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**Key words:** Double-Blind Dutch Auction, Poisson processes, demand and supply, alternating renewal processes

**Abstract**: In this paper, we re-examine Mendelson's model for the equilibrium price of a double-blind Dutch auction with Poisson-distributed stochastic demand and supply. We present a number of new results. We focus on the various ways that demand and supply cross. We identify four different categories of crossing, extending Mendelson's results which are based on a single category of crossing. Secondly, conditioning on quantity, we derive the joint distribution of the relevant demand and supply prices associated with such two-sided markets originally described by Bohm-Bawerk (1891). The distributional result is extended to the case where the limit orders on different sides of the market arrive at different rates. Finally, we derive the distributional properties of the price elasticities.

## 1. Introduction

This paper provides a detailed analysis of an important micro-structure model based on the analysis of a double-blind Dutch auction. The (single) Dutch Auction is different from a standard auction in that the auction starts with the highest selling price for the asset, then successively lowers it until a bidder accepts the price. MacAfee and McMillan (1987) attest that "The Dutch Auction is used, for example, for selling cut flowers in the Netherlands, fish in Israel and tobacco in Canada." Double-blind Dutch auctions are where buying and selling prices ascend or descend and where prices are constructed without prior knowledge of the other prices.

Whilst this structure is not frequently used in financial markets, it has been used as a structure for modelling market fragmentation and consolidation, see Mendelson (1987). Furthermore, it has wider relevance, the notion used here of market-clearing equilibrium is much more in line with how economists think about markets; these two-sided market models have a central role in a great deal of market theory. We can but quote Shapley and Shubik (1971); "Two-sided market models are important, as COURNOT, EDGEWORTH, BOHM-BAWERK and others have observed, not only for the insights they may give into more general economic situations with many types of traders, consumers, and producers, but also for the simple reason that in real life many markets and most actual transactions are in fact bilateral--i.e. bring together a buyer and a seller of a single commodity."

Our starting point is that both demand and supply are generated by buy and sell orders for one unit of commodity; the numbers of which arriving in time up to $T$ are generated by independent Poisson processes with a common parameter $\lambda$. The framework follows Mendelson (1982). The impact of his assumptions is that these processes become homogeneous spatial Poisson processes (see Cressie, 1993) in price space where the analogue of the inter-arrival times is the non-negative change in price for supply and minus the non-positive price decrease for demand. Equilibrium occurs when the two paths cross. Mendelson solves for the volume (quantity) and the prices in equilibrium.

There are a number of reasons why we revisit his particular model; firstly the original paper provides an accomplished and interesting analysis in its own right. However, it is rather terse and difficult to follow in parts; by extending and spelling out in detail some of the technical arguments, we hope to make it more accessible. Secondly, market fragmentation has become an important topic in recent times and, as mentioned above, one of the few fragmentation models, also by Mendelson (1987), builds directly on this structure. Thirdly, there is an expanding literature on modelling trading processes, which is also based on Poisson processes. This includes research on limit order markets, for an example, see Parlour (1998). This aims to capture the nature of contemporary trading venues, whereby traders submit a flow of orders, often cancelling some of them immediately. Trading is determined by the stock of live orders, the book, plus the arrival of further orders. Here the emphasis is on price discovery and strategic equilibrium, rather than market-clearing price equilibrium, as in Mendelson. However, there are numerous similarities of approach, which we shall comment on throughout our paper. Fourthly, there is an emerging area of financial econometrics based on work by Easley et. al. (2008) , which uses Poisson trade arrivals in the modelling of the probability of information-based trading (PIN).

In what follows, we assume, as does Mendelson, that we start with a bid (the highest), then we follow with a sell (the lowest) and so on, working our way along the respective curves, until the paths cross. In price space, we have what is called an alternating renewal process; these processes are widely used in reliability analysis. Where we differ is that we look at different ways in which the curves cross. These different crossings lead to different marginal pairs of prices which bound the equilibrium price and in turn these different marginal pairs lead to different clearing –price ranges. In section 2 we analyse the different crossings. In section 3, we present results on the joint distribution of the equilibrium prices in the two cases of equal or different arrival rates. Section 4 looks at the distribution of the price elasticities; section 5 presents some numerical results, and conclusions are presented in section 6.

## 2. Demand and Supply and Market Clearing

We consider a Poisson realisation of demand and supply; our interest is the range of possible prices where the curves cross. Neither ourselves, nor Mendelson discuss the case where limit bids are to the left of limit asks but the curves do not cross; this warrants further analysis. Assuming that the number of limit orders is *k*, which will be the quantity traded, we can envisage four prices, corresponding to moving up the supply curve and down the demand curve; $P_{sk} < P_{sk+1}$ and $P_{dk} > P_{dk+1}$. In Figure 1(below), we show the four possible orderings of these prices subject to the above constraints.

The situation considered by Mendelson is Figure 1; case(a), see Mendelson (1982, Figure 1, page 1508). However, the other three cases occur with finite probability and have equal economic meaning.

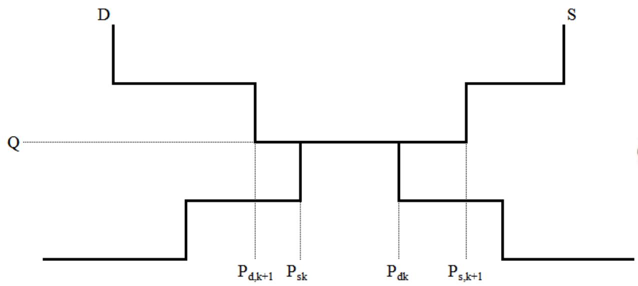


Fig. 1 case (a)

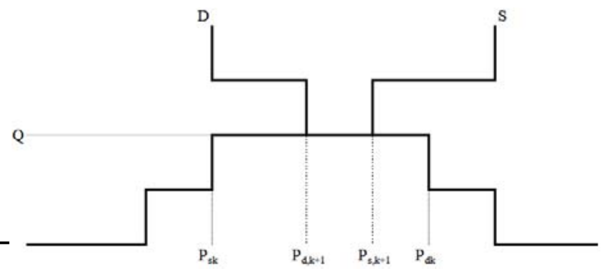


Fig. 1 case (b)

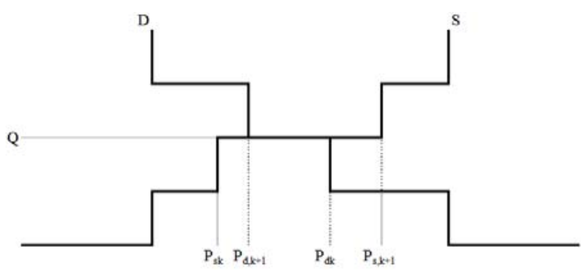


Fig. 1 case (c)

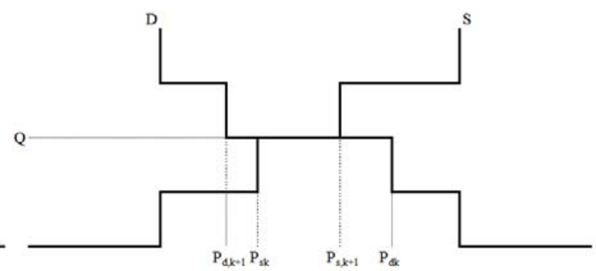


Fig. 1 case (d)

Using the properties of the Poisson process, we consider the probabilities that randomly-generated demand and supply will have market-clearing of these four kinds. We present these later but first we need the joint *pdf* of the four prices.

There is an issue, see Mendelson,(1982, footnote 9) to do with the treatment of prices which are restricted to lie in (0, *m*) but which are not restricted if we assume inter-arrival times to be negative exponential. It is worth thinking about what this might mean in terms of auctioneer behaviour. Restricting offers in excess of m seems intuitively reasonable as they add nothing .A similar thing could be said for bids that are negative. Precisely how the auctioneer treats these will have statistical implications. If she discards them, we are, in effect, truncating the price distributions. If she scales them back to be m for offers and puts the bids at zero, we will have censoring. Mendelson does not discuss this point explicitly but we will show, by examination of one of his results, that at least some of the time, he assumes censoring. This is mentioned in the discussion below figure 5, page 1516, where he refers to the probability mass of the execution price. We shall first provide, in proposition 1, a calculation for a quantity he refers to as the expected value of the market-clearing price range. In the notation below this is $E(Y\text{-}y)$. Later these variables are referred to as $U$ and $L$ , to help comparison with the original paper,and represent upper and lower values for the market-clearing price, more detail is given at the start of section 3.

Let $x = P_{sk+1}, X = P_{dk+1}, y = P_{sk}, Y = P_{dk}$ .We wish to calculate the joint *pdf*' of the four variables, conditional that the volume traded is $k$. We call this *pdf*($x$, $X$, $y$, $Y$); this can be thought of as a truncated density (differing from Mendelson) as any bids in excess of $m$ or asks less than 0 are ignored and our *pdf* is re-scaled to integrate to 1. Before we progress to the calculation of this *pdf*, we note that assumptions about censoring or truncation are not innocuous. Proposition 1 shows the impact of different assumptions on the expected market –clearing price range.

**Proposition 1**

The expected market –clearing price range under censoring is

$$E_c(\Delta) = (1 - \exp(-2\lambda m)) / 2\lambda$$

The expected market –clearing price range under truncation is

$$E_T(\Delta) = (1 - \exp(-2\lambda m) - 2\lambda m \exp(-\lambda m)) / 2\lambda(1 - \exp(-\lambda m)).$$

Furthermore, $E_c(\Delta) - E_T(\Delta)$ is always non-negative.

**Proof**: Under censoring we note that the result is given by Mendelson, see his theorem 4.1.

The impact of censoring is that $P(L \leq 0) = P(m < U) = \exp(-\lambda m)$.

It therefore follows that

$$E_T(U) = (E_c(U) - m \exp(-\lambda m)) / (1 - \exp(-\lambda m))$$
$$E_T(L) = E_c(L) / (1 - \exp(-\lambda m))$$

$$E_T(U - L) = (E_c(U) - E_c(L) - m \exp(-\lambda m)) / (1 - \exp(-\lambda m))$$

The formula follows from substituting in the censoring result and simplifying.

The inequality can be easily established by noting that the two terms are equal when $m = 0$ but the censoring term has a larger derivative in $m$ for all positive $m$.

**Proposition 2**

The truncated joint density $pdf(x, X, y, Y|k)$ is given by the following, subject to the necessary inequality $X \le x, y \le Y$ corresponding to moving along the curves plus the constraint imposed by $Q = k$, $Y > X$, $y < x$.

$$pdf(x, X, y, Y|k) = \lambda^2 c^2 y^{k-1}(m-Y)^{k-1} e^{-\lambda x} e^{-\lambda(m-X)}$$

where $c = \lambda^k/\gamma(k, \lambda m)$ and $\gamma(k, \lambda m)$ is the incomplete gamma function.

Proof: See Appendix.

We could, of course, provide a censored joint *pdf* as well. This turns out to be very messy as can seen by Proposition 3.

**Proposition 3**

The censored joint density $pdf(x, X, y, Y|)$ is given by the following, subject to the necessary inequality $X \le x, y \le Y$ corresponding to moving along the curves plus the constraint imposed by $Q = k$, $Y > X$, $y < x$. Under censoring we have that

$$pdf(x, y|k) = \frac{\lambda^{k+1}}{\gamma(k,\lambda m)} y^{k-1} e^{-\lambda x}, \quad 0 < y < x < m$$

$$= \frac{\lambda^{k}}{\gamma(k,\lambda m)} y^{k-1} e^{-\lambda m}, \quad x = m,\ 0 < y < m$$

and

$$pdf(X, Y|k) = \frac{\lambda^{k+1}}{\gamma(k,\lambda m)} (m-Y)^{k-1} e^{-\lambda(m-X)}, \quad 0 < X < Y < m$$

$$= \frac{\lambda^{k}}{\gamma(k,\lambda m)} (m-Y)^{k-1} e^{-\lambda m}, \quad X = 0,\ 0 < Y < m$$

Therefore, the joint $pdf(x, X, y, Y|k)$ is given by the following four expressions:

$$pdf(x, X, y, Y|k) = \lambda^2 c^2 y^{k-1}(m-Y)^{k-1} e^{-\lambda x} e^{-\lambda(m-X)}, \quad 0 < y < x < m,\ 0 < X < Y < m$$

$$= \lambda c^2 e^{-\lambda m} y^{k-1}(m-Y)^{k-1} e^{-\lambda x}, \quad 0 < y < x < m,\ X = 0,\ 0 < Y < m$$

$$= \lambda c^2 e^{-\lambda m} y^{k-1}(m-Y)^{k-1} e^{-\lambda(m-X)}, \quad x = m,\ 0 < y < m,\ 0 < X < Y < m$$

$$= c^2 e^{-\lambda m} y^{k-1}(m-Y)^{k-1}, \qquad x = m,\ 0 < y < m,\ X = 0,\ 0 < Y < m$$

where $c = \dfrac{\lambda^k}{\gamma(k, \lambda m)}$.

**Remark** It is worth noting that our bivariate censored *pdf* has two components. When we consider truncated *pdfs* in one dimension, our results simplify as above and we have only one component. It seems clear that the censored *pdf* is much more complex, with no obvious gains in realism; this suggests to us, at least, that we should work, where appropriate, with truncated distributions. In fact, it seems that Mendelson sometimes does not impose both constraints, for example when discussing the distribution of quantities traded. We shall make the same assumptions for quantities traded.

Having computed the truncated density in proposition 2, it is now possible to calculate the probabilities of the different crossings; these are given in proposition 4.

**Proposition 4.**

The probabilities of state, *a*, *b*, *c*, and *d* , all conditional upon quantity *k*, are given by the following expressions $P(a|k)$, $P(b|k)$, $P(c|k)$, $P(d|k)$

$$P(a\,|\,k) = e^{-\lambda m} \frac{\left[ \sum_{j=0}^{\infty} \frac{(\lambda m)^j (k)_j}{j!(2k+1)_j} F_1(k, 2k+1+j; \lambda m) - 2\,{}_1F_1(k, 2k+1; \lambda m) + 1 \right]}{\left[ 1 + e^{-\lambda m} + \frac{\lambda m}{2k+1} - 2\,{}_1F_1(k, 2k+1; -\lambda m) \right]}$$

$$P(b\,|\,k) = \frac{(\lambda m)^2 e^{-\lambda m}}{(2k+1)_2} \frac{\left[ \sum_{j=0}^{\infty} \frac{(\lambda m)^j (k+1)_j}{j!(2k+3)_j} {}_1F_1(k+1, 2k+3+j; \lambda m) \right]}{\left[ 1 + e^{-\lambda m} + \frac{\lambda m}{2k+1} - 2\,{}_1F_1(k, 2k+1; -\lambda m) \right]}$$

$$P(c\mid k)=\frac{(\lambda m)e^{-\lambda m}}{(2k+1)}\frac{\left[\sum_{j=0}^{\infty}\frac{(\lambda m)^{j}(k+1)_{j}}{j!(2k+2)_{j}}\,{}_1F_1(k,2k+2+j;\lambda m)-{}_1F_1(k+1,2k+2;\lambda m)\right]}{\left[1+e^{-\lambda m}+\frac{\lambda m}{2k+1}-2\,{}_1F_1(k,2k+1;-\lambda m)\right]}$$

$= P(d|k)$

Proof: See Appendix. The ${}_1F_1$ function above is the confluent hypergeometric function, see Slater (1960).

We recall the expression for $Prob(Q = k)$ derived by Mendelson (1982, equation 3.3,1511). As already noted, he makes no censoring or truncating assumptions for this calculation. We shall follow his procedure without attempting to adjust it.

$$Prob(Q=k)=\exp(-\lambda m)\frac{(\lambda m)^{2k}}{\Gamma(2k+1)}(1+\frac{\lambda m}{2k+1}) \qquad (1)$$

For $k > 0$,

$Prob(Q = k) = Prob(Q = k, \text{state} = a) + Prob(Q = k, \text{state} = b) + Prob(Q = k, \text{state} = c) +$

$Prob(Q = k, \text{state} = d)$.

Furthermore, we can employ Bayes's rule to infer

$Prob(\text{state} = a|Q = k) = Prob(Q = k, \text{state} = a)/Prob(Q = k)$,

with corresponding results for the other states.

We now discuss extensions of our results when the arrival rates of buy and sell limit orders differ; we include this case because of its added realism.We derive the joint pdf under truncation in proposition 5.

**Proposition 5.**

Extending the result stated in Proposition 2 to the case where demand and supply have different rates of arrival given by $\lambda_d$ and $\lambda_s$ respectively, we have the joint truncated conditional $pdf(x, X, y, Y|k)$ given by

$$pdf(x, X, y, Y) \propto \frac{(\lambda_s \lambda_d)^{k+1}}{\gamma(k,\lambda_d m)\gamma(k,\lambda_s m)} y^{k-1} e^{-\lambda_s x(m-Y)^{k-1}} \cdot \exp(-\lambda_d (m-X))$$

for $0 < y < x < m$ and $0 < X < Y < m$

To help make comparisons with Mendelson's results, we present marginal pdfs for the case of different arrival rates. Results for his case can be found based on arguments presented on page 1521.

**Proposition 6.**

The marginal distributions of $L$ and $U$ (under censoring) which bound the market- clearing price in case (a), for unequal arrival rates are given by

$$f_l(L = x) = e^{-\lambda_d m}, \qquad \text{when } x = 0$$

$$= e^{-\lambda_d m}\left[ \lambda_d \sum_{k=0}^{\infty} \frac{(m\lambda_s m\lambda_d)^k}{\Gamma(2k+2)} e^{(\lambda_d - \lambda_s)x} \cdot \beta(k+1, k+1; x/m) \right.$$

$$\left. + \lambda_s \sum_{k=0}^{\infty} \frac{(m\lambda_s)^k (m\lambda_d)^{k+1}}{\Gamma(2k+3)} e^{(\lambda_d - \lambda_s)x} \cdot \beta(k+1; k+2; x/m) \right], \text{ for } 0 < x \le m$$

$$f_u(U = x) = e^{-\lambda_d m}\left[ \lambda_s \sum_{k=0}^{\infty} \frac{(m\lambda_s)^k (m\lambda_d)^k}{\Gamma(2k+2)} e^{(\lambda_d - \lambda_s)x} \beta(k+1, k+1; x/m) \right.$$

$$\left. + \lambda_d \sum_{k=0}^{\infty} \frac{(m\lambda_s)^{k+1} (m\lambda_d)^k}{\Gamma(2k+3)} e^{(\lambda_d - \lambda_s)x} \beta(k+2, k+1; (x/m)) \right], \text{ for } 0 < x \le m$$

$$= e^{-\lambda_s m}, \text{ when } x = m$$

Where $\beta(p,q;w) = \frac{1}{B(p,q)} w^{p-1}(1-w)^{q-1}$, the Beta *pdf* with $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

Once we have these two sets of results, we can, in principle, compare the two *pdf*'s for $U$ and $L$ in the four crossing case versus the ($a$) crossing case. It could well be possible to provide some analytic results here; we could easily present some numerical results for our various densities. However the issue of censoring versus truncation makes comparisons more complicated.

## 3. Prices

Here we can compute Δ, the clearing price range, where, as previously mentioned, we define *U* and *L* to be the upper and lower prices of the price equilibrium respectively. Both prices will depend upon the quantity traded, *Q*, both are clearing prices, as are any of the prices between them. In state(*a*), by inspection of Figure 1 in Mendelson,(1982), $L = P_{sk}$ and $U = P_{dk}$ . Inspecting the three other cases in Figure 1 in this paper, we see that the same relationships hold. The point to bear in mind is that these are limit orders, thus the buyer will buy at $P_{dk}$ or any price cheaper whilst the seller will sell at $P_{sk}$ or any price higher. The calculations for the clearing price range distribution, and the comparisons with Mendelson's results, are very lengthy and we shall leave them for a companion paper.

### The Elasticities of Supply and Demand

In this section, we provide some new results on the elasticity of supply and demand. Since we are computing elasticities at arbitrary points on the demand and supply curves and we do not consider issues of truncation; we use the symbol *i* for an arbitrary quantity(not necessarily traded). Here, we take the quantity, *i*, as given and the price $p_i$ as random; the (inverse) elasticity of supply, $\varepsilon_s$ (with a corresponding definition for the inverse elasticity of demand, $\varepsilon_d$ ) will be defined as:

$$\varepsilon_s = \left. \frac{\Delta p_i}{p_{i-1}} \middle/ \frac{1}{(i-1)} \right.$$

Noting the independence of the price increments whose common distribution is negative exponential with parameter $\lambda_s$, which we denote by $NE_i(\lambda_s)$ ; for the *i*th price change, we see that the distribution of the elasticity of supply is that of the following ratio;

$$\frac{(i-1)NE_i(\lambda_s)}{\sum_{j=1}^{i-1} NE_j(\lambda_s)}$$

From the independence of the elements in the series, we see that the numerator and denominator are independent.

The moment-generating function of the numerator (divided by (i-1)) is given by

$$m_n(t) = 1/(1-\lambda_s t)$$

The moment-generating function of the denominator is given by

$$m_d(t) = 1/(1-\lambda_s t)^{i-1}$$

On inspection, we see that the distribution of the numerator (divided by i-1) is $\frac{\lambda_s}{2}$ times a chi-squared with 2 degrees of freedom. Likewise, the distribution of the denominator is $\frac{\lambda_s}{2}$ times a chi-squared with 2($i$-1) degrees of freedom. Putting these together, we see that $\varepsilon_s$ is distributed as $F(2,2(i-1))$. In particular, it does not depend upon $\lambda_s$, a surprising result. This is for the distribution of elasticity, conditional upon $i$; the unconditional result could be calculated, but it seems to have little economic content, being a value of elasticity, averaged over quantity

Using results for the inverse moments of the chi-squared distribution, we see that $E(\varepsilon_s)=1+\frac{1}{(i-2)}$, so that inverse elasticity of supply will on average be above 1 whilst inverse elasticity of demand will, on average, be below-1. Furthermore, Var($\varepsilon_s$) will be O($\frac{1}{i}$). Thus as we move up the supply curve and back up the demand curve our inverse elasticities converge in mean square, and hence probability, to 1 and -1 respectively. Finally, if we wanted the distribution of elasticity, rather than inverse elasticity, they are easily recoverable since the inverse of $F(n, m)$ is distributed as $F(m, n)$.

**5. Numerical Results**

We present some illustrative results for the parameter $\lambda = .1$ and $m = 20$, and small values of $k$ from 1 to 5, see Table 2.We note in passing that

$$E(Q) = \frac{\lambda m}{2} - (1-\exp(-2\lambda m))/4$$

And thus, for this example, $E(Q)$=.754.

**Table 2**

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| P(Q=k\|a) | 0.6030067156 | 0.3365814773 | 0.05569860961 | 0.004490380057 | 0.0002157514455 |
| P(Q=k\|b) | 0.8646276964 | 0.1254818855 | 0.00944194220 | 0.000434657865 | 0.0000135093537 |
| P(Q=k\|c) | 0.7589533435 | 0.2155159849 | 0.02401154626 | 0.001461167516 | 0.0000564181404 |

We can discern from Table 2 that for $k = 1$, $P(Q = k|a)$ is not the highest conditional but this changes as $k$ increases. This can be further illustrated by using Bayes rule as discussed above and graphing all four functions in terms of $k$, i.e., *Prob*(state|$k$) see Figure 2, below.

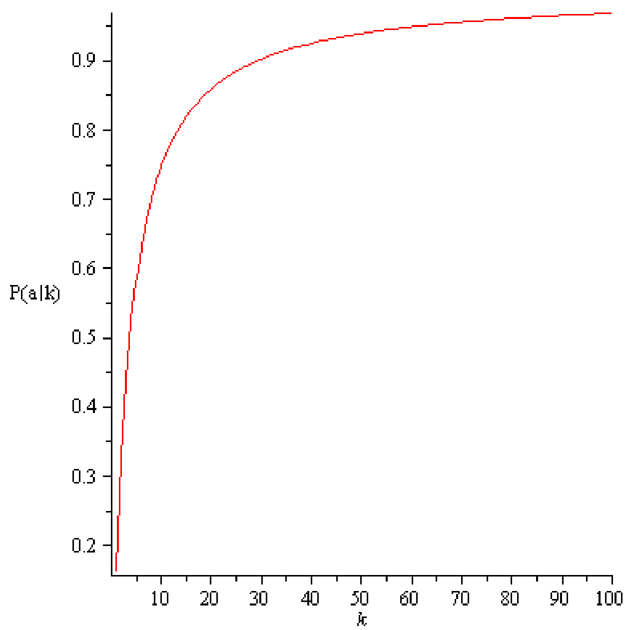


**Figure 2(a)**

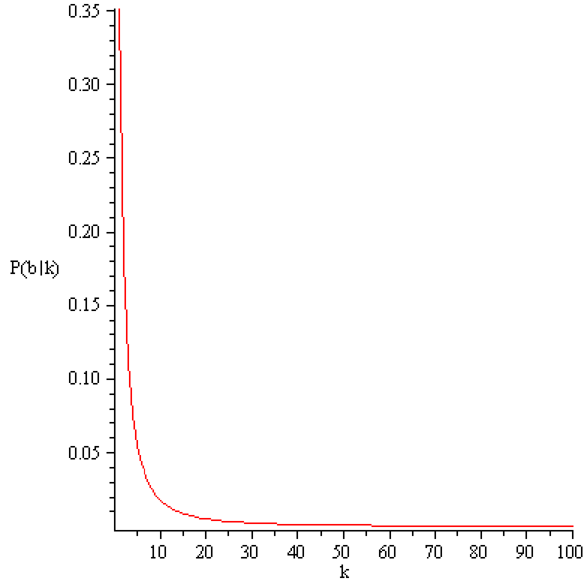


**Figure 2(b)**

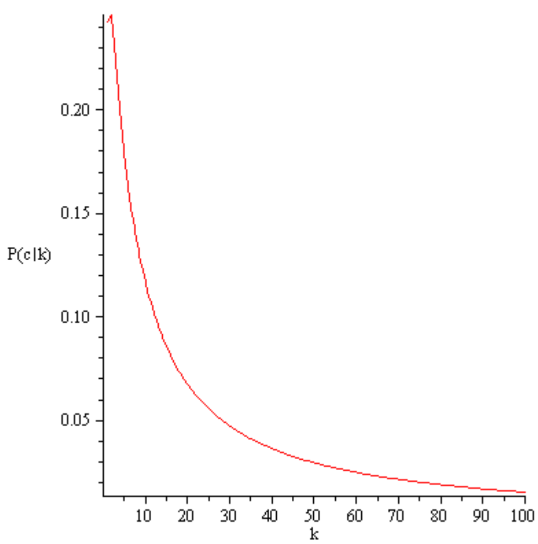


**Figure 2(c)**

As $k$ tends to infinity, the probability of state $a$, conditional on $k$ increases whilst the other three states all decrease; thus we would expect Mendelson's model to be highly accurate for large $k$, even in our general case of four states. However, large $k$, especially relative to small $\lambda m$, is a low probability event. The most pertinent calculation is that of the unconditional probabilities of the four states which can be easily calculated from Proposition 2 and equation 1. For the parameters, $\lambda$ =.1 and $m$ = 20, we find that $Prob(a)$ = .122, $Prob(b)$ =.183, $Prob(c)$ = $Prob(d)$ =.144,whilst

$Prob(Q = 0) = .406$. Thus, for this example, which is based on an example used by Mendelson, state($a$) is the least likely.

## 6. Conclusions

We have demonstrated in this paper that Mendelson's model of double-blind Dutch auctions with Poisson arrivals can be analysed in terms of four different ways in which demand and supply curves cross. We have shown that the probabilities of the three "new" states are non-negligible. Indeed, for the example when $\lambda = .1$ and $m = 20$ we might expect, on average, 2 buy and 2 sell orders per day (unit time). For this case, state (a) is the event least likely relative to the other three states or the probability of no trade. For the second case considered by Mendelsohn, where $\lambda = 1$ and m= 20, Prob (Q=0) is approximately 0, and all four states are approximately equally likely.

Furthermore analysis will depend upon the assumptions we make as to how the auctioneer treats extremely large limit order offers and extremely small limit order bids .We argue for a truncation approach rather than a censoring one favoured by Mendelson. This leads to considerable simplifications in the joint distribution of prices. We also demonstrate numerically that as volume traded increases the Mendelson state becomes the most likely but the probability of high volume in the context of Poisson demand and supply is likely to be very small. These findings will have an impact on market-clearing prices which we leave for a subsequent paper. Finally we derive the distribution of demand and supply price elasticity and show that, conditional on quantity, they follow $F$ distributions.

## Bibliography

Bohm-Bawerk, E.V., *The Positive Theory of Capital*, Macmillan, London, 1891.

Cressie, Noel A. C., *Statistics for Spatial Data*, revised edition, New York: John Wiley & Sons, Inc. (1993).

Easley, D., R. Engle, M. O'Hara, and Liuren Wu, "Time-Varying Arrival Rates of Informed and Uninformed Trades," *Journal of Financial Economics*, 171-207 (2008).

Mendelson, H., "Market Behaviour in a Clearing House," *Econometrica*, Vol. 50, No. 6 1505-1524 (Nov 1982).

Mendelson, H., "Consolidation, fragmentation, and market performance," *Journal of Quantitative and Financial Analysis*, Vol. 22, No. 2, 189-207 (June 1987).

Parlour, Ch. A., "Price dynamics in limit order markets," *Review of Financial Studies* 11(4) 789–816 (1998).

Shapley, L.S. and Shubik, M. "The assignment Game I: The core," *International Journal of Game Theory,*112-130 (1972).

Slater,L. *Confluent Hypergeometric Functions*; Cambridge University Press, Cambridge.(1960).

## Appendix

### Proof of Proposition 2

Since $x = P_{sk+1}$, $y = P_{sk}$, $X = P_{dk+1}$, $Y = P_{dk}$ and these prices are the occurrence time of our respective spatial Poisson processes we have:

$x = y + s_{k+1}$ and $X = Y - d_{k+1}$

with $s_{k+1}$ and $d_{k+1}$ *iid* $EX(\lambda)$. Therefore

$$pdf(x|y) \alpha\, \lambda\, exp(-\lambda(x - y)), \quad y < x < m$$

and

$$pdf(y) = \frac{\lambda^k}{\gamma(k, \lambda m)} y^{k-1} e^{-\lambda y},\ 0 < y < m$$

where

$$\gamma(n, z) = \int_0^z w^{n-1} e^{-w} dw,$$

the incomplete Gamma function. Consequently,

$$pdf(x,y) = pdf(x|y)pdf(y)$$

$$\alpha \frac{\lambda^{k+1}}{\gamma(k, \lambda m)} y^{k-1} e^{-\lambda x}, \quad 0 < y < x < m$$

For demand we have

$$Y = P_{dk} = m - \Sigma\, dj$$

resulting in

$$X = P_{dk+1} = Y - d_{k+1}.$$

Consequently,

$$pdf(X|Y)\ \alpha\ \lambda\, exp(-\lambda(Y - X))$$

and

$$pdf\,(Y) = \frac{\lambda^k}{\gamma(k,\lambda m)}(m-y)^{k-1}\exp(-\lambda(m-y)), \quad 0 < y < m$$

Thus

$$pdf\,(X,Y)\alpha\frac{\lambda^{k+1}}{\gamma(k,\lambda m)}(m-y)^{k-1}\exp(-\lambda(m-X))$$

and since demand and supply are independent we have

$$pdf\,(x,X,y,Y)\ \alpha\,\frac{\lambda^{2(k+1)}}{\gamma^2(k,\lambda m)}y^{k-1}e^{-\lambda x}(m-y)^{k-1}\exp(-\lambda(m-X))\ ,\ 0< y < x < m,\ 0 < X < Y < m$$

**Proof of Proposition 3**

Proceeding as in the proof of Proposition 2 we have

$$x = y + s_{k+1}$$

giving

$$pdf(x|y) \quad = \quad \lambda\exp(-\lambda(x-y)), \qquad y < x < m$$

and

$$pdf(x = m|y) \quad = \quad \int_m^\infty \lambda\exp(-\lambda(x-y))dx$$

$$= \quad \exp(-\lambda(m-y)), \qquad 0 < y < m$$

Thus

$$pdf(x|y) \quad = \quad \lambda\exp(-\lambda(x-y)), \qquad 0 < y < x < m$$

$$= \quad \exp(-\lambda(m-y)), \qquad x = m,\ 0 < y < m$$

and since

$$pdf(y, y < m) = \frac{\lambda^k}{\gamma(k,m\lambda)}y^{k-1}e^{-\lambda y}, \qquad 0 < y < m$$

we have

$$pdf(x, y) = pdf(x|y)pdf(y)$$

$$= \frac{\lambda^{k+1}}{\gamma(k, m\lambda)} y^{k-1} e^{-\lambda x}, \qquad 0 < y < x < m$$

$$= \frac{\lambda^{k}}{\gamma(k, m\lambda)} y^{k-1} e^{-\lambda m}, \qquad x = m,\ 0 < y < m$$

Similarly, since

$$X = Y - d_{k+1}$$

we have

$$pdf(X|Y) = \lambda \exp(-\lambda(Y - X)), \qquad 0 < X < Y < m$$

$$pdf(X = 0|\mathrm{Y}) = \int_Y^\infty \lambda \exp(-\lambda w) dx = \exp(-\lambda Y)$$

Thus

$$pdf(X|Y) = \lambda \exp(-\lambda(Y - X)), \qquad 0 < X < Y < m$$

$$= \exp(-\lambda Y), \qquad X = 0,\ 0 < Y < m$$

and since

$$pdf(Y, Y < m) = \frac{\lambda^{k}}{\gamma(k, m\lambda)} (m - y)^{k-1} e^{-\lambda(m-Y)}$$

we have

$$pdf(X, Y) = \frac{\lambda^{k+1}}{\gamma(k, m\lambda)} (m - y)^{k-1} \exp(-\lambda(m - X)), \quad 0 < X < Y < m$$

$$= \frac{\lambda^{k}}{\gamma(k, m\lambda)} e^{-\lambda m}(m - Y)^{k-1}, \qquad X = 0,\ 0 < Y < m$$

Finally

$$pdf(x, X, y, Y) = pdf(x,y) \cdot pdf(X,Y)$$

giving the four expressions stated in the theorems.

**Proof of Proposition 4**

The various probabilities are readily found by straightforward but tedious integration of the joint *pdf*($x$, $X$, $y$, $Y$) as specified in Proposition 2 and subject to the particular ordering of $X$, $X$, $y$, $Y$ for the respective crossing. Thus

$$P(a|k) = \int_0^m \int_0^x \int_0^Y \int_0^y pdf(x, X, y, Y) dX dy dY dx$$

$$P(b|k) = \int_0^m \int_0^Y \int_0^x \int_0^X pdf(x, X, y, Y) dy dX dx dY$$

$$P(c|k) = \int_0^m \int_0^x \int_0^Y \int_0^X pdf(x, X, y, Y) dy dX dY dx$$

$$= P(d|k)$$

**Proof of Proposition 5**

We proceed as in the proof of Proposition 2 except we now allow $s_{k+1}$ and $d_{k+1}$ to be distributed, respectively, as $EX(\lambda_s)$ and $EX(\lambda_d)$. Thus

$$pdf(x|y) \propto \lambda_s \; e^{-\lambda_s(x-y)}, \qquad y < x < m$$

with

$$pdf(y) = \frac{\lambda_s^k}{\gamma(k, \lambda_s m)} y^{k-1} e^{-\lambda_s y} \qquad 0 < y < m$$

Thus

$$pdf(x,y) \propto \frac{\lambda_s^{k+1}}{\gamma(k, \lambda_s m)} y^{k-1} e^{-\lambda_s x} \qquad 0 < y < x < m$$

and

$$pdf(X,Y) \propto \frac{\lambda_d^{k+1}}{\gamma(k, \lambda_d m)} (m-y)^{k-1} \exp(-\lambda_d (m-X)) \qquad 0 < X < Y < m$$

giving

$$pdf(x, X, y, Y) \propto \frac{(\lambda_s \lambda_d)^{k+1}}{\gamma(k, \lambda_d m)\gamma(k, \lambda_s m)} y^{k-1} e^{-\lambda_s x (m-Y)^{k-1}} \cdot \exp(-\lambda_d (m - X))$$

$0 < y < x < m$ and $0 < X < Y < m$

**Proof of Proposition 6**

From Figure 1 in Mendelson (1982) it is clear that

$L > x \Leftrightarrow S(x) = k$ and $D(m - x) = j,\ j \geq k + 1$

Consequently

$$P(L > x) = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} P(S(x) = k,\ D(m - x) = j)$$

$$= \sum_{k=0}^{\infty} P(S(x) = k) \sum_{j=k+1}^{\infty} P(D(m - x) = j)$$

$$= e^{-\lambda_d m + (\lambda_d - \lambda_s)x} \sum_{k=0}^{\infty} \frac{(\lambda_s x)^k}{k!} \sum_{j=k+1}^{\infty} \frac{(\lambda_d (m - x))^j}{j!}$$

Letting

$a = \lambda_s x$

$b = \lambda_d m - (\lambda_d - \lambda_s) x$

with $(b - a) = \lambda_d (m - x)$

we have

$$P(L > x) = e^{-b} \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{j=k+1}^{\infty} \frac{(b - a)^j}{j!}$$

Now

$$\sum_{j=k+1}^{\infty} \frac{(b - a)^j}{j!} = e^{b-a} - \sum_{j=0}^{k} \frac{(b - a)^j}{j!}$$

$$= e^{b-a}(1-e^{-(b-a)}\sum_{j=0}^{k}\frac{(b-a)^j}{j!})$$

Thus

$$P(L > x) = 1 - e^{-b}\sum_{k=0}^{\infty}\frac{a^k}{k!}\sum_{j=0}^{k}\frac{(b-a)^j}{j!}$$

Giving

$$P(L \le x) = e^{-b}\sum_{k=0}^{\infty}\frac{a^k}{k!}\sum_{j=0}^{k}\frac{(b-a)^j}{j!}$$

And we notice immediately that

$$P(L \le 0) = e^{-\lambda_d m}$$

and$P(L \leq m) = 1$

and further, since $0 \leq L \leq m$ we have a point mass at $x = 0$ of $e^{-\lambda_d m}$.

The pdf, $f_l(x)$ is readily found via differentiation

i.e., $f_l(x) = \frac{\partial}{\partial x} P(L \le x)$

and since

$$\frac{\partial a}{\partial x} = \lambda_s, \quad \frac{\partial b}{\partial x} = -(\lambda_d - \lambda_s) \text{ and } \frac{\partial (b-a)}{\partial x} = -\lambda_d$$

We have, after some simplification

$$f_l(x) = \lambda_d e^{-b}\sum_{k=0}^{\infty}\frac{a^k(b-a)^k}{k!k!} + \lambda_s e^{-b}\sum_{k=0}^{\infty}\frac{a^k(b-a)^{k+1}}{k!(k+1)!}$$

Recalling the definitions of $a$ and $b$ we note

$$\frac{a^k(b-a)^k}{\Gamma(k+1)\Gamma(k+1)} = \frac{\lambda_s^k \lambda_d^k m^{2k}}{\Gamma(2k+2)} \cdot \beta(k+1, k+1; x/m)$$

where $\beta(p,q;z) = \dfrac{1}{B(p,q)} \cdot z^{p-1}(1-z)^{q-1}$, the Beta pdf

with $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

Using a similar approach for the second sum we get the expression for $f_l(x)$ given in the theorem.

For the distribution of $U$ we now note that

$$U \le x \Leftrightarrow S(x) = j \text{ and } D(m-x) = k,\ j \ge k+1$$

Thus

$$P(U < x) = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} P(S(x) = j,\ D(m-x) = k)$$

and with the same definitions of $a$ and $b$ we have

$$P(U < x) = 1 - e^{-b} \sum_{k=0}^{\infty} \frac{(b-a)^k}{k!} \sum_{j=0}^{k} \frac{a^j}{j!}$$

giving

$$P(U < 0) = 0$$

and $P(U < m) = 1 - e^{-\lambda_s m}$.

Further, since $P(U \le m) = 1$ we have a point mass at $x = m$ of $e^{-\lambda_s m}$.

Again we have

$$f_u(x) = \frac{\partial}{\partial x} P(U < x)$$

$$= \lambda_s e^{-b} \sum_{k=0}^{\infty} \frac{(b-a)^k a^k}{k!\,k!} + \lambda_d e^{-b} \sum_{k=0}^{\infty} \frac{(b-a)^k a^{k+1}}{k!(k+1)!}$$

which upon substitution for $a$ and $b$ we get the expression for $f_u(x)$ given in the theorem.