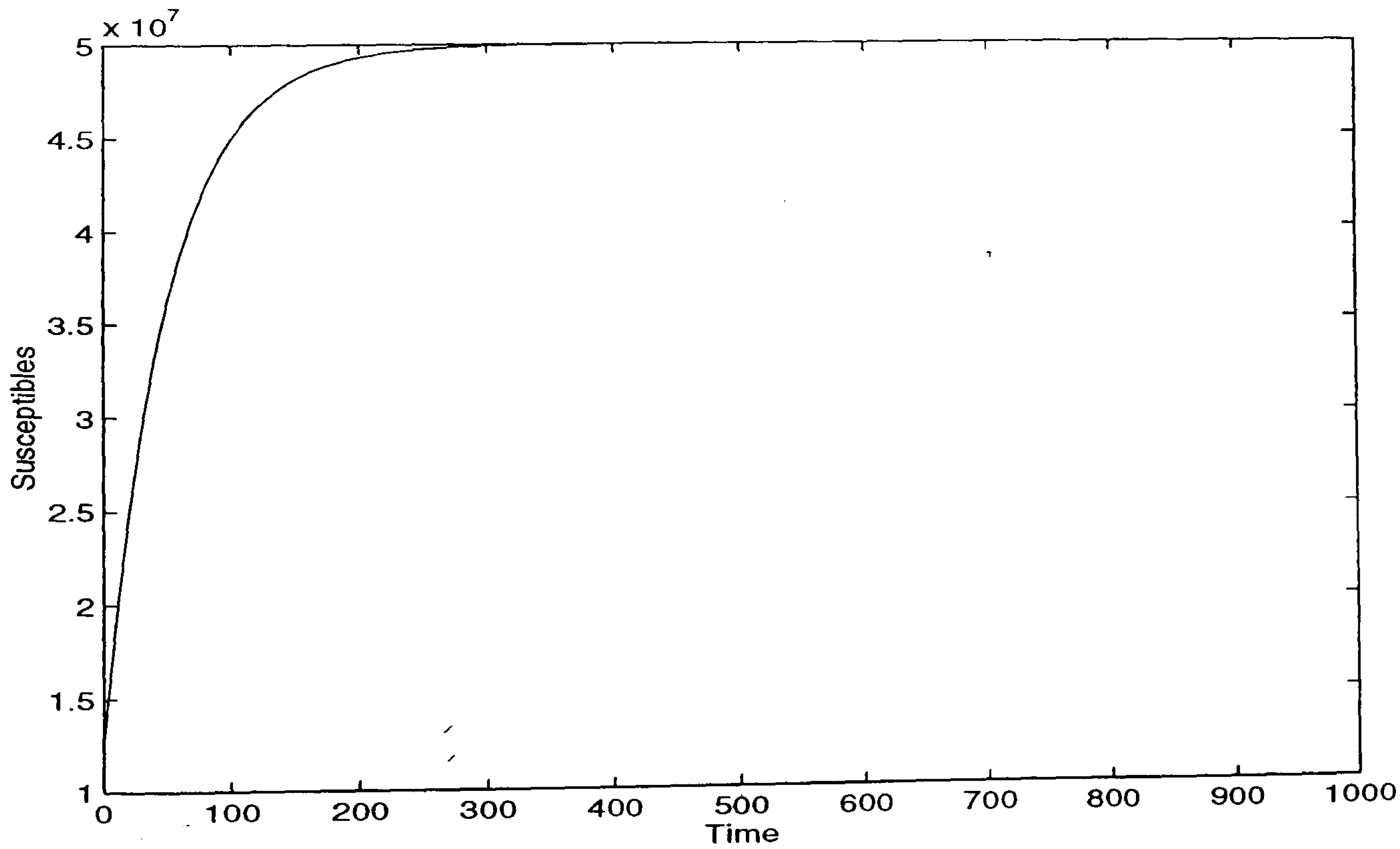

Figure 3.28: *Method 2, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 10$.

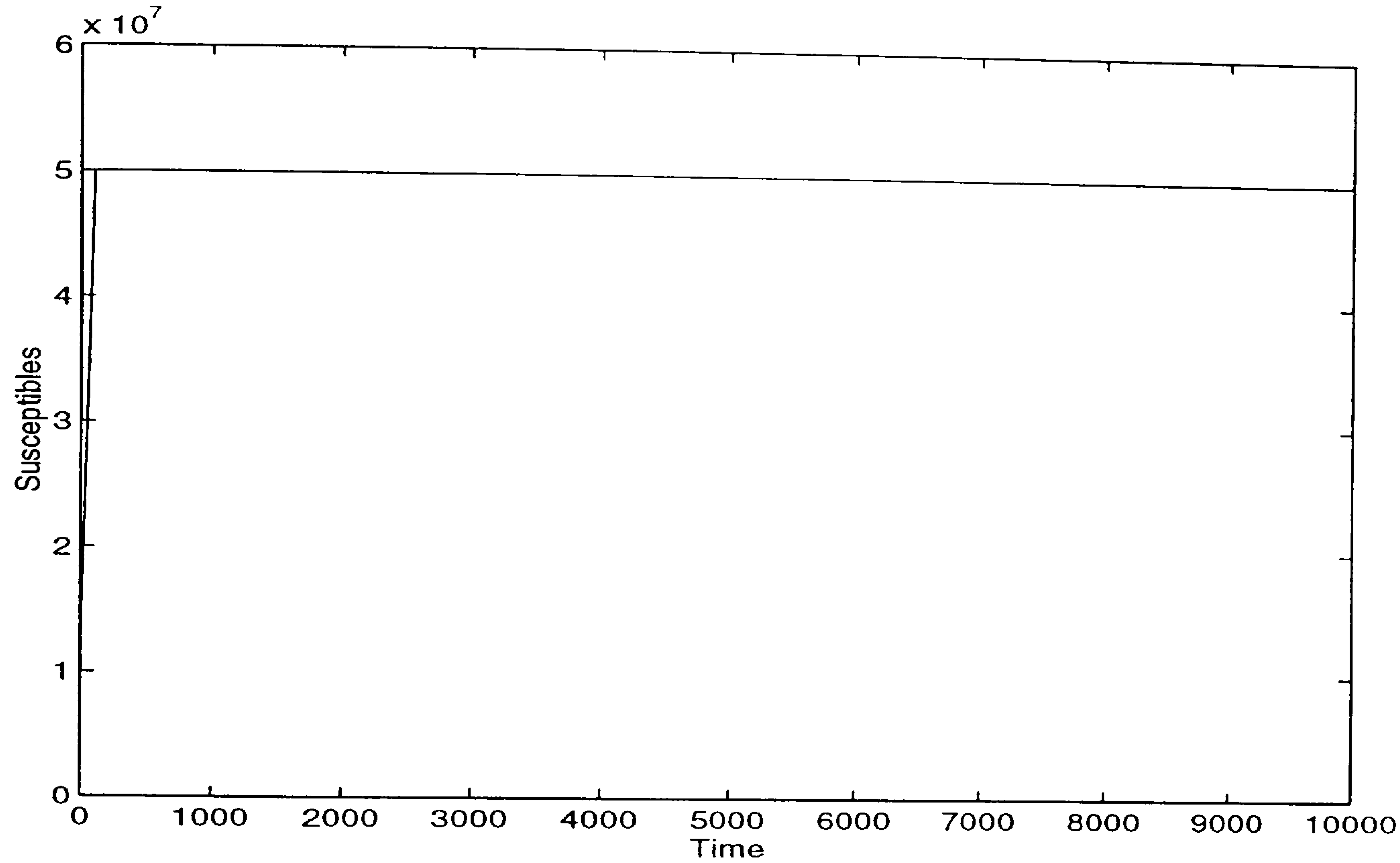

Figure 3.29: *Method 2, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 100.$

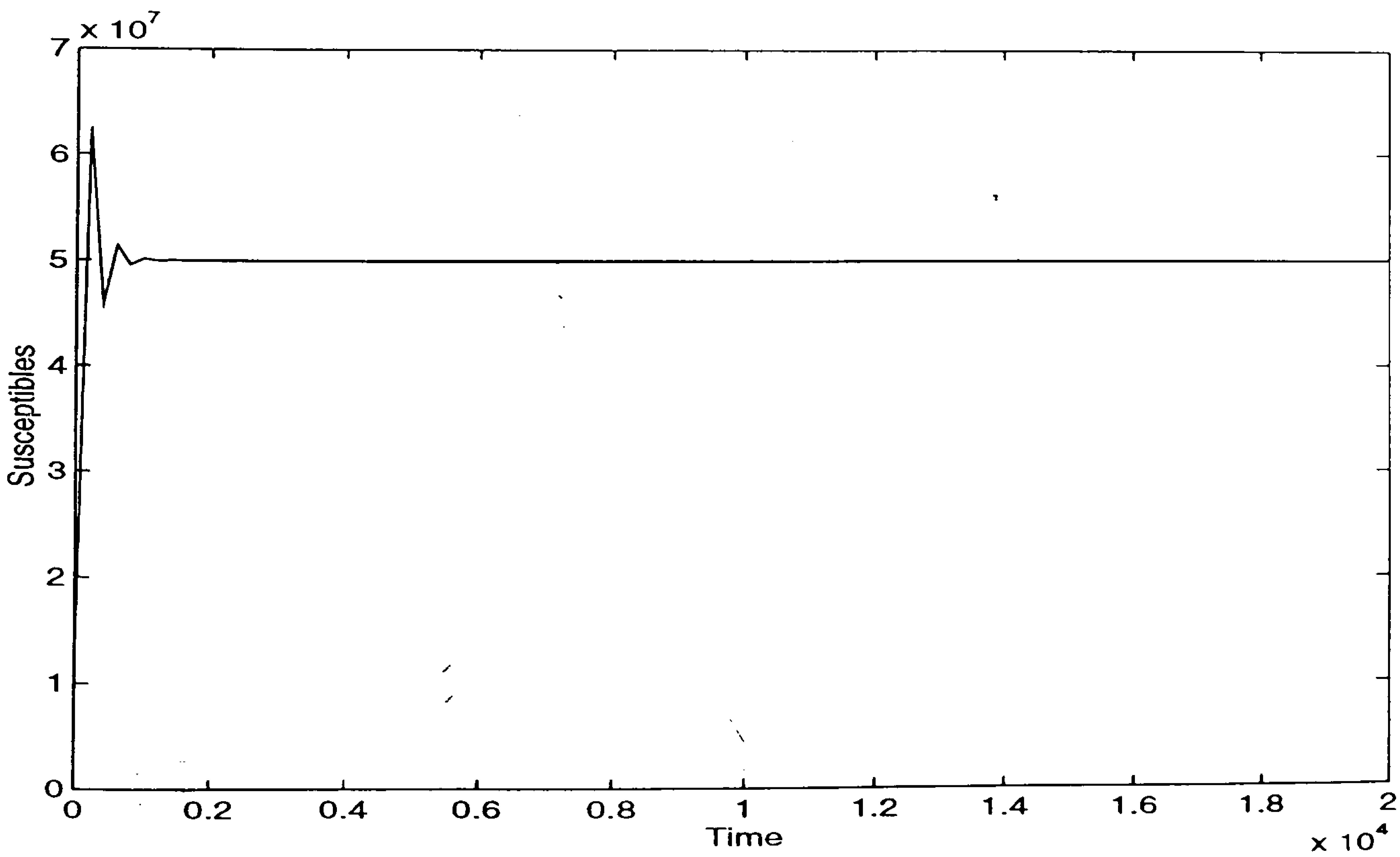

Figure 3.30: *Method 2, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 200.$

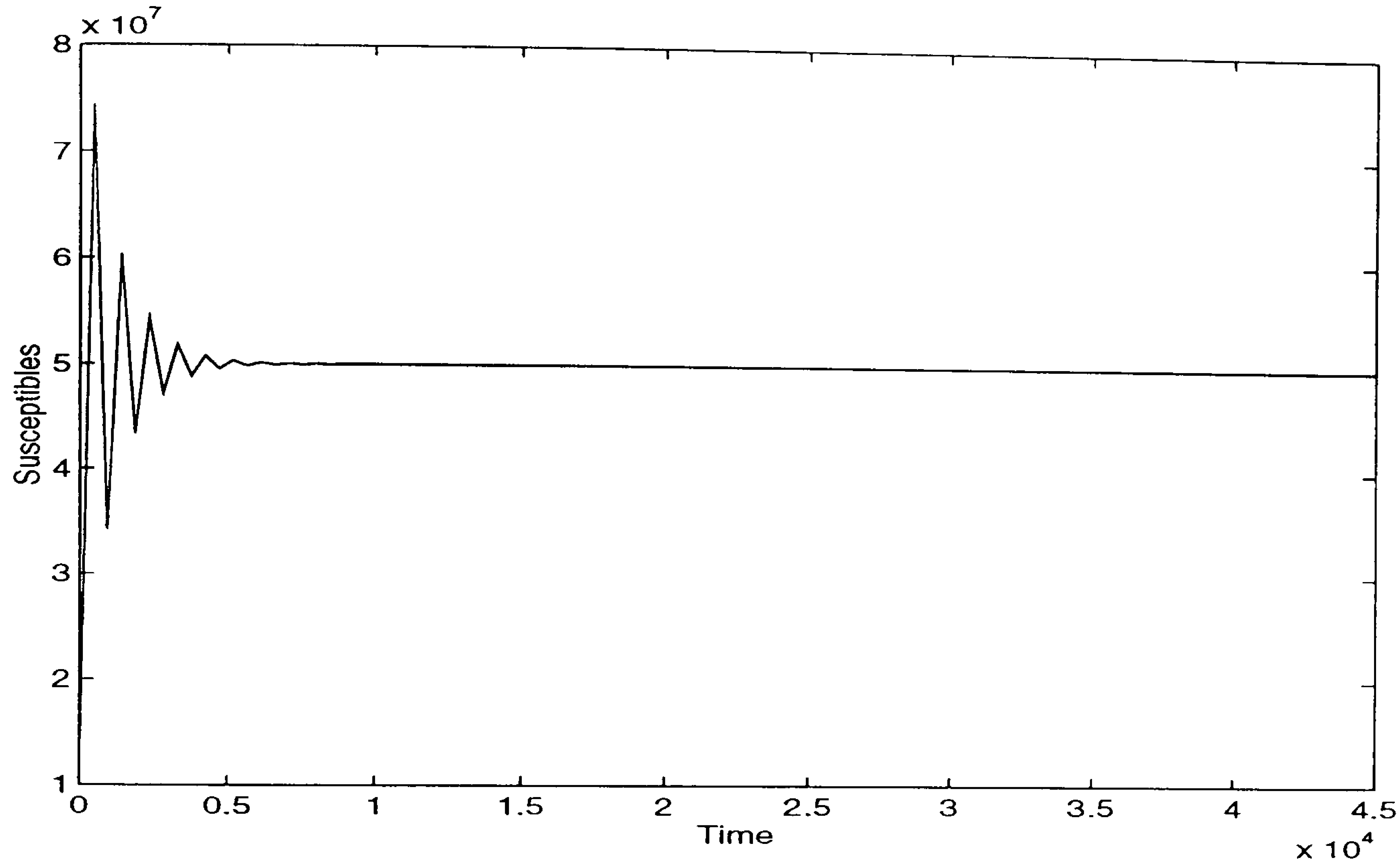

Figure 3.31: *Method 2, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 470$.

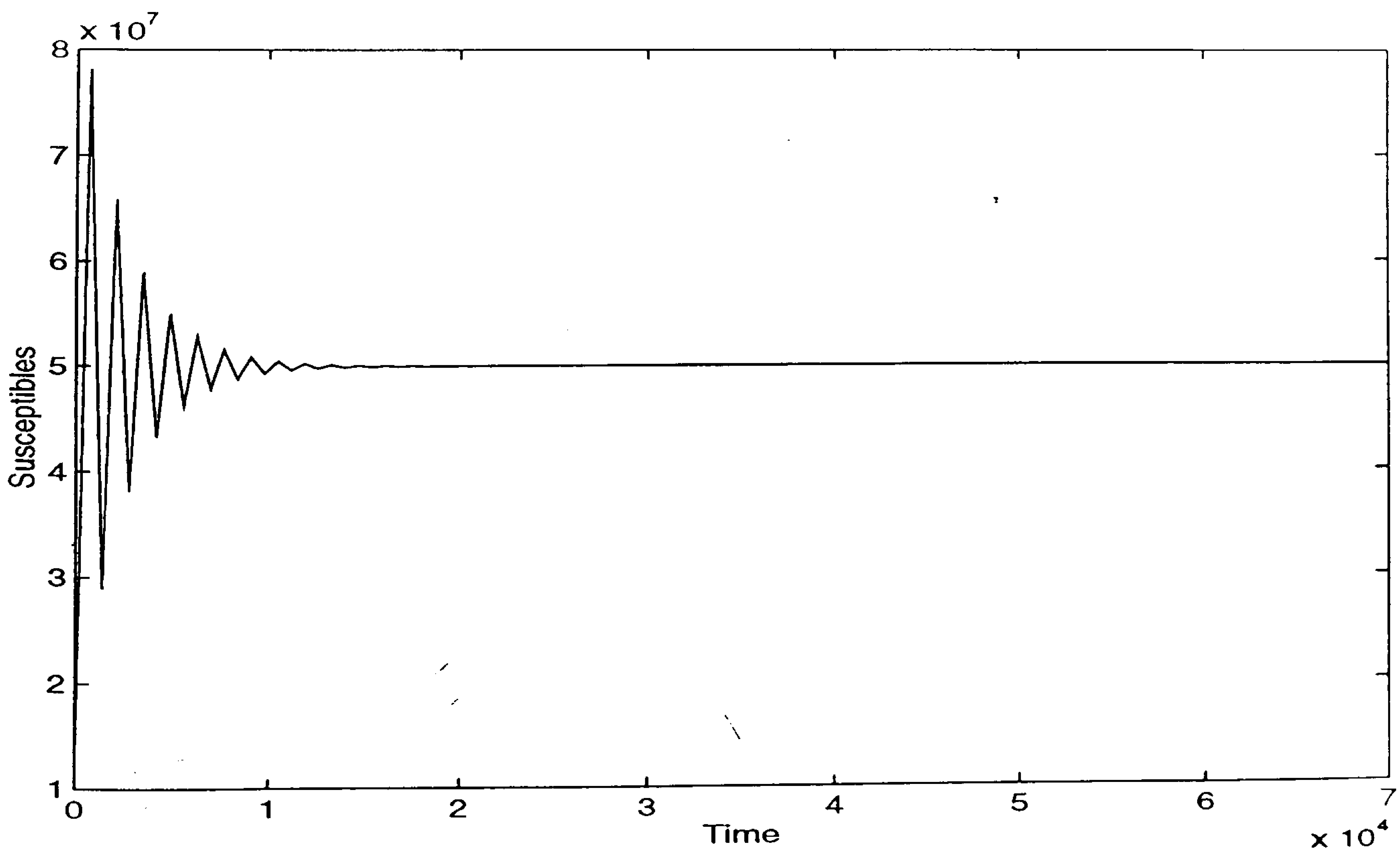

Figure 3.32: *Method 2, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 700$.

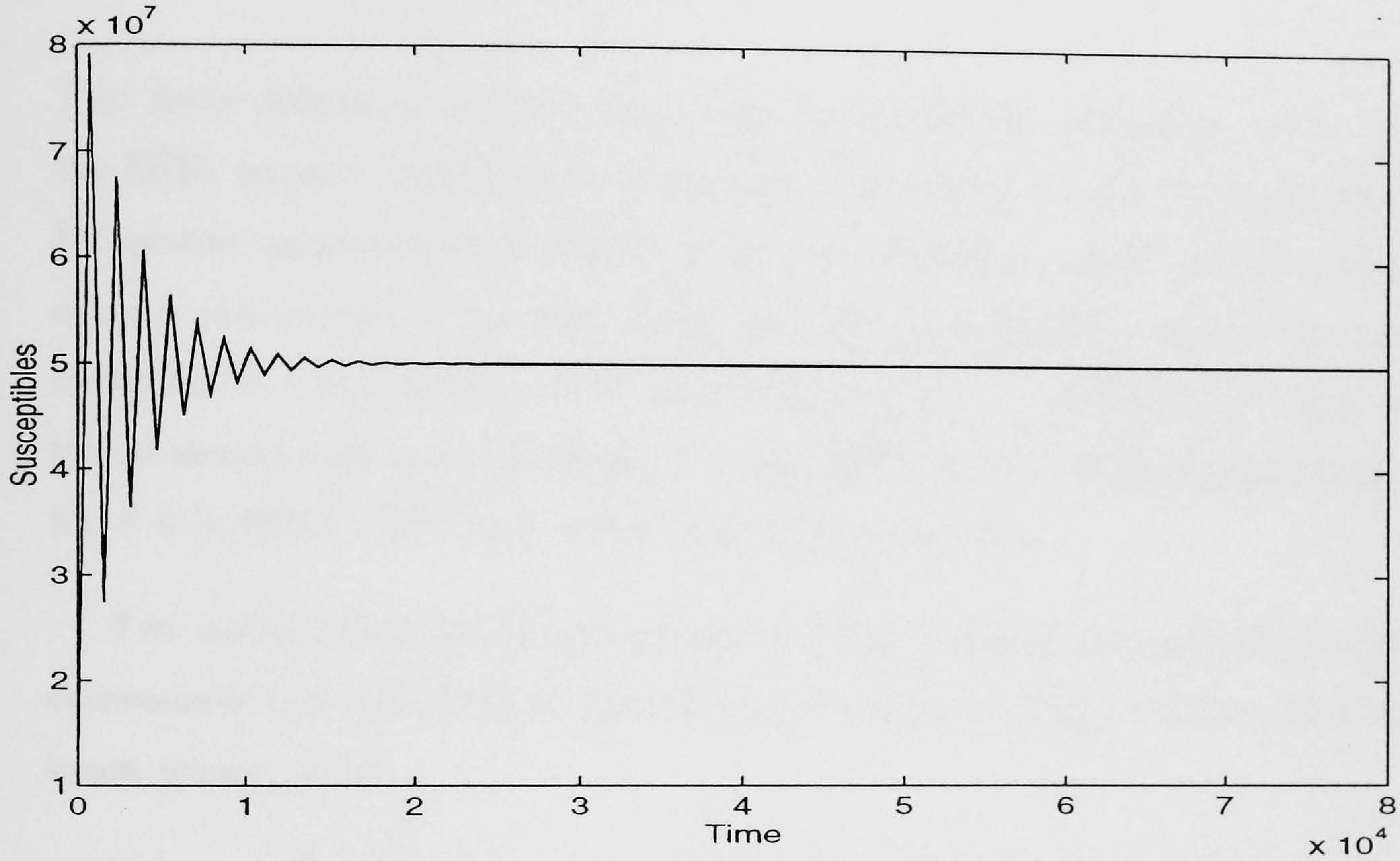

Figure 3.33: *Method 2, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 794$.

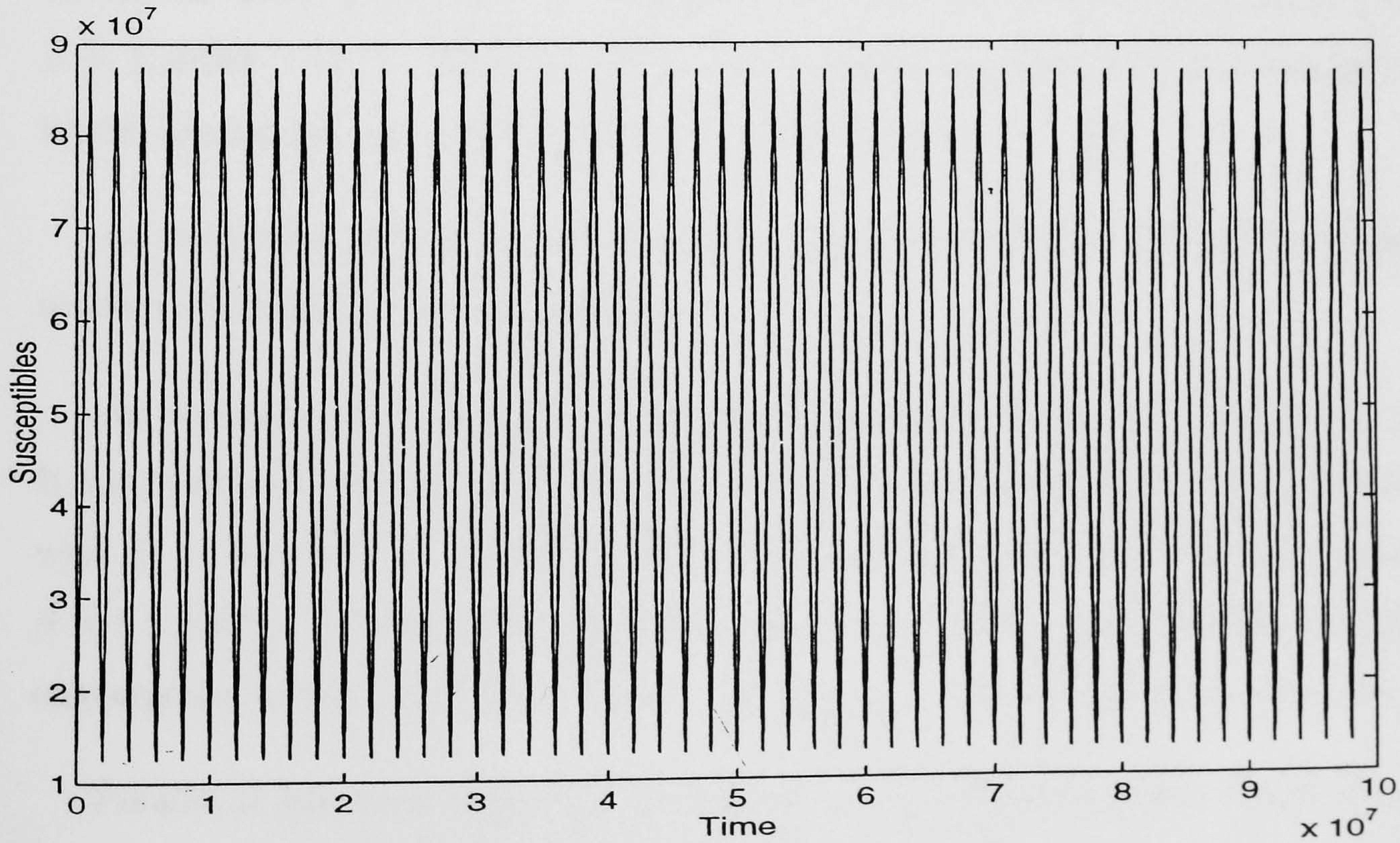

Figure 3.34: *Method 2, experiment B,* $\beta = 1 \times 10^{-6}$ *or* $\beta = 5 \times 10^{-4}, \ell = 1 \times 10^6$.

3.8 Conclusion

Two finite-difference methods have been considered for computing the solutions of the SEIR measles model which is given by a non-linear system of equations (3.4.1). Bifurcation analysis showed that there are two stationary points; one is trivial and the other is non-trivial. It has been shown that the trivial steady-state is asymptotically stable for $\beta \in [0, 1.46104 \times 10^{-6})$ and unstable for $\beta \geq 1.46104 \times 10^{-6}$ and the non-trivial steady-state is unstable for $\beta \in [0, 1.45975 \times 10^{-6})$ and asymptotically stable for $\beta \in [1.45975 \times 10^{-6}, \infty)$, where β is the infection rate.

Two initial conditions (experiments) have been used to test the methods; the first corresponds to a mid-epidemic situation and the second refers to the case where measles is not present at all.

Using a sufficiently small value of the time step ℓ, in the first experiment, with the first method (the familiar explicit Euler method), the solutions obtained converged to the trivial fixed point for $\beta < 1.46104 \times 10^{-6}$ and to the non-trivial fixed point for $\beta > 1.46104 \times 10^{-6}$. While, in the second experiment, the method converged to the trivial fixed point for both values $\beta = 1 \times 10^{-6}$ and $\beta = 5 \times 10^{-4}$.

As with most problems solved using it, the first-order explicit method requires a severe restriction on the time stepsize.

An alternative second-order method has been used to solve the SEIR measles model. It has been seen that this method is very restrictive on stepsize, for the first experiment, with $\beta = 5 \times 10^{-4} > 1.46104 \times 10^{-6}$ and converged to the trivial fixed point for $\beta = 1 \times 10^{-6} < 1.46104 \times 10^{-6}$ and large values of stepsize. In the second experiment, convergence to the correct fixed point has been seen for large values of the time step.

Finally, it has been seen that the second method behaves much better than that of the well-known Euler method, as the Euler method is first-order accurate, whereas the second method is second-order accurate. The maximum time step permitted with method 2, especially in experiment B, is larger than that of the Euler method. Moreover, method 2 dose not produce overflow, in experiment B, as in the Euler method. with a very large time step.

Chapter 4

One-Dimensional Measles Dynamics

4.1 Introduction

In the previous chapter the SEIR model which consists of three non-linear ordinary differential equations (ODEs) was considered. In this chapter and the next two chapters some realistic phenomena (complexities), such as one- and two-space dimension distributions, will be considered. In this chapter a spatially-structured (reaction-diffusion) measles model will be studied. A two-dimensional measles model will be considered in the following chapter and a one-dimensional-structure measles model of hyperbolic type will be given in Chapter 6.

Recent extensions of the SEIR model have included heterogeneities in terms of age (Anderson & May[1]; Dietz & Schenzle[13]; Schenzle[57]; Tudor[65]), seasonality (Aron & Schwartz[3]; Schenzle[57]; Schwartz & Smith [59]; Schwartz[58]), and spatial structure (Bartlett[5]; Murray & Cliff[49]; Schwartz[58]).

4.2 The Reaction-Diffusion System

The partial differential equation (PDE) model which encompasses a variety of the models in the literature and gives a formal structure for the reaction-diffusion system is given below. It captures the basic feature of measles epidemiology discussed in the previous chapter and allows for differences in transmission according to space and time. In order to proceed, the epidemic is assumed to diffuse through space. Also, it is assumed that all births are into the susceptible class, and that births exactly balance deaths so that the total population size, N, is constant.

The reaction-diffusion equations are given by

$$\begin{aligned} \frac{\partial S}{\partial t} &= \mu N - (\mu + \beta I) S + \alpha \frac{\partial^2 S}{\partial x^2} \\ \frac{\partial E}{\partial t} &= \beta I S - (\mu + \sigma) E + \alpha \frac{\partial^2 E}{\partial x^2} \\ \frac{\partial I}{\partial t} &= \sigma E - (\mu + \gamma) I + \alpha \frac{\partial^2 I}{\partial x^2} \end{aligned} \qquad (4.2.1)$$

in which $S = S(x,t)$, $E = E(x,t)$ and $I = I(x,t)$ are the number of susceptibles, exposed and infectious individuals, respectively, at time t and distance x from the origin; $\alpha > 0$ is the diffusion rate. The parameters μ, β, σ and γ are as in the previous chapter.

The initial conditions are of the form

$$S(x,0) = S^0, \quad E(x,0) = E^0, \quad I(x,0) = I^0; \quad -L \leq x \leq L \qquad (4.2.2)$$

and the boundary conditions are

$$\frac{\partial S(\pm L,t)}{\partial x} = \frac{\partial E(\pm L,t)}{\partial x} = \frac{\partial I(\pm L,t)}{\partial x} = 0; \quad t > 0. \qquad (4.2.3)$$

On differentiating with respect to t, the equations in (4.2.1) give[1]

$$\begin{aligned} S_{tt} - \alpha\, S_{xxt} + (\mu + \beta\, I)\, S_t + \beta\, I_t\, S &= 0\,, \\ E_{tt} - \alpha\, E_{xxt} + (\mu + \sigma)\, E_t - \beta\, I\, S_t - \beta\, I_t\, S &= 0\,, \\ I_{tt} - \alpha\, I_{xxt} + (\mu + \gamma)\, I_t - \sigma\, E_t &= 0\,, \end{aligned} \qquad (4.2.4)$$

It will be assumed that the PDEs in (4.2.1) are defined for $-L < x < L$, $t > 0$ and so, for these ranges, the initial/boundary-value problem (IBVP) $\{(4.2.1)\text{-}(4.2.3)\}$ is symmetric about the line $x = 0$. This may be exploited in deriving a numerical method by taking

$$\begin{aligned} S_x(0,t) = E_x(0,t) = I_x(0,t) &= 0\,; \qquad t > 0\,, \\ S_x(L,t) = E_x(L,t) = I_x(L,t) &= 0\,; \qquad t > 0\,, \end{aligned} \qquad (4.2.5)$$

as the boundary conditions and equations (4.2.2) as the initial conditions for $0 \le x \le L$ and solving the PDEs in (4.2.1) for $0 \le x \le L$, $t > 0$ subject to (4.2.2) and (4.2.5).

4.3 Discretization and Notations

The interval $0 \le x \le L$ is divided into $M + 1$ subintervals each of width h so that $(M + 1)\, h = L$ and the time interval $t \ge 0$ is discretized in steps of length ℓ. The open region $\Omega = [0 < x < L] \times [t > 0]$ and its boundary $\partial\Omega$ consisting of the lines $x = 0$, $x = L$ and $t = 0$ have thus been covered by a rectangular mesh having coordinates of the form (x_m, t_n) where $x_m = m\, h \quad (m = 0, 1, 2, \ldots, M, M + 1)$ and $t_n = n\, \ell \quad (n = 0, 1, 2, \ldots)$.

[1] Note that from now on the notations S_t and S_{xx} will be used to represent $\frac{\partial S}{\partial t}$ and $\frac{\partial^2 S}{\partial x^2}$, respectively. Similarly for other functions.

The solutions of (4.2.1), (4.2.2) and (4.2.5) at the typical mesh point (x_m, t_n) are, of course, $S(x_m, t_n)$, $E(x_m, t_n)$ and $I(x_m, t_n)$: these may be denoted by S_m^n, E_m^n and I_m^n, respectively. The theoretical solutions of numerical approximations to (4.2.1) at the same mesh point will be denoted by A_m^n, B_m^n and C_m^n, respectively, while the values actually obtained, which may be subject, for example, to round-off errors, will be denoted by $\widetilde{A_m^n}$, $\widetilde{B_m^n}$ and $\widetilde{C_m^n}$, respectively.

4.4 Numerical Methods

4.4.1 Numerical Method for S

Finite-difference methods are developed by approximating the time derivative in the first equation in (4.2.1) by the first-order forward-difference replacement

$$S_t(x,t) \approx [\,S(x,t+\ell) - S(x,t)\,]/\ell \qquad (4.4.1)$$

and the space derivative by the weighted approximant

$$\begin{aligned} S_{xx} \;&\approx\; h^{-2}\,[\,\phi\,\{S(x-h,t+\ell) - 2\,S(x,t+\ell) + S(x+h,t+\ell)\} \\ &+\; (1-\phi)\,\{S(x-h,t) - 2\,S(x,t) + S(x+h,t)\}\,]\,, \end{aligned} \qquad (4.4.2)$$

in which $x = x_m\,(m = 0,1,2,\ldots,M,M+1)$, $t = t_n\,(n = 0,1,2,\ldots)$ and $\phi\,(0 \le \phi \le 1)$ is a parameter. When $\phi = 0$, (4.4.2) is $O(h^2)$ as $h \longrightarrow 0$ and $O(h^2+\ell)$ as $h, \ell \longrightarrow 0$ when $0 < \phi \le 1$.

Equations (3.6.13) and (3.6.16) will be used to obtain approximations to $S(x_m, t_{n+1})$ for use in the first equation in (4.2.1). Now, using equations (3.6.13), (4.4.1) and (4.4.2) in the first equation in (4.2.1) gives

$$\frac{A_m^{n+1} - A_m^n}{\ell} + (\mu + \beta\, C_m^n)\, A_m^{n+1} - \frac{\alpha}{h^2}\left[\phi\left\{A_{m-1}^{n+1} - 2\, A_m^{n+1} + A_{m+1}^{n+1}\right\}\right.$$
$$\left. + (1-\phi)\left\{A_{m-1}^n - 2\, A_m^n + A_{m+1}^n\right\}\right] - \mu\, N = 0\,, \qquad (4.4.3)$$

while using equations (3.6.16), (4.4.1) and (4.4.2) gives

$$\frac{A_m^{n+1} - A_m^n}{\ell} + \left(\mu + \beta\, C_m^{n+1}\right) A_m^n - \frac{\alpha}{h^2}\left[\phi\left\{A_{m-1}^{n+1} - 2\, A_m^{n+1} + A_{m+1}^{n+1}\right\}\right.$$
$$\left. + (1-\phi)\left\{A_{m-1}^n - 2\, A_m^n + A_{m+1}^n\right\}\right] - \mu\, N = 0\,. \qquad (4.4.4)$$

In order to obtain a second-order approximation to $S(x_m, t_{n+1})$, equations (4.4.3) and (4.4.4) should be added, as in Section 3.6.2. Averaging equations (4.4.3) and (4.4.4) gives

$$\frac{A_m^{n+1} - A_m^n}{\ell} + \frac{1}{2}(\mu + \beta\, C_m^n)\, A_m^{n+1} + \frac{1}{2}(\mu + \beta\, C_m^{n+1})\, A_m^n - \frac{\alpha}{h^2}\phi\left\{A_{m-1}^{n+1} - 2\, A_m^{n+1} + A_{m+1}^{n+1}\right\}$$
$$- \frac{\alpha}{h^2}(1-\phi)\left\{A_{m-1}^n - 2\, A_m^n + A_{m+1}^n\right\} - \mu\, N = 0\,, \qquad (4.4.5)$$

which, after rearranging, becomes

$$-\alpha\,\phi\, p\, A_{m-1}^{n+1} + \left[1 + 2\,\alpha\,\phi\, p + \frac{1}{2}\ell\,(\mu + \beta\, C_m^n)\right] A_m^{n+1} - \alpha\,\phi\, p\, A_{m+1}^{n+1}$$
$$= (1-\phi)\,\alpha\, p\, A_{m-1}^n + \left[1 - 2\,(1-\phi)\,\alpha\, p - \frac{1}{2}\ell\,\left(\mu + \beta\, C_m^{n+1}\right)\right] A_m^n$$
$$+ (1-\phi)\,\alpha\, p\, A_{m+1}^n + \ell\,\mu\, N\,, \qquad (4.4.6)$$

where $p = \ell/h^2$.

The local truncation error $\mathcal{L}_S = \mathcal{L}_S\,[S(x,t), E(x,t), I(x,t); h, \ell]$ associated with (4.4.6) at the point $(x,t) = (x_m, t_n)$ may be written down from (4.4.5): it is

$$
\begin{aligned}
\mathcal{L}_S \;=\; & \frac{S(x,t+\ell)-S(x,t)}{\ell}+\frac{1}{2}\left(\mu+\beta\, I(x,t)\right) S(x,t+\ell) \\
+\; & \frac{1}{2}\left(\mu+\beta\, I(x,t+\ell)\right) S(x,t)-\frac{\alpha\,\phi}{h^2}\left\{S(x-h,t+\ell)-2\,S(x,t+\ell)+S(x+h,t+\ell)\right\} \\
-\; & (1-\phi)\,\frac{\alpha}{h^2}\left\{S(x-h,t)-2\,S(x,t)+S(x+h,t)\right\}-\mu\, N \\
-\; & \left\{S_t(x,t)+(\mu+\beta\, I(x,t))\, S(x,t)-\alpha\, S_{xx}(x,t)-\mu\, N\right\}. \qquad (4.4.7)
\end{aligned}
$$

Expanding $S(x,t+\ell)$, $S(x\pm h,t+\ell)$, $S(x\pm h,t)$ and $I(x,t+\ell)$ in (4.4.7) about (x,t) leads to

$$
\begin{aligned}
\mathcal{L}_S \;=\; & \left[\frac{1}{2}\,S_{tt}+\frac{1}{2}(\mu+\beta\, I)\, S_t+\frac{1}{2}\,\beta\, S\, I_t-\phi\,\alpha\, S_{xxt}\right]\ell-\frac{1}{12}\,\alpha\, h^2\, S_{xxxx} \\
+\; & \left[\frac{1}{6}\,S_{ttt}+\frac{1}{4}(\mu+\beta\, I)\, S_{tt}+\frac{1}{4}\,\beta\, S\, I_{tt}-\frac{1}{2}\,\phi\,\alpha\, S_{xxtt}\right]\ell^2 \\
+\; & \cdots. \qquad (4.4.8)
\end{aligned}
$$

Clearly, $\mathcal{L}_S = O(h^2+\ell)$ as $h,\, \ell \longrightarrow 0$ except when $\phi=\frac{1}{2}$ for then the term in ℓ vanishes in (4.4.8), see the first equation in (4.2.4), leaving

$$
\mathcal{L}_S=-\frac{1}{12}\,\alpha\, h^2\, S_{xxxx}+\left[\frac{1}{6}\,S_{ttt}+\frac{1}{4}(\mu+\beta\, I)\, S_{tt}+\frac{1}{4}\,\beta\, S\, I_{tt}-\frac{1}{4}\,\alpha\, S_{xxtt}\right]\ell^2+\cdots \qquad (4.4.9)
$$

which is $O(h^2+\ell^2)$ as $h,\, \ell \longrightarrow 0$. It may be concluded that the unique $O(h^2+\ell^2)$ method, as $h,\, \ell \longrightarrow 0$, contained in the family (4.4.6) arises when $\phi=\frac{1}{2}$ and is given by

$$
\begin{aligned}
-\frac{1}{2}\,\alpha\, p\, A_{m-1}^{n+1} \;+\; & \left[1+\alpha\, p+\frac{1}{2}\,\ell\,(\mu+\beta\, C_m^n)\right] A_m^{n+1}-\frac{1}{2}\,\alpha\, p\, A_{m+1}^{n+1}+\frac{1}{2}\,\ell\,\beta\, C_m^{n+1}\, A_m^n \\
=\; & \frac{1}{2}\,\alpha\, p\, A_{m-1}^n+\left[1-\alpha\, p-\frac{1}{2}\,\ell\,\mu\right] A_m^n+\frac{1}{2}\,\alpha\, p\, A_{m+1}^n \\
+\; & \ell\,\mu\, N. \qquad (4.4.10)
\end{aligned}
$$

The finite-difference method (4.4.10) may be applied for $m=1,2,\ldots M$ and $n=0,1,2,\ldots$. In the case $m=0$ it requires some modification and may be simplified a little when $m = M + 1$. Applying (4.4.10) with $m = 0$ introduces the terms A_{-1}^{n+1} and A_{-1}^{n}. Now, the points (x_{-1}, t_{n+1}) and (x_{-1}, t_n) are outside the grid superimposed on $\Omega \cup \partial\Omega$. However, the boundary conditions (4.2.5) give, to second order, $A_{-1}^{n} = A_1^n$ and $A_{-1}^{n+1} = A_1^{n+1}$ so that, for $m = 0$, equation (4.4.10) may be modified to give

$$\begin{aligned}[1 + \alpha p \; + \; & \frac{1}{2}\ell\,(\mu + \beta\, C_0^n)\,]\, A_0^{n+1} - \alpha\, p\, A_1^{n+1} + \frac{1}{2}\ell\,\beta\, C_0^{n+1}\, A_0^n \\ = \; & \left[1 - \alpha\, p - \frac{1}{2}\ell\,\mu\right]\, A_0^n + \alpha\, p\, A_1^n + \ell\,\mu\, N\,. \end{aligned} \qquad (4.4.11)$$

Now with $m = M + 1$, it follows from (4.2.5) that $A_{M+2}^{n} = A_M^n$ and $A_{M+2}^{n+1} = A_M^{n+1}$ and so (4.4.10) may be written as

$$\begin{aligned}-\alpha\, p\, A_M^{n+1} \; + \; & \left[1 + \alpha\, p + \frac{1}{2}\ell\,\left(\mu + \beta\, C_{M+1}^n\right)\right]\, A_{M+1}^{n+1} + \frac{1}{2}\ell\,\beta\, C_{M+1}^{n+1}\, A_{M+1}^n \\ = \; & \alpha\, p\, A_M^n + \left[1 - \alpha\, p - \frac{1}{2}\ell\,\mu\right]\, A_{M+1}^n + \ell\,\mu\, N\,. \end{aligned} \qquad (4.4.12)$$

4.4.2 Numerical Method for E

The time derivative in the second equation in (4.2.1) is approximated by the first-order forward-difference replacement

$$E_t \approx [\,E(x, t + \ell) - E(x, t)\,]/\ell \qquad (4.4.13)$$

and the space derivative by the weighted approximant

$$\begin{aligned} E_{xx} \;\approx\; & h^{-2}\,[\,\phi\,\{\,E(x - h, t + \ell) - 2\,E(x, t + \ell) + E(x + h, t + \ell)\,\} \\ + \; & (1 - \phi)\,\{\,E(x - h, t) - 2\,E(x, t) + E(x + h, t)\,\}\,]\,, \end{aligned} \qquad (4.4.14)$$

in which $x = x_m \, (m = 0, 1, 2, \ldots, M, M+1)$, $t = t_n \, (n = 0, 1, 2, \ldots)$ and $\phi \, (0 \le \phi \le 1)$ is a parameter. When $\phi = 0$, (4.4.14) is $O(h^2)$ as $h \longrightarrow 0$ and is $O(h^2 + \ell)$ as $h, \ell \longrightarrow 0$ when $0 < \phi \le 1$.

Using equations (3.6.25), (4.4.13) and (4.4.14) in the second equation in (4.2.1) gives

$$\begin{aligned} \frac{B_m^{n+1} - B_m^n}{\ell} &- \beta \, C_m^n \, A_m^{n+1} + (\mu + \sigma) \, B_m^{n+1} - \frac{\alpha}{h^2} \left[\phi \left\{ B_{m-1}^{n+1} - 2 \, B_m^{n+1} + B_{m+1}^{n+1} \right\} \right. \\ &+ \left. (1 - \phi) \left\{ B_{m-1}^n - 2 \, B_m^n + B_{m+1}^n \right\} \right] = 0 \, , \end{aligned} \qquad (4.4.15)$$

while using (3.6.28), (4.4.13) and (4.4.14) in the second equation in (4.2.1) gives

$$\begin{aligned} \frac{B_m^{n+1} - B_m^n}{\ell} &- \beta \, C_m^{n+1} \, A_m^n + (\mu + \sigma) \, B_m^n - \frac{\alpha}{h^2} \left[\phi \left\{ B_{m-1}^{n+1} - 2 \, B_m^{n+1} + B_{m+1}^{n+1} \right\} \right. \\ &+ \left. (1 - \phi) \left\{ B_{m-1}^n - 2 \, B_m^n + B_{m+1}^n \right\} \right] = 0 \, . \end{aligned} \qquad (4.4.16)$$

Following Section 4.4.1, to obtain a second-order approximation to $E(x_m, t_{n+1})$, equations (4.4.15) and (4.4.16) should be added. Averaging equations (4.4.15) and (4.4.16) gives

$$\begin{aligned} \frac{B_m^{n+1} - B_m^n}{\ell} &- \frac{1}{2} \beta \, C_m^n \, A_m^{n+1} - \frac{1}{2} \beta \, C_m^{n+1} \, A_m^n + \frac{1}{2} (\mu + \sigma) \, B_m^{n+1} + \frac{1}{2} (\mu + \sigma) \, B_m^n \\ &- \frac{\alpha}{h^2} \left[\phi \left\{ B_{m-1}^{n+1} - 2 \, B_m^{n+1} + B_{m+1}^{n+1} \right\} + (1 - \phi) \left\{ B_{m-1}^n - 2 \, B_m^n \right. \right. \\ &+ \left. \left. B_{m+1}^n \right\} \right] = 0 \, , \end{aligned} \qquad (4.4.17)$$

which, after rearranging, becomes

$$\begin{aligned} -\alpha \, \phi \, p \, B_{m-1}^{n+1} &+ \left[1 + 2 \, \alpha \, \phi \, p + \frac{1}{2} \ell \, (\mu + \sigma) \right] \, B_m^{n+1} - \alpha \, \phi \, p \, B_{m+1}^{n+1} - \frac{1}{2} \ell \, \beta \, C_m^n \, A_m^{n+1} \\ &- \frac{1}{2} \ell \, \beta \, C_m^{n+1} \, A_m^n = (1 - \phi) \, \alpha \, p \, B_{m-1}^n + \left[1 - 2 \, (1 - \phi) \, \alpha \, p \right. \\ &- \left. \frac{1}{2} \ell \, (\mu + \sigma) \right] \, B_m^n + (1 - \phi) \, \alpha \, p \, B_{m+1}^n \, , \end{aligned} \qquad (4.4.18)$$

where $p = \ell / h^2$.

The local truncation error $\mathcal{L}_E = \mathcal{L}_E[S(x,t), E(x,t), I(x,t); h, \ell]$ associated with (4.4.18) at the point $(x,t) = (x_m, t_n)$ may be written down from (4.4.17): it is

$$\begin{aligned}
\mathcal{L}_E &= \frac{E(x,t+\ell) - E(x,t)}{\ell} - \frac{1}{2}\beta\, I(x,t)\, S(x,t+\ell) \\
&- \frac{1}{2}\beta\, I(x,t+\ell)\, S(x,t) + \frac{1}{2}(\mu+\sigma)\, E(x,t+\ell) + \frac{1}{2}(\mu+\sigma)\, E(x,t) \\
&- \frac{\alpha}{h^2}\left[\phi\left\{ E(x-h,t+\ell) - 2\,E(x,t+\ell) + E(x+h,t+\ell)\right\}\right. \\
&+ \left.(1-\phi)\left\{ E(x-h,t) - 2\,E(x,t) + E(x+h,t)\right\}\right] \\
&- \left\{ E_t(x,t) - \beta\, I(x,t)\, S(x,t) + (\mu+\sigma)\, E(x,t) - \alpha\, E_{xx}\right\} . \qquad (4.4.19)
\end{aligned}$$

Expanding $E(x,t+\ell)$, $E(x \pm h, t+\ell)$, $E(x \pm h, t)$, $I(x,t+\ell)$ and $S(x,t+\ell)$ in (4.4.19), using Taylor's expansion, about the point (x,t) leads to

$$\begin{aligned}
\mathcal{L}_E &= \left[\frac{1}{2}\, E_{tt} + \frac{1}{2}(\mu+\sigma)\, E_t - \frac{1}{2}\beta\, S\, I_t - \frac{1}{2}\beta\, S_t\, I - \alpha\,\phi\, E_{xxt}\right]\ell \\
&- \frac{1}{12}\alpha\, h^2\, E_{xxxx} + \left[\frac{1}{6}\, E_{ttt} + \frac{1}{4}(\mu+\sigma)\, E_{tt} - \frac{1}{4}\beta\, I\, S_{tt}\right. \\
&- \left.\frac{1}{4} S\, I_{tt} - \frac{1}{2}\alpha\,\phi\, E_{xxtt}\right]\ell^2 + \cdots . \qquad (4.4.20)
\end{aligned}$$

The second equation in (4.2.4) reveals that the coefficient of ℓ in (4.4.20) vanishes provided $\phi = \frac{1}{2}$, leaving

$$\begin{aligned}
\mathcal{L}_E &= -\frac{1}{12}\alpha\, h^2\, E_{xxxx} + \left[\frac{1}{6}\, E_{ttt} + \frac{1}{4}(\mu+\sigma)\, E_{tt}\right. \\
&- \left.\frac{1}{4}\beta\, I\, S_{tt} - \frac{1}{4}\beta\, S\, I_{tt} - \frac{1}{4}\alpha\, E_{xxtt}\right]\ell^2 + \cdots , \qquad (4.4.21)
\end{aligned}$$

which is $O(h^2 + \ell^2)$ as $h, \ell \longrightarrow 0$. It may be concluded that the unique $O(h^2 + \ell^2)$ method, as $h, \ell \longrightarrow 0$, contained in the family (4.4.18) arises when $\phi = \frac{1}{2}$ and is given by

$$
\begin{aligned}
-\frac{1}{2}\alpha\, p\, B_{m-1}^{n+1} &+ \left[1+\alpha\, p+\frac{1}{2}\ell\,(\mu+\sigma)\right] B_m^{n+1} - \frac{1}{2}\alpha\, p\, B_{m+1}^{n+1} - \frac{1}{2}\ell\,\beta\, C_m^n\, A_m^{n+1} \\
&- \frac{1}{2}\ell\,\beta\, C_m^{n+1}\, A_m^n = \frac{1}{2}\alpha\, p\, B_{m-1}^n + \left[1-\alpha\, p-\frac{1}{2}\ell\,(\mu+\sigma)\right] B_m^n \\
&+ \frac{1}{2}\alpha\, p\, B_{m+1}^n\,, \qquad (4.4.22)
\end{aligned}
$$

with $m = 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots$.

As in the case of S, equation (4.4.22) may be modified for use with $m = 0$ and $m = M+1$ using equations in (4.2.5) to obtain, respectively,

$$
\begin{aligned}
[\,1+\alpha\, p &+ \frac{1}{2}\ell\,(\mu+\sigma)\,]\, B_0^{n+1} - \alpha\, p\, B_1^{n+1} - \frac{1}{2}\ell\,\beta\, C_0^n\, A_0^{n+1} - \frac{1}{2}\ell\,\beta\, C_0^{n+1}\, A_0^n \\
&= \left[1-\alpha\, p-\frac{1}{2}\ell\,(\mu+\sigma)\right] B_0^n + \alpha\, p\, B_1^n\,, \qquad (4.4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
-\alpha\, p\, B_M^{n+1} &+ \left[1+\alpha\, p+\frac{1}{2}\ell\,(\mu+\sigma)\right] B_{M+1}^{n+1} - \frac{1}{2}\ell\,\beta\, C_{M+1}^n\, A_{M+1}^{n+1} - \frac{1}{2}\ell\,\beta\, C_{M+1}^{n+1}\, A_{M+1}^n \\
&= \alpha\, p\, B_M^n + \left[1-\alpha\, p-\frac{1}{2}\ell\,(\mu+\sigma)\right] B_{M+1}^n\,. \qquad (4.4.24)
\end{aligned}
$$

4.4.3 Numerical Method for I

The time deivative in the third equation in (4.2.1) is approximated by the first-order forward-difference replacement

$$
I_t \approx [\,I(x,t+\ell) - I(x,t)\,]/\ell \qquad (4.4.25)
$$

and the space derivative by the weighted approximant

$$
\begin{aligned}
I_{xx} &\approx h^{-2}\,[\,\phi\,\{\,I(x-h,t+\ell) - 2\,I(x,t+\ell) + I(x+h,t+\ell)\,\} \\
&+ (1-\phi)\,\{\,I(x-h,t) - 2\,I(x,t) + I(x+h,t)\,\}\,]\,, \qquad (4.4.26)
\end{aligned}
$$

in which $x = x_m\,(m = 0, 1, 2, \ldots, M, M+1)$, $t = t_n\,(n = 0, 1, 2, \ldots)$ and $\phi\,(0 \le \phi \le 1)$ is a parameter. As before, when $\phi = 0$, (4.4.26) is $O(h^2)$ as $h \longrightarrow 0$ and $O(h^2 + \ell)$ as $h\,,\ell \longrightarrow 0$ when $0 < \phi \le 1$. Now equations (3.6.34) and (3.6.37) will be used to obtain an approximation to $I(x_m, t_{n+1})$ for use in the third equation in (4.2.1). To reach this approximation, equations (3.6.34), (4.4.25) and (4.4.26) are used in the third equation in (4.2.1) to give

$$\begin{aligned} \frac{C_m^{n+1} - C_m^n}{\ell} - \sigma\,B_m^{n+1} &+ (\mu + \gamma)\,C_m^{n+1} - \frac{\alpha}{h^2}\left[\phi\left\{C_{m-1}^{n+1} - 2\,C_m^{n+1} + C_{m+1}^{n+1}\right\}\right. \\ &+ \left.(1-\phi)\left\{C_{m-1}^n - 2\,C_m^n + C_{m+1}^n\right\}\right] = 0\,, \qquad (4.4.27) \end{aligned}$$

and equations (3.6.37), (4.4.25) and (4.4.26) are used in the third equation in (4.2.1) to give

$$\begin{aligned} \frac{C_m^{n+1} - C_m^n}{\ell} - \sigma\,B_m^n &+ (\mu + \gamma)\,C_m^n - \frac{\alpha}{h^2}\left[\phi\left\{C_{m-1}^{n+1} - 2\,C_m^{n+1} + C_{m+1}^{n+1}\right\}\right. \\ &+ \left.(1-\phi)\left\{C_{m-1}^n - 2\,C_m^n + C_{m+1}^n\right\}\right] = 0\,. \qquad (4.4.28) \end{aligned}$$

In order to obtain a second-order approximation to $I(x_m, t_{n+1})$, equations (4.4.27) and (4.4.28) are averaged to obtain

$$\begin{aligned} \frac{C_m^{n+1} - C_m^n}{\ell} &- \frac{1}{2}\sigma\,B_m^{n+1} - \frac{1}{2}\sigma\,B_m^n + \frac{1}{2}(\mu+\gamma)\,C_m^{n+1} + \frac{1}{2}(\mu+\gamma)\,C_m^n \\ &- \frac{\alpha}{h^2}\left[\phi\left\{C_{m-1}^{n+1} - 2\,C_m^{n+1} + C_{m+1}^{n+1}\right\} + (1-\phi)\left\{C_{m-1}^n\right.\right. \\ &- \left.\left.2\,C_m^n + C_{m+1}^n\right\}\right] = 0\,, \qquad (4.4.29) \end{aligned}$$

which, after rearranging, becomes

$$\begin{aligned} -\alpha\,\phi\,p\,C_{m-1}^{n+1} &+ \left[1 + 2\,\alpha\,\phi\,p + \frac{1}{2}\ell\,(\mu+\gamma)\right]C_m^{n+1} - \alpha\,\phi\,p\,C_{m+1}^{n+1} - \frac{1}{2}\ell\,\sigma\,B_m^{n+1} \\ &= (1-\phi)\,\alpha\,p\,C_{m-1}^n + \left[1 - 2\,(1-\phi)\,\alpha\,p - \frac{1}{2}\ell\,(\mu+\gamma)\right]C_m^n \\ &+ (1-\phi)\,\alpha\,p\,C_{m+1}^n + \frac{1}{2}\ell\,\sigma\,B_m^n\,, \qquad (4.4.30) \end{aligned}$$

where $p = \ell / h^2$.

The local truncation error $\mathcal{L}_I = \mathcal{L}_I[S(x,t), E(x,t), I(x,t); h, \ell]$ associated with (4.4.30) at the point $(x,t) = (x_m, t_n)$ may be written down from (4.4.29): it is

$$\begin{aligned}
\mathcal{L}_I &= \frac{I(x,t+\ell) - I(x,t)}{\ell} - \frac{1}{2}\sigma\, E(x,t+\ell) - \frac{1}{2}\sigma\, E(x,t) \\
&+ \frac{1}{2}(\mu+\gamma)\, I(x,t+\ell) + \frac{1}{2}(\mu+\gamma)\, I(x,t) - \frac{\alpha}{h^2}\,[\phi\,\{\, I(x-h,t+\ell) - 2\,I(x,t+\ell) \\
&+ I(x+h,t+\ell)\} + (1-\phi)\,\{\, I(x-h,t) - 2\,I(x,t) + I(x+h,t)\,\}\,] \\
&- \{\, I_t(x,t) - \sigma\, E(x,t) + (\mu+\gamma)\, I(x,t) - \alpha\, I_{xx}\,\}\,. \qquad (4.4.31)
\end{aligned}$$

Expanding $I(x,t+\ell)$, $I(x \pm h, t+\ell)$, $I(x \pm h, t)$ and $E(x,t+\ell)$ in (4.4.31), using Taylor's expansion, about the point (x,t), gives

$$\begin{aligned}
\mathcal{L}_I &= \left[\frac{1}{2}\, I_{tt} + \frac{1}{2}(\mu+\gamma)\, I_t - \frac{1}{2}\sigma\, E_t - \alpha\,\phi\, I_{xxtt}\right]\ell \\
&- \frac{1}{12}\alpha\, h^2\, I_{xxxx} + \left[\frac{1}{6}\, I_{ttt} + \frac{1}{4}(\mu+\gamma)\, I_{tt} - \frac{1}{4}\sigma\, E_{tt}\right. \\
&- \left.\frac{1}{2}\alpha\,\phi\, I_{xxtt}\right]\ell^2 + \cdots\,. \qquad (4.4.32)
\end{aligned}$$

The third equation in (4.2.4) reveals that the coefficient of ℓ in (4.4.32) vanishes provided $\phi = \frac{1}{2}$, leaving

$$\mathcal{L}_I = -\frac{1}{12}\alpha\, h^2\, I_{xxxx} + \left[\frac{1}{6}\, I_{ttt} + \frac{1}{4}(\mu+\gamma)\, I_{tt} - \frac{1}{4}\sigma\, E_{tt} - \frac{1}{4}\alpha\, I_{xxtt}\right]\ell^2 + \cdots\,, \qquad (4.4.33)$$

which is $O(h^2 + \ell^2)$ as $h, \ell \longrightarrow 0$. The unique $O(h^2 + \ell^2)$ method in (4.4.30), when $\phi = \frac{1}{2}$, is thus given by

$$\begin{aligned}
-\frac{1}{2}\alpha\, p\, C_{m-1}^{n+1} &+ \left[1 + \alpha\, p + \frac{1}{2}\ell(\mu+\gamma)\right] C_m^{n+1} - \frac{1}{2}\alpha\, p\, C_{m+1}^{n+1} - \frac{1}{2}\ell\,\sigma\, B_m^{n+1} \\
&= \frac{1}{2}\alpha\, p\, C_{m-1}^{n} + \left[1 - \alpha\, p - \frac{1}{2}\ell(\mu+\gamma)\right] C_m^{n} + \frac{1}{2}\alpha\, p\, C_{m+1}^{n} \\
&+ \frac{1}{2}\ell\,\sigma\, B_m^{n}\,, \qquad (4.4.34)
\end{aligned}$$

with $m = 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots$.

Equation (4.4.34) may be modified, as in the case of S and E, for use with $m = 0$ and $m = M + 1$ using the boundary conditions in (4.2.5), respectively, to obtain

$$\left[1 + \alpha p + \frac{1}{2}\ell(\mu+\gamma)\right] C_0^{n+1} - \alpha p C_1^{n+1} - \frac{1}{2}\ell\sigma B_0^{n+1} = \left[1 - \alpha p - \frac{1}{2}\ell(\mu+\gamma)\right] C_0^n + \alpha p C_0^n + \frac{1}{2}\ell\sigma B_0^n, \quad (4.4.35)$$

and

$$-\alpha p C_M^{n+1} + \left[1 + \alpha p + \frac{1}{2}\ell(\mu+\gamma)\right] C_{M+1}^{n+1} - \frac{1}{2}\ell\sigma B_{M+1}^{n+1} = \alpha p C_M^n + \left[1 - \alpha p - \frac{1}{2}\ell(\mu+\gamma)\right] C_{M+1}^n + \frac{1}{2}\ell\sigma B_{M+1}^n. \quad (4.4.36)$$

4.5 Implementation

Let $\mathbf{A}^{n+1} = \left[A_0^{n+1}, A_1^{n+1}, \cdots, A_{M+1}^{n+1}\right]^T$, $\mathbf{B}^{n+1} = \left[B_0^{n+1}, B_1^{n+1}, \cdots, B_{M+1}^{n+1}\right]^T$ and $\mathbf{C}^{n+1} = \left[C_0^{n+1}, C_1^{n+1}, \cdots, C_{M+1}^{n+1}\right]^T$, where T denotes transpose. Then the price to be paid in using (4.4.10)-(4.4.12), (4.4.22)-(4.4.24), (4.4.34), (4.4.35) and (4.4.36) to obtain $O(h^2 + \ell^2)$ solutions to (4.2.1) as $h, \ell \longrightarrow 0$ is that $\mathbf{A}^{n+1}$, $\mathbf{B}^{n+1}$ and $\mathbf{C}^{n+1}$ cannot be obtained by solving three linear algebraic systems of order $(M + 2)$ at each time step, either in sequence or in parallel (see, analogously, Twizell et al.[69]). Instead, because of the appearance of the elements of $\mathbf{C}^{n+1}$ in (4.4.10)-(4.4.12), the elements of $\mathbf{A}^{n+1}$ and $\mathbf{C}^{n+1}$ in (4.4.22)-(4.4.24) and the elements of $\mathbf{B}^{n+1}$ in (4.4.34)-(4.4.36), the vectors $\mathbf{A}^{n+1}$, $\mathbf{B}^{n+1}$ and $\mathbf{C}^{n+1}$ must be obtained simultaneously by solving a linear algebraic system of order $(3M + 6)$ at each time step.

Let $\mathbf{U}^{n+1} = \left[(\mathbf{A}^{n+1})^T, (\mathbf{B}^{n+1})^T, (\mathbf{C}^{n+1})^T\right]^T$ and $\mathbf{U}^n = \left[(\mathbf{A}^n)^T, (\mathbf{B}^n)^T, (\mathbf{C}^n)^T\right]^T$, where T denotes transpose. Then it may be seen that (4.4.10)-(4.4.12), (4.4.22)-(4.4.24), (4.4.34), (4.4.35) and (4.4.36) may be written in matrix-vector form as

$$W^n\,\mathbf{U}^{n+1} = M^n\,\mathbf{U}^n + \mathbf{b} \tag{4.5.1}$$

in which

$$W^n = \begin{pmatrix} X^n & O & H_1^n \\ F_1^n & Y^n & F_2^n \\ O & H_2^n & Z^n \end{pmatrix}, \tag{4.5.2}$$

$$M^n = \begin{pmatrix} P^n & O & O \\ O & Q^n & O \\ O & H_3^n & R^n \end{pmatrix}, \tag{4.5.3}$$

where O is the zero matrix of order $(M+2)$ and the vector $\mathbf{b}$ is a column-vector of order $(3\,M+6)$ and is given by

$$\mathbf{b} = \Big[\underbrace{\ell\,\mu\,N,\ldots,\ell\,\mu\,N}_{(M+2)\,times},0,\ldots,0\Big]^T . \tag{4.5.4}$$

The matrices W^n and M^n are both of order $(3\,M+6)$ and their submatrices are of orders $(M+2)$ and are given by

$$X^n = \begin{pmatrix} v_0 & -\alpha\,p & & & & \\ -\frac{1}{2}\,\alpha\,p & v_1 & -\frac{1}{2}\,\alpha\,p & & & \\ & -\frac{1}{2}\,\alpha\,p & v_2 & -\frac{1}{2}\,\alpha\,p & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{2}\,\alpha\,p & v_M & -\frac{1}{2}\,\alpha\,p \\ & & & -\alpha\,p & v_{M+1} \end{pmatrix}, \tag{4.5.5}$$

$$Y^n = \begin{pmatrix} v_Y & -\alpha p & & & & \\ -\frac{1}{2}\alpha p & v_Y & -\frac{1}{2}\alpha p & & & \\ & -\frac{1}{2}\alpha p & v_Y & -\frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & & -\frac{1}{2}\alpha p & v_Y & -\frac{1}{2}\alpha p \\ & & & & -\alpha p & v_Y \end{pmatrix}, \qquad (4.5.6)$$

$$Z^n = \begin{pmatrix} v_Z & -\alpha p & & & & \\ -\frac{1}{2}\alpha p & v_Z & -\frac{1}{2}\alpha p & & & \\ & -\frac{1}{2}\alpha p & v_Z & -\frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & & -\frac{1}{2}\alpha p & v_Z & -\frac{1}{2}\alpha p \\ & & & & -\alpha p & v_Z \end{pmatrix}, \qquad (4.5.7)$$

$$P^n = \begin{pmatrix} v_P & \alpha p & & & & \\ \frac{1}{2}\alpha p & v_P & \frac{1}{2}\alpha p & & & \\ & \frac{1}{2}\alpha p & v_P & \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & & \frac{1}{2}\alpha p & v_P & \frac{1}{2}\alpha p \\ & & & & \alpha p & v_P \end{pmatrix}, \qquad (4.5.8)$$

$$Q^n = \begin{pmatrix} v_Q & \alpha p & & & & \\ \frac{1}{2}\alpha p & v_Q & \frac{1}{2}\alpha p & & & \\ & \frac{1}{2}\alpha p & v_Q & \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & & \frac{1}{2}\alpha p & v_Q & \frac{1}{2}\alpha p \\ & & & & \alpha p & v_Q \end{pmatrix}, \qquad (4.5.9)$$

$$R^n = \begin{pmatrix} v_R & \alpha p & & & & \\ \frac{1}{2}\alpha p & v_R & \frac{1}{2}\alpha p & & & \\ & \frac{1}{2}\alpha p & v_R & \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & \frac{1}{2}\alpha p & v_R & \frac{1}{2}\alpha p \\ & & & \alpha p & v_R \end{pmatrix}, \qquad (4.5.10)$$

$$H_1^n = diag\left\{\frac{1}{2}\ell\beta A_m^n\right\}, \quad H_2^n = diag\left\{-\tfrac{1}{2}\ell\sigma I_{(M+2)}\right\}, \quad H_3^n = diag\left\{\frac{1}{2}\ell\sigma I_{(M+2)}\right\},$$

$$F_1^n = diag\left\{-\frac{1}{2}\ell\beta C_m^n\right\} \ \ and \ \ F_2^n = diag\left\{-\tfrac{1}{2}\ell\beta A_m^n\right\}, \qquad (4.5.11)$$

where

$$\begin{aligned} v_i &= 1+\alpha p+\frac{1}{2}\ell\,(\mu+\beta C_i^n); \quad i=0,1,2,\ldots,M,M+1, \\ v_Y &= 1+\alpha p+\frac{1}{2}\ell(\mu+\sigma), \\ v_Z &= 1+\alpha p+\frac{1}{2}\ell(\mu+\gamma), \\ v_P &= 1-\alpha p-\frac{1}{2}\ell\mu, \\ v_Q &= 1-\alpha p-\frac{1}{2}\ell(\mu+\sigma), \\ v_R &= 1-\alpha p-\frac{1}{2}\ell(\mu+\gamma), \end{aligned} \qquad (4.5.12)$$

$I_{(M+2)}$ is the identity matrix of order $(M+2)$ and $p=\ell/h^2$.

Further research reveals that it is not necessary to compute $\mathbf{A}^{n+1}$, $\mathbf{B}^{n+1}$ and $\mathbf{C}^{n+1}$ simultaneously by solving the linear algebraic system (4.5.1), which is of order $(3N+6)$, on a single processor. It *is* possible after all to compute $\mathbf{A}^{n+1}$, $\mathbf{B}^{n+1}$ and $\mathbf{C}^{n+1}$ in parallel on an architecture with three processors; this is made possible because F_1^n, $X^n Y^n$. F_2^n, Z^n, $X^n Z^n F_1^n$, F_2^n and Y^n commute with X^n, H_2^n, Z^n, H_1^n, $F_1^n Z^n X^n$, H_1^n and H_2^n, respectively.

Equation (4.5.1) may be split to give the following equations

$$\begin{array}{llllll} X^n\,\mathbf{A}^{n+1} & & +H_1^n\,\mathbf{C}^{n+1} & = & P^n\,\mathbf{A}^n + \mathbf{b}\,, \\ F_1^n\,\mathbf{A}^{n+1} & +Y^n\,\mathbf{B}^{n+1} & +F_2^n\,\mathbf{C}^{n+1} & = & Q^n\,\mathbf{B}^n\,, \\ & H_2^n\,\mathbf{B}^{n+1} & +Z^n\,\mathbf{C}^{n+1} & = & H_3^n\,\mathbf{B}^n + R^n\,\mathbf{C}^n\,, \end{array} \qquad (4.5.13)$$

which may be solved for $\mathbf{A}^{n+1}$, $\mathbf{B}^{n+1}$ and $\mathbf{C}^{n+1}$. These vectors may then be obtained using an architecture with three processors on which to implement the following algorithm.

Algorithm :

Processor 1: Solve

$$\begin{aligned}[Y^n Z^n X^n &- H_2^n\,(F_2^n X^n - H_1^n F_1^n)]\,\mathbf{A}^{n+1} = [Y^n Z^n - H_2^n F_2^n]\,P^n\,\mathbf{A}^n \\ &+ [Y^n Z^n - H_2^n F_2^n]\,\mathbf{b} + [H_2^n H_1^n Q^n - Y^n H_1^n H_3^n]\,\mathbf{B}^n - Y^n H_1^n R^n\,\mathbf{C}^n\end{aligned}$$

for $\mathbf{A}^{n+1}$.

Processor 2: Solve

$$\begin{aligned}[X^n\,(Z^n Y^n - F_2^n H_2^n) &+ F_1^n H_1^n H_2^n]\,\mathbf{B}^{n+1} = [X^n\,(Z^n Q^n - F_2^n H_3^n) + F_1^n H_1^n H_3^n]\,\mathbf{B}^n \\ &+ [F_1^n H_1^n - X^n F_2^n]\,R^n\,\mathbf{C}^n - F_1^n Z^n P^n\,\mathbf{A}^n - F_1^n Z^n\,\mathbf{b}\end{aligned}$$

for $\mathbf{B}^{n+1}$.

Processor 3: Solve

$$\begin{aligned}[H_2^n\,(X^n F_2^n - F_1^n H_1^n) &- X^n Y^n Z^n]\,\mathbf{C}^{n+1} = [H_2^n X^n Q^n - X^n Y^n H_3^n]\,\mathbf{B}^n \\ &- H_2^n F_1^n P^n\,\mathbf{A}^n - X^n Y^n R^n\,\mathbf{C}^n - H_2^n F_1^n\,\mathbf{b}\end{aligned}$$

for $\mathbf{C}^{n+1}$.

All the three processors solve a linear algebraic system of order $(M+2)$ at each time step.

4.6 Stability Analysis

The von Neumann and matrix stability analysis methods both failed to give a criterion for the stability of the methods involved in this chapter as shown below.

To discuss the von Neumann method of analysing stability, consider a small error Z_m^n of the form

$$\begin{aligned} Z_{S,m}^n = A_m^n - \widetilde{A_m^n} &= e^{\alpha_1 n\ell}\, e^{i\nu_1 mh}\,, \\ Z_{E,m}^n = B_m^n - \widetilde{B_m^n} &= e^{\alpha_2 n\ell}\, e^{i\nu_2 mh}\,, \\ Z_{I,m}^n = C_m^n - \widetilde{C_m^n} &= e^{\alpha_3 n\ell}\, e^{i\nu_3 mh}\,, \end{aligned}$$

where α_i are complex and $\nu_i\,(i = 1, 2, 3)$ are real and $i = +\sqrt{-1}$.

Substituting $Z_{S,m}^n$ and $Z_{I,m}^n$ into (4.4.10) gives

$$\begin{aligned} -\frac{1}{2}\alpha p\, e^{\alpha_1 (n+1)\ell}\, e^{i\nu_1 (m-1)h} &+ \left[1 + \alpha p + \frac{1}{2}\ell\left(\mu + \beta\, e^{\alpha_3 n\ell}\, e^{i\nu_3 mh}\right)\right] e^{\alpha_1 (n+1)\ell}\, e^{i\nu_1 mh} \\ &- \frac{1}{2}\alpha p\, e^{\alpha_1 (n+1)\ell}\, e^{i\nu_1 (m+1)h} + \underline{\frac{1}{2}\ell\beta\, e^{\alpha_3 (n+1)\ell}\, e^{i\nu_3 mh}\, e^{\alpha_1 n\ell}\, e^{i\nu_1 mh}} \\ &= \frac{1}{2}\alpha p\, e^{\alpha_1 n\ell}\, e^{i\nu_1 (m-1)h} + \left[1 - \alpha p - \frac{1}{2}\ell\mu\right] e^{\alpha_1 n\ell}\, e^{i\nu_1 mh} \\ &+ \frac{1}{2}\alpha p\, e^{\alpha_1 n\ell}\, e^{i\nu_1 (m+1)h} + \ell\mu N\,. \end{aligned}$$

As the underlined term cannot be uncoupled, an explicit expression for $\xi_S = e^{\alpha_1 \ell}$ cannot be found; likewise, it is not possibe to find $\xi_E = e^{\alpha_2 \ell}$ and $\xi_I = e^{\alpha_3 \ell}$ explicitly. Hence the von Neumann method is not appropriate for the stability analysis.

When the matrix stability analysis is applied to equation (4.5.1), the quantity

$$\left\| (W^n)^{-1} M^n \right\| \le 1$$

is required in some norm. That is, stability of the finite-difference approximation is ensured if all the eigenvalues of $(W^n)^{-1} M^n$ are, in absolute value, less than or equal to 1. But W^n and M^n are both block matrices and it is very difficult to find the eigenvalues of $(W^n)^{-1} M^n$. So the matrix stability analysis is not practical either.

Therefore, the maximum principle analysis will be used to discuss the stability of the finite-difference approximations (4.4.10)–(4.4.12), (4.4.22)–(4.4.24), (4.4.34), (4.4.35) and (4.4.36).

Assume that a solution of (4.2.1), (4.2.2) and (4.2.5) exists in the closed region $[CR : 0 \leq x \leq L, 0 \leq t \leq T]$ such that $\partial^4 S/\partial x^4$, $\partial^4 E/\partial x^4$, $\partial^4 I/\partial x^4$, $\partial^2 S/\partial x^2$, $\partial^2 E/\partial x^2$ and $\partial^2 I/\partial x^2$ exist and are bounded in CR.

In order to use the maximum principle analysis to examine convergence and stability, the first equation in (4.2.1) may be written as

$$\begin{aligned} \frac{1}{2}\alpha\,(S_{xx}+S_{xx}) &= S_t - \mu N + \frac{1}{2}(\mu+\beta I)\,S + \frac{1}{2}(\mu+\beta I)\,S \\ &= S_t - \mu N + G_1 + G_2\,, \end{aligned} \qquad (4.6.1)$$

with initial and boundary conditions are as in (4.2.2) and (4.2.5) and where G_1 and G_2 are assumed to be boundedly-differentiable with respect to S and I.

The difference equation to be studied to approximate (4.6.1) is ($\phi = 1/2$)

$$\begin{aligned} \frac{1}{2}\alpha\nabla^2\left(A_m^{n+1}+A_m^n\right) &= \nabla_t A_m^n - \mu N + \frac{1}{2}(\mu+\beta C_m^n)\,A_m^{n+1} \\ &+ \frac{1}{2}\left(\mu+\beta C_m^{n+1}\right)A_m^n\,, \quad n \geq 0\,, \end{aligned} \qquad (4.6.2)$$

where A_m^n is an approximation to S_m^n at the point $(x,t) = (x_m, t_n)$ defined on CR and agrees with $S(x,t)$ on the boundaries and ∇^2 and ∇_t are defined by

$$\begin{aligned} \nabla^2 A_m^n &= \left(A_{m-1}^n - 2A_m^n + A_{m+1}^n\right)/h^2\,, \\ \nabla_t A_m^n &= \left(A_m^{n+1} - A_m^n\right)/\ell\,. \end{aligned} \qquad (4.6.3)$$

It is known that

$$\left.\begin{aligned}
G_1 = G_1\left(x_m, t_{n+1}, S_m^{n+1}, I_m^n\right) &= G_1\left(x_m, t_{n+1}, S_m^n, I_m^n\right) + \left(S_m^{n+1} - S_m^n\right)\overline{\frac{\partial G_1}{\partial S}} \\
&= \tfrac{1}{2}\left(\mu + \beta\, I_m^n\right) S_m^n + \left(S_m^{n+1} - S_m^n\right)\overline{\frac{\partial G_1}{\partial S}}, \\
G_2 = G_2\left(x_m, t_{n+1}, S_m^n, I_m^{n+1}\right) &= G_2\left(x_m, t_{n+1}, S_m^n, I_m^n\right) + \left(I_m^{n+1} - I_m^n\right)\overline{\frac{\partial G_2}{\partial I}} \\
&= \tfrac{1}{2}\left(\mu + \beta\, I_m^n\right) S_m^n + \left(I_m^{n+1} - I_m^n\right)\overline{\frac{\partial G_2}{\partial I}}, \\
\frac{\partial S_m^{n+1}}{\partial t} &= \frac{S_m^{n+1} - S_m^n}{\ell} + \tfrac{1}{2}\,\ell\,\overline{\frac{\partial^2 S_m^{n+1}}{\partial t^2}}, \\
\tfrac{1}{2}\nabla^2\left(S_m^{n+1} + S_m^n\right) &= \tfrac{1}{2}\frac{\partial^2 S_m^{n+1}}{\partial x^2} + \tfrac{1}{2}\frac{\partial^2 S_m^n}{\partial x^2} + \tfrac{1}{24}h^2\left(\overline{\frac{\partial^4 S_m^{n+1}}{\partial x^4}} + \overline{\frac{\partial^4 S_m^n}{\partial x^4}}\right),
\end{aligned}\right\} \quad (4.6.4)$$

where the barred derivatives are evaluated at intermediate argument values as called for by the Mean Value Theorem.

Substituting (4.6.4) into (4.6.1) gives

$$\begin{aligned}
\frac{1}{2}\alpha\nabla^2\left(S_m^{n+1} + S_m^n\right) &= \nabla_t S_m^n - \mu N + \frac{1}{2}\left(\mu + \beta\, I_m^n\right) S_m^{n+1} + \frac{1}{2}\left(\mu + \beta\, I_m^{n+1}\right) S_m^n \\
&+ \left\{\frac{1}{24}h^2\left(\overline{\frac{\partial^4 S_m^{n+1}}{\partial x^4}} + \overline{\frac{\partial^4 S_m^n}{\partial x^4}}\right) + \frac{1}{2}\ell\,\overline{\frac{\partial^2 S_m^{n+1}}{\partial t^2}} - \left(S_m^{n+1} - S_m^n\right)\overline{\frac{\partial G_1}{\partial S}}\right. \\
&- \left.\left(I_m^{n+1} - I_m^n\right)\overline{\frac{\partial G_2}{\partial I}}\right\}. \qquad (4.6.5)
\end{aligned}$$

The assumption on S above requires the boundedness of all the derivatives appearing inside the bracket along with $(S_m^{n+1} - S_m^n)$ and $(I_m^{n+1} - I_m^n)$ in the region $0 \leq x \leq L, 0 \leq t \leq T$. Hence, in this region,

$$\begin{aligned}
\frac{1}{2}\alpha\nabla^2\left(S_m^{n+1} + S_m^n\right) &= \nabla_t S_m^n - \mu N + \frac{1}{2}\left(\mu + \beta\, I_m^n\right) S_m^{n+1} \\
&+ \frac{1}{2}\left(\mu + \beta\, I_m^{n+1}\right) S_m^n + g_m^n \qquad (4.6.6)
\end{aligned}$$

with

$$g_m^n = O(h^2 + \ell). \qquad (4.6.7)$$

Now, let

$$\left.\begin{aligned} Z_{1_m}^n &= S_m^n - A_m^n \\ Z_{2_m}^n &= E_m^n - B_m^n \\ Z_{3_m}^n &= I_m^n - C_m^n \end{aligned}\right\} . \qquad (4.6.8)$$

Then, if (4.6.2) is subtracted from (4.6.6),

$$\begin{aligned} \frac{1}{2}\alpha\nabla^2\left(Z_{1_m}^{n+1} + Z_{1_m}^n\right) &= \nabla_t Z_{1_m}^n + \frac{1}{2}\left(\mu + \beta I_m^n\right) S_m^{n+1} - \frac{1}{2}\left(\mu + \beta C_m^n\right) A_m^{n+1} \\ &+ \frac{1}{2}\left(\mu + \beta I_m^{n+1}\right) S_m^n - \frac{1}{2}\left(\mu + \beta C_m^{n+1}\right) A_m^n + g_m^n \qquad (4.6.9) \end{aligned}$$

with $Z_{1_m}^n$ and $Z_{1_m}^{n+1}$ vanishing on the boundary. As

$$\begin{aligned} G_1\left(x_m, t_{n+1}, S_m^{n+1}, I_m^n\right) &= G_1\left(x_m, t_{n+1}, A_m^{n+1}, C_m^n\right) + \overline{\frac{\partial G_1}{\partial S}}\left(S_m^{n+1} - A_m^{n+1}\right) \\ &+ \overline{\frac{\partial G_1}{\partial I}}\left(I_m^n - C_m^n\right) \qquad (4.6.10) \end{aligned}$$

and

$$\begin{aligned} G_2\left(x_m, t_{n+1}, S_m^n, I_m^{n+1}\right) &= G_2\left(x_m, t_{n+1}, A_m^n, C_m^{n+1}\right) + \overline{\frac{\partial G_2}{\partial S}}\left(S_m^n - A_m^n\right) \\ &+ \overline{\frac{\partial G_2}{\partial I}}\left(I_m^{n+1} - C_m^{n+1}\right), \qquad (4.6.11) \end{aligned}$$

it follows that equation (4.6.9) may be written as

$$\begin{aligned} \frac{1}{2}\alpha\nabla^2\left(Z_{1_m}^{n+1} + Z_{1_m}^n\right) &= \nabla_t Z_{1_m}^n + \frac{1}{2}\overline{\frac{\partial G_1}{\partial S}} Z_{1_m}^{n+1} + \frac{1}{2}\overline{\frac{\partial G_1}{\partial I}} Z_{3_m}^n + \frac{1}{2}\overline{\frac{\partial G_2}{\partial S}} Z_{1_m}^n \\ &+ \frac{1}{2}\overline{\frac{\partial G_2}{\partial I}} Z_{3_m}^{n+1} + g_m^n . \qquad (4.6.12) \end{aligned}$$

Assume that $Z_{3_m}^n$ and $Z_{3_m}^{n+1}$ are bounded. Then equation (4.6.12) may be written in the form

$$\begin{aligned} \frac{1}{2}\alpha\nabla^2\left(Z_{1_m}^{n+1} + Z_{1_m}^n\right) &= \nabla_t Z_{1_m}^n + \frac{1}{2}M_S\left(Z_{1_m}^{n+1} + Z_{1_m}^n\right) \\ &+ \frac{1}{2}M_I\left(Z_{3_m}^{n+1} + Z_{3_m}^n\right) + g_m^n \qquad (4.6.13) \end{aligned}$$

where $M_S = \max\left\{\overline{\frac{\partial G_1}{\partial S}}, \overline{\frac{\partial G_2}{\partial S}}\right\}$ and $M_I = \max\left\{\overline{\frac{\partial G_1}{\partial I}}, \overline{\frac{\partial G_2}{\partial I}}\right\}$.

It is known that g_m^n is bounded and $Z_{1_m}^n$ and $Z_{1_m}^{n+1}$ vanish on the boundary. Hence, by Theorem 1.4.1, A_m^n and A_m^{n+1} converge uniformly to S_m^n and S_m^{n+1}.

Now, the second equation in (4.2.1) may be written as

$$\begin{aligned}\frac{1}{2}\alpha\,(E_{xx}+E_{xx}) &= E_t - \frac{1}{2}\beta\, I\, S - \frac{1}{2}\beta\, I\, S + \frac{1}{2}(\mu+\sigma)\,E + \frac{1}{2}(\mu+\sigma)\,E \\ &= E_t - G_3 - G_4 + G_5 + \frac{1}{2}(\mu+\sigma)\,E\,, \qquad (4.6.14)\end{aligned}$$

with initial and boundary conditions as given in (4.2.2) and (4.2.5) and where G_3, G_4 and G_5 are assumed to be boundedly-differentiable with respect to S, E and I. The difference equation to approximate (4.6.14) is ($\phi = 1/2$)

$$\begin{aligned}\frac{1}{2}\alpha\,\nabla^2\left(B_m^{n+1}+B_m^n\right) &= \nabla_t B_m^n - \frac{1}{2}\beta\, C_m^n\, A_m^{n+1} - \frac{1}{2}\beta\, C_m^{n+1}\, A_m^n + \frac{1}{2}(\mu+\sigma)\,B_m^{n+1} \\ &+ \frac{1}{2}(\mu+\sigma)\,B_m^n\,, \quad n \geq 0\,, \qquad (4.6.15)\end{aligned}$$

where ∇^2 and ∇_t are defined in (4.6.3).

It is easy to see that

$$\left.\begin{aligned}G_3 = G_3(x_m, t_{n+1}, S_m^{n+1}, I_m^n) &= G_3(x_m, t_{n+1}, S_m^n, I_m^n) + (S_m^{n+1} - S_m^n)\,\overline{\frac{\partial G_3}{\partial S}} \\ &= \tfrac{1}{2}\beta\, I_m^n\, S_m^n + (S_m^{n+1} - S_m^n)\,\overline{\frac{\partial G_3}{\partial S}}\,, \\ G_4 = G_4(x_m, t_{n+1}, S_m^n, I_m^{n+1}) &= G_4(x_m, t_{n+1}, S_m^n, I_m^n) + (I_m^{n+1} - I_m^n)\,\overline{\frac{\partial G_4}{\partial I}} \\ &= \tfrac{1}{2}\beta\, I_m^n\, S_m^n + (I_m^{n+1} - I_m^n)\,\overline{\frac{\partial G_4}{\partial I}}\,; \\ G_5 = G_5(x_m, t_{n+1}, E_m^{n+1}) &= G_5(x_m, t_{n+1}, E_m^n) + (E_m^{n+1} - E_m^n)\,\overline{\frac{\partial G_5}{\partial E}} \\ &= \tfrac{1}{2}(\mu+\sigma)\,E_m^n + (E_m^{n+1} - E_m^n)\,\overline{\frac{\partial G_5}{\partial E}}\,, \\ \frac{\partial E_m^{n+1}}{\partial t} &= \frac{E_m^{n+1} - E_m^n}{\ell} + \tfrac{1}{2}\,\ell\,\overline{\frac{\partial^2 E_m^{n+1}}{\partial t^2}}\,, \\ \tfrac{1}{2}\nabla^2(E_m^{n+1} + E_m^n) &= \tfrac{1}{2}\frac{\partial^2 E_m^{n+1}}{\partial x^2} + \tfrac{1}{2}\frac{\partial^2 E_m^n}{\partial x^2} + \tfrac{1}{24}h^2\left(\overline{\frac{\partial^4 E_m^{n+1}}{\partial x^4}} + \overline{\frac{\partial^4 E_m^n}{\partial x^4}}\right).\end{aligned}\right\} \qquad (4.6.16)$$

where the barred derivatives are evaluated at intermediate argument values as called for by the Mean Value Theorem.

Substituting (4.6.16) into equation (4.6.14) gives

$$\begin{aligned}
\frac{1}{2}\alpha\nabla^2\left(E_m^{n+1}+E_m^n\right) &= \nabla_t E_m^n - \frac{1}{2}\beta\, I_m^n\, S_m^{n+1} - \frac{1}{2}\beta\, I_m^{n+1}\, S_m^n + \frac{1}{2}(\mu+\sigma)\, E_m^{n+1} \\
&+ \frac{1}{2}(\mu+\sigma)\, E_m^n + \left\{ \frac{1}{24}h^2\left(\frac{\overline{\partial^4 E_m^{n+1}}}{\partial x^4} + \frac{\overline{\partial^4 E_m^n}}{\partial x^4}\right) + \frac{1}{2}\ell\frac{\overline{\partial^2 E_m^{n+1}}}{\partial t^2}\right. \\
&+ \left(S_m^{n+1}-S_m^n\right)\overline{\frac{\partial G_3}{\partial S}} + \left(I_m^{n+1}-I_m^n\right)\overline{\frac{\partial G_4}{\partial I}} \\
&- \left.\left(E_m^{n+1}-E_m^n\right)\overline{\frac{\partial G_5}{\partial E}}\right\}. \qquad (4.6.17)
\end{aligned}$$

The assumption on E requires the boundedness of all the derivatives appearing inside the bracket along with $(S_m^{n+1}-S_m^n)$, $(E_m^{n+1}-E_m^n)$ and $(I_m^{n+1}-I_m^n)$ in the region $0 \le x \le L, 0 \le t \le T$. Hence, in this region,

$$\begin{aligned}
\frac{1}{2}\alpha\nabla^2\left(E_m^{n+1}+E_m^n\right) &= \nabla_t E_m^n - \frac{1}{2}\beta\, I_m^n\, S_m^{n+1} - \frac{1}{2}\beta\, I_m^{n+1}\, S_m^n + \frac{1}{2}(\mu+\sigma)E_m^{n+1} \\
&+ \frac{1}{2}(\mu+\sigma)\, E_m^n + g_m^n\,, \qquad (4.6.18)
\end{aligned}$$

with

$$g_m^n = O(h^2+\ell)\,. \qquad (4.6.19)$$

If equation (4.6.15) is subtracted from equation (4.6.18) and using the definitions in (4.6.8) for the truncation errors,

$$\begin{aligned}
\frac{1}{2}\alpha\nabla^2\left(Z_{2_m}^{n+1}+Z_{2_m}^n\right) &= \nabla_t Z_{2_m}^n - \frac{1}{2}\beta\, I_m^n\, S_m^{n+1} + \frac{1}{2}\beta\, C_m^n\, A_m^{n+1} \\
&- \frac{1}{2}\beta\, I_m^{n+1}\, S_m^n + \frac{1}{2}\beta\, C_m^{n+1}\, A_m^n + \frac{1}{2}(\mu+\sigma)\, Z_{2_m}^{n+1} \\
&+ \frac{1}{2}(\mu+\sigma)\, Z_{2_m}^n + g_m^n\,, \qquad (4.6.20)
\end{aligned}$$

with $Z_{2_m}^n$ and $Z_{2_m}^{n+1}$ vanishing on the boundary. As G_3 and G_4 can be written as in the form of (4.6.10) and (4.6.11), if follows that equation (4.6.20) becomes

$$\frac{1}{2}\alpha\nabla^2\left(Z_{2_m}^{n+1}+Z_{2_m}^n\right) = \nabla_t Z_{2_m}^n - \frac{1}{2}\overline{\frac{\partial G_3}{\partial S}}\, Z_{1_m}^{n+1} - \frac{1}{2}\overline{\frac{\partial G_3}{\partial I}}\, Z_{3_m}^n - \frac{1}{2}\overline{\frac{\partial G_4}{\partial S}}\, Z_{1_m}^n$$

$$- \frac{1}{2}\overline{\frac{\partial G_4}{\partial I}}\, Z_{3_m}^{n+1} + \frac{1}{2}(\mu+\sigma)\, Z_{2_m}^{n+1} + \frac{1}{2}(\mu+\sigma)\, Z_{2_m}^{n} + g_m^n \,. \qquad (4.6.21)$$

Let $M_S = \max\left\{\overline{\frac{\partial G_3}{\partial S}}, \overline{\frac{\partial G_4}{\partial S}}\right\}$ and $M_I = \max\left\{\overline{\frac{\partial G_3}{\partial I}}, \overline{\frac{\partial G_4}{\partial I}}\right\}$. Then equation (4.6.21) may be written in the form

$$\begin{aligned} \frac{1}{2}\alpha\nabla^2\left(Z_{2_m}^{n+1} + Z_{2_m}^{n}\right) &= \nabla_t Z_{2_m}^n - \frac{1}{2}M_S\left(Z_{1_m}^{n+1} + Z_{1_m}^{n}\right) - \frac{1}{2}M_I\left(Z_{3_m}^{n+1} + Z_{3_m}^{n}\right) \\ &+ \frac{1}{2}(\mu+\sigma)\, Z_{2_m}^{n+1} + \frac{1}{2}(\mu+\sigma)\, Z_{2_m}^{n} + g_m^n \end{aligned} \qquad (4.6.22)$$

Assume that $Z_{1_m}^n$, $Z_{1_m}^{n+1}$, $Z_{3_m}^n$ and $Z_{3_m}^{n+1}$ are bounded. Since $Z_{2_m}^n$ and $Z_{2_m}^{n+1}$ vanish on the boundary, it follows, by Theorem 1.4.1, that B_m^n and B_m^{n+1} converge to E_m^n and E_m^{n+1} uniformly.

The third equation in (4.2.1) can be written in the form

$$\begin{aligned} \frac{1}{2}\alpha\,(I_{xx} + I_{xx}) &= I_t - \frac{1}{2}\sigma E - \frac{1}{2}\sigma E + \frac{1}{2}(\mu+\gamma)I + \frac{1}{2}(\mu+\gamma)I \\ &= I_t - G_6 - \frac{1}{2}\sigma E + G_7 + \frac{1}{2}(\mu+\gamma)I\,, \end{aligned} \qquad (4.6.23)$$

with initial and boundary conditions as given in (4.2.2) and (4.2.5) and where, as before, G_6 and G_7 are assumed to be boundedly-differentiable with respect to E and I, respectively. The difference equation to approximate (4.6.23) is ($\phi = 1/2$)

$$\begin{aligned} \frac{1}{2}\alpha\nabla^2\left(C_m^{n+1} + C_m^n\right) &= \nabla_t C_m^n - \frac{1}{2}\sigma B_m^n - \frac{1}{2}\sigma B_m^{n+1} + \frac{1}{2}(\mu+\gamma)C_m^{n+1} \\ &+ \frac{1}{2}(\mu+\gamma)C_m^n\,, \quad n \geq 0\,, \end{aligned} \qquad (4.6.24)$$

where ∇^2 and ∇_t are as in (4.6.3).

It is easy to show that

$$\left.\begin{aligned}
G_6 = G_6\,(x_m, t_{n+1}, E_m^{n+1}) &= G_6\,(x_m, t_{n+1}, E_m^n) + (E_m^{n+1} - E_m^n)\,\overline{\tfrac{\partial G_6}{\partial E}} \\
&= \tfrac{1}{2}\,\sigma\,E_m^n + (E_m^{n+1} - E_m^n)\,\overline{\tfrac{\partial G_6}{\partial E}}\,, \\
G_7 = G_7\,(x_m, t_{n+1}, I_m^{n+1}) &= G_7\,(x_m, t_{n+1}, I_m^n) + (I_m^{n+1} - I_m^n)\,\overline{\tfrac{\partial G_7}{\partial I}} \\
&= \tfrac{1}{2}\,(\mu + \gamma)\,I_m^n + (I_m^{n+1} - I_m^n)\,\overline{\tfrac{\partial G_7}{\partial I}}\,, \\
\tfrac{\partial I_m^{n+1}}{\partial t} &= \tfrac{I_m^{n+1} - I_m^n}{\ell} + \tfrac{1}{2}\,\ell\,\overline{\tfrac{\partial^2 I_m^{n+1}}{\partial t^2}}\,, \\
\tfrac{1}{2}\,\nabla^2\,(I_m^{n+1} + I_m^n) &= \tfrac{1}{2}\,\tfrac{\partial^2 I_m^{n+1}}{\partial x^2} + \tfrac{1}{2}\,\tfrac{\partial^2 I_m^n}{\partial x^2} + \tfrac{1}{24}\,h^2\left(\overline{\tfrac{\partial^4 I_m^{n+1}}{\partial x^4}} + \overline{\tfrac{\partial^4 I_m^n}{\partial x^4}}\right)\,,
\end{aligned}\right\} \qquad (4.6.25)$$

where the barred derivatives are evaluated at intermediate argument values as called for by the Mean Value Theorem.

Substituting (4.6.25) into equation (4.6.23) gives

$$\begin{aligned}
\frac{1}{2}\,\alpha\,\nabla^2\left(I_m^{n+1} + I_m^n\right) &= \nabla_t I_m^n - \frac{1}{2}\,\sigma\,E_m^n - \frac{1}{2}\,\sigma\,E_m^{n+1} + \frac{1}{2}\,(\mu + \gamma)\,I_m^{n+1} + \frac{1}{2}\,(\mu + \gamma)\,I_m^n \\
&+ \left\{\frac{1}{24}\,h^2\left(\overline{\frac{\partial^4 I_m^{n+1}}{\partial x^4}} + \overline{\frac{\partial^4 I_m^n}{\partial x^4}}\right) + \frac{1}{2}\,\ell\,\overline{\frac{\partial^2 I_m^{n+1}}{\partial t^2}} - \left(E_m^{n+1} - E_m^n\right)\overline{\frac{\partial G_6}{\partial E}}\right. \\
&- \left.\left(I_m^{n+1} - I_m^n\right)\overline{\frac{\partial G_7}{\partial I}}\right\}. \qquad (4.6.26)
\end{aligned}$$

The assumption on I requires the boundedness of all the derivatives appearing inside the bracket along with $(E_m^{n+1} - E_m^n)$ and $(I_m^{n+1} - I_m^n)$ in the region $0 \le x \le L, 0 \le t \le T$. Hence, in this region,

$$\begin{aligned}
\frac{1}{2}\,\alpha\,\nabla^2\left(I_m^{n+1} + I_m^n\right) &= \nabla_t I_m^n - \frac{1}{2}\,\sigma\,E_m^n - \frac{1}{2}\,\sigma\,E_m^{n+1} + \frac{1}{2}\,(\mu + \gamma)\,I_m^{n+1} \\
&+ \frac{1}{2}\,(\mu + \gamma)\,I_m^n + g_m^n\,, \qquad (4.6.27)
\end{aligned}$$

with, again,

$$g_m^n = O(h^2 + \ell)\,. \qquad (4.6.28)$$

If equation (4.6.24) is subtracted from equation (4.6.27) and using the definitions in (4.6.8) for the truncaion errors,

$$\begin{aligned}\frac{1}{2}\alpha\nabla^2\left(Z_{3_m}^{n+1}+Z_{3_m}^{n}\right) &= \nabla_t Z_{3_m}^{n}-\frac{1}{2}\sigma Z_{2_m}^{n}-\frac{1}{2}\sigma Z_{2_m}^{n+1}\\ &+ \frac{1}{2}(\mu+\gamma)Z_{3_m}^{n+1}+\frac{1}{2}(\mu+\gamma)Z_{3_m}^{n}+g_m^n\,, \qquad (4.6.29)\end{aligned}$$

with $Z_{3_m}^{n}$ and $Z_{3_m}^{n+1}$ vanishing on the boundary.

Assume that $Z_{2_m}^{n}$ and $Z_{2_m}^{n+1}$ are bounded. Moreover, $Z_{3_m}^{n}$ and $Z_{3_m}^{n+1}$ vanish on the boundary. Therefore, by Theorem 1.4.1, C_m^n and C_m^{n+1} converge uniformly to I_m^n and I_m^{n+1}.

4.7 Numerical Results and Discussions

To test the second-order methods (4.4.10)-(4.4.12), (4.4.22)-(4.4.24) and (4.4.34)-(4.4.36) for susceptibles, exposed and infectives, respectively, the initial/boundary-value problem

$$\begin{aligned}S_t &= \mu N-(\mu+\beta I)S+\alpha S_{xx}\\ E_t &= \beta I S-(\mu+\sigma)E+\alpha E_{xx} \qquad (4.7.1)\\ I_t &= \sigma E-(\mu+\gamma)E+\alpha I_{xx}\end{aligned}$$

with the initial conditions

$$S(x,0)=S^0(x),\quad E(x,0)=E^0(x),\quad I(x,0)=I^0(x)\,;\,0\le x\le 1 \qquad (4.7.2)$$

and the boundary conditions

$$\begin{aligned}S_x(0,t) &= E_x(0,t) = I_x(0,t) = 0; \quad t>0,\\ S_x(1,t) &= E_x(1,t) = I_x(1,t) = 0; \quad t>0,\end{aligned} \qquad (4.7.3)$$

was solved using the set of parameters given in (3.4.4) for N, μ, σ and γ with the infection rate, β, chosen to be $\beta = 5\times10^{-4}$ and the diffusion rate, α, to be $\alpha = 0.01$.

In the following numerical experiments the total numbers of susceptibles, exposed and infected individuals are taken to be 1.25×10^7, 5×10^4 and 3×10^4, respectively. The ways in which each is distributed over the interval $0 \le x \le 1$ give the functions $S^0(x)$, $E^0(x)$ and $I^0(x)$ in (4.7.2).

- **Experiment A**

In this experiment, hat-shaped initial distributions are used for S, E and I. Taking $h = 0.025$ so that $M = 39$, giving the discretization $x_i\,(i = 0, 1, \ldots, 40)$ of the interval $0 \le x \le 1$, the initial conditions in (4.7.2) are distributed as follows (see Figures 4.1 and 4.2)

$$S(x_i, 0) = \begin{cases} 31250\, i & , \quad 0 \le i \le \frac{M+1}{2} \\ 31250\,(M+1-i) & , \quad \frac{M+1}{2} < i \le M+1\,, \end{cases}$$

$$E(x_i, 0) = \begin{cases} 125\, i & , \quad 0 \le i \le \frac{M+1}{2} \\ 125\,(M+1-i) & , \quad \frac{M+1}{2} < i \le M+1\,, \end{cases}$$

$$I(x_i, 0) = \begin{cases} 75\, i & , \quad 0 \le i \le \frac{M+1}{2} \\ 75\,(M+1-i) & , \quad \frac{M+1}{2} < i \le M+1\,. \end{cases}$$

Initially, the maximum value of each class of individuals is concentrated at the middle of the interval ($0 \le x \le 1$) and the numbers decrease linearly to zero at the boundaries $x = 0$ and $x = 1$.

As time is increased, the number of susceptibles is decreased whereas the numbers of both exposed and infectious individuals are increased until the time $t = 0.09$ after which the number of susceptibles becomes less than the number of exposed individuals, near the middle of the interval, see Figure 4.3. This reveals the dynamic behaviour of measles and is as would be expected.

Figure 4.3 shows the distribution of susceptibles, exposed and infectious individuals at time $t = 0.1$. Figures 4.4–4.7, respectively, give three-dimensional plots of susceptible, exposed, infectious and recovered individuals for $0 \le x \le 1$ and $0 \le t \le 0.1$.

The profiles in Figure 4.3 can be seen clearly in Figures 4.4–4.6 by locating the plane $t = 0.1$ in each figure.

As the diffusion rate, α, increases, the dynamic behaviour of measles changes as shown in Figures 4.8–4.13; as the diffusion rate, α, increases, the number of susceptibles becomes larger than both the numbers of exposed and infected individuals and both the exposed and infected individuals spread on the x-axis.

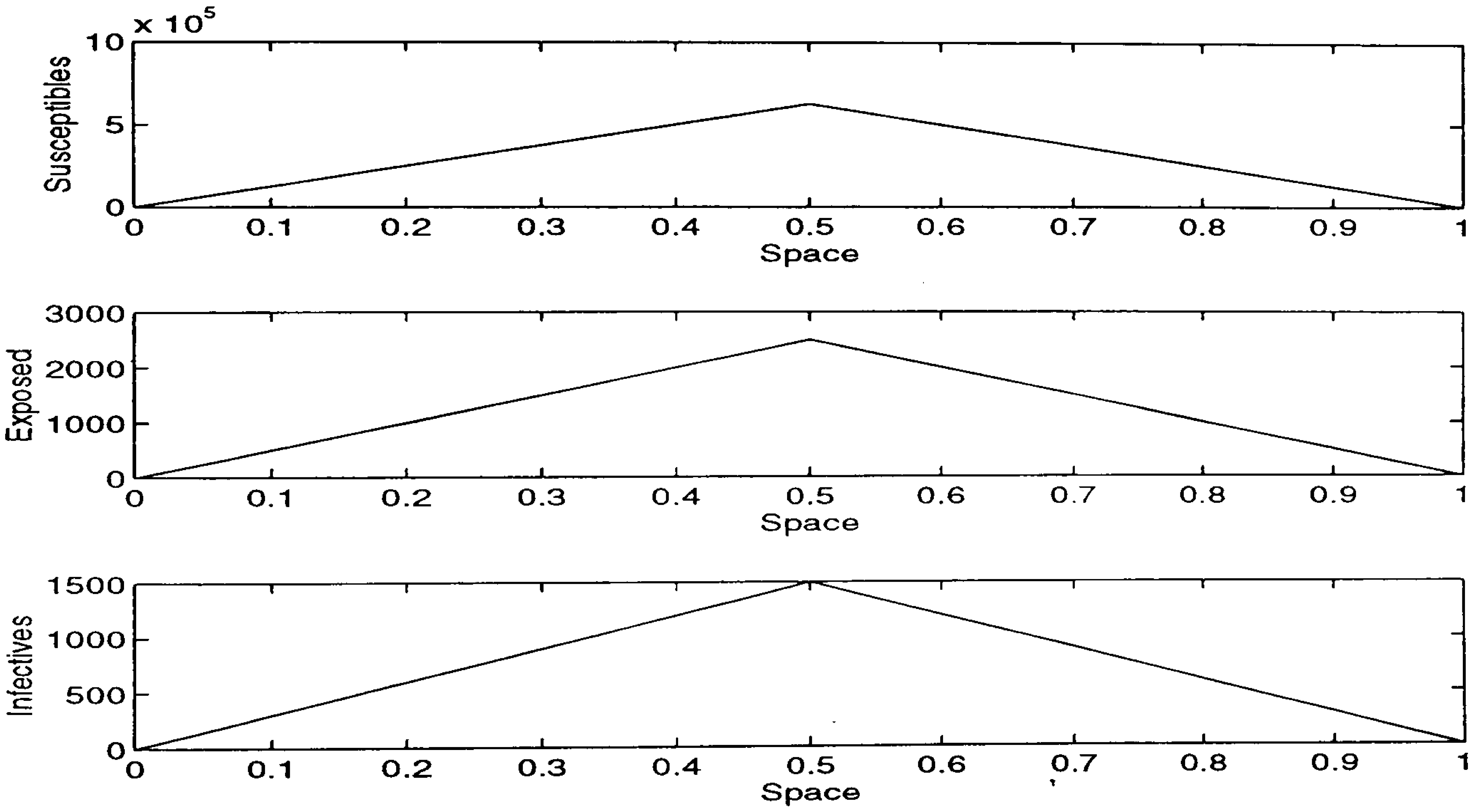

Figure 4.1: *Experiment A, initial distributions of susceptibles, exposed and infectives.*

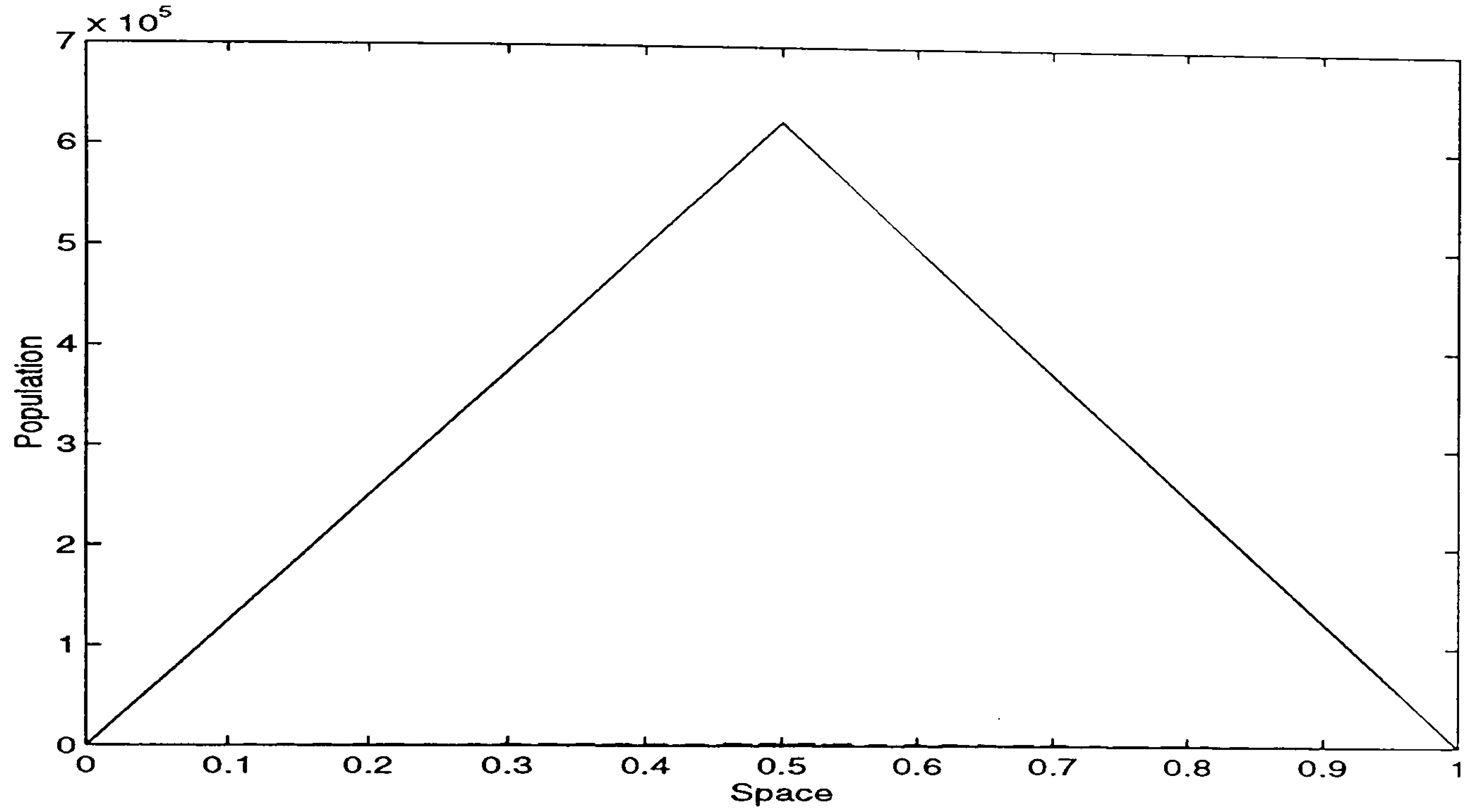

Figure 4.2: *Experiment A, initial distributions of susceptibles(—), exposed(— —) and infectives(—.).*

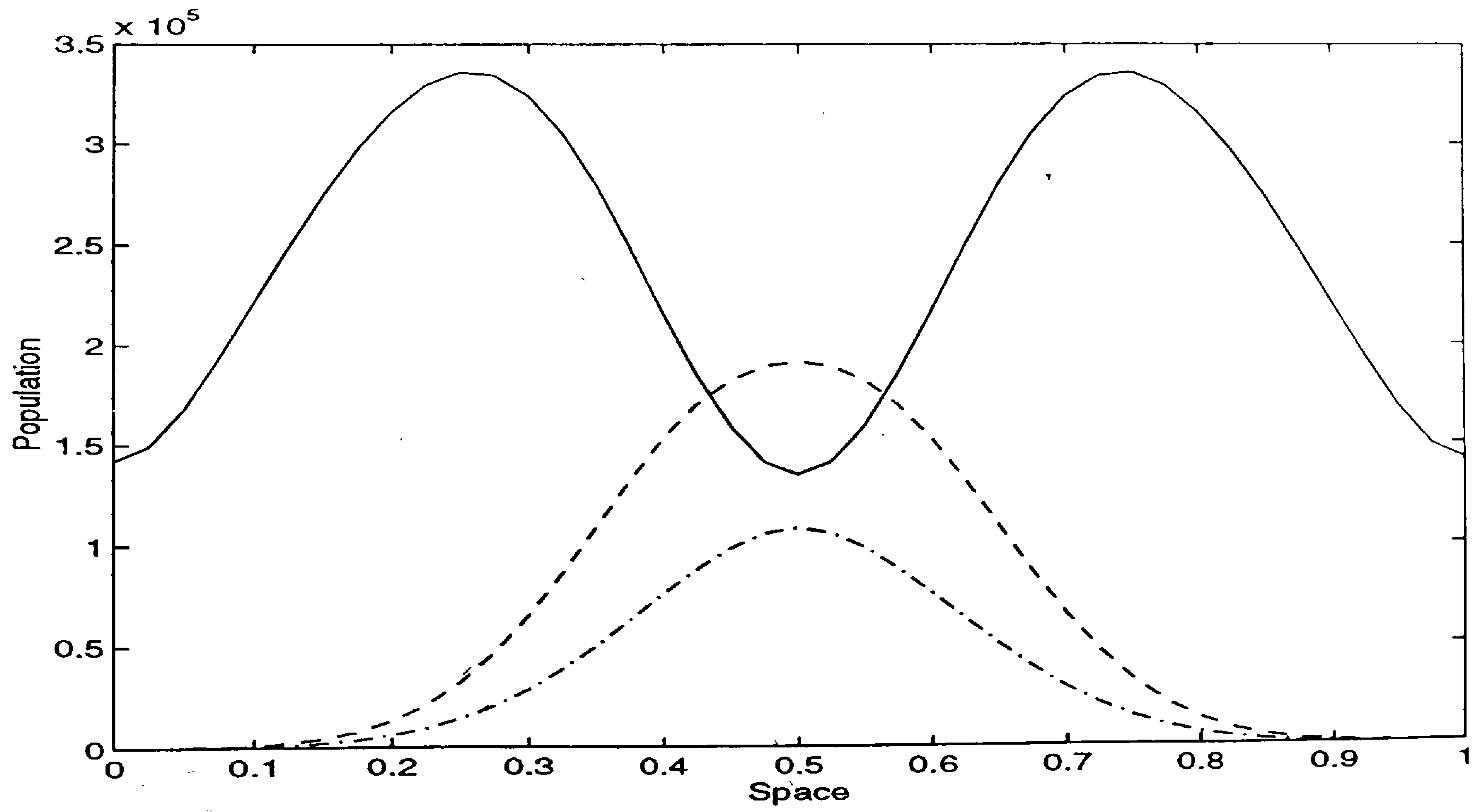

Figure 4.3: *Experiment A, dynamics of measles at time* $t = 0.1$, $\alpha = 0.01$, $\ell = 0.001$ *and* $h = 0.025$; *susceptibles (—), exposed (— —) and infectives (—.).*

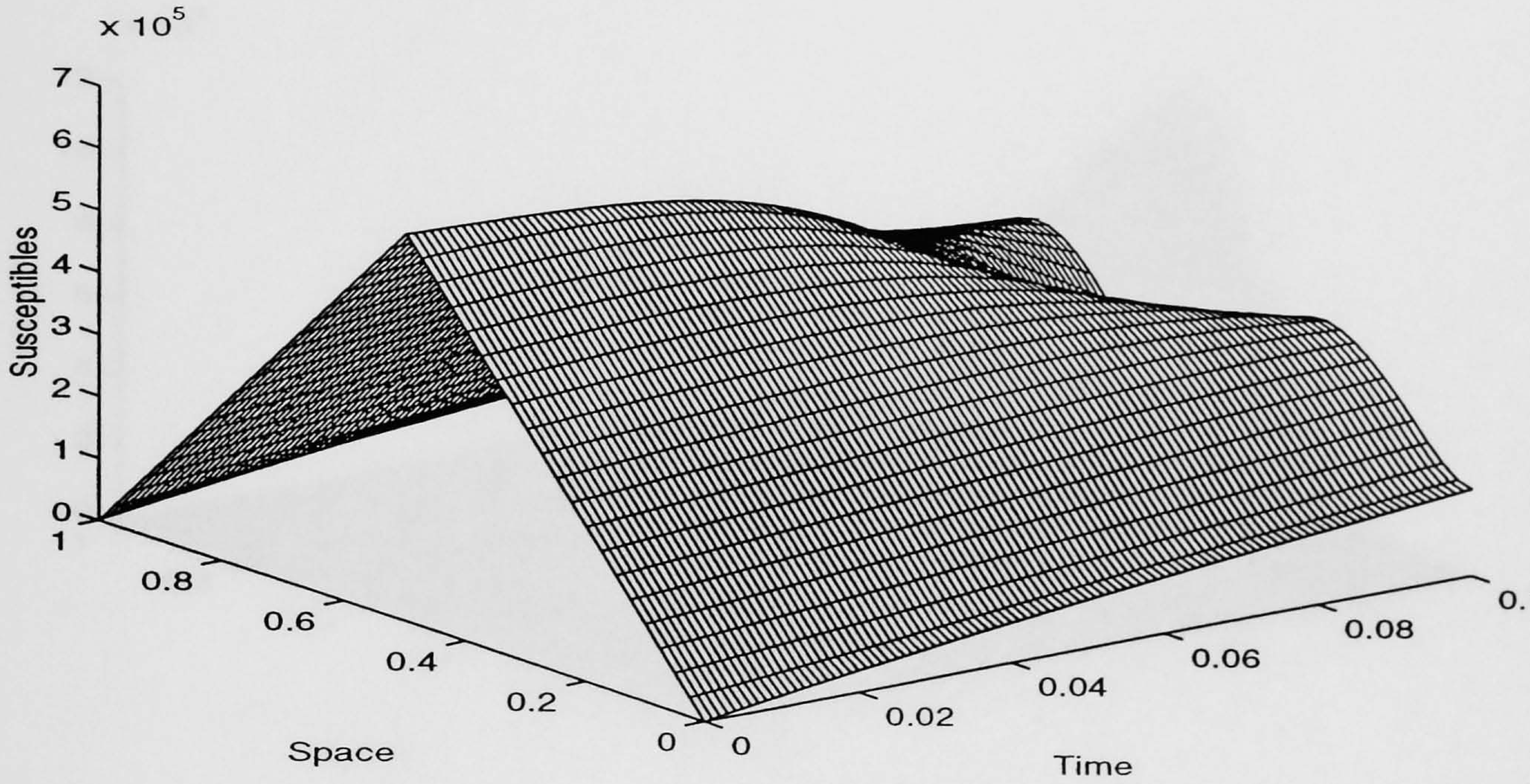

Figure 4.4: *Experiment A, three-dimensional distribution of susceptibles;* $\ell = 0.001$ *and* $h = 0.025$.

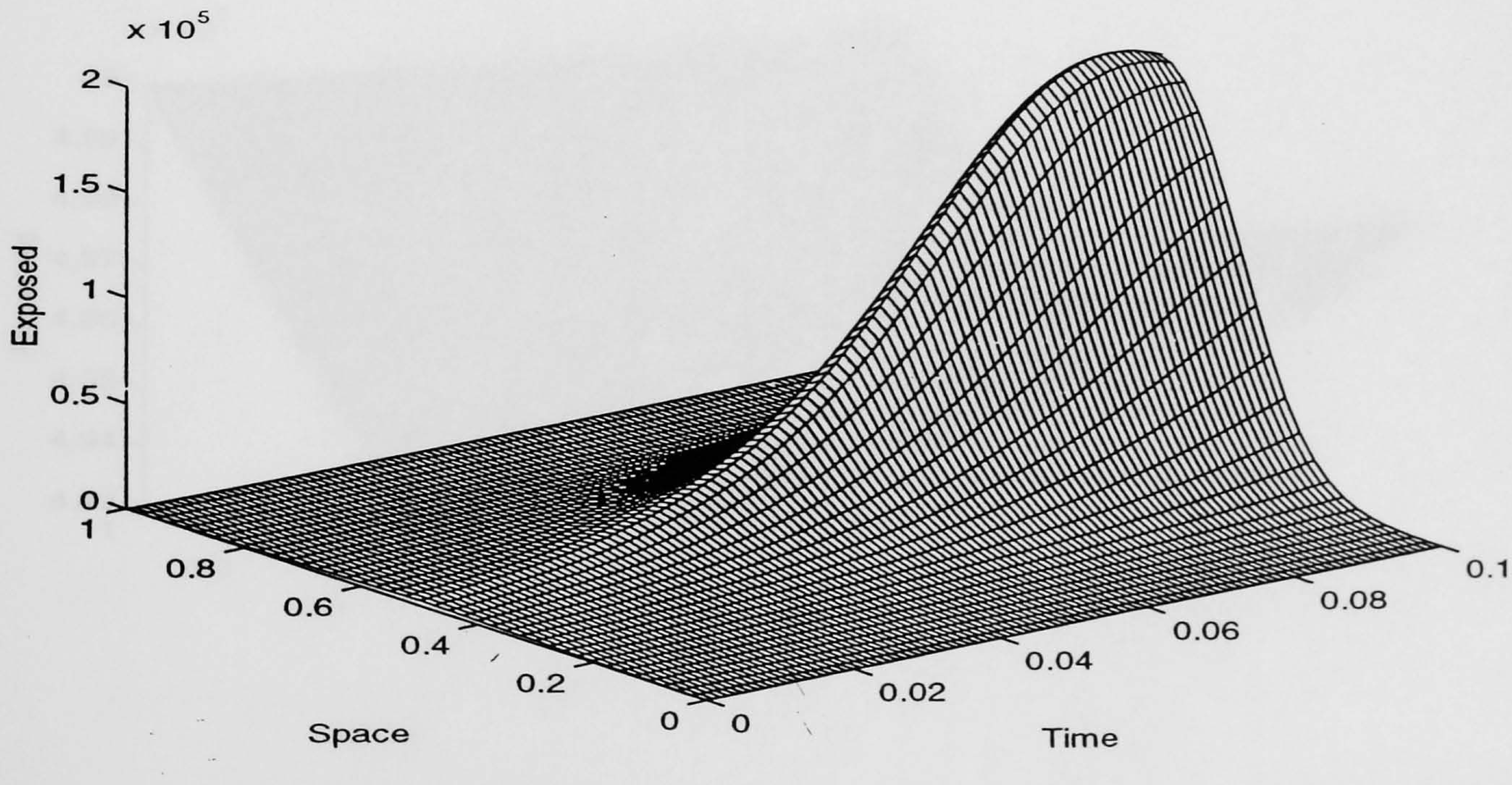

Figure 4.5: *Experiment A, three-dimensional distribution of exposed individuals;* $\ell = 0.001$ *and* $h = 0.025$.

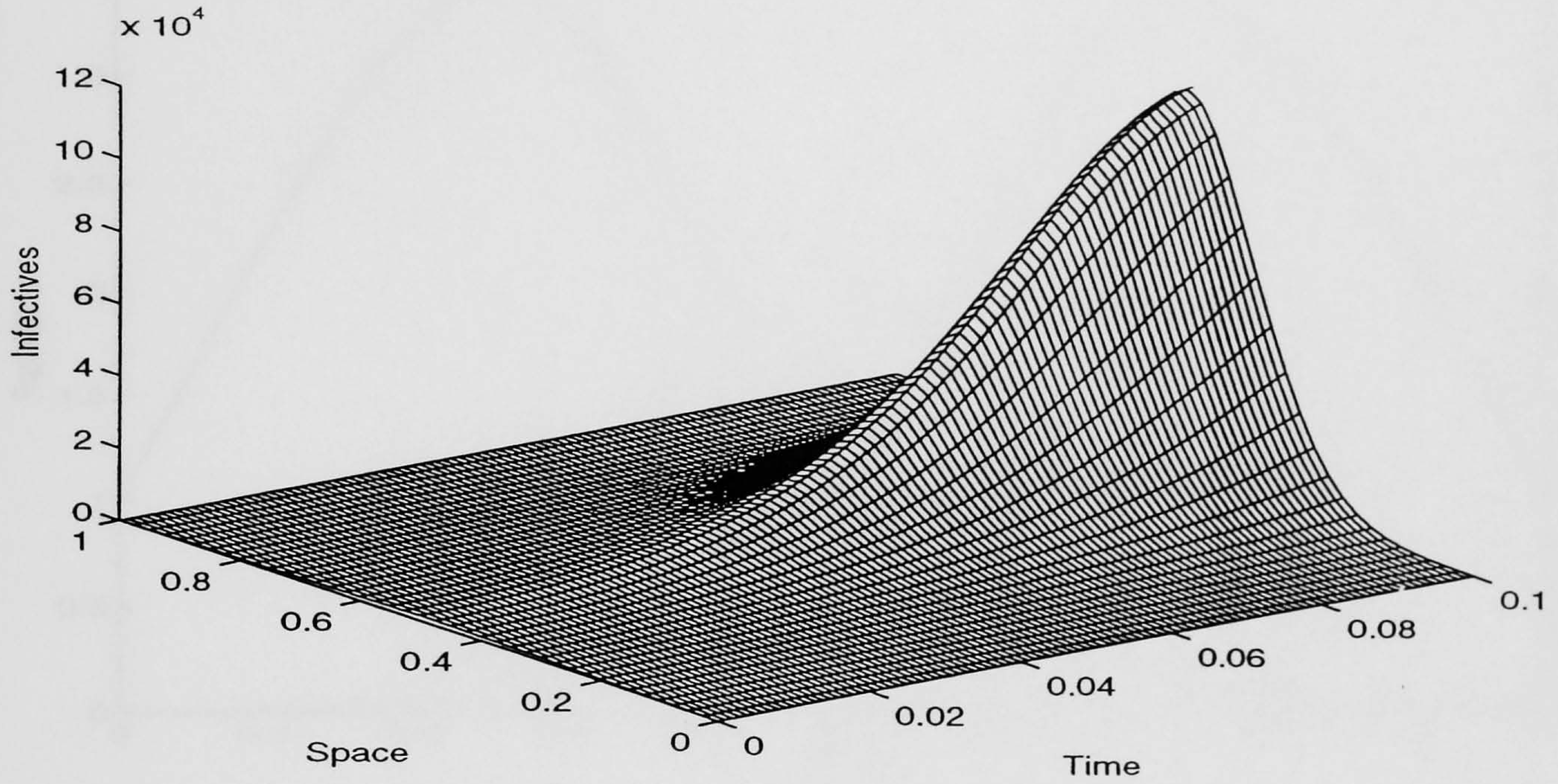

Figure 4.6: *Experiment A, three-dimensional distribution of infectives;* $\ell = 0.001$ *and* $h = 0.025$.

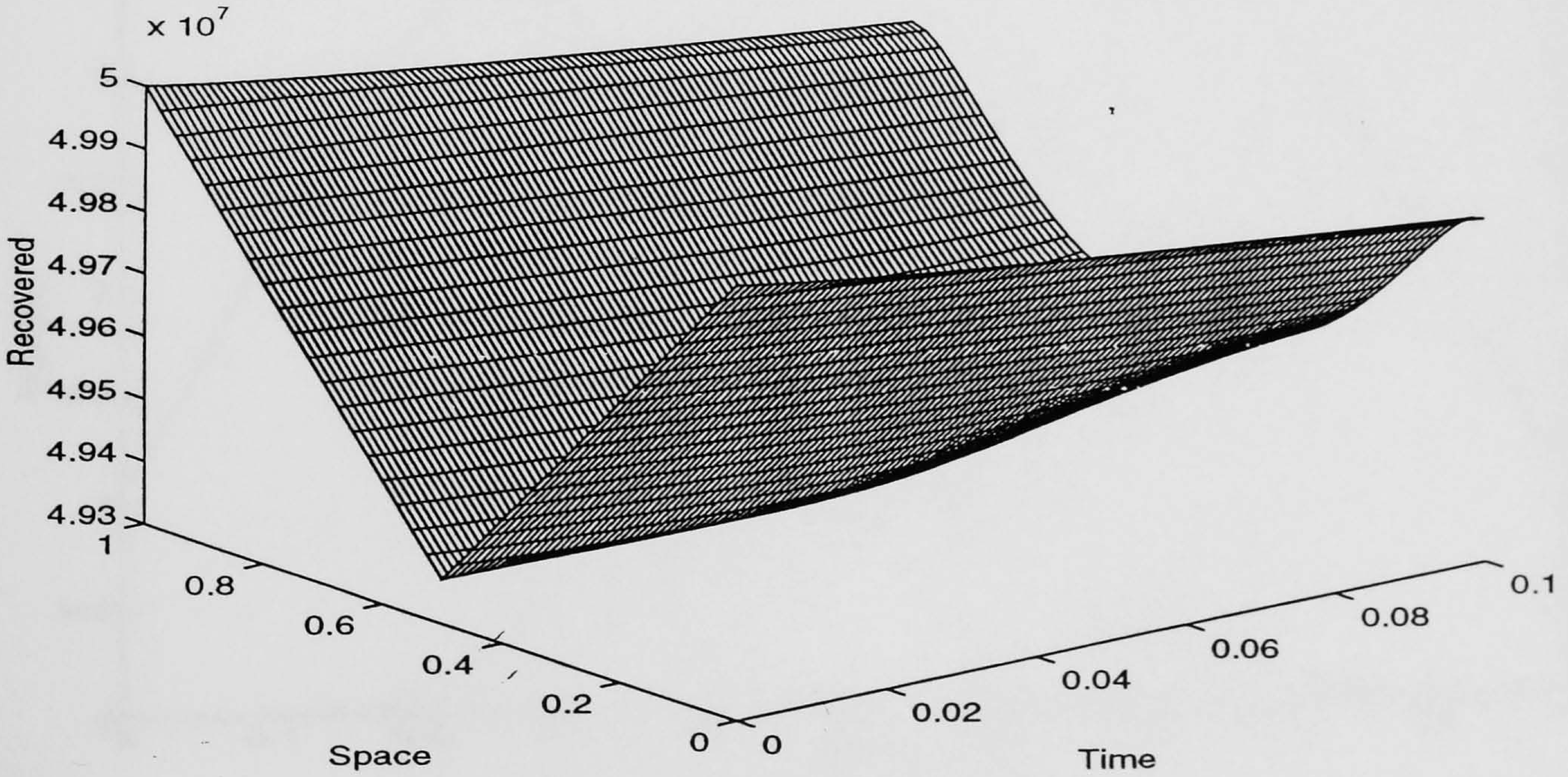

Figure 4.7: *Experiment A, three-dimensional distribution of recovered individuals;* $\ell = 0.001$ *and* $h = 0.025$.

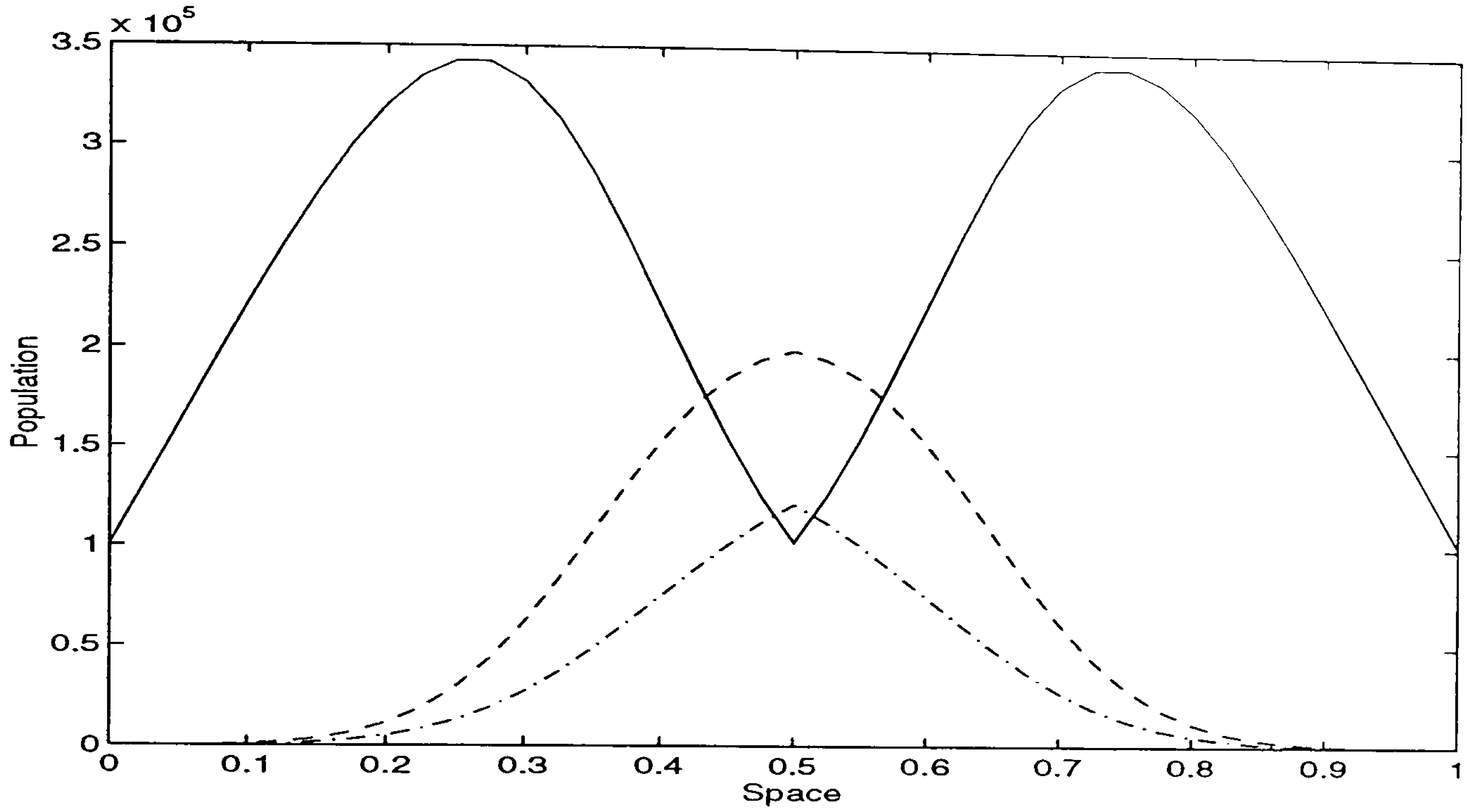

Figure 4.8: *Experiment A, dynamics of measles at time* $t = 0.1$, $\alpha = 0.0001$, $\ell = 0.001$ *and* $h = 0.025$; *susceptibles(–), exposed(– –) and infectives(–.).*

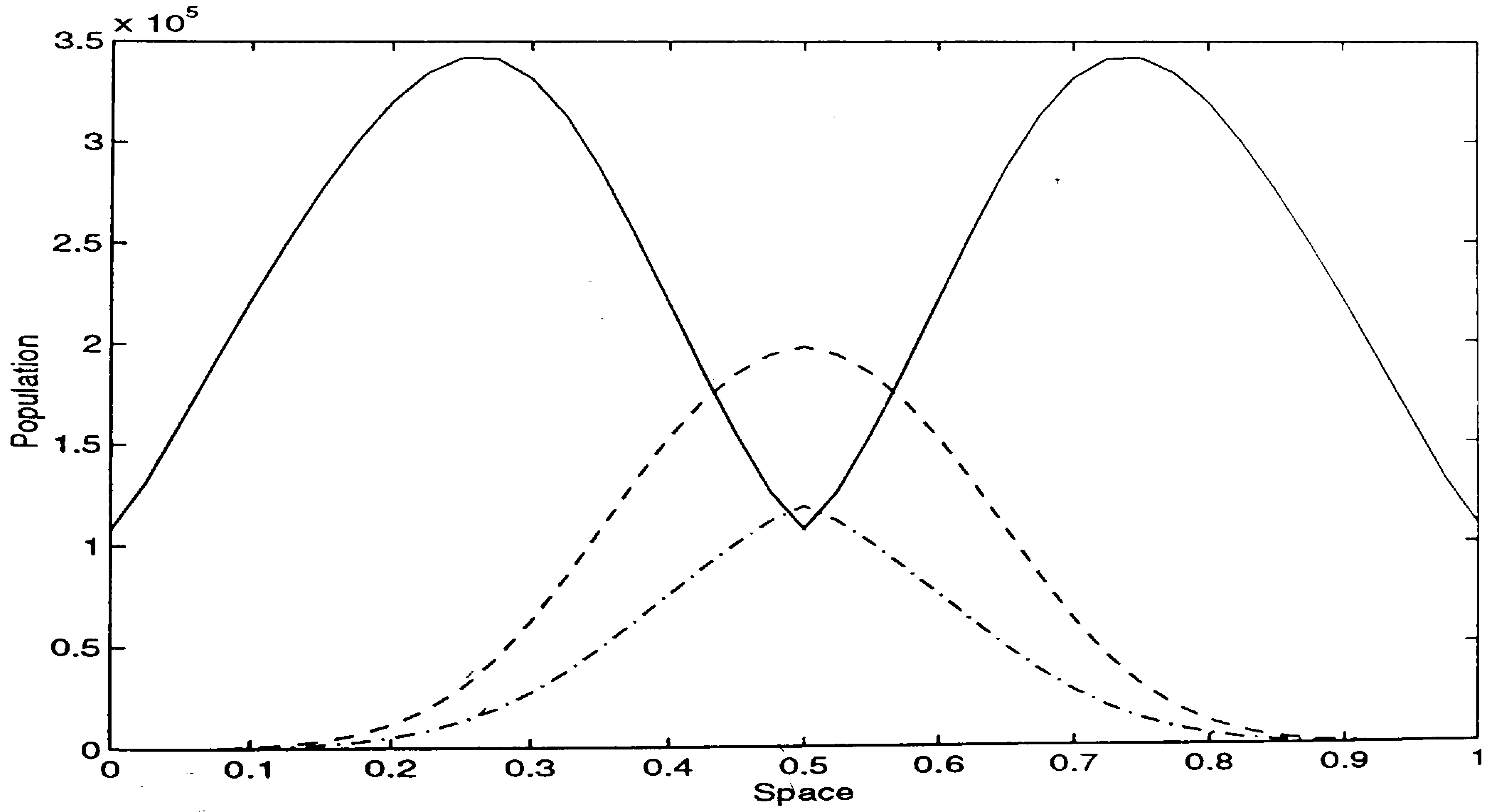

Figure 4.9: *Experiment A, dynamics of measles at time* $t = 0.1$, $\alpha = 0.001$, $\ell = 0.001$ *and* $h = 0.025$; *susceptibles(–), exposed(– –) and infectives(–.).*

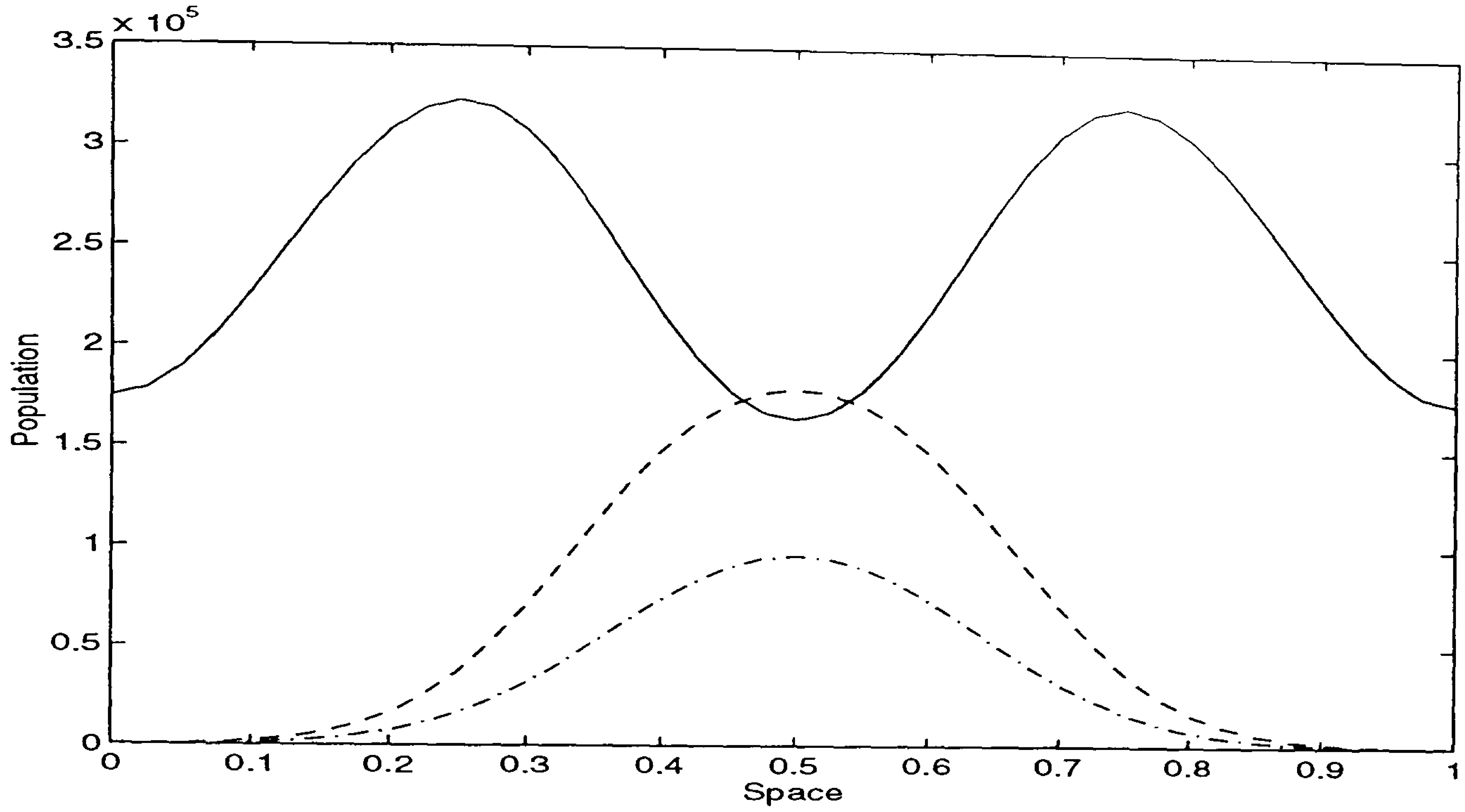

Figure 4.10: *Experiment A, dynamics of measles at time* $t = 0.1$, $\alpha = 0.03$, $\ell = 0.001$ *and* $h = 0.025$; *susceptibles(–), exposed(– –) and infectives(–.).*

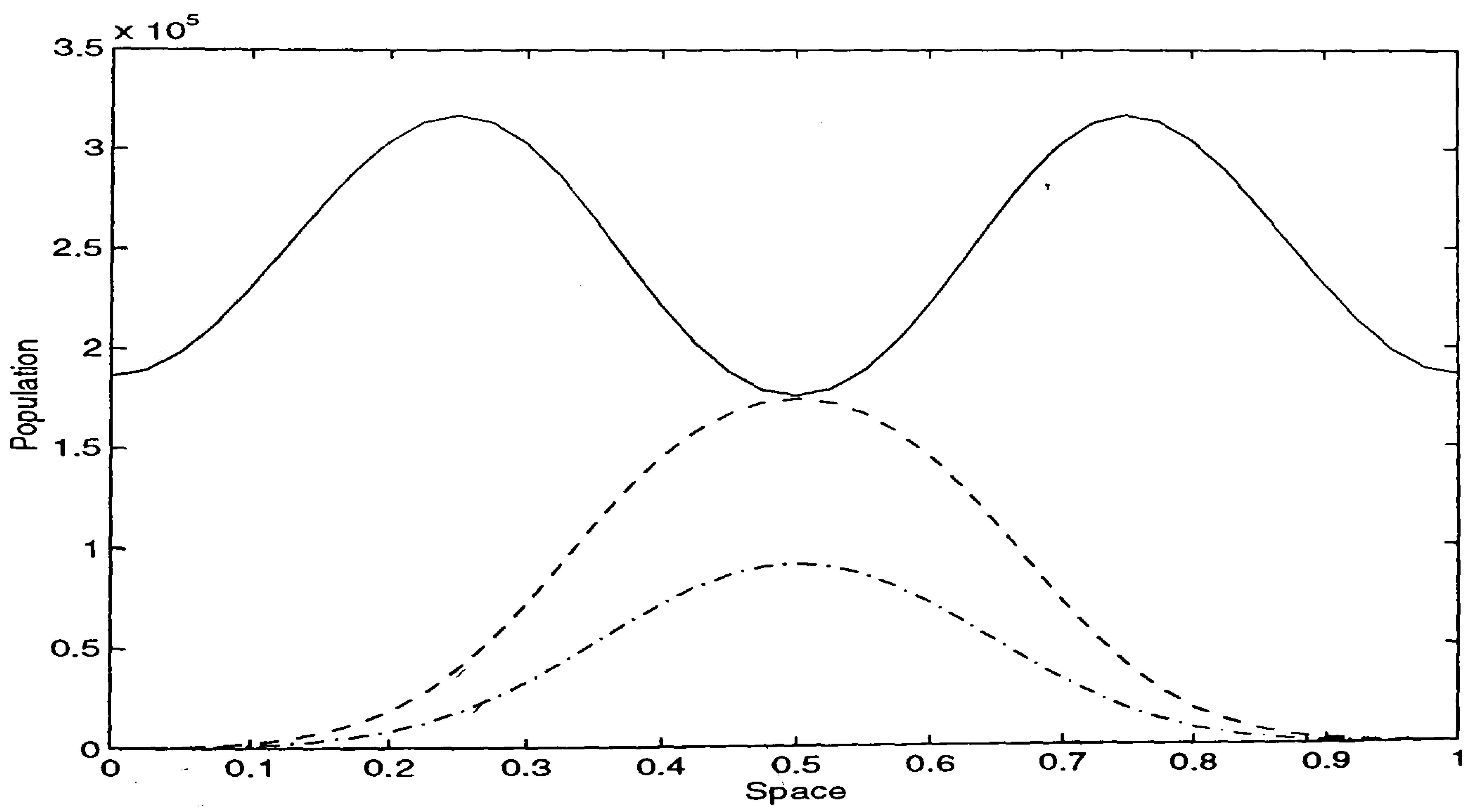

Figure 4.11: *Experiment A, dynamics of measles at time* $t = 0.1$, $\alpha = 0.04$, $\ell = 0.001$ *and* $h = 0.025$; *susceptibles(–), exposed(– –) and infectives(–.).*

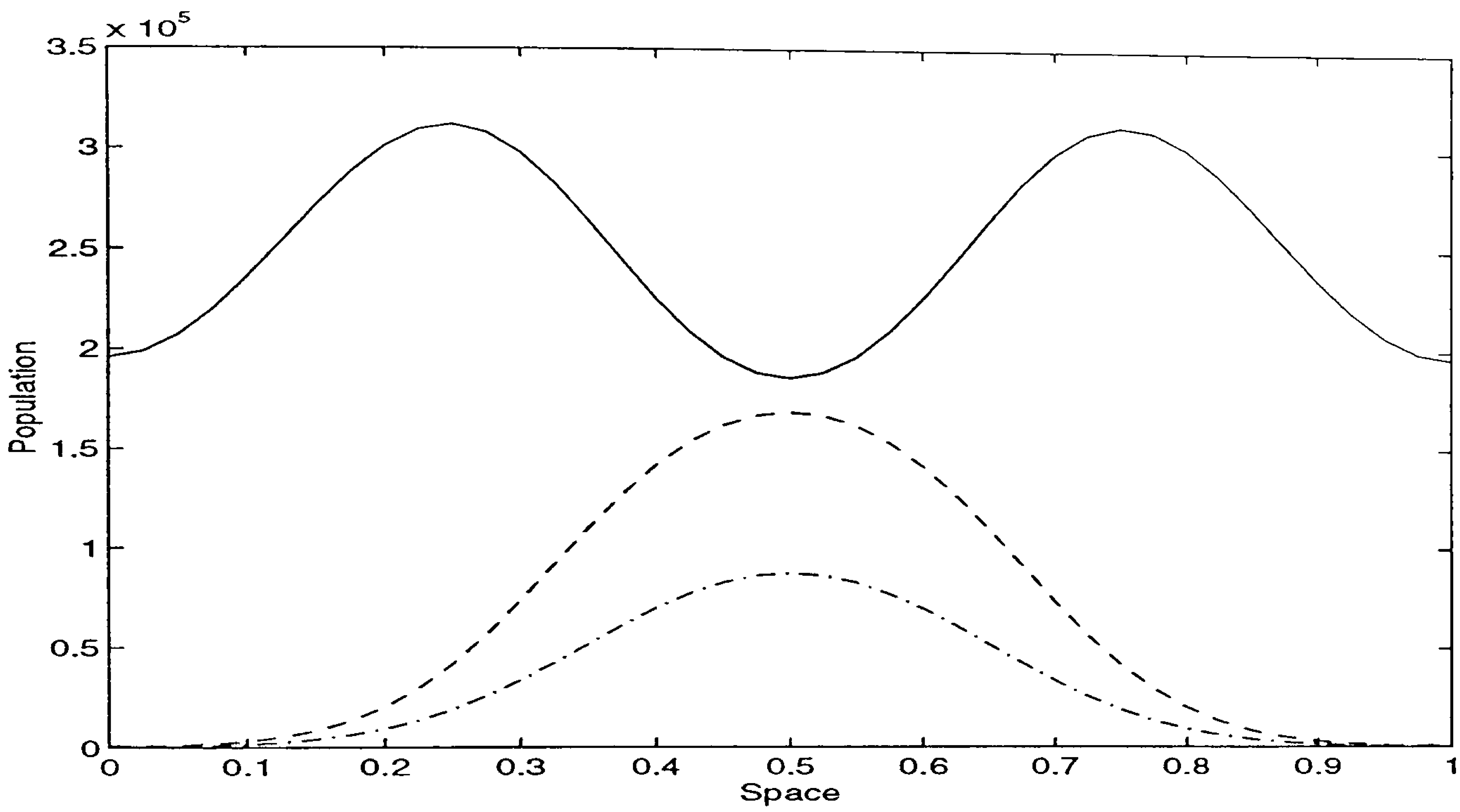

Figure 4.12: *Experiment A, dynamics of measles at time* $t = 0.1$, $\alpha = 0.05$, $\ell = 0.001$ *and* $h = 0.025$; *susceptibles(–), exposed(– –) and infectives(–.).*

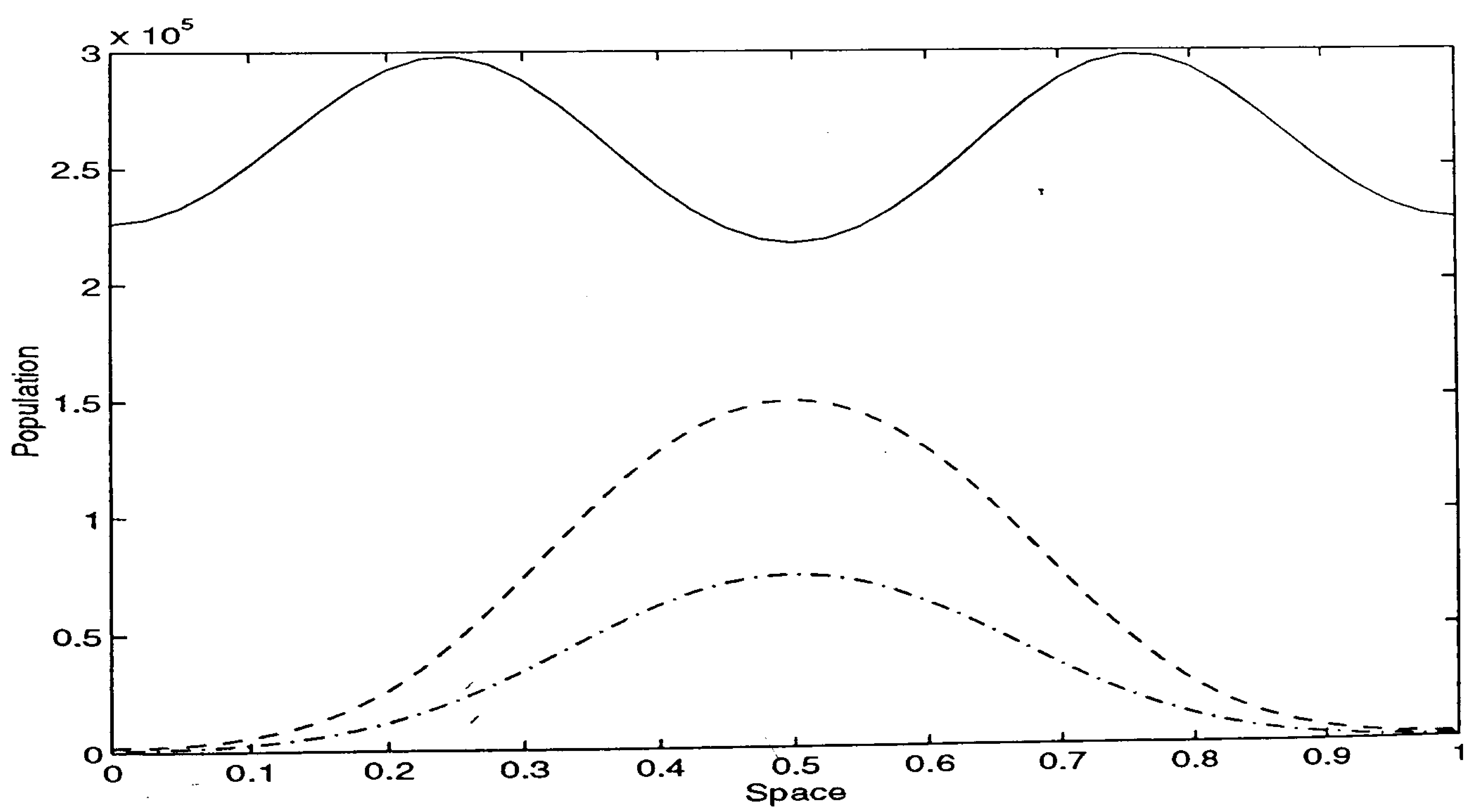

Figure 4.13: *Experiment A, dynamics of measles at time* $t = 0.1$, $\alpha = 0.09$, $\ell = 0.001$ *and* $h = 0.025$; *susceptibles(–), exposed(– –) and infectives(–.).*

- **Experiment B**

In this experiment, h is chosen to be 0.05 so that $M = 19$ and the initial conditions are distributed as in Figures 4.14 and 4.15; they are of the form

$$S(x_i, 0) = \begin{cases} 569048 & , \quad i = \frac{M+1}{2} \\ 589048 & , \quad i = \frac{M-3}{2}, \frac{M-1}{2}, \frac{M+3}{2}, \frac{M+5}{2} \\ 599048 & , \quad 0 \leq i \leq \frac{M-5}{2} \quad \& \quad \frac{M+7}{2} \leq i \leq M+1 \end{cases}$$

$$E(x_i, 0) = \begin{cases} 10^4 & , \quad \frac{M-3}{2} \leq i \leq \frac{M+5}{2} \\ 0 & , \quad 0 \leq i < \frac{M-3}{2} \quad \& \quad \frac{M+5}{2} < i \leq M+1 \end{cases}$$

$$I(x_i, 0) = \begin{cases} 2 \times 10^4 & , \quad i = \frac{M+1}{2} \\ 0.5 \times 10^4 & , \quad i = \frac{M-1}{2}, \frac{M+3}{2} \\ 0 & , \quad 0 \leq i < \frac{M-1}{2} \quad \& \quad \frac{M+3}{2} < i \leq M+1 \end{cases}$$

where the exposed and infectious individuals are concentrated in the middle of the interval ($0 \leq x \leq 1$) and the susceptibles are distributed along the whole interval such that the number of susceptible individuals in the middle of the interval is less than the other parts of the interval.

As in experiment A, the number of susceptibles is seen to decrease and those of exposed and infectious individuals are increased as time is increased. This behaviour continues till the time $t = 0.06$ after which the numbers of exposed and infectious individuals become greater than that of susceptibles, as may be expected from the dynamics of the disease. The profiles of the three classes of individual, as predicted by the model, at time $t = 0.1$ are shown in Figure 4.16. Three-dimensional plots of susceptible, exposed, infectious and recovered individuals are shown, respectively, in Figures 4.17–4.20.

As the diffusion rate, α, increases, the number of susceptibles decreases and the numbers of exposed and infected individuals increase and take a larger area on the x-axis, see Figures 4.21–4.24.

From these experiments, it is seen that the dynamic behaviour of measles depends on the initial distributions and the diffusion rate.

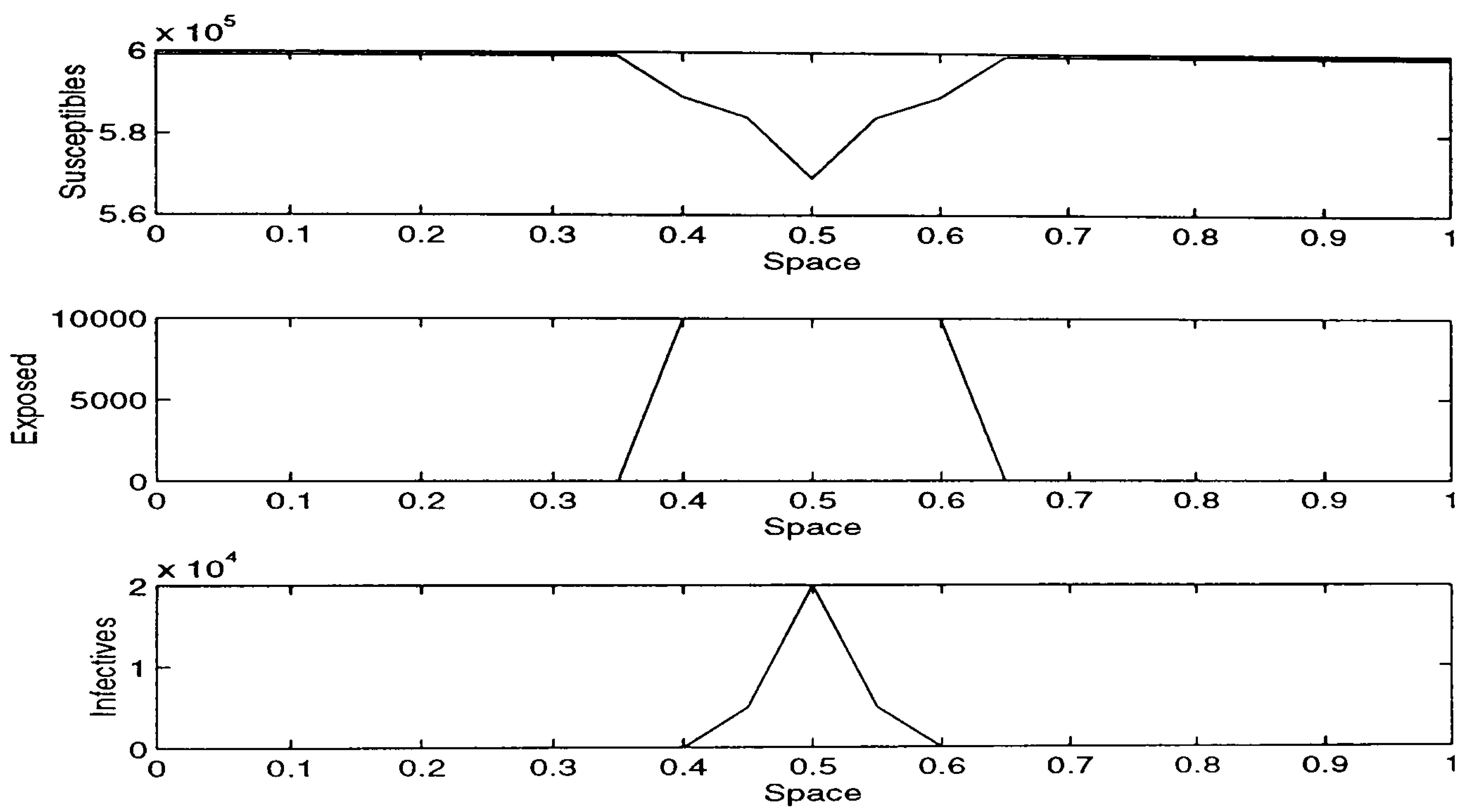

Figure 4.14: *Experiment B, initial disributions of susceptibles, exposed and infectives.*

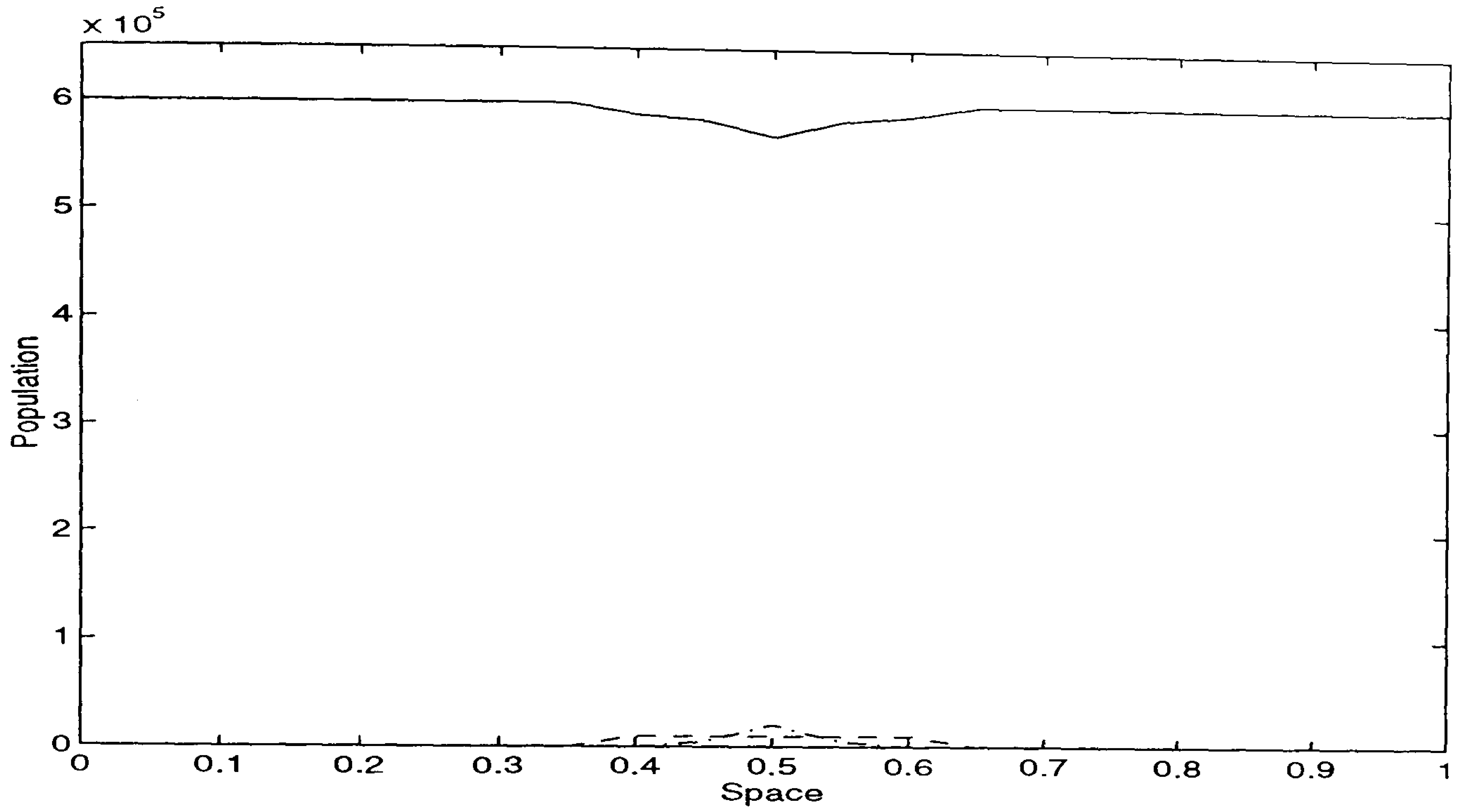

Figure 4.15: *Experiment B, initial distributions of susceptibles (–), exposed (– –) and infectious (–.) individuals.*

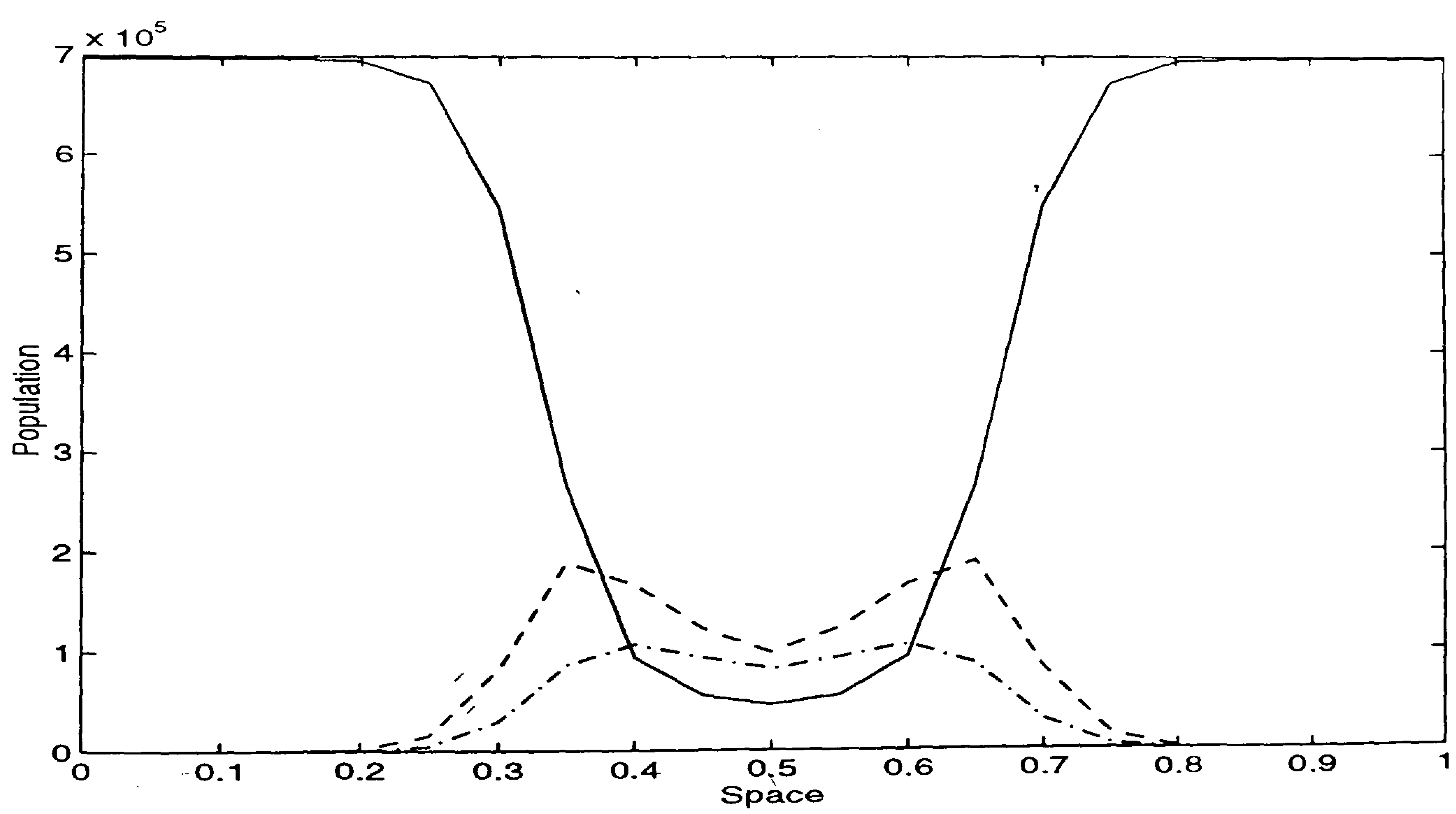

Figure 4.16: *Experiment B, distribution of susceptibles (–), exposed (– –) and infectives (–.) after 100 iterations ($t = 0.1$); $\ell = 0.001$ and $h = 0.05$.*

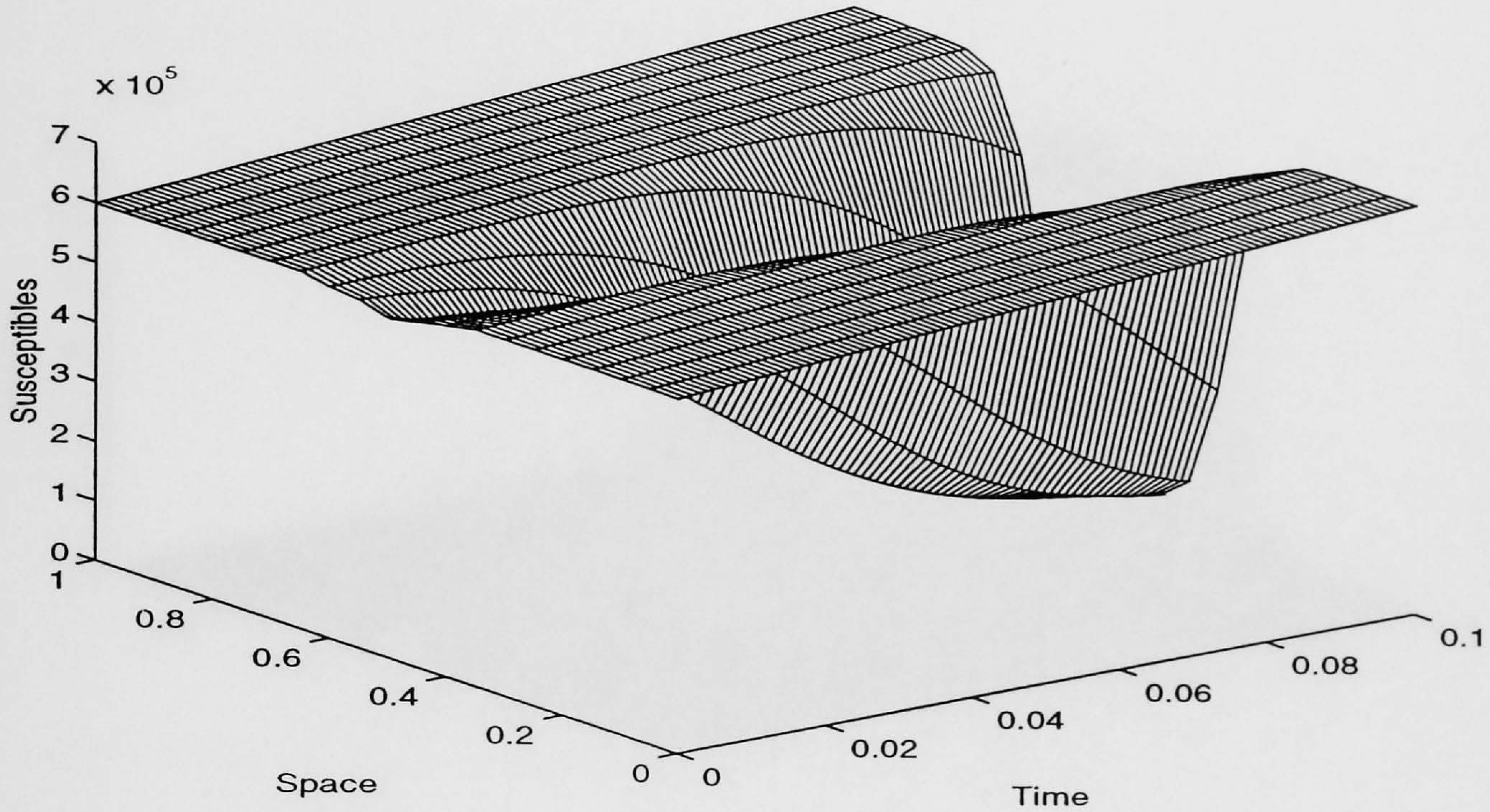

Figure 4.17: *Experiment B, three-dimensional distribution of susceptibles;* $\ell = 0.001$ *and* $h = 0.05$.

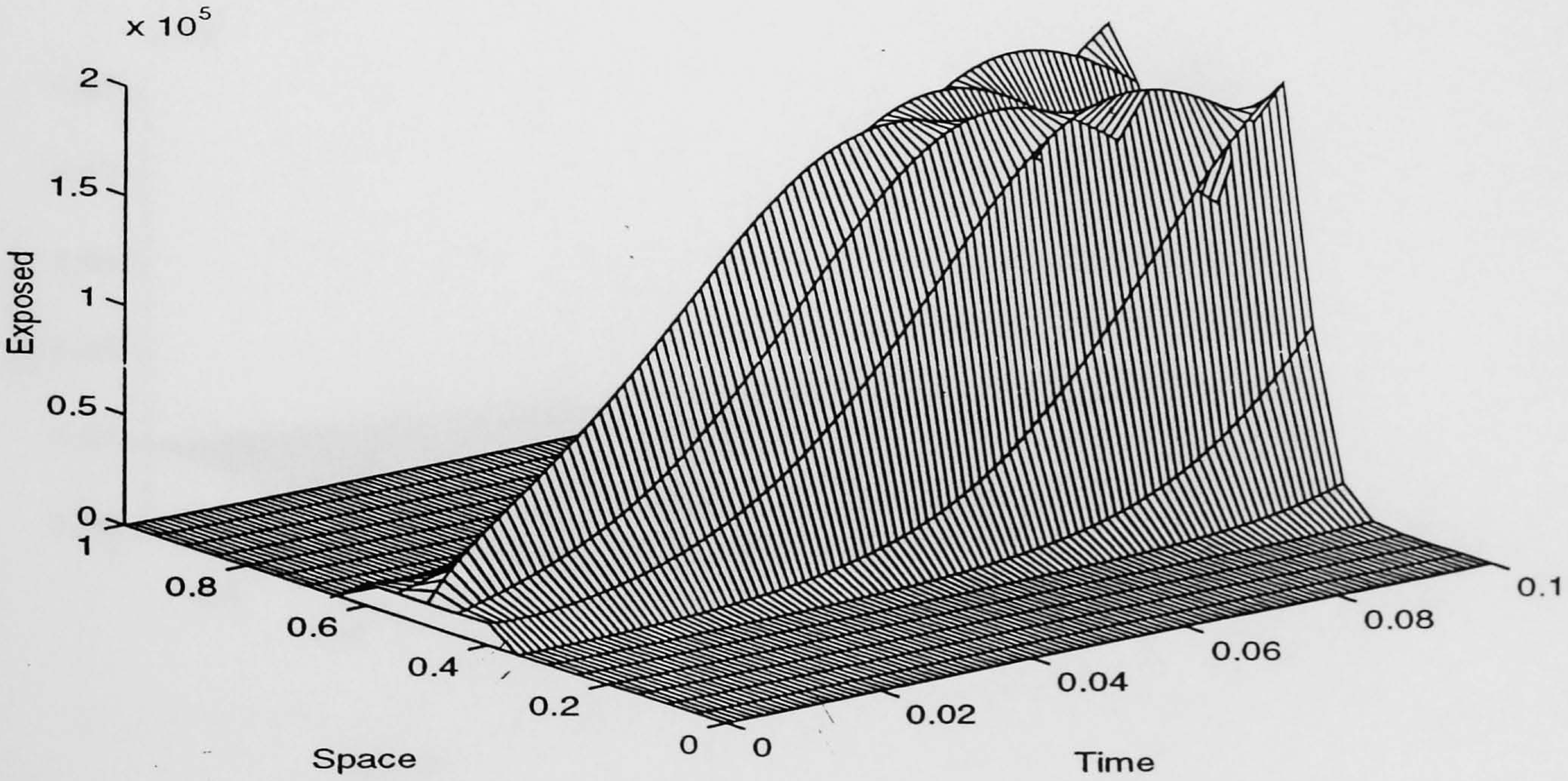

Figure 4.18: *Experiment B, three-dimensional distribution of exposed;* $\ell = 0.001$ *and* $h = 0.05$.

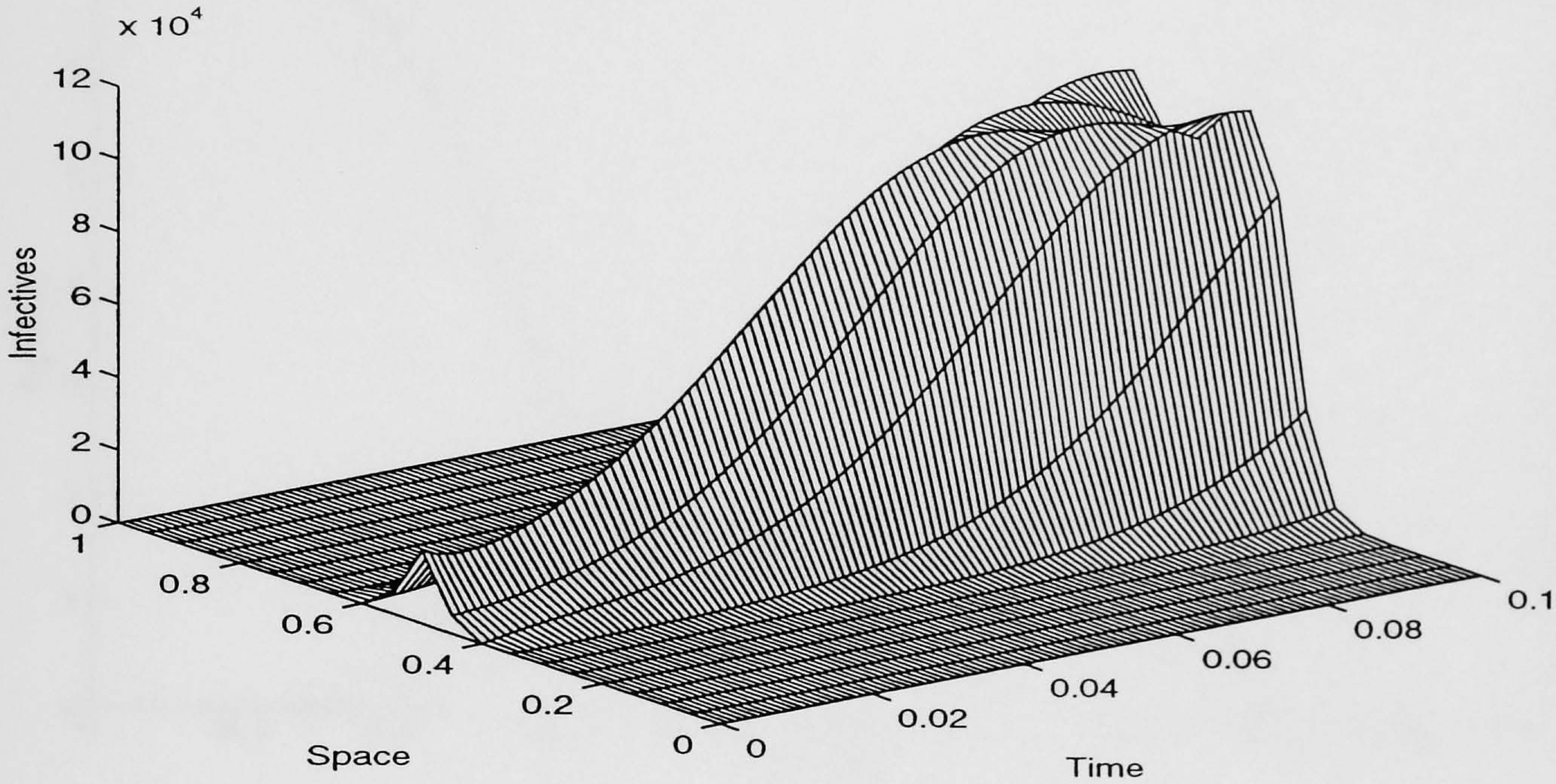

Figure 4.19: *Experiment B, three-dimensional distribution of infectives;* $\ell = 0.001$ *and* $h = 0.05$.

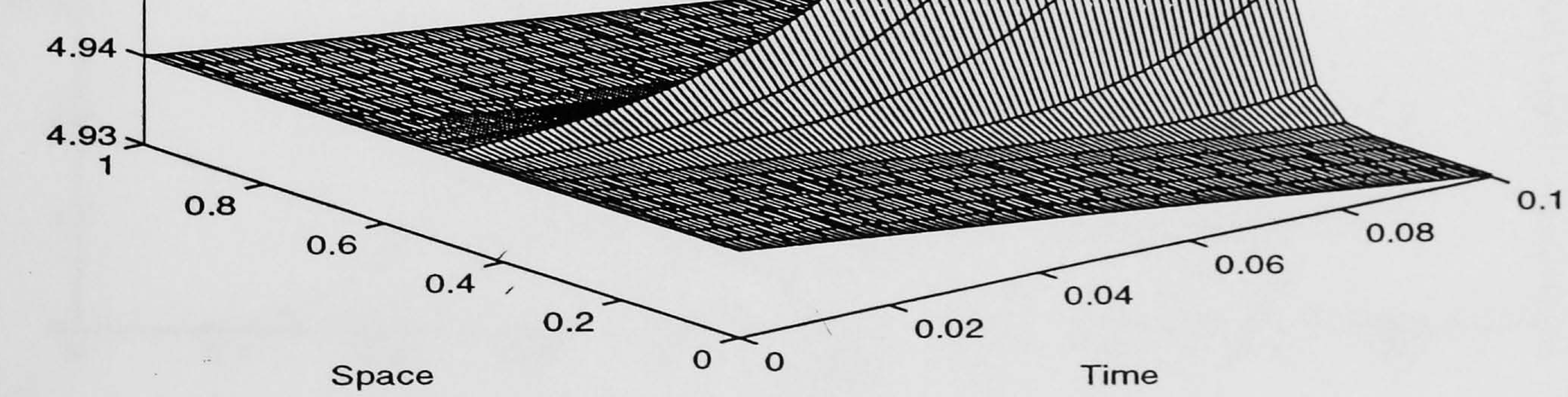

Figure 4.20: *Experiment B, three-dimensional distribution of recovered;* $\ell = 0.001$ *and* $h = 0.05$.

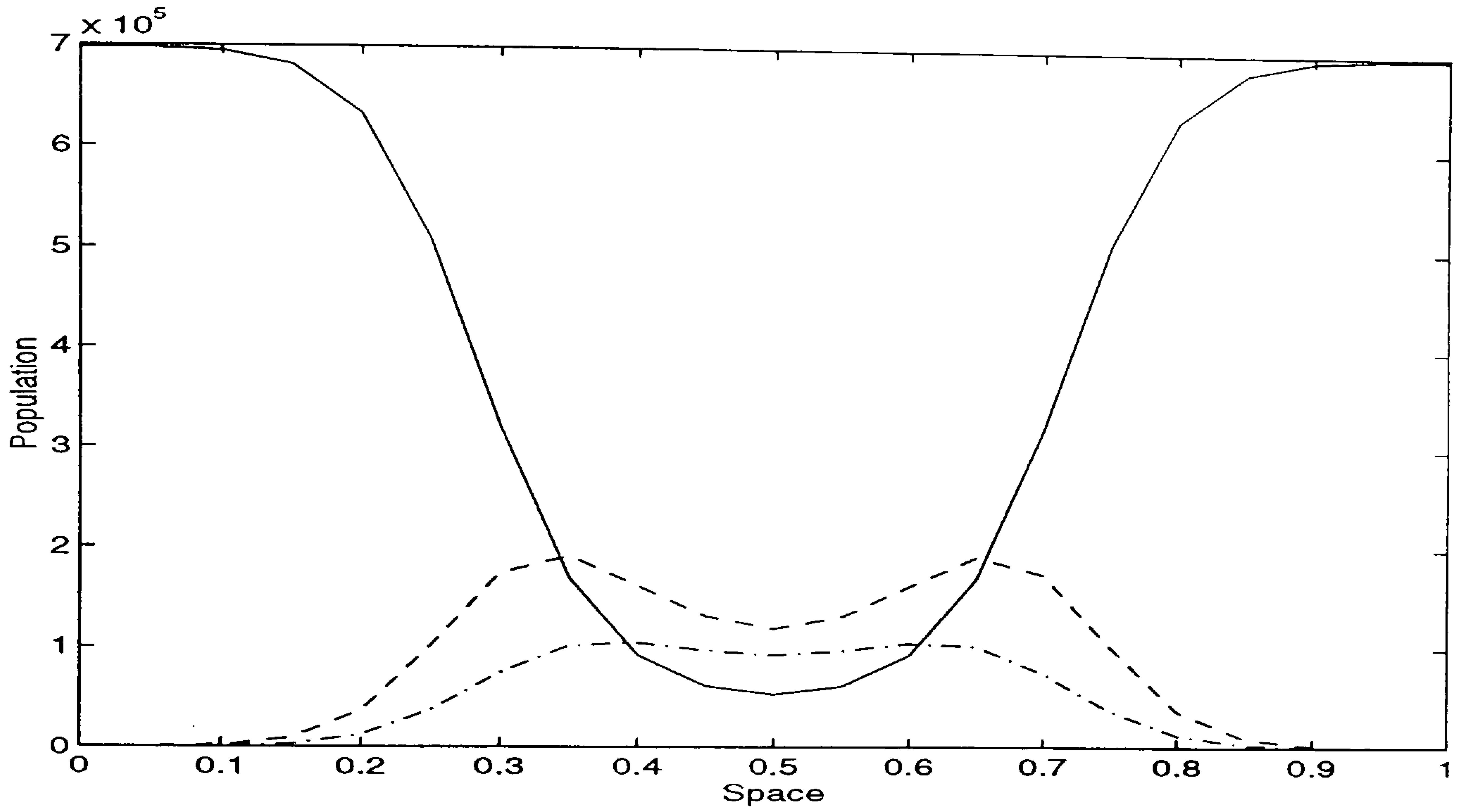

Figure 4.21: *Experiment B, dynamics of measles at time* $t = 0.1$, $\alpha = 0.03$, $\ell = 0.001$ *and* $h = 0.05$; *susceptibles (–), exposed (– –) and infectives (–.).*

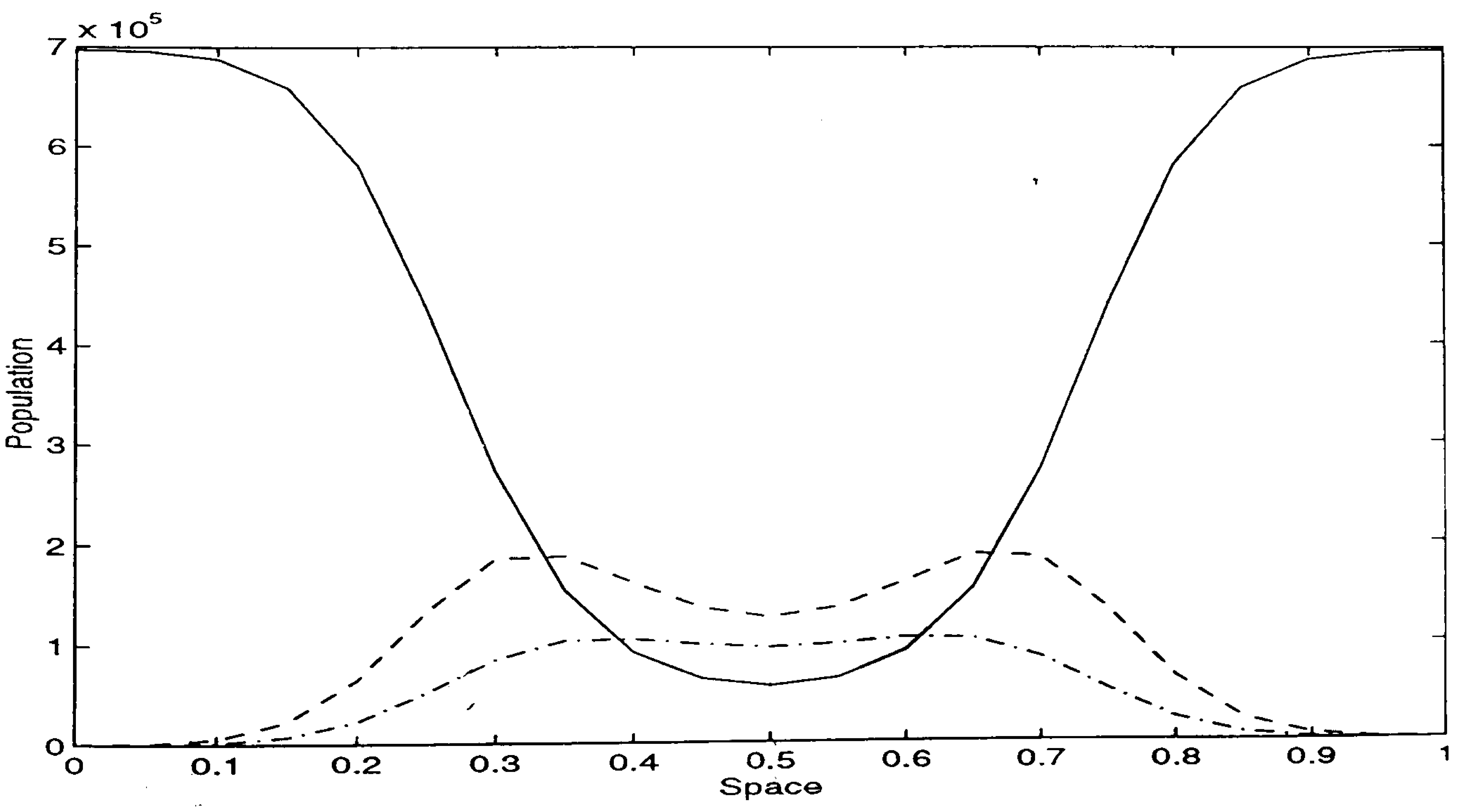

Figure 4.22: *Experiment B, dynamics of measles at time* $t = 0.1$, $\alpha = 0.04$, $\ell = 0.001$ *and* $h = 0.05$; *susceptibles (–), exposed (– –) and infectives (–.).*

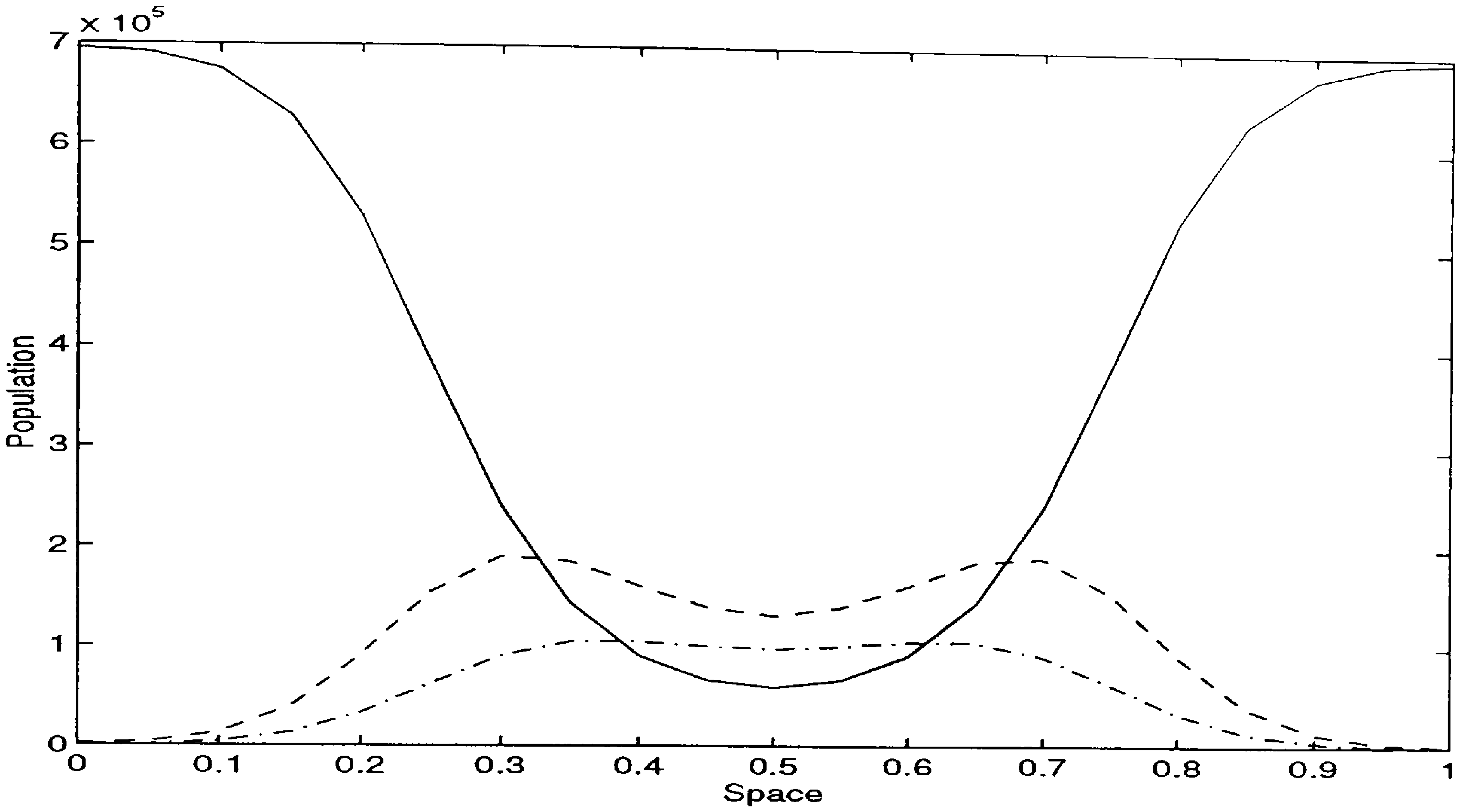

Figure 4.23: *Experiment B, dynamics of measles at time* $t = 0.1$, $\alpha = 0.05$, $\ell = 0.001$ *and* $h = 0.05$; *susceptibles (–), exposed (– –) and infectives (–.).*

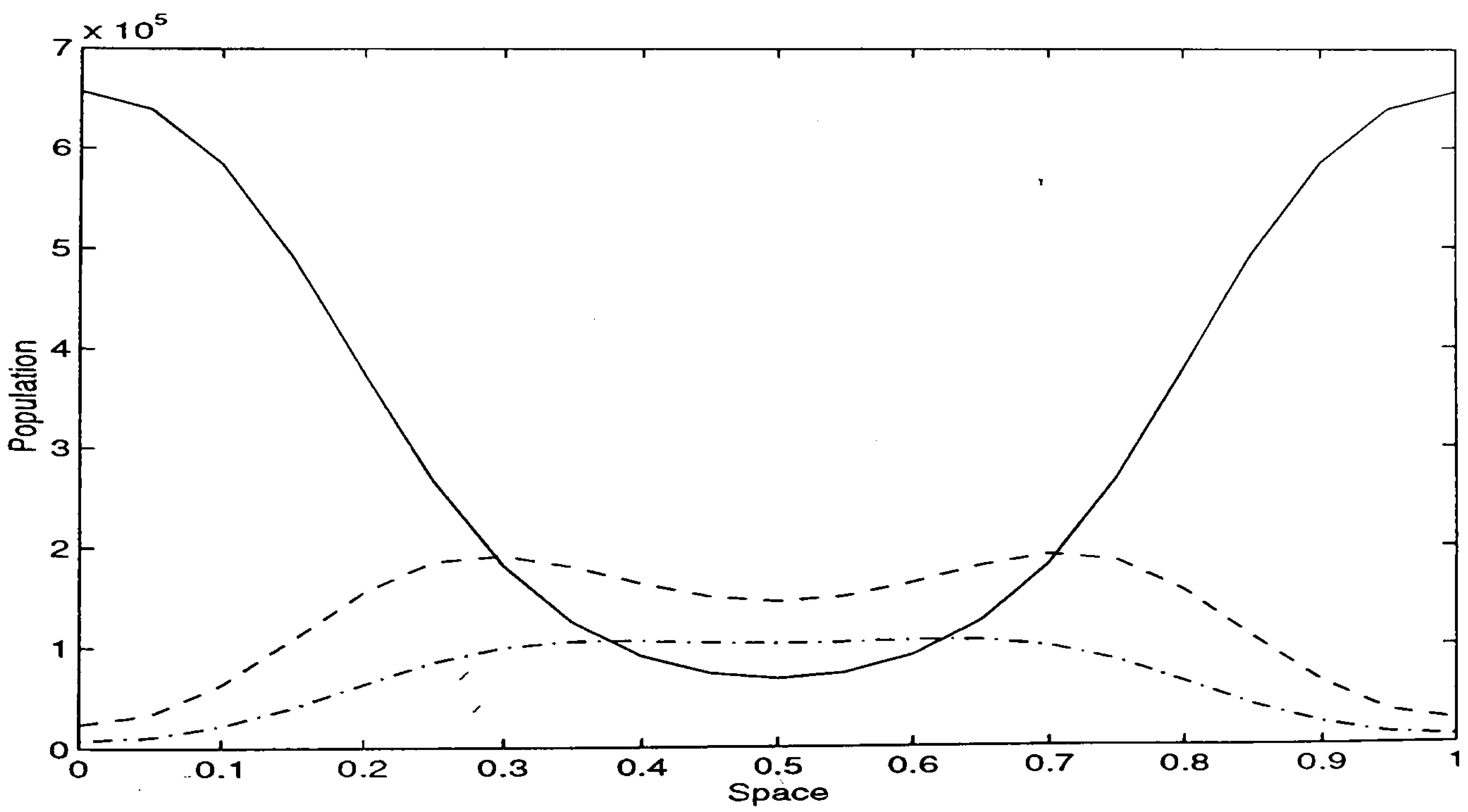

Figure 4.24: *Experiment B, dynamics of measles at time* $t = 0.1$, $\alpha = 0.09$, $\ell = 0.001$ *and* $h = 0.05$; *susceptibles (–), exposed (– –) and infectives (–.).*

4.8 Conclusion

A second-order finite-difference method has been developed and implemented in this chapter for computing the solutions of the SEIR measles model in one dimension (4.2.1). A stability analysis revealed that both the von Neumann and matrix stability methods failed to give a criterion for the stability of the method so the maximum principle analysis was used to show that the method is stable and consistent and thus convergent. Two numerical experiments were chosen to investigate the dynamic behaviour of the model for different steplengths and diffusion rates. It was seen that the dynamic behaviour of measles depends on the initial distributions and the diffusion rate.

Chapter 5

Two-Dimensional Measles Dynamics

5.1 Introduction

In this chapter the SEIR model of measles dynamics will be extended to consider the spread of the disease in two space dimensions. Till the work on this thesis the literature has contained no references to this aspect of the disease.

The partial differential equation (PDE) model in two dimensions retains the features of measles epidemiology discussed in Chapters 3 and 4. Of course, like Chapter 4, in order to proceed, the epidemic is assumed to diffuse through space in two dimensions. Also it is assumed that all births are into the susceptible class, and that births exactly balance deaths so that the total population size, N, is constant.

The reaction-diffusion equations are given by

$$\begin{aligned} S_t &= \mu N - (\mu + \beta I) S + \alpha S_{xx} + \lambda S_{yy} \\ E_t &= \beta S I - (\mu + \sigma) E + \alpha E_{xx} + \lambda E_{yy} \\ I_t &= \sigma E - (\mu + \gamma) I + \alpha I_{xx} + \lambda I_{yy} \end{aligned} \qquad (5.1.1)$$

in which $S = S(x, y, t)$, $E = E(x, y, t)$ and $I = I(x, y, t)$ are the number of susceptibles, exposed and infectious individuals, respectively, at time t and distances x and y from the origin; $\alpha > 0$ and $\lambda > 0$ are the diffusion rates. The parameters μ, σ, γ and β are as defined previously.

The initial conditions are of the forms

$$S(x,y,0) = S^0, \quad E(x,y,0) = E^0, \quad I(x,y,0) = I^0; \quad 0 \le x, y \le L \qquad (5.1.2)$$

and the boundary conditions are

$$u_x(0,y,t) = u_x(L,y,t) = 0; \quad t > 0 \qquad (5.1.3)$$

$$u_y(x,0,t) = u_y(x,L,t) = 0; \quad t > 0 \qquad (5.1.4)$$

where $u(x, y, t)$ represents $S(x, y, t)$, $E(x, y, t)$ or $I(x, y, t)$.

It will be assumed that the PDEs in (5.1.1) are defined for $0 \le x, y \le L$, $t > 0$ and the initial/boundary-value problem (5.1.1)$-$(5.1.4) will be solved in these ranges.

Following Twizell[66], both intervals $0 \le x \le L$ and $0 \le y \le L$ are divided into $M + 1$ subintervals each of width h, so that $(M + 1)h = L$ and the time variable t is incremented in steps of length ℓ. At each level $t = t_n = n\ell\,(n = 0, 1, 2, \ldots)$ the square Ω, defined by the lines $x = 0$, $y = 0$, $x = L$, $y = L$, together with its boundary $\partial\Omega$, have thus been superimposed by a square of M^2 points within Ω and $M + 2$ equally-spaced points along each side of $\partial\Omega$.

The solutions $S(x, y, t), E(x, y, t)$ and $I(x, y, t)$ of (5.1.1)$-$(5.1.4) are sought at each point $(kh, mh, n\ell)$ in $\Omega \cup \partial\Omega$ for $t > 0$ where $k, m = 0, 1, 2, \ldots, M, M + 1$ and $n = 1, 2, \ldots$. As in the one-dimensional case, the solutions $S(x, y, t), E(x, y, t)$ and $I(x, y, t)$ of (5.1.1) at the mesh point $(kh, mh, n\ell)$ will be denoted by $S^n_{k,m}$, $E^n_{k,m}$ and $I^n_{k,m}$, respectively, while the theoretical solutions of an approximating scheme will be denoted by $A^n_{k,m}$, $B^n_{k,m}$ and $C^n_{k,m}$, respectively. The values actually obtained, which may be subject, for example, to round-off errors, will be denoted by $\widetilde{A^n_{k,m}}$, $\widetilde{B^n_{k,m}}$ and $\widetilde{C^n_{k,m}}$, respectively.

Introduce, now, three vectors $\mathbf{A}^n$, $\mathbf{B}^n$ and $\mathbf{C}^n$ the elements of which will be ordered in rows parallel to the x-axis. The vector $\mathbf{A}^n$ which is of order $(M+2)^2$ thus takes the form

$$\mathbf{A}^n = \Big[A^n_{0,0}, A^n_{1,0}, \quad \cdots \quad, A^n_{M,0}, A^n_{M+1,0}; A^n_{0,1}, A^n_{1,1}, \cdots, A^n_{M,1}, A^n_{M+1,1}; \\ \cdots \quad ; A^n_{0,M+1}, A^n_{1,M+1}, \cdots, A^n_{M,M+1}, A^n_{M+1,M+1}\Big]^T ,$$

with equivalent definitions for $\mathbf{B}^n$ and $\mathbf{C}^n$. It will be convenient to use the vector $\mathbf{U}^n$ of order $3(M+2)^2$ given by

$$\mathbf{U}^n = \Big((\mathbf{A}^n)^T, (\mathbf{B}^n)^T, (\mathbf{C}^n)^T\Big)^T$$

where T denotes transpose.

On differentiating with respect to t, the equations in (5.1.1) give

$$\begin{aligned} S_{tt} - \alpha\, S_{xxt} - \lambda\, S_{yyt} + (\mu + \beta\, I)\, S_t + \beta\, S\, I_t &= 0\,, \\ E_{tt} - \alpha\, E_{xxt} - \lambda\, E_{yyt} + (\mu + \sigma)\, E_t - \beta\, S\, I_t - \beta\, S_t\, I &= 0\,, \\ I_{tt} - \alpha\, I_{xxt} - \lambda\, I_{yyt} + (\mu + \gamma)\, I_t - \sigma\, E_t &= 0\,. \end{aligned} \qquad (5.1.5)$$

5.2 Numerical Methods

5.2.1 Numerical Method for S

Finite-difference methods are developed by approximating the time derivative in the first equation in (5.1.1) by the first-order forward-difference replacement

$$S_t(x, y, t) \approx [S(x, y, t + \ell) - S(x, y, t)]/\ell \qquad (5.2.1)$$

and the space derivatives by the weighted approximants

$$\begin{aligned} S_{xx} &\approx h^{-2}\left[\phi\{S(x-h,y,t+\ell)-2\,S(x,y,t+\ell)+S(x+h,y,t+\ell)\}\right. \\ &+ \left.(1-\phi)\{S(x-h,y,t)-2\,S(x,y,t)+S(x+h,y,t)\}\right], \end{aligned} \qquad (5.2.2)$$

and

$$\begin{aligned} S_{yy} &\approx h^{-2}\left[\phi\{S(x,y-h,t+\ell)-2\,S(x,y,t+\ell)+S(x,y+h,t+\ell)\}\right. \\ &+ \left.(1-\phi)\{S(x,y-h,t)-2\,S(x,y,t)+S(x,y+h,t)\}\right], \end{aligned} \qquad (5.2.3)$$

in which $x = x_k$, $y = y_m\,(k,m = 0,1,2,\ldots,M,M+1)$, $t = t_n\,(n = 0,1,2,\ldots)$ and $\phi\,(0 \le \phi \le 1)$ is a parameter. When $\phi = 0$, (5.2.2) and (5.2.3) are $O(h^2)$ approximants as $h \longrightarrow 0$ and are $O(h^2+\ell)$ approximants as $h,\ell \longrightarrow 0$ when $0 < \phi \le 1$.

As in Section 4.4.1, equations (3.6.13) and (3.6.16) are used to obtain approximations to $S(x_k,y_m,t_{n+1})$ for use in the first equation in (5.1.1). Using equations (3.6.13) and (5.2.1) in the first equation in (5.1.1) gives

$$\frac{A^{n+1}_{k,m}-A^{n}_{k,m}}{\ell}+\left(\mu+\beta\,C^{n}_{k,m}\right)A^{n+1}_{k,m}-\alpha\,S_{xx}-\lambda\,S_{yy}-\mu\,N=0\,, \qquad (5.2.4)$$

while using equations (3.6.16) and (5.2.1) gives

$$\frac{A^{n+1}_{k,m}-A^{n}_{k,m}}{\ell}+\left(\mu+\beta\,C^{n+1}_{k,m}\right)A^{n}_{k,m}-\alpha\,S_{xx}-\lambda\,S_{yy}-\mu\,N=0\,. \qquad (5.2.5)$$

In order to obtain a second-order approximation to $S(x_k,y_m,t_{n+1})$, equations (5.2.4) and (5.2.5) should be added. Averaging the equations and substituting for S_{xx} and S_{yy} from (5.2.2) and (5.2.3), respectively, gives

$$\begin{aligned} \frac{A^{n+1}_{k,m}-A^{n}_{k,m}}{\ell} &+ \frac{1}{2}\left(\mu+\beta\,C^{n}_{k,m}\right)A^{n+1}_{k,m}+\frac{1}{2}\left(\mu+\beta\,C^{n+1}_{k,m}\right)A^{n}_{k,m} \\ &- \frac{\alpha}{h^2}\,\phi\left\{A^{n+1}_{k-1,m}-2\,A^{n+1}_{k,m}+A^{n+1}_{k+1,m}\right\}-\frac{\alpha}{h^2}(1-\phi)\left\{A^{n}_{k-1,m}-2\,A^{n}_{k,m}+A^{n}_{k+1,m}\right\} \\ &- \frac{\lambda}{h^2}\,\phi\left\{A^{n+1}_{k,m-1}-2\,A^{n+1}_{k,m}+A^{n+1}_{k,m+1}\right\}-\frac{\lambda}{h^2}(1-\phi)\left\{A^{n}_{k,m-1}-2\,A^{n}_{k,m}+A^{n}_{k,m+1}\right\} \\ &- \mu\,N=0 \end{aligned} \qquad (5.2.6$$

which, after rearranging, becomes

$$\begin{aligned}
-\lambda\,\phi\,p\,A^{n+1}_{k,m-1} &- \alpha\,\phi\,p\,A^{n+1}_{k-1,m} + \left[1 + \frac{1}{2}\,\ell\,(\mu + \beta\,C^{n}_{k,m}) + 2\,\phi\,(\alpha\,p + \lambda\,p)\right]\,A^{n+1}_{k,m} \\
&- \alpha\,\phi\,p\,A^{n+1}_{k+1,m} - \lambda\,\phi\,p\,A^{n+1}_{k,m+1} = (1-\phi)\,\lambda\,p\,A^{n}_{k,m-1} + (1-\phi)\,\alpha\,p\,A^{n}_{k-1,m} \\
&+ \left[1 - \frac{1}{2}\,\ell\,(\mu + \beta\,C^{n+1}_{k,m}) - 2\,(1-\phi)(\alpha\,p + \lambda\,p)\right]\,A^{n}_{k,m} \\
&+ (1-\phi)\,\alpha\,p\,A^{n}_{k+1,m} + (1-\phi)\,\lambda\,p\,A^{n}_{k,m+1} + \ell\,\mu\,N\,, \qquad (5.2.7)
\end{aligned}$$

where $p = \ell/h^2$.

The local truncation error $\mathcal{L}_S = \mathcal{L}_S[S(x,y,t), E(x,y,t), I(x,y,t); h, \ell]$ associated with (5.2.7) at the point $(x,y,t) = (x_k, y_m, t_n)$ may be written down from (5.2.6): it is

$$\begin{aligned}
\mathcal{L}_S &= \frac{S(x,y,t+\ell) - S(x,y,t)}{\ell} + \frac{1}{2}\,(\mu + \beta\,I(x,y,t))\,S(x,y,t+\ell) \\
&+ \frac{1}{2}\,(\mu + \beta\,I(x,y,t+\ell))\,S(x,y,t) - \frac{\alpha\,\phi}{h^2}\,\{S(x-h,y,t+\ell) - 2\,S(x,y,t+\ell) \\
&+ S(x+h,y,t+\ell)\} - \frac{(1-\phi)\,\alpha}{h^2}\,\{S(x-h,y,t) - 2\,S(x,y,t) + S(x+h,y,t)\} \\
&- \frac{\lambda\,\phi}{h^2}\,\{S(x,y-h,t+\ell) - 2\,S(x,y,t+\ell) + S(x,y+h,t+\ell)\} \\
&- \frac{(1-\phi)\,\lambda}{h^2}\,\{S(x,y-h,t) - 2\,S(x,y,t) + S(x,y+h,t)\} - \mu\,N \\
&- \{S_t(x,y,t) + (\mu + \beta\,I(x,y,t))\,S(x,y,t) - \alpha\,S_{xx}(x,y,t) \\
&- \lambda\,S_{yy}(x,y,t) - \mu\,N\}\,. \qquad (5.2.8)
\end{aligned}$$

Expanding $S(x,y,t+\ell)$, $S(x\pm h,y,t+\ell)$, $S(x\pm h,y,t)$, $S(x,y\pm h,t+\ell)$, $S(x,y\pm h,t)$ and $I(x,y,t+\ell)$ about (x,y,t) leads to

$$\begin{aligned}
\mathcal{L}_S &= \left[\frac{1}{2}\,S_{tt} + \frac{1}{2}\,(\mu + \beta\,I)\,S_t + \frac{1}{2}\,\beta\,S\,I_t - \phi\,\alpha\,S_{xxt} - \phi\,\lambda\,S_{yyt}\right]\,\ell \\
&- \frac{h^2}{12}\,(\alpha\,S_{xxxx} + \lambda\,S_{yyyy}) + \left[\frac{1}{6}\,S_{ttt} + \frac{1}{4}\,(\mu + \beta\,I)\,S_{tt} + \frac{1}{4}\,\beta\,S\,I_{tt}\right. \\
&- \left.\frac{1}{2}\,\phi\,\alpha\,S_{xxtt} - \frac{1}{2}\,\phi\,\lambda\,S_{yytt}\right]\,\ell^2 + \cdots\,. \qquad (5.2.9)
\end{aligned}$$

Clearly, $\mathcal{L}_S = O(h^2 + \ell)$ as $h,\ \ell \longrightarrow 0$ except when $\phi = 1/2$ for then the term in ℓ

in (5.2.9) vanishes (see the first equation in (5.1.5)) leaving

$$\begin{aligned}\mathcal{L}_S &= -\frac{h^2}{12}(\alpha S_{xxxx} + \lambda S_{yyyy}) + \left[\frac{1}{6} S_{ttt} + \frac{1}{4}(\mu + \beta I) S_{tt} + \frac{1}{4}\beta S I_{tt} \right.\\ &- \left. \frac{1}{4}\alpha S_{xxtt} - \frac{1}{4}\lambda S_{yytt}\right] \ell^2 + \cdots \end{aligned} \qquad (5.2.10)$$

which is $O(h^2 + \ell^2)$ as $h, \ell \longrightarrow 0$. It may be concluded that the unique $O(h^2 + \ell^2)$ method, as $h, \ell \longrightarrow 0$, contained in the family (5.2.7) arises when $\phi = 1/2$ and is given by

$$\begin{aligned} -\frac{1}{2}\lambda p A^{n+1}_{k,m-1} &- \frac{1}{2}\alpha p A^{n+1}_{k-1,m} + \left[1 + \frac{1}{2}\ell(\mu + \beta C^n_{k,m}) + \alpha p + \lambda p\right] A^{n+1}_{k,m} \\ &- \frac{1}{2}\alpha p A^{n+1}_{k+1,m} - \frac{1}{2}\lambda p A^{n+1}_{k,m+1} = \frac{1}{2}\lambda p A^n_{k,m-1} + \frac{1}{2}\alpha p A^n_{k-1,m} \\ &- \frac{1}{2}\ell\beta C^{n+1}_{k,m} A^n_{k,m} + (1 - \frac{1}{2}\ell\mu - \alpha p - \lambda p) A^n_{k,m} + \frac{1}{2}\alpha p A^n_{k+1,m} \\ &+ \frac{1}{2}\lambda p A^n_{k,m+1} + \ell\mu N . \end{aligned} \qquad (5.2.11)$$

The finite-difference method (5.2.11) may be applied for $k, m = 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots$; for k or $m = 0$ and for k or $m = M + 1$ it requires some modification. Applying (5.2.11) with $k = 0$ and $m = 0$ introduces the terms $A^{n+1}_{-1,0}$, $A^n_{-1,0}$, $A^{n+1}_{0,-1}$ and $A^n_{0,-1}$. Now the points (x_{-1}, y_0, t_{n+1}), (x_{-1}, y_0, t_n), (x_0, y_{-1}, t_{n+1}) and (x_0, y_{-1}, t_n) are outside the grid superimposed on $\Omega \cup \partial\Omega$. However, the boundary conditions in (5.1.3) and (5.1.4) give, to second order, $A^n_{-1,0} = A^n_{1,0}$, $A^{n+1}_{-1,0} = A^{n+1}_{1,0}$, $A^n_{0,-1} = A^n_{0,1}$ and $A^{n+1}_{0,-1} = A^{n+1}_{0,1}$. Thus, for $k = 0$ and $m = 0$, equation (5.2.11) may be modified to give

$$\begin{aligned} \left[1 + \frac{1}{2}\ell(\mu\right. &+ \left.\beta C^n_{0,0}) + \alpha p + \lambda p\right] A^{n+1}_{0,0} - \alpha p A^{n+1}_{1,0} - \lambda p A^{n+1}_{0,1} + \frac{1}{2}\ell\beta C^{n+1}_{0,0} A^n_{0,0} \\ &= \left[1 - \frac{1}{2}\ell\mu - \alpha p - \lambda p\right] A^n_{0,0} + \alpha p A^n_{1,0} + \lambda p A^n_{0,1} + \ell\mu N . \end{aligned} \qquad (5.2.12)$$

For $m = 0$ and $k = 1, 2, \ldots, M$, equation (5.2.11) gives

$$\begin{aligned} -\frac{1}{2}\alpha p A^{n+1}_{k-1,0} &+ \left[1 + \frac{1}{2}\ell(\mu + \beta C^n_{k,0}) + \alpha p + \lambda p\right] A^{n+1}_{k,0} - \frac{1}{2}\alpha p A^{n+1}_{k+1,0} - \lambda p A^{n+1}_{k,1} \\ &+ \frac{1}{2}\ell\beta C^{n+1}_{k,0} A^n_{k,0} = \frac{1}{2}\alpha p A^n_{k-1,0} + \left[1 - \frac{1}{2}\ell\mu - \alpha p - \lambda p\right] A^n_{k,0} \\ &+ \frac{1}{2}\alpha p A^n_{k+1,0} + \lambda p A^n_{k,0} + \ell\mu N . \end{aligned} \qquad (5.2.13)$$

When $m = 0$, $k = M + 1$, it follows from (5.1.3) and (5.1.4) that $A^{n+1}_{M,0} = A^{n+1}_{M+2,0}$ and $A^{n}_{M,0} = A^{n}_{M+2,0}$ and so equation (5.2.11) may be written as

$$\begin{aligned} -\alpha\, p\, A^{n+1}_{M,0} &+ \left[1 + \frac{1}{2}\ell\,(\mu + \beta\, C^{n}_{M+1,0}) + \alpha\, p + \lambda\, p\right] A^{n+1}_{M+1,0} - \lambda\, p\, A^{n+1}_{M+1,1} \\ &+ \frac{1}{2}\ell\,\beta\, C^{n+1}_{M+1,0}\, A^{n}_{M+1,0} = \alpha\, p\, A^{n}_{M,0} + \left[1 - \frac{1}{2}\ell\,\mu - \alpha\, p - \lambda\, p\right] A^{n}_{M+1,0} \\ &+ \lambda\, p\, A^{n}_{M+1,1} + \ell\,\mu\, N\,. \end{aligned} \qquad (\ 5.2.14\ )$$

For $k = 0$ and $m = 1, 2, \ldots, M$, equation (5.2.11) can be written as

$$\begin{aligned} -\frac{1}{2}\lambda\, p\, A^{n+1}_{0,m-1} &+ \left[1 + \frac{1}{2}\ell\,(\mu + \beta\, C^{n}_{0,m}) + \alpha\, p + \lambda\, p\right] A^{n+1}_{0,m} - \frac{1}{2}\lambda\, p\, A^{n+1}_{0,m+1} \\ &- \alpha\, p\, A^{n+1}_{1,m} + \frac{1}{2}\ell\,\beta\, C^{n+1}_{0,m}\, A^{n}_{0,m} = \frac{1}{2}\lambda\, p\, A^{n}_{0,m-1} + \left[1 - \frac{1}{2}\ell\,\mu - \alpha\, p - \lambda\, p\right] A^{n}_{0,m} \\ &+ \frac{1}{2}\lambda\, p\, A^{n}_{0,m+1} + \alpha\, p\, A^{n}_{1,m} + \ell\,\mu\, N\,. \end{aligned} \qquad (\ 5.2.15\ )$$

For $k = 0$ and $m = M + 1$, equation (5.2.11) gives

$$\begin{aligned} \Big[1 + \frac{1}{2}\ell\,(\mu &+ \beta\, C^{n}_{0,M+1}) + \alpha\, p + \lambda\, p\Big] A^{n+1}_{0,M+1} - \lambda\, p\, A^{n+1}_{0,M} - \alpha\, p\, A^{n+1}_{1,M+1} \\ &+ \frac{1}{2}\ell\,\beta\, C^{n+1}_{0,M+1}\, A^{n}_{0,M+1} = \left[1 - \frac{1}{2}\ell\,\mu - \alpha\, p - \lambda\, p\right] A^{n}_{0,M+1} + \lambda\, p\, A^{n}_{0,M} \\ &+ \alpha\, p\, A^{n}_{1,M+1} + \ell\,\mu\, N\,. \end{aligned} \qquad (\ 5.2.16\ )$$

When $m = M + 1$ and $k = 1, 2, \ldots, M$, equation (5.2.11) becomes

$$\begin{aligned} -\frac{1}{2}\alpha\, p\, A^{n+1}_{k-1,M+1} &+ \left[1 + \frac{1}{2}\ell\,(\mu + \beta\, C^{n}_{k,M+1}) + \alpha\, p + \lambda\, p\right] A^{n+1}_{k,M+1} - \frac{1}{2}\alpha\, p\, A^{n+1}_{k+1,M+1} \\ &- \lambda\, p\, A^{n+1}_{k,M} + \frac{1}{2}\ell\,\beta\, C^{n+1}_{k,M+1}\, A^{n}_{k,M+1} = \frac{1}{2}\alpha\, p\, A^{n}_{k-1,M+1} \\ &+ \left[1 - \frac{1}{2}\ell\,\mu - \alpha\, p - \lambda\, p\right] A^{n}_{k,M+1} + \frac{1}{2}\alpha\, p\, A^{n}_{k+1,M+1} \\ &+ \lambda\, p\, A^{n}_{k,M} + \ell\,\mu\, N\,. \end{aligned} \qquad (\ 5.2.17\ )$$

For $k = M + 1$ and $m = 1, 2, \ldots, M$, equation (5.2.11) gives

$$-\frac{1}{2}\lambda\, p\, A^{n+1}_{M+1,m-1} + \left[1 + \frac{1}{2}\ell\,(\mu + \beta\, C^{n}_{M+1,m}) + \alpha\, p + \lambda\, p\right] A^{n+1}_{M+1,m} - \frac{1}{2}\lambda\, p\, A^{n+1}_{M+1,m+1}$$

$$\begin{aligned} &- \alpha\, p\, A^{n+1}_{M,m} + \frac{1}{2}\,\ell\,\beta\, C^{n+1}_{M+1,m}\, A^{n}_{M+1,m} = \frac{1}{2}\,\lambda\, p\, A^{n}_{M+1,m-1} \\ &+ \left[1 - \frac{1}{2}\,\ell\,\mu - \alpha\, p - \lambda\, p\right] A^{n}_{M+1,m} + \frac{1}{2}\,\lambda\, p\, A^{n}_{M+1,m+1} \\ &+ \alpha\, p\, A^{n}_{M,m} + \ell\,\mu\, N\,, \end{aligned} \qquad (5.2.18)$$

and, when $k = M+1$ and $m = M+1$, it gives

$$\begin{aligned} -\lambda\, p\, A^{n+1}_{M+1,M} &+ \left[1 + \frac{1}{2}\,\ell\,(\mu + \beta\, C^{n}_{M+1,M+1}) + \alpha\, p + \lambda\, p\right] A^{n+1}_{M+1,M+1} - \alpha\, p\, A^{n+1}_{M,M+1} \\ &+ \frac{1}{2}\,\ell\,\beta\, C^{n+1}_{M+1,M+1}\, A^{n}_{M+1,M+1} = \lambda\, p\, A^{n}_{M+1,M} \\ &+ \left[1 - \frac{1}{2}\,\ell\,\mu - \alpha\, p - \lambda\, p\right] A^{n}_{M+1,M+1} + \alpha\, p\, A^{n}_{M,M+1} + \ell\,\mu\, N\,. \end{aligned} \qquad (5.2.19)$$

5.2.2 Numerical Method for E

Now the time derivative in the second equation in (5.1.1) is approximated by the first-order forward-difference replacement

$$E_t(x,y,t) \approx [E(x,y,t+\ell) - E(x,y,t)]/\ell\,. \qquad (5.2.20)$$

The x-derivative is approximated by the weighted approximant

$$\begin{aligned} E_{xx} \;\approx\; & h^{-2}\,[\phi\,\{E(x-h,y,t+\ell) - 2\,E(x,y,t+\ell) + E(x+h,y,t+\ell)\} \\ +\; & (1-\phi)\,\{E(x-h,y,t) - 2\,E(x,y,t) + E(x+h,y,t)\}]\,, \end{aligned} \qquad (5.2.21)$$

and the y-derivative is approximated by the weighted approximant

$$\begin{aligned} E_{yy} \;\approx\; & h^{-2}\,[\phi\,\{E(x,y-h,t+\ell) - 2\,E(x,y,t+\ell) + E(x,y+h,t+\ell)\} \\ +\; & (1-\phi)\,\{E(x,y-h,t) - 2\,E(x,y,t) + E(x,y+h,t)\}]\,, \end{aligned} \qquad (5.2.22)$$

in which $x = x_k$, $y = y_m\,(k,m = 0,1,2,\ldots,M,M+1)$, $t = t_n\,(n = 0,1,2,\ldots)$ and $\phi\,(0 \le \phi \le 1)$ is a parameter. Both equations (5.2.21) and (5.2.22) are $O(h^2)$ as $h \longrightarrow 0$ when $\phi = 0$ and are $O(h^2 + \ell)$ as $h,\ell \longrightarrow 0$ when $0 < \phi \le 1$.

Using equations (3.6.25) and (5.2.20) in the second equation in (5.1.1) gives

$$\frac{B^{n+1}_{k,m} - B^{n}_{k,m}}{\ell} - \beta\, C^{n}_{k,m}\, A^{n+1}_{k,m} + (\mu+\sigma)\, B^{n+1}_{k,m} - \alpha\, E_{xx} - \lambda\, E_{yy} = 0\,. \qquad (5.2.23)$$

while using equations (3.6.28) and (5.2.20) in the second equation in (5.1.1) gives

$$\frac{B^{n+1}_{k,m} - B^{n}_{k,m}}{\ell} - \beta\, C^{n+1}_{k,m}\, A^{n}_{k,m} + (\mu+\sigma)\, B^{n}_{k,m} - \alpha\, E_{xx} - \lambda\, E_{yy} = 0\,. \qquad (5.2.24)$$

In order to obtain a second-order approximation to $E(x_k, y_m, t_{n+1})$, as in Section 5.2.1, equations (5.2.23) and (5.2.24) should be added. Averaging equations (5.2.23) and (5.2.24) and substituting for E_{xx} and E_{yy} from (5.2.21) and (5.2.22), respectively, gives

$$\begin{aligned}
\frac{B^{n+1}_{k,m} - B^{n}_{k,m}}{\ell} &- \frac{1}{2}\beta\, C^{n}_{k,m}\, A^{n+1}_{k,m} - \frac{1}{2}\beta\, C^{n+1}_{k,m}\, A^{n}_{k,m} + \frac{1}{2}(\mu+\sigma)\, B^{n+1}_{k,m} \\
&+ \frac{1}{2}(\mu+\sigma)\, B^{n}_{k,m} - \frac{\alpha\,\phi}{h^2}\left\{B^{n+1}_{k-1,m} - 2\,B^{n+1}_{k,m} + B^{n+1}_{k+1,m}\right\} \\
&- (1-\phi)\,\frac{\alpha}{h^2}\left\{B^{n}_{k-1,m} - 2\,B^{n}_{k,m} + B^{n}_{k+1,m}\right\} - \frac{\lambda\,\phi}{h^2}\left\{B^{n+1}_{k,m-1} - 2\,B^{n+1}_{k,m} + B^{n+1}_{k,m+1}\right\} \\
&- (1-\phi)\,\frac{\lambda}{h^2}\left\{B^{n}_{k,m-1} - 2\,B^{n}_{k,m} + B^{n}_{k,m+1}\right\} = 0 \qquad (5.2.25)
\end{aligned}$$

which, after rearranging, becomes

$$\begin{aligned}
-\lambda\,\phi\, p\, B^{n+1}_{k,m-1} &- \alpha\,\phi\, p\, B^{n+1}_{k-1,m} + \left[1 + \frac{1}{2}\ell\,(\mu+\sigma) + 2\,\phi\,(\alpha\, p + \lambda\, p)\right] B^{n+1}_{k,m} \\
&- \alpha\,\phi\, p\, B^{n+1}_{k+1,m} - \lambda\,\phi\, p\, B^{n+1}_{k,m+1} - \frac{1}{2}\ell\,\beta\, C^{n}_{k,m}\, A^{n+1}_{k,m} \\
&- \frac{1}{2}\ell\,\beta\, C^{n+1}_{k,m}\, A^{n}_{k,m} = (1-\phi)\,\lambda\, p\, B^{n}_{k,m-1} + (1-\phi)\,\alpha\, p\, B^{n}_{k-1,m} \\
&+ \left[1 - \frac{1}{2}\ell\,(\mu+\sigma) - 2\,(1-\phi)(\alpha\, p + \lambda\, p)\right] B^{n}_{k,m} + (1-\phi)\,\alpha\, p\, B^{n}_{k+1,m} \\
&+ (1-\phi)\,\lambda\, p\, B^{n}_{k,m+1}\,, \qquad (5.2.26)
\end{aligned}$$

where $p = \ell/h^2$.

The local truncation error $\mathcal{L}_E = \mathcal{L}_E[S(x,y,t), E(x,y,t), I(x,y,t); h, \ell]$ associated with (5.2.26) at the point $(x,y,t) = (x_k, y_m, t_n)$ may be written down from (5.2.25): it is

$$
\begin{aligned}
\mathcal{L}_E &= \frac{E(x,y,t+\ell)-E(x,y,t)}{\ell} - \frac{1}{2}\beta\, I(x,y,t)\, S(x,y,t+\ell) \\
&- \frac{1}{2}\beta\, I(x,y,t+\ell)\, S(x,y,t) + \frac{1}{2}(\mu+\sigma)\, E(x,y,t+\ell) + \frac{1}{2}(\mu+\sigma)\, E(x,y,t) \\
&- \frac{\alpha\,\phi}{h^2}\,\{E(x-h,y,t+\ell) - 2\,E(x,y,t+\ell) + E(x+h,y,t+\ell)\} \\
&- (1-\phi)\,\frac{\alpha}{h^2}\,\{E(x-h,y,t) - 2\,E(x,y,t) + E(x+h,y,t)\} \\
&- \frac{\lambda\,\phi}{h^2}\,\{E(x,y-h,t+\ell) - 2\,E(x,y,t+\ell) + E(x,y+h,t+\ell)\} \\
&- (1-\phi)\,\frac{\lambda}{h^2}\,\{E(x,y-h,t) - 2\,E(x,y,t) + E(x,y+h,t)\} \\
&- \{E_t(x,y,t) - \beta\, I(x,y,t)\, S(x,y,t) + (\mu+\sigma)\, E(x,y,t) \\
&- \alpha\, E_{xx}(x,y,t) - \lambda\, E_{yy}(x,y,t)\}\,. \qquad (5.2.27)
\end{aligned}
$$

Expanding $E(x,y,t+\ell)$, $E(x\pm h,y,t+\ell)$, $E(x\pm h,y,t)$, $E(x,y\pm h,t+\ell)$, $E(x,y\pm h,t)$, $S(x,y,t+\ell)$ and $I(x,y,t+\ell)$ about (x,y,t) leads to

$$
\begin{aligned}
\mathcal{L}_E &= \left[\frac{1}{2}\, E_{tt} + \frac{1}{2}(\mu+\sigma)\, E_t - \frac{1}{2}\beta\, S_t\, I - \frac{1}{2}\beta\, S\, I_t - \phi\,\alpha\, E_{xxt} - \phi\,\lambda\, E_{yyt}\right]\ell \\
&- \frac{h^2}{12}(\alpha\, E_{xxxx} + \lambda\, E_{yyyy}) + \left[\frac{1}{6}\, E_{ttt} + \frac{1}{4}(\mu+\sigma)\, E_{tt} - \frac{1}{4}\beta\, S\, I_{tt}\right. \\
&- \left.\frac{1}{4}\beta\, S_{tt}\, I - \frac{1}{2}\alpha\,\phi\, E_{xxtt} - \frac{1}{2}\lambda\, E_{yytt}\right]\ell^2 + \cdots\,. \qquad (5.2.28)
\end{aligned}
$$

The second equation in (5.1.5) reveals that the coefficient of ℓ in (5.2.28) vanishes provided $\phi = 1/2$, leaving

$$
\begin{aligned}
\mathcal{L}_E &= -\frac{h^2}{12}(\alpha\, E_{xxxx} + \lambda\, E_{yyyy}) + \left[\frac{1}{6}\, E_{ttt} + \frac{1}{4}(\mu+\sigma)\, E_{tt} - \frac{1}{4}\beta\, S\, I_{tt}\right. \\
&- \left.\frac{1}{4}\beta\, S_{tt}\, I - \frac{1}{4}\alpha\, E_{xxtt} - \frac{1}{4}\lambda\, E_{yytt}\right]\ell^2 + \cdots \qquad (5.2.29)
\end{aligned}
$$

which is $O(h^2+\ell^2)$ as $h,\,\ell \longrightarrow 0$. It may be concluded that the unique $O(h^2+\ell^2)$ method, as $h,\,\ell \longrightarrow 0$, contained in the family (5.2.26) arises when $\phi = 1/2$ and is given by

$$
\begin{aligned}
-\frac{1}{2}\lambda\, p\, B^{n+1}_{k,m-1} &- \frac{1}{2}\alpha\, p\, B^{n+1}_{k-1,m} + \left[1 + \frac{1}{2}\ell\,(\mu+\sigma) + \alpha\, p + \lambda\, p\right]\, B^{n+1}_{k,m} \\
&- \frac{1}{2}\alpha\, p\, B^{n+1}_{k+1,m} - \frac{1}{2}\lambda\, p\, B^{n+1}_{k,m+1} - \frac{1}{2}\ell\,\beta\, C^{n}_{k,m}\, A^{n+1}_{k,m} \\
&- \frac{1}{2}\ell\,\beta\, C^{n+1}_{k,m}\, A^{n}_{k,m} = \frac{1}{2}\lambda\, p\, B^{n}_{k,m-1} + \frac{1}{2}\alpha\, p\, B^{n}_{k-1,m} \\
&+ \left[1 - \frac{1}{2}\ell\,(\mu+\sigma) - \alpha\, p - \lambda\, p\right]\, B^{n}_{k,m} + \frac{1}{2}\alpha\, p\, B^{n}_{k+1,m} \\
&+ \frac{1}{2}\lambda\, p\, B^{n}_{k,m+1}\,, \qquad (5.2.30)
\end{aligned}
$$

with $k,\ m = 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots$.

As in the case of S, Section 5.2.1, equation (5.2.30) must be modified for use with k or $m = 0$ and k or $m = M+1$. Applying (5.2.30) with $k,\ m = 0$ introduces the terms $B^{n}_{-1,0}$, $B^{n+1}_{-1,0}$, $B^{n}_{0,-1}$ and $B^{n+1}_{0,-1}$. Now, as noted earlier, the points (x_{-1}, y_0, t_n), (x_{-1}, y_0, t_{n+1}), (x_0, y_{-1}, t_n) and (x_0, y_{-1}, t_{n+1}) are outside the grid superimposed on $\Omega \cup \partial\Omega$. Nevertheless, the boundary conditions in (5.1.3) and (5.1.4) give, to second order, $B^{n}_{-1,0} = B^{n}_{1,0}$, $B^{n+1}_{-1,0} = B^{n+1}_{1,0}$, $B^{n}_{0,-1} = B^{n}_{0,1}$ and $B^{n+1}_{0,-1} = B^{n+1}_{0,1}$. Thus applying equation (5.2.30) with $k,\ m = 0$ gives

$$
\begin{aligned}
\left[1 + \frac{1}{2}\ell\,(\mu+\sigma)\right. &+ \left.\alpha\, p + \lambda\, p\right]\, B^{n+1}_{0,0} - \alpha\, p\, B^{n+1}_{1,0} - \lambda\, p\, B^{n+1}_{0,1} - \frac{1}{2}\ell\,\beta\, C^{n}_{0,0}\, A^{n+1}_{0,0} \\
&- \frac{1}{2}\ell\,\beta\, C^{n+1}_{0,0}\, A^{n}_{0,0} = \left[1 - \frac{1}{2}\ell\,(\mu+\sigma) - \alpha\, p - \lambda\, p\right]\, B^{n+1}_{0,0} \\
&+ \alpha\, p\, B^{n}_{1,0} + \lambda\, p\, B^{n}_{0,1}\,. \qquad (5.2.31)
\end{aligned}
$$

Applying equation (5.2.30) with $m = 0$ and $k = 1, 2, \ldots, M$ gives

$$
\begin{aligned}
-\frac{1}{2}\alpha\, p\, B^{n+1}_{k-1,0} &+ \left[1 + \frac{1}{2}\ell\,(\mu+\sigma) + \alpha\, p + \lambda\, p\right]\, B^{n+1}_{k,0} - \frac{1}{2}\alpha\, p\, B^{n+1}_{k+1,0} - \lambda\, p\, B^{n+1}_{k,1} \\
&- \frac{1}{2}\ell\,\beta\, C^{n}_{k,0}\, A^{n+1}_{k,0} - \frac{1}{2}\ell\,\beta\, C^{n+1}_{k,0}\, A^{n}_{k,0} = \frac{1}{2}\alpha\, p\, B^{n}_{k-1,0} \\
&+ \left[1 - \frac{1}{2}\ell\,(\mu+\sigma) - \alpha\, p - \lambda\, p\right]\, B^{n}_{k,0} + \frac{1}{2}\alpha\, p\, B^{n}_{k+1,0} \\
&+ \lambda\, p\, B^{n}_{k,1}\,. \qquad (5.2.32)
\end{aligned}
$$

For $m = 0$ and $k = M+1$, equation (5.2.30) gives

$$\begin{aligned} -\alpha\, p\, B^{n+1}_{M,0} &+ \left[1 + \frac{1}{2}\ell(\mu+\sigma) + \alpha\, p + \lambda\, p\right] B^{n+1}_{M+1,0} - \lambda\, p\, B^{n+1}_{M+1,1} - \frac{1}{2}\ell\,\beta\, C^{n}_{M+1,0}\, A^{n+1}_{M+1,0} \\ &- \frac{1}{2}\ell\,\beta\, C^{n+1}_{M+1,0}\, A^{n}_{M+1,0} = \alpha\, p\, B^{n}_{M,0} + \left[1 - \frac{1}{2}\ell(\mu+\sigma) - \alpha\, p - \lambda\, p\right] B^{n}_{M+1,0} \\ &+ \lambda\, p\, B^{n}_{M+1,1}\,. \end{aligned} \qquad (5.2.33)$$

When $k = 0$ and $m = 1, 2, \ldots, M$, equation (5.2.30) gives

$$\begin{aligned} -\frac{1}{2}\lambda\, p\, B^{n+1}_{0,m-1} &+ \left[1 + \frac{1}{2}\ell(\mu+\sigma) + \alpha\, p + \lambda\, p\right] B^{n+1}_{0,m} - \frac{1}{2}\lambda\, p\, B^{n+1}_{0,m+1} \\ &- \alpha\, p\, B^{n+1}_{1,m} - \frac{1}{2}\ell\,\beta\, C^{n}_{0,m}\, A^{n+1}_{0,m} - \frac{1}{2}\ell\,\beta\, C^{n+1}_{0,m}\, A^{n}_{0,m} \\ &= \frac{1}{2}\lambda\, p\, B^{n}_{0,m-1} + \left[1 - \frac{1}{2}\ell(\mu+\sigma) - \alpha\, p - \lambda\, p\right] B^{n}_{0,m} \\ &+ \frac{1}{2}\lambda\, p\, B^{n}_{0,m+1} + \alpha\, p\, B^{n}_{1,m}\,. \end{aligned} \qquad (5.2.34)$$

Equation (5.2.30) gives, for $k = 0$ and $m = M+1$,

$$\begin{aligned} -\lambda\, p\, B^{n+1}_{0,M} &+ \left[1 + \frac{1}{2}\ell(\mu+\sigma) + \alpha\, p + \lambda\, p\right] B^{n+1}_{0,M+1} - \alpha\, p\, B^{n+1}_{1,M+1} - \frac{1}{2}\ell\,\beta\, C^{n}_{0,M+1}\, A^{n+1}_{0,M+1} \\ &- \frac{1}{2}\ell\,\beta\, C^{n+1}_{0,M+1}\, A^{n}_{0,M+1} = \lambda\, p\, B^{n}_{0,M} + \left[1 - \frac{1}{2}\ell(\mu+\sigma) - \alpha\, p - \lambda\, p\right] B^{n}_{0,M+1} \\ &+ \alpha\, p\, B^{n}_{1,M+1}\,. \end{aligned} \qquad (5.2.35)$$

For $m = M+1$ and $k = 1, 2, \ldots, M$, equation (5.2.30) gives

$$\begin{aligned} -\frac{1}{2}\alpha\, p\, B^{n+1}_{k-1,M+1} &+ \left[1 + \frac{1}{2}\ell(\mu+\sigma) + \alpha\, p + \lambda\, p\right] B^{n+1}_{k,M+1} - \frac{1}{2}\alpha\, p\, B^{n+1}_{k+1,M+1} \\ &- \lambda\, p\, B^{n+1}_{k,M} - \frac{1}{2}\ell\,\beta\, C^{n}_{k,M+1}\, A^{n+1}_{k,M+1} - \frac{1}{2}\ell\,\beta\, C^{n+1}_{k,M+1}\, A^{n}_{k,M+1} \\ &= \frac{1}{2}\alpha\, p\, B^{n}_{k-1,M+1} + \left[1 - \frac{1}{2}\ell(\mu+\sigma) - \alpha\, p - \lambda\, p\right] B^{n}_{k,M+1} \\ &+ \frac{1}{2}\alpha\, p\, B^{n}_{k+1,M+1} + \lambda\, p\, B^{n}_{k,M}\,. \end{aligned} \qquad (5.2.36)$$

When $k = M+1$ and $m = 1, 2, \ldots, M$, equation (5.2.30) gives

$$
\begin{aligned}
-\frac{1}{2}\lambda\, p\, B^{n+1}_{M+1,m-1} &+ \left[1+\frac{1}{2}\ell(\mu+\sigma)+\alpha\, p+\lambda\, p\right] B^{n+1}_{M+1,m} - \frac{1}{2}\lambda\, p\, B^{n+1}_{M+1,m+1} \\
&- \alpha\, p\, B^{n+1}_{M,m} - \frac{1}{2}\ell\,\beta\, C^{n}_{M+1,m}\, A^{n+1}_{M+1,m} - \frac{1}{2}\ell\,\beta\, C^{n+1}_{M+1,m}\, A^{n}_{M+1,m} \\
&= \frac{1}{2}\lambda\, p\, B^{n}_{M+1,m-1} + \left[1-\frac{1}{2}\ell(\mu+\sigma)-\alpha\, p-\lambda\, p\right] B^{n}_{M+1,m} \\
&+ \frac{1}{2}\lambda\, p\, B^{n}_{M+1,m+1} + \alpha\, p\, B^{n}_{M,m}\,, \qquad (5.2.37)
\end{aligned}
$$

and finally, when $k = M+1$ and $m = M+1$, equation (5.2.30) gives

$$
\begin{aligned}
-\lambda\, p\, B^{n+1}_{M+1,M} &- \alpha\, p\, B^{n+1}_{M,M+1} + \left[1+\frac{1}{2}\ell(\mu+\sigma)+\alpha\, p+\lambda\, p\right] B^{n+1}_{M+1,M+1} \\
&- \frac{1}{2}\ell\,\beta\, C^{n}_{M+1,M+1}\, A^{n+1}_{M+1,M+1} - \frac{1}{2}\ell\,\beta\, C^{n+1}_{M+1,M+1}\, A^{n}_{M+1,M+1} = \lambda\, p\, B^{n}_{M+1,M} \\
&+ \alpha\, p\, B_{M,M+1} + \left[1-\frac{1}{2}\ell(\mu+\sigma)-\alpha\, p-\lambda\, p\right] B^{n}_{M+1,M+1}\,. \qquad (5.2.38)
\end{aligned}
$$

5.2.3 Numerical Method for I

The time derivative in the third equation in (5.1.1) is approximated by the first-order forward-difference replacement

$$
I_t(x,y,t) \approx [I(x,y,t+\ell) - I(x,y,t)]/\ell\,. \qquad (5.2.39)
$$

The x-derivative is approximated by the weighted approximant

$$
\begin{aligned}
I_{xx} &\approx h^{-2}\,[\phi\{I(x-h,y,t+\ell) - 2\,I(x,y,t+\ell) + I(x+h,y,t+\ell)\} \\
&+ (1-\phi)\{I(x-h,y,t) - 2\,I(x,y,t) + I(x+h,y,t)\}]\,, \qquad (5.2.40)
\end{aligned}
$$

and the y-derivative is approximated by the weighted approximant

$$
\begin{aligned}
I_{yy} &\approx h^{-2}\,[\phi\{I(x,y-h,t+\ell) - 2\,I(x,y,t+\ell) + I(x,y+h,t+\ell)\} \\
&+ (1-\phi)\{I(x,y-h,t) - 2\,I(x,y,t) + I(x,y+h,t)\}]\,, \qquad (5.2.41)
\end{aligned}
$$

in which $x = x_k$, $y = y_m\,(k,m = 0,1,2,\ldots,M,M+1)$, $t = t_n\,(n = 0,1,2,\ldots)$ and $\phi\,(0 \leq \phi \leq 1)$ is a parameter. Both equations (5.2.40) and (5.2.41) are $O(h^2)$ as $h \longrightarrow 0$ when $\phi = 0$ and are $O(h^2 + \ell)$ as $h, \ell \longrightarrow 0$ when $0 < \phi \leq 1$.

Now equations (3.6.34) and (3.6.37) will be used to obtain an approximation to $I(x_k, y_m, t_{n+1})$ for use in the third equation in (5.1.1). To reach this approximation, equations (3.6.34) and (5.2.39) are used in the third equation in (5.1.1) to give

$$\frac{C^{n+1}_{k,m} - C^{n}_{k,m}}{\ell} - \sigma\, B^{n+1}_{k,m} + (\mu + \gamma)\, C^{n+1}_{k,m} - \alpha\, I_{xx} - \lambda\, I_{yy} = 0 \qquad (5.2.42)$$

while using equations (3.6.37) and (5.2.39) gives

$$\frac{C^{n+1}_{k,m} - C^{n}_{k,m}}{\ell} - \sigma\, B^{n}_{k,m} + (\mu + \gamma)\, C^{n}_{k,m} - \alpha\, I_{xx} - \lambda\, I_{yy} = 0\,. \qquad (5.2.43)$$

In order to obtain a second-order approximation to $I(x_k, y_m, t_{n+1})$, equations (5.2.42) and (5.2.43) must be added. Averaging equations (5.2.42) and (5.2.43) and substituting for I_{xx} and I_{yy} from (5.2.40) and (5.2.41), respectively, gives

$$\begin{aligned}
\frac{C^{n+1}_{k,m} - C^{n}_{k,m}}{\ell} &- \frac{1}{2}\sigma\, B^{n+1}_{k,m} - \frac{1}{2}\sigma\, B^{n}_{k,m} + \frac{1}{2}(\mu+\gamma)\, C^{n+1}_{k,m} \\
&+ \frac{1}{2}(\mu+\gamma)\, C^{n}_{k,m} - \frac{\alpha\,\phi}{h^2}\left\{C^{n+1}_{k-1,m} - 2\,C^{n+1}_{k,m} + C^{n+1}_{k+1,m}\right\} \\
&- (1-\phi)\,\frac{\alpha}{h^2}\left\{C^{n}_{k-1,m} - 2\,C^{n}_{k,m} + C^{n}_{k+1,m}\right\} \\
&- \frac{\lambda\,\phi}{h^2}\left\{C^{n+1}_{k,m-1} - 2\,C^{n+1}_{k,m} + C^{n+1}_{k,m+1}\right\} \\
&- (1-\phi)\,\frac{\lambda}{h^2}\left\{C^{n}_{k,m-1} - 2\,C^{n}_{k,m} + C^{n}_{k,m+1}\right\} = 0
\end{aligned} \qquad (5.2.44)$$

which, after rearranging, becomes

$$\begin{aligned}
-\lambda\,\phi\,p\,C^{n+1}_{k,m-1} &- \alpha\,\phi\,p\,C^{n+1}_{k-1,m} + \left[1 + \frac{1}{2}\ell\,(\mu+\gamma) + 2\,\phi\,(\alpha\,p + \lambda\,p)\right] C^{n+1}_{k,m} \\
&- \alpha\,\phi\,p\,C^{n+1}_{k+1,m} - \lambda\,\phi\,p\,C^{n+1}_{k,m+1} - \frac{1}{2}\ell\,\sigma\, B^{n+1}_{k,m} = (1-\phi)\,\lambda\,p\,C^{n}_{k,m-1} \\
&+ (1-\phi)\,\alpha\,p\,C^{n}_{k-1,m} + \left[1 - \frac{1}{2}\ell\,(\mu+\gamma) - 2\,(1-\phi)(\alpha\,p + \lambda\,p)\right] C^{n}_{k,m} \\
&+ (1-\phi)\,\alpha\,p\,C^{n}_{k+1,m} + (1-\phi)\,\lambda\,p\,C^{n}_{k,m+1} + \frac{1}{2}\ell\,\sigma\, B^{n}_{k,m}\,,
\end{aligned} \qquad (5.2.45)$$

where $p = \ell/h^2$.

The local truncation error $\mathcal{L}_I = \mathcal{L}_I[S(x,y,t), E(x,y,t), I(x,y,t); h, \ell]$ associated with (5.2.45) at the point $(x,y,t) = (x_k, y_m, t_n)$ may be written down from (5.2.44): it is

$$\begin{aligned}
\mathcal{L}_I &= \frac{I(x,y,t+\ell) - I(x,y,t)}{\ell} - \frac{1}{2}\sigma E(x,y,t+\ell) - \frac{1}{2}\sigma E(x,y,t) + \frac{1}{2}(\mu+\gamma) I(x,y,t+\ell) \\
&+ \frac{1}{2}(\mu+\gamma) I(x,y,t) - \frac{\alpha\phi}{h^2}\{I(x-h,y,t+\ell) - 2I(x,y,t+\ell) + I(x+h,y,t+\ell)\} \\
&- (1-\phi)\frac{\alpha}{h^2}\{I(x-h,y,t) - 2I(x,y,t) + I(x+h,y,t)\} \\
&- \frac{\lambda\phi}{h^2}\{I(x,y-h,t+\ell) - 2I(x,y,t+\ell) + I(x,y+h,t+\ell)\} \\
&- (1-\phi)\frac{\lambda}{h^2}\{I(x,y-h,t) - 2I(x,y,t) + I(x,y+h,t)\} \\
&- \{I_t(x,y,t) - \sigma E(x,y,t) + (\mu+\gamma) I(x,y,t) \\
&- \alpha I_{xx}(x,y,t) - \lambda I_{yy}(x,y,t)\}\,. \qquad (5.2.46)
\end{aligned}$$

Expanding $I(x,y,t+\ell)$, $I(x \pm h, y, t+\ell)$, $I(x \pm h, y, t)$, $I(x, y \pm h, t+\ell)$, $I(x, y \pm h, t)$ and $E(x,y,t+\ell)$ about (x,y,t) leads to

$$\begin{aligned}
\mathcal{L}_I &= \left[\frac{1}{2} I_{tt} - \frac{1}{2}\sigma E_t + \frac{1}{2}(\mu+\gamma) I_t - \phi\alpha I_{xxt} - \phi\lambda I_{yyt}\right]\ell \\
&- \frac{h^2}{12}(\alpha I_{xxxx} + \lambda I_{yyyy}) + \left[\frac{1}{6} I_{ttt} - \frac{1}{4}\sigma E_{tt} + \frac{1}{4}(\mu+\gamma) I_{tt}\right. \\
&- \left.\frac{1}{2}\phi\alpha I_{xxtt} - \frac{1}{2}\phi\lambda I_{yytt}\right]\ell^2 + \cdots\,. \qquad (5.2.47)
\end{aligned}$$

The third equation in (5.1.5) shows that, for $\phi = 1/2$, the coefficient of ℓ in (5.2.47) vanishes leaving

$$\begin{aligned}
\mathcal{L}_I &= -\frac{h^2}{12}(\alpha I_{xxxx} + \lambda I_{yyyy}) + \left[\frac{1}{6} I_{ttt} - \frac{1}{4}\sigma E_{tt} + \frac{1}{4}(\mu+\gamma) I_{tt}\right. \\
&- \left.\frac{1}{4}\alpha I_{xxtt} - \frac{1}{4}\lambda I_{yytt}\right]\ell^2 + \cdots \qquad (5.2.48)
\end{aligned}$$

which is $O(h^2 + \ell^2)$ as $h, \ell \longrightarrow 0$. It may be concluded that the unique $O(h^2 + \ell^2)$ method, as $h, \ell \longrightarrow 0$, contained in the family (5.2.45) arises when $\phi = 1/2$ and is given by

$$
\begin{aligned}
-\frac{1}{2}\lambda\, p\, C^{n+1}_{k,m-1} &- \frac{1}{2}\alpha\, p\, C^{n+1}_{k-1,m} + \left[1 + \frac{1}{2}\ell\,(\mu+\gamma) + \alpha\, p + \lambda\, p\right] C^{n+1}_{k,m} - \frac{1}{2}\alpha\, p\, C^{n+1}_{k+1,m} \\
&- \frac{1}{2}\lambda\, p\, C^{n+1}_{k,m+1} - \frac{1}{2}\ell\,\sigma\, B^{n+1}_{k,m} = \frac{1}{2}\lambda\, p\, C^{n}_{k,m-1} + \frac{1}{2}\alpha\, p\, C^{n}_{k-1,m} \\
&+ \left[1 - \frac{1}{2}\ell\,(\mu+\gamma) - \alpha\, p - \lambda\, p\right] C^{n}_{k,m} + \frac{1}{2}\alpha\, p\, C^{n}_{k+1,m} \\
&+ \frac{1}{2}\lambda\, p\, C^{n}_{k,m+1} + \frac{1}{2}\ell\,\sigma\, B^{n}_{k,m}\,. \qquad (5.2.49)
\end{aligned}
$$

The finite-difference method (5.2.49) may be applied for $k,\ m = 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots$. For k or $m = 0$ and k or $m = M+1$ it requires some modification. Applying (5.2.49) with k and $m = 0$ introduces the terms $C^{n}_{-1,0}$, $C^{n+1}_{-1,0}$, $C^{n}_{0,-1}$ and $C^{n+1}_{0,-1}$. Now, as before, the points (x_{-1}, y_0, t_n), (x_{-1}, y_0, t_{n+1}), (x_0, y_{-1}, t_n) and (x_0, y_{-1}, t_{n+1}) are outside the grid superimposed on $\Omega \cup \partial\Omega$. However, the boundary conditions in (5.1.3) and (5.1.4) give, to second order, $C^{n}_{-1,0} = C^{n}_{1,0}$, $C^{n+1}_{-1,0} = C^{n+1}_{1,0}$, $C^{n}_{0,-1} = C^{n}_{0,1}$ and $C^{n+1}_{0,-1} = C^{n+1}_{0,1}$. Thus applying equation (5.2.49) with $k,\ m = 0$ gives

$$
\begin{aligned}
\left[1 + \frac{1}{2}\ell\,(\mu\right. &+ \left.\gamma) + \alpha\, p + \lambda\, p\right] C^{n+1}_{0,0} - \alpha\, p\, C^{n+1}_{1,0} - \lambda\, p\, C^{n+1}_{0,1} - \frac{1}{2}\ell\,\sigma\, B^{n+1}_{0,0} \\
&= \left[1 - \frac{1}{2}\ell\,(\mu+\gamma) - \alpha\, p - \lambda\, p\right] C^{n}_{0,0} + \alpha\, p\, C^{n}_{1,0} + \lambda\, p\, C^{n}_{0,1} \\
&+ \frac{1}{2}\ell\,\sigma\, B^{n}_{0,0}\,. \qquad (5.2.50)
\end{aligned}
$$

Applying equation (5.2.49) with $m = 0$ and $k = 1, 2, \ldots, M$ gives

$$
\begin{aligned}
-\frac{1}{2}\alpha\, p\, C^{n+1}_{k-1,0} &+ \left[1 + \frac{1}{2}\ell\,(\mu+\gamma) + \alpha\, p + \lambda\, p\right] C^{n+1}_{k,0} - \frac{1}{2}\alpha\, p\, C^{n+1}_{k+1,0} - \lambda\, p\, C^{n+1}_{k,1} \\
&- \frac{1}{2}\ell\,\sigma\, B^{n+1}_{k,0} = \frac{1}{2}\alpha\, p\, C^{n}_{k-1,0} + \left[1 - \frac{1}{2}\ell\,(\mu+\gamma) - \alpha\, p - \lambda\, p\right] C^{n}_{k,0} \\
&+ \frac{1}{2}\alpha\, p\, C^{n}_{k+1,0} + \lambda\, p\, C^{n}_{k,1} + \frac{1}{2}\ell\,\sigma\, B^{n}_{k,0}\,. \qquad (5.2.51)
\end{aligned}
$$

For $m = 0$ and $k = M+1$, equation (5.2.49) gives

$$
-\alpha\, p\, C^{n+1}_{M,0} + \left[1 + \frac{1}{2}\ell\,(\mu+\gamma) + \alpha\, p + \lambda\, p\right] C^{n+1}_{M+1,0} - \lambda\, p\, C^{n+1}_{M+1,1} - \frac{1}{2}\ell\,\sigma\, B^{n+1}_{M+1,0}
$$

$$= \alpha p C^n_{M,0} + \left[1 - \frac{1}{2}\ell(\mu+\gamma) - \alpha p - \lambda p\right] C^n_{M+1,0} + \lambda p C^n_{M+1,1}$$
$$+ \frac{1}{2}\ell\sigma B^n_{M+1,0}. \qquad (5.2.52)$$

When $k = 0$ and $m = 1, 2, \ldots, M$, equation (5.2.49) gives

$$-\frac{1}{2}\lambda p C^{n+1}_{0,m-1} + \left[1 + \frac{1}{2}\ell(\mu+\gamma) + \alpha p + \lambda p\right] C^{n+1}_{0,m} - \frac{1}{2}\lambda p C^{n+1}_{0,m+1} - \alpha p C^{n+1}_{1,m}$$
$$- \frac{1}{2}\ell\sigma B^{n+1}_{0,m} = \frac{1}{2}\lambda p C^n_{0,m-1} + \left[1 - \frac{1}{2}\ell(\mu+\gamma) - \alpha p - \lambda p\right] C^n_{0,m}$$
$$+ \frac{1}{2}\lambda p C^n_{0,m+1} + \alpha p C^n_{1,m} + \frac{1}{2}\ell\sigma B^n_{0,m}. \qquad (5.2.53)$$

Equation (5.2.49), for $k = 0$ and $m = M + 1$, gives

$$-\lambda p C^{n+1}_{0,M} + \left[1 + \frac{1}{2}\ell(\mu+\gamma) + \alpha p + \lambda p\right] C^{n+1}_{0,M+1} - \alpha p C^{n+1}_{1,M+1} - \frac{1}{2}\ell\sigma B^{n+1}_{0,M+1}$$
$$= \lambda p C^n_{0,M} + \left[1 - \frac{1}{2}\ell(\mu+\gamma) - \alpha p - \lambda p\right] C^n_{0,M+1} + \alpha p C^n_{1,M+1}$$
$$+ \frac{1}{2}\ell\sigma B^n_{0,M+1}. \qquad (5.2.54)$$

When $m = M + 1$ and $k = 1, 2, \ldots, M$, equation (5.2.49) gives

$$-\frac{1}{2}\alpha p C^{n+1}_{k-1,M+1} + \left[1 + \frac{1}{2}\ell(\mu+\gamma) + \alpha p + \lambda p\right] C^{n+1}_{k,M+1} - \frac{1}{2}\alpha p C^{n+1}_{k+1,M+1} - \lambda p C^{n+1}_{k,M}$$
$$- \frac{1}{2}\ell\sigma B^{n+1}_{k,M+1} = \frac{1}{2}\alpha p C^n_{k-1,M+1} + \left[1 - \frac{1}{2}\ell(\mu+\gamma) - \alpha p - \lambda p\right] C^n_{k,M+1}$$
$$+ \frac{1}{2}\alpha p C^n_{k+1,M+1} + \lambda p C^n_{k,M} + \frac{1}{2}\ell\sigma B^n_{k,M+1}. \qquad (5.2.55)$$

For $k = M + 1$ and $m = 1, 2, \ldots, M$, equation (5.2.49) gives

$$-\frac{1}{2}\lambda p C^{n+1}_{M+1,m-1} + \left[1 + \frac{1}{2}\ell(\mu+\gamma) + \alpha p + \lambda p\right] C^{n+1}_{M+1,m} - \frac{1}{2}\lambda p C^{n+1}_{M+1,m+1} - \alpha p C^{n+1}_{M,m}$$
$$- \frac{1}{2}\ell\sigma B^{n+1}_{M+1,m} = \frac{1}{2}\lambda p C^n_{M+1,m-1} + \left[1 - \frac{1}{2}\ell(\mu+\gamma) - \alpha p - \lambda p\right] C^n_{M+1,m}$$
$$+ \frac{1}{2}\lambda p C^n_{M+1,m+1} + \alpha p C^n_{M,m} + \frac{1}{2}\ell\sigma B^n_{M+1,m}, \qquad (5.2.56)$$

and finally, when k and $m = M+1$, equation (5.2.49) gives

$$\begin{aligned} -\lambda\, p\, C^{n+1}_{M+1,M} &- \alpha\, p\, C^{n+1}_{M,M+1} + \left[1 + \frac{1}{2}\ell\,(\mu+\gamma) + \alpha\, p + \lambda\, p\right] C^{n+1}_{M+1,M+1} - \frac{1}{2}\ell\,\sigma\, B^{n+1}_{M+1,M+1} \\ &= \lambda\, p\, C^{n}_{M+1,M} + \alpha\, p\, C^{n}_{M,M+1} + \left[1 - \frac{1}{2}\ell\,(\mu+\gamma) - \alpha\, p - \lambda\, p\right] C^{n}_{M+1,M+1} \\ &+ \frac{1}{2}\ell\,\sigma\, B^{n}_{M+1,M+1}\,. \qquad (5.2.57) \end{aligned}$$

5.3 Implementation

Let $\mathbf{U}^{n+1} = \left[(\mathbf{A}^{n+1})^T, (\mathbf{B}^{n+1})^T, (\mathbf{C}^{n+1})^T\right]^T$ and $\mathbf{U}^{n} = \left[(\mathbf{A}^{n})^T, (\mathbf{B}^{n})^T, (\mathbf{C}^{n})^T\right]^T$, where $\mathbf{A}^n$, $\mathbf{B}^n$, $\mathbf{C}^n$, $\mathbf{A}^{n+1}$, $\mathbf{B}^{n+1}$ and $\mathbf{C}^{n+1}$ are as defined in Section 5.1 and T denotes transpose. Then equations (5.2.11)–(5.2.19), (5.2.30)– (5.2.38) and (5.2.49)–(5.2.57) may be written in matrix-vector form as

$$W^n\,\mathbf{U}^{n+1} = M^n\,\mathbf{U}^n + \mathbf{b} \qquad (5.3.1)$$

in which

$$W^n = \begin{pmatrix} X^n & O & H^n \\ F^n & Y^n & G^n \\ O & J_1^n & Z^n \end{pmatrix}, \qquad (5.3.2)$$

$$M^n = \begin{pmatrix} P^n & O & O \\ O & Q^n & O \\ O & J_2^n & R^n \end{pmatrix}, \qquad (5.3.3)$$

where O is the zero matrix of order $(M+2)^2$ and the vector $\mathbf{b}$ is a column-vector of order $3(M+2)^2$ and is given by

$$\mathbf{b} = \Big[\underbrace{\ell\,\mu\,N, \ldots, \ell\,\mu\,N}_{(M+2)^2\ times}, 0, \ldots, 0\Big]^T. \qquad (5.3.4)$$

The matrices W^n and M^n are both of order $3(M+2)^2$ and their submatrices are as given below. The submatrices X^n, Y^n, Z^n, P^n, Q^n and R^n are block tridiagonal of order $(M+2)^2$ with tridiagonal blocks on the diagonal and block diagonal off the diagonal and are given by

$$X^n = \begin{pmatrix} XD_0 & X1 & & & & \\ X2 & XD_1 & X2 & & & \\ & X2 & XD_2 & X2 & & \\ & \ddots & \ddots & \ddots & & \\ & & X2 & XD_M & X2 \\ & & & X1 & XD_{M+1} \end{pmatrix}, \qquad (5.3.5)$$

where

$$XD_i = \begin{pmatrix} v_{0,i} & -\alpha p & & & & \\ -\frac{1}{2}\alpha p & v_{1,i} & -\frac{1}{2}\alpha p & & & \\ & -\frac{1}{2}\alpha p & v_{2,i} & -\frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{2}\alpha p & v_{M,i} & -\frac{1}{2}\alpha p \\ & & & -\alpha p & v_{M+1,i} \end{pmatrix}_{(M+2)\times(M+2)}, \qquad (5.3.6)$$

in which

$$v_{j,i} = 1 + \frac{1}{2}\ell(\mu + \beta C^n_{j,i}) + \alpha p + \lambda p \quad (i, j, = 0, 1, 2, \dots, M, M+1), \qquad (5.3.7)$$

$X1 = -\lambda p I_{(M+2)}$, $X2 = -\frac{1}{2}\lambda p I_{(M+2)}$ and $I_{(M+2)}$ is the identity matrix of order $(M+2)$,

$$Y^n = \begin{pmatrix} YD & Y1 & & & & \\ Y2 & YD & Y2 & & & \\ & Y2 & YD & Y2 & & \\ & \ddots & \ddots & \ddots & & \\ & & Y2 & YD & Y2 \\ & & & Y1 & YD \end{pmatrix}, \qquad (5.3.8)$$

where

$$YD = \begin{pmatrix} v_Y & -\alpha\, p & & & & \\ -\frac{1}{2}\,\alpha\, p & v_Y & -\frac{1}{2}\,\alpha\, p & & & \\ & -\frac{1}{2}\,\alpha\, p & v_Y & -\frac{1}{2}\,\alpha\, p & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{2}\,\alpha\, p & v_Y & -\frac{1}{2}\,\alpha\, p & \\ & & & -\alpha\, p & v_Y & \end{pmatrix}_{(M+2)\times(M+2)}, \qquad (5.3.9)$$

in which

$$v_Y = 1 + \frac{1}{2}\,\ell\,(\mu + \sigma) + \alpha\, p + \lambda\, p, \qquad (5.3.10)$$

$Y1 = -\lambda\, p\, I_{(M+2)}$, $Y2 = -\frac{1}{2}\,\lambda\, p\, I_{(M+2)}$,

$$Z^n = \begin{pmatrix} ZD & Z1 & & & \\ Z2 & ZD & Z2 & & \\ & Z2 & ZD & Z2 & \\ & \ddots & \ddots & \ddots & \\ & & Z2 & ZD & Z2 \\ & & & Z1 & ZD \end{pmatrix}, \qquad (5.3.11)$$

where

$$ZD = \begin{pmatrix} v_Z & -\alpha\, p & & & & \\ -\frac{1}{2}\,\alpha\, p & v_Z & -\frac{1}{2}\,\alpha\, p & & & \\ & -\frac{1}{2}\,\alpha\, p & v_Z & -\frac{1}{2}\,\alpha\, p & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{2}\,\alpha\, p & v_Z & -\frac{1}{2}\,\alpha\, p & \\ & & & -\alpha\, p & v_Z & \end{pmatrix}_{(M+2)\times(M+2)}, \qquad (5.3.12)$$

in which

$$v_Z = 1 + \frac{1}{2}\,\ell\,(\mu + \gamma) + \alpha\, p + \lambda\, p, \qquad (5.3.13)$$

$Z1 = -\lambda\, p\, I_{(M+2)}$, $Z2 = -\frac{1}{2}\,\lambda\, p\, I_{(M+2)}$,

$$P^n = \begin{pmatrix} PD & P1 & & & & \\ P2 & PD & P2 & & & \\ & P2 & PD & P2 & & \\ & \ddots & \ddots & \ddots & & \\ & & & P2 & PD & P2 \\ & & & & P1 & PD \end{pmatrix}, \tag{5.3.14}$$

where

$$PD = \begin{pmatrix} v_P & \alpha\, p & & & & \\ \frac{1}{2}\alpha\, p & v_P & \frac{1}{2}\alpha\, p & & & \\ & \frac{1}{2}\alpha\, p & v_P & \frac{1}{2}\alpha\, p & & \\ & \ddots & \ddots & \ddots & & \\ & & & \frac{1}{2}\alpha\, p & v_P & \frac{1}{2}\alpha\, p \\ & & & & \alpha\, p & v_P \end{pmatrix}_{(M+2)\times(M+2)}, \tag{5.3.15}$$

in which

$$v_P = 1 - \frac{1}{2}\,\ell - \alpha\, p - \lambda\, p, \tag{5.3.16}$$

$P1 = \lambda\, p\, I_{(M+2)}$, $P2 = \frac{1}{2}\,\lambda\, I_{(M+2)}$,

$$Q^n = \begin{pmatrix} QD & Q1 & & & & \\ Q2 & QD & Q2 & & & \\ & Q2 & QD & Q2 & & \\ & \ddots & \ddots & \ddots & & \\ & & & Q2 & QD & Q2 \\ & & & & Q1 & QD \end{pmatrix}, \tag{5.3.17}$$

where

$$QD = \begin{pmatrix} v_Q & \alpha p & & & & \\ \frac{1}{2}\alpha p & v_Q & \frac{1}{2}\alpha p & & & \\ & \frac{1}{2}\alpha p & v_Q & \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & & \frac{1}{2}\alpha p & v_Q & \frac{1}{2}\alpha p \\ & & & & \alpha p & v_Q \end{pmatrix}_{(M+2)\times(M+2)}, \qquad (5.3.18)$$

in which

$$v_Q = 1 - \frac{1}{2}\ell(\mu+\sigma) - \alpha p - \lambda p, \qquad (5.3.19)$$

$Q1 = \lambda p I_{(M+2)}$, $Q2 = \frac{1}{2}\lambda p I_{(M+2)}$,

$$R^n = \begin{pmatrix} RD & R1 & & & & \\ R2 & RD & R2 & & & \\ & R2 & RD & R2 & & \\ & \ddots & \ddots & \ddots & & \\ & & & R2 & RD & R2 \\ & & & & R1 & RD \end{pmatrix}, \qquad (5.3.20)$$

where

$$RD = \begin{pmatrix} v_R & \alpha p & & & & \\ \frac{1}{2}\alpha p & v_R & \frac{1}{2}\alpha p & & & \\ & \frac{1}{2}\alpha p & v_R & \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & & \frac{1}{2}\alpha p & v_R & \frac{1}{2}\alpha p \\ & & & & \alpha p & v_R \end{pmatrix}_{(M+2)\times(M+2)}, \qquad (5.3.21)$$

in which

$$v_R = 1 - \frac{1}{2}\ell(\mu+\gamma) - \alpha p - \lambda p, \qquad (5.3.22)$$

$R1 = \lambda p I_{(M+2)}$ and $R2 = \frac{1}{2}\lambda p I_{(M+2)}$.

The matrices F^n, G^n, H^n, J_1^n and J_2^n are block-diagonal of order $(M+2)^2$ and are of the form

$$F^n = diag\left\{-\frac{1}{2}\,\ell\,\beta\,C_{k,m}^n\right\}, \qquad G^n = diag\left\{-\tfrac{1}{2}\,\ell\,\beta\,\mathcal{A}_{k,m}^n\right\}, \qquad H^n = diag\left\{\frac{1}{2}\,\ell\,\beta\,\mathcal{A}_{k,m}^n\right\},$$

$$J_1^n = diag\left\{-\frac{1}{2}\,\ell\,\beta\,I_{(M+2)}\right\} \quad and \quad J_2^n = diag\left\{\tfrac{1}{2}\,\ell\,\sigma\,I_{(M+2)}\right\}, \qquad (5.3.23)$$

where $\mathcal{A}_{k,m}^n$ and $\mathcal{C}_{k,m}^n$ are diagonal matrices of orders $(M+2)$ and are given by

$$\mathcal{A}_{k,m}^n = diag\left\{A_{k,m}^n\right\} \quad and \quad \mathcal{C}_{k,m}^n = diag\left\{C_{k,m}^n\right\}. \qquad (5.3.24)$$

5.4 Stability Analysis

As in the previous chapter, the maximum principle analysis will be used to discuss the stability of the finite-difference approximation as both the von Neumann and matrix stability methods are not practical.

In order to use the maximum principle analysis to examine convergence and stability, assume that a solution of the initial/boundary-value problem (5.1.1)$-$(5.1.4) exists in the closed region $[CR: 0 \le x \le 1, 0 \le y \le 1, 0 \le t \le T]$ and that the solution is six times continuously differentiable in the closed region.

Let the first equation in (5.1.1) be written in the form

$$\begin{aligned} \alpha\,S_{xx} + \lambda\,S_{yy} &= S_t - \mu\,N + \frac{1}{2}(\mu + \beta\,I)\,S + \frac{1}{2}(\mu + \beta\,I)\,S \\ &= S_t - \mu\,N + G_1 + G_2 \qquad (5.4.1) \end{aligned}$$

with initial and boundary conditions given in (5.1.2)$-$(5.1.4). The terms G_1 and G_2 are assumed to be boundedly-differentiable with respect to S and I.

Consider, as a direct extension of the procedure given in (4.6.2), the difference equations

$$\left.\begin{array}{rcl} A^0_{k,m} & = & S^0 \\ \frac{\alpha}{2}\,\nabla^2_x\left(A^{n+1}_{k,m}+A^n_{k,m}\right) & + & \frac{\lambda}{2}\,\nabla^2_y\left(A^{n+1}_{k,m}+A^n_{k,m}\right)=\frac{A^{n+1}_{k,m}-A^n_{k,m}}{\ell}-\mu\,N \\ & + & \frac{1}{2}\left(\mu+\beta\,C^n_{k,m}\right)A^{n+1}_{k,m}+\frac{1}{2}\left(\mu+\beta\,C^{n+1}_{k,m}\right)A^n_{k,m}\,, \quad n\geq 0\,, \end{array}\right\} \qquad (\ 5.4.2\ )$$

where $A^n_{k,m}$ is an approximation to $S(x,y,t)$ at the typical point $(kh, mh, n\ell)$ which agrees with S on the boundaries. The operators ∇^2_x and ∇^2_y are defined as follows

$$\begin{array}{rcl} \nabla^2_x A^n_{k,m} & = & \left(A^n_{k-1,m}-2\,A^n_{k,m}+A^n_{k+1,m}\right)/h^2\,, \\ \nabla^2_y A^n_{k,m} & = & \left(A^n_{k,m-1}-2\,A^n_{k,m}+A^n_{k,m+1}\right)/h^2\,. \end{array} \qquad (\ 5.4.3\ )$$

Following Section 4.6 of Chapter 4, it is easy to see that

$$\begin{array}{rcl} \frac{\alpha}{2}\,\nabla^2_x\left(S^{n+1}_{k,m}\right. & + & \left.S^n_{k,m}\right)+\frac{\lambda}{2}\,\nabla^2_y(S^{n+1}_{k,m}+S^n_{k,m})=\frac{S^{n+1}_{k,m}-S^n_{k,m}}{\ell}-\mu\,N \\ & + & \frac{1}{2}\left(\mu+\beta\,I^n_{k,m}\right)S^{n+1}_{k,m}+\frac{1}{2}\left(\mu+\beta\,I^{n+1}_{k,m}\right)S^n_{k,m}+g^n_{k,m} \end{array} \qquad (\ 5.4.4\ )$$

with

$$g^n_{k,m}=O(h^2+\ell)\,. \qquad (\ 5.4.5\ )$$

Let

$$\left.\begin{array}{rcl} Z^n_{1_{k,m}} & = & S^n_{k,m}-A^n_{k,m} \\ Z^n_{2_{k,m}} & = & E^n_{k,m}-B^n_{k,m} \\ Z^n_{3_{k,m}} & = & I^n_{k,m}-C^n_{k,m} \end{array}\right\}\,, \qquad (\ 5.4.6\ )$$

then the basic truncation error equation is obtained, by subtracting equation (5.4.2) from (5.4.4), as follows

$$\left.\begin{array}{rcl} Z^0_{1_{k,m}} & = & 0\,, \\ \frac{\alpha}{2}\,\nabla^2_x\left(Z^{n+1}_{1_{k,m}}+Z^n_{1_{k,m}}\right) & + & \frac{\lambda}{2}\,\nabla^2_y\left(Z^{n+1}_{1_{k,m}}+Z^n_{1_{k,m}}\right)=\frac{Z^{n+1}_{1_{k,m}}-Z^n_{1_{k,m}}}{\ell} \\ & + & \underline{\frac{1}{2}\left(\mu+\beta\,I^n_{k,m}\right)S^{n+1}_{k,m}}-\frac{1}{2}\left(\mu+\beta\,C^n_{k,m}\right)A^{n+1}_{k,m} \\ & + & \underline{\frac{1}{2}\left(\mu+\beta\,I^{n+1}_{k,m}\right)S^n_{k,m}}-\frac{1}{2}\left(\mu+\beta\,C^{n+1}_{k,m}\right)A^n_{k,m}+g^n_{k,m}\,. \\ Z^n_{1_{k,m}} & = & 0 \qquad (\textit{on boundary})\,. \end{array}\right\} \qquad (\ 5.4.7\ )$$

As the underlined terms can be written in the form of equations (4.6.10) and (4.6.11), respectively, it follows that equation (5.4.7) can be written as

$$\frac{\alpha}{2}\nabla_x^2\left(Z_{1_{k,m}}^{n+1} + Z_{1_{k,m}}^{n}\right) + \frac{\lambda}{2}\nabla_y^2\left(Z_{1_{k,m}}^{n+1} + Z_{1_{k,m}}^{n}\right) = \frac{Z_{1_{k,m}}^{n+1} - Z_{1_{k,m}}^{n}}{\ell} + \frac{1}{2}\overline{\frac{\partial G_1}{\partial S}}\,Z_{1_{k,m}}^{n+1}$$
$$+ \frac{1}{2}\overline{\frac{\partial G_1}{\partial I}}\,Z_{3_{k,m}}^{n} + \frac{1}{2}\overline{\frac{\partial G_2}{\partial S}}\,Z_{1_{k,m}}^{n} + \frac{1}{2}\overline{\frac{\partial G_2}{\partial I}}\,Z_{3_{k,m}}^{n+1} + g_{k,m}^{n}\,. \qquad (5.4.8)$$

Assume that $Z_{3_{k,m}}^{n}$ and $Z_{3_{k,m}}^{n+1}$ are bounded, then the above equation may be written as

$$\alpha\nabla_x^2\left(Z_{1_{k,m}}^{n+1} + Z_{1_{k,m}}^{n}\right) + \lambda\nabla_y^2\left(Z_{1_{k,m}}^{n+1} + Z_{1_{k,m}}^{n}\right) = 2\,\frac{Z_{1_{k,m}}^{n+1} - Z_{1_{k,m}}^{n}}{\ell}$$
$$+ \overline{\frac{\partial G_1}{\partial S}}\,Z_{1_{k,m}}^{n+1} + \overline{\frac{\partial G_2}{\partial S}}\,Z_{1_{k,m}}^{n} + M_I + 2\,g_{k,m}^{n} \qquad (5.4.9)$$

where

$$M_I = \max\left\{\overline{\frac{\partial G_1}{\partial I}}\,Z_{3_{k,m}}^{n},\ \overline{\frac{\partial G_3}{\partial I}}\,Z_{3_{k,m}}^{n+1}\right\}.$$

The following lemma and theorem which may be found in Douglas[15] will be used in relating the magnitude of $Z_{1_{k,m}}^{n+1}$ to that of $Z_{1_{k,m}}^{n}$.

Lemma 5.4.1 *If $\nabla_x^2 Z_{k,m} + \nabla_y^2 Z_{k,m} - \rho(x,y)\,Z_{k,m} = g_{k,m}$; $0 \le x, y \le 1$, $\rho > 0$, and $Z = 0$ on the boundary, then*

$$\max|\,Z(x,y)\,| \le \max\left|\frac{g(x,y)}{\rho(x,y)}\right|.$$

Theorem 5.4.1 (Douglas) *If there exists a solution u of (5.4.1) such that u is six times continuously differentiable in the region $0 \le x, y \le 1$, $0 \le t \le T$, then the solution of (5.4.2) converges uniformly to u in the region with the truncation error being not worse than $O(\ell)$.*

Douglas[15] has expressed $Z^n_{1_{k,m}} = Z_n$ as a sum of functions satisfying homogeneous equations of the form

$$Z_n = \sum_{i=0}^{n-1} Z_n^{(i)} \qquad (5.4.10)$$

where

$$\left.\begin{aligned}
Z_m^{(i)} &= 0 \qquad (m \le i) \\
\alpha\,\nabla_x^2 Z_{i+1}^{(i)} + \lambda\,\nabla_y^2 Z_{i+1}^{(i)} - \tfrac{2}{\ell}\,Z_{i+1}^{(i)} &= 2\,g_i + M_I \qquad (m = i+1) \\
\alpha\,\nabla_x^2 Z_m^{(i)} + \lambda\,\nabla_y^2 Z_m^{(i)} + \alpha\,\nabla_x^2 Z_{m-1}^{(i)} &+ \lambda\,\nabla_y^2 Z_{m-1}^{(i)} = 2\,\frac{Z_m^{(i)} - Z_{m-1}^{(i)}}{\ell} \\
&+ \overline{\tfrac{\partial G_1}{\partial S}}\,Z_m^{(i)} + \overline{\tfrac{\partial G_2}{\partial S}}\,Z_{m-1}^{(i)} \qquad (m \ge i+2) \\
Z_m^{(i)} &= 0 \qquad (\text{on boundary})\,.
\end{aligned}\right\} \qquad (5.4.11)$$

To show that Z_n as defined by (5.4.10) satisfies the difference equation (5.4.9), substituting directly yields

$$\begin{aligned}
\alpha\,\nabla_x^2 Z_{n+1} + \lambda\,\nabla_y^2 Z_{n+1} &+ \alpha\,\nabla_x^2 Z_n + \lambda\,\nabla_y^2 Z_n - 2\,\frac{Z_{n+1} - Z_n}{\ell} - \overline{\frac{\partial G_1}{\partial S}}\,Z_{n+1} - \overline{\frac{\partial G_2}{\partial S}}\,Z_n \\
&= \left(\alpha\,\nabla_x^2 + \lambda\,\nabla_y^2\right)\left\{\sum_{i=0}^{n} Z_{n+1}^{(i)} + \sum_{i=1}^{n-1} Z_n^{(i)}\right\} \\
&- \frac{2}{\ell}\left\{\sum_{i=1}^{n} Z_{n+1}^{(i)} - \sum_{i=0}^{n-1} Z_n^{(i)}\right\} - \sum_{i=0}^{n-1}\left\{\overline{\frac{\partial G_1}{\partial S}}\,Z_{n+1}^{(i)} + \overline{\frac{\partial G_2}{\partial S}}\,Z_n^{(i)}\right\} \\
&= \alpha\,\nabla_x^2 Z_{n+1}^{(n)} + \lambda\,\nabla_y^2 Z_{n+1}^{(n)} - \frac{2}{\ell}\,Z_{n+1}^{(n)} \\
&+ \sum_{i=0}^{n-1}\left\{\left(\alpha\,\nabla_x^2 + \lambda\,\nabla_y^2\right)\left(Z_{n+1}^{(i)} + Z_n^{(i)}\right) - 2\,\frac{Z_{n+1}^{(i)} - Z_n^{(i)}}{\ell}\right. \\
&- \left.\overline{\frac{\partial G_1}{\partial S}}\,Z_{n+1}^{(i)} - \overline{\frac{\partial G_2}{\partial S}}\,Z_n^{(i)}\right\} = 2\,g_n + M_I\,, \qquad (5.4.12)
\end{aligned}$$

by (5.4.11). Thus, (5.4.9) is satisfied. Next, a bound for $Z_{n+1}^{(n)}$ will be obtained using Lemma 5.4.1.

$$\max_i \left| Z_{i+1}^{(i)} \right| \le \ell.\max_i |\,g_n\,| + \frac{1}{2}.\ell.\max_i |\,M_I\,| \qquad (5.4.13)$$

Since each term on the right-hand side of (5.4.13) is bounded, $\max_i \left| Z_{i+1}^{(i)} \right|$ is bounded, too.

Applying Theorem 5.4.1, it follows that the solution of (5.4.2) converges uniformly to S in the region.

The second equation in (5.1.1) may be written in the form

$$\begin{aligned} \alpha E_{xx} + \lambda E_{yy} &= E_t - \frac{1}{2}\beta S I - \frac{1}{2}\beta S I + \frac{1}{2}(\mu+\sigma) E + \frac{1}{2}(\mu+\sigma) E \\ &= E_t - G_3 - G_4 + \frac{1}{2}(\mu+\sigma) E + G_5 \qquad (5.4.14) \end{aligned}$$

with initial and boundary conditions given in (5.1.2)–(5.1.4). The terms G_3 and G_4 are assumed to be boundedly-differentiable with respect to S and I and G_5 is assumed to be boundedly-differentiable with respect to E.

Consider the difference equations

$$\left.\begin{aligned} B^0_{k,m} &= E^0 \\ \frac{\alpha}{2}\nabla_x^2\left(B^{n+1}_{k,m} + B^n_{k,m}\right) &+ \frac{\lambda}{2}\nabla_y^2\left(B^{n+1}_{k,m} + B^n_{k,m}\right) = \frac{B^{n+1}_{k,m} - B^n_{k,m}}{\ell} - \frac{1}{2}\beta C^n_{k,m} A^{n+1}_{k,m} \\ &- \frac{1}{2}\beta C^{n+1}_{k,m} A^n_{k,m} + \frac{1}{2}(\mu+\sigma) B^{n+1}_{k,m} + \frac{1}{2}(\mu+\sigma) B^n_{k,m}, \quad n \geq 0, \end{aligned}\right\} \qquad (5.4.15)$$

where $B^n_{k,m}$ is an approximation to $E(x,y,t)$ at the typical point $(kh, mh, n\ell)$ which agrees with E on the boundaries. The operators ∇_x^2 and ∇_y^2 are defined as in (5.4.3).

It is easy to see that

$$\begin{aligned} \frac{\alpha}{2}\nabla_x^2\left(E^{n+1}_{k,m} + E^n_{k,m}\right) + \frac{\lambda}{2}\nabla_y^2(E^{n+1}_{k,m} + E^n_{k,m}) &= \frac{E^{n+1}_{k,m} - E^n_{k,m}}{\ell} - \frac{1}{2}\beta I^n_{k,m} S^{n+1}_{k,m} \\ - \frac{1}{2}\beta I^{n+1}_{k,m} + \frac{1}{2}(\mu+\sigma) E^{n+1}_{k,m} &+ \frac{1}{2}(\mu+\sigma) E^n_{k,m} + g^n_{k,m} \qquad (5.4.16) \end{aligned}$$

with

$$g^n_{k,m} = O(h^2 + \ell). \qquad (5.4.17)$$

Subtracting equation (5.4.15) from equation (5.4.16) and using the definitions for the truncation errors in (5.4.6) gives

$$\left.\begin{aligned} Z^0_{2_{k,m}} &= 0\,, \\ \tfrac{\alpha}{2}\nabla^2_x\left(Z^{n+1}_{2_{k,m}} + Z^n_{2_{k,m}}\right) &+ \tfrac{\lambda}{2}\nabla^2_y\left(Z^{n+1}_{2_{k,m}} + Z^n_{2_{k,m}}\right) = \tfrac{Z^{n+1}_{2_{k,m}} - Z^n_{2_{k,m}}}{\ell} - \underline{\tfrac{1}{2}\beta\, I^n_{k,m}\, S^{n+1}_{k,m}} \\ &+ \tfrac{1}{2}\beta\, C^n_{k,m}\, A^{n+1}_{k,m} - \underline{\tfrac{1}{2}\beta\, I^{n+1}_{k,m}\, S^n_{k,m}} + \tfrac{1}{2}\beta\, C^{n+1}_{k,m}\, A^n_{k,m} \\ &+ \tfrac{1}{2}(\mu+\sigma)\, Z^{n+1}_{2_{k,m}} + \tfrac{1}{2}(\mu+\sigma)\, Z^n_{2_{k,m}} + g^n_{k,m} \\ Z^n_{2_{k,m}} &= 0 \quad (on\ boundary)\,. \end{aligned}\right\} \qquad (5.4.18)$$

As the underlined terms can be written in the form of equations (4.6.10) and (4.6.11), respectively, it follows that equation (5.4.18) may be written as

$$\begin{aligned} \frac{\alpha}{2}\nabla^2_x\left(Z^{n+1}_{2_{k,m}} + Z^n_{2_{k,m}}\right) &+ \frac{\lambda}{2}\nabla^2_y\left(Z^{n+1}_{2_{k,m}} + Z^n_{2_{k,m}}\right) = \frac{Z^{n+1}_{2_{k,m}} - Z^n_{2_{k,m}}}{\ell} - \frac{1}{2}\overline{\frac{\partial G_3}{\partial S}}\, Z^{n+1}_{1_{k,m}} \\ &- \frac{1}{2}\overline{\frac{\partial G_3}{\partial I}}\, Z^n_{3_{k,m}} - \frac{1}{2}\overline{\frac{\partial G_4}{\partial S}}\, Z^n_{1_{k,m}} + \frac{1}{2}\overline{\frac{\partial G_4}{\partial I}}\, Z^{n+1}_{3_{k,m}} + \frac{1}{2}(\mu+\sigma)\, Z^{n+1}_{2_{k,m}} \\ &+ \frac{1}{2}(\mu+\sigma)\, Z^n_{2_{k,m}} + g^n_{k,m}\,. \end{aligned} \qquad (5.4.19)$$

Assume that $Z^n_{1_{k,m}}$, $Z^{n+1}_{1_{k,m}}$, $Z^n_{3_{k,m}}$ and $Z^{n+1}_{3_{k,m}}$ are bounded. Then equation (5.4.19) may be written in the form

$$\begin{aligned} \alpha\nabla^2_x\left(Z^{n+1}_{2_{k,m}} + Z^n_{2_{k,m}}\right) &+ \lambda\nabla^2_y\left(Z^{n+1}_{2_{k,m}} + Z^n_{2_{k,m}}\right) = 2\,\frac{Z^{n+1}_{2_{k,m}} - Z^n_{2_{k,m}}}{\ell} + M_S + M_I \\ &+ (\mu+\sigma)\, Z^{n+1}_{2_{k,m}} + (\mu+\sigma)\, Z^n_{2_{k,m}} + 2\, g^n_{k,m}\,, \end{aligned} \qquad (5.4.20)$$

where $M_S = \max\left\{\overline{\frac{\partial G_3}{\partial S}}\, Z^n_{1_{k,m}}, \overline{\frac{\partial G_4}{\partial S}}\, Z^{n+1}_{1_{k,m}}\right\}$ and $M_I = \max\left\{\overline{\frac{\partial G_3}{\partial I}}\, Z^n_{3_{k,m}}, \overline{\frac{\partial G_4}{\partial I}}\, Z^{n+1}_{3_{k,m}}\right\}$.

Following Douglas[15], $Z^n_{2_{k,m}} = Z_n$ may be written in the form of (5.4.10)–(5.4.12) and applying Lemma 5.4.1 and Theorem 5.4.1 shows that the solution of (5.4.15) converges uniformly to E in the region.

Similar analysis may be applied to the third equation in (5.1.1) to conclude that the approximate solution, $C^n_{k,m}$, converges uniformly to I.

5.5 Numerical Results

The second-order methods (5.2.11)–(5.2.19), (5.2.30)–(5.2.38) and (5.2.49)–(5.2.57) were tested on the initial/boundary-value problem

$$\begin{aligned} S_t &= \mu N - (\mu + \beta I) S + \alpha S_{xx} + \lambda S_{yy} \\ E_t &= \beta I S - (\mu + \sigma) E + \alpha E_{xx} + \lambda E_{yy} \\ I_t &= \sigma E - (\mu + \gamma) I + \alpha I_{xx} + \lambda I_{yy} \end{aligned} \quad (5.5.1)$$

with the initial conditions

$$S(x,y,0) = S^0(x,y), \quad E(x,y,0) = E^0(x,y), \quad I(x,y,0) = I^0(x,y)\,; 0 \le x,y \le 1 \quad (5.5.2)$$

and the boundary conditions

$$\begin{aligned} u_x(0,y,t) &= u_x(1,y,t) = 0; \quad t > 0 \\ u_y(x,0,t) &= u_y(x,1,t) = 0; \quad t > 0\,, \end{aligned} \quad (5.5.3)$$

where $u(x,y,t)$ represents $S(x,y,t)$, $E(x,y,t)$ or $I(x,y,t)$. The set of parameters given in (3.4.4) for N, μ, σ and γ was used. The infection rate, β, was chosen to be $\beta = 5 \times 10^{-4}$; the diffusion rates were $\alpha = 0.01$ and $\lambda = 0.01$.

In the following numerical experiment the total numbers of susceptibles, exposed and infected individuals are taken to be 1.25×10^7, 5×10^4 and 3×10^4, respectively. The way in which each is distributed over the square $0 \le x,y \le 1$ gives the functions $S^0(x,y)$, $E^0(x,y)$ and $I^0(x,y)$ in (5.5.2).

Taking $h = 0.1$ so that $M = 9$, giving the discretizations x_i and y_i $(i = 0, 1, 2, \ldots, 10)$ of the square $0 \le x,y \le 1$. The initial conditions are distributed as in Figures 5.1–5.3, where the numbers of exposed and infected individuals are concentrated in the centre of the square $0 \le x,y \le 1$ and the number of susceptibles is distributed over the whole square such that the number of susceptible individuals in the centre of the square is less than the other parts of the square; they are distributed as follows

$$S(x_i, y_j, 0) = \begin{cases} 103967 & , \; 0 \le i \le M+1 \;\; \& \;\; 0 \le j \le \frac{M-5}{2}; \\ & \;\;\; 0 \le i \le \frac{M-5}{2} \;\; \& \;\; \frac{M-3}{2} \le j \le \frac{M+5}{2}; \\ & \;\;\; \frac{M+7}{2} \le i \le M+1 \;\; \& \;\; \frac{M-3}{2} \le j \le \frac{M+5}{2}; \\ & \;\;\; 0 \le i \le M+1 \;\; \& \;\; \frac{M+7}{2} \le j \le M+1 \\ 101967 & , \; \frac{M-1}{2} \le i \le \frac{M+3}{2} \;\; \& \;\; j = \frac{M-3}{2}, \frac{M+5}{2}; \\ & \;\;\; i = \frac{M-3}{2}, \frac{M+5}{2} \;\; \& \;\; \frac{M-3}{2} \le j \le \frac{M+5}{2} \\ 100717 & , \; \frac{M-1}{2} \le i \le \frac{M+3}{2} \;\; \& \;\; j = \frac{M-1}{2}, \frac{M+3}{2}; \\ & \;\;\; i = \frac{M-1}{2} \;\; \& \;\; j = \frac{M+1}{2}; \\ & \;\;\; i = \frac{M+3}{2} \;\; \& \;\; j = \frac{M+1}{2} \\ 81967 & , \; i = \frac{M+1}{2} \;\; \& \;\; j = \frac{M+1}{2}, \end{cases}$$

$$E(x_i, y_j, 0) = 2000 \;\; , \;\; \frac{M-3}{2} \le i, j \le \frac{M+5}{2},$$

$$I(x_i, y_j, 0) = \begin{cases} 1250 & , \; \frac{M-1}{2} \le i \le \frac{M+3}{2} \;\; \& \;\; j = \frac{M-1}{2}; \\ & \;\;\; \frac{M-1}{2} \le i \le \frac{M+3}{2} \;\; \& \;\; j = \frac{M+3}{2}; \\ & \;\;\; i = \frac{M-1}{2} \;\; \& \;\; j = \frac{M+1}{2}; \\ & \;\;\; i = \frac{M+3}{2} \;\; \& \;\; j = \frac{M+1}{2} \\ 20000 & , \; i, j = \frac{M+1}{2}. \end{cases}$$

Figure 5.4 shows the distribution of susceptibles, exposed and infected individuals in one dimension at $t = 0.1$. Figures 5.5−5.7 give three-dimensional plots of susceptibles, exposed and infected individuals, respectively, for $0 \le x, y \le 1$ and $t = 0.3$. Comparing Figure 5.4, which was compiled using the one-dimension model of Chapter 4, and Figures 5.5−5.7 in two dimensions, it is seen that the profiles in Figures 5.5−5.7 are similar to those in Figure 5.4. The time taken to compile Figures 5.5−5.7 is larger than that taken for Figure 5.4; this is expected because in two dimensions, the time taken for the disease to spread out is larger than that in one dimension. The profiles in Figure 5.4 can be seen clearly in Figures 5.5−5.7 by taking the cross-sections $x = 0.5$ or $y = 0.5$.

Figures 5.8—5.10 show the dynamic behaviour of measles in two dimensions at time $t = 0.4$.

Although M was chosen small, $M = 9$, the computer required extensive time to compute the results. For $M > 9$, the computer used was unable to compute the results since the matrices involved are block matrices and very large so the factorization of these block-matrices into lower- and upper- (LU) matrices and finding the inverses of L and U at every time step need more advanced architecture.

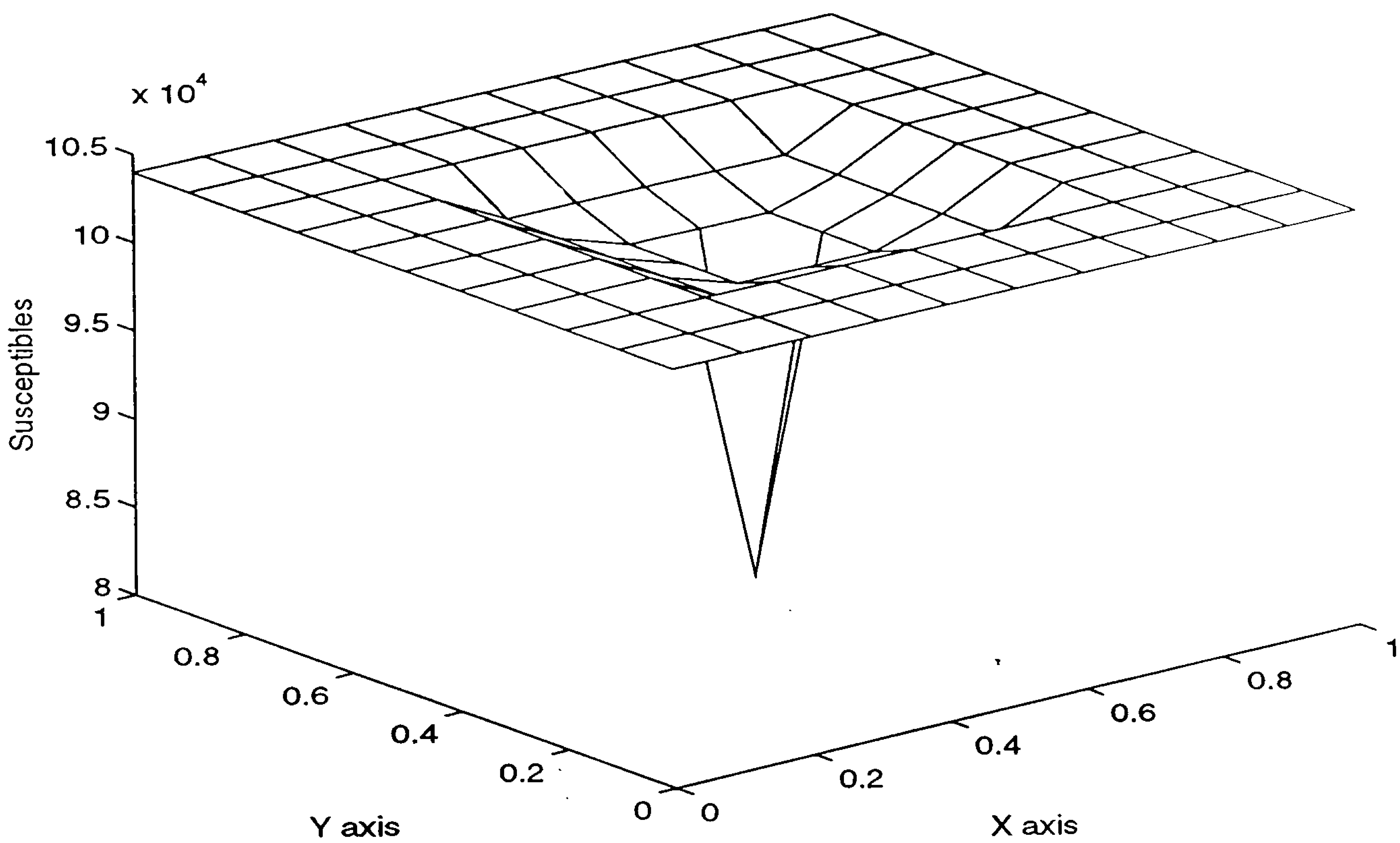

Figure 5.1: *Initial distribution of susceptibles.*

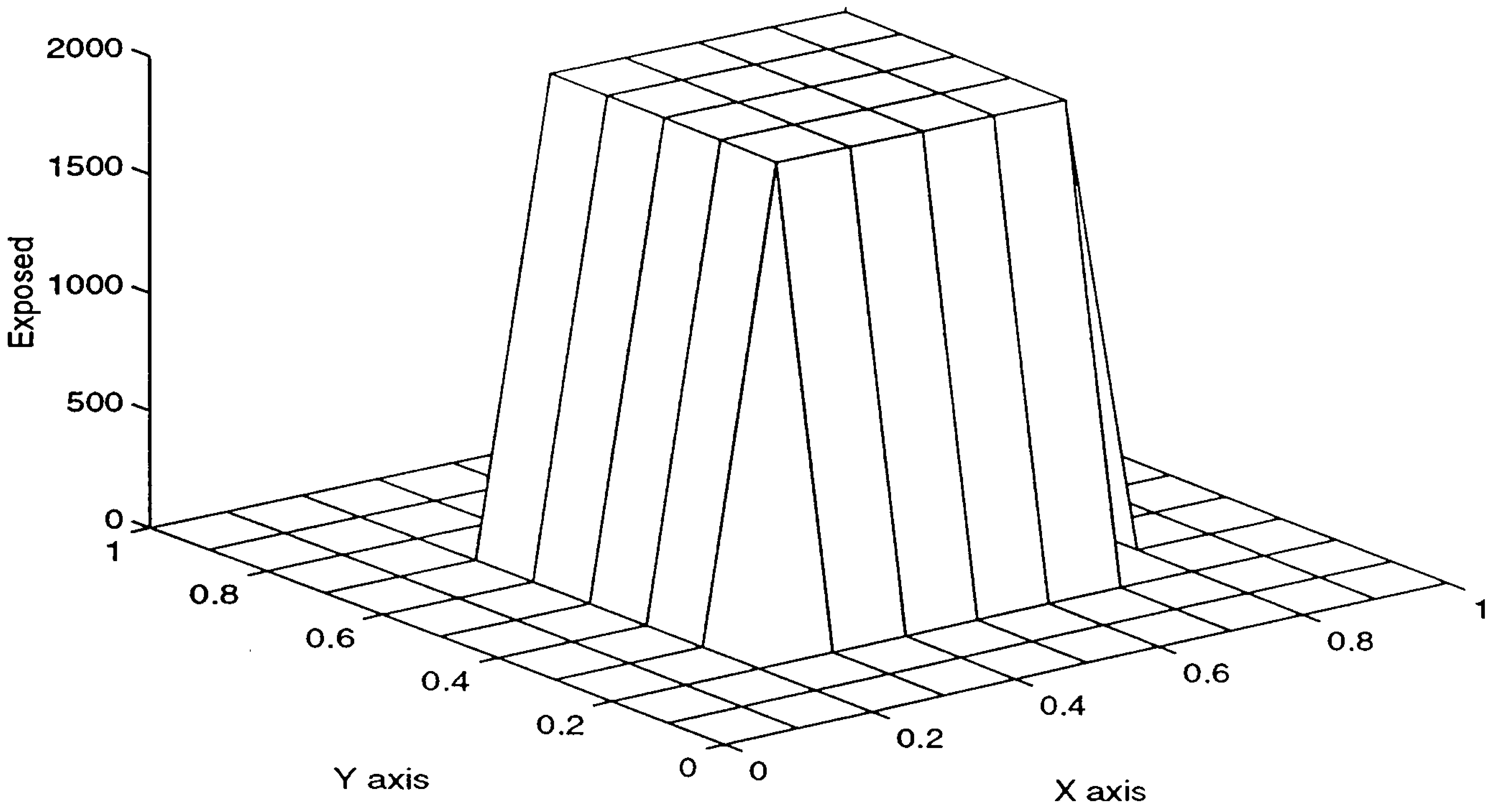

Figure 5.2: *Initial distribution of exposed individuals.*

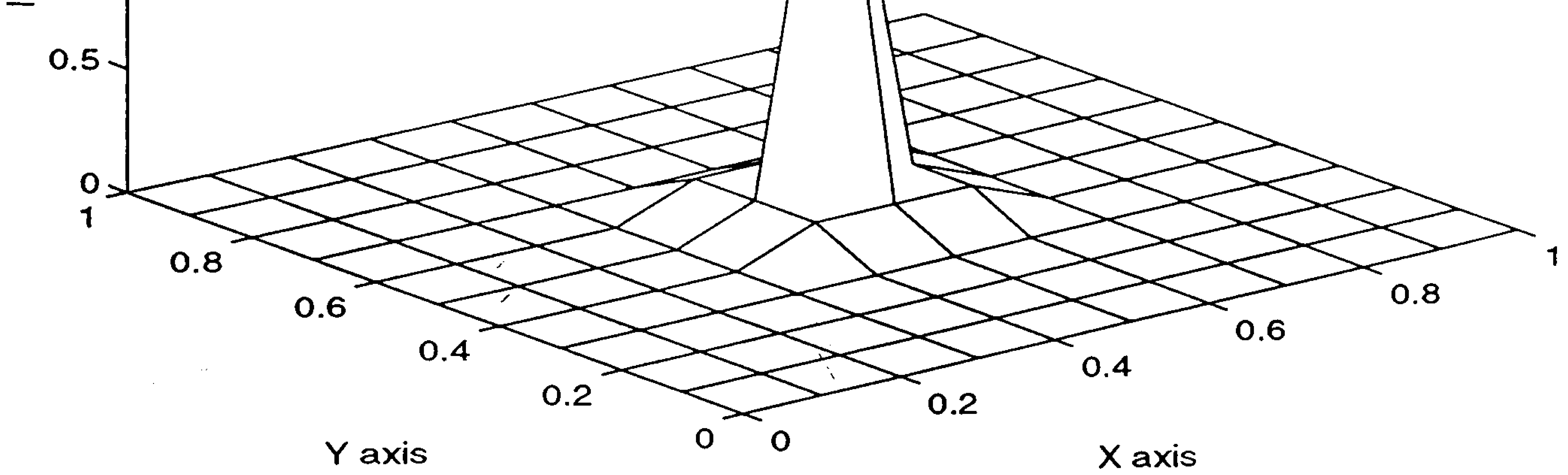

Figure 5.3: *Initial distribution of infectives.*

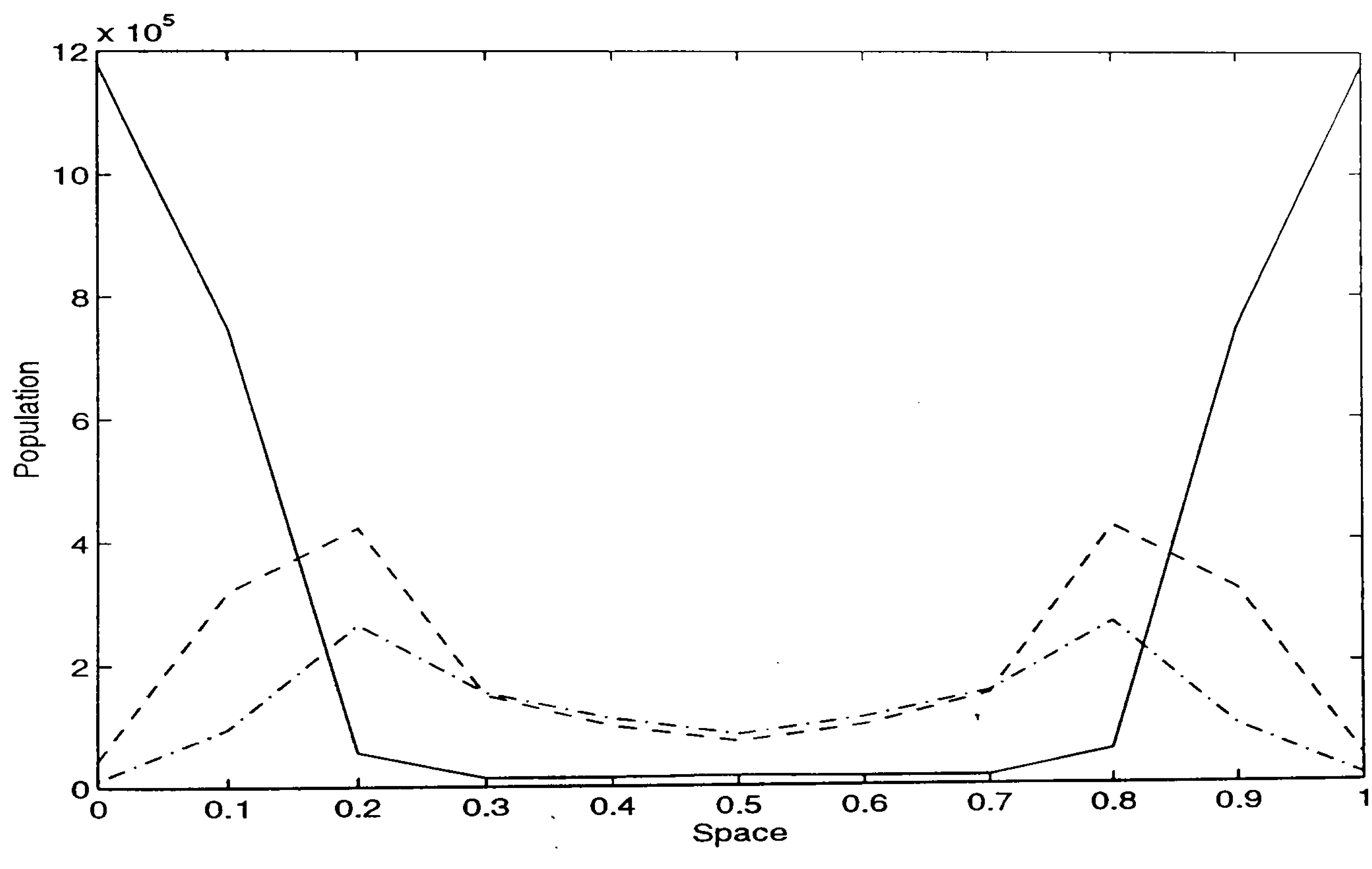

Figure 5.4: *Dynamical behaviour of measles in one dimension at* $t = 0.1$ *using* $h = 0.1$*; susceptibles(–), exposed(– –) and infectives(–.).*

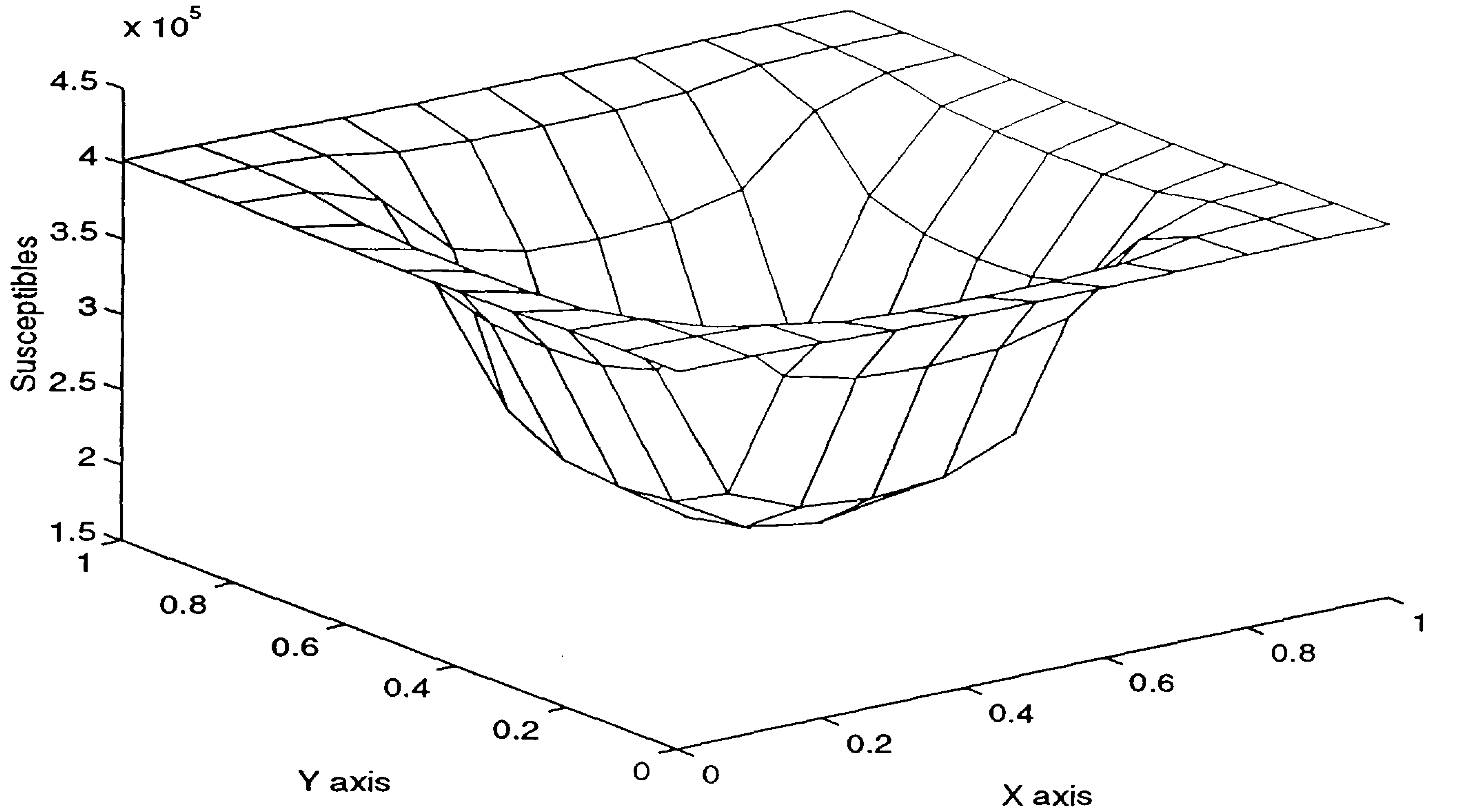

Figure 5.5: *Distribution of susceptibles after 300 time steps* ($t = 0.3$); $\ell = 0.001$ *and* $h = 0.1$.

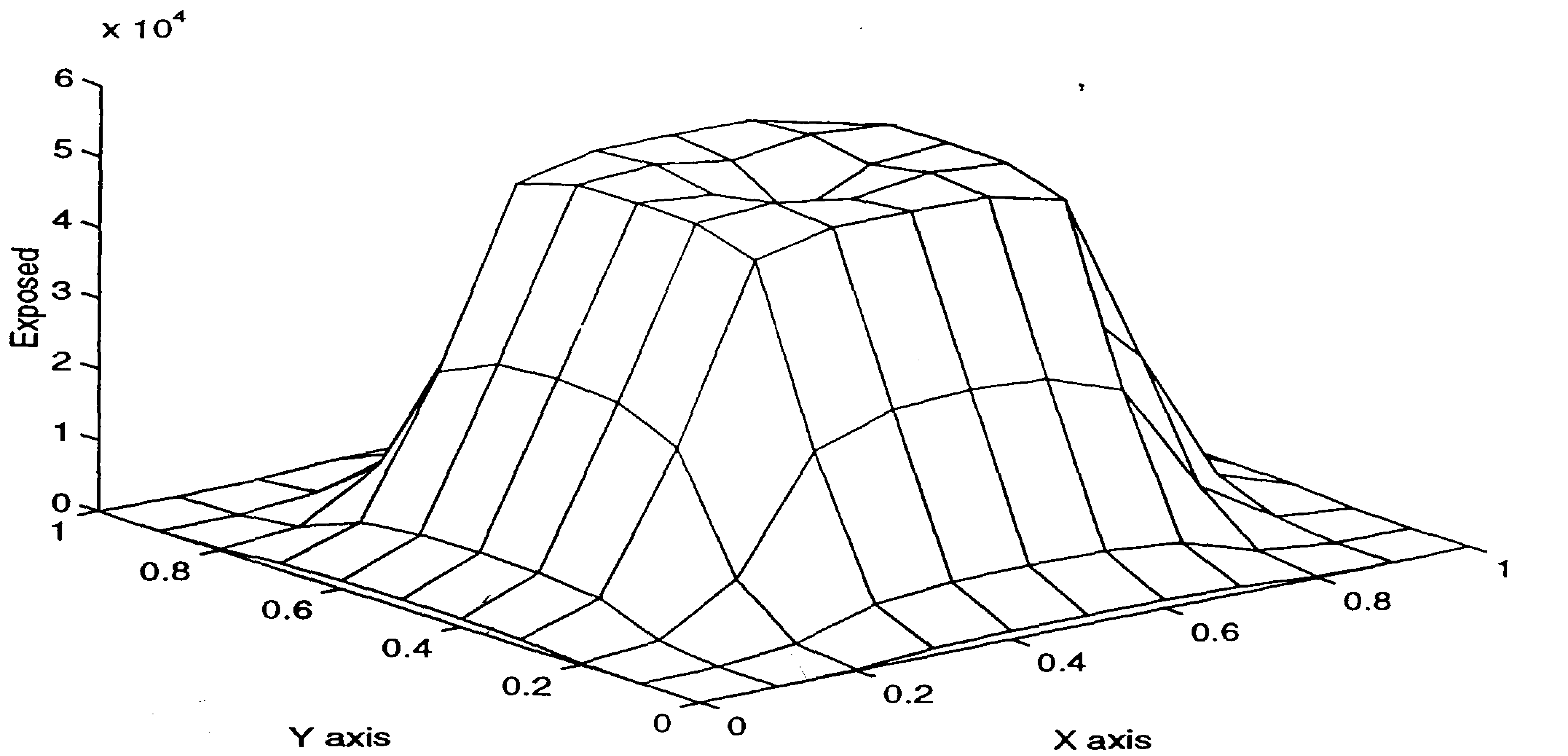

Figure 5.6: *Distribution of exposed individuals after 300 time steps* ($t = 0.3$); $\ell = 0.001$ *and* $h = 0.1$.

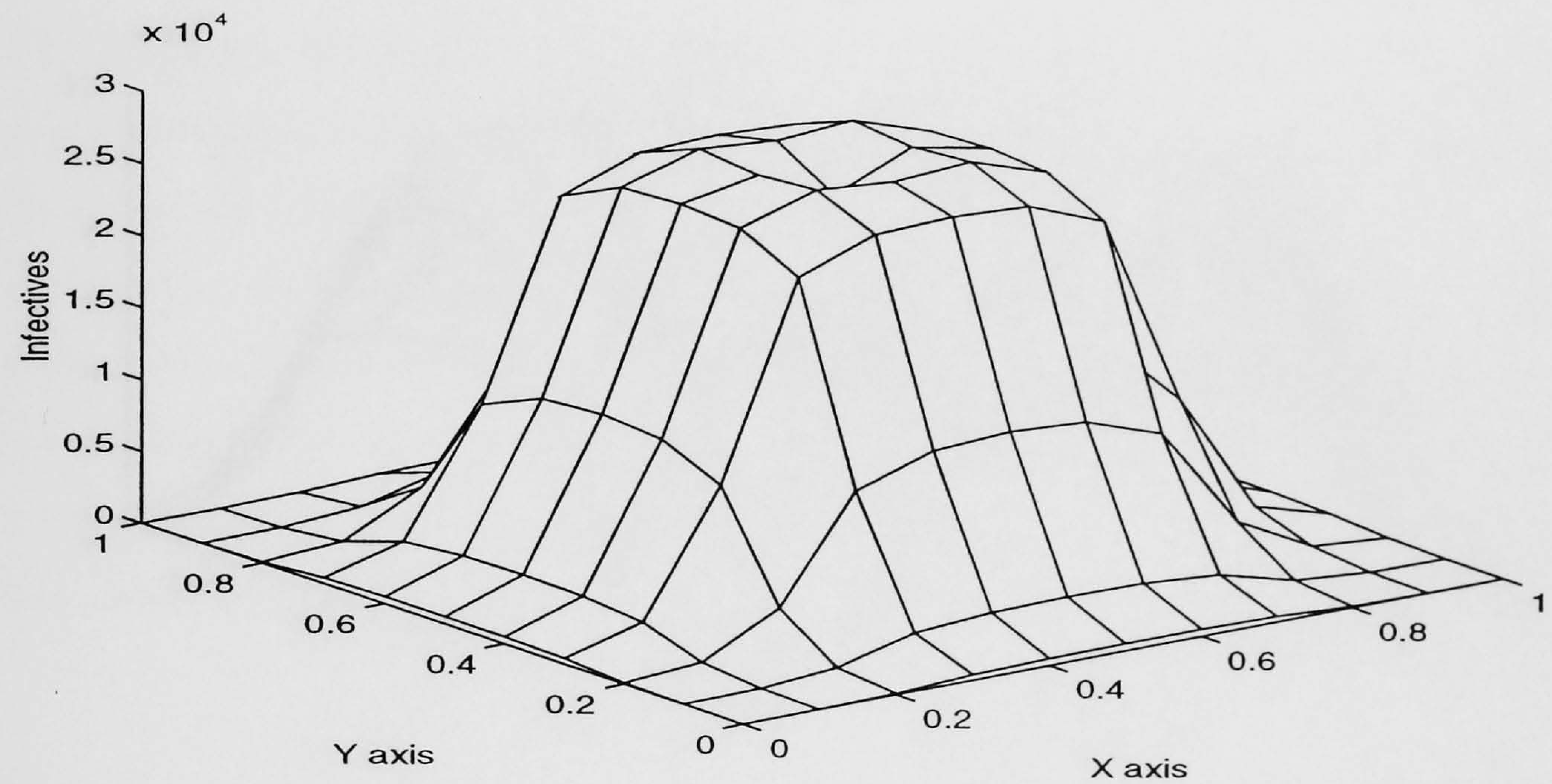

Figure 5.7: *Distribution of infectives after 300 time steps ($t = 0.3$); $\ell = 0.001$ and $h = 0.1$.*

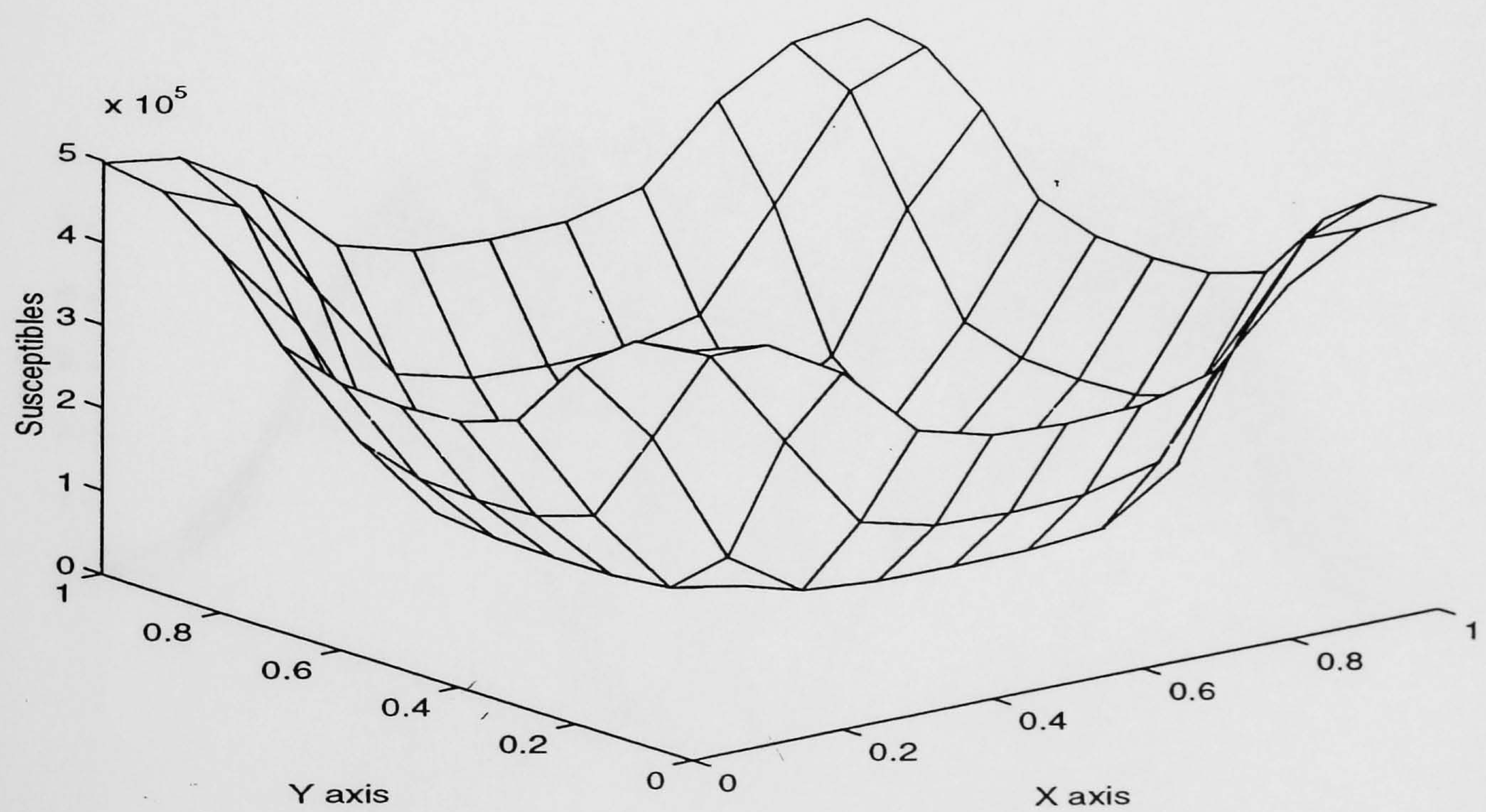

Figure 5.8: *Distribution of susceptibles after 400 time steps ($t = 0.4$); $\ell = 0.001$ and $h = 0.1$.*

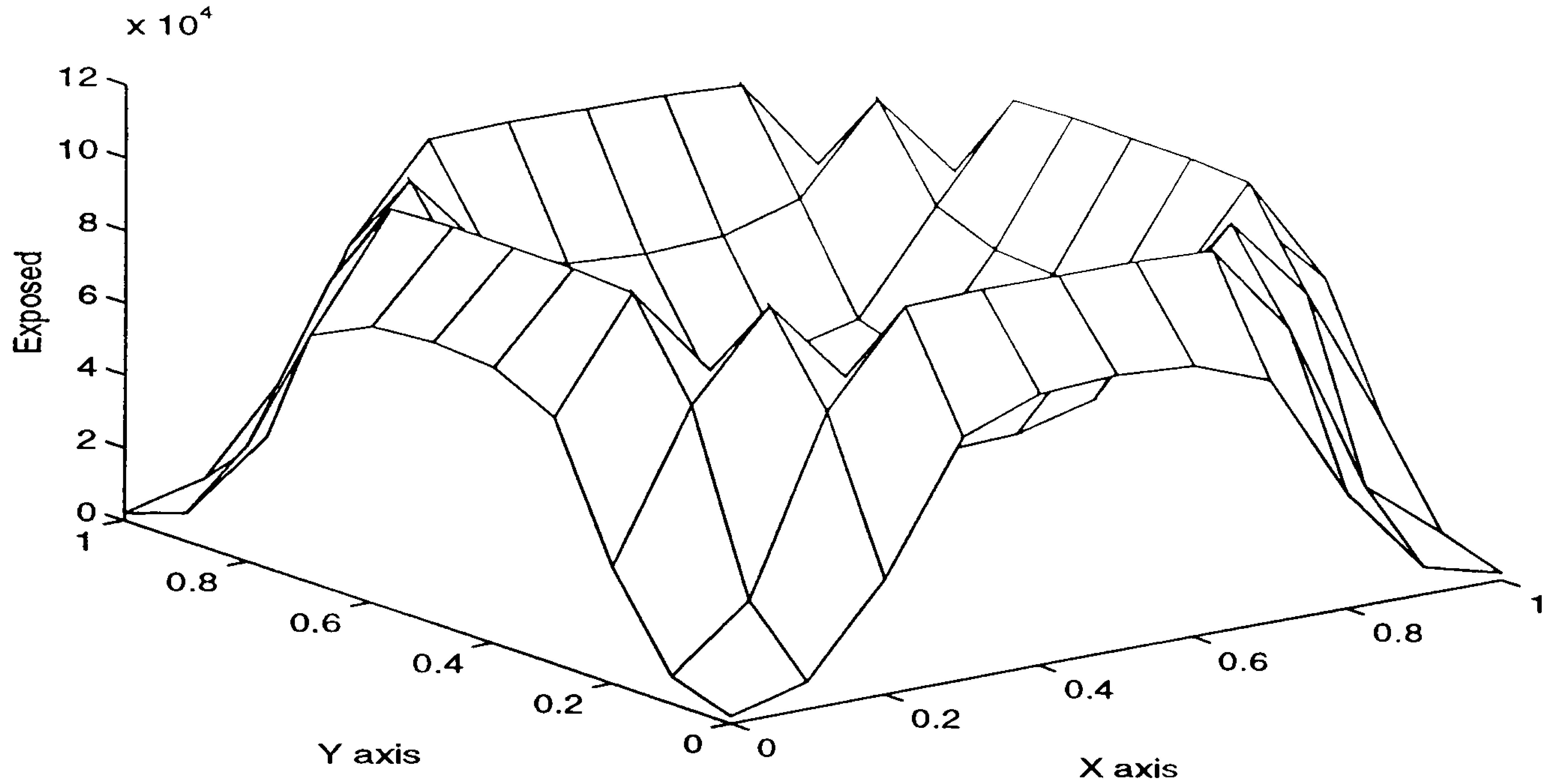

Figure 5.9: *Distribution of exposed individuals after 400 time steps ($t = 0.4$); $\ell = 0.001$ and $h = 0.1$.*

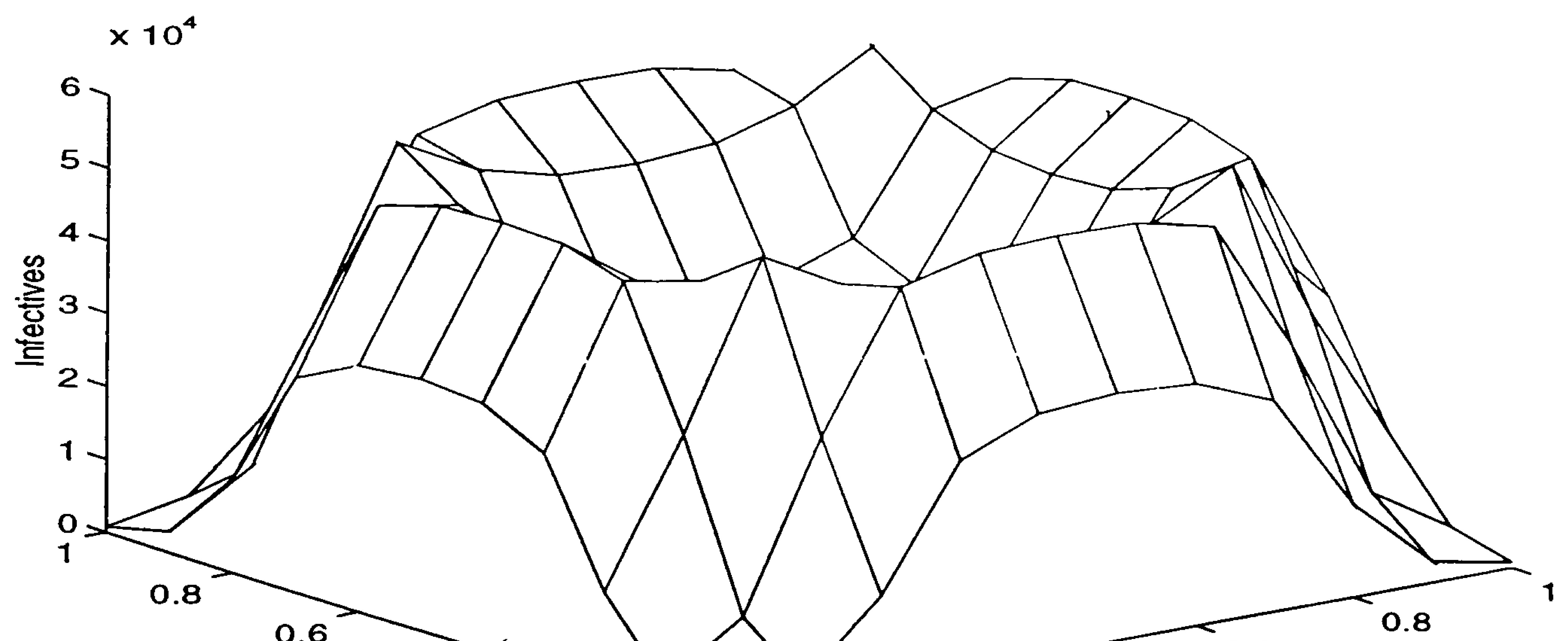

Figure 5.10: *Distribution of infectives after 400 time steps ($t = 0.4$); $\ell = 0.001$ and $h = 0.1$.*

5.6 Conclusion

The SEIR measles model of Chapter 3 has been extended to two dimensions and the same parameter values have been used in numerical experiments. The model equations have not been seen in the literature and are a mathematical extension to the SEIR measles model. Block-tridiagonal matrices were obtained after implementing a second-order finite-difference method and the maximum principle analysis was used to reveal that the developed numerical method is convergent. Numerical results have shown that the dynamic behaviour of measles in two dimensions is similar to that in one dimension but takes more time to spread.

Chapter 6

One-Dimensional Measles Dynamics of Convection Type

6.1 Introduction

In this chapter, a one-dimensional model of measles of hyperbolic type will be discussed, which assumes that the measles might be spread in a wave form.

The system to be considered is

$$\begin{aligned} S_t + \rho S_x &= \mu N - (\mu + \beta I) S \\ E_t + \rho E_x &= \beta I S - (\mu + \sigma) E \\ I_t + \rho I_x &= \sigma E - (\mu + \gamma) I \end{aligned} \qquad (6.1.1)$$

in which $S = S(x,t)$, $E = E(x,t)$ and $I = I(x,t)$ are the numbers of susceptibles, exposed and infectious individuals, respectively, at time t and distance x from the origin; $\rho > 0$ is the convection rate. The other parameters are as before.

The initial conditions are of the form

$$S(x,0) = S^0(x),\ E(x,0) = E^0(x),\ I(x,0) = I^0(x); \quad 0 \le x \le L \qquad (6.1.2)$$

and the boundary conditions are

$$S(0,t) = S(t),\ E(0,t) = E(t),\ I(0,t) = I(t); \quad t > 0 \qquad (6.1.3)$$

Suppose that the solutions $S(x,t)$, $E(x,t)$ and $I(x,t)$ of (6.1.1)–(6.1.3) are sought in some region $\Omega = [0 < x < L,\ t > 0]$ of the first quadrant. The interval $0 \leq x \leq L$ is divided into M subintervals each of width h so that $Mh = L$ and the time variable t is discretized in steps of length ℓ such that $t_n = n\,\ell\,(n = 0,1,2,\ldots)$. The x-coordinates of the M points of this discretization are $x_k = k\,h\,(k = 0,1,2,\ldots,M)$; clearly, $x_0 = 0$ is the x-coordinate of every point on the t-axis.

The solutions of (6.1.1), (6.1.2) and (6.1.3) at the typical point (x_k, t_n) are $S(x_k,t_n)$, $E(x_k,t_n)$ and $I(x_k,t_n)$: these will be denoted by S_k^n, E_k^n and I_k^n, respectively. The theoretical solutions of numerical approximations to (6.1.1) at the same mesh point will be denoted by A_k^n, B_k^n and C_k^n, respectively, while the values actually obtained, which may be subject, for example, to round-off errors, will be denoted by $\widetilde{A_k^n}$, $\widetilde{B_k^n}$ and $\widetilde{C_k^n}$, respectively. These were the notations used in Chapter 4.

6.2 Numerical Methods

6.2.1 Numerical Method for S

Finite-difference methods are developed by approximating the time derivative in the first equation in (6.1.1) by the first-order forward-difference replacement

$$S_t \approx [S(x,t+\ell) - S(x,t)]/\ell \qquad (6.2.1)$$

and the space derivative by the approximant

$$S_x \approx \frac{1}{2}\left\{\frac{S(x,t) - S(x-h,t)}{h} + \frac{S(x,t+\ell) - S(x-h,t+\ell)}{h}\right\} \qquad (6.2.2)$$

in which $x = x_k\,(k = 1,2,\ldots,M)$ and $t = t_n\,(n = 0,1,2,\ldots)$.

By substituting equations (6.2.1) and (6.2.2) in the first equation in (6.1.1) and approximating as follows

$$\begin{aligned} \frac{A_k^{n+1} - A_k^n}{\ell} &+ \frac{\rho}{2\,h}\left\{A_k^n - A_{k-1}^n + A_k^{n+1} - A_{k-1}^{n+1}\right\} + (\mu + \beta\,C_k^n)(\frac{A_k^{n+1} + A_k^n}{2}) \\ &- \mu\,N = 0 \end{aligned} \qquad (6.2.3)$$

gives

$$\begin{aligned} -\frac{1}{2}\rho\,r\,A_{k-1}^{n+1} &+ \left[1 + \frac{1}{2}\ell\,(\mu + \beta\,C_k^n) + \frac{1}{2}\rho\,r\right]\,A_k^{n+1} = \frac{1}{2}\rho\,r\,A_{k-1}^n \\ &+ \left[1 - \frac{1}{2}\ell\,(\mu + \beta\,C_k^n) - \frac{1}{2}\rho\,r\right]\,A_k^n + \ell\,\mu\,N\,, \end{aligned} \qquad (6.2.4)$$

where $r = \ell/h$, which is a four-point, two-time level, implicit, finite-difference method. Note that, although the method in (6.2.4) is implicit it may be implemented explicitly.

The local truncation error $\mathcal{L}_S = \mathcal{L}_S[S(x,t), E(x,t), I(x,t); h, \ell]$ associated with (6.2.4) at the point $(x,t) = (x_k, t_n)$ may be written down from (6.2.3): it is

$$\begin{aligned} \mathcal{L}_S &= \frac{S(x,t+\ell) - S(x,t)}{\ell} + \frac{\rho}{2\,h}\,\{S(x,t) - S(x-h,t) + S(x,t+\ell) - S(x-h,t+\ell)\} \\ &+ \frac{1}{2}(\mu + \beta\,I(x,t))\,S(x,t+\ell) + \frac{1}{2}(\mu + \beta\,I(x,t))\,S(x,t) - \mu\,N \\ &- \{S_t(x,t) + \rho\,S_x(x,t) + (\mu + \beta\,I(x,t))\,S(x,t) - \mu\,N\}\,. \end{aligned} \qquad (6.2.5)$$

Expanding $S(x-h,t)$, $S(x,t+\ell)$ and $S(x-h,t+\ell)$ as Taylor series about (x,t) results in

$$\begin{aligned} \mathcal{L}_S &= \left[\frac{1}{2}S_{tt} + \frac{1}{2}(\mu + \beta\,I)\,S_t + \frac{1}{2}\rho\,S_{xt}\right]\,\ell - \frac{1}{4}\rho\,h\,\ell\,S_{xxt} \\ &- \frac{1}{2}\rho\,h\,S_{xx} + \left[\frac{1}{6}S_{ttt} + \frac{1}{4}(\mu + \beta\,I)\,S_{tt} + \frac{1}{4}\rho\,S_{xtt}\right]\,\ell^2 + \cdots\,. \end{aligned} \qquad (6.2.6)$$

Equation (6.2.6) reveals that the implicit method in (6.2.4) is first order because $\mathcal{L}_S = O(h + \ell)$ as $h, \ell \longrightarrow 0$.

6.2.2 Numerical Method for E

The time derivative in the second equation in (6.1.1) is approximated by the first-order forward-difference replacement

$$E_t \approx [E(x,t+\ell) - E(x,t)]/\ell \qquad (6.2.7)$$

and the space derivative by the approximant

$$E_x \approx \frac{1}{2}\left\{\frac{E(x,t)-E(x-h,t)}{h} + \frac{E(x,t+\ell)-E(x-h,t+\ell)}{h}\right\} \qquad (6.2.8)$$

in which $x = x_k\,(k = 1,2,\ldots,M)$ and $t = t_n\,(n = 0,1,2,\ldots)$.

By substituting equations (6.2.7) and (6.2.8) in the second equation in (6.1.1) and approximating as follows

$$\begin{aligned} \frac{B_k^{n+1}-B_k^n}{\ell} &+ \frac{\rho}{2\,h}\left\{B_k^n - B_{k-1}^n + B_k^{n+1} - B_{k-1}^{n+1}\right\} - \beta\,A_k^n\,C_k^n \\ &+ (\mu+\sigma)\left(\frac{B_k^{n+1}+B_k^n}{2}\right) = 0 \end{aligned} \qquad (6.2.9)$$

gives

$$\begin{aligned} -\frac{1}{2}\rho\,r\,B_{k-1}^{n+1} &+ \left[1+\frac{1}{2}\ell\,(\mu+\sigma)+\frac{1}{2}\rho\,r\right]\,B_k^{n+1} = \frac{1}{2}\rho\,r\,B_{k-1}^n \\ &+ \left[1-\frac{1}{2}\ell\,(\mu+\sigma)-\frac{1}{2}\rho\,r\right]\,B_k^n + \ell\,\beta\,A_k^n\,C_k^n\,, \end{aligned} \qquad (6.2.10)$$

where $r = \ell/h$, which is a four-point, two-time level, implicit, finite-difference method. As before, this method, although implicit, may be implemented explicitly.

The local truncation error $\mathcal{L}_E = \mathcal{L}_E[S(x,t), E(x,t), I(x,t); h, \ell]$ associated with (6.2.10) at the point $(x,t) = (x_k, t_n)$ may be written down from (6.2.9): it is

$$\begin{aligned} \mathcal{L}_E &= \frac{E(x,t+\ell)-E(x,t)}{\ell} + \frac{\rho}{2\,h}\,\{E(x,t)-E(x-h,t)+E(x,t+\ell)-E(x-h,t+\ell)\} \\ &- \beta\,S(x,t)\,I(x,t) + \frac{1}{2}(\mu+\sigma)\,E(x,t+\ell) + \frac{1}{2}(\mu+\sigma)\,E(x,t) \\ &- \{E_t(x,t) + \rho\,E_x(x,t) - \beta\,S(x,t)\,I(x,t) + (\mu+\sigma)\,E(x,t)\}\,. \end{aligned} \qquad (6.2.11)$$

Expanding $E(x-h,t)$, $E(x,t+\ell)$ and $E(x-h,t+\ell)$ as Taylor series about (x,t) leads to

$$\begin{aligned}\mathcal{L}_E &= \left[\frac{1}{2}E_{tt} + \frac{1}{2}(\mu+\sigma)E_t + \frac{1}{2}\rho E_{xt}\right]\ell - \frac{1}{4}\rho h \ell E_{xxt} - \frac{1}{2}\rho h E_{xx} \\ &+ \left[\frac{1}{6}E_{ttt} + \frac{1}{4}(\mu+\sigma)E_{tt} + \frac{1}{4}\rho E_{xtt}\right]\ell^2 + \cdots \end{aligned} \qquad (6.2.12)$$

which is $O(h+\ell)$ as $h,\ell \longrightarrow 0$. This reveals that the implicit method (6.2.10) is first-order accurate in time as well as in space.

6.2.3 Numerical Method for I

Now, the time derivative in the third equation in (6.1.1) is approximated by the first-order forward-difference replacement

$$I_t \approx [I(x,t+\ell) - I(x,t)]/\ell \qquad (6.2.13)$$

and the space derivative by the approximant

$$I_x \approx \frac{1}{2}\left\{\frac{I(x,t)-I(x-h,t)}{h} + \frac{I(x,t+\ell)-I(x-h,t+\ell)}{h}\right\} \qquad (6.2.14)$$

in which $x = x_k\,(k=1,2,\ldots,M)$ and $t=t_n\,(n=0,1,2,\ldots)$.

Using the above finite-difference approximations for I_t and I_x in the third equation in (6.1.1) and approximating as follows

$$\begin{aligned}\frac{C_k^{n+1}-C_k^n}{\ell} &+ \frac{\rho}{2h}\left\{C_k^n - C_{k-1}^n + C_k^{n+1} - C_{k-1}^{n+1}\right\} - \sigma B_k^n \\ &+ (\mu+\gamma)\left(\frac{C_k^{n+1}+C_k^n}{2}\right)\end{aligned} \qquad (6.2.15)$$

gives

$$\begin{aligned}-\frac{1}{2}\rho r C_{k-1}^{n+1} &+ \left[1+\frac{1}{2}\ell(\mu+\gamma)+\frac{1}{2}\rho r\right]C_k^{n+1} = \frac{1}{2}\rho r C_{k-1}^n \\ &+ \left[1-\frac{1}{2}\ell(\mu+\gamma)-\frac{1}{2}\rho r\right]C_k^n + \ell\sigma B_k^n,\end{aligned} \qquad (6.2.16)$$

where $r = \ell/h$, which is a four-point, two-time level, implicit, finite-difference method. This method may also be implemented explicitly, although it is, by nature, implicit.

The local truncation error $\mathcal{L}_I = \mathcal{L}_I[S(x,t), E(x,t), I(x,t); h, \ell]$ associated with (6.2.16) at the point $(x,t) = (x_k, t_n)$ may be written down from (6.2.15): it is

$$\begin{aligned} \mathcal{L}_I &= \frac{I(x,t+\ell) - I(x,t)}{\ell} + \frac{\rho}{2h}\{I(x,t) - I(x-h,t) + I(x,t+\ell) - I(x-h,t+\ell)\} \\ &- \sigma E(x,t) + \frac{1}{2}(\mu+\gamma)\, I(x,t+\ell) + \frac{1}{2}(\mu+\gamma)\, I(x,t) \\ &- \{I_t(x,t) + \rho I_x(x,t) - \sigma E(x,t) + (\mu+\gamma)\, I(x,t)\}\,. \end{aligned} \qquad (6.2.17)$$

Expanding $I(x-h,t)$, $I(x,t+\ell)$ and $I(x-h,t+\ell)$ as Taylor series about (x,t) results in

$$\begin{aligned} \mathcal{L}_I &= \left[\frac{1}{2} I_{tt} + \frac{1}{2}(\mu+\gamma)\, I_t + \frac{1}{2}\rho\, I_{xt}\right]\ell - \frac{1}{4}\rho\, h\,\ell\, I_{xxt} - \frac{1}{2}\rho\, h\, I_{xx} \\ &+ \left[\frac{1}{6} I_{ttt} + \frac{1}{4}(\mu+\gamma)\, I_{tt} + \frac{1}{4}\rho\, I_{xtt}\right]\ell^2 + \cdots \end{aligned} \qquad (6.2.18)$$

which reveals that the implicit method in (6.2.16) is first-order accurate in time and space since $\mathcal{L}_I = O(h+\ell)$ as $h, \ell \longrightarrow 0$.

6.3 Stability Analysis

As in Section 4.6 of Chapter 4, because the term $I\,S$ in the first and second equations in (6.1.1) cannot be uncoupled, it follows that the von Neumann and the matrix methods are not appropriate for the stability analysis. Therefore, the maximum principle analysis will be used to discuss the stability of the finite-difference approximations (6.2.4), (6.2.10) and (6.2.16).

To use the maximum principle analysis to examine convergence, assume that a solution of (6.1.1)–(6.1.3) exists in the closed region [CR: $0 \le x \le L$, $0 \le t \le T$] such that $\frac{\partial^2 S}{\partial x^2}$, $\frac{\partial^2 E}{\partial x^2}$ and $\frac{\partial^2 I}{\partial x^2}$ exist and are bounded in CR.

Let the first equation in (6.1.1)–(6.1.3) be written as

$$\begin{aligned} S_t + \frac{1}{2}\,\rho\,(S_x + S_x) &= \mu\,N - \frac{1}{2}\,(\mu + \beta\,I)\,S - \frac{1}{2}\,(\mu + \beta\,I)\,S \\ &= \mu\,N - G_1 - G_2 \end{aligned} \qquad (6.3.1)$$

with

$$S(x,0) = S^0(x), \quad S(0,t) = S(t)\,, \qquad (6.3.2)$$

where G_1 and G_2 are assumed to be boundedly-differentiable with respect to S and I.

Consider the following difference equations to approximate equations (6.3.1) and (6.3.2)

$$\begin{aligned} \nabla_t\,A_k^n + \frac{1}{2}\,\rho\,\nabla\,\left(A_k^{n+1} + A_k^n\right) &= \mu\,N - \frac{1}{2}\,(\mu + \beta\,C_k^n)\,A_k^n \\ &- \frac{1}{2}\,(\mu + \beta\,C_k^n)\,A_k^{n+1}\,; \quad n \geq 0 \end{aligned} \qquad (6.3.3)$$

with

$$A_k^0 = S^0(x), \quad A_0^n = S(t) \qquad (6.3.4)$$

where ∇_t and ∇ are, respectively, the forward- and backward- difference operators with

$$\begin{aligned} \nabla_t\,A_k^n &= \left(A_k^{n+1} - A_k^n\right)/\ell\,, \\ \nabla\,A_k^n &= \left(A_k^n - A_{k-1}^n\right)/h\,. \end{aligned} \qquad (6.3.5)$$

It is known that

$$\left.\begin{aligned} G_2 = G_2(x_k, t_{n+1}, S_k^{n+1}, I_k^n) &= G_2(x_k, t_{n+1}, S_k^n, I_k^n) + \left(S_k^{n+1} - S_k^n\right)\overline{\tfrac{\partial G_2}{\partial S}} \\ &= \tfrac{1}{2}\,(\mu + \beta\,I_k^n)\,S_k^n + \left(S_k^{n+1} - S_k^n\right)\overline{\tfrac{\partial G_2}{\partial S}}\,, \\ \tfrac{\partial S_k^{n+1}}{\partial t} &= \tfrac{S_k^{n+1} - S_k^n}{\ell} + \tfrac{1}{2}\,\ell\,\overline{\tfrac{\partial^2 S_k^{n+1}}{\partial t^2}}\,, \\ \tfrac{1}{2}\,\nabla\,\left(S_k^{n+1} + S_k^n\right) &= \tfrac{1}{2}\,\tfrac{\partial S_k^{n+1}}{\partial x} + \tfrac{1}{2}\,\tfrac{\partial S_k^n}{\partial x} - \tfrac{1}{4}\,h\,\left(\overline{\tfrac{\partial^2 S_k^{n+1}}{\partial x^2}} + \overline{\tfrac{\partial^2 S_k^n}{\partial x^2}}\right)\,. \end{aligned}\right\} \qquad (6.3.6)$$

where the barred derivatives are evaluated at intermediate argument values as called for by the Mean Value Theorem.

Substituting (6.3.6) into (6.3.1) gives

$$
\begin{aligned}
\nabla_t S_k^n + \frac{1}{2}\rho\nabla\left(S_k^{n+1} + S_k^n\right) &= \mu N - \frac{1}{2}(\mu + \beta I_k^n) S_k^n - \frac{1}{2}(\mu + \beta I_k^n) S_k^{n+1} \\
&+ \left\{-\frac{1}{4}h\left(\overline{\frac{\partial^2 S_k^{n+1}}{\partial x^2}} + \overline{\frac{\partial^2 S_k^n}{\partial x^2}}\right) - \frac{1}{2}\ell\,\overline{\frac{\partial^2 S_k^{n+1}}{\partial t^2}}\right. \\
&+ \left.\left(S_k^{n+1} - S_k^n\right)\overline{\frac{\partial G_2}{\partial S}}\right\}. \qquad (6.3.7)
\end{aligned}
$$

The assumption on S above requires the boundedness of all derivatives appearing inside the bracket along with $S_k^{n+1} - S_k^n$ in the region $0 \le x \le L$, $0 \le t \le T$. Hence, in this region,

$$
\begin{aligned}
\nabla_t S_k^n + \frac{1}{2}\rho\nabla\left(S_k^{n+1} + S_k^n\right) &= \mu N - \frac{1}{2}(\mu + \beta I_k^n) S_k^n \\
&- \frac{1}{2}(\mu + \beta I_k^n) S_k^{n+1} + g_k^n \qquad (6.3.8)
\end{aligned}
$$

with

$$
g_k^n = O(h + \ell)\,. \qquad (6.3.9)
$$

Now let

$$
\left.\begin{aligned}
Z_{1_k}^n &= S_k^n - A_k^n \\
Z_{2_k}^n &= E_k^n - B_k^n \\
Z_{3_k}^n &= I_k^n - C_k^n
\end{aligned}\right\}. \qquad (6.3.10)
$$

Then, if (6.3.3) is subtracted from (6.3.8),

$$
\begin{aligned}
\nabla_t Z_{1_k}^n + \frac{1}{2}\rho\nabla(Z_{1_k}^{n+1} + Z_{1_k}^n) &= -\frac{1}{2}(\mu + \beta I_k^n) S_k^n + \frac{1}{2}(\mu + \beta C_k^n) A_k^n - \frac{1}{2}(\mu + \beta I_k^n) S_k^{n+1} \\
&+ \frac{1}{2}(\mu + \beta C_k^n) A_k^{n+1} + g_k^n \qquad (6.3.11)
\end{aligned}
$$

with

$$
Z_{1_k}^0 = 0\,, \quad Z_{1_0}^n = 0
$$

As

$$\begin{aligned} G_1(x_k, t_{n+1}, S_k^n, I_k^n) &= G_1(x_k, t_{n+1}, A_k^n, C_k^n) + \overline{\frac{\partial G_1}{\partial S}}(S_k^n - A_k^n) \\ &+ \overline{\frac{\partial G_1}{\partial I}}(I_k^n - C_k^n) \end{aligned} \quad (6.3.12)$$

and

$$\begin{aligned} G_2(x_k, t_{n+1}, S_k^{n+1}, I_k^n) &= G_2(x_k, t_{n+1}, A_k^{n+1}, C_k^n) + \overline{\frac{\partial G_2}{\partial S}}(S_k^{n+1} - A_k^{n+1}) \\ &+ \overline{\frac{\partial G_2}{\partial I}}(I_k^n - C_k^n), \end{aligned} \quad (6.3.13)$$

it follows that equation (6.3.11) may be written as

$$\begin{aligned} \nabla_t Z_{1_k}^n + \frac{1}{2}\rho \nabla (Z_{1_k}^{n+1} + Z_{1_k}^n) &= -\frac{1}{2}\overline{\frac{\partial G_1}{\partial S}} Z_{1_k}^n - \frac{1}{2}\overline{\frac{\partial G_1}{\partial I}} Z_{3_k}^n \\ &- \frac{1}{2}\overline{\frac{\partial G_2}{\partial S}} Z_{1_k}^{n+1} - \frac{1}{2}\overline{\frac{\partial G_2}{\partial I}} Z_{3_k}^n + g_k^n . \end{aligned} \quad (6.3.14)$$

Rewriting equation (6.3.14) yields

$$\nabla_t Z_{1_k}^n + \rho \nabla \widehat{Z_{1_k}^n} + M_1 \widehat{Z_{1_k}^n} = -\frac{1}{2}\left(\overline{\frac{\partial G_1}{\partial I}} + \overline{\frac{\partial G_2}{\partial I}}\right) Z_{3_k}^n + g_k^n \quad (6.3.15)$$

where

$$\begin{aligned} \widehat{Z_{1_k}^n} &= \frac{1}{2}\left(Z_{1_k}^{n+1} + Z_{1_k}^n\right), \\ M_1 &= \max\left\{\overline{\frac{\partial G_1}{\partial S}}, \overline{\frac{\partial G_2}{\partial S}}\right\}. \end{aligned} \quad (6.3.16)$$

The following lemma, the proof of which may be found in Thomée[63], will be used to find a bound for equation (6.3.15).

Lemma 6.3.1 (Thomée) *Assume that Z_k^n satisfies*

$$\begin{gathered} \nabla_t Z_k^n \quad -D \nabla Z_k^n \quad +G_k^n \widehat{Z_k^n} = F_k^n \\ Z_k^0 = f_k^0 \\ Z_0^n = f^-, \quad Z_L^n = f^+ \end{gathered} \quad (6.3.17)$$

where D can be written as

$$D = \begin{pmatrix} D^- & 0 \\ 0 & D^+ \end{pmatrix}$$

and for some $K_0 > 0$, $-K_0 I^- \leq D^- \leq -K_0^{-1} I^-$, $K_0^{-1} I^+ \leq D^+ \leq -K_0 I^+$, $|G| \leq K_0$; $\left| D_k^n - D_{k+1}^n \right| + \left| D_k^n - D_k^{n+1} \right| \leq K_0\, h$. Then there are constants K and δ independent of h, ℓ, Z, F, f^0, f^- and f^+ such that for $h \leq \delta$

$$\| Z_k^n \|_\infty \leq K \left\{ \| f^0 \| + \| f^- \| + \| f^+ \| + \| F \| \right\} . \qquad (6.3.18)$$

Now, assume that $Z_{3_k}^n$ is bounded and applying Lemma 6.3.1 to equation (6.3.15), with $Z_{1_k}^0 = 0$ and $Z_{1_0}^n = Z_{1_L}^n = 0$, gives

$$\left\| Z_{1_k}^n \right\|_\infty = \max_k \left| Z_{1_k}^n \right| \leq K_1 , \qquad (6.3.19)$$

where $K_1 > 0$ depends on the bounds of $\overline{\frac{\partial G_1}{\partial I}}$, $\overline{\frac{\partial G_2}{\partial I}}$ and g_k^n. Therefore the theoretical solution A_k^n of the approximating method (6.3.3) converges to the solution S_k^n of the equation (6.3.1) as $h \longrightarrow 0$.

Now, the second equation in (6.1.1)–(6.1.3) may be written as

$$\begin{aligned} E_t + \frac{1}{2}\rho\,(E_x + E_x) &= \beta\, I\, S - \frac{1}{2}(\mu + \sigma)\, E - \frac{1}{2}(\mu + \sigma)\, E \\ &= G_3 - \frac{1}{2}(\mu + \sigma)\, E - G_4 \end{aligned} \qquad (6.3.20)$$

with

$$E(x, 0) = E^0(x) , \quad E(0, t) = E(t) , \qquad (6.3.21)$$

where G_3 is assumed to be boundedly-differentiable with respect to S and I and G_4 is assumed to be boundedly-differentiable with respect to E only.

Consider the following difference equations

$$\begin{aligned} \nabla_t B_k^n + \rho \nabla \widehat{B_k^n} &= \beta\, C_k^n\, A_k^n - \frac{1}{2}(\mu + \sigma)\, B_k^n \\ &- \frac{1}{2}(\mu + \sigma)\, B_k^{n+1} ; \quad n \geq 0 , \end{aligned} \qquad (6.3.22)$$

$$B_k^0 = E^0(x)\,, \quad B_0^n = E(t) \qquad (6.3.23)$$

where $\nabla_t B_k^n$ and $\widehat{B_k^n}$ are as in (6.3.5) and (6.3.16), respectively.

It is easy to see that

$$\left.\begin{aligned} G_4 = G_4(x_k, t_{n+1}, E_k^{n+1}) &= G_4(x_k, t_{n+1}, E_k^n) + \left(E_k^{n+1} - E_k^n\right) \overline{\tfrac{\partial G_4}{\partial E}} \\ &= \tfrac{1}{2}(\mu+\sigma)\,E_k^n + \left(E_k^{n+1} - E_k^n\right) \overline{\tfrac{\partial G_4}{\partial E}}\,, \\ \tfrac{\partial E_k^{n+1}}{\partial t} &= \nabla_t\, E_k^n + \tfrac{1}{2}\,\ell\, \overline{\tfrac{\partial^2 E_k^{n+1}}{\partial t^2}}\,, \\ \nabla\, \widehat{E_k^n} &= \tfrac{1}{2}\tfrac{\partial E_k^{n+1}}{\partial x} + \tfrac{1}{2}\tfrac{\partial E_k^n}{\partial x} - \tfrac{1}{4}\,h \left(\overline{\tfrac{\partial^2 E_k^{n+1}}{\partial x^2}} + \overline{\tfrac{\partial^2 E_k^n}{\partial x^2}} \right), \end{aligned}\right\} \qquad (6.3.24)$$

where the barred derivatives are evaluated at intermediate argument values as called for by the Mean Value Theorem.

Substituting (6.3.24) into (6.3.20) yields

$$\begin{aligned} \nabla_t\, E_k^n + \rho\, \nabla\, \widehat{E_k^n} &= \beta\, I_k^n\, S_k^n - \frac{1}{2}(\mu+\sigma)\,E_k^n - \frac{1}{2}(\mu+\sigma)\,E_k^{n+1} \\ &+ \left\{ -\frac{1}{4}\,h \left(\overline{\frac{\partial^2 E_k^{n+1}}{\partial x^2}} + \overline{\frac{\partial^2 E_k^n}{\partial x^2}} \right) - \frac{1}{2}\,\ell\, \overline{\frac{\partial^2 E_k^{n+1}}{\partial t^2}} \right. \\ &+ \left. \left(E_k^{n+1} - E_k^n\right) \overline{\frac{\partial G_4}{\partial E}} \right\}. \end{aligned} \qquad (6.3.25)$$

As before, $\widehat{E_k^n}$ is defined in (6.3.16). The assumption on E above requires the boundedness of all the derivatives appearing inside the bracket along with $E_k^{n+1} - E_k^n$ in the region CR. Hence, in this region,

$$\begin{aligned} \nabla_t\, E_k^n + \rho\, \nabla\, \widehat{E_k^n} &= \beta\, I_k^n\, S_k^n - \frac{1}{2}(\mu+\sigma)\,E_k^n \\ &- \frac{1}{2}(\mu+\sigma)\,E_k^{n+1} + g_k^n\,, \end{aligned} \qquad (6.3.26)$$

with

$$g_k^n = O(h+\ell)\,. \qquad (6.3.27)$$

If equation (6.3.22) is subtracted from equation (6.3.26) and using the definitions in (6.3.10) for the truncation errors, then

$$\begin{aligned} \nabla_t Z_{2_k}^n + \rho \nabla \widehat{Z_{2_k}^n} &= \beta I_k^n S_k^n - \beta C_k^n A_k^n - \frac{1}{2}(\mu+\sigma) Z_{2_k}^n \\ &- \frac{1}{2}(\mu+\sigma) Z_{2_k}^{n+1} + g_k^n . \end{aligned} \qquad (6.3.28)$$

As the term $\beta I_k^n S_k^n$ can be written down in the form of equation (6.3.12), it follows that equation (6.3.28) becomes

$$\begin{aligned} \nabla_t Z_{2_k}^n + \rho \nabla \widehat{Z_{2_k}^n} &= \overline{\frac{\partial G_3}{\partial S}} Z_{1_k}^n + \overline{\frac{\partial G_3}{\partial I}} Z_{3_k}^n \\ &- (\mu+\sigma) \widehat{Z_{2_k}^n} + g_k^n . \end{aligned} \qquad (6.3.29)$$

where $\widehat{Z_{2_k}^n}$ is defined as in (6.3.16).

Assume that $Z_{1_k}^n$ and $Z_{3_k}^n$ are bounded. Moreover, $Z_{2_k}^n = 0$ and $Z_{2_0}^n = Z_{2_L}^n = 0$. Then, by Lemma 6.3.1,

$$\left\| Z_{2_k}^n \right\|_\infty = \max_k \left| Z_{2_k}^n \right| \le K_2 \qquad (6.3.30)$$

for some $K_2 > 0$ which depends on the bounds of $\overline{\frac{\partial G_3}{\partial S}}$, $\overline{\frac{\partial G_3}{\partial I}}$ and g_k^n.

Therefore the theoretical solution B_k^n of the approximating method (6.3.22) converges to the solution E_k^n of the equation (6.3.2) as $h \longrightarrow 0$.

The third equation in (6.1.1)–(6.1.3) may be written in the form

$$\begin{aligned} I_t + \frac{1}{2}\rho(I_x + I_x) &= \sigma E - \frac{1}{2}(\mu+\gamma) I - \frac{1}{2}(\mu+\gamma) I \\ &= \sigma E - \frac{1}{2}(\mu+\gamma) I - G_5 \end{aligned} \qquad (6.3.31)$$

with

$$I(x,0) = I^0(x), \quad I(0,t) = I(t), \qquad (6.3.32)$$

where, as before, G_5 is assumed to be boundedly-differentiable with respect to I.

Consider the following difference approximations to equations (6.3.31)–(6.3.32)

$$\nabla_t C_k^n + \rho \nabla \widehat{C_k^n} = \sigma B_k^n - \frac{1}{2}(\mu+\gamma) C_k^n - \frac{1}{2}(\mu+\gamma) C_k^{n+1}; \quad n \geq 0, \qquad (6.3.33)$$

with

$$C_k^0 = I^0(x), \quad C_0^n = I(t) \qquad (6.3.34)$$

where $\nabla_t C_k^n$ and $\widehat{C_k^n}$ are as in (6.3.5) and (6.3.16), respectively.

Writing I_t, I_x and G_5 at the point $(kh,(n+1)\ell)$ as in (6.3.24) and substituting into (6.3.31) gives

$$\nabla_t I_k^n + \rho \nabla \widehat{I_k^n} = \sigma E_k^n - \frac{1}{2}(\mu+\gamma) I_k^n - \frac{1}{2}(\mu+\gamma) I_k^{n+1} + \left\{ -\frac{1}{4} h \left(\overline{\frac{\partial^2 I_k^{n+1}}{\partial x^2} + \frac{\partial^2 I_k^n}{\partial x^2}} \right) - \frac{1}{2} \ell \overline{\frac{\partial^2 I_k^{n+1}}{\partial t^2}} + \left(I_k^{n+1} - I_k^n \right) \overline{\frac{\partial G_5}{\partial I}} \right\}. \qquad (6.3.35)$$

The assumption on I above requires the boundedness of all the derivatives appearing inside the bracket along with $I_k^{n+1} - I_k^n$ in the region CR. Hence, in this region,

$$\nabla_t I_k^n + \rho \nabla \widehat{I_k^n} = \sigma E_k^n - \frac{1}{2}(\mu+\gamma) I_k^n - \frac{1}{2}(\mu+\gamma) I_k^{n+1} + g_k^n \qquad (6.3.36)$$

with, again,

$$g_k^n = O(h+\ell). \qquad (6.3.37)$$

If equation (6.3.33) is subtracted from equation (6.3.36) and using the definitions in (6.3.10) for the truncation errors, then

$$\nabla_t Z_{3_k}^n + \rho \nabla \widehat{Z_{3_k}^n} = \sigma Z_{2_k}^n - \frac{1}{2}(\mu+\gamma) Z_{3_k}^n - \frac{1}{2}(\mu+\gamma) Z_{3_k}^{n+1} + g_k^n. \qquad (6.3.38)$$

This equation may be rearranged to obtain

$$\nabla_t Z_{3_k}^n + \rho \nabla \widehat{Z_{3_k}^n} = \sigma Z_{2_k}^n + (\mu + \gamma) \widehat{Z_{3_k}^n} + g_k^n . \qquad (6.3.39)$$

Since $Z_{3_k}^0 = 0$ and $Z_{3_0}^n = Z_{3_L}^n = 0$, it follows that, by applying Lemma 6.3.1 to equation (6.3.39), assuming $Z_{2_k}^n$ is bounded,

$$\left\| Z_{3_k}^n \right\|_\infty = \max_k \left| Z_{3_k}^n \right| \leq K_3 \qquad (6.3.40)$$

for some $K_3 > 0$ which depends on g_k^n only.

Therefore the theoretical solution C_k^n of the difference method (6.3.33) converges to the solution I_k^n of the equation (6.3.31) as $h \longrightarrow 0$.

6.4 Numerical Results

In this section, the initial/boundary-value problem (6.1.1)–(6.1.3) was solved using the set of parameters given in (3.4.4) for N, μ, σ and γ with the infection rate, β, chosen to be $\beta = 5 \times 10^{-4}$ and the convection rate $\rho = 0.01$. Taking $h = 0.025$ so that $M = 40$, giving the discretization $x_i \, (i = 0, 1, 2, \ldots, 40)$ of the interval $0 \leq x \leq 1$, the initial conditions in (6.1.2) were distributed as follows (see Figure 6.4)

$$\begin{aligned} S(x_i, 0) &= 625000(1.0 - x_i) , \\ E(x_i, 0) &= 2500(1.0 - x_i) , \\ I(x_i, 0) &= 1500(1.0 - x_i) ; \end{aligned} \qquad (6.4.1)$$

$i = 0, 1, 2, \ldots, 40$, and the boundary conditions are given by

$$\left.\begin{aligned} S(0,t) &= 624984.53 - 307593.99\, t - 19050772.40\, t^2 + 1166543986.44\, t^3 \\ &\quad -36484106091.98\, t^4 + 221651249075.76\, t^5 , \\ E(0,t) &= 1492.96 + 650903.17\, t - 19644625.80\, t^2 + 388568240.19\, t^3 \\ &\quad +8182794371.50\, t^4 - 88931566812.11\, t^5 , \\ I(0,t) &= 1703.66 - 107358.36\, t + 19484101.83\, t^2 - 675809159.84\, t^3 \\ &\quad +13016185133.41\, t^4 - 70376307841.96\, t^5 . \end{aligned}\right\} \qquad (6.4.2)$$

The boundary conditions given in (6.4.2) were obtained from experiment A in Chapter 4 by taking the numbers of susceptibles, exposed and infectives at $x = 0.5$ and $0 \leq t \leq 0.1$ from Figures 4.4, 4.5 and 4.6, respectively, and fitting[1] them as in Figures 6.1−6.3.

As time increases, the number of susceptibles decreases whereas the numbers of exposed individuals and infectives increase near $x = 0$, where, initially, the number of each individual is large. Although some negative values for susceptibles were seen (see Figure 6.5) for small number of iterations ($t \approx 0.005$), the method works satisfactorily, as shown in Figures 6.5−6.7.

[1] *Fitting the curves were done by the software package "Cricket Graph", version 3.1, ©Microsoft Corporation.*

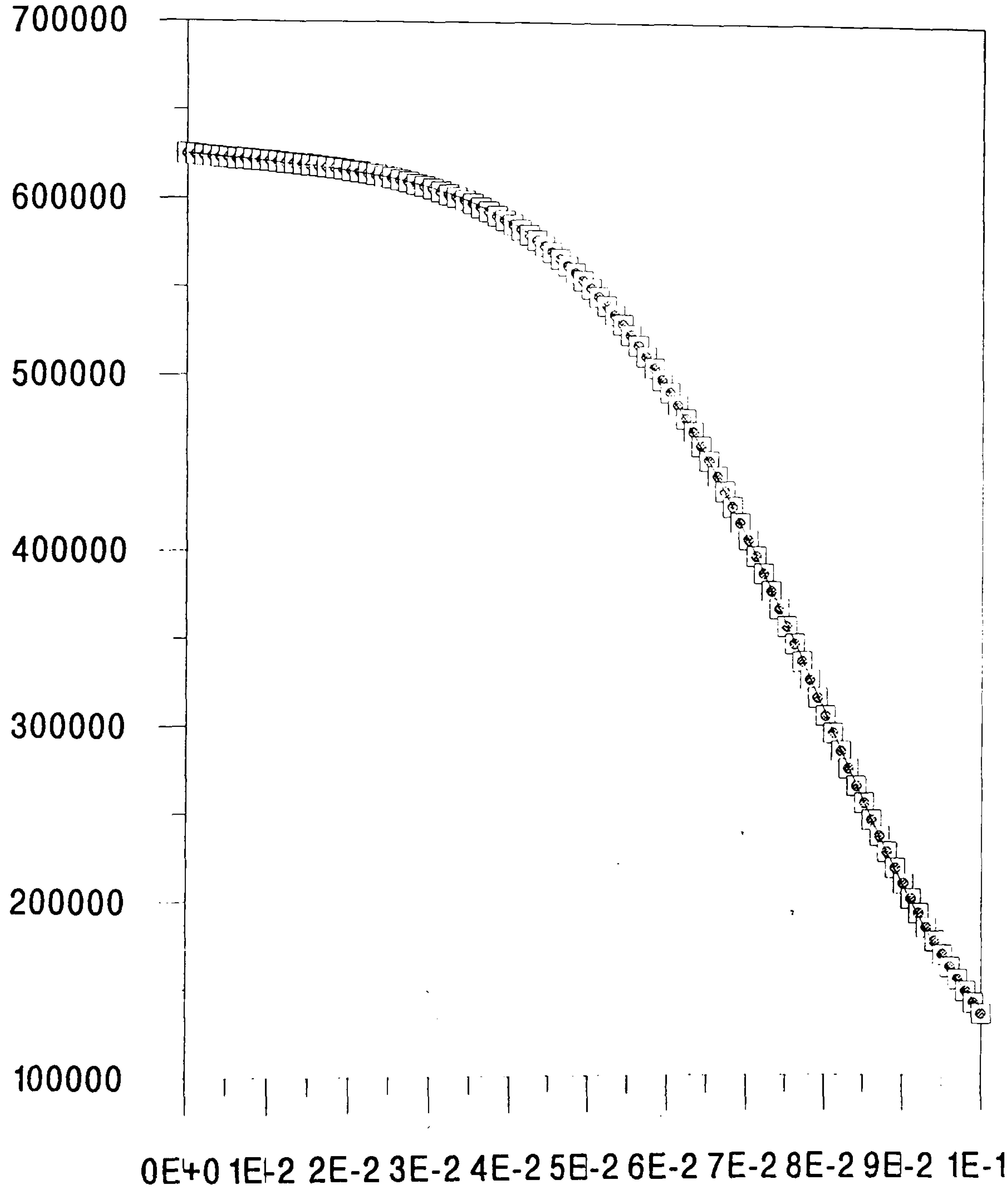

Figure 6.1: *Fifth-degree polynomial fitting of susceptibles; the regression coefficient is 1.0.* □ *data points;* – *polynomial fitting.*

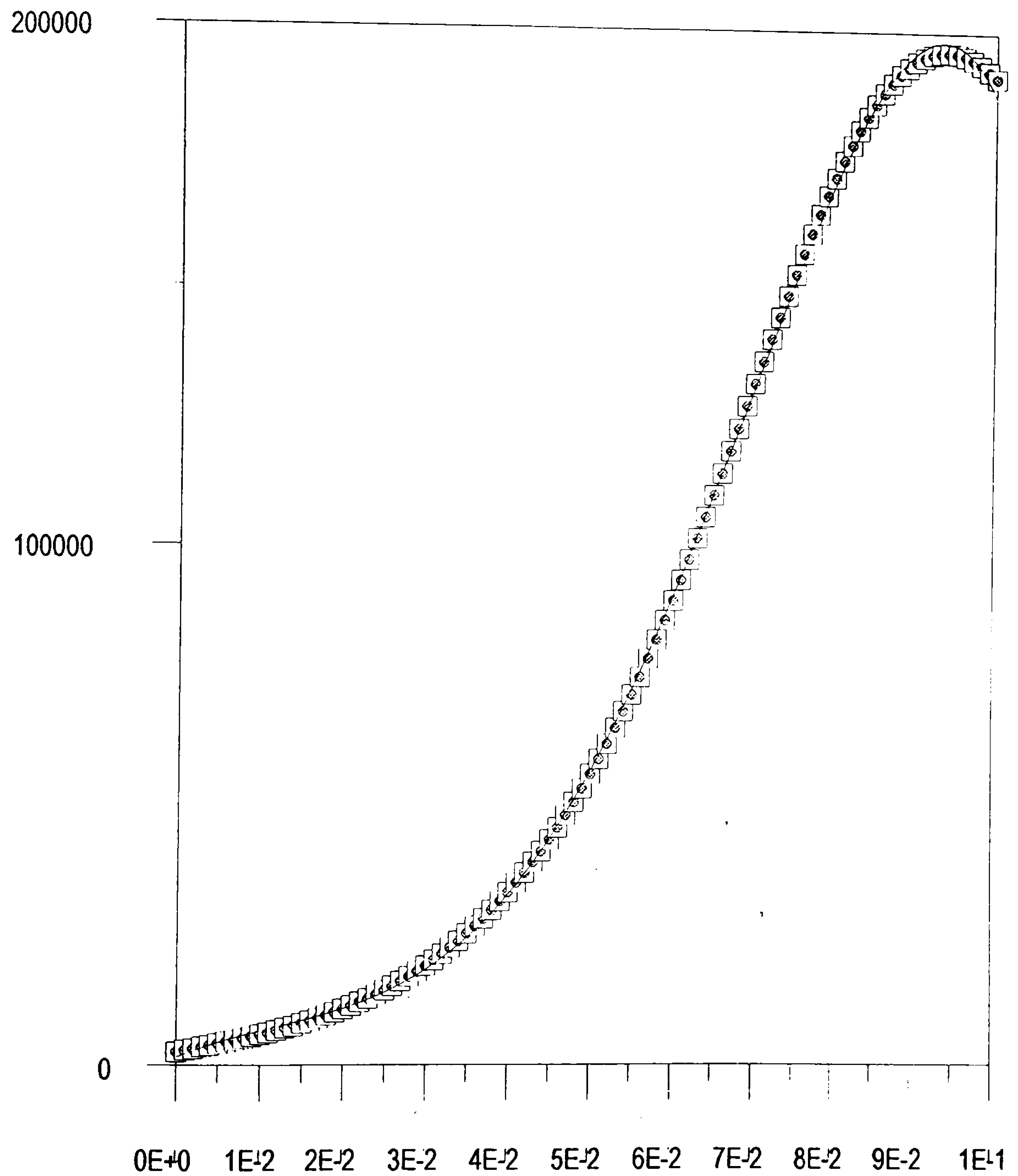

Figure 6.2: *Fifth-degree polynomial fitting of exposed; the regression coefficient is 1.0.* □ *data points; – polynomial fitting.*

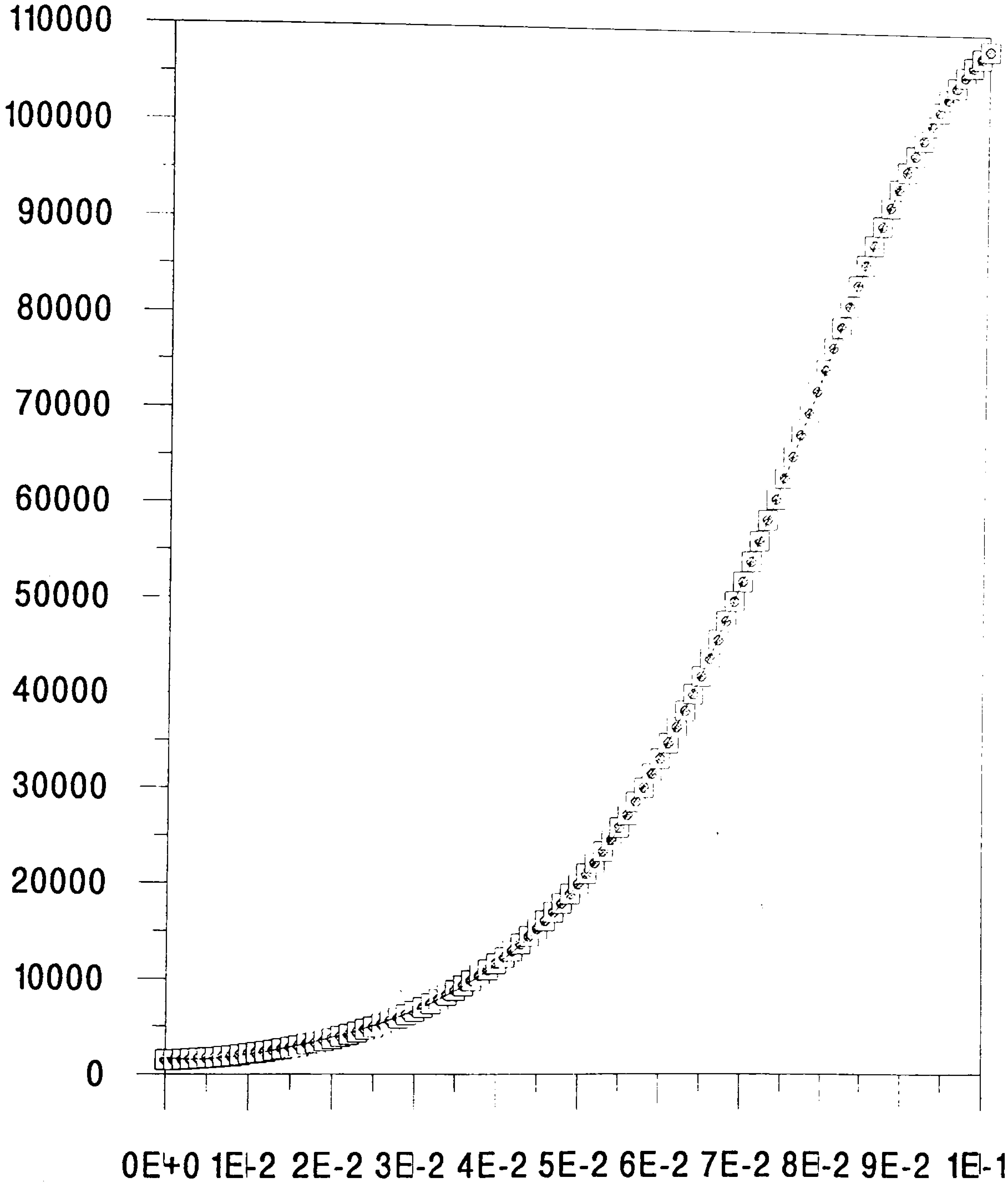

Figure 6.3: *Fifth-degree polynomial fitting of infectives; the regression coefficient is 1.0.* □ *data points; – polynomial fitting.*

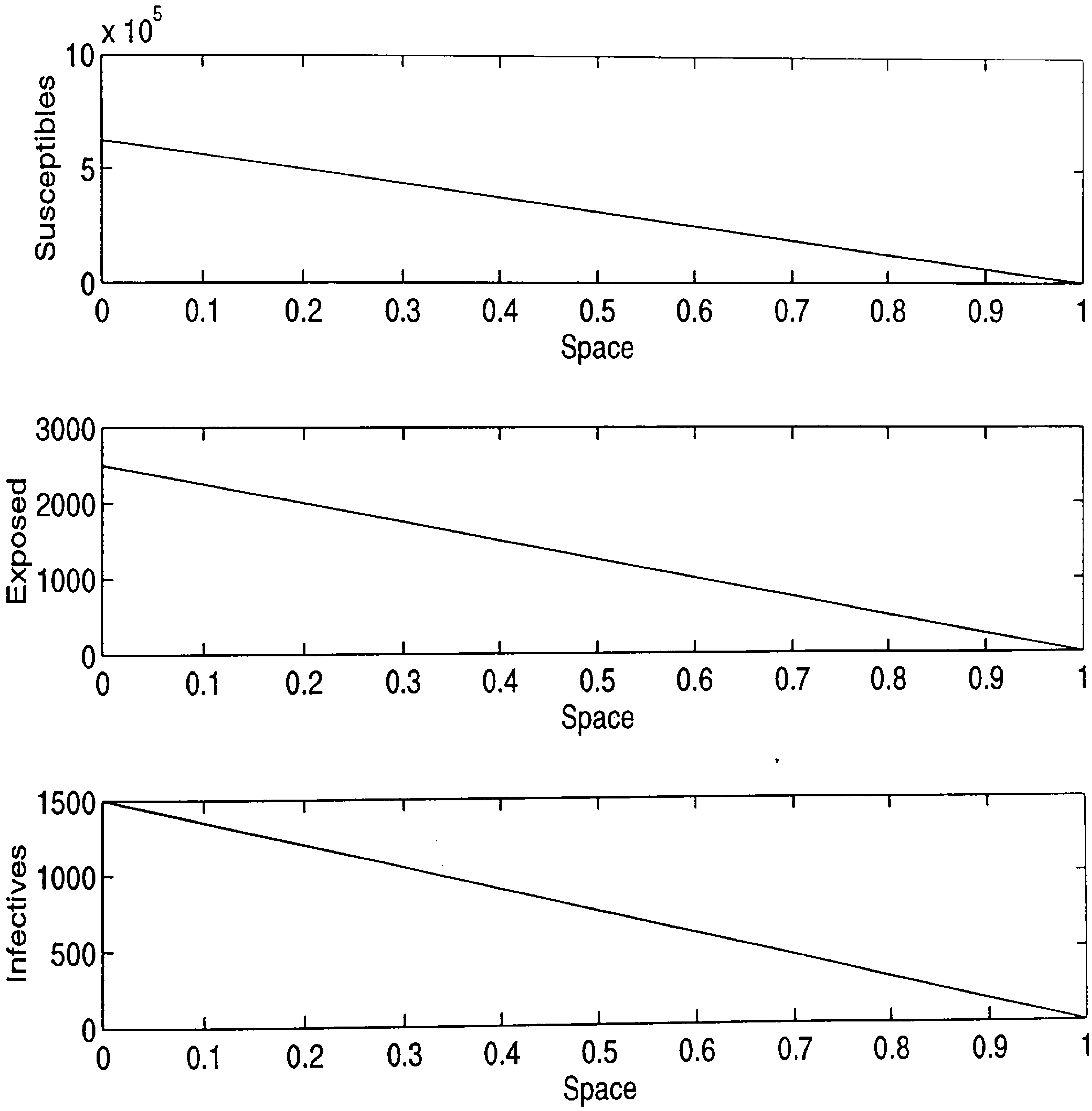

Figure 6.4: *Initial distributions for susceptibles, exposed and infectives.*

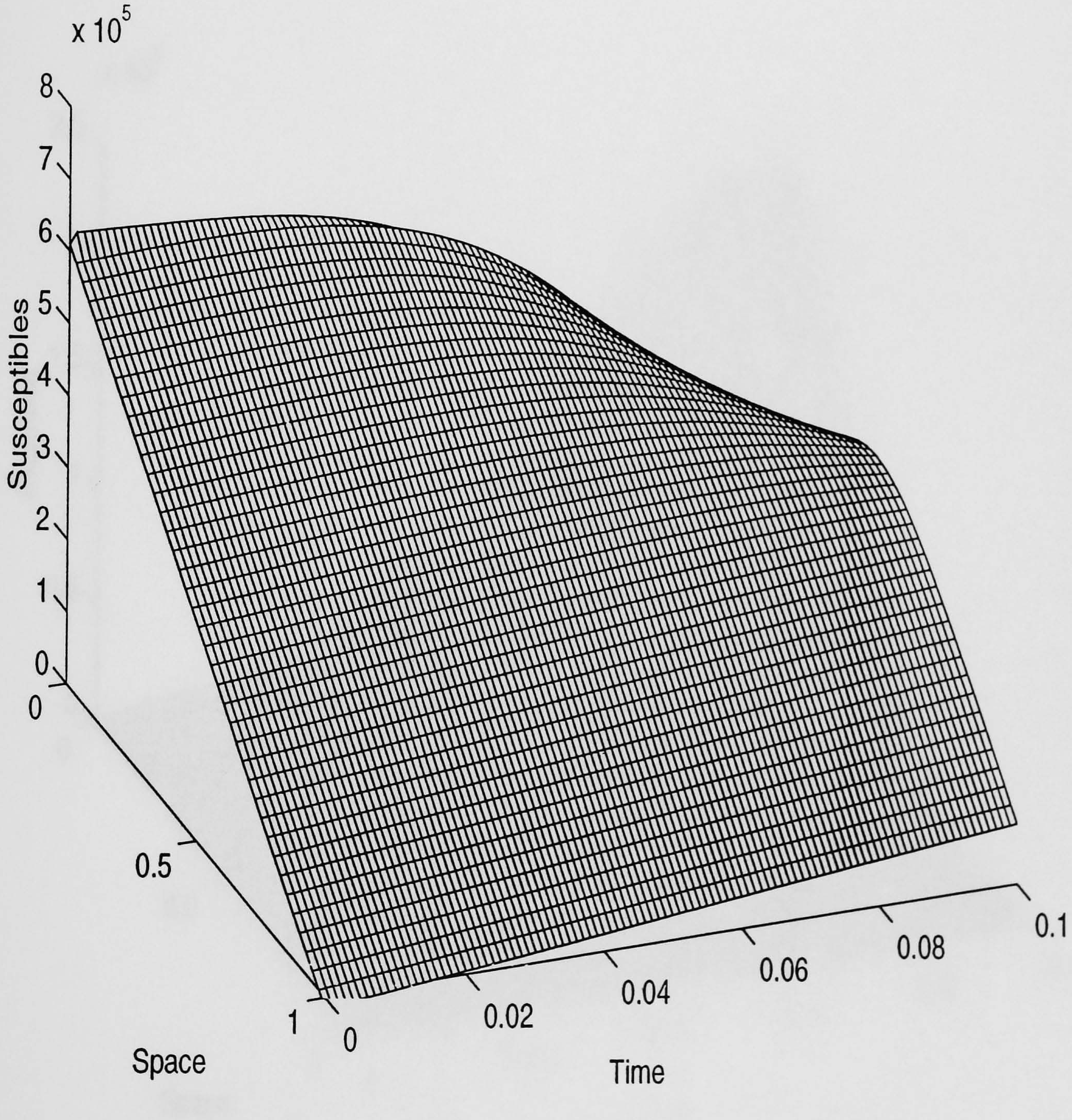

Figure 6.5: *Three-dimensional distribution of susceptibles;* $\ell = 0.001$ *and* $h = 0.025$.

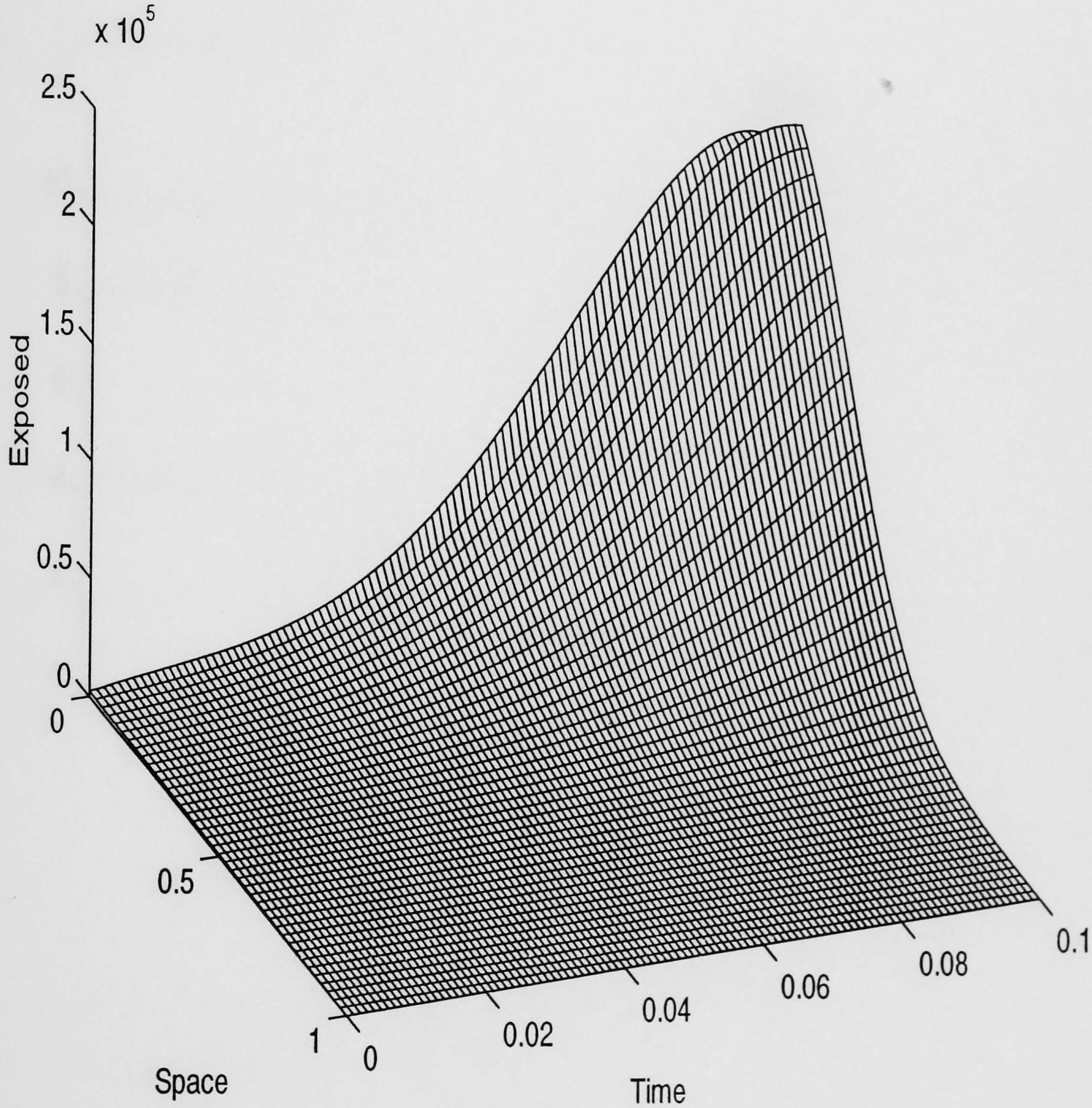

Figure 6.6: *Three-dimensional distribution of exposed;* $\ell = 0.001$ *and* $h = 0.025$.

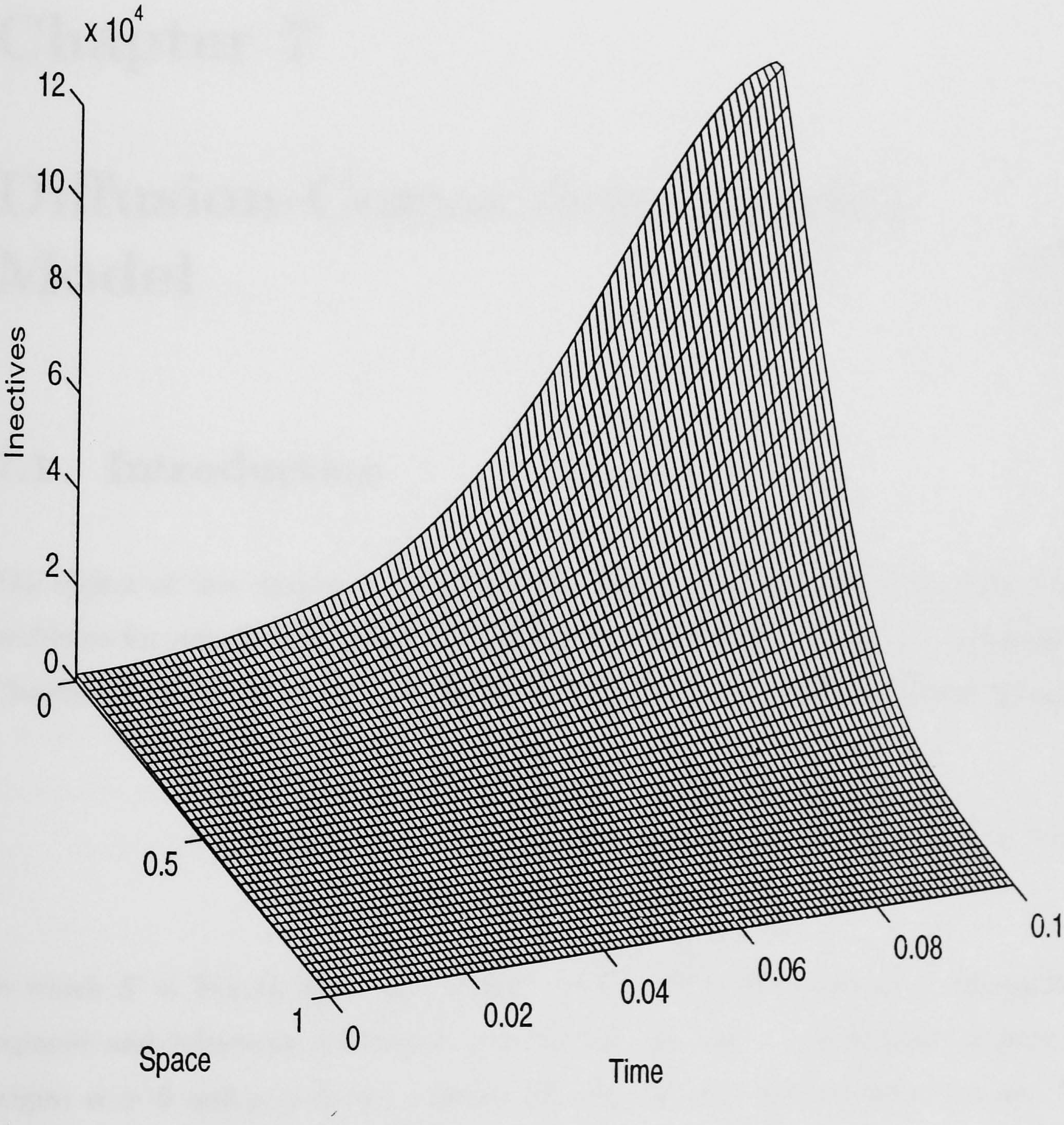

Figure 6.7: *Three-dimensional distribution of infectives;* $\ell = 0.001$ *and* $h = 0.025$.

Chapter 7

Diffusion-Convection Measles Model

7.1 Introduction

The object of this chapter is to study the solutions of mixed initial/boundary-value problems for measles dynamics of diffusion-convection type, which is a composite of Chapters 4 and 6. The system is the class of non-linear parabolic equations given by

$$\begin{aligned} \alpha\, S_{xx} &= S_t + \rho\, S_x - \mu\, N + (\mu + \beta\, I)\, S \\ \alpha\, E_{xx} &= E_t + \rho\, E_x - \beta\, I\, S + (\mu + \sigma)\, E \\ \alpha\, I_{xx} &= I_t + \rho\, I_x - \sigma\, E + (\mu + \gamma)\, I \end{aligned} \qquad (7.1.1)$$

in which $S = S(x,t)$, $E = E(x,t)$ and $I = I(x,t)$ are the number of susceptibles, exposed and infectious individuals, respectively, at time t and distance x from the origin; $\alpha > 0$ and $\rho > 0$ are, respectively, the diffusion and convection rates. The parameters μ, β, σ and γ are as before.

The initial conditions are of the form

$$S(x,0) = S^0(x), \quad E(x,0) = E^0(x), \quad I(x,0) = I^0(x); \quad 0 \le x \le L \qquad (7.1.2)$$

and the boundary conditions are

$$S_x(0,t) = E_x(0,t) = I_x(0,t) = 0\,; \qquad t > 0\,,$$

$$S_x(L,t) = E_x(L,t) = I_x(L,t) = 0\,; \qquad t > 0\,. \qquad (7.1.3)$$

Differentiating the three equations in (7.1.1) with respect to t gives

$$\begin{aligned} \alpha\, S_{xxt} - S_{tt} - \rho\, S_{xt} - (\mu + \beta\, I)\, S_t - \beta\, I_t\, S &= 0 \\ \alpha\, E_{xxt} - E_{tt} - \rho\, E_{xt} + \beta\, I\, S_t + \beta\, I_t\, S - (\mu + \sigma)\, E_t &= 0 \\ \alpha\, I_{xxt} - I_{tt} - \rho\, I_{xt} + \sigma\, E_t - (\mu + \gamma)\, I_t &= 0\,. \end{aligned} \qquad (7.1.4)$$

Let the open region $\Omega = [0 < x < L, t > 0]$ be bounded by the lines $x = 0$, $t = 0$ and $x = L$; the closure of Ω will be denoted by $\overline{\Omega}$. The set composed of the segments $\partial\Omega_0(0 < x < L, t = 0)$, $\partial\Omega_1(x = 0, t > 0)$ and $\partial\Omega_2(x = L, t > 0)$ will be denoted by $\partial\Omega$ and called the boundary of Ω. The interval $0 \leq x \leq L$ is divided into $M + 1$ subintervals each of width h so that $(M + 1)\, h = L$ and the time interval $t \geq 0$ is discretized in steps of length ℓ. The open region Ω and its boundary have thus been covered by a rectangular mesh having coordinates of the form (x_m, t_n) where $x_m = m\, h\, (m = 0, 1, 2, \ldots, M, M + 1)$ and $t_n = n\, \ell\, (n = 0, 1, 2, \ldots)$.

The solutions of (7.1.1)–(7.1.3) at the typical point (x_m, t_n) are, of course, $S(x_m, t_n)$, $E(x_m, t_n)$ and $I(x_m, t_n)$: these may be denoted by S_m^n, E_m^n and I_m^n, respectively. The theoretical solutions of numerical approximations to (7.1.1) at the same mesh point will be denoted, respectively, by A_m^n, B_m^n and C_m^n, while the values actually obtained, which may be subject, for example, to round-off errors, will be denoted by $\widetilde{A_m^n}$, $\widetilde{B_m^n}$ and $\widetilde{C_m^n}$, respectively.

7.2 Numerical Methods

7.2.1 Numerical Method for S

The time derivative in the first equation in (7.1.1) is approximated by the first-order forward-difference replacement in (4.4.1), S_{xx} by the weighted approximant as given in (4.4.2) with $\phi = \frac{1}{2}$ and S_x by the central-difference approximant

$$S_x \approx \frac{1}{2}\left\{\frac{S(x+h,t+\ell)-S(x-h,t+\ell)}{2h}+\frac{S(x+h,t)-S(x-h,t)}{2h}\right\} \qquad (7.2.1)$$

in which $x = x_m\,(m = 0,1,2,\ldots,M,M+1)$ and $t = t_n\,(n = 0,1,2,\ldots)$.

Equations (4.4.1), (4.4.2) ($\phi = \frac{1}{2}$) and (7.2.1) are used in the first equation in (7.1.1) and approximating that equation as follows

$$\begin{aligned}\frac{A_m^{n+1}-A_m^n}{\ell} &+ \frac{1}{2}\rho\left\{\frac{A_{m+1}^{n+1}-A_{m-1}^{n+1}}{2h}+\frac{A_{m+1}^{n}-A_{m-1}^{n}}{2h}\right\}+\frac{1}{2}(\mu+\beta\,C_m^n)\,A_m^{n+1}\\ &+ \frac{1}{2}(\mu+\beta\,C_m^{n+1})\,A_m^n-\frac{\alpha}{2h^2}\left\{A_{m-1}^{n+1}-2A_m^{n+1}+A_{m+1}^{n+1}\right.\\ &+ \left.A_{m-1}^n-2A_m^n+A_{m+1}^n\right\}-\mu N=0, \qquad (7.2.2)\end{aligned}$$

gives, after rearranging,

$$\begin{aligned}-\left(\frac{1}{4}\rho r+\frac{1}{2}\alpha p\right)A_{m-1}^{n+1} &+ \left[1+\frac{1}{2}\ell(\mu+\beta C_m^n)+\alpha p\right]A_m^{n+1}+\left(\frac{1}{4}\rho r-\frac{1}{2}\alpha p\right)A_{m+1}^{n+1}\\ &+ \frac{1}{2}\ell\beta C_m^{n+1}A_m^n=\left(\frac{1}{4}\rho r+\frac{1}{2}\alpha p\right)A_{m-1}^n+\left[1-\frac{1}{2}\ell\mu-\alpha p\right]A_m^n\\ &+ \left(-\frac{1}{4}\rho r+\frac{1}{2}\alpha p\right)A_{m+1}^n+\ell\mu N, \qquad (7.2.3)\end{aligned}$$

where $r = \ell/h$ and $p = \ell/h^2$.

The local truncation error $\mathcal{L}_S = \mathcal{L}_S[S(x,t),E(x,t),I(x,t);h,\ell]$ associated with (7.2.3) at the point $(x,t) = (x_m,t_n)$ may be written down from (7.2.2): it is

$$\begin{aligned}\mathcal{L}_S &= \frac{S(x,t+\ell)-S(x,t)}{\ell}+\frac{\rho}{4h}\{S(x+h,t+\ell)-S(x-h,t+\ell)\\ &+ S(x+h,t)-S(x-h,t)\}+\frac{1}{2}(\mu+\beta I(x,t))\,S(x,t+\ell)\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}(\mu+\beta\, I(x,t+\ell))\, S(x,t) - \frac{\alpha}{2\,h^2}\,\{S(x-h,t+\ell) - 2\,S(x,t+\ell) \\
&+ \quad S(x+h,t+\ell) + S(x-h,t) - 2\,S(x,t) + S(x+h,t)\} - \mu\, N \\
&- \quad \{S_t(x,t) + \rho\, S_x(x,t) + (\mu+\beta\, I(x,t))\, S(x,t) \\
&- \quad \alpha\, S_{xx}(x,t) - \mu\, N\}\, . \qquad (7.2.4)
\end{aligned}
$$

Expanding $S(x,t+\ell)$, $S(x\pm h,t+\ell)$, $S(x\pm h,t)$ and $I(x,t+\ell)$ as Taylor series about (x,t) leads to

$$
\begin{aligned}
\mathcal{L}_S &= \left[\frac{1}{2}\,S_{tt} + \frac{1}{2}\,\rho\, S_{xt} + \frac{1}{2}\,(\mu+\beta\, I)\, S_t + \frac{1}{2}\,\beta\, I_t\, S - \frac{1}{2}\,\alpha\, S_{xxt}\right]\,\ell + \frac{1}{12}\,\rho\, h^2\,\ell\, S_{xxxt} \\
&+ \left(\frac{1}{6}\,\rho\, S_{xxx} - \frac{1}{12}\,\alpha\, S_{xxxx}\right)\, h^2 + \left[\frac{1}{6}\,S_{ttt} + \frac{1}{4}\,\rho\, S_{xtt} + \frac{1}{4}\,(\mu+\beta\, I)\, S_{tt}\right. \\
&+ \left.\frac{1}{4}\,\beta\, I_{tt}\, S - \frac{1}{4}\,\alpha\, S_{xxtt}\right]\,\ell^2 + \cdots\, . \qquad (7.2.5)
\end{aligned}
$$

The first quation in (7.1.4) reveals that the term in ℓ in the equation (7.2.5) vanishes. Thus $\mathcal{L}_S = O(h^2+\ell^2)$ as $h,\ell \longrightarrow 0$, leaving

$$
\begin{aligned}
\mathcal{L}_S &= \frac{1}{12}\,\rho\, h^2\,\ell\, S_{xxxt} + \left(\frac{1}{6}\,\rho\, S_{xxx} - \frac{1}{12}\,\alpha\, S_{xxxx}\right)\, h^2 + \left[\frac{1}{6}\,S_{ttt} + \frac{1}{4}\,\rho\, S_{xtt}\right. \\
&+ \left.\frac{1}{4}\,(\mu+\beta\, I)\, S_{tt} + \frac{1}{4}\,\beta\, I_{tt}\, S - \frac{1}{4}\,\alpha\, S_{xxtt}\right]\,\ell^2 + \cdots\, . \qquad (7.2.6)
\end{aligned}
$$

The finite-difference method (7.2.3) may be applied for $m = 1,2,\ldots,M$ and $n = 0,1,2,\ldots$. When $m=0$ it requires some modification and may be simplified a little when $m = M+1$. Applying (7.2.3) with $m=0$ introduces the terms A_{-1}^{n+1} and A_{-1}^{n} but the points (x_{-1},t_{n+1}) and (x_{-1},t_n) are outside the grid superimposed on $\Omega \cup \partial\Omega$. However, applying the central-difference approximant

$$
S_x \approx \frac{S(x+h,t) - S(x-h,t)}{2\,h}
$$

to the boundary conditions (7.1.3) give, to second order, $A_{-1}^{n} = A_1^{n}$ and $A_{-1}^{n+1} = A_1^{n+1}$ so that equation (7.2.3) yields

$$
\begin{aligned}
[1+\frac{1}{2}\,\ell\,(\mu &+ \beta\, C_0^{n}) + \alpha\, p]\, A_0^{n+1} - \alpha\, p\, A_1^{n+1} + \frac{1}{2}\,\ell\,\beta\, C_0^{n+1}\, A_0^{n} \\
&= \left[1 - \frac{1}{2}\,\ell\,\mu - \alpha\, p\right]\, A_0^{n} + \alpha\, p\, A_1^{n} + \ell\,\mu\, N\, . \qquad (7.2.7)
\end{aligned}
$$

With $m = M+1$, the terms A_{M+2}^{n+1} and A_{M+2}^{n} are obtained which may be replaced by, using similar discussion as above, A_M^{n+1} and A_M^n and so equation (7.2.3) gives, for $m = M+1$,

$$\begin{aligned} -\alpha\, p\, A_M^{n+1} &+ \left[1 + \frac{1}{2}\ell(\mu + \beta\, C_{M+1}^{n}) + \alpha\, p\right] A_{M+1}^{n+1} + \frac{1}{2}\ell\,\beta\, C_{M+1}^{n+1}\, A_{M+1}^{n} \\ &= \alpha\, p\, A_M^n + \left[1 - \frac{1}{2}\ell\,\mu - \alpha\, p\right] A_{M+1}^{n} + \ell\,\mu\, N\,. \end{aligned} \quad (7.2.8)$$

7.2.2 Numerical Method for E

In this section the time derivative in the second equation in (7.1.1) is approximated using (4.4.13), E_{xx} is approximated as in (4.4.14) ($\phi = \frac{1}{2}$) and E_x is approximated by

$$E_x \approx \frac{1}{2}\left\{\frac{E(x+h, t+\ell) - E(x-h, t+\ell)}{2\,h} + \frac{E(x+h,t) - E(x-h,t)}{2\,h}\right\} \quad (7.2.9)$$

in which $x = x_m\,(m = 0, 1, 2, \ldots, M, M+1)$ and $t = t_n\,(n = 0, 1, 2, \ldots)$.

Using equations (4.4.13), (4.4.14) ($\phi = \frac{1}{2}$) and (7.2.9) in the second equation in (7.1.1) and approximating as follows

$$\begin{aligned} \frac{B_m^{n+1} - B_m^n}{\ell} &+ \frac{1}{2}\rho\left\{\frac{B_{m+1}^{n+1} - B_{m-1}^{n+1}}{2\,h} + \frac{B_{m+1}^{n} - B_{m-1}^{n}}{2\,h}\right\} - \frac{1}{2}\beta\, C_m^n\, A_m^{n+1} \\ &- \frac{1}{2}\beta\, C_m^{n+1}\, A_m^n + \frac{1}{2}(\mu+\sigma)\, B_m^{n+1} + \frac{1}{2}(\mu+\sigma)\, B_m^n - \frac{\alpha}{2\,h^2}\left\{B_{m-1}^{n+1}\right. \\ &- \left. 2\,B_m^{n+1} + B_{m+1}^{n+1} + B_{m-1}^n - 2\,B_m^n + B_{m+1}^n\right\} = 0\,, \end{aligned} \quad (7.2.10)$$

gives, after rearranging,

$$\begin{aligned} -\left(\frac{1}{4}\rho\, r + \frac{1}{2}\alpha\, p\right) B_{m-1}^{n+1} &+ \left[1 + \frac{1}{2}\ell(\mu+\sigma) + \alpha\, p\right] B_m^{n+1} + \left(\frac{1}{4}\rho\, r - \frac{1}{2}\alpha\, p\right) B_{m+1}^{n+1} \\ &- \frac{1}{2}\ell\,\beta\, C_m^n\, A_m^{n+1} - \frac{1}{2}\ell\,\beta\, C_m^{n+1}\, A_m^n = \left(\frac{1}{4}\rho\, r + \frac{1}{2}\alpha\, p\right) B_{m-1}^n \\ &+ \left[1 - \frac{1}{2}\ell(\mu+\sigma) - \alpha\, p\right] B_m^n \\ &+ \left(-\frac{1}{4}\rho\, r + \frac{1}{2}\alpha\, p\right) B_{m+1}^n\,, \end{aligned} \quad (7.2.11)$$

where $r = \ell/h$ and $p = \ell/h^2$.

The local truncation error $\mathcal{L}_E = \mathcal{L}_E[S(x,t), E(x,t), I(x,t); h, \ell]$ associated with (7.2.11) at the point $(x,t) = (x_m, t_n)$ may be written down from (7.2.10): it is

$$\begin{aligned}
\mathcal{L}_E &= \frac{E(x,t+\ell) - E(x,t)}{\ell} + \frac{\rho}{4\,h}\,\{E(x+h,t+\ell) - E(x-h,t+\ell) + E(x+h,t) \\
&- E(x-h,t)\} - \frac{1}{2}\beta\, I(x,t)\, S(x,t+\ell) - \frac{1}{2}\beta\, I(x,t+\ell)\, S(x,t) \\
&+ \frac{1}{2}(\mu+\sigma)\, E(x,t+\ell) + \frac{1}{2}(\mu+\sigma)\, E(x,t) - \frac{\alpha}{2\,h^2}\,\{E(x-h,t+\ell) - 2\,E(x,t+\ell) \\
&+ E(x+h,t+\ell) + E(x-h,t) - 2\,E(x,t) + E(x+h,t)\} - \{E_t(x,t) + \rho\, E_x(x,t) \\
&- \beta\, I(x,t)\, S(x,t) + (\mu+\sigma)\, E(x,t) - \alpha\, E_{xx}(x,t)\}\ . \qquad (7.2.12)
\end{aligned}$$

After expanding $E(x,t+\ell)$, $E(x \pm h, t+\ell)$, $E(x \pm h, t)$, $S(x,t+\ell)$ and $I(x,t+\ell)$, using Taylor's expansion, about the point (x,t), equation (7.2.12) becomes

$$\begin{aligned}
\mathcal{L}_E &= \left[\frac{1}{2}\,E_{tt} + \frac{1}{2}\,\rho\, E_{xt} - \frac{1}{2}\,\beta\, I_t\, S - \frac{1}{2}\,\beta\, I\, S_t + \frac{1}{2}\,(\mu+\sigma)\, E_t - \frac{1}{2}\,\alpha\, E_{xxt}\right]\ell \\
&+ \frac{1}{12}\,\rho\, h^2\, \ell\, E_{xxxt} + \left(\frac{1}{6}\,\rho\, E_{xxx} - \frac{1}{12}\,\alpha\, E_{xxxx}\right) h^2 + \left[\frac{1}{6}\,E_{ttt} + \frac{1}{4}\,\rho\, E_{xtt} - \frac{1}{4}\,\beta\, I\, S_{tt} \right. \\
&- \left. \frac{1}{4}\,\beta\, I_{tt}\, S + \frac{1}{4}\,(\mu+\sigma)\, E_{tt} - \frac{1}{4}\,\alpha\, E_{xxtt}\right]\ell^2 + \cdots\,. \qquad (7.2.13)
\end{aligned}$$

The second equation in (7.1.4) reveals that the coefficient of ℓ in (7.2.13) vanishes, leaving

$$\begin{aligned}
\mathcal{L}_E &= \frac{1}{12}\,\rho\, h^2\, \ell\, E_{xxxt} + \left(\frac{1}{6}\,\rho\, E_{xxx} - \frac{1}{12}\,\alpha\, E_{xxxx}\right) h^2 + \left[\frac{1}{6}\,E_{ttt} + \frac{1}{4}\,\rho\, E_{xtt} \right. \\
&- \left. \frac{1}{4}\,\beta\, I\, S_{tt} - \frac{1}{4}\,\beta\, I_{tt}\, S + \frac{1}{4}\,(\mu+\sigma)\, E_{tt} - \frac{1}{4}\,\alpha\, E_{xxtt}\right]\ell^2 + \cdots\,, \qquad (7.2.14)
\end{aligned}$$

which is $O(h^2 + \ell^2)$ as $h, \ell \longrightarrow 0$.

The finite-difference method (7.2.11) may be applied for $m = 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots$. When $m = 0$ it requires some modification and may be simplified a little when $m = M+1$. As in the case of S, with $m = 0$, equation (7.2.11) produces the terms

B_{-1}^{n+1} and B_{-1}^{n} which may be replaced to second order by B_{1}^{n+1} and B_{1}^{n}, respectively, so that

$$\begin{aligned} -\frac{1}{2}\,\ell\,\beta\,C_0^n\,A_0^{n+1} &+ \left[1+\frac{1}{2}\,\ell\,(\mu+\sigma)+\alpha\,p\right]\,B_0^{n+1} - \alpha\,p\,B_1^{n+1} - \frac{1}{2}\,\ell\,\beta\,C_0^{n+1}\,A_0^n \\ &= \left[1-\frac{1}{2}\,\ell\,(\mu+\sigma)-\alpha\,p\right]\,B_0^n + \alpha\,p\,B_1^n \qquad (7.2.15) \end{aligned}$$

and with $m = M+1$ equation (7.2.11) becomes

$$\begin{aligned} -\frac{1}{2}\,\ell\,\beta\,C_{M+1}^n\,A_{M+1}^{n+1} &- \alpha\,p\,B_M^{n+1} + \left[1+\frac{1}{2}\,\ell\,(\mu+\sigma)+\alpha\,p\right]\,B_{M+1}^{n+1} - \frac{1}{2}\,\ell\,\beta\,C_{M+1}^{n+1}\,A_{M+1}^n \\ &= \alpha\,p\,B_M^n + \left[1-\frac{1}{2}\,\ell\,(\mu+\sigma)-\alpha\,p\right]\,B_{M+1}^n\,. \qquad (7.2.16) \end{aligned}$$

7.2.3 Numerical Method for I

The time derivative in the third equation in (7.1.1) is approximated by (4.4.25), I_{xx} is approximated by (4.4.26) ($\phi = \frac{1}{2}$) and I_x is approximated by the central-difference replacement

$$I_x \approx \frac{1}{2}\left\{\frac{I(x+h,t+\ell)-I(x-h,t+\ell)}{2\,h} + \frac{I(x+h,t)-I(x-h,t)}{2\,h}\right\} \qquad (7.2.17)$$

in which $x = x_m\,(m = 0,1,2,\ldots,M,M+1)$ and $t = t_n\,(n = 0,1,2,\ldots)$.

Using equations (4.4.25), (4.4.26) ($\phi = \frac{1}{2}$) and (7.2.17) in the third equation in (7.1.1) and approximating as follows

$$\begin{aligned} \frac{C_m^{n+1}-C_m^n}{\ell} &+ \frac{1}{2}\,\rho\left\{\frac{C_{m+1}^{n+1}-C_{m-1}^{n+1}}{2\,h} + \frac{C_{m+1}^n-C_{m-1}^n}{2\,h}\right\} - \frac{1}{2}\,\sigma\,B_m^{n+1} - \frac{1}{2}\,\sigma\,B_m^n \\ &+ \frac{1}{2}\,(\mu+\gamma)\,C_m^{n+1} + \frac{1}{2}\,(\mu+\gamma)\,C_m^n - \frac{\alpha}{2\,h^2}\left\{C_{m-1}^{n+1} - 2\,C_m^{n+1} + C_{m+1}^{n+1}\right. \\ &+ \left. C_{m-1}^n - 2\,C_m^n + C_{m+1}^n\right\} = 0\,, \qquad (7.2.18) \end{aligned}$$

yields, after rearranging,

$$\begin{aligned} -\left(\frac{1}{4}\,\rho\,r+\frac{1}{2}\,\alpha\,p\right)\,C_{m-1}^{n+1} &+ \left[1+\frac{1}{2}\,\ell\,(\mu+\gamma)+\alpha\,p\right]\,C_m^{n+1} + \left(\frac{1}{4}\,\rho\,r-\frac{1}{2}\,\alpha\,p\right)\,C_{m+1}^{n+1} \\ &- \frac{1}{2}\,\ell\,\sigma\,B_m^{n+1} = \left(\frac{1}{4}\,\rho\,r+\frac{1}{2}\,\alpha\,p\right)\,C_{m-1}^n + \left[1-\frac{1}{2}\,\ell\,(\mu+\gamma)-\alpha\,p\right]\,B_m^n \\ &+ \left(-\frac{1}{4}\,\rho\,r+\frac{1}{2}\,\alpha\,p\right)\,C_{m+1}^n + \frac{1}{2}\,\ell\,\sigma\,B_m^n\,, \qquad (7.2.19) \end{aligned}$$

where $r = \ell/h$ and $p = \ell/h^2$.

The local truncation error $\mathcal{L}_I = \mathcal{L}_I[S(x,t), E(x,t), I(x,t); h, \ell]$ associated with (7.2.19) at the point $(x,t) = (x_m, t_n)$ may be written down from (7.2.18): it is

$$\begin{aligned}
\mathcal{L}_I &= \frac{I(x,t+\ell) - I(x,t)}{\ell} + \frac{\rho}{4h}\{I(x+h,t+\ell) - I(x-h,t+\ell) + I(x+h,t) \\
&- I(x-h,t)\} - \frac{1}{2}\sigma E(x,t+\ell) - \frac{1}{2}\sigma E(x,t) + \frac{1}{2}(\mu+\gamma) I(x,t+\ell) \\
&+ \frac{1}{2}(\mu+\gamma) I(x,t) - \frac{\alpha}{2h^2}\{I(x-h,t+\ell) - 2I(x,t+\ell) + I(x+h,t+\ell) \\
&+ I(x-h,t) - 2I(x,t) + I(x+h,t)\} - \{I_t(x,t) + \rho I_x(x,t) \\
&- \sigma E(x,t) + (\mu+\gamma) I(x,t) - \alpha I_{xx}(x,t)\}\,. \qquad (7.2.20)
\end{aligned}$$

Expanding $I(x,t+\ell)$, $I(x\pm h,t+\ell)$, $I(x\pm h,t)$ and $E(x,t+\ell)$, using Taylor's expansion, about the point (x,t) gives

$$\begin{aligned}
\mathcal{L}_I &= \left[\frac{1}{2} I_{tt} + \frac{1}{2}\rho I_{xt} - \frac{1}{2}\sigma E_t + \frac{1}{2}(\mu+\gamma) I_t - \frac{1}{2}\alpha I_{xxt}\right]\ell + \frac{1}{12}\rho h^2 \ell I_{xxxt} \\
&+ \left(\frac{1}{6}\rho I_{xxx} - \frac{1}{12}\alpha I_{xxxx}\right) h^2 + \left[\frac{1}{6} I_{ttt} + \frac{1}{4}\rho I_{xtt} - \frac{1}{4}\sigma E_{tt}\right. \\
&+ \left.\frac{1}{4}(\mu+\gamma) I_{tt} - \frac{1}{4}\alpha I_{xxtt}\right]\ell^2 + \cdots\,. \qquad (7.2.21)
\end{aligned}$$

This reveals that $\mathcal{L}_I = O(h^2 + \ell^2)$ as $h, \ell \longrightarrow 0$ because the term in ℓ vanishes (see the third equation in (7.1.4)).

The finite-difference method (7.2.19) may be applied for $m = 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots$. When $m = 0$ it needs some modification and may be simplified a little when $m = M+1$. As in Subsections 7.2.1 and 7.2.2, with $m = 0$, equation (7.2.19) becomes

$$\begin{aligned}
-\frac{1}{2}\ell\sigma B_0^{n+1} &+ \left[1 + \frac{1}{2}\ell(\mu+\gamma) + \alpha p\right] C_0^{n+1} - \alpha p C_1^{n+1} = \frac{1}{2}\ell\sigma B_0^n \\
&+ \left[1 - \frac{1}{2}\ell(\mu+\gamma) - \alpha p\right] C_0^n + \alpha p C_1^n\,, \qquad (7.2.22)
\end{aligned}$$

while for $m = M+1$ it gives

$$-\frac{1}{2}\ell\sigma B_{M+1}^{n+1} - \alpha p C_M^{n+1} + \left[1 + \frac{1}{2}\ell(\mu+\gamma) + \alpha p\right] C_{M+1}^{n+1} = \alpha p C_M^n$$

$$+ \left[1 - \frac{1}{2}\ell(\mu+\gamma) - \alpha p\right] C^n_{M+1} + \frac{1}{2}\ell\sigma B^n_{M+1} \,. \qquad (7.2.23)$$

7.3 Implementation

Adapting the notations of Section 4.5 it is easy to implement equations (7.2.3), (7.2.11) and (7.2.19) which may be written in matrix-vector form as

$$W^n \mathbf{U}^{n+1} = M^n \mathbf{U}^n + \mathbf{b} \qquad (7.3.1)$$

in which W^n and M^n are both of order $3(M+2)$ and are of the forms

$$W^n = \begin{pmatrix} X^n & O & H_1^n \\ F_1^n & Y^n & F_2^n \\ O & H_2^n & Z^n \end{pmatrix}, \qquad (7.3.2)$$

$$M^n = \begin{pmatrix} P^n & O & O \\ O & Q^n & O \\ O & H_3^n & R^n \end{pmatrix}, \qquad (7.3.3)$$

where O is the zero matrix of order $(M+2)$ and the vector $\mathbf{b}$ is a column-vector of order $3(M+2)$ given by

$$\mathbf{b} = \Big[\underbrace{\ell\mu N, \ldots, \ell\mu N}_{(M+2)\, times}, 0, \ldots, 0,\Big]^T, \qquad (7.3.4)$$

T denoting transpose.

The sub-matrices of W^n and M^n are each of order $(M+2)$ and are given by

$$X^n = \begin{pmatrix} v_0 & -\alpha p & & & & \\ -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_1 & \frac{1}{4}\rho r - \frac{1}{2}\alpha p & & & \\ & -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_2 & \frac{1}{4}\rho r - \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_M & \frac{1}{4}\rho r - \frac{1}{2}\alpha p \\ & & & -\alpha p & v_{M+1} \end{pmatrix}, \qquad (7.3.5)$$

$$Y^n = \begin{pmatrix} v_Y & -\alpha p & & & & \\ -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_Y & \frac{1}{4}\rho r - \frac{1}{2}\alpha p & & & \\ & -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_Y & \frac{1}{4}\rho r - \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_Y & \frac{1}{4}\rho r - \frac{1}{2}\alpha p \\ & & & -\alpha p & v_Y \end{pmatrix}, \quad (7.3.6)$$

$$Z^n = \begin{pmatrix} v_Z & -\alpha p & & & & \\ -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_Z & \frac{1}{4}\rho r - \frac{1}{2}\alpha p & & & \\ & -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_Z & \frac{1}{4}\rho r - \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{4}\rho r - \frac{1}{2}\alpha p & v_Z & \frac{1}{4}\rho r - \frac{1}{2}\alpha p \\ & & & -\alpha p & v_Z \end{pmatrix}, \quad (7.3.7)$$

$$P^n = \begin{pmatrix} v_P & \alpha p & & & & \\ \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_P & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p & & & \\ & \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_P & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_P & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p \\ & & & \alpha p & v_P \end{pmatrix}, \quad (7.3.8)$$

$$Q^n = \begin{pmatrix} v_Q & \alpha p & & & & \\ \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_Q & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p & & & \\ & \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_Q & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_Q & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p \\ & & & \alpha p & v_Q \end{pmatrix}. \quad (7.3.9)$$

$$R^n = \begin{pmatrix} v_R & \alpha p & & & & \\ \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_R & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p & & & \\ & \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_R & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p & & \\ & \ddots & \ddots & \ddots & & \\ & & \frac{1}{4}\rho r + \frac{1}{2}\alpha p & v_R & -\frac{1}{4}\rho r + \frac{1}{2}\alpha p \\ & & & \alpha p & v_R \end{pmatrix}; \quad (7.3.10)$$

H_1^n, H_2^n, H_3^n, F_1^n and F_2^n are given in (4.5.11) and $v_i\,(i = 0, 1, 2, \ldots, M, M+1)$, v_Y, v_Z, v_P, v_Q and v_R are as in (4.5.12).

7.4 Stability Analysis

As in Sections 4.6, 5.4 and 6.3, the maximum principle analysis will be used in this section to prove convergence of the numerical method developed in this chapter. The theorem which will be used as a tool to prove convergence is due to Rose[53]. Theorem 1.4.1.

Now, by comparing system (7.1.1)–(7.1.3) with equations (1.4.6) and (1.4.8), it is easily seen that $u = S, E \, or \, I$, $\beta_0 = -1, \alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$. Also conditions (1.4.7) are satisfied for all functions S, E and I.

The first equation in (7.1.1) is written in the form

$$\begin{aligned}\frac{\alpha}{2}\,(S_{xx}+S_{xx}) &= S_t+\frac{\rho}{2}\,(S_x+S_x)-\mu\,N+\frac{1}{2}\,(\mu+\beta\,I)\,S+\frac{1}{2}\,(\mu+\beta\,I)\,S \\ &= S_t+\frac{\rho}{2}\,(S_x+S_x)-\mu\,N+G_1+G_2 \qquad (7.4.1)\end{aligned}$$

with initial and boundary conditions

$$S(x,0)=S^0(x)\,;\qquad 0\le x\le L$$

$$(7.4.2)$$

$$S_x(0,t)=S_x(L,t)=0\,;\qquad t>0$$

where G_1 and G_2 are assumed to be boundedly-differentiable with respect to S and I.

The solution $S(x,t)$ is approximated by A_m^n defined on Ω which agrees with $S(x,t)$ on $\partial\Omega_i\,(i=0,1,2)$ and satisfies the difference equation

$$\begin{aligned}\frac{\alpha}{2}\,\nabla^2\left(A_m^{n+1}+A_m^n\right) &= \nabla_t A_m^n+\frac{\rho}{2}\,\nabla_x\left(A_m^{n+1}+A_m^n\right)-\mu\,N+\frac{1}{2}\,(\mu+\beta\,C_m^n)\,A_m^{n+1} \\ &+ \frac{1}{2}\,(\mu+\beta\,C_m^{n+1})\,A_m^n\,,\quad n\ge 0\,, \qquad (7.4.3)\end{aligned}$$

where

$$\begin{aligned}\nabla^2 A_m^n &= \left(A_{m-1}^n-2\,A_m^n+A_{m+1}^n\right)/h^2\,, \\ \nabla_x A_m^n &= \left(A_{m+1}^n-A_{m-1}^n\right)/2\,h\,, \qquad (7.4.4) \\ \nabla_t A_m^n &= \left(A_m^{n+1}-A_m^n\right)/\ell\,.\end{aligned}$$

It is known that

$$\left.\begin{aligned}
G_1 = G_1(x_m, t_{n+1}, S_m^{n+1}, I_m^n) &= G_1(x_m, t_{n+1}, S_m^n, I_m^n) + (S_m^{n+1} - S_m^n)\,\overline{\tfrac{\partial G_1}{\partial S}} \\
&= \tfrac{1}{2}(\mu + \beta\, I_m^n)\, S_m^n + (S_m^{n+1} - S_m^n)\,\overline{\tfrac{\partial G_1}{\partial S}}\,, \\
G_2 = G_2(x_m, t_{n+1}, S_m^n, I_m^{n+1}) &= G_2(x_m, t_{n+1}, S_m^n, I_m^n) + (I_m^{n+1} - I_m^n)\,\overline{\tfrac{\partial G_2}{\partial I}} \\
&= \tfrac{1}{2}(\mu + \beta\, I_m^n)\, S_m^n + (I_m^{n+1} - I_m^n)\,\overline{\tfrac{\partial G_2}{\partial I}}\,, \\
\tfrac{1}{2}\nabla^2 (S_m^{n+1} + S_m^n) &= \tfrac{1}{2}\tfrac{\partial^2 S_m^{n+1}}{\partial x^2} + \tfrac{1}{2}\tfrac{\partial^2 S_m^n}{\partial x^2} + \tfrac{1}{24}h^2\left(\overline{\tfrac{\partial^4 S_m^{n+1}}{\partial x^4}} + \overline{\tfrac{\partial^4 S_m^n}{\partial x^4}}\right), \\
\tfrac{1}{2}\nabla_x (S_m^{n+1} + S_m^n) &= \tfrac{1}{2}\tfrac{\partial S_m^{n+1}}{\partial x} + \tfrac{1}{2}\tfrac{\partial S_m^n}{\partial x} + \tfrac{1}{12}h^2\left(\overline{\tfrac{\partial^3 S_m^{n+1}}{\partial x^3}} + \overline{\tfrac{\partial^3 S_m^n}{\partial x^3}}\right), \\
\nabla_t S_m^n &= \tfrac{\partial S_m^{n+1}}{\partial t} - \tfrac{1}{2}\ell\,\overline{\tfrac{\partial^2 S_m^{n+1}}{\partial t^2}}\,,
\end{aligned}\right\} \tag{7.4.5}$$

where the barred derivatives are evaluated at intermediate argument values as called for by the Mean Value Theorem.

Substituting (7.4.5) into (7.4.1) yields

$$\begin{aligned}
\frac{\alpha}{2}\nabla^2\left(S_m^{n+1} + S_m^n\right) &= \nabla_t S_m^n + \frac{\rho}{2}\nabla_x\left(S_m^{n+1} + S_m^n\right) - \mu\, N + \frac{1}{2}(\mu + \beta\, I_m^n)\, S_m^{n+1} \\
&+ \frac{1}{2}(\mu + \beta\, I_m^{n+1})\, S_m^n + \left\{\frac{1}{24}h^2\left(\overline{\frac{\partial^4 S_m^{n+1}}{\partial x^4}} + \overline{\frac{\partial^4 S_m^n}{\partial x^4}}\right) + \frac{1}{2}\ell\,\overline{\frac{\partial^2 S_m^{n+1}}{\partial t^2}}\right. \\
&- \frac{1}{12}h^2\left(\overline{\frac{\partial^3 S_m^{n+1}}{\partial x^3}} + \overline{\frac{\partial^3 S_m^n}{\partial x^3}}\right) - \left(S_m^{n+1} - S_m^n\right)\overline{\frac{\partial G_1}{\partial S}} \\
&- \left.\left(I_m^{n+1} - I_m^n\right)\overline{\frac{\partial G_2}{\partial I}}\right\}.
\end{aligned} \tag{7.4.6}$$

The assumption on S above requires the boundedness of all the derivatives appearing inside the bracket along with $S_m^{n+1} - S_m^n$ and $I_m^{n+1} - I_m^n$ in the region $\overline{\Omega}$. Hence, in this region,

$$\begin{aligned}
\frac{\alpha}{2}\nabla^2\left(S_m^{n+1} + S_m^n\right) &= \nabla_t S_m^n + \frac{\rho}{2}\nabla_x\left(S_m^{n+1} + S_m^n\right) - \mu\, N + \frac{1}{2}(\mu + \beta\, I_m^n)\, S_m^{n+1} \\
&+ \frac{1}{2}(\mu + \beta\, I_m^{n+1})\, S_m^n + g_m^n
\end{aligned} \tag{7.4.7}$$

with

$$g_m^n = O(h^2 + \ell)\,. \tag{7.4.8}$$

Now, if (7.4.3) is subtracted from (7.4.7) and using the definitions of the truncation errors in (4.6.8),

$$\begin{aligned}\frac{\alpha}{2}\nabla^2\left(Z_{1_m}^{n+1}+Z_{1_m}^{n}\right) &= \nabla_t Z_{1_m}^{n}+\frac{\rho}{2}\nabla_x\left(Z_{1_m}^{n+1}+Z_{1_m}^{n}\right)+\frac{1}{2}(\mu+\beta\, I_m^n)\,S_m^{n+1}\\ &- \frac{1}{2}(\mu+\beta\, C_m^n)\,A_m^{n+1}+\frac{1}{2}(\mu+\beta\, I_m^{n+1})\,S_m^n\\ &- \frac{1}{2}(\mu+\beta\, C_m^{n+1})\,A_m^n+g_m^n \qquad (7.4.9)\end{aligned}$$

with $Z_{1_m}^n$ and $Z_{1_m}^{n+1}$ vanishing on $\partial\Omega_i\,(i=0,1,2)$. As

$$\begin{aligned}G_1(x_m,t_{n+1},S_m^{n+1},I_m^n) &= G_1(x_m,t_{n+1},A_m^{n+1},C_m^n)+\overline{\frac{\partial G_1}{\partial S}}\left(S_m^{n+1}-A_m^{n+1}\right)\\ &+ \overline{\frac{\partial G_1}{\partial I}}\,(I_m^n-C_m^n) \qquad (7.4.10)\end{aligned}$$

and

$$\begin{aligned}G_2(x_m,t_{n+1},S_m^{n},I_m^{n+1}) &= G_2(x_m,t_{n+1},A_m^{n},C_m^{n+1})+\overline{\frac{\partial G_2}{\partial S}}\,(S_m^{n}-A_m^{n})\\ &+ \overline{\frac{\partial G_2}{\partial I}}\left(I_m^{n+1}-C_m^{n+1}\right), \qquad (7.4.11)\end{aligned}$$

it follows that equation (7.4.9) may be written as

$$\begin{aligned}\frac{\alpha}{2}\nabla^2\left(Z_{1_m}^{n+1}+Z_{1_m}^{n}\right) &= \nabla_t Z_{1_m}^{n}+\frac{\rho}{2}\nabla_x\left(Z_{1_m}^{n+1}+Z_{1_m}^{n}\right)+\frac{1}{2}\overline{\frac{\partial G_1}{\partial S}}\,Z_{1_m}^{n+1}+\frac{1}{2}\overline{\frac{\partial G_1}{\partial I}}\,Z_{3_m}^{n}\\ &+ \frac{1}{2}\overline{\frac{\partial G_2}{\partial S}}\,Z_{1_m}^{n}+\frac{1}{2}\overline{\frac{\partial G_2}{\partial I}}\,Z_{3_m}^{n+1}+g_m^n\,. \qquad (7.4.12)\end{aligned}$$

Rewriting equation (7.4.12) gives

$$\begin{aligned}\frac{\alpha}{2}\nabla^2\left(Z_{1_m}^{n+1}+Z_{1_m}^{n}\right) &= \nabla_t Z_{1_m}^{n}+\frac{\rho}{2}\nabla_x\left(Z_{1_m}^{n+1}+Z_{1_m}^{n}\right)+M_S\,\widehat{Z_{1_m}^n}\\ &+ M_I\,\widehat{Z_{3_m}^n}+g_m^n \qquad (7.4.13)\end{aligned}$$

where $M_S=\max\left\{\overline{\frac{\partial G_1}{\partial S}},\overline{\frac{\partial G_2}{\partial S}}\right\}$, $M_I=\max\left\{\overline{\frac{\partial G_1}{\partial I}},\overline{\frac{\partial G_2}{\partial I}}\right\}$ and $\widehat{Z_{1_m}^n}$ and $\widehat{Z_{3_m}^n}$ are as in (6.3.16). Assume that $\widehat{Z_{3_m}^n}$ is bounded. It is known that g_m^n is bounded. Moreover, $Z_{1_m}^n$ and $Z_{1_m}^{n+1}$ vanish on $\partial\Omega$. Hence, by Theorem1.4.1, A_m^n and A_m^{n+1} converge to S_m^n and S_m^{n+1} uniformly.

The second equation in (7.1.1) may be written in the form

$$\begin{aligned}\frac{\alpha}{2}\left(E_{xx}+E_{xx}\right) &= E_t+\frac{\rho}{2}\left(E_x+E_x\right)-\frac{1}{2}\beta I S-\frac{1}{2}\beta I S+\frac{1}{2}(\mu+\sigma)E+\frac{1}{2}(\mu+\sigma)E \\ &= E_t+\frac{\rho}{2}\left(E_x+E_x\right)-G_3-G_4+G_5+\frac{1}{2}(\mu+\sigma)E \qquad (7.4.14)\end{aligned}$$

with initial and boundary conditions

$$\begin{aligned} E(x,0)=E^0(x)\,; &\qquad 0\le x\le L \\ & \qquad\qquad (7.4.15) \\ E_x(0,t)=E_x(L,t)=0\,; &\qquad t>0\end{aligned}$$

where G_3 and G_4 are assumed to have bounded derivatives with respect to S and I and G_5 is assumed to be boundedly-differentiable with respect to E only.

The solution $E(x,t)$ is approximated by B_m^n defined on Ω, agrees with $E(x,t)$ on $\partial\Omega_i\,(i=0,1,2)$ and satisfies the difference equation

$$\begin{aligned}\frac{\alpha}{2}\nabla^2\left(B_m^{n+1}+B_m^n\right) &= \nabla_t B_m^n+\frac{\rho}{2}\nabla_x\left(B_m^{n+1}+B_m^n\right)-\frac{1}{2}\beta\,C_m^n\,A_m^{n+1}-\frac{1}{2}\beta\,C_m^{n+1}\,A_m^n \\ &+ (\mu+\sigma)\,\widehat{B_m^n}\,,\quad n\ge 0\,, \qquad (7.4.16)\end{aligned}$$

where ∇^2, ∇_x and ∇_t are defined in (7.4.4) and $\widehat{B_m^n}$ is defined as in (6.3.16).

It is easy to see that

$$\left.\begin{aligned}\tfrac{1}{2}\nabla^2\left(E_m^{n+1}+E_m^n\right) &= \tfrac{1}{2}\frac{\partial^2 E_m^{n+1}}{\partial x^2}+\tfrac{1}{2}\frac{\partial^2 E_m^n}{\partial x^2}+\tfrac{1}{24}h^2\left(\overline{\frac{\partial^4 E_m^{n+1}}{\partial x^4}}+\overline{\frac{\partial^4 E_m^n}{\partial x^4}}\right), \\ \tfrac{1}{2}\nabla_x\left(E_m^{n+1}+E_m^n\right) &= \tfrac{1}{2}\frac{\partial E_m^{n+1}}{\partial x}+\tfrac{1}{2}\frac{\partial E_m^n}{\partial x}+\tfrac{1}{12}h^2\left(\overline{\frac{\partial^3 E_m^{n+1}}{\partial x^3}}+\overline{\frac{\partial^3 E_m^n}{\partial x^3}}\right), \\ \nabla_t E_m^n &= \frac{\partial E_m^{n+1}}{\partial t}-\tfrac{1}{2}\ell\,\overline{\frac{\partial^2 E_m^{n+1}}{\partial t^2}}\,,\end{aligned}\right\} \qquad (7.4.17)$$

where the barred derivatives are evaluated at intermediate argument values as called for by the Mean Value Theorem. Moreover, G_3, G_4 and G_5 may be written in the form of (4.6.16).

Substituting the expressions for G_3, G_4, G_5 from (4.6.16) and the expressions in (7.4.17) into (7.4.14) gives

$$\begin{aligned}
\frac{\alpha}{2}\nabla^2\left(E_m^{n+1}+E_m^n\right) &= \nabla_t E_m^n + \frac{\rho}{2}\nabla_x\left(E_m^{n+1}+E_m^n\right) - \frac{1}{2}\beta\, I_m^n\, S_m^{n+1} - \frac{1}{2}\beta\, I_m^{n+1}\, S_m^n \\
&+ (\mu+\sigma)\,\widehat{E_m^n} + \left\{\frac{1}{24}h^2\left(\overline{\frac{\partial^4 E_m^{n+1}}{\partial x^4}} + \overline{\frac{\partial^4 E_m^n}{\partial x^4}}\right) + \frac{1}{2}\ell\,\overline{\frac{\partial^2 E_m^{n+1}}{\partial t^2}}\right. \\
&- \frac{1}{12}h^2\left(\overline{\frac{\partial^3 E_m^{n+1}}{\partial x^3}} + \overline{\frac{\partial^3 E_m^n}{\partial x^3}}\right) + \left(S_m^{n+1}-S_m^n\right)\overline{\frac{\partial G_3}{\partial S}} \\
&+ \left.\left(I_m^{n+1}-I_m^n\right)\overline{\frac{\partial G_4}{\partial I}} - \left(E_m^{n+1}-E_m^n\right)\overline{\frac{\partial G_5}{\partial E}}\right\}. \qquad (7.4.18)
\end{aligned}$$

The assumption on E above requires the boundedness of all the derivatives appearing inside the bracket along with $S_m^{n+1}-S_m^n$, $E_m^{n+1}-E_m^n$ and $I_m^{n+1}-I_m^n$ in the region $\overline{\Omega}$. Hence, in this region,

$$\begin{aligned}
\frac{\alpha}{2}\nabla^2\left(E_m^{n+1}+E_m^n\right) &= \nabla_t E_m^n + \frac{\rho}{2}\nabla_x\left(E_m^{n+1}+E_m^n\right) - \frac{1}{2}\beta\, I_m^n\, S_m^{n+1} - \frac{1}{2}\beta\, I_m^{n+1}\, S_m^n \\
&+ (\mu+\sigma)\,\widehat{E_m^n} + g_m^n, \qquad (7.4.19)
\end{aligned}$$

with

$$g_m^n = O(h^2+\ell). \qquad (7.4.20)$$

Now, if (7.4.16) is subtracted from (7.4.19) and using the definitions of the truncation errors in (4.6.8),

$$\begin{aligned}
\frac{\alpha}{2}\nabla^2\left(Z_{2m}^{n+1}+Z_{2m}^n\right) &= \nabla_t Z_{2m}^n + \frac{\rho}{2}\nabla_x\left(Z_{2m}^{n+1}+Z_{2m}^n\right) - \frac{1}{2}\beta\, I_m^n\, S_m^{n+1} + \frac{1}{2}\beta\, C_m^n\, A_m^{n+1} \\
&- \frac{1}{2}\beta\, I_m^{n+1}\, S_m^n + \frac{1}{2}\beta\, C_m^{n+1}\, A_m^n + (\mu+\sigma)\,\widehat{Z_{2m}^n} + g_m^n. \qquad (7.4.21)
\end{aligned}$$

As

$$\begin{aligned}
G_3 = G_3(x_m, t_{n+1}, S_m^{n+1}, I_m^n) &= G_3(x_m, t_{n+1}, A_m^{n+1}, C_m^n) + \overline{\frac{\partial G_3}{\partial S}}\left(S_m^{n+1}-A_m^{n+1}\right) \\
&+ \overline{\frac{\partial G_3}{\partial I}}\left(I_m^n - C_m^n\right) \qquad (7.4.22)
\end{aligned}$$

and

$$G_4 = G_4(x_m, t_{n+1}, S_m^n, I_m^{n+1}) = G_4(x_m, t_{n+1}, A_m^n, C_m^{n+1}) + \overline{\frac{\partial G_4}{\partial S}}\,(S_m^n - A_m^n) + \overline{\frac{\partial G_4}{\partial I}}\left(I_m^{n+1} - C_m^{n+1}\right), \qquad (7.4.23)$$

it follows that equation (7.4.21) may be written as

$$\frac{\alpha}{2}\nabla^2\left(Z_{2_m}^{n+1} + Z_{2_m}^n\right) = \nabla_t Z_{2_m}^n + \frac{\rho}{2}\nabla_x\left(Z_{2_m}^{n+1} + Z_{2_m}^n\right) - \frac{1}{2}\overline{\frac{\partial G_3}{\partial S}}\,Z_{1_m}^{n+1} - \frac{1}{2}\overline{\frac{\partial G_3}{\partial I}}\,Z_{3_m}^n - \frac{1}{2}\overline{\frac{\partial G_4}{\partial S}}\,Z_{1_m}^n - \frac{1}{2}\overline{\frac{\partial G_4}{\partial I}}\,Z_{3_m}^{n+1} + (\mu+\sigma)\,\widehat{Z_{2_m}^n} + g_m^n \qquad (7.4.24)$$

with $Z_{2_m}^n$ and $Z_{2_m}^{n+1}$ vanishing on $\partial\Omega_i$.

Let $M_S = \max\left\{\overline{\frac{\partial G_3}{\partial S}}, \overline{\frac{\partial G_4}{\partial S}}\right\}$ and $M_I = \max\left\{\overline{\frac{\partial G_3}{\partial I}}, \overline{\frac{\partial G_4}{\partial I}}\right\}$. Then equation (7.4.24) may be written in the form

$$\frac{\alpha}{2}\nabla^2\left(Z_{2_m}^{n+1} + Z_{2_m}^n\right) = \nabla_t Z_{2_m}^n + \frac{\rho}{2}\nabla_x\left(Z_{2_m}^{n+1} + Z_{2_m}^n\right) - M_S\,\widehat{Z_{1_m}^n} - M_I\,\widehat{Z_{3_m}^n} + (\mu+\sigma)\,\widehat{Z_{2_m}^n} + g_m^n\,. \qquad (7.4.25)$$

Assume that $Z_{1_m}^n$, $Z_{1_m}^{n+1}$, $Z_{3_m}^n$ and $Z_{3_m}^{n+1}$ are bounded. Since $Z_{2_m}^n$, $Z_{2_m}^{n+1}$ vanish on $\partial\Omega_i$, it follows that, by Theorem1.4.1, B_m^n and B_m^{n+1} converge uniformly to E_m^n and E_m^{n+1}.

Let the third equation in (7.1.1) be written in the form

$$\begin{aligned}\frac{\alpha}{2}(I_{xx} + I_{xx}) &= I_t + \frac{\rho}{2}(I_x + I_x) - \frac{1}{2}\sigma E - \frac{1}{2}\sigma E + \frac{1}{2}(\mu+\gamma)I + \frac{1}{2}(\mu+\gamma)I \\ &= I_t + \frac{\rho}{2}(I_x + I_x) - G_6 + \frac{1}{2}(\mu+\gamma)I + G_7 \qquad (7.4.26)\end{aligned}$$

with initial and boundary conditions

$$I(x,0) = I^0(x)\,; \qquad 0 \le x \le L$$

$$(7.4.27)$$

$$I_x(0,t) = I_x(L,t) = 0\,; \qquad t > 0$$

where, as before, G_6 and G_7 are assumed to be boundedly-differentiable with respect to E and I, respectively.

Consider the difference equation which approximates equation (7.4.26)

$$\begin{aligned}\frac{\alpha}{2}\nabla^2\left(C_m^{n+1}+C_m^n\right) &= \nabla_t C_m^n+\frac{\rho}{2}\nabla_x\left(C_m^{n+1}+C_m^n\right)-\frac{1}{2}\sigma\,B_m^n-\frac{1}{2}\sigma\,B_m^{n+1}\\ &+\ (\mu+\gamma)\,\widehat{C_m^n}\,,\quad n\geq 0\,,\end{aligned} \qquad (7.4.28)$$

where ∇^2, ∇_x and ∇_t are defined in (7.4.4) and $\widehat{C_m^n}$ is as in (6.3.16). The term C_m^n is an approximate solution for I at the point (x,t) in Ω which agrees with I on the boundaries. The terms G_6 and G_7 may be written as in (4.6.25).

It is easy to show that

$$\left.\begin{aligned}\tfrac{1}{2}\nabla^2\left(I_m^{n+1}+I_m^n\right) &= \tfrac{1}{2}\frac{\partial^2 I_m^{n+1}}{\partial x^2}+\tfrac{1}{2}\frac{\partial^2 I_m^n}{\partial x^2}+\tfrac{1}{24}h^2\left(\overline{\frac{\partial^4 I_m^{n+1}}{\partial x^4}}+\overline{\frac{\partial^4 I_m^n}{\partial x^4}}\right),\\ \tfrac{1}{2}\nabla_x\left(I_m^{n+1}+I_m^n\right) &= \tfrac{1}{2}\frac{\partial I_m^{n+1}}{\partial x}+\tfrac{1}{2}\frac{\partial I_m^n}{\partial x}+\tfrac{1}{12}h^2\left(\overline{\frac{\partial^3 I_m^{n+1}}{\partial x^3}}+\overline{\frac{\partial^3 I_m^n}{\partial x^3}}\right),\\ \nabla_t I_m^n &= \frac{\partial I_m^{n+1}}{\partial t}-\tfrac{1}{2}\ell\,\overline{\frac{\partial^2 I_m^{n+1}}{\partial t^2}}\,,\end{aligned}\right\} \qquad (7.4.29)$$

where the barred derivatives are evaluated at intermediate argument values as called for by the Mean Value Theorem.

Substituting the expressions for G_6 and G_7 from (4.6.25) and the expressions in (7.4.29) into (7.4.26) yields

$$\begin{aligned}\frac{\alpha}{2}\nabla^2\left(I_m^{n+1}+I_m^n\right) &= \nabla_t I_m^n+\frac{\rho}{2}\nabla_x\left(I_m^{n+1}+I_m^n\right)-\frac{1}{2}\sigma\,E_m^n-\frac{1}{2}\sigma\,E_m^{n+1}+(\mu+\gamma)\,\widehat{I_m^n}\\ &+\ \left\{\frac{1}{24}h^2\left(\overline{\frac{\partial^4 I_m^{n+1}}{\partial x^4}}+\overline{\frac{\partial^4 I_m^n}{\partial x^4}}\right)-\frac{1}{12}h^2\left(\overline{\frac{\partial^3 I_m^{n+1}}{\partial x^3}}+\overline{\frac{\partial^3 I_m^n}{\partial x^3}}\right)\right.\\ &+\ \frac{1}{2}\ell\,\overline{\frac{\partial^2 I_m^{n+1}}{\partial t^2}}+\left(E_m^{n+1}-E_m^n\right)\overline{\frac{\partial G_6}{\partial E}}\\ &-\ \left.\left(I_m^{n+1}-I_m^n\right)\overline{\frac{\partial G_7}{\partial I}}\right\}.\end{aligned} \qquad (7.4.30)$$

The assumption on I above requires the boundedness of all the derivatives appearing inside the bracket along with $E_m^{n+1}-E_m^n$ and $I_m^{n+1}-I_m^n$ in the region $\overline{\Omega}$. Hence, in this region,

$$\frac{\alpha}{2}\nabla^2\left(I_m^{n+1}+I_m^n\right) = \nabla_t I_m^n + \frac{\rho}{2}\nabla_x\left(I_m^{n+1}+I_m^n\right) - \frac{1}{2}\sigma\,E_m^n - \frac{1}{2}\sigma\,E_m^{n+1} + (\mu+\gamma)\,\widehat{I_m^n} + g_m^n\,, \qquad (7.4.31)$$

with

$$g_m^n = O(h^2+\ell)\,. \qquad (7.4.32)$$

If equation (7.4.28) is subtracted from (7.4.31) and using the definitions of the truncation errors in (4.6.8),

$$\frac{\alpha}{2}\nabla^2\left(Z_{3_m}^{n+1}+Z_{3_m}^n\right) = \nabla_t Z_{3_m}^n + \frac{\rho}{2}\nabla_x\left(Z_{3_m}^{n+1}+Z_{3_m}^n\right) - \frac{1}{2}\sigma\,Z_{2_m}^n - \frac{1}{2}\sigma\,Z_{2_m}^{n+1} + (\mu+\gamma)\,\widehat{Z_{3_m}^n} + g_m^n \qquad (7.4.33)$$

which may be rearranged to yield

$$\frac{\alpha}{2}\nabla^2\left(Z_{3_m}^{n+1}+Z_{3_m}^n\right) = \nabla_t Z_{3_m}^n + \frac{\rho}{2}\nabla_x\left(Z_{3_m}^{n+1}+Z_{3_m}^n\right) - \sigma\,\widehat{Z_{2_m}^n} + (\mu+\gamma)\,\widehat{Z_{3_m}^n} + g_m^n \qquad (7.4.34)$$

where $\widehat{Z_{2_m}^n}$ and $\widehat{Z_{3_m}^n}$ are defined as in (6.3.16).

Assume $Z_{2_m}^n$ and $Z_{2_m}^{n+1}$ in equation (7.4.34) are bounded. Moreover, $Z_{3_m}^n$ and $Z_{3_m}^{n+1}$ vanish on $\partial\Omega_i$. Therefore, by Theorem 1.4.1, C_m^n and C_m^{n+1} converge to I_m^n and I_m^{n+1} uniformly.

7.5 Numerical Results

Following Chapter 4, experiment A was carried out to solve the initial/boundary-value problem (7.1.1)–(7.1.3) using the second-order methods (7.2.3), (7.2.11) and (7.2.19) for susceptibles, exposed and infectives, respectively, with the same set of parameters given in (3.4.4) for N, μ, σ and γ and with the infection rate, β, chosen to be $\beta = 5\times10^{-4}$. Four sets of results were obtained for different values of α and ρ.

In the first set with $\alpha = \rho = 0.01$, the methods were seen to give reasonable results when $\ell < 0.04$ and negative values otherwise. When $\ell = 0.001$, similar results to those found in Chapter 4 were obtained (see Figures 4.4–4.7). When ℓ was increased, the number of iterations was kept as before (100 iterations), the dynamics of the model became clearer; as the number of exposed and infectives was increased the number of susceptibles was decreased. These findings are shown in Figures 7.1, 7.3, and 7.5 while Figures 7.2, 7.4 and 7.6 are different angle views of susceptibles, exposed and infectives, respectively.

In the second set of results with $\alpha = \rho = 2.0$ and $\ell = 0.01$ the profiles are as shown in Figures 7.7–7.11. In this case, it is seen from profiles 7.7, 7.8 and 7.10 that as the numbers of exposed and infectious individuals increase the number of susceptible individuals decreases. This is seen clearly at $t \approx 0.2$ (approximately) after which the number of susceptible individuals increases because of birth and the numbers of exposed and infectious individuals decrease because of death. This behaviour continues until the dynamics of the model reach a steady state as t gets large.

When the convection rate was increased to $\rho = 2.0$ and the diffusion rate kept at $\alpha = 0.01$, the behaviour of the measles dynamics changed as may be seen in Figures 7.12–7.17. In this case the symmetry about the line $x = 0.5$ was completely changed as t increased.

In the fourth set with $\alpha = 2.0$ and $\rho = 0.01$, the methods were seen to give negative results when $\ell \geq 0.05$. With $\ell = 0.04$ it is seen from Figures 7.18–7.23 that all susceptible, exposed and infectious individuals converge in a damped oscillatory manner to the steady state.

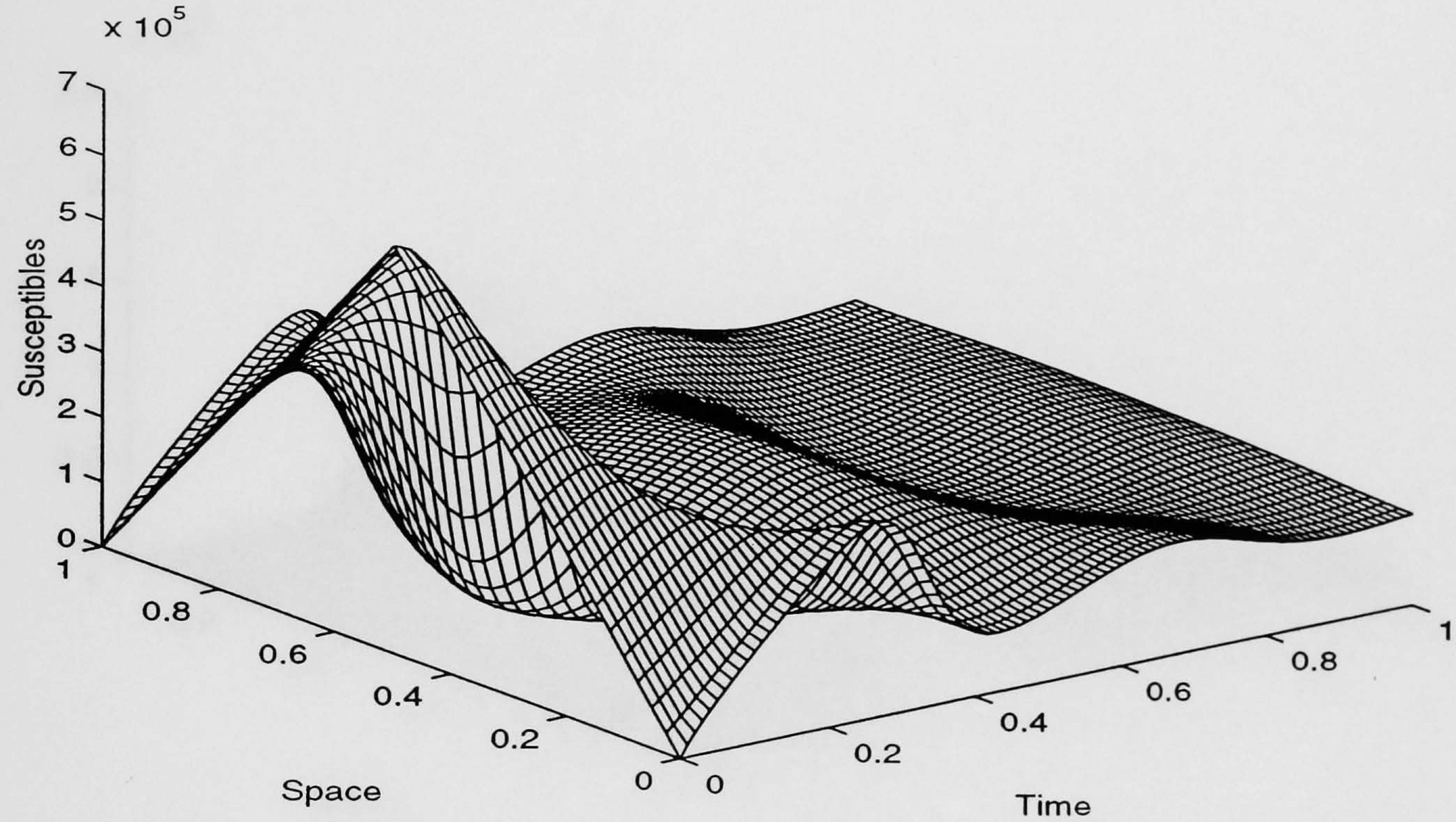

Figure 7.1: *Three-dimensional profile of susceptibles;* $\alpha = \rho = 0.01$, $\ell = 0.01$ *and* $h = 0.025$.

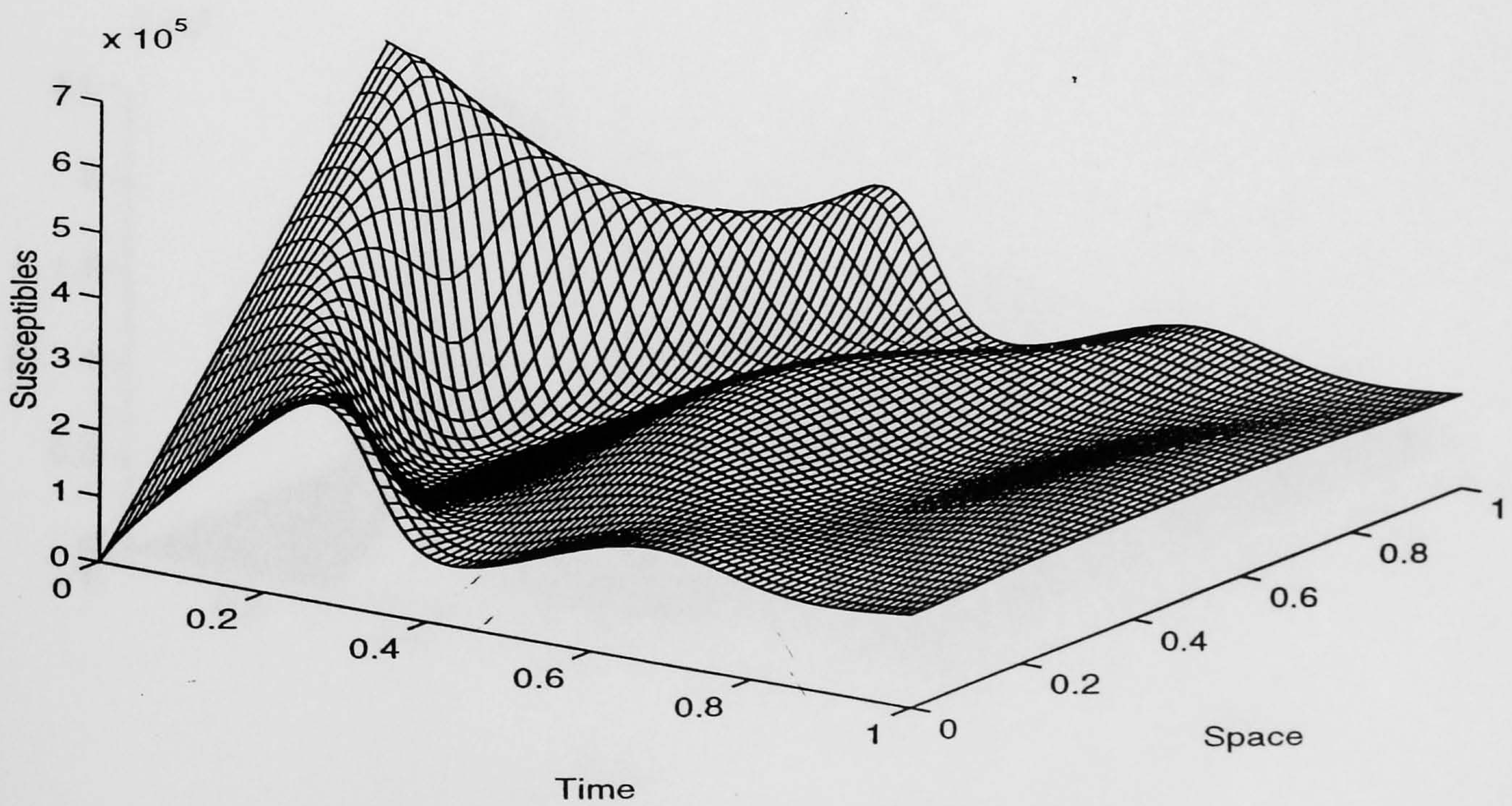

Figure 7.2: *Different angle view of Figure 7.1; view([35,30]).*

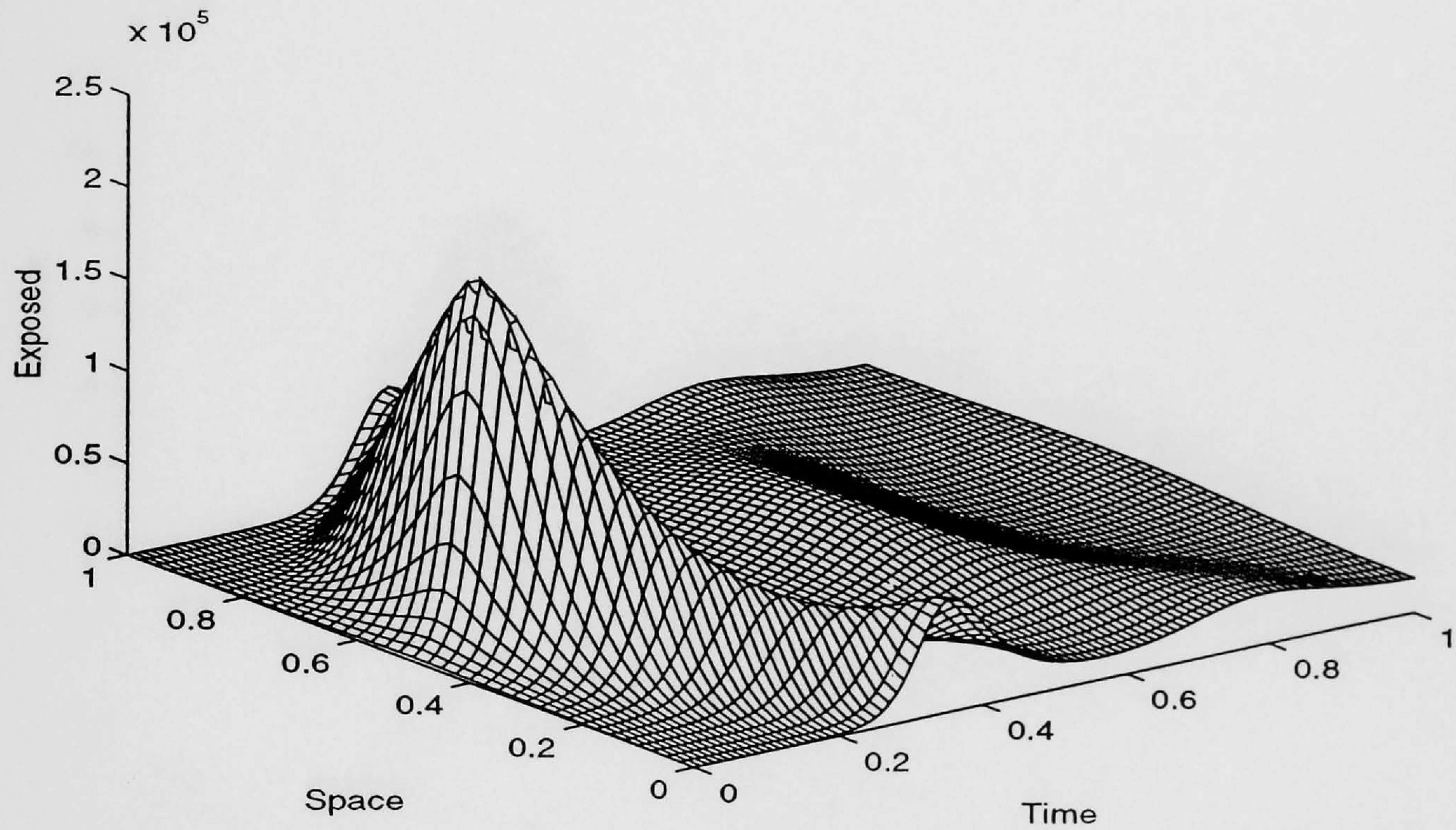

Figure 7.3: *Three-dimensional profile of exposed;* $\alpha = \rho = 0.01$, $\ell = 0.01$ *and* $h = 0.025$.

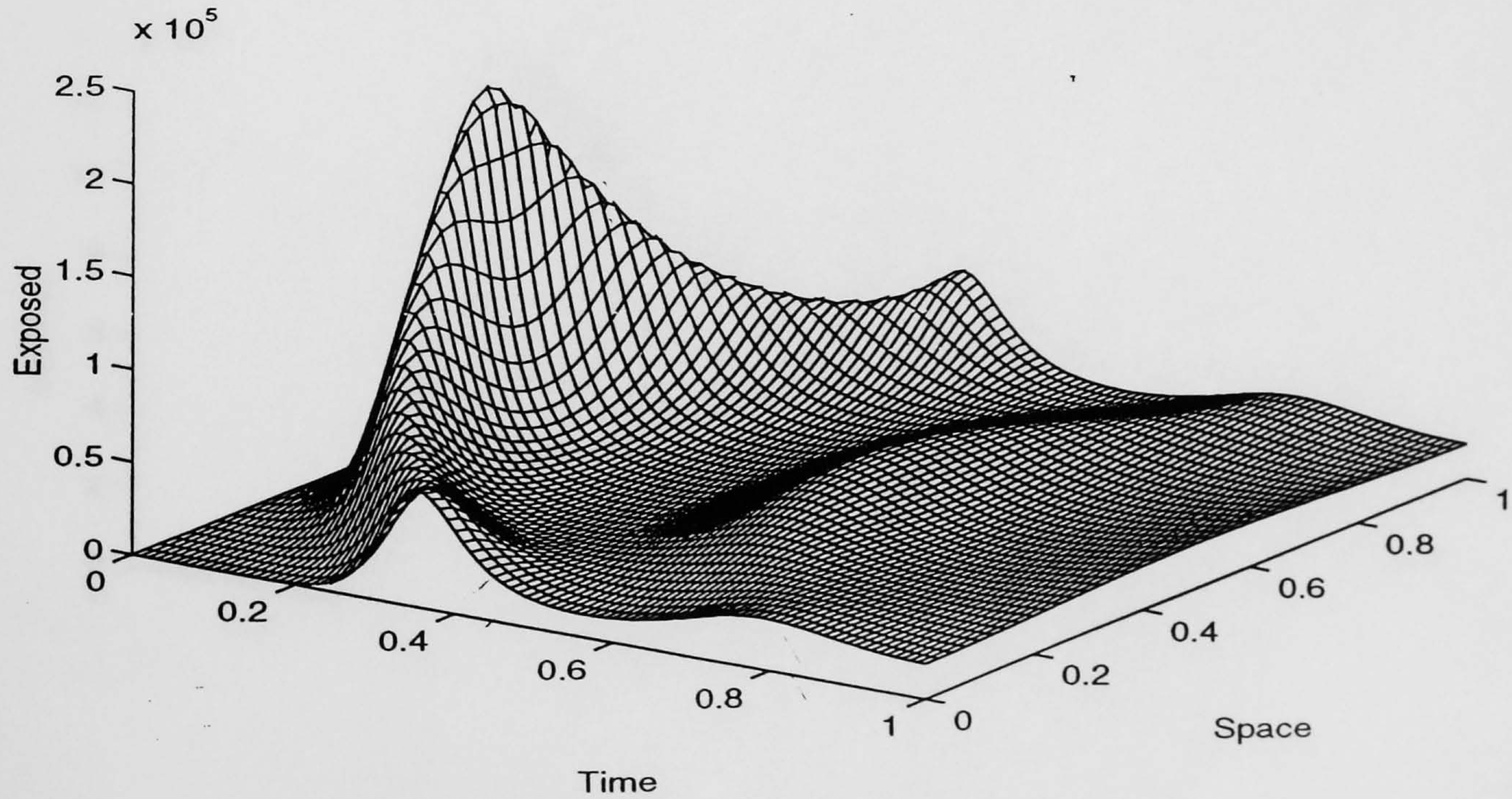

Figure 7.4: *Different angle view of Figure 7.3; view([35,30]).*

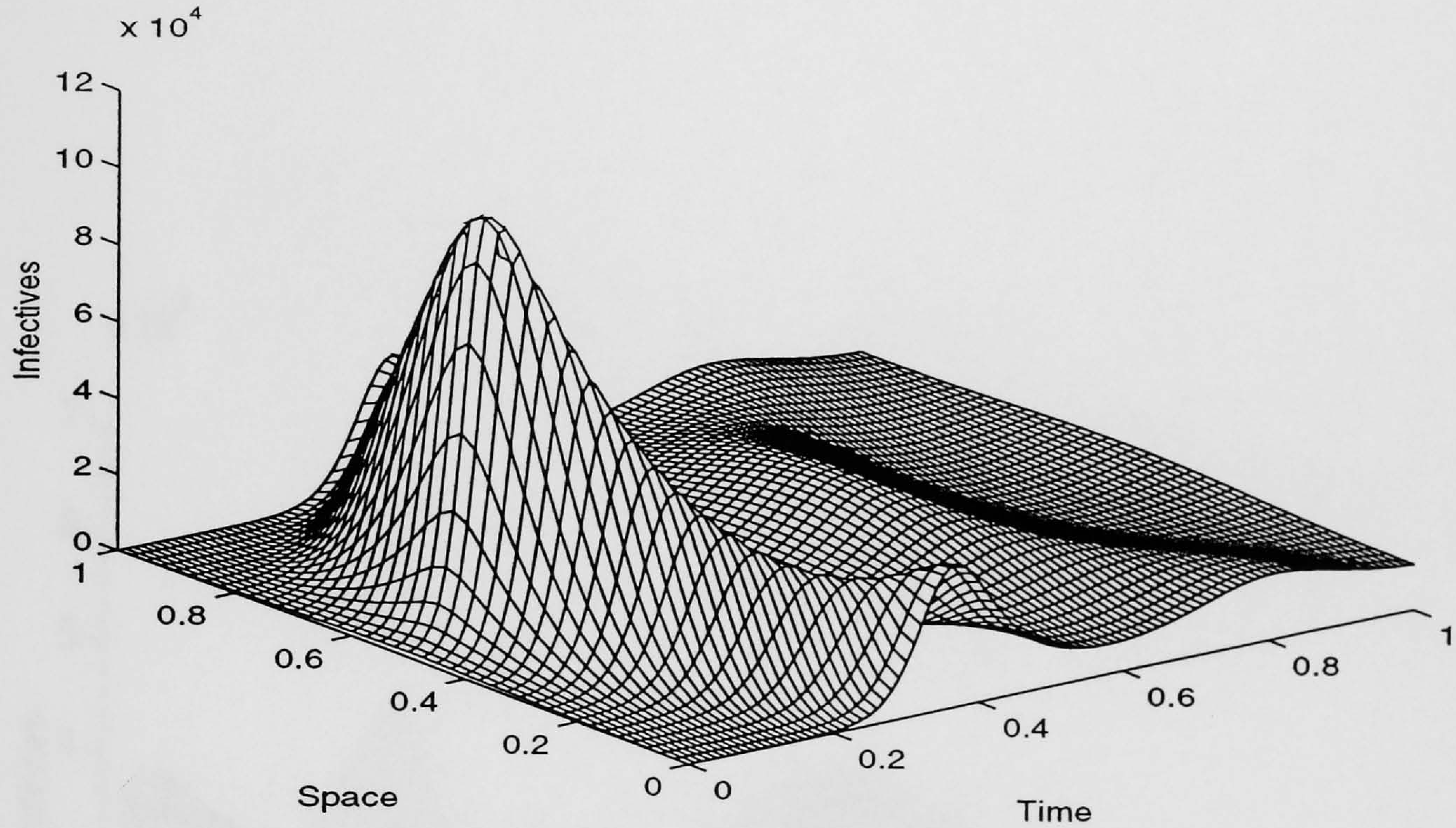

Figure 7.5: *Three-dimensional profile of infectives;* $\alpha = \rho = 0.01$, $\ell = 0.01$ *and* $h =$ 0.025.

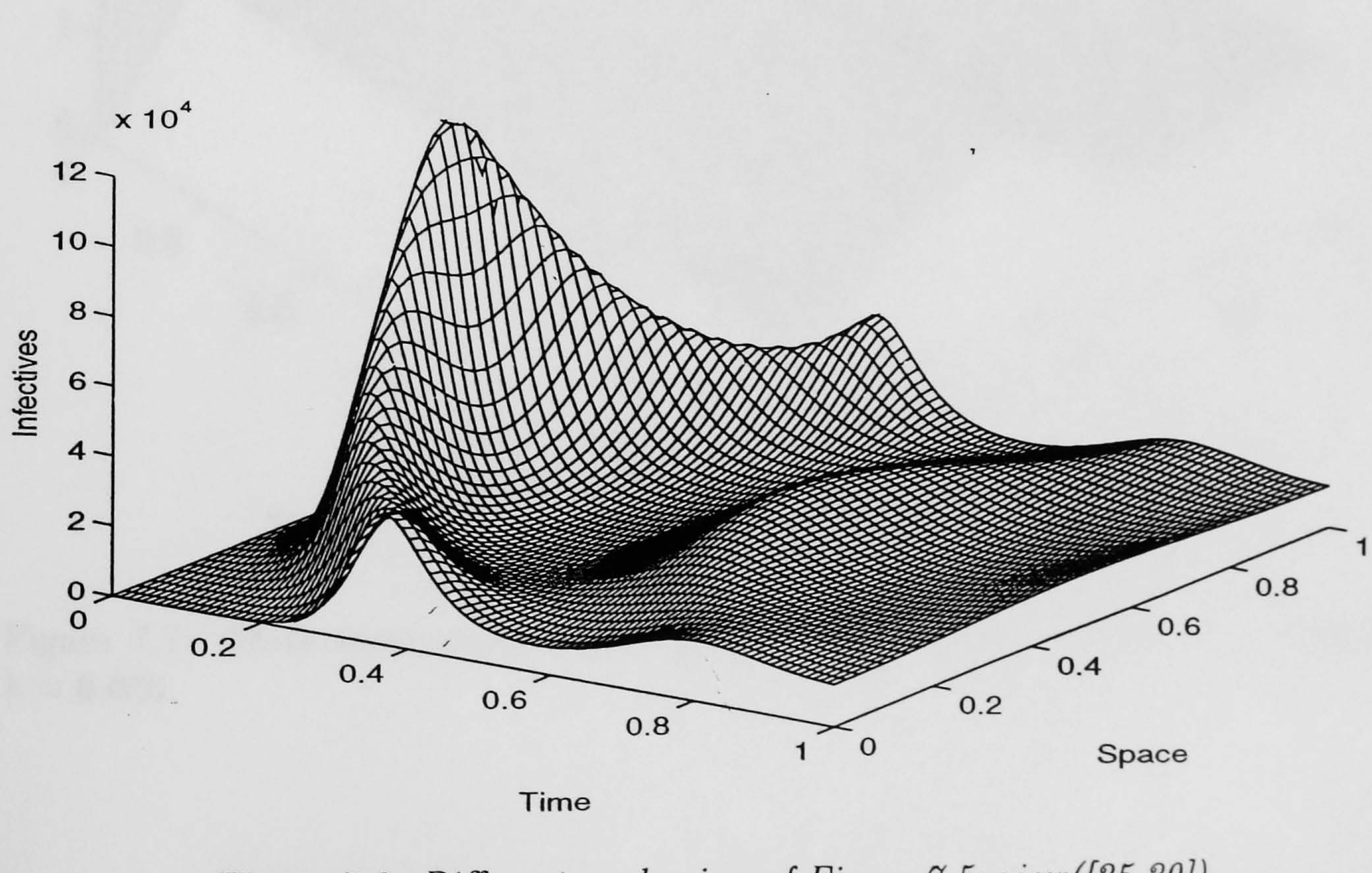

Figure 7.6: *Different angle view of Figure 7.5; view([35,30]).*

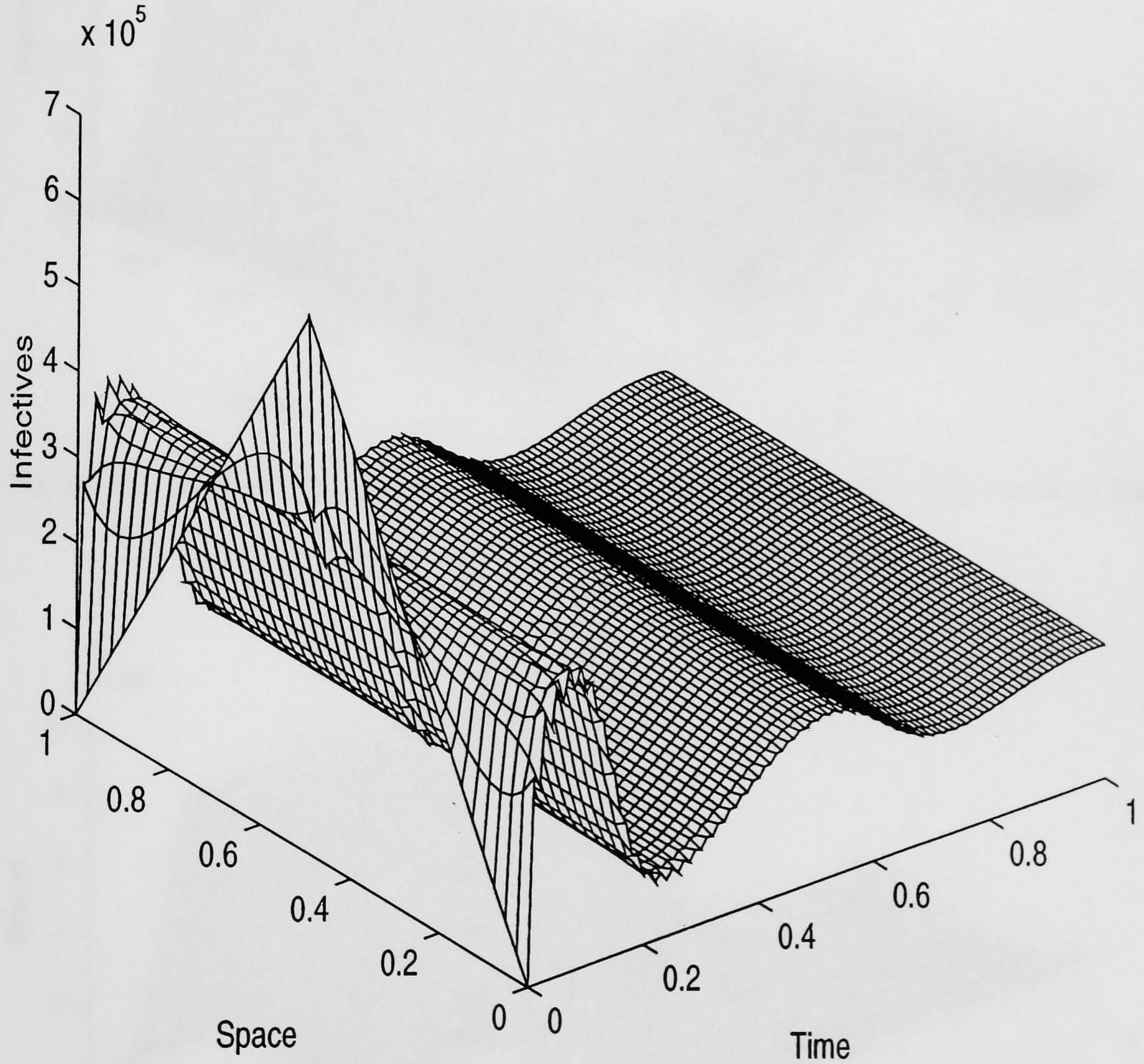

Figure 7.7: *Three-dimensional profile of susceptibles;* $\alpha = \rho = 2.0$, $\ell = 0.01$ *and* $h = 0.025$.

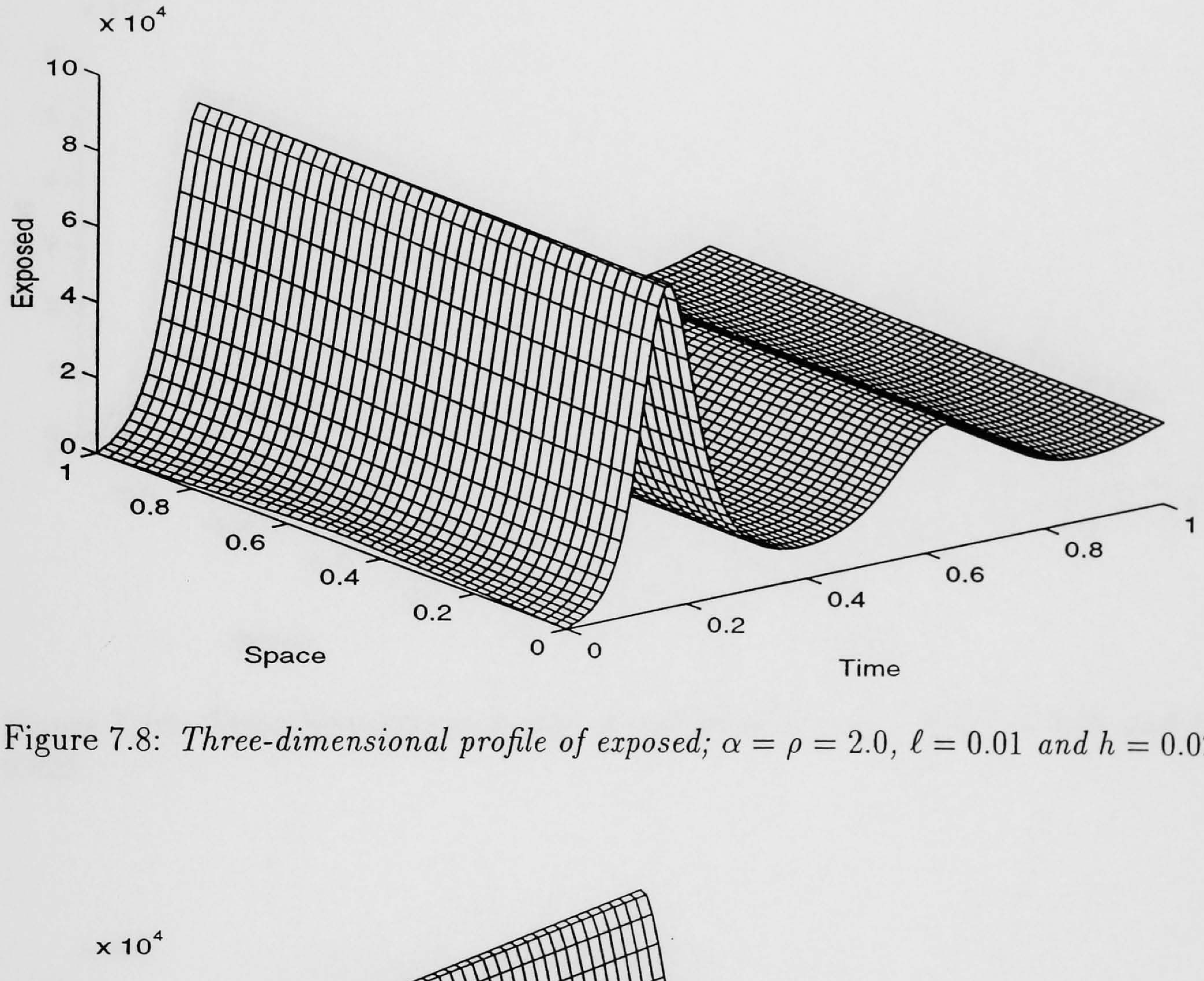

Figure 7.8: *Three-dimensional profile of exposed;* $\alpha = \rho = 2.0$, $\ell = 0.01$ *and* $h = 0.025$.

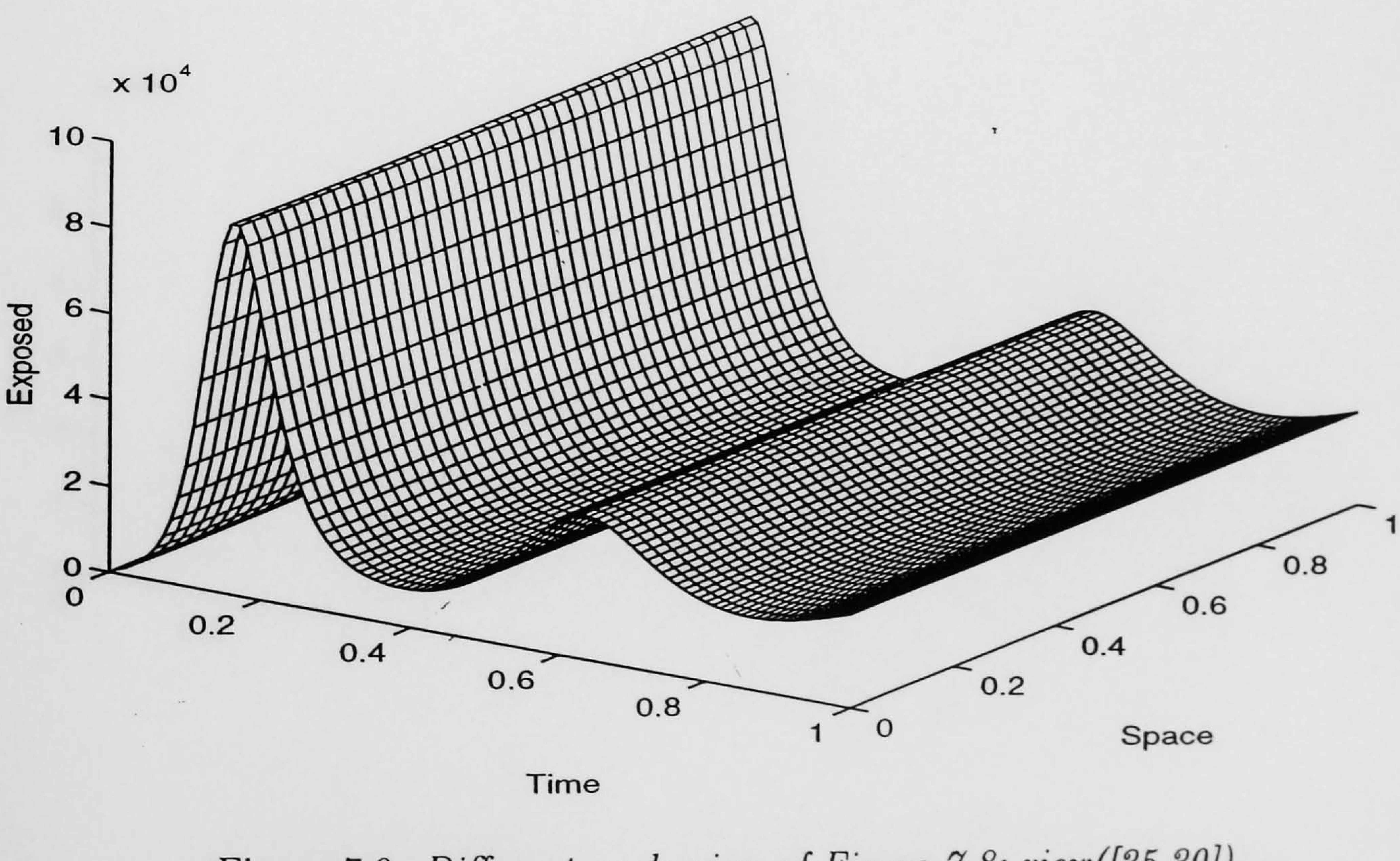

Figure 7.9: *Different angle view of Figure 7.8; view([35,30]).*

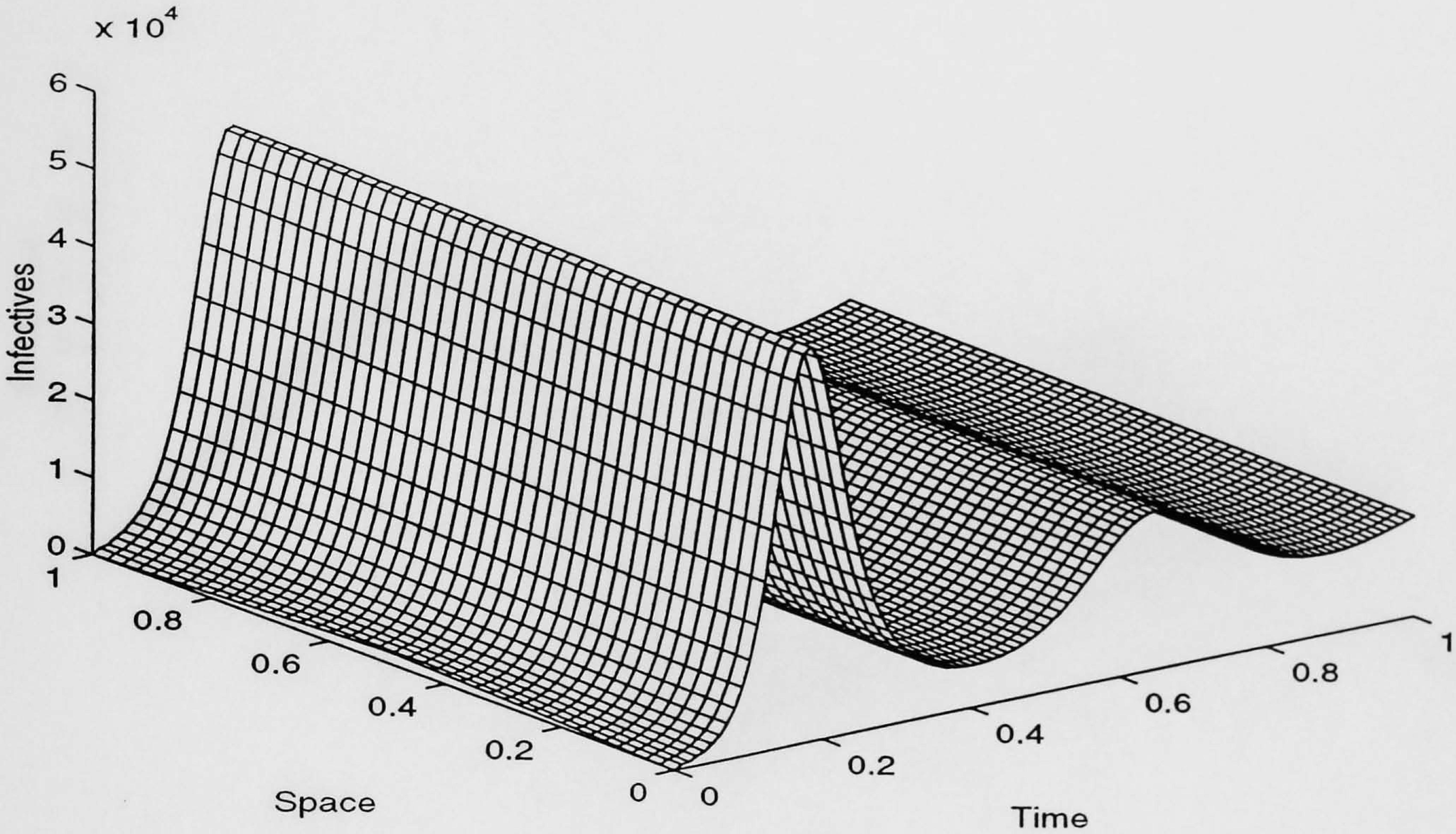

Figure 7.10: *Three-dimensional profile of infectives;* $\alpha = \rho = 2.0$, $\ell = 0.01$ *and* $h = 0.025$.

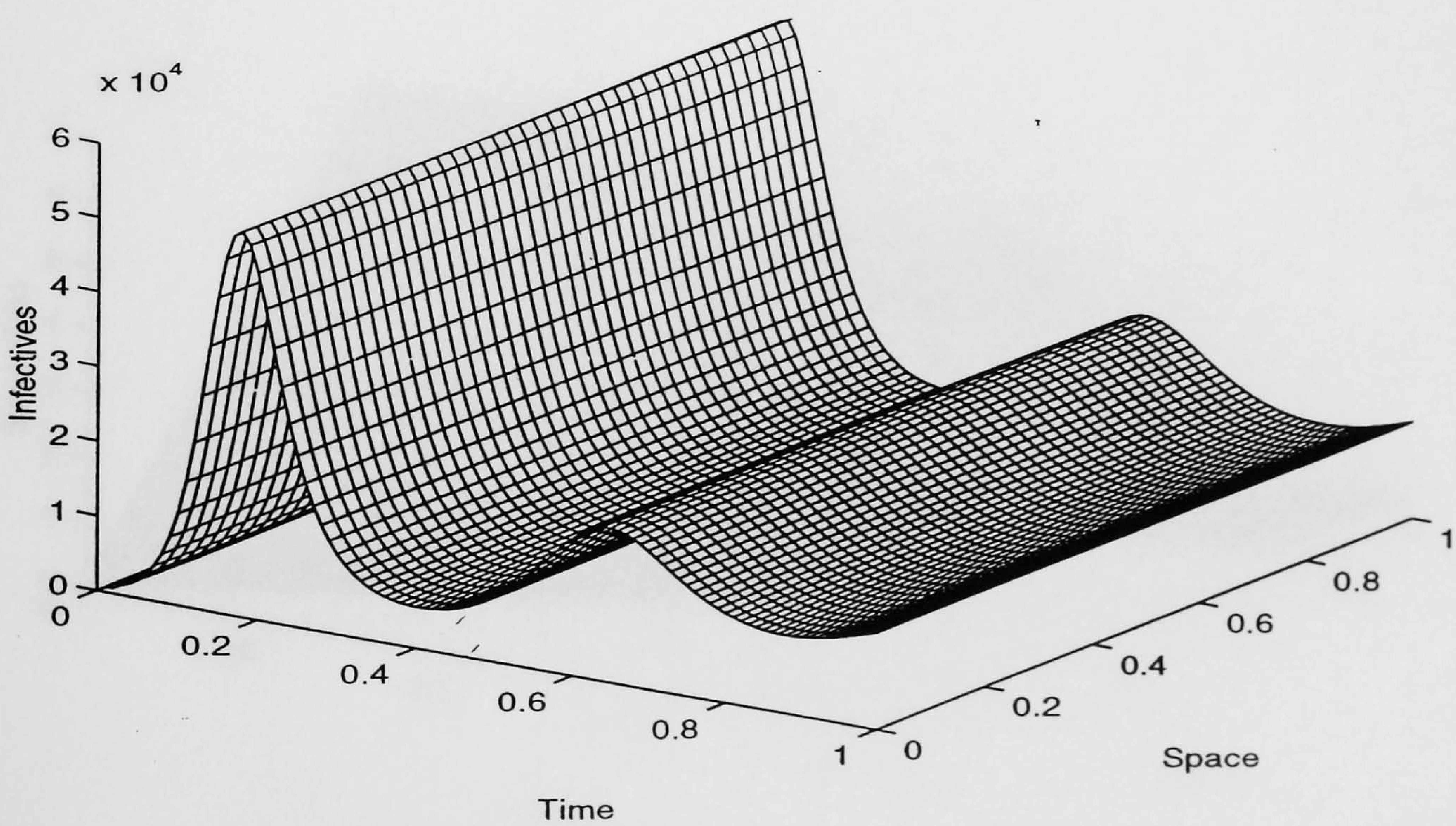

Figure 7.11: *Different angle view of Figure 7.10; view([35,30]).*

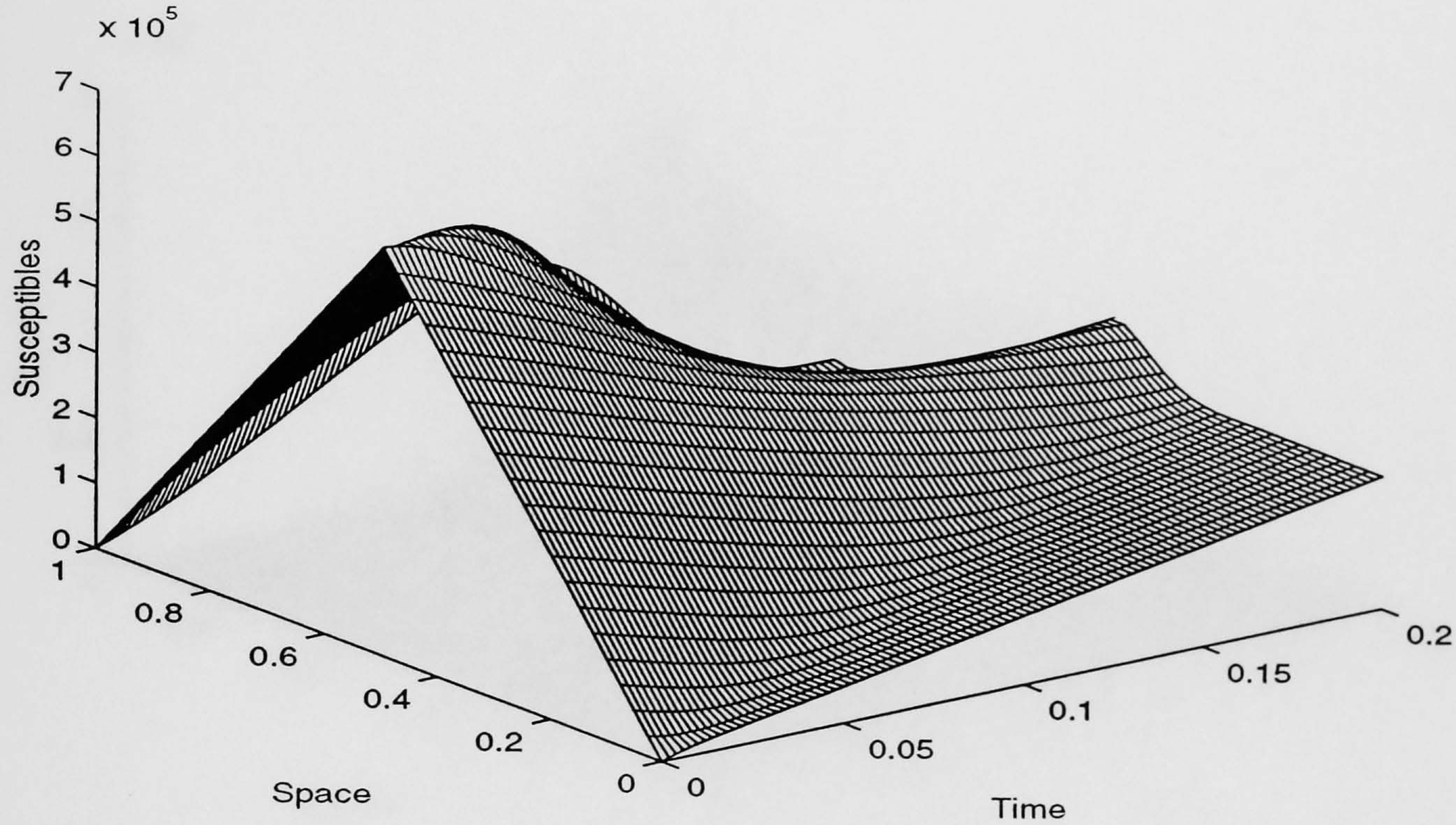

Figure 7.12: *Three-dimensional profile of susceptibles;* $\alpha = 0.01$, $\rho = 2.0$, $\ell = 0.002$ *and* $h = 0.025$.

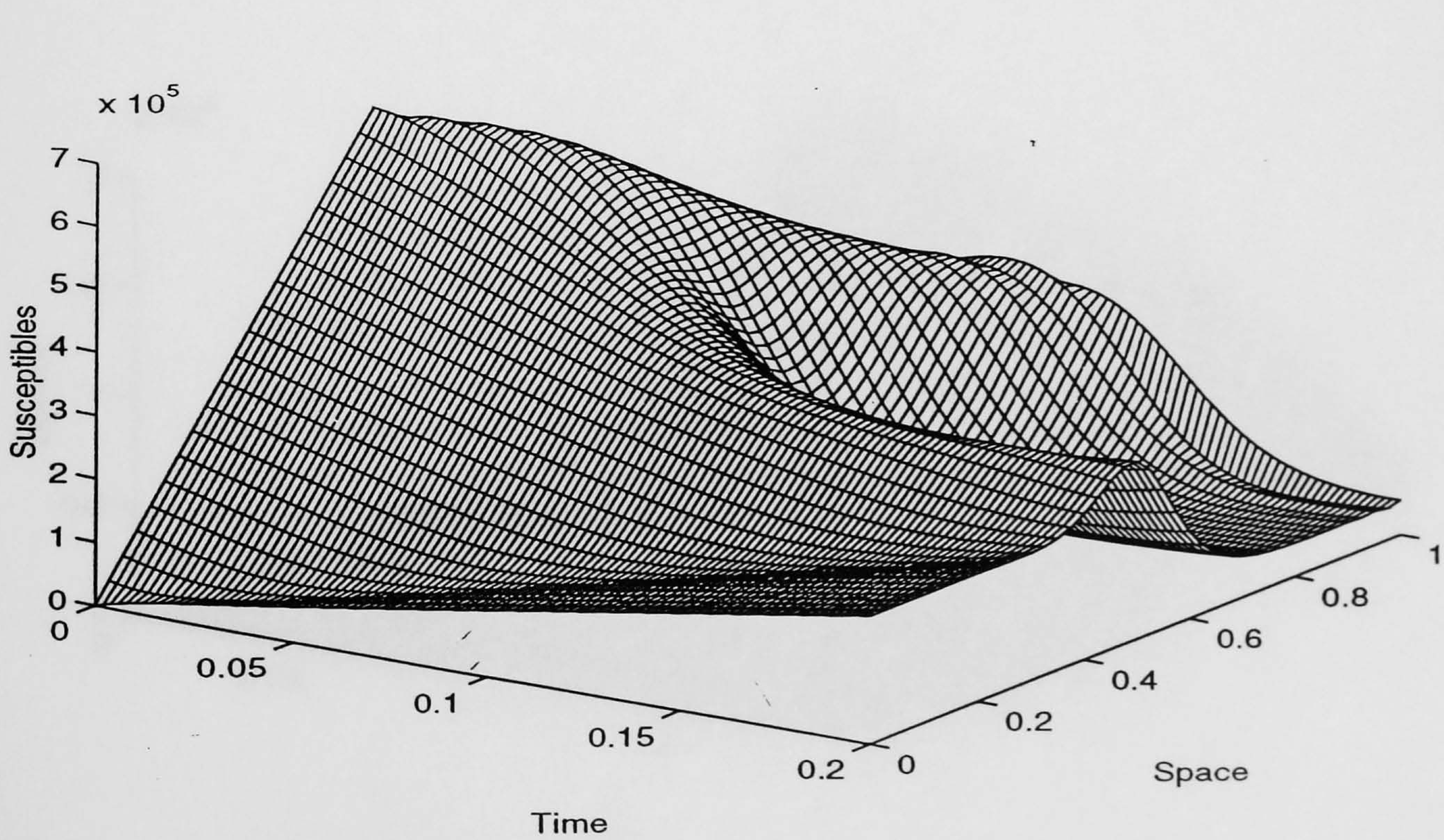

Figure 7.13: *Different angle view of Figure 7.12; view([35,30]).*

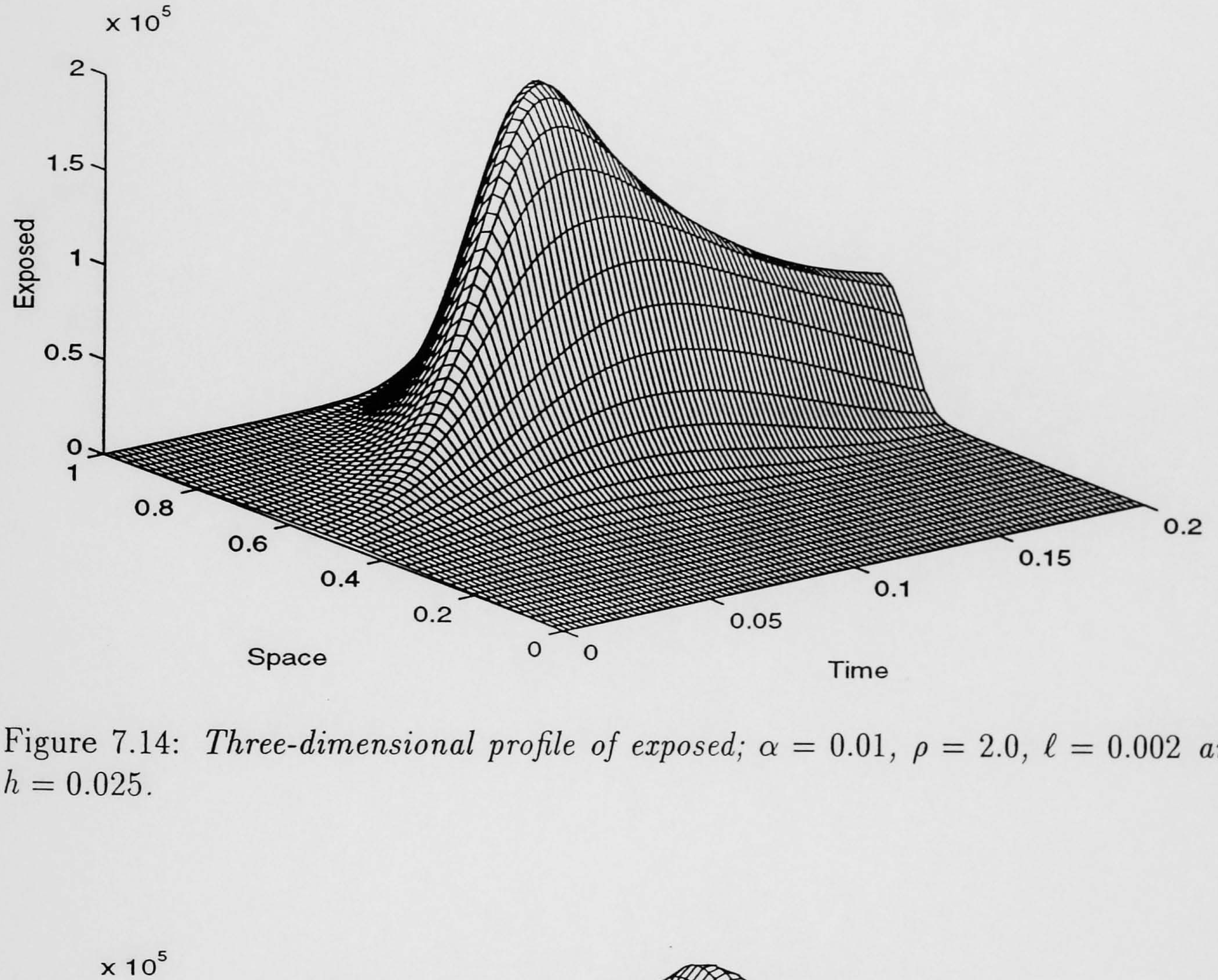

Figure 7.14: *Three-dimensional profile of exposed;* $\alpha = 0.01$, $\rho = 2.0$, $\ell = 0.002$ *and* $h = 0.025$.

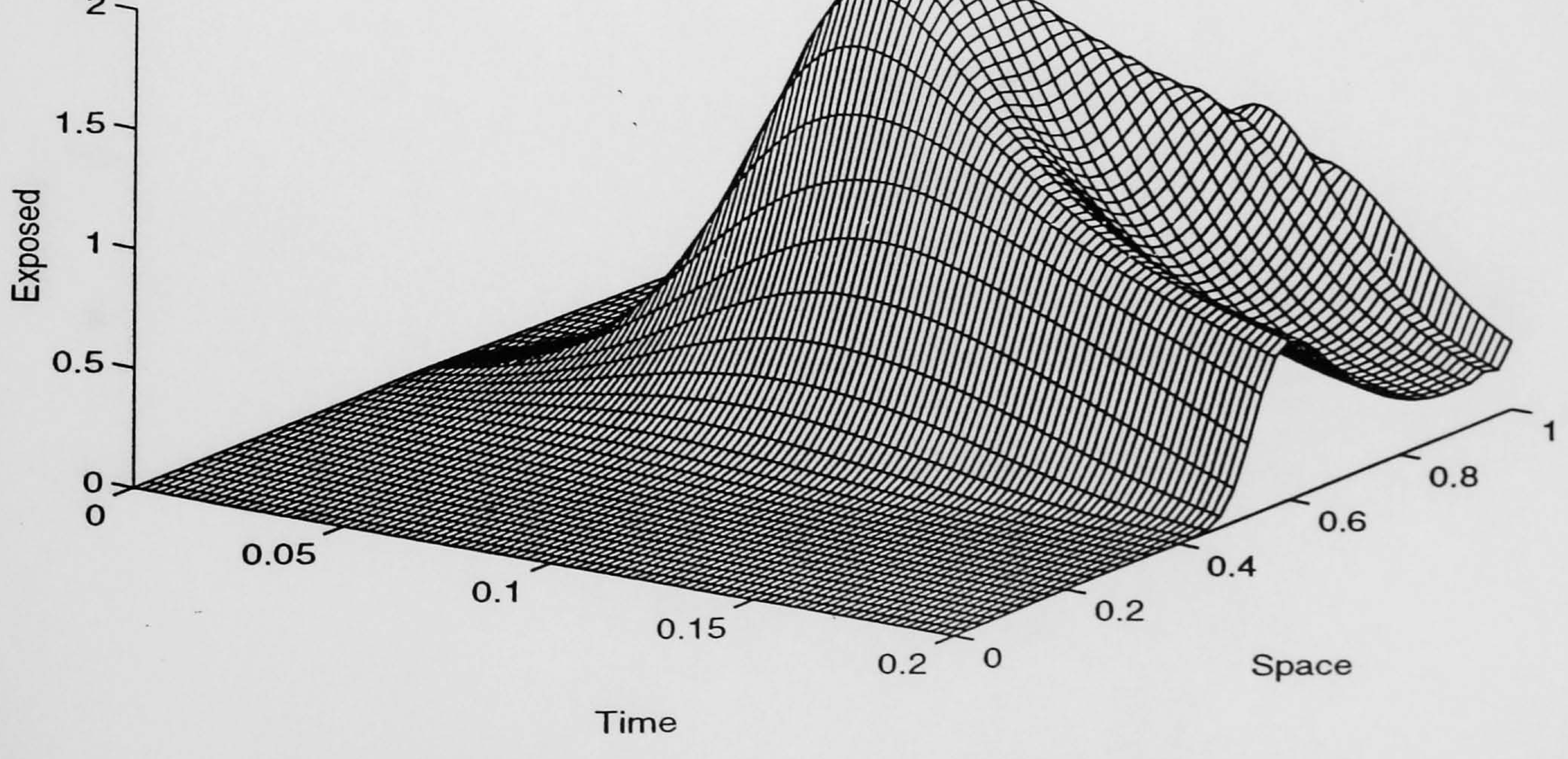

Figure 7.15: *Different angle view of Figure 7.14; view([35,30]).*

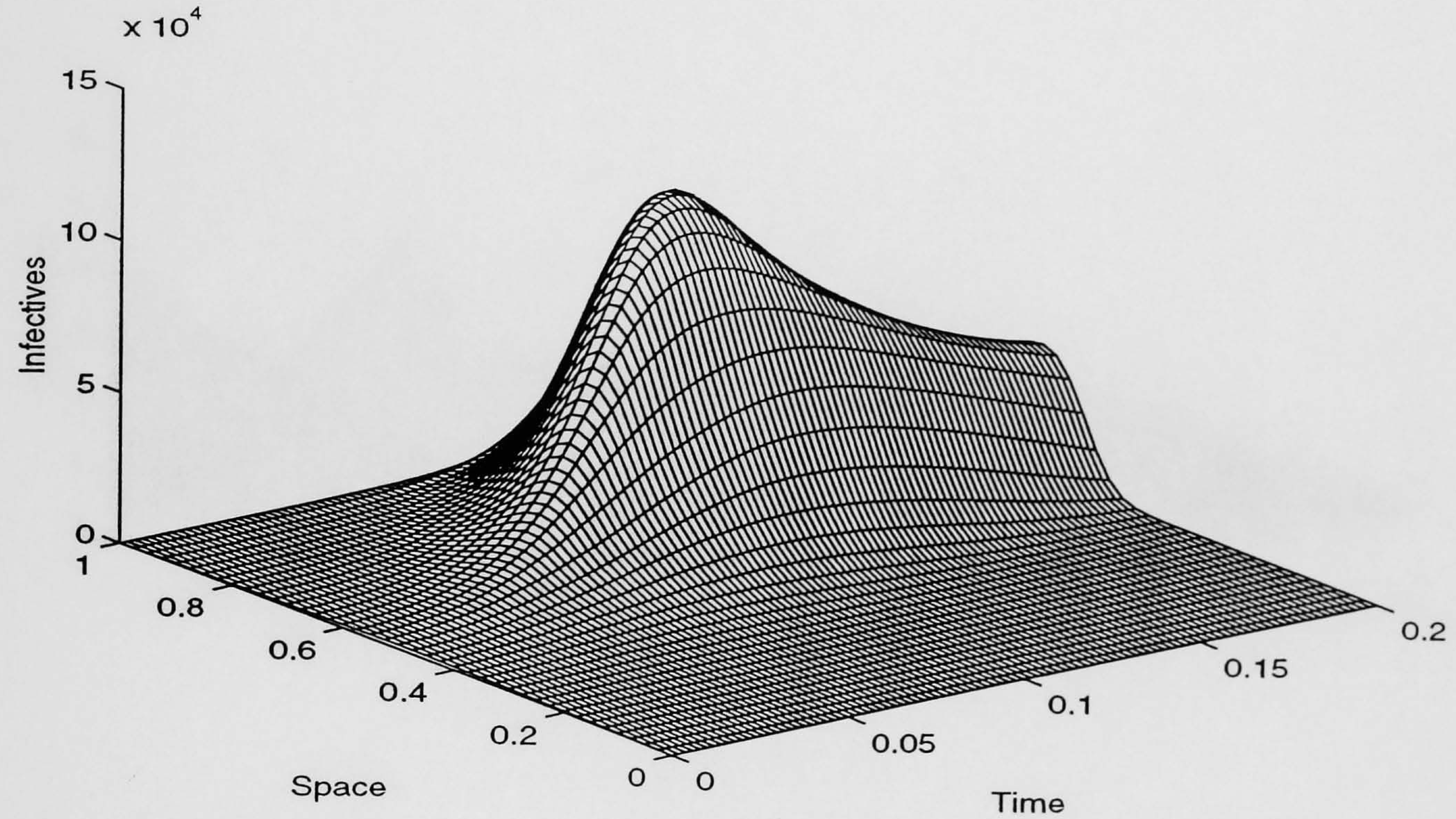

Figure 7.16: *Three-dimensional profile of infectives;* $\alpha = 0.01$, $\rho = 2.0$, $\ell = 0.002$ *and* $h = 0.025$.

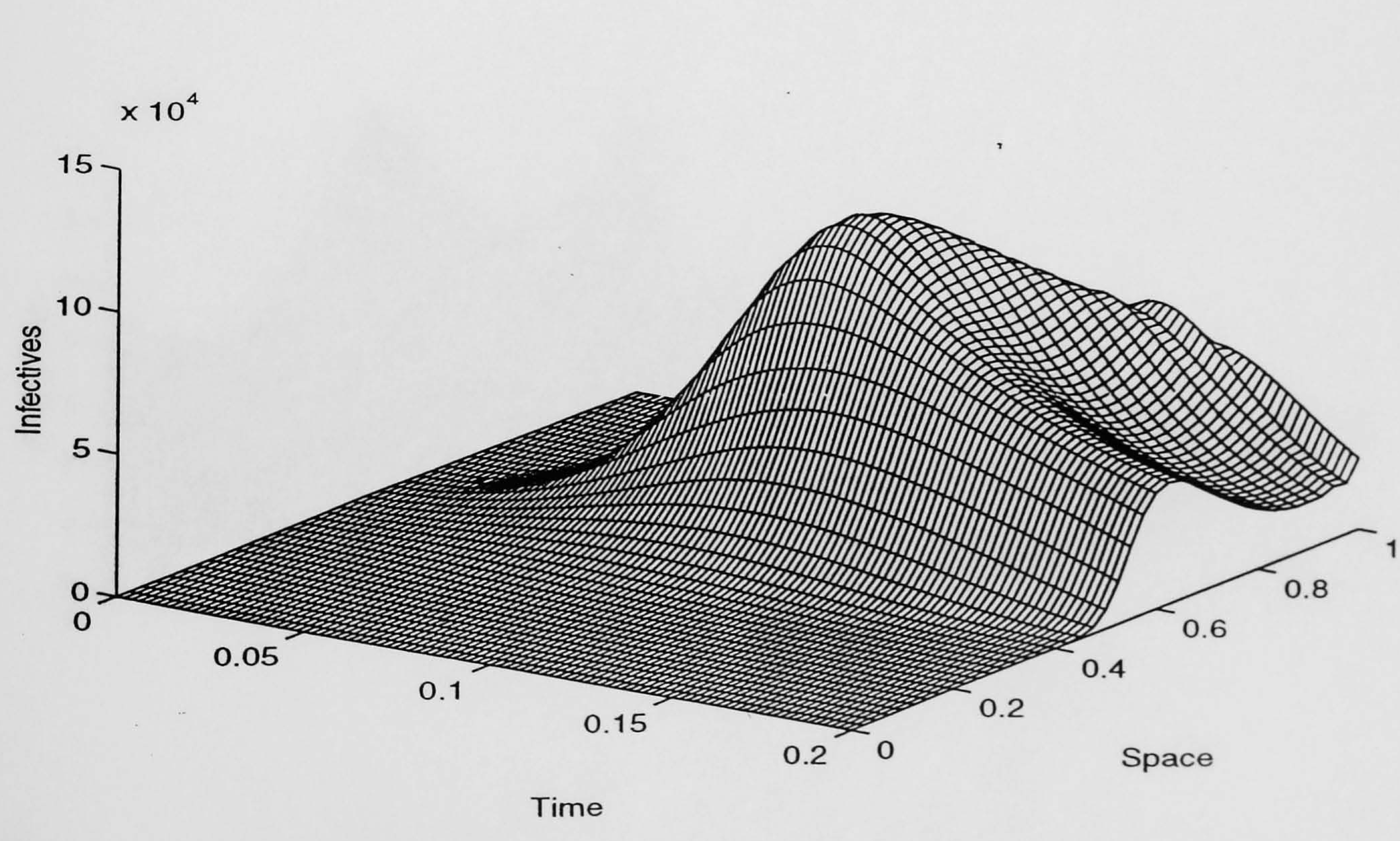

Figure 7.17: *Different angle view of Figure 7.16; view([35,30]).*

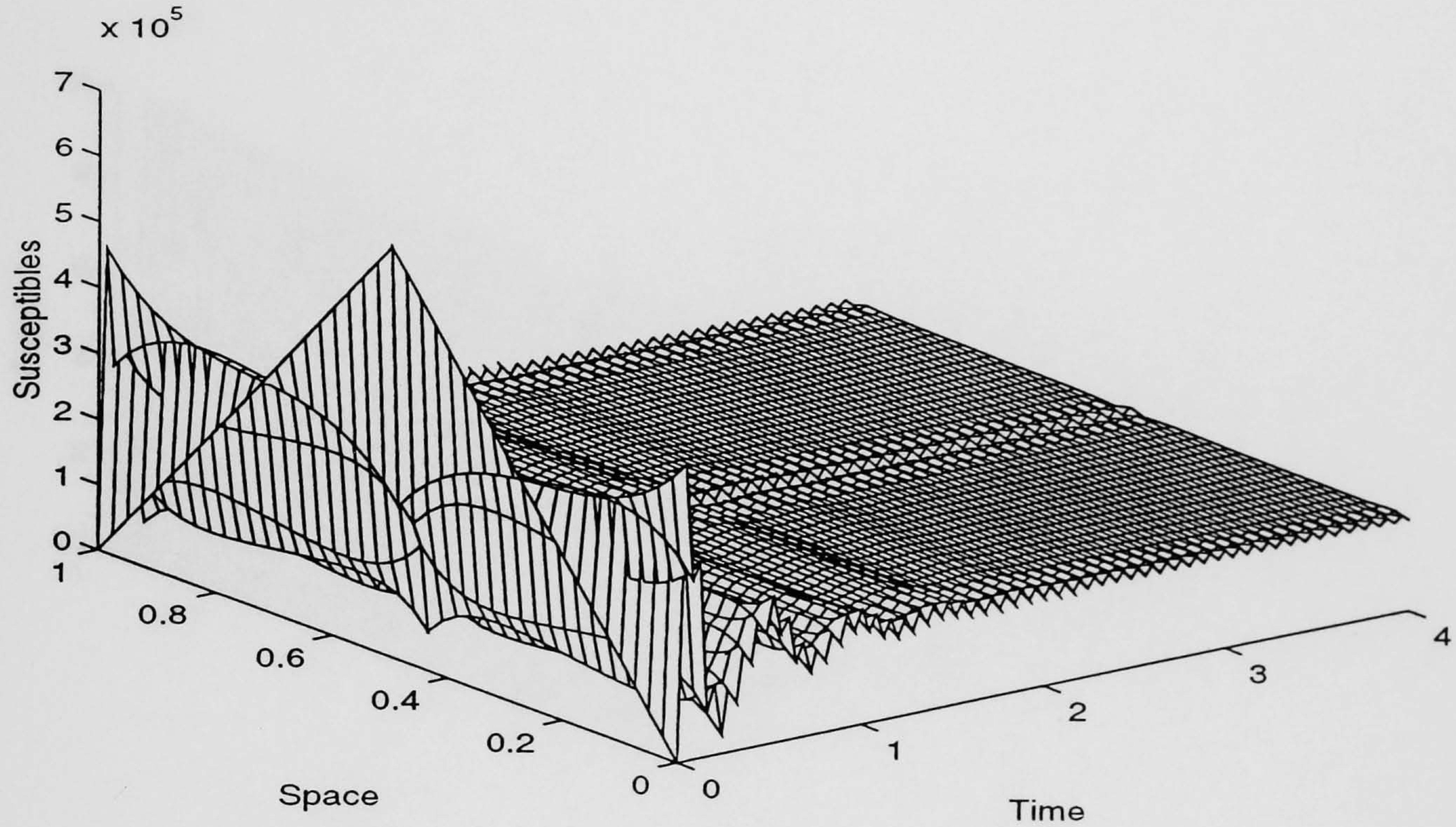

Figure 7.18: *Three-dimensional profile of susceptibles;* $\alpha = 2.0$, $\rho = 0.01$, $\ell = 0.04$ *and* $h = 0.025$.

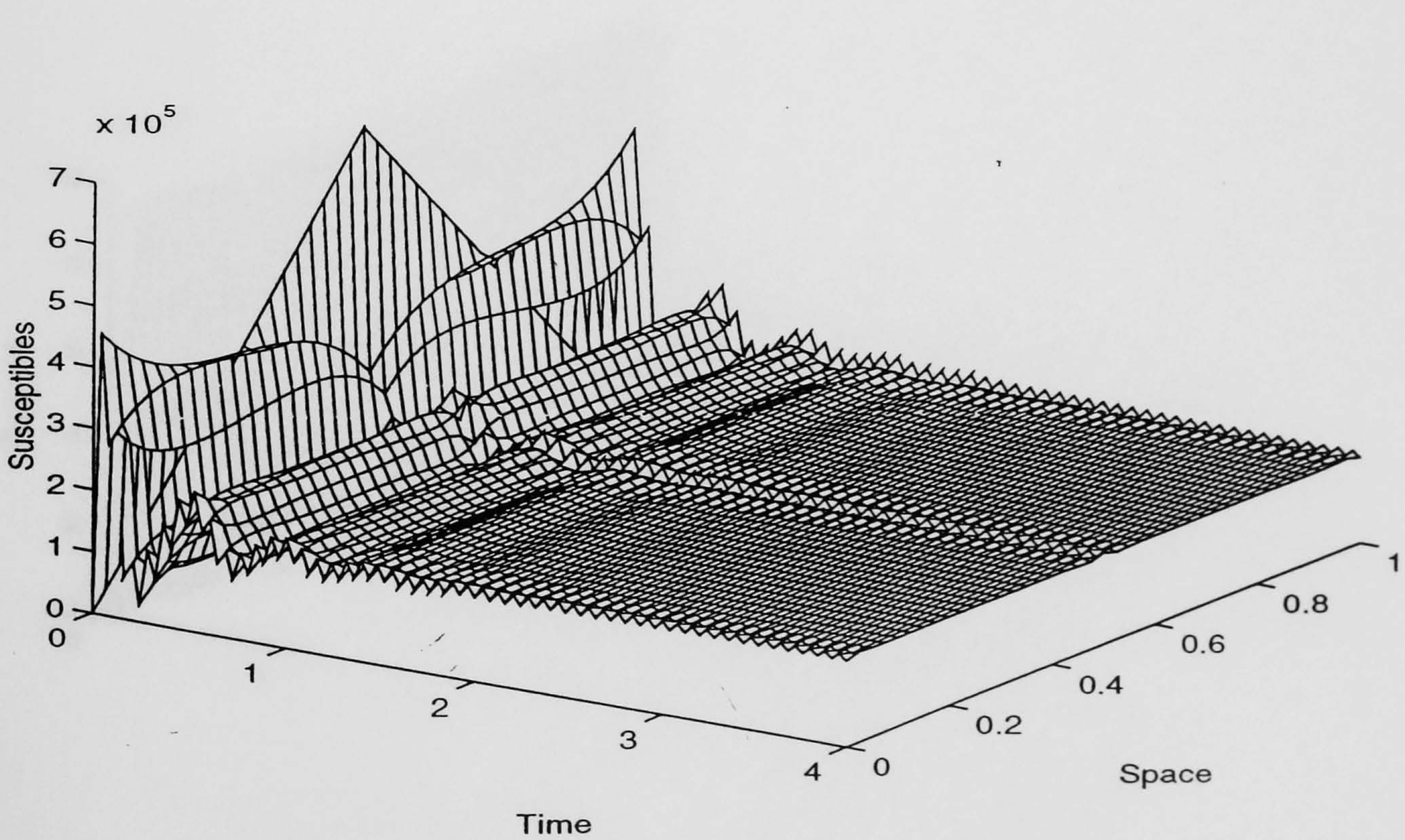

Figure 7.19: *Different angle view of Figure 7.18; view([35,30]).*

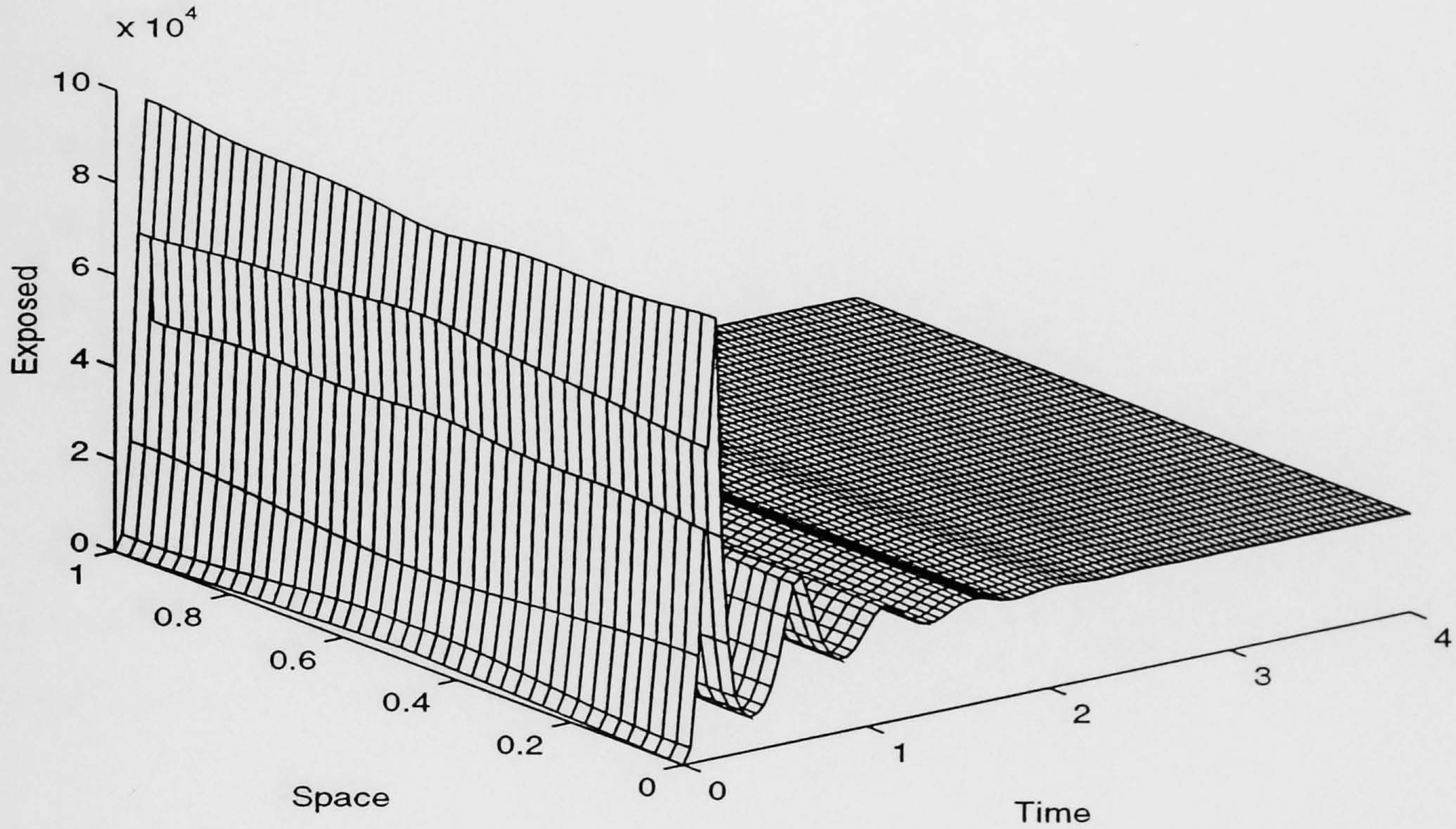

Figure 7.20: *Three-dimensional profile of exposed;* $\alpha = 2.0$, $\rho = 0.01$, $\ell = 0.04$ *and* $h = 0.025$.

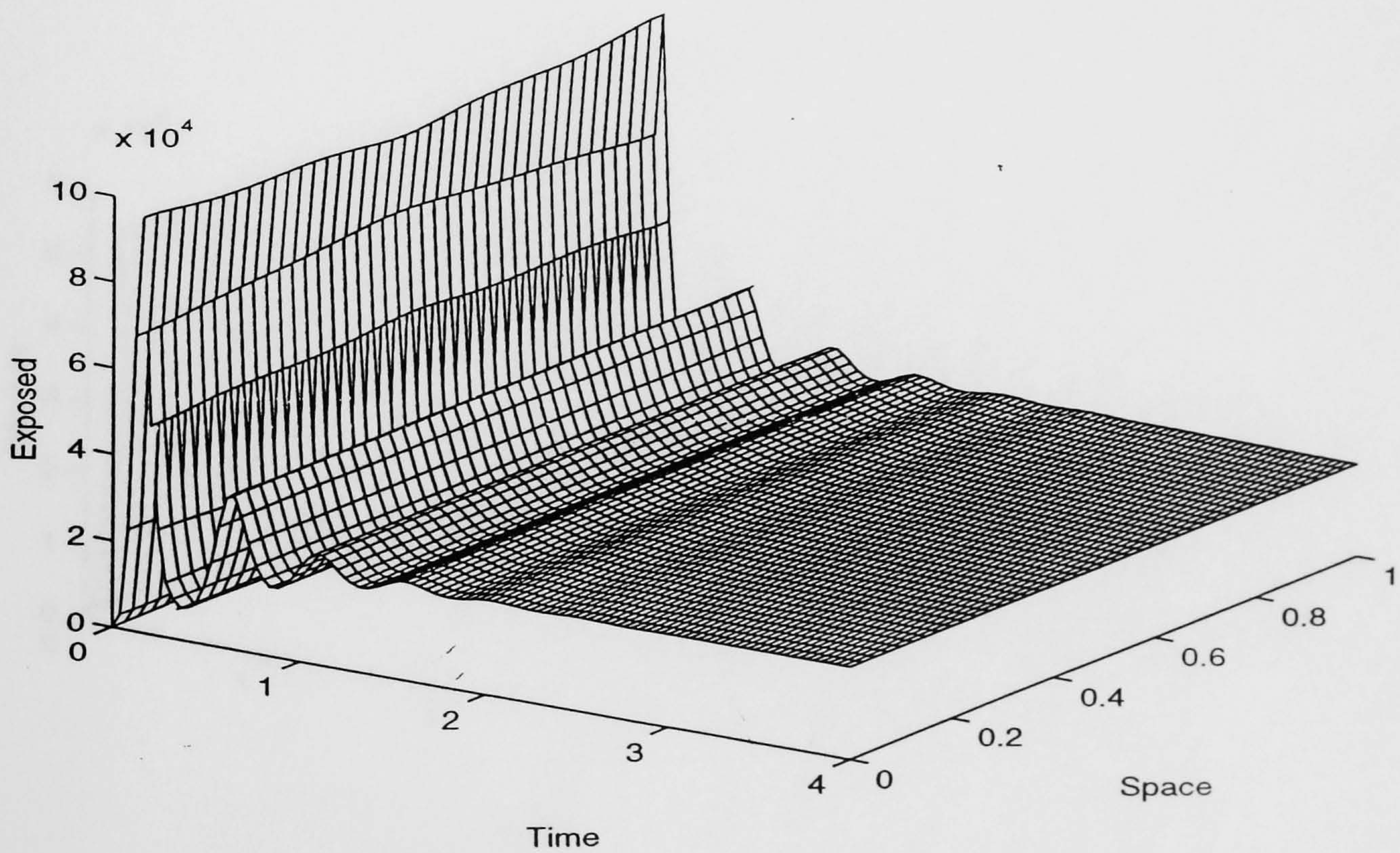

Figure 7.21: *Different angle view of Figure 7.20; view([35,30]).*

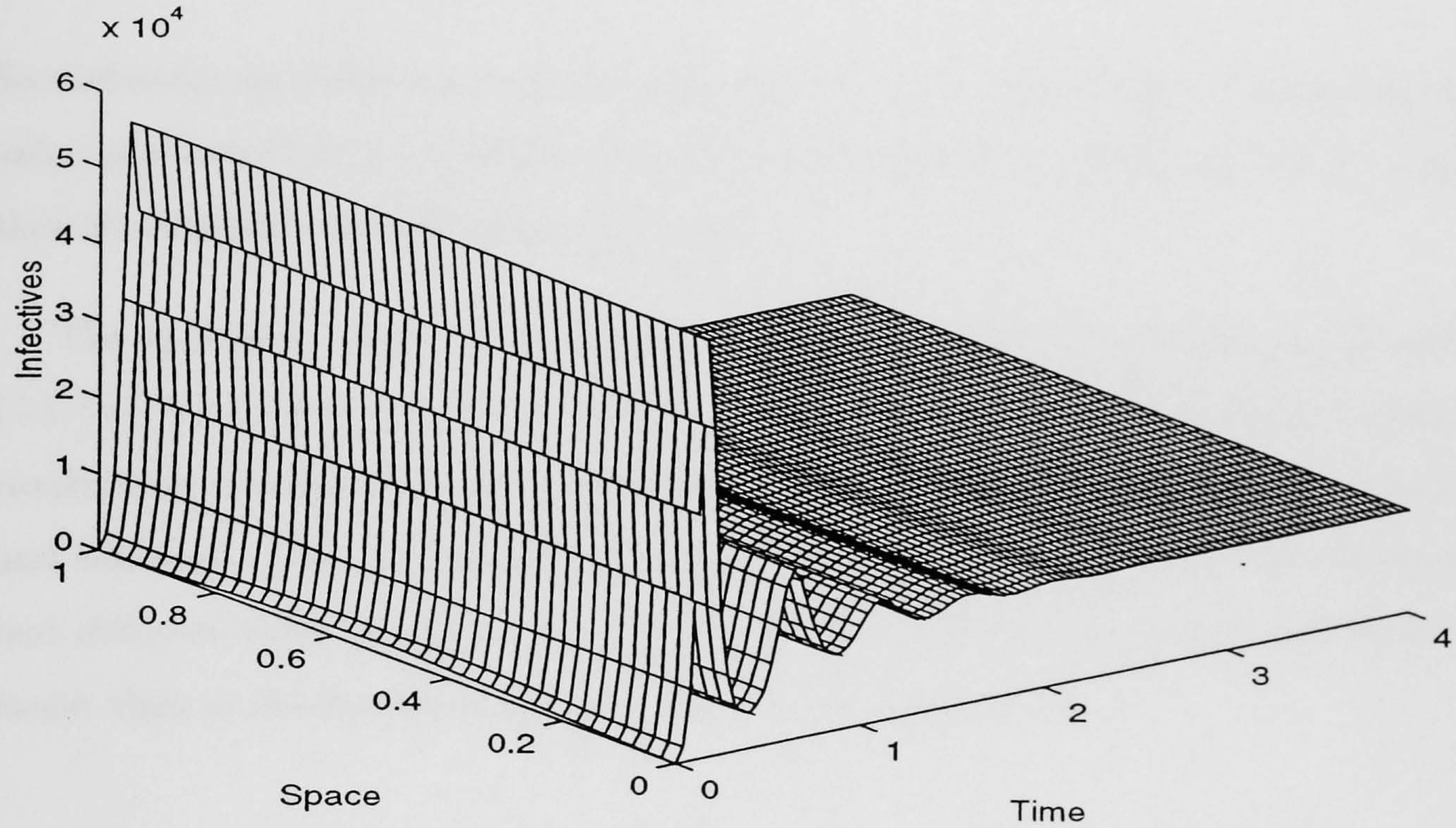

Figure 7.22: *Three-dimensional profile of infectives;* $\alpha = 2.0$, $\rho = 0.01$, $\ell = 0.04$ *and* $h = 0.025$.

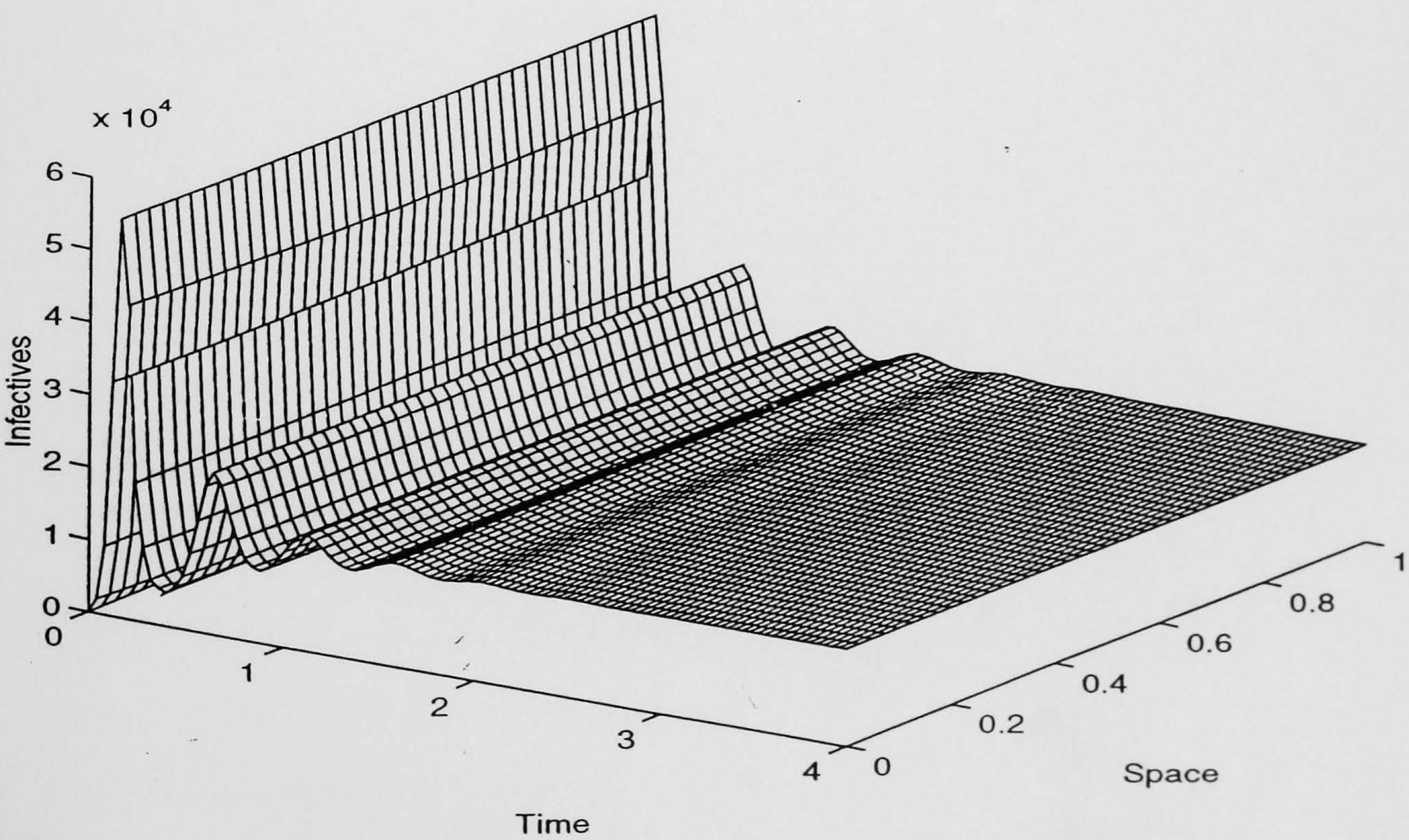

Figure 7.23: *Different angle view of Figure 7.22; view([35,30]).*

7.6 Conclusion

Second-order methods have been developed for the numerical solution of the initial/boundary-value problem (7.1.1)−(7.1.3) and the maximum principle analysis was used to prove that the methods developed are convergent.

The approach adapted was used to solve the non-linear partial differential system (7.1.1)−(7.1.3) with appropriate initial and boundary conditions specified. Numerical results were obtained for the adapted SEIR model for different values of the convection and diffusion rates. It was seen from the results obtained that when the convection and diffusion rates ρ and α are equal the model behaved as expected and when ρ is larger than α the dynamics of the model change unexpectedly.

Chapter 8

Conclusion

Finite-difference numerical methods have been developed for the solution of some systems in the biomedical sciences; namely, a predator-prey model and the SEIR (Susceptibles/ Exposed/ Infectious/ Recovered) measles model.

The familiar Euler forward-difference method is well known to induce chaos in the solution of certain initial-value problems whenever the parameter of the time-discretization exceeds a certain value. To avoid this, while retaining the use of a large time-step, two alternative explicit finite-difference methods were proposed each of which gave convergence to the non-trivial stationary point for a range of values of the parameters in the differential equations of the predator-prey model. The findings relating to all three numerical methods were seen to carry over to a study of the behaviour of the numerical solution of the of the reaction-diffusion equations of the predator-prey model.

A second-order explicit finite-difference method was proposed and compared to the familiar Euler method for the solution of the SEIR measles model using two numerical experiments. It has been seen that this method is very restrictive on stepsize for the first of two experiments and converged to the trivial stationary point for large values of the stepsize and with the infection rate $\beta = 10^{-6}$. Convergence to the correct stationary point has been seen, in the second experiment, for large values of the time step and for any value of the infection rate.

The SEIR measles model has been extended to a one- and two-space dimension diffusion model, a one dimensional convection type and a one-space dimension diffusion-convection type. For the reaction-diffusion problems, Chapters 4, 5 and 7, a second-order method, based on the proposed numerical method to solve the ordinary differential equations of the SEIR model, has been developed, analysed and implemented. A first-order method has been developed for the numerical solution of the SEIR measles model of convection type. In all the numerical methods developed to solve the partial differential equations of the SEIR measles model, the maximum principle analysis has been used to to prove convergence and stability.

It has been seen, from the numerical results, that all the numerical methods developed to solve the partial differential equations for the models depend on the initial conditions and the set of parameters. The major benefit of the numerical methods developed for solving the non-linear partial differential equations was the need to solve only a linear algebraic system at each time step in order to obtain the solution rather than a non-linear algebraic system as often happens.

Overall, these results illustrate the general point that introducing extra biological complexity into the biomedical systems can simplify rather than complicate their behaviour.

The proposed SEIR measles model can be refined further in a number of ways, namely, by adding realistic age structure, maternal antibodies, natural mortality and non-constant population size.

References

[1] Anderson, R.M. & May, R.M. (1985). Age-related changes in the rate of disease transmission: Implications for the design of vaccination programmes. *J. Hyg. (Camb.)*, **94**, 365–436.

[2] Anderson, R.M. & May, R.M. (1991). *Infectious Diseases of Humans: Dynamics and Control.* Oxford University Press, Oxford.

[3] Aron, J.L. & Schwartz, I.B. (1984). Seasonality and period-doubling bifurcations in an epidemic model. *J. Theor. Biol.*, **110**, 665–79.

[4] Bailey, N.J.T. (1975). *The Mathematical Theory of Infectious Diseases and its Applications.* 2nd edn. Griffin, London.

[5] Bartlett, M.S. (1957). Measles periodicity and community size. *Jl R. statist. Soc.*, A **120**, 48–70.

[6] Bolker, B.M. & Grenfell, B.T. (1993). Chaos and biological complexity in measles dynamics. *Proc. R. Soc. Lond.* B. **251**, 75–81.

[7] Britton, N.F. (1986). *Reaction-Diffusion Equations and Their Applications to Biology.* Academic Press, London.

[8] Brownlee, J. (1906). Statistical studies in immunity: the theory of an epidemic. *Proc.Roy.Soc.Edn.*, **26**, 484–521.

[9] Burden, R.L. & Faires, J.D. (1993). *Numerical Analysis.* 5th edn. Prindle, Weber & Schmidt, Boston.

[10] Chewing, W.C. (1975). Migratory effects in predator-prey models. *Math. Biosci.*, **23**, 253–62.

[11] Comins, H.N. & Blatt, D.W.E. (1974). Prey-predator models in spatially heterogeneous environments. *J. Theor. Biol.*, **48**, 75–83.

[12] Dietz, K. & Schenzle, D. (1985a). Proportionate mixing models for age-dependent infection transmission. *J. Math. Biol.*, **22**, 117–120.

[13] Dietz, K. & Schenzle, D. (1985b). Mathematical models for infectious disease statistics. In *A celebration of Statistics* (ed. A.C. Atkinson & S.E. Feinberg), 167–204, Springer, New York.

[14] Dietz, K. (1988). Density-dependence in parasite transmission dynamics. *Parasitol. Today*, **4**, 91–7.

[15] Douglas, Jim JR. (1955). On the numerical integration of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$ by implicit methods. *J. Soc. Indust. Appl. Math.*, **3**(1), 42–65.

[16] Douglas, Jim JR. (1956). On the numerical integration of quasi-linear parabolic differential equations. *Pacific J. Math.*, **6**, 35–42.

[17] Faires, J.D. & Faires, B.T. (1988). *Calculus.* 2nd edn. Random House, New York.

[18] Fine, P.E.M. & Clarkson, J.A. (1982). Measles in England and Wales-I: An analysis of factors underlying seasonal patterns. *Int. J. Epidemiol.*, **11**, 5–15.

[19] Gleick, J. (1987). *Chaos: Making a New Science.* Cardinal/Sphere Books. New York.

[20] Hamer, W.H. (1906). Epidemic disease in England. *The Lancet*, **i**, 733–9.

[21] Hassell, M.P., Comins, H.N. & May, R.M. (1991). Spatial structure and chaos in insect population dynamics. *Nature, Lond.*, **353**, 255–58.

[22] Hastings, A. (1977). Spatial heterogeneity and the stability of predator-prey systems. *Theor. Popul. Biol.*, **12**, 37–48.

[23] Hastings, A. & Powell, T. (1991). Chaos in a three-species food chain. *Ecology*, **72** (3), 896–903.

[24] Henrici, P.H. (1962). *Discrete Variable Methods in Ordinary Differential Equations.* John Wiley, New York.

[25] Herges, U.S. (1995). *Qualitative and Quantitative Analysis of Ordinary Differential Equations and Models of Chemical Reactions Involving Arrhenius Temperature Dependence.* Ph.D. thesis, Brunel University.

[26] Jansen, H. (1994). *The Numerical Modelling of Measles Dynamics.* M.Sc. dissertation, Brunel University.

[27] Khaliq, A.Q.M. (1983). *Numerical Methods for Ordinary Differential Equations with Applications to Partial Differential Equations.* Ph.D. thesis, Brunel University.

[28] Kermack, W.O. & McKendrick, A.G. (1927). A contribution to the mathematical theory of epidemics. *Proc. R. Soc.*, A **115**, 700–21.

[29] Kolmogorov, A.N. (1936). Sulla Teoria di Volterra della Lotta per l'Esisttenza. *Giorn. Instituto Ital. Attuari*, **7**, 74–80.

[30] Kot, M. (1989). Diffusion-driven periodic-doubling bifurcations. *Biosystems*, **22**, 279–87.

[31] Lambert, J.D. (1991). *Numerical Methods for Ordinary Differential Systems: The Inintial-Value Problem.* John Wiley & Sons, Chichester.

[32] Lapidus, L. & Pinder, G.F. (1982). *Numerical Solution of Partial Differential Equations in Science and Engineering.* Wiley-Interscience, New York.

[33] Lees, M. (1959). Approximate solutions of parabolic equations. *J. Soc. Indust. Appl. Math.*, **7**(2), 167–183.

[34] Levin, S. & Segel, L.A. (1985). Pattern generation in space and aspect. *SIAM Rev.*, **27**(1), 45–67.

[35] Levins, R. (1968). *Evolution in Changing Environments: some theoretical explorations.* Princeton University Press,

[36] London, W.P. & Yorke, J.A. (1973). Recurrent outbreaks of measles, chickenpox and mumps. I. Seasonal variation in contact rates. *Am. J. Epidemiol.*, **98**, 453–68.

[37] Lotka, A.J. (1925). *Elements of physical biology.* Baltimore: Williams and Wilkins. (Reissued as *Elements of mathematical biology* by Dover, 1956.)

[38] Luenberger, D.G. (1979). *Introduction to Dynamical Systems Theory, Models and Applications.* John Wiley & Sons., New York.

[39] Lynch, R.V., Ostberg, D.R. & Kuller, R.G. (1973). *Calculus with Computer Applications.* Xerox College Publishing, Massachusetts.

[40] Mathews, Frank P. (1996). *Measles*, vol.**15**, Colliers Encyclopedia CD-Rom, 28.2.1996.

[41] May, R.M. (1974). *Stabibility and Complexity in Model Ecosystems.* Princeton University Press, New Jersy.

[42] May, R.M. (1981). Models for two iteracting populations in "Theoretical Ecology" (R.M. May, ed.) 2nd edn., 78–104, Sinauer, Sunderland, MA.

[43] Maynard Smith, J. (1968). *Mathematical Ideas in Biology.* Cambridge University Press, Cambridge.

[44] McLaughlin, J.F. & Roughgarden, J. (1991). Pattern and stability in predator-prey communities: how diffusion in spatially variable environments affects the Lotka-Volterra model. *Theor. Popul. Biol.*, **40**, 148–72.

[45] McMurtie, R. (1978). Persistence and stability of single-species and prey-predator systems in spatially heterogeneous environments. *Math. Biosci.*, **39**, 11–51.

[46] Minorsky, N. (1962). *Non-Linear Oscillations*, Princeton: D. Van Nostrand Co.,Inc.

[47] Moshkovskii, Sh. D. (1950). Basic laws of the epidemiology of malaria. *AMN, Moscow*, (In Russian).

[48] Murray, J.D. (1993). *Mathematical Biology.* 2nd edn. Springer-Verlag.

[49] Murray, G.D. & Cliff, A.D. (1975). A stochastic model for measles epidemics in a multi-region setting. *Inst. Brit. Geog.*, **2**, 158–74.

[50] Olsen, L.F. & Schaffer, W.M. (1990). Chaos versus noisy periodicity: alternative hypotheses for childhood epidemics. *Science, Wash.*, **249**, 499–504.

[51] Ortega, J.M. (1972). *Numerical Analysis: A Second Course.* Academic Press, New York.

[52] Pascual, M. (1993). Diffusion-induced chaos in a spatial predator-prey system. *Proc. R. Soc. Lond.*, B **251**, 1–7.

[53] Rose, M.E. (1956). On the integration of non-linear parabolic equations by implicit difference methods. *Quart. Appl. Math.*, **14**(3), 237–248.

[54] Ross, R. (1908). *Report on the prevention of malaria in Mauritius*, London.

[55] Ross, R. & Hudson, H.P. (1917). An application of the theory of probabilities to the study of *a priori* pathometry, III. *Proc. R. Soc.*, A**93**, 225–40.

[56] Sandefur, J. (1990). *Discrete Dynamical Systems. Theory and Applications.* Clarendon Press, Oxford.

[57] Schenzle, D. (1984). An age-structured model of pre- and post-vaccination measles transmission. *IMA J. Math. Appl. Med. Biol.*, **1**, 169–91.

[58] Schwartz, I.B. (1985). Multiple recurrent outbreaks and predictability in seasonally forced nonlinear epidemic models. *J. Math. Biol.*, **21**, 347–61.

[59] Schwartz, I.B. & Smith, H.L. (1984). Infinite subharmonic bifurcation in an SEIR model. *J. Math. Biol.*, **18**, 233–53.

[60] Segel, L.A. & Jackson, J.L. (1972). Dissipative structure: an explanation and an ecological example. *J. Theor. Biol.*, **37**, 545–59.

[61] Solé, R.V. & Valls, J. (1992). On structural stability and chaos in biological systems. *J. Theor. Biol.*, **155**, 87–102.

[62] Soper, M.A. (1929). The interpretation of periodicity in disease prevalence. *J.R. Stat. Soc.*, A**92**, 34–61.

[63] Thomée, V. (1962). A stable difference scheme for the mixed boundary problem for a hyperbolic, first-order system in two dimensions. *J. Soc. Indust. Appl. Math.*, **10**(2), 229–45.

[64] Tidd, C.W., Olsen, L.F. & Schaffer, W.M. (1993). The case for chaos in childhood epidemics II. Predicting historical epidemics from mathematical models. *Proc. R. Soc. Lond.*, B**254**, 257–73.

[65] Tudor, D.W. (1985). An age-dependent epidemic model with application to measles. *Math. Biosci.*, **73**, 131–47.

[66] Twizell, E.H. (1984). *Computational Methods for Partial Differential Equations.* Ellis Horwood, Chichester.

[67] Twizell, E.H. (1988). *Numerical Methods with Applications in the Biomedical Sciences.* Ellis Horwood, Chichester.

[68] Twizell, E.H., Yigong Wangs & Price, W.G. (1990). Chaos-free numerical solutions of reaction-diffusion equations. *Proc. R. Soc. Lond.*, A **430**, 541–76.

[69] Twizell, E.H., Yigong Wangs, Price, W.G. & Fakhr, F. (1994). Finite-difference methods for solving the reaction-diffusion equations of a simple isothermal chemical system. *Numerical Methods for Partial Differential Equations*, **10**, 435–54.

[70] Volterra, V. (1926). Variazioni e fluttuazioni del numero d'individui in speice animali conviventi. *Mem. Acad. Lincei.*, **2**, 31–113. (Translation in an appendix to Chapman's *Animal ecology*, New York, 1931.)

[71] Wiktor, E. (1979). *Asymptotic Analysis of Singular Perturbations.* North-Holland, Amesterdam.

Bibliography

1. Ames, W.F. (1977). *Numerical Methods for Partial Differenial Equations.* 2nd. edn. Acamedic Press, New York.

2. Arrowsmith, D.K. & Place, C.M. (1992). *Dynamical Systems.* Chapman & Hall, London.

3. Campbell, Neil A. (1993). *Biology.* 3rd edn. The Benjamin/Cummings Publishing Company, Inc., California.

4. Conte, S.D. & de Boor, C. (1965). *Elementary Numerical Analysis: An Algorithmic Approach.* 2nd edn., McGraw-Hill, New York.

5. Epstein, B. (1962). *Partial Differential Equations: An Introduction.* McGraw-Hill, New York.

6. Friedman, A. (1964). *Partial Differential Equations of Parabolic Type.* Prentice-Hall, Inc., Englewood Cliffs, N.J.

7. Golub, G.H. & Ortega J.M. (1992). *Scientific Computing and Differential Equations.* Academic Press, Inc., San Diego.

8. Golub, G.H. & Van Loan, C.F. (1983). *Matrix Computations.* North Oxford Academic, Oxfod.

9. Isaacson, E. & Keller, H.B. (1966). *Analysis of Numerical Methods.* John Wiley & Sons, New York.

10. Iserles, A. (1996). *A First Course in the Numerical Analysis of Differential Equations.* Cambridge University Press.

11. Jones, D.S. & Sleeman, B.D. (1983). *Differential Equations and Mathematical Biology.* George Allen & Unwin, London.

12. Mandell, G.L., Douglas, R.G. & Bennett, J.E. (1990). *Principles and Practice of Infectious Diseases.* 3rd edn. Churchill Livingstone, New York.

13. Martin Braun (1993). *Differential Equations and Their Applications: An Introduction to Applied Mathematics.* 4th edn. Springer-Verlag.

14. Protter, Murray H. & Weinberger, Hans F. (1967). *Maximum Principles in Differential Equations.* Prentice-Hall, New Jersey.

15. Redheffer, R. (1991). *Differential Equations: Theory and Applications.* Jones and Bartlett Publishers, Boston.

16. Richtmyer, R.D. & Morton, K.W. (1967). *Difference Methods for Initial-Value Problems.* 2nd edn. John Wiley & Sons, Inc., New York.

17. Thompson, J.M.T. & Stewart, H.B. (1986). *Nonlinear Dynamics and Chaos.* John Wiley & Sons, Chichester.

18. Wessells, Norman K. & Hopson, Janet L. (1988). *Biology.* Random House, Inc., New York.