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SYMMETRIC PRESENTATIONS OF COXETER GROUPS

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Abstract We apply the techniques of symmetric generation to establish the standard presentations of the finite simply laced irreducible finite Coxeter groups, that is, the Coxeter groups of types A_n, D_n and E_n, and show that these are naturally arrived at purely through consideration of certain natural actions of symmetric groups. We go on to use these techniques to provide explicit representations of these groups.

Keywords: symmetric generation; Coxeter group

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1. Introduction

A *Coxeter diagram* of a presentation is a graph in which the vertices correspond to involutory generators and an edge is labelled with the order of the product of its two endpoints. Commuting vertices are not joined and an edge is left unlabelled if the corresponding product has order 3. A Coxeter diagram and its associated group are said to be *simply laced* if all of the edges of the graph are unlabelled. Curtis has noted [**8**] that if such a diagram has a 'tail' of length at least 2, as in Figure 1, then we see that the generator corresponding to the terminal vertex, a_r, commutes with the subgroup generated by the subgraph $\mathcal{G}_0$.

We slightly generalize the notion of a 'graph with a tail' and, in doing so, provide symmetric presentations for all the simply laced irreducible finite Coxeter groups with the aid of little more than a single short relation. These in turn readily give rise to natural representations of these groups.

Presentations of groups having certain types of symmetry properties have been considered since at least Coxeter's work [**7**] of 1959 and they have proved useful for providing natural and elementary definitions of groups and also for having great computational use. In [**11**] Curtis and Fairbairn used one kind of symmetric presentation for the Conway group $\cdot 0$ obtained by Bray and Curtis in [**2**] to represent elements of $\cdot 0$ as a string of at most 64 symbols and typically far fewer. This represents a considerable saving compared

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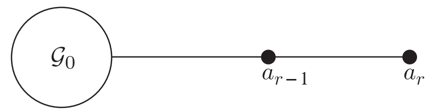

Figure 1. A Coxeter diagram with a tail.

with representing an element of $\cdot 0$ as a permutation of 196 560 symbols or as a 24×24 matrix (i.e. as a string of $24^2 = 576$ symbols). More in-depth discussions of symmetric generation more generally may be found in [**8**, **10**, **13**].

The presentations given here, while not new, do provide an excellent example of how the techniques of symmetric generation may be used to arrive at very natural constructions of groups. These presentations may in turn lead to highly symmetric representations of these groups. While recent results of Fairbairn and Müller [**14**] generalize our main theorem to a wider class of Coxeter groups, the symmetric presentations there are not well motivated (indeed, it is the results presented here that provide the main motivation for the results of [**14**]), may not be arrived at as naturally as those presented here and do not easily lead to explicit representations (the matrices we are naturally led to for the representations of the groups considered here being strikingly simple in nature).

For the basic definitions and notation for Coxeter groups used throughout this paper, we refer the reader to [**15**]. Throughout we shall use the standard Atlas notation for groups found in [**6**].

This paper is organized as follows. In § 2 we outline the basic techniques of involutory symmetric generation. In § 3 we state our main theorem and the barriers to further extension. In § 4 we show how general results in symmetric generation naturally lead us straight to the presentations considered in this paper. In § 5 we perform a coset enumeration that is necessary for proving our main theorem. In § 6 we use the symmetric presentations of the main theorem to construct real representations of the groups concerned and, in doing so, complete the proof. In § 7 we construct $\mathbb{Z}_2$-representations from our real representations in the E_n cases to identify these groups as $\mathbb{Z}_2$ matrix groups.

2. Involutory symmetric generation

We shall describe here only the case when the symmetric generators are involutions as originally discussed by Bray *et al*. [**3**]. For a discussion of the more general case, see [**10**, § III].

Let $2^{\star n}$ denote the free product of n involutions. We write $\{t_1, \dots, t_n\}$ for a set of generators of this free product. A permutation $\pi \in S_n$ induces an automorphism of this free product $\hat{\pi}$ by permuting its generators, i.e. $t_i^{\hat{\pi}} = t_{\pi(i)}$. Given a subgroup $N \leqslant S_n$, we can form a semi-direct product $\mathcal{P} = 2^{\star n} : N$ where, for $\pi \in N$, $\pi^{-1} t_i \pi = t_{\pi(i)}$. When N is transitive, we call $\mathcal{P}$ a *progenitor*. We call N the *control group* of $\mathcal{P}$ and the t_i the *symmetric generators*. Elements of $\mathcal{P}$ can all be written in the form πw with $\pi \in N$ and w is a word in the symmetric generators, so any homomorphic image of the progenitor can be obtained by factoring out relations of the form $\pi w = 1$. We call such a homomorphic image that is finite a *target group*. If G is the target group obtained by factoring the progenitor $2^{\star n} : N$ by the relators $\pi_1 w_1, \pi_2 w_2, \ldots$, we write

$$\frac{2^{\star n} : N}{\pi_1 w_1, \pi_2 w_2, \ldots} \cong G.$$

In keeping with the now traditional notational conventions used in works discussing symmetric generation, we write N both for the control group and its image in G and refer to both simply as 'the control group'. Similarly we shall write t_i both for a symmetric generator and its image in G and we shall refer to both as a 'symmetric generator'.

To decide whether a given homomorphic image of a progenitor is finite, we shall perform a coset enumeration. Given a word in the symmetric generators, w, we define the *coset stabilizing subgroup* of the coset Nw to be the subgroup

$$N^{(w)} := \{\pi \in N \mid Nw\pi = Nw\} \leqslant N.$$

This is clearly a subgroup of N and there are $|N : N^{(w)}|$ right cosets of $N^{(w)}$ in N contained in the double coset $NwN \subset G$. We will write $[w]$ for the double coset NwN and $[\star]$ will denote the coset $[\mathrm{id}_N] = N$. We shall write $w \sim w'$ to mean $[w] = [w']$. We can enumerate these cosets using procedures such as the Todd–Coxeter Algorithm, which can readily be programmed into a computer. The sum of the numbers $|N : N^{(w)}|$ then gives the index of N in G, and we are thus able to determine the order of G and, in doing so, prove it is finite.

In particular, if the target group corresponds to the group defined by a Coxeter diagram with a tail, then removing the vertex at the end of the tail provides a control group for a symmetric presentation with the vertex itself acting as a symmetric generator.

A family of results suggest that this approach lends itself to the construction of groups with low index perfect subgroups. For instance, we have the following.

Lemma 2.1. *If N is perfect and primitive, then $|\mathcal{P} : \mathcal{P}'| = 2$ and $\mathcal{P}'' = \mathcal{P}'$.*

Corollary 2.2. *If N is perfect and primitive, then any image of $\mathcal{P}$ possesses a perfect subgroup of index at most 2. In particular, any homomorphic image of $\mathcal{P}$ satisfying a relation of odd length is perfect.*

For proofs of these results, see [**9**, Theorem 1, p. 356].

The next lemma, while easy to state and prove, has turned out to be extremely powerful in leading to constructions of groups in terms of symmetric generating sets, most notably a majority of the sporadic simple groups [**10**].

Lemma 2.3.

$$\langle t_i, t_j \rangle \cap N \leqslant C_N(\mathrm{Stab}_N(i, j)).$$

Given a pair of symmetric generators t_1 and t_2, Lemma 2.3 tells us which permutations $\pi \in N$ may be written as a word in t_1 and t_2, but gives us no indication of the length of such a word. Naturally, we wish to factor a given progenitor by the shortest and most easily understood relation possible. The following lemma shows that, in many circumstances, a relation of the form $\pi t_1 t_2 t_1$ is of great interest.

Lemma 2.4. *Let $G = \langle \mathcal{T} \rangle$, where $\mathcal{T} = \{t_1, \dots, t_n\} \subseteq G$ is a set of involutions in G with $N = N_G(\mathcal{T})$ acting primitively on $\mathcal{T}$ by conjugation. (Thus, G is a homomorphic image of the progenitor $2^{\star n} : N$.) If $t_1 t_i \in N$, $t_1 \notin N$ for some $i \neq 1$, then $|G| = 2|N|$.*

For proofs of these results, see [**10**, pp. 58–59].

3. The main theorem

Using the notation of the last section we will prove the following.

Theorem 3.1. *Let S_n be the symmetric group acting on n objects and let $W(\Phi)$ denote the Weyl group of the root system Φ. Then*

(i) $\dfrac{2^{\star\binom{n}{1}} : S_n}{(t_1(12))^3} \cong W(A_n)$,

(ii) $\dfrac{2^{\star\binom{n}{2}} : S_n}{(t_{12}(23))^3} \cong W(D_n)$ *for* $n \geqslant 4$,

(iii) $\dfrac{2^{\star\binom{n}{3}} : S_n}{(t_{123}(34))^3} \cong W(E_n)$ *for* $n = 6, 7, 8$.

In case (i) the action of S_n defining the progenitor is the natural action of S_n on $X := \{1, \dots, n\}$, in case (ii) the action of S_n defining the progenitor is the action of S_n on the 2-element subsets of X, and in case (iii) the action of S_n defining the progenitor is the action of S_n on the 3-element subsets of X.

Case (i) of Theorem 3.1 has been noted by various authors [**10**, Theorem 3.2, p. 63], but we include it here for completeness.

More suggestively, we can express these symmetric presentations as Coxeter diagrams as given in Figure 2. (Note that, from the presentations given in this theorem, without even drawing any Coxeter diagrams, the exceptional coincidences of $D_3 = A_3$ and $E_5 = D_5$ are immediate since $\binom{3}{2} = \binom{3}{1}$ and $\binom{5}{3} = \binom{5}{2}$.)

We remark that the natural pattern of applying the relation $(t_{1,\dots,k}(k, k+1))^3$ to the progenitor $2^{\star\binom{n}{k}} : S_n$ to produce a finite image does not extend further. In [**4**], Bray *et al*. prove the symmetric presentation

$$\frac{2^{\star\binom{n}{4}} : S_8}{(t_{1234}(45))^3, t_{1234}t_{5678}} \cong W(E_7) \cong S_6(2) \times 2.$$

The second relation, which simply identifies a 4-element subset with its complement so that the symmetric generators correspond to partitions of the eight points into two fours, is necessary for the coset enumeration to terminate; hence the pattern does not continue when the control group is the full symmetric group. However, using a control group smaller than the full symmetric group can resolve this problem. In [**2**] Bray and Curtis prove that

$$\frac{2^{\star\binom{24}{4}} : M_{24}}{\pi t_{ab} t_{ac} t_{ad}} \cong \cdot 0,$$

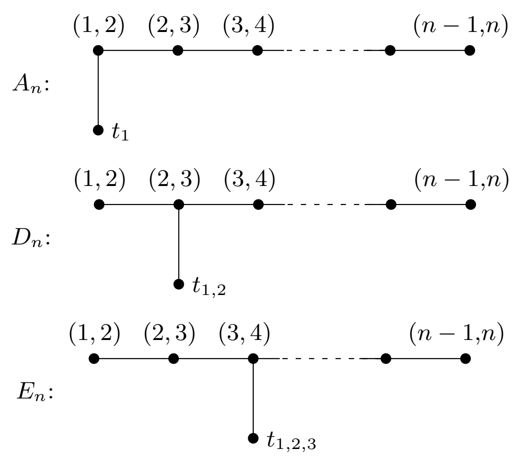

Figure 2. Symmetric presentations as Coxeter diagrams.

where M_{24} denotes the largest of the sporadic simple Mathieu groups, where a, b, c and d are pairs of points, the union of which is a block of the $\mathcal{S}(5, 8, 24)$ Steiner system on which M_{24} naturally acts [**6**, p. 94], where $\cdot 0$ is the full cover group of the largest sporadic simple Conway group [**6**, p. 180] and where $\pi \in M_{24}$ is the unique permutation of M_{24} set-wise fixing the sextets defined by each of the symmetric generators whose use is motivated by Lemma 2.3.

The proof of Theorem 3.1 is obtained as follows. In § 5 we enumerate the double cosets NwN in each case to verify that the orders of the target groups are at most the orders claimed in Theorem 3.1. In § 6 we exhibit elements of the target groups that generate them and satisfy the additional relations, thereby providing lower bounds for the orders and verifying the presentations.

4. Motivating the relations of Theorem 3.1

In this section we will show how the relators used in Theorem 3.1 may be arrived at naturally by considering the natural actions of the control group used to define the progenitors appearing in the main theorem.

Given Lemma 2.3 it is natural to want to compute $C_{S_n}(\mathrm{Stab}_{S_n}(1, 2))$. In the A_n case we find

$$\mathrm{Stab}_{S_n}(1,2) = \begin{cases} \langle \mathrm{id} \rangle & \text{if } n \in \{2,3\}, \\ \langle (3,4), (3,\dots,n) \rangle & \text{if } n \geqslant 4. \end{cases}$$

Calculating $C_{S_n}(\mathrm{Stab}_{S_n}(1, 2))$ thus gives us

$$C_{S_n}(\mathrm{Stab}_{S_n}(1,2)) = \begin{cases} \langle (1,2) \rangle & \text{if } n = 2 \text{ or } n \geqslant 5, \\ \langle (1,2), (1,2,3) \rangle & \text{if } n = 3, \\ \langle (1,2), (3,4) \rangle & \text{if } n = 4. \end{cases}$$

For $n \geqslant 5$ we see that $\langle t_1, t_2\rangle \cap N \leqslant \langle (1,2)\rangle$. Lemma 2.4 now tells us that the shortest natural relator worth considering is $(1,2)t_1t_2t_1$, which we rewrite more succinctly as $(t_1(12))^3$. We are thus naturally led to considering the factored progenitor

$$\frac{2^{\star\binom{n}{1}} : S_n}{(t_1(12))^3}.$$

Recall that S_n is the symmetric group acting on n objects. The high transitivity of the natural action of S_n on n objects enables us to form the progenitors

$$\mathcal{P}_1 := 2^{\star\binom{n}{1}} : S_n, \qquad \mathcal{P}_2 := 2^{\star\binom{n}{2}} : S_n, \qquad \mathcal{P}_3 := 2^{\star\binom{n}{3}} : S_n.$$

Arguments similar to those used in the case $\mathcal{P}_1$ may be applied in the other two cases, naturally leading us to consider the factored progenitors

$$\frac{2^{\star\binom{n}{2}} : S_n}{(t_{12}(23))^3} \text{ for } n \geqslant 4 \quad \text{and} \quad \frac{2^{\star\binom{n}{3}} : S_n}{(t_{123}(34))^3} \text{ for } n \geqslant 6.$$

In all three cases the exceptional stabilizers and centralizers encountered for small values of n can be shown to lead straight to interesting presentations of various finite groups [**12**, § 3.8], but we shall not make use of these results here.

5. Coset enumeration

To prove that the homomorphic images under the relations appearing in Theorem 3.1 are finite, we need to perform a double coset enumeration to place an upper bound on the order of the target group in each case.

The orders of all finite irreducible Coxeter groups, including those of types A_n, D_n and E_n, may be found listed in Humphreys [**15**, Table 2, p. 44].

5.1. A_n

For $\mathcal{P}_1$ we enumerate the cosets by hand. Since $t_it_j = (ij)t_i$ for $i, j \in \{1, \dots, n\}$, $i \neq j$, any coset representative must have length at most one. Since the stabilizer of a symmetric generator in our control group $S_n^{(t_1)}$ clearly contains a subgroup isomorphic to S_{n-1} (namely the stabilizer in S_n of the point 1), we have that

$$|S_n : S_n^{(t_1)}| \leqslant n \quad \text{and} \quad |S_n : S_n^{(\star)}| = 1,$$

so the target group must contain the image of S_n to index at most $n+1$.

5.2. D_n

We shall prove the following.

Lemma 5.1. *Let*

$$G := \frac{2^{\star\binom{n}{2}} : S_n}{(t_{12}(23))^3} \quad \textit{for } n \geqslant 4.$$

The representatives for the double cosets $S_n w S_n \subset G$ with w a word in the symmetric generators are

$$[\star], [t_{12}], [t_{12}t_{34}], \dots, [t_{12}t_{34}\dots t_{2k-1,2k}],$$

where k is the largest integer such that $2k \leqslant n$. We thus have $|G : S_n| \leqslant 2^{n-1}$.

We shall prove this by using the following two lemmata.

Lemma 5.2. *For the group G as above, the double coset represented by the word $t_{ab}\cdots t_{ij}\cdots t_{ik}\cdots t_{cd}$ may be represented by a shorter word (i.e. if two symmetric generators in a given word have some index in common, then that word can be replaced by a shorter word).*

Proof. The relation immediately tells us $t_{12}t_{13} = (23)t_{12}$ and so $[t_{12}t_{13}] = [t_{12}]$; thus, we can suppose our word has length at least 3. Using the high transitivity of the action of S_n on n points, we may assume that our word contains a subword of the form $t_{12}\cdots t_{34}t_{15}$ with no other occurrence of the index '1' and no other repetitions appearing anywhere between the symmetric generators t_{12} and t_{15} of this subword. Now,

$$\begin{aligned} t_{12}\cdots t_{34}t_{15} &= t_{12}\cdots t_{34}t_{13}^2 t_{15} \\ &= t_{12}\cdots ((14)t_{34})((35)t_{13}) \\ &= (14)(35)t_{24}\cdots t_{45}t_{13}, \end{aligned}$$

and so the repeated indices can be 'moved closer together'. Repeating the above, the two symmetric generators with the common index can eventually be placed side by side, at which point our relation immediately shortens this word since $t_{12}t_{13} = (23)t_{12}$. Since our word has finite length, we can easily repeat this procedure to eliminate all repetitions. □

Lemma 5.3. $t_{12}t_{34} \sim t_{13}t_{24}$.

Proof.

$$t_{12}t_{34} = t_{12}t_{34}t_{24}^2 = t_{12}(23)t_{34}t_{24} = (23)t_{13}t_{34}t_{24} = (23)(14)t_{13}t_{24} \sim t_{13}t_{24}.$$

□

Proof of Lemma 5.1. By Lemma 5.2, the indices appearing in any coset representative must be distinct. By Lemma 5.3, the indices appearing in a word of length 2 may be reordered. Since the indices are all distinct, it follows that the indices appearing in a coset representative of any length may be reordered. The double cosets must therefore be $[\star], [t_{12}], \dots, [t_{12}\cdots t_{2k-1,2k}]$, where k is the largest integer such that $2k \leqslant n$. There is therefore no more than one double coset for each subset of $\{1, \dots, n\}$ of even size and so $|G : S_n| \leqslant 2^{n-1}$. □

Table 1. *The coset enumeration for E_6.*

Label $[w]$	Coset stabilizing subgroup	$\lvert N : N^{(w)}\rvert$
$[\star]$	N	1
$[t_{123}]$	$N^{(t_{123})} \cong S_3 \times S_3$	20
$[t_{123}t_{145}]$	$N^{(t_{123}t_{145})} \cong S_4$ since $t_{123}t_{145} = t_{123}t_{124}^2t_{145} \sim t_{123}(25)t_{124} \sim t_{135}t_{124}$	30
$[t_{123}t_{456}]$ $[t_{123}t_{456}t_{124}] = [t_{356}t_{245}]$	$N^{(t_{123}t_{456})} \cong S_3 \times S_3$ since $t_{123}t_{456}t_{124} = t_{123}t_{456}t_{145}^2t_{124} = t_{123}(16)t_{456}(25)t_{145} \sim t_{356}t_{245}t_{145} \sim t_{356}t_{245}$	20
$[t_{123}t_{456}t_{123}]$	$N^{(t_{123}t_{456}t_{123})} \cong S_6$ since $t_{123}t_{456}t_{123} = t_{123}(34)t_{456}t_{356}t_{123} \sim t_{123}(34)t_{456}t_{356}t_{235}^2t_{123} = t_{124}t_{456}(26)t_{356}(15)t_{235} = t_{456}t_{124}t_{136}t_{235} = t_{456}t_{146}^2t_{124}t_{136}t_{235} = (15)t_{456}(62)t_{146}t_{136}t_{235} = t_{245}t_{146}t_{136}t_{235} = t_{245}(34)t_{146}t_{235} \sim t_{235}t_{146}t_{235}$	1

5.3. E_6

The coset enumeration in this case may also be performed by hand. We list the cosets in Table 1. Not every case is considered in this table; however, all remaining cases may be deduced from them as follows. Since $t_{123}t_{145} \sim t_{124}t_{135}$, the S_4 permuting these indices ensures that, for any three-element subset $\{a,b,c\} \subset \{1,\ldots,6\}$, the word $t_{123}t_{145}t_{abc}$ will shorten. Since the only non-collapsing word of length 3 is of the form $t_{123}t_{456}t_{123}$ and $t_{123}t_{456}t_{123} \sim t_{124}t_{356}t_{124}$, the S_6 permuting these indices ensures that, for any three-element subset $\{a,b,c\} \subset \{1,\ldots,6\}$, the word $t_{123}t_{456}t_{123}t_{abc}$ will shorten and so all words of length 4 shorten.

From this double coset enumeration we see that $|W(E_6) : S_6| \leqslant 1+20+30+20+1 = 72$. Our target group must therefore have order at most $72 \times |S_6| = 51\,840$.

5.4. E_7

Since we expect both the index and the number of cosets to be much larger in this case than in the E_6 case (and in particular to be too unwieldy for a 'by hand' approach to work), we use a computer, and in particular the algebra package Magma [**5**], to determine the index.

```
> S:=Sym(7);
> stab:=Stabilizer(S,{1,2,3});
> f,nn:=CosetAction(S,stab);
```

Here we have defined a copy of the symmetric group S_7 (now named '`nn`') in its permutation representation defined by the action on the $\binom{7}{3} = 35$ subsets of cardinality 3 via the natural representation, and a homomorphism `f` from a copy of S_7 that acts on seven points to our new copy `nn`.

```
> 1^f(S!(3,4));
22
```

The computer has labelled the set $\{1,2,3\}$ `1` and, to find the label the computer has given to the set $\{1,2,4\}$, we find the image of `1` under the action of the permutation `f((1,2))` $\in$ `nn`, finding that, on this occasion, the computer has given the set $\{1,2,4\}$ the label 22.

```
> RR:=[<[1,22,1],f(S!(3,4))>];
> CT:=DCEnum(nn,RR,nn:Print:=5,Grain:=100);

Index: 576 = Rank: 10 = Edges: 40 = Status: Early closed = Time: 0.150
```

The ordered sequence `RR` contains the sequence of symmetric generators $t_{123}t_{124}t_{123}$ and the permutation (34) that we are equating with this word to input our additional relation into the computer. The command `DCEnum` simply calls the double coset enumeration program of Bray and Curtis as described in [**1**].

The computer has found there to be at most 10 distinct double cosets and that $|W(E_7) : S_7| \leqslant 576$. Our target group must therefore have order at most $576 \times |S_7| = 2\,903\,040$.

5.5. $\mathbf{E_8}$

Again, we use the computer to determine the index, with each of the Magma commands below being the same as those used in the previous section.

```
> S:=Sym(8);
> stab:=Stabilizer(S,{1,2,3});
> f,nn:=CosetAction(S,stab);
> 1^f(S!(3,4));
28
> RR:=[<[1,28,1],f(S!(3,4))>];
> CT:=DCEnum(nn,RR,nn:Print:=5,Grain:=100);
Index: 17280 = Rank: 35 = Edges: 256 = Status: Early closed = Time: 0.940
```

We see that $|W(E_8) : S_8| \leqslant 17\,280$. Our target group must therefore have order at most $17\,280 \times |S_8| = 696\,729\,600$.

6. Representations

In this section we use the symmetric presentations of Theorem 3.1 to construct representations of the target groups and in doing so we verify that we have the structures that we claim. In the A_n and D_n cases this is sufficient to show that the groups are what we expect them to be.

6.1. $W(A_n)$

Since these groups are most naturally viewed as permutation groups we shall construct the natural permutation representation. The lowest degree of a permutation representation in which the control group, S_n, acts faithfully is n, so the lowest degree of a permutation representation in which the target group acts faithfully is n. Since the control group already contains all possible permutations of n objects, the target group must be a permutation group of at least $n+1$ objects. A permutation corresponding to a symmetric generator must commute with its stabilizer in the control group, namely S_{n-1}. There is only one such permutation satisfying this: $t_i = (i, n+1)$. Since this has order 2 and satisfies the relation, we must therefore have that our target group is isomorphic to $S_{n+1} \cong W(A_n)$.

6.2. $W(D_n)$

We shall use our symmetric generators to construct an elementary abelian 2-group lying outside our control group and thus to verify that our target group has structure $2^{n-1} : S_n$.

Lemma 6.1. $t_{12}t_{34} = t_{34}t_{12}$.

Proof.

$$\begin{aligned} t_{12}t_{34}t_{12} &= t_{12}t_{34}t_{13}^2t_{12} \\ &= t_{12}(14)t_{34}(23)t_{13} \\ &= (14)(23)t_{34}t_{24}t_{13} \\ &= (14)(t_{34}t_{24})t_{24}t_{13} \\ &= t_{34}. \end{aligned}$$

□

Lemma 6.2. *The elements $e_{ij} := (ij)t_{ij}$ for $1 \leqslant i, j \leqslant n$ generate an elementary abelian 2-group.*

Proof. Each of the elements e_{ij} have order 2 since the symmetric generators have order 2. If $i, j \notin \{k, l\}$, then by Lemma 6.1, $e_{ij}e_{kl} = e_{kl}e_{ij}$. Suppose $i = l$; then

$$\begin{aligned} e_{ij}e_{ik}e_{ij}e_{ik} &= (ij)t_{ij}(ik)t_{ik}(ij)t_{ij}(ik)t_{ik} \\ &= (ij)(ik)(ij)(ik)t_{ik}t_{ij}t_{jk}t_{ik} \\ &= (ij)(ik)(ij)(ik)(jk)t_{ik}t_{jk}t_{ik} \\ &= (ij)(ik)(ij)(ik)(jk)(ij) \\ &= \mathrm{id}_{S_n} \end{aligned}$$

□

Lemma 6.3. *If e_{ij} is as defined in Lemma 6.2, then $e_{ij}e_{ik} = e_{jk}$ for $i \neq j \neq k \neq i$.*

Proof.

$$\begin{aligned} e_{ij}e_{ik} &= (ij)t_{ij}(ik)t_{ik} \\ &= (ij)(ik)t_{jk}t_{ik} \\ &= (ij)(ik)(ij)t_{jk} \\ &= (jk)t_{jk} \\ &= e_{jk}. \end{aligned}$$

□

We have thus shown that there is an elementary abelian group of order 2^{n-1} lying outside the control group: the elements e_{ij} defined in Lemma 6.2 each have order 2 (since the symmetric generators each have order 2), by Lemma 6.2 any two of the elements e_{ij} commute and by Lemma 6.3 the subgroup generated by these elements is clearly generated by the $n-1$ elements $e_{12}, e_{13}, \ldots, e_{1n}$.

It is natural to represent the elements e_{ij} as diagonal matrices with -1 entries in the i and j positions. Using the natural n-dimensional representation of S_n as permutation matrices, we have been naturally led to

$$t_{12} = \begin{pmatrix} & -1 & & & \\ -1 & & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

The control group naturally acts on the group generated by the elements e_{ij} by permuting the indices. In particular, recalling from the double coset enumeration of § 5.2 that N has index at most 2^{n-1} in the target group, the above lemmas together show that our target group is isomorphic to the group $2^{n-1}\colon S_n \cong W(D_n)$.

6.3. $W(E_6)$

In the case of E_6, we shall construct a six-dimensional real representation in which the control group acts as permutation matrices. In such a representation, the matrix corresponding to the symmetric generator t_{123} must

(i) commute with the stabilizer of t_{123},

(ii) have order 2,

(iii) satisfy the relation.

By condition (i) such a matrix is of the form

$$t_{123} = \begin{pmatrix} aI_3 + bJ_3 & cJ_3 \\ c'J_3 & a'I_3 + b'J_3 \end{pmatrix},$$

where I_3 denotes the 3×3 identity matrix and J_3 denotes a 3×3 matrix, all the entries of which are 1. Now, condition (ii) requires

$$(aI_3 + bJ_3)^2 + 3cc'J_3 = (a'I_3 + b'J_3)^2 + 3cc'J = I_3,$$

implying that

$$c(a + a' + 3b + 3b') = c'(a + a' + 3b + 3b') = 0,$$
$$a^2 = a'^2 = 1$$

and

$$2ab + 3b^2 + 3cc' = 2a'b' + 3b'^2 + 3cc' = 0.$$

If our control group acts as permutation matrices, then condition (iii) implies that the determinant of the matrix for the symmetric generators must be -1. This requires that

$$(a + 3b)(a' + 3b') = -1.$$

From these relations we are naturally led to matrices of the form

$$t_{123} = \begin{pmatrix} I_3 - \frac{2}{3}J_3 & \frac{1}{3}J_3 \\ 0_3 & I_3 \end{pmatrix}.$$

The representation of the control group we have used is not irreducible and splits into two irreducible representations: the subspace spanned by the vector $v := (1^6)$ and the subspace $v^{\perp}$. The above matrices do not respect this decomposition since they map v to vectors of the form $(0^3, 1^3)$. Consequently, the above representation of $W(E_6)$ is irreducible.

6.4. $W(E_7)$

Using arguments entirely analogous to those appearing in the previous section there is a seven-dimensional representation of $W(E_7)$ in which the control group acts as permutation matrices and we can represent the symmetric generators for $W(E_7)$ with matrices of the form

$$t_{123} = \begin{pmatrix} I_3 - \frac{2}{3}J_3 & \frac{1}{3}J_{3\times 4} \\ 0_{4\times 3} & I_4 \end{pmatrix},$$

which is again irreducible.

6.5. $W(E_8)$

Again using arguments entirely analogous to those used in the E_6 case, there is an eight-dimensional representation of $W(E_8)$ in which the control group acts as permutation matrices and we can represent the symmetric generators for E_8 with matrices of the form

$$t_{1,2,3} = \begin{pmatrix} I_3 - \frac{2}{3}J_3 & \frac{1}{3}J_{3\times 5} \\ 0_{5\times 3} & I_5 \end{pmatrix},$$

which is again irreducible.

7. $\mathbb{Z}_2$-representations of the groups $W(E_n)$

In this section we use the matrices obtained in § 6 for representing the Weyl groups of types E_6, E_7 and E_8 to exhibit representations of these groups over $\mathbb{Z}_2$ and, in doing so, we identify the structure of the groups in question.

7.1. $W(E_6)$

Multiplying the matrices for our symmetric generators found in the last section by 3 ($\equiv 1 \pmod 2$) we find that these matrices, working over $\mathbb{Z}_2$, are of the form

$$t_{123} = \begin{pmatrix} I_3 & J_3 \\ 0_3 & I_3 \end{pmatrix}.$$

These matrices still satisfy the relation and the representation is still irreducible for the same reason as in the real case, as is easily verified by Magma. Consequently, we see the isomorphism $W(E_6) \cong O_6^-(2) : 2$ since all of our matrices preserve the non-singular quadratic form $\sum_{i\neq j} x_i x_j$.

7.2. $W(E_7)$

Similarly, we obtain a representation of $2 \times O_7(2)$ in the E_7 case, accepting that the central involution must clearly act trivially here. In this case, the matrices preserve the non-singular quadratic form defined by xJ_7y^{T}.

From [**16**, p. 110] we see that there is no irreducible $\mathbb{Z}_2$ representation of $O_7(2)$ in seven dimensions, and this is precisely what we find here. The matrices for the symmetric generators and the whole of the control group fix the vector $v := (1^7)$. The space $v^\perp$ thus gives us a six-dimensional $\mathbb{Z}_2$-module for this group to act on. It may be easily verified with the aid of Magma that this representation is irreducible.

Since the above form is symplectic when restricted to this subspace, we immediately recover the classical exceptional isomorphism $O_7(2) \cong S_6(2)$.

(It is worth noting that in both the E_6 and E_7 cases the symmetric generators may be interpreted as 'bifid maps' acting on the 27 lines of Schläfli's general cubic surface and Hesse's 28 bitangents to the plane quartic curve, respectively. See [**6**, pp. 26, 46] for details.)

7.3. $W(E_8)$

Similarly, we obtain a representation of $2^{\cdot}O_8^+(2)$ in the E_8 case, again accepting that the central involution must clearly act trivially. As in the E_6 case, the matrices preserve the non-singular quadratic form $\sum_{i\neq j} x_i x_j$.

Note that working in an even number of dimensions removes the irreducibility problem encountered with E_7, since the image of (1^8) under the action of a symmetric generator is of the form $(0^3, 1^5)$.

Remark 7.1. Here we have focused our attention on the simply laced Coxeter groups. Analogous results may be obtained for other Coxeter groups, but are much less enlightening. For example,

$$\frac{2^{\star 2n} : W(B_{n-1})}{(t_1(12)(n+1, n+2))^3} \cong W(B_n),$$

$$\frac{2^{\star n} : S_n}{(t_1(12))^5} \cong W(H_n) \quad \text{for } n = 3, 4.$$

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