ISSN 1745-8587

Birkbeck
UNIVERSITY OF LONDON

School of Economics, Mathematics and Statistics

Birkbeck Working Papers in Economics & Finance

BWPEF 1203

Modeling Style Rotation: Switching and Re-Switching

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January 2012

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Keywords: Market dynamics, asset prices, style rotation, momentum investing.

Abstract:

The purpose of this paper is to investigate the dynamics and statistics of style rotation based on the Barberis-Shleifer model of style switching. Investors in stocks regard the forecasting of style-relative performance, especially style rotation, as highly desirable but difficult to achieve in practice. Whilst we do not claim to be able to do this in an empirical sense, we do provide a framework for addressing these issues. We develop some new results from the Barberis-Shleifer model which allows us to understand some of the time series properties of style relative price performance and determine the statistical properties of the time until a switch between styles. We apply our results to a set of empirical data to get estimates of some of the model parameters including the level of risk aversion of market participants.

1. Introduction.

Dynamic style rotation or "style switching" is one of those themes that is often addressed in conferences and sell-side papers but is, to our knowledge, fairly rarely implemented by practitioners. This reflects the difficulty involved in forecasting when value, growth, momentum, or indeed, some other style, may do well or badly. The idea one would want to implement is to determine when you would want to tilt your portfolio towards or away from a particular style before the market moved, that is to anticipate when one style starts or stops outperforming the other. Attempts to understand this using macro-economic conditioning variables, have been published, see, for example Black et al(2009), Zhang et al (2009), and these show some links between style returns and macro-economic variables, but these links usually lack clear theoretical motivation and do not provide accurate enough predictive power to encourage investment.

It is well understood that various styles have differing levels of autocorrelation over different time horizons; for example, momentum has a pattern which, broadly, seems to be negatively correlated over very short periods (short-term reversal), followed by positive correlation over medium periods of about a year, followed by negative autocorrelation over longer periods. The task we have set ourselves in this paper is not to explain the autocorrelation of individual stocks or factors but that of individual styles as well as their relative returns. More interestingly, we look for a model which is capable of providing a structure for not just when styles switch, but when they might also switch back or re-switch. From a theoretical perspective, the Barberis-Shleifer model ("BS Model"), based on market equilibrium between style switchers (or "momentum" traders) and rational agents and with a strong behavioral basis, provides a much more appealing framework in providing "micro-foundations" for this problem. We analyse this model, especially its time-series properties to develop some predictions about the expected time until a style switch as well as the autocorrelation structure of style relative returns. We then apply our finding to a set of empirical data, namely the returns on two popular styles: developed equities vs emerging equities, in order to derive estimates of some of the model parameters, including the level of risk aversion of fundamental traders.

In section 2, we briefly discuss the model and the dynamic equation that determines style relative returns in equilibrium. In section 3 we compute and examine the autocorrelation function of relative returns and the dynamic equation determining expected relative returns and solve these to determine when, and how many times, the autocorrelation function, as well as the relative return, changes sign and when we can expect the style relative returns to reverse sign. The later times are examined and its comparative statics reveal their dependence on model parameters. In section 4 we extend the model to understand the dynamic of prices, as opposed to returns, and demonstrate that, with one additional assumption, the prices in the BS Model follow a process similar to the process for relative returns. In section 5, we apply our results to returns f two popular "competing" styles: investment in developed vs emerging markets in the period from 1993 till 2011 and derive estimates of a number of model parameters including estimates of risk aversion of the fundamental traders, which in he model perform the market-making (or "clearing") function. The empirical data demonstrates that the times of increased levels of risk aversion broadly coincide with the times of negative returns of one or both styles, thus confirming the common intuition. Section 6 concludes the paper, with the references provided in section 7 and the proof of the most important analytical results provided in the Appendix.

2. The Model.

The BS Model considers two kinds of investors: "switchers", who allocate their resources to a particular style based on that style's past performance relative to other styles, and "fundamental traders", who act as arbitrageurs and try to prevent the price of an asset from deviating too far from what is expected on the basis of available information.

For simplicity, the model has only two styles although a multi-style generalisation can be easily accommodated. The model has $2n$ risky assets in fixed supply, and a risk-free asset – cash, in perfectly elastic supply with zero net return. All risky assets belong to one of the two styles, the first n risky assets are in style X and the other n risky assets belong to style Y. Each risky asset i is modeled as a claim on a single liquidating dividend $D_{i,T}$ to be paid at some later time T, with the eventual dividend being

$$D_{i,T} = D_{i,0} + \varepsilon_{i,1} + \ldots + \varepsilon_{i,T}, \tag{1}$$

where $\varepsilon_{i,t}$ represents news about the final cashflow released at time t.

The first group, "switchers", invest in a style based on an Exponentially-weighted Moving Average (or "EWMA") calculation of past relative returns of the two styles, the type of averaging widely used in technical analysis (refer e.g. to Achelis (2001)). In particular, the demand from "switchers" for shares of an asset *i* in style *X* is

$$N_{i,t}^{S} = \frac{1}{n}\left(A_X + \sum_{k=1}^{t-1} \theta^{k-1}\left(\frac{\Delta P_{X,t-k} - \Delta P_{Y,t-k}}{2}\right)\right), \quad (2)$$

where A_X and $\theta$are constants, with 0 <θ< 1. This parameter constraint is standard in EWMA and is uncontroversial. Here

$$\Delta P_{X,t} = P_{X,t} - P_{X,t-1} \quad \text{and} \quad \Delta P_{Y,t} = P_{Y,t} - P_{Y,t-1} \quad (3)$$

is the return on style *X* between time t– 1 and time *t*, and $P_{X,t}$ is defined as the average price of a share across all assets in style *X:*

$$P_{X,t} = \frac{1}{n}\sum_{i \in X} P_{i,t} \quad \text{and} \quad P_{Y,t} = \frac{1}{n}\sum_{j \in Y} P_{j,t} \quad (4)$$

Symmetrically, the demand from "switchers" for shares of an asset *j* in style *Y* is

$$N_{j,t}^{S} = \frac{1}{n}\left(A_Y + \sum_{k=1}^{t-1} \theta^{k-1}\left(\frac{\Delta P_{Y,t-k} - \Delta P_{X,t-k}}{2}\right)\right). \quad (5)$$

In their December 2000 version of the paper "Style investing" (refer to Barberis and Shleifer (2000)), the authors demonstrate formally how adaptive expectations combined with a constraint on overall equity holdings lead to an exponentially decaying demand feature like the one provided in (2) and (5).

The second group of investors, rational or "fundamental" investors, maximize expected utility of a usual kind $\mu_p - \frac{\lambda}{2}\sigma_p^2$, in particular they solve for

$$\max_{N_t} E_t^F(-\exp[-\gamma(W_t + N_t'(P_{t+1} - P_t))]), \quad (6)$$

Where

$$N_t = (N_{1,t}, \dots, N_{2n,t})'$$

$$P_t = \left(P_{1,t}, \dots, P_{2n,t}\right)',$$

and where $N_{i,t}$ is the number of shares allocated to risky asset *i*, γ governs the degree of risk aversion of the fundamental traders, E_t^F denotes fundamental traders' expectations at time *t*, and W_t is time *t* wealth.

If fundamental traders assume a Normal distribution for conditional price changes, optimal holding N_t^F are given by

$$N_t^F = \frac{\left(V_t^F\right)^{-1}}{\gamma}\left(E_t^F(P_{t+1}) - P_t\right), \tag{7}$$

where

$$V_t^F = \mathrm{var}_t^F(P_{t+1} - P_t), \tag{8}$$

with the F superscript denoting a forecast made by fundamental traders.

The fundamental traders serve as market makers and treat the demand from switchers as a supply shock. If the total supply of the *2n* assets is given by the vector *Q*, equation (7) implies

$$P_t = E_t^F(P_{t+1}) - \gamma V_t^F(Q - N_t^F). \tag{9}$$

As shown in the Barberis-Shleifer article, for a particular form of *V* conjectured by fundamental traders, which is

$$V^{ij} = \begin{cases} \sigma^2, \; i = j \\ \sigma^2\rho_1, i, j \text{ in the same style and } i \neq j \\ \sigma^2\rho_2, \; i, j \text{ in different styles,} \end{cases} \tag{10}$$

this simplifies even further. Up to a constant, the price of an asset *i* in style *X* is

$$P_{i,t} = D_{i,t} + \gamma\sigma^2(1 - \rho_1 + n(\rho_1 - \rho_2))\frac{N_{X,t}^S}{n} = \tag{11}$$

$$D_{i,t} + \frac{1}{\phi}\sum_{k=1}^{t-1} \theta^{k-1} \left(\frac{\Delta P_{X,t-k} - \Delta P_{Y,t-k}}{2}\right), \tag{12}$$

where

$$\phi = \frac{n}{\gamma\sigma^2(1-\rho_1 + n(\rho_1 - \rho_2))}, \tag{13}$$

which is positive and likely to be larger than 1 for large n. The price of an asset *j* in style *Y* is

$$P_{j,t} = D_{\mathrm{j},t} + \frac{1}{\phi}\sum_{k=1}^{t-1} \theta^{k-1} \left(\frac{\Delta P_{Y,t-k} - \Delta P_{X,t-k}}{2}\right). \tag{14}$$

Furthermore, equations (12) and (14) can be aggregated over all stocks in each style using equations (4). These equations are fundamental to the BS Model. They show that the equilibrium prices of assets in the model deviate from $D_{i,t}$and $D_{j,t}$, which are the prices based purely on "fundamentals", by the amount based on demand from the "switchers", the traders who follow momentum investing. The degree of such deviation is driven by two parameters: θ, "persistence" or the degree of decay of the demand from "switchers", and ϕ, a parameter relating to the characteristics of demand from the "fundamental" traders. As it can be seen from (13), this parameter in turn is determined largely by γ, the degree of risk aversion of the "fundamental" traders. It is clear from equations (12) and (14) that the deviation of prices from their fundamental values is smaller if

(i) θ is smaller, i.e. the demand from "switchers" decays faster with time, or

(ii) γ is smaller, i.e. the "fundamental" traders are less risk averse and are willing to commit more of their private wealth to eliminating the arbitrage opportunity caused by the demand from "switchers".

It follows from equations (12) and (14), after aggregating these equations over all stocks in their respective styles, that the excess return of style *X* over style *Y* in period t +1, which we denote as Y_{t+1}, can be expressed as

$$Y_{t+1} = \Delta P_{X,t+1} - \Delta P_{Y,t+1} = (\varepsilon_{X,t+1} - \varepsilon_{Y,t+1}) + \frac{\Delta P_{X,t} - \Delta P_{Y,t}}{\phi}$$

$$-\frac{(1-\theta)}{\phi}\sum_{k=1}^{t-1}\theta^{k-1}(\Delta P_{X,t-k} - \Delta P_{Y,t-k}), \qquad (15)$$

which in turn implies the following times-series model for excess return between styles:

$$Y_t = \left(\theta + \frac{1}{\phi}\right)Y_{t-1} - \frac{1}{\phi}Y_{t-2} + \varepsilon_t - \theta\varepsilon_{t-1} \qquad (16)$$

As it is clear from (16), when the market is cleared, the resulting prices turn out to follow an ARMA(2,1) model with restrictions on coefficients. According to the standard time-series theory, Y_t is a stable process as long as the roots of the auxiliary equation

$$\alpha^2 - \alpha\left(\theta + \frac{1}{\varphi}\right) + \frac{1}{\varphi} = 0 \qquad (17)$$

are all less than one in absolute magnitude. As pointed out in the Barberis-Shleifer article, within the range $\theta > 0, \varphi > 1$ this will be true as long as

$$0 < \theta < 1, \varphi > 1. \qquad (17^*)$$

Here the white noise innovation in equation (16), ε_t, is defined as

$$\varepsilon_t = \varepsilon_X - \varepsilon_Y, \qquad (18)$$

with $\varepsilon_{X,t} = \frac{1}{n}\sum_{i\in X}\varepsilon_{i,t}$ and $\varepsilon_{Y,t} = \frac{1}{n}\sum_{j\in Y}\varepsilon_{j,t}$, and is assumed to be distributed as $N(0,\text{var}(\varepsilon_t))$, with $\text{var}(\varepsilon_t)$ to be easily found based on the definition in (4) and the following cash-flow covariance structure assumed in the BS Model:

$$\Sigma_D^{ij} = \begin{cases} 1, \ i = j \\ \psi_M^2 + \psi_S^2, i,j \text{ in the same style and } i \neq j \\ \psi_M^2, \ i,j \text{ in different styles.} \end{cases} \qquad (19)$$

Here constantsψ_M and ψ_S simply control the relative importance of the market-wide cash flow variance factor over the style-specific cash flow variance factor, with the asset's idiosyncratic variance factor having a weight of $\sqrt{(1-\psi_M^2+\psi_S^2)}$, as all assets are assumed to have the total cash flow news variance of precisely one. It should be noted that the covariance structure (19) is similar in form to the asset covariance structure (10) assumed by the "fundamental" traders. According to (17), the parameter φ is greater than 1 and θ lies between 0 and 1. These conditions, which follow from the economics of the model, imply that the resulting process is stationary.

3. Results.

3.1 Autocovariance Structure.

In this section we derive the autocorrelation function of the model given by (16). Using the following notation,

$$\Gamma_k = \mathrm{cov}(Y_t, Y_{t+k}) = \mathrm{cov}(\Delta P_{X,t} - \Delta P_{Y,t}, \Delta P_{X,t+k} - \Delta P_{Y,t+k}) \tag{20}$$

$$\hat{\Gamma}_k = \mathrm{cov}\big(\Delta P_{X,t}, \Delta P_{X,t+k}\big) = \frac{1}{4}\Gamma_k\ , \tag{21}$$

$$a \equiv \left(\theta + \frac{1}{\varphi}\right), b \equiv \frac{1}{\varphi} \tag{22}$$

The following results are proven in the Barberis-Shleifer article for the autocovariances at first three lags:

$$\Gamma_0 = a\Gamma_1 - b\Gamma_2 + \big(1 - b(a-b)\big)(2\psi_S^2 + k_0), \tag{23}$$

$$\Gamma_1(1+b) = a\Gamma_0 - (a-b)(2\psi_S^2 + k_0), \tag{24}$$

$$\Gamma_2 = a\Gamma_1 - b\Gamma_0, \tag{25}$$

$$\Gamma_1 = \frac{(2\psi_S^2 + k_0)(1+a-b)}{\left(\frac{1}{b} - 1\right)(1+a+b)}\quad , \tag{26}$$

where $k_0 \equiv \frac{2}{n}(1 - \psi_M^2 + \psi_S^2)$.

The article also shows that the autocovariance does turn negative at some unknown time lag but does not go into further details. We are interested in exploring the autocovariance structure of excess returns of one style over the other and, through the relationship (21), the autocovariance structure of returns on a single style. In particular, we would like to derive the general formula for the autocovariance structure at lag *k* as well as determining the lag k^* at which the autocovariance changes sign.

By computing the covariance of equation (16) with $Y_{t-k}$where $k \geq 2$, we have the following difference equation for Γ_k, autocovariance at lag *k:*

$$\Gamma_k = a\Gamma_{k-1} - b\Gamma_{k-2}, \qquad (27)$$

As we will see from the analysis below, the time-series dependence of the type provide by equation (27) is the key dependence in this model: it governs not only the dynamics of autocovariance but also dynamics of forecasted returns as well as the coefficients in the infinite moving average (MA) representation of the time series (16) for excess returns. The corresponding *auxiliary equation* is

$$\alpha^2 - a\alpha + b = 0, \qquad (28)$$

which is the same as equation (17) but now rewritten using definitions for *a* and *b*, and it's general solution is a sum of power functions of the two roots of equation (28):

$$\alpha_1 = \frac{a+\sqrt{a^2 - 4b}}{2}, \text{ and} \qquad (28^*)$$

$$\alpha_2 = \frac{a-\sqrt{a^2 - 4b}}{2}. \qquad (28^{**})$$

Depending on the relationship between *a* and *b*, which in turn are determined entirely by model parameters θ and φ, we have the case of either real (two distinct ones or a single one) or complex roots, depending on whether the discriminant of equation (28), $D = a^2 - 4b$, is positive, zero or negative. Remembering the definition of *a* and *b,* the case of real roots corresponds to the case $\theta \geq \frac{2}{\sqrt{\varphi}} - \frac{1}{\varphi}$ and the case of complex roots to the case $\theta < \frac{2}{\sqrt{\varphi}} - \frac{1}{\varphi}$. We start with the autocovariance function.

Proposition 1: The autocovariance function of the relative return process given by equation (16) has the following properties:

1. $\Gamma_0 = \frac{(2\psi_S^2 + k_0)(1+\frac{1}{b}+(a-b)(\frac{1}{b}-1))}{\left(\frac{1}{b}-1\right)(1+a+b)}$ (29)

2. *The autocovariance at lag k, where* $k \geq 0$ *,is determined as:*

(a) If $a^2 - 4b > 0$,

$$\Gamma_k = A_1\left(\frac{a+\sqrt{a^2-4b}}{2}\right)^k + A_2\left(\frac{a-\sqrt{a^2-4b}}{2}\right)^k, \quad (30)$$

$$\textit{where } A_1 = \frac{\Gamma_0}{2} + \frac{(\Gamma_1 - \frac{a\Gamma_0}{2})}{\sqrt{a^2-4b}}, A_2 = \frac{\Gamma_0}{2} - \frac{(\Gamma_1 - \frac{a\Gamma_0}{2})}{\sqrt{a^2-4b}} \quad (31)$$

(b) If $a^2 - 4b = 0$,

$$\Gamma_k = \left(\sqrt{b}\right)^k (A_3 + A_4 k), (30^*)$$

$$\textit{where } A_3 = \Gamma_0, A_4 = \frac{1}{\sqrt{b}}\Gamma_1 - \Gamma_0 \quad (31^*)$$

(c) If $a^2 - 4b < 0$,

$$\Gamma_k = \left(\sqrt{b}\right)^k (A_5 \cos(\gamma k) - A_6 \sin(\gamma k)), \quad (32)$$

$$\textit{where } A_5 = \Gamma_0, A_6 = \frac{(a\Gamma_0 - 2\Gamma_1)}{\sqrt{4b-a^2}} \quad (32^*)$$

$$\textit{and } \gamma = \arctan\left(\sqrt{\frac{4b}{a^2} - 1}\right) \quad (33)$$

It is clear from Proposition 1 that depending on whether the discriminant of (28) is positive, zero or negative, the solution for the autocovariance function at lag k is either a sum of two power functions in (a) or (b) or an oscillating solution in (c) with period $T = 2\pi/\gamma$ with γ determined in (33).

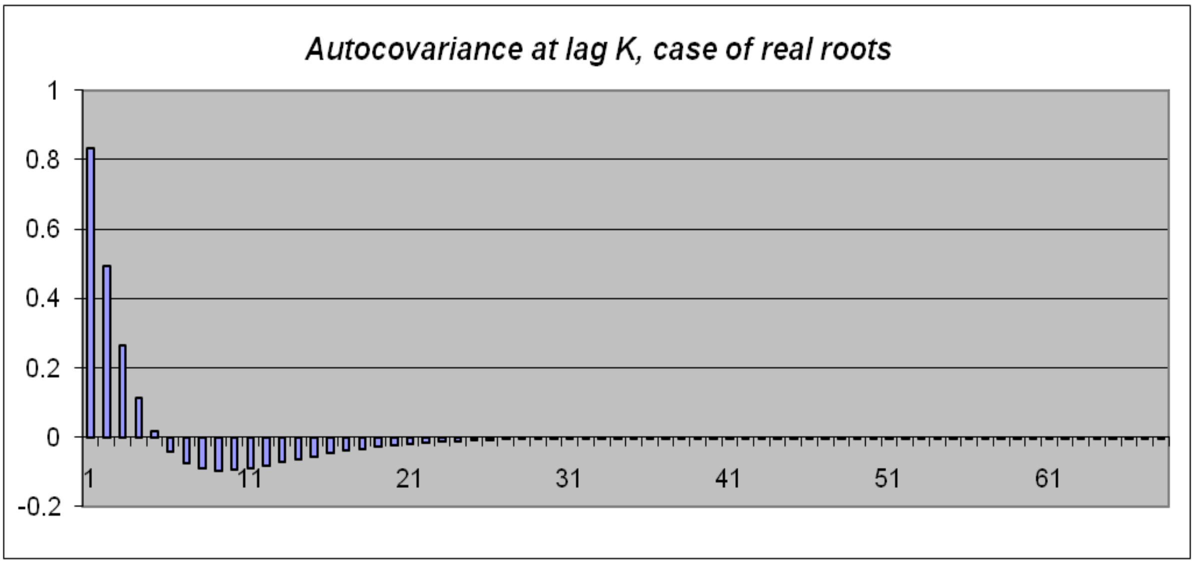
In all cases the magnitude of the autocovariance (i.e. its absolute value) falls exponentially as $k \rightarrow \infty$. Each of the cases (a), (b) and (c) is also rich with different dynamics depending on the relative magnitude of a and b. Given the conditions of stationarity in (17*),we will always have

$$a \geq b, b < 1, \tag{34}$$

which means $\Gamma_0$and $\Gamma_1$in this model given by formulas (29) and (26) are always greater than zero. Autocovariances for further lags given by formulas (30) – (33) can be either positive or negative.

3.2 Analysis: time to first "switch" in autocovariance.

Looking at the case (a) in Proposition 1 of both real roots, we see that at all times $\alpha_1 > 0, \alpha_2 > 0$ and $\alpha_1 > \alpha_2$. It is also clear from the definitions in (30) that $A_1$is always negative while A_2 is always positive. Therefore, as the lag k increases, the autocovariance changes sign from positive to negative, and that switch happens just once. The typical autocovariance behavior with lag is presented at the diagram below.

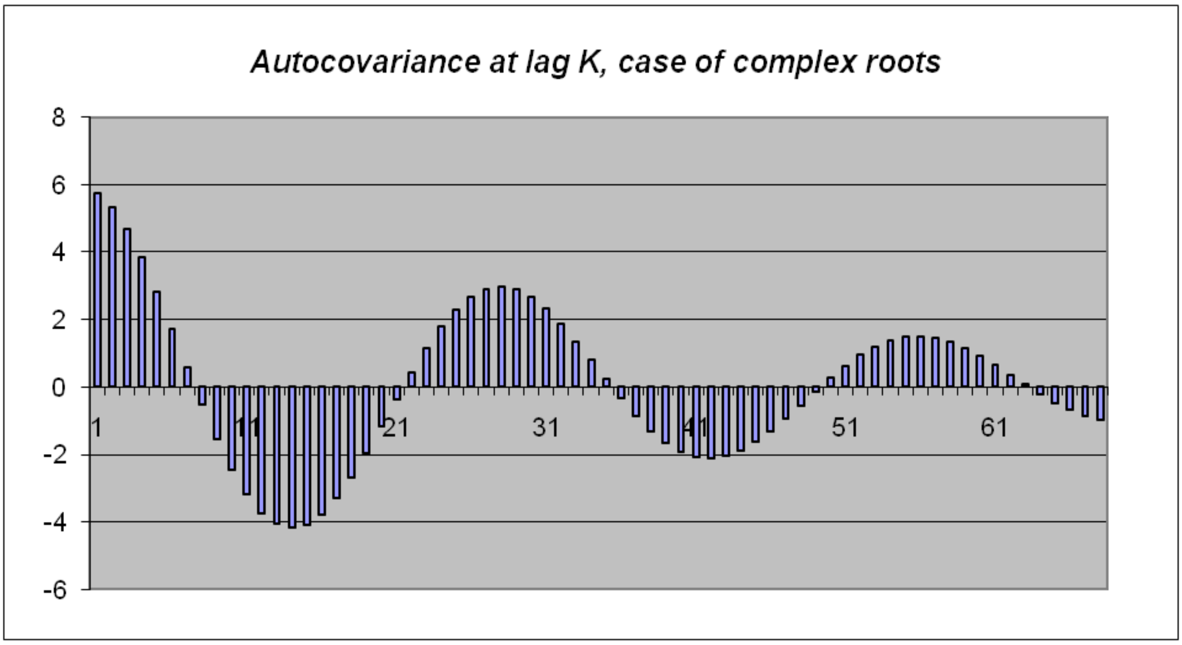


If we look for the lag at which the autocovariance turns from positive to negative, it can be found as follows. Defining k^* as the last lag at which the autocovariance is still positive, it can be found, defining *trunc(x)* as the greatest integer function, as

$$k^* = trunc\left(\frac{\ln\left(-\frac{A_2}{A_1}\right)}{\ln\left(\frac{\alpha_1}{\alpha_2}\right)}\right), \tag{35}$$

Correspondingly, $k^* + 1$ is the first lag at which the autocovariance turns negative. The required k^*defined by (35) is well defined given that $A_1$is always negative while A_2 is always positive as mentioned above.

Turning now to the case (c) of Proposition 1, it is clear that the solution is an oscillating function of k with the magnitude, or the absolute value, falling exponentially with k as $k \to \infty$.(It is interesting to note that the rate of the exponential decay is determined entirely by $b \equiv \frac{1}{\varphi}$, i.e. it does not depend on θ, the rate of the decay of "switchers'" demand). As before, Γ_0 and Γ_1, determined by (29) and (26) respectively, are always positive and further autocovariances can be either positive or negative depending on a and b. The period of oscillation is $T = \frac{2\pi}{\gamma}$ where γ is determined by equation (33). The typical autocovariance behavior with lags is presented at the diagram below, with the main difference from the real root case being that the change in sign happens an infinite number of times.



Here we can again ask for the *first* lag at which autocovariance changes sign from positive to negative. Again, defininining$k$*as the last lag at which the auto-covariance is still positive, we can find it as:

$$k^* = trunc(\frac{1}{\gamma} arctan\left(\frac{\Gamma_0\sqrt{4b- a^2}}{(a\Gamma_0 - 2\Gamma_1)}\right)), \tag{36}$$

where γ is determined by equation (33), with k* + 1 being the first lag at which the autocovariance turns negative.

The formula (32) corresponding to the case of complex roots provides for the possibility of "re-switching" i.e. the case where, having changed its sign ones, the autocovariance change the sign back again. The time until such "re-switching" can be easily found using formulas (32) and (33) above.

3.3Model for excess return.

The fact that the autocovariance changes sign from positive to negative is not surprising and was demonstrated in the BS article, albeit without deriving the exact formula for autocovariance or the lag at which it changes sign. What is less well-known and understood is that the expected excess return in the model, based on today's information, also follows similar dynamics with time lags.

Since the excess return follows an ARMA (2,1) model defined by equation (16) or, using the definitions of a and b in equation (22), then

$$Y_t = aY_{t-1} - bY_{t-2} + \varepsilon_t - (a-b)\varepsilon_{t-1}, \tag{37}$$

We define an h-step ahead forecast as:

$$Y_T(h) \equiv E[Y_{T+h}|H_T], \tag{38}$$

where H_T stands for all information accumulated up to time T. Having defined

$$\hat{Y}_T(0) = Y_T, \tag{39}$$

We substitute (37) into (38), and we have the following equations for $\hat{Y}_T(h)$:

$$h = 1: \quad \hat{Y}_T(1) = aY_T - bY_{T-1} - (a-b)\varepsilon_T, \tag{40}$$

$$h \geq 2: \quad \hat{Y}_T(h) = a\hat{Y}_T(h-1) - b\hat{Y}_T(h-2). \tag{41}$$

As equation (41) produces the same dependence of $\hat{Y}_T(h)$ on h as Γ_k on k in equation (27), the general solution has the same form:

$$\hat{Y}_T(h) = B_1\left(\frac{a+\sqrt{a^2-4b}}{2}\right)^h + B_2\left(\frac{a-\sqrt{a^2-4b}}{2}\right)^h, \tag{42}$$

with equations (39) and (40) serving as boundary conditions. Although ε_T in (40) is not known, it can be expressed through the past values of $\hat{Y}_T$: $\hat{Y}_T, \hat{Y}_{T-1}, \hat{Y}_{T-2}$ ... etc using the following lemma:

Lemma 1:

For the ARMA (2,1) process defined by equation (37),

$$\varepsilon_T = Y_T - bY_{T-1} + b(1-a+b)\sum_{k=2}^{\infty}(a-b)^{k-2}Y_{T-k}. \tag{43}$$

Using this lemma, we can now prove the following proposition:

Proposition 2 (the h-step ahead forecast for the excess return processY_t):

The h-step ahead forecast $\hat{Y}_T(h)$can be determined as

(a) If $a^2 - 4b > 0$,

$$\hat{Y}_T(h) = B_1\left(\frac{a+\sqrt{a^2-4b}}{2}\right)^h + B_2\left(\frac{a-\sqrt{a^2-4b}}{2}\right)^h, \tag{44}$$

where

$$B_1 = \frac{1}{2}\left(1 - \frac{a-2b}{\sqrt{a^2 - 4b}}\right) Y_T - \frac{b(1-a+b)}{\sqrt{a^2-4b}} \sum_{k=1}^{\infty} (a-b)^{k-1} Y_{T-k}, \tag{45}$$

$$B_2 = \frac{1}{2}\left(1 + \frac{a-2b}{\sqrt{a^2 - 4b}}\right) Y_T + \frac{b(1-a+b)}{\sqrt{a^2-4b}} \sum_{k=1}^{\infty} (a-b)^{k-1} Y_{T-k}, \tag{46}$$

(b) If $a^2 - 4b = 0$,

$$\hat{Y}_T(h) = \left(\sqrt{b}\right)^h (B_3 + B_4 h), \tag{47}$$

where

$$B_3 = Y_T, \tag{48}$$

$$B_4 = \left(\frac{2b}{a} - 1\right) Y_T - \frac{2b}{a}(1 - a + b) \sum_{k=1}^{\infty} (a-b)^{k-1} Y_{T-k} = \left(\frac{a}{2} - 1\right) Y_T - \frac{a}{2}\left(1 - \frac{a}{2}\right)^2 \sum_{k=1}^{\infty} \left(a\left(1 - \frac{a}{4}\right)\right)^{k-1} Y_{T-k}, \tag{49}$$

and $b = \frac{a^2}{4}$,

(c) If $a^2 - 4b < 0$,

$$\hat{Y}_T(h) = \left(\sqrt{b}\right)^h (B_5 \cos(\gamma h) + B_6 \sin(\gamma h)), \tag{50}$$

where

$$B_5 = Y_T, \tag{51}$$

$$B_6 = \frac{2}{\sqrt{4b - a^2}} \left[\left(b - \frac{a}{2}\right) Y_T - b(b - a + 1) \sum_{k=1}^{\infty} (a-b)^{k-1} Y_{T-k}\right], \tag{52}$$

where $\gamma = \arctan\left(\sqrt{\frac{4b}{a^2} - 1}\right)$ (53)

As we now have a formula for the *h-step* ahead forecast of the excess return Y_t, we may ask ourselves about the mean square error of the forecast or the forecast error variance, $V(h)$. In order to find it, we need the infinite moving average representation for the process Y_t, which as we know, is an ARMA(2,1) process defined by equation (37). Thus, we need to find coefficients $\psi_i$in the representation

$$Y_t = \sum_{i=0}^{\infty} \psi_i \, \varepsilon_{t-i} \tag{54}$$

The following lemma provides the result:

Lemma 2 (the infinite moving average representation of the excess return processY_t) :

$$\psi_i = F_1 \left(\frac{a+\sqrt{a^2-4b}}{2}\right)^i + F_2 \left(\frac{a-\sqrt{a^2-4b}}{2}\right)^i, \tag{55}$$

where

$$F_1 = \frac{1}{2} + \frac{(2b-a)}{2\sqrt{a^2-4b}}, F_2 = \frac{1}{2} - \frac{(2b-a)}{2\sqrt{a^2-4b}}. \tag{56}$$

Using this result, we can now prove the following proposition.

Proposition 3 (the mean square error of the forecast$\widehat{Y}_T(h)$):

The mean square error (or the forecast error variance) $V(h)$ of the forecast $\hat{Y}_T(h)$ is

$$V(h) = \sum_{k=0}^{h-1} {\psi_i}^2 Var(\varepsilon_t), \tag{57}$$

where $\psi_i$are given by Lemma 2.

3.4 Analysis: time to first switch in excess return

As was the case for autocovariances, having at our disposal an explicit time series model for the forecast of the excess return between the two styles, we are naturally interested in finding the expected time to the first "switch" i.e., the time lag at which the forecast of the excess return changes sign. Having looked at formulae (44) to (53) for the forecast of the excess return, we immediately observe that they are more complicated than formulas (30) to (33) for autocovariances. The reason is that our forecast for the excess returns takes into account information about all prior returns (from the "dawn of time"), not just the last few, although the last observed excess return Y_T does appear more prominently. We also observe that the weights of prior returns follow the familiar EWMA decay with the same degree of decay $\theta = a - b$, which is coming from the formula for the demand from "switchers". This has an obvious resonance with prevalence of moving average and EWMA rules in the use of market practitioners.

We also notice that the formulae for the forecast of excess return produce a greater variety of different cases than that for autocovariances. Given the conditions of stationarity, $\Gamma_0$and Γ_1 in this model are given by formulas (29) and (26) and are always positive while the autocovariances for further lags given by formulas (30) – (33) can be either positive or negative depending on the relative magnitude of the model parameters θ and φ. In particular, in cases (a) and (c) for autocovariances at least one switch of the sign is always guaranteed. In the case of the forecast of the excess return, this is no longer the case and depends on the relative values of the excess return realized to the present time *T.*As before, let's consider the three cases corresponding to cases (a), (b) and (c) in the Proposition 2.

Looking at case (a) in Proposition 2 of two real roots set as before by formulas (28*) and (28**), we again notice that $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 > \alpha_2$. Yet, unlike the case of autocovariances, the relative magnitudes of B_1 and B_2 are uncertain as they are dependent on all prior realized values of the excess return. The following lemma imposes *sufficient* conditions of having at least one "switch" in the sign of the forecast of the excess return at a future time lag $h^* \geq 1$ (where as before h^* is defined as the last time lag at which the forecast of the excess return $\hat{Y}_T(h^*)$ has the same sign as Y_T with $\hat{Y}_T(h^* + 1)$ having the opposite sign):

Lemma 3 (Sufficient Conditions of a switch in the case of real roots):

If the forecast of the excess return is determined by formula (44) where both roots α_1 and α_2 defined by formulas (28) and (28**) are real and satisfy the following conditions: $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 > \alpha_2$. Then for any Y_T there will be $h^* \geq 1$ such that $sign(Y_T) \neq sign\left(\hat{Y}_T(h)\right)$ where $\hat{Y}_T(h)$ is set by formulas (44) – (46) if and only if*

(i) $\quad sign(B_1) \neq sign(B_2),$ (58)

(ii) $\quad |B_1| < |B_2|$, *and* (59)

(iii) $\quad ln\left(-{}^{B_2}/_{B_1}\right) > ln({}^{\alpha_1}/_{\alpha_2}).$ (60)

In that case h^* *will be determined as* $h^* = trunc(\frac{\ln\left(-\frac{B_2}{B_1}\right)}{\ln\left(\frac{\alpha_1}{\alpha_2}\right)}),$ (61)

While the case (a) of both real roots of the auxiliary equation (28) (i.e. the case $D = a^2 - 4b > 0$) does not guarantee the existence of a switch in the sign of the forecast for the excess return, the case (c) of complex roots produces an oscillating solution with guaranteed switches occurring with period $T = \frac{2\pi}{\gamma}$. As it is clear from formulas (50) – (52), the first switch is expected to occur at lag

$$h^* = trunc(\frac{1}{\gamma}\arctan\left(-{}^{B_5}/_{B_6} + 2\pi m\right)), \quad (62)$$

where B_5 and B_6 are determined by equations (51) and (52) and m is defined as the minimum $m \in N$ such that the expression $(-{}^{B_5}/_{B_6} + 2\pi m) > 0$.

3.5 Dependencies on the model parameters

In this paragraph we would like explore the sensitivities of the expected time to first switch found in paragraph 3.4, in particular the sensitivity of h^* corresponding to the case of two real roots (i.e. $D = a^2 - 4b > 0$) to the model parameters θ, φ as well as the last known value of the excess return Y_T. Looking at formula (61)

$h^* = trunc(\frac{\ln\left(-\frac{B_2}{B_1}\right)}{\ln\left(\frac{\alpha_1}{\alpha_2}\right)})$, we immediately notice that if we use the following new notations:

$$S \equiv \frac{b(1-a+b)}{\sqrt{a^2-4b}} \sum_{k=1}^{\infty} (a-b)^{k-1} Y_{T-k}, \quad (63)$$

$$q \equiv \frac{a-2b}{\sqrt{a^2-4b}}, \quad (64)$$

the coefficients B_1 and B_2 from formulas (45) and (46) can be presented as

$$B_1 = -\frac{1}{2}(q-1)Y_T - S, \tag{65}$$

$$B_2 = \frac{1}{2}(q+1)Y_T + S, \tag{66}$$

and h^* can be written as $h^* = trunc[\frac{1}{\ln\left(\frac{\alpha_1}{\alpha_2}\right)} \ln\left(1 + \frac{1}{\frac{1}{2}(q-1) + \frac{S}{Y_T}}\right)]$, (67)

For the purposes of calculating the sensitivities to various parameters we will ignore the truncation in the formula for h^*. Taking the derivative by Y_T we have

$$\frac{\partial h^*}{\partial Y_T} = \frac{1}{\ln\left(\frac{\alpha_1}{\alpha_2}\right)} * \left(-\frac{B_1}{B_2}\right) * \left(\frac{1}{\frac{1}{2}(q-1) + \frac{S}{Y_T}}\right)^2 * \left(\frac{S}{Y_T}\right)^2, \tag{68}$$

which based on Lemma 3 and definitions of α_1 and α_2 is always positive. Thus we notice that

(i) the time to the first switch depends only on the relative size of Y_T (relative compared to Y_{T-1}, Y_{T-2}etc), and

(ii) the time to the first switch always increases with an increase in Y_T.

(iii) Formally, our solution for the time is homogeneous of degree zero in Y_T , Y_{T-1}, Y_{T-2}.....This means we can apply Euler's theorem to find relationships between partial derivatives.

4. Price Dynamics

So far, we have focused our investigation on the relative changes in prices of the two styles, as they can serve as proxies for relative returns. Equations (15) and (16) describe the time series dynamics of the difference between the price changes of the two styles. Yet it is logical to ask if we can derive any conclusions directly regarding the dynamics of price levels, as opposed to their changes. Practitioners may find conclusions about price levels more useful: after all, the price levels can be observed directly. Besides, as further discussion shows, applying our modeling to the levels of prices will let us test our results empirically with a greater degree of accuracy.

Lemma 4: *the demand from "switchers" set by formula (2) and (5) can be re-written respectively as*

$$N_{i,t}^{S} = \frac{1}{n}\Big(A_X + \frac{1}{2}\big(P_{X,t-1} - P_{Y,t-1}\big) - \frac{(1-\theta)}{2}\big(\sum_{k=2}^{t-1}\theta^{k-2}P_{X,t-k} - \sum_{k=2}^{t-1}\theta^{k-2}P_{Y,t-k}\Big),$$

(69)

and

$$N_{j,t}^{S} = \frac{1}{n}\Big(A_Y + \frac{1}{2}\big(P_{Y,t-1} - P_{X,t-1}\big) - \frac{(1-\theta)}{2}\big(\sum_{k=2}^{t-1}\theta^{k-2}P_{Y,t-k} - \sum_{k=2}^{t-1}\theta^{k-2}P_{X,t-k}\Big). \quad (70)$$

The interpretation of the result of this lemma is simple: if the "switchers" form demand for equities in the two styles as described by formulas (2) and (5), their demand at time t in fact is proportionate to the difference between (i) the current difference between the price levels of the two styles, $P_{X,t-1}$ and $P_{Y,t-1}$, and (ii) an EWMA of their prior differences. In particular, assuming for simplicity that the constants A_X and A_Y are set to nil (the assumption made in Barberis and Shleifer (2003)), a positive difference between the latest price levels and their EWMA creates a positive demand for one style at the expense of the other style, while the opposite case reverses the situation. This lemma allows us to prove the following proposition:

Proposition 4: The difference between the price levels of the two styles evolves over time according to the following process:

$$P_{X,t+1} - P_{Y,t+1} = \frac{1}{n}\left(\sum_{i\in X} D_{i,t+1} - \sum_{j\in Y} D_{i,t+1}\right) - \frac{\theta}{n}\left(\sum_{i\in X} D_{i,t} - \sum_{j\in Y} D_{i,t}\right) +$$
$$\left(\theta + \frac{1}{\phi}\right)(P_{X,t} - P_{Y,t}) - \frac{1}{\phi}(P_{X,t-1} - P_{Y,t-1}). \quad (71)$$

It is clear from the above proposition that the difference between price levels evolves according to a process which is very similar to the process defined by (16) for the difference between in price changes of the two styles. The only difference is that the role of random innovation ε_t here is played by the difference between the two dividend streams $\sum_{i\in X} D_{i,t} = D_{X,t}$ and $\sum_{j\in Y} D_{i,t} = D_{Y,t}$. If these two dividend streams can be treated as cointegrated, i.e. their difference is stationary, the framework and solutions developed for the difference in the changes of price levels can be applied verbatim to the difference in price levels.

Let's pause for a second and consider what additional assumptions can be made in respect of the difference $D_{X,t} - D_{Y,t}$.

1) First, in accordance with the assumptions made of the BS model, the first moment of such difference is zero i.e. $E[D_{X,t} - D_{Y,t}] = 0$.
2) Second, the variance of the difference is constant, it does not depend on t. This assumption can be accepted from equilibrium considerations: if the two styles are truly two competing equity styles, then even if one style might happen to dominate the other one fundamentally over a considerate period of time, we would not expect such domination to continue happening indefinitely, as such domination of one group of equities over another group of equities would present a certain misbalance in the economy. Instead, we would reasonably expect that the two dividend streams, albeit deviating from each other over time, from the two styles would return to an equilibrium from time to time, as the economy progresses through different stages of its cycle (would be great to add some references to support our logic here) so that any such imbalance would eventually be rectified.

Therefore, we model $D_{X,t} - D_{Y,t}$as a random innovation ε_t distributed as N(0,var(ε_t)), and the difference between the price levels of the two styles at time t can be expressed as

$$(P_{X,t} - P_{Y,t}) = \left(\theta + \frac{1}{\phi}\right)(P_{X,t-1} - P_{Y,t-1}) - \frac{1}{\phi}(P_{X,t-2} - P_{Y,t-2}) + \varepsilon_t - \theta\varepsilon_{t-1}. \quad (72)$$

As a result, the price difference follows the restricted ARMA(2,1) process of the kind set by equation (16).

5. Empirical Results

In order to test whether the theoretical results above conform to the behavior of prices observable in the market, we applied the model to two very popular "competing" equity styles: "developed equities" vs. "emerging equities". In order to satisfy various assumptions underlying the model, we have selected two broad indices representing each of these two styles from the same family of indices calculated and published by FTSE International Limited("FTSE"):

- Developed equity markets were represented by ***FTSE Developed Index***, a total return index of circa 2000 large and midsize companies located in major developed countries.
- Emerging equity markets are represented by ***FTSE Emerging Index,*** a total return index of close to 2000 large and midsize companies located in major emerging markets.

According to FTSE, the stocks included in both indices are free-float adjusted and screened for liquidity to make sure only the investible opportunity is included in the index. Daily levels for both indices are available since 31 December 1993 thus giving us more than 17.5 years of daily data. The following properties of the indices rendered them suitable for fitting the model:

- The number of components in each index is large and approximately the same (circa 2000).
- Both indices are calculated and rebalanced by the same index provider according to the same methodology.
- Both indices are "free-float" adjusted and their composition is screened for liquidity in order to make sure that only the investible opportunity is included in the indices.
- For both indices a long history (17.5 years) of daily returns is available.

Below is the summary of performance for both indices over the entire period, 31-Dec-1993 to 19-Jul-2011:

	Annualised Return, 1993-2011	Annualised Volatility, 1993-2011	Annualised Return, 1993-2001	Annualised Volatility, 1993-2001	Annualised Return, 2001-2011	Annualised Volatility, 2001-2011
FTSE Developed Markets	4.67%	15.54%	6.67%	12.62%	3.00%	17.62%

FTSE Emerging Markets	4.65%	19.67%	-6.03%	17.65%	13.61%	21.20%

It may come as a surprise but both indices generated roughly the same performance over the whole 17.5-year period, although that performance was much more unevenly distributed for the emerging market index, which is also evidenced by its higher volatility.

Although the model is rich with scenarios of price behavior, it only has only two internal parameters which cannot be observed directly, the coefficients θ and φ defined in formulas (2) and (13) respectively, and therefore would have to be estimated empirically. In order to do so, we fit the price data for both FTSE indices into the restricted ARMA (2,1) of either equation (16) or (72) which will give us estimates of θ and φ in a particular period. Coefficient θ is interesting in its own right, as it measures the rate at which the switchers' demand for shares decays over time. As it can be seen from formula (13), all parameters in the formula for coefficient φ can be estimated independently except for the coefficient of risk aversion of fundamental traders (or "arbitragers") γ. Thus, estimating φ empirically would give as an opportunity to estimate risk aversion γ, which is not directly observable otherwise. As the number of index constituents is large, circa 2000, the formula for γ can be simplified as follows:

$$\gamma = \frac{n}{\varphi\sigma^2(1-\rho_1 + n(\rho_1 - \rho_2))} \approx \frac{1}{\varphi\sigma^2(\rho_1 - \rho_2)}, \tag{73}$$

For the purposes of obtaining γ, we have made the following assumptions, which in general were consistent with the data at hand:

- ρ_1, the correlation between prices of two stocks in the same style, equals 0.4,
- ρ_2, the correlation between prices of two stocks in different styles is 0.28.

As far as the other important parameter, the volatility of a single stock σ, is concerned, we have applied two different treatments: (i) first, where we estimated average volatility of a single stock over the entire 17-year period and used this value for estimating risk aversion γ in every calendar year, and (ii) second, where the volatility was estimated each year and applied to estimating risk aversion in that year only. The reason is that, in addition to risk aversion, volatility is another parameter associated with the level of risk aversion of market participants. Hence, we "controlled" for volatility by running estimates of γ with and without volatility kept constant: if the changes in the volatility level fully reflect the changes in the level of risk aversion, we would expect the estimates of γ in two treatments to behave very differently and *vice versa.*

First, we did not find any empirical evidence that the difference between price changes of the two indices follows ARMA (2,1) process of equation (16) in the time period we considered. We do not provide the outcome of the fitting procedure here but the results can be made available upon request.

Below is the table summarizing the results of fitting the restricted ARMA(2,1) model set by equation (72) in each one-year period from 1994 to 2011. The results include the performance of the two indices, coefficient φ estimated from the model, the coefficient of risk aversion γ obtained using formula (73), as well as the ratio of γ in the current period to the γ obtained from fitting the model to the whole 17.5 year period (with γ (1994-2011) = 1.08).

i) **Table A:** Volatility estimate based on the whole period

Year	1994	1995	1996	1997	1998	1999	2000	2001	2002
FTSE Developed	3.97%	17.57%	11.54%	14.71%	21.86%	23.50%	11.54%	-16.22%	-20.78%
FTSE Emerging	-7.37%	-2.46%	4.26%	-21.98%	-24.60%	65.69%	4.26%	0.98%	-7.95%
Θ	0.998	1.000	0.924	1.000	1.000	0.991	0.968	1.000	0.999
ϕ	2.39	9.31	26.83	2.76	2.65	18.72	646.41	15.83	3.48
γ	2.796	0.719	0.249	2.423	2.529	0.358	0.010	0.423	1.923
γ /γ (1994-2011)	3.23	0.83	0.29	2.80	2.92	0.41	0.01	0.49	2.22

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011 (H1)
FTSE Developed	30.54%	13.18%	7.84%	18.64%	7.64%	-43.18%	27.56%	9.54%	2.01%

FTSE Emerging	49.83%	23.63%	31.05%	29.90%	36.70%	-54.89%	78.24%	16.86%	-3.85%
Θ	0.999	0.953	1.000	0.929	1.000	1.000	1.000	0.996	0.989
ϕ	125.00	1000.00	3.65	418.83	8.03	1.54	29.99	38.51	11.20
γ	0.054	0.007	1.832	0.016	0.834	2.168	0.223	0.174	0.896
γ /γ (1994-2011)	0.06	0.01	2.12	0.02	0.96	2.50	0.26	0.20	1.03

ii) **Table B**: Volatility re-estimated every calendar year

Year	1994	1995	1996	1997	1998	1999	2000	2001	2002
FTSE Developed	3.97%	17.57%	11.54%	14.71%	21.86%	23.50%	11.54%	-16.22%	-20.78%
FTSE Emerging	-7.37%	-2.46%	4.26%	-21.98%	-24.60%	65.69%	4.26%	0.98%	-7.95%
Θ	0.998	1.000	0.924	1.000	1.000	0.991	0.968	1.000	0.999
ϕ	2.39	9.31	26.83	2.76	2.65	18.72	646.41	15.83	3.48
γ	5.315	1.604	0.750	2.532	1.721	0.310	0.011	0.432	1.614
γ /γ (1994-2011)	6.14	1.85	0.87	2.92	1.99	0.36	0.01	0.50	1.86

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011 (H1)
FTSE Developed	30.54%	13.18%	7.84%	18.64%	7.64%	-43.18%	27.56%	9.54%	2.01%
FTSE Emerging	49.83%	23.63%	31.05%	29.90%	36.70%	-54.89%	78.24%	16.86%	-3.85%
Θ	0.999	0.953	1.000	0.929	1.000	1.000	1.000	0.996	0.989
ϕ	125.00	1000.0	3.65	418.83	8.03	1.54	29.99	38.51	11.20

γ	0.062	0.010	4.073	0.017	0.563	2.640	0.070	0.155	2.172
γ /γ (1994-2011)	0.07	0.01	4.70	0.02	0.65	3.05	0.08	0.18	2.51

By looking at the tables, we can make a number of observations.

First, as one would expect, the spikes of the empirically obtained coefficient of risk aversion γ largely correspond to the periods of negative performance of one or both indices. Yet, perhaps what is less expected is that the risk aversion may remain elevated in subsequent periods too, even if the markets experience positive performance. For example, the risk aversion went up at the time of Mexican crisis in 1994-95 yet it remained high in 1996 even though both indices had positive returns in that year. Equally, after the market sold off in 2008 causing a spike in risk aversion, both indices had large positive performance in both 2009, 2010, and the first half of 2011, yet, as the table indicates, the risk aversion in these periods remained high. One explanation of this could be that the risk aversion tends to be "sticky" i.e. not only does it rise during the periods of negative returns but it also tends to remain high in the periods that immediately follow notwithstanding the fact that the markets perform well during these periods.

Second, many estimates of θ are either 1 or very close to 1, the border of the stationarity region for the price process (see equation (17*)). From the econometrics point of view, it indicates that the process is likely to have a unit root and therefore is non-stationary, which of course is no surprise as we fit an ARMA model to a process created by the difference in price levels. From the economics point of view, θ close to 1 in the equations (2) and (5) for the demand from "switchers" indicates that in those periods, the "switchers" do not "discount" past returns in forming their demand but instead focus on the difference between cumulative long-term returns on two styles. The latter can be seen from the considering the following limit:

$$\lim_{\theta \to 1} \sum_{k=1}^{t-1} \theta^{k-1} \left(\frac{\Delta P_{X,t-k} - \Delta P_{Y,t-k}}{2}\right) = \frac{P_{X,t-1} - P_{X,1}}{2} - \frac{P_{Y,t-1} - P_{Y,1}}{2} \qquad (74)$$

During these times, in making their allocation decisions, the "switchers" as a group look at the long term cumulative outperformance of one style over the other style, as opposed to being driven by short term gains and losses.

Third, as expected, regardless whether we used the same estimate of the volatility of a single stock for every annual period (Table A) or we re-estimated volatility for each year as the realized volatility in that year, the picture of changes in the coefficient of risk aversion broadly remains the same with spikes and troughs as described above. What is also apparent from comparing the tables is that Table A produced estimates of the coefficient of risk aversion which appear more consistent with what one would expected based on the return realised in each year. That may indicate that traders, when making their investment decisions, do not instantaneously adjust their volatility assumption σ and instead the level of their risk aversion can be better assessed through the coefficient of risk aversion γ.

6. Conclusions.

This paper made the following contribution to the literature on the Barberis-Shleifer model. First, we have explored in greater detail the autocovariance structure generated by the model and classified different regimes in which the changes in prices of the styles can evolve in the model. In particular, we confirmed the statement from the original paper (Barberis and Shleifer (2003)) that the autocovariance structure within the model is capable of changing sign (i.e. "switching"), which we have done by deriving the exact analytical expression for the aucovariance function at arbitrary lags. Using that formula, we have derived estimates for the expected "time to first switch" under different regimes. The same analysis was repeated for the model of excess returns and we also provided a sufficient condition for a switch in case of a real roots - the case where the occurrence of a switch is not guaranteed. We subsequently explored the dependencies of the "time to first switch" on model parameters, confirming the intuitive conclusions based on intuition.

We subsequently developed the model further by exploring the behavior of the prices (as opposed to their changes) in the model. Having made an additional assumption about the dividend process, we have concluded that the prices follow a stochastic process of the same kind as their changes, therefore the conclusions of the model can be applied directly to prices. We subsequently applied the model for prices to 17.5 years of historical data for prices of two popular equity styles: emerging vs. developed equities and derived yearly estimates of the coefficient of risk aversion of the "fundamental" traders or market makers. Such estimates broadly confirmed the intuition that risk aversion is negatively correlated with market returns.

7. References.

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8. Appendix.

Proof of Proposition 1.

1. Proving part 1 of the proposition is an easy exercise as equations (23) to (25) are a system of three linear equations over the first three autocovariancesΓ_0, Γ_1, and Γ_2, which can be easily resolved to confirm formula (26) and derive formula (29).

2. (a) If $a^2 > 4b$, given the auxiliary equation (28), the general solution of equation (27) is given by formula (30), by fitting the boundary conditions on Γ_0 and Γ_1, we have a system of equations

$$\mathit{\Gamma}_0 = A_1 + A_2 \tag{A.1}$$

$$\mathit{\Gamma}_1 = A_1\left(\frac{a+\sqrt{a^2-4b}}{2}\right) + A_2\left(\frac{a-\sqrt{a^2-4b}}{2}\right) , \tag{A.2}$$

which, when resolved, produces formula (30) for coefficients $A_1$and A_2.

(b) If $a^2 = 4b$, the auxiliary equation (28) has only one real root and the general solution of equation (27) is given by

$$\mathit{\Gamma}_k = A_3\left(\frac{a}{2}\right)^k + A_4 k\left(\frac{a}{2}\right)^k, \tag{A.3}$$

or remembering that $a/2 = \sqrt{b}$ and fitting the boundary conditions on $\mathit{\Gamma}_0$ and $\mathit{\Gamma}_1$, we have a system

$$\mathit{\Gamma}_0 = A_3 \tag{A.4}$$

$$\mathit{\Gamma}_1 = A_3\sqrt{b} + A_4 k\sqrt{b}, \tag{A.5}$$

which, when resolved, produces formula (30*) for coefficients A_1 and A_2.

(d) If $a^2 < 4b$, the auxiliary equation (28) has two complex conjugate roots and the general solution is given by

$$\mathit{\Gamma}_k = A_5\left(\frac{1}{2}\rho e^{i\gamma}\right)^k + A_6\left(\frac{1}{2}\rho e^{-i\gamma}\right)^k, \tag{A.6}$$

where$A_5$and $A_6$are complex conjugates too (see e.g. chapter 1.2 of J.D.Hamilton “*Time Series Analysis”,* 1994), and

$$\rho = 2\sqrt{b} \tag{A.7}$$

$$\gamma = \arctan\left(\sqrt{\frac{4b}{a^2} - 1}\,\right), \tag{A.8}$$

Substituting$A_5 = p + iq$, $A_6 = p - iq$, we have the general solution to have the form

$$\mathit{\Gamma}_k = \left(\sqrt{b}\right)^k (2p\cos(\gamma k) - 2q\sin(\gamma k)), \tag{A.9}$$

which after remembering the boundary conditions on Γ_0 and Γ_1, produces solution of the form given in equation (32).

Proof of Lemma 1:

Starting with equation (37) and using the lag operator L, we have

$$(1 - aL + bL^2)Y_t = (1 - (a-b)L)\varepsilon_t, \tag{A.10}$$

which can be rewritten as $\varepsilon_t = \frac{(1-aL+bL^2)}{(1-(a-b)L)} Y_t = (1 - aL + bL^2)\sum_{i=0}^{\infty} [(a-b)L]^i Y_t =$

$$= (1 - aL + bL^2)(Y_t + (a-b)Y_{t-1} + (a-b)^2 Y_t + \cdots) = Y_t + (a-b)Y_{t-1} +$$

$$(a-b)^2 Y_{t-2} + (a-b)^3 Y_{t-3} + \ldots - aY_{t-1} - a(a-b)Y_{t-2} - a(a-b)^2 Y_{t-3} + \cdots$$

$$+ bY_{t-2} + b(a-b)Y_{t-3} + b(a-b)^2 Y_{t-4} + = Y_t - bY_{t-1} + b(b - a +$$

$$+ 1)\sum_{i=2}^{\infty} [(a-b)]^{i-2} Y_{t-i} \quad , \tag{A.11}$$

i.e.$\varepsilon_T = Y_T - bY_{T-1} + b(b-a+1)\sum_{k=2}^{\infty}(a-b)^{k-2} Y_{T-k}$, which proves the lemma.

Proof of Proposition 2:

The proof is similar to the proof of Proposition 1, in that we have the same general equation (41) for the forecast of excess return as we had in equation (27) for the autocovariance, which means that the solutions to the three cases (a), (b) and (c) which correspond to three different levels of discriminant $D = a^2 - 4b$ (positive, nill or negative) will have the same general form. The difference in solutions comes from different boundary (or initial) conditions and is demonstrated below.

(a) If $a^2 > 4b$, the general solution is given by formula (42), with the boundary conditions provided in (39) and (40).We have a system of equations

$$\hat{Y}_T(0) = B_1 + B_2 = Y_T \tag{A.12}$$

$$\hat{Y}_T(1) = B_1\left(\frac{a+\sqrt{a^2-4b}}{2}\right) + B_2\left(\frac{a-\sqrt{a^2-4b}}{2}\right) = aY_T - bY_{T-1} - (a-b)\varepsilon_T \quad , \text{(A.13)}$$

where$\varepsilon_T$is set by Lemma 1. Therefore we have a system of equations for $B_1$and B_2:

$$B_1 + B_2 = Y_T, \tag{A.14}$$

$$B_1\left(\frac{a+\sqrt{a^2-4b}}{2}\right) + B_2\left(\frac{a-\sqrt{a^2-4b}}{2}\right) = aY_T - bY_{T-1} - (a-b)\varepsilon_T \quad , \tag{A.15}$$

which, when solved, produces formulas (45) and (46) for coefficients B_1 and B_2.

(b) If $a^2 = 4b$, the auxiliary equation (28) has only one real root and the general solution of equation (41) is given by

$$\hat{Y}_T(h) = B_3\left(\frac{a}{2}\right)^h + B_4 h\left(\frac{a}{2}\right)^h, \tag{A.16}$$

or remembering that $a/2 = \sqrt{b}$ and fitting the boundary conditions on $\hat{Y}_T(0)$ and $\hat{Y}_T(1)$, we have a system

$$\hat{Y}_T(0) = Y_T = B_3 \tag{A.17}$$

$$\hat{Y}_T(1) = \frac{a}{2}(B_3 + B_4) = aY_T - \frac{a^2}{4}Y_{T-1} - (a - \frac{a^2}{4})\varepsilon_T, \tag{A.18}$$

which, when resolved, produces formula (48) and (49) for coefficients B_3 and B_4.

(c) If $a^2 < 4b$, the auxiliary equation (28) has two complex conjugate roots and the general solution is given by

$$\Gamma_k = A_5\left(\frac{1}{2}\rho e^{i\gamma}\right)^k + A_6\left(\frac{1}{2}\rho e^{-i\gamma}\right)^k, \tag{A.6}$$

where$A_5$and $A_6$are complex conjugates too (see e.g. chapter 1.2 of J.D.Hamilton "*Time Series Analysis",* 1994), and

$$\rho = 2\sqrt{b} \tag{A.7}$$

$$\gamma = \arctan\left(\sqrt{\frac{4b}{a^2} - 1}\,\right), \tag{A.8}$$

Substituting $A_5 = p + iq$, $A_6 = p - iq$, we have the general solution to have the form

$$\Gamma_k = \left(\sqrt{b}\right)^k (2p\cos(\gamma k) - 2q\sin(\gamma k)), \tag{A.9}$$

which after remembering the boundary conditions on Γ_0 and Γ_1, produces solution of the form given in equation (32).

Proof of Lemma 2:

We are looking for a moving average representation of the kind

$$Y_t = \sum_{i=0}^{\infty} \psi_i \, \varepsilon_{t-i} \tag{54}$$

for the process defined by equation (37). Putting formula (54) into equation (37) we obtain

$$\sum_{i=0}^{\infty} \psi_i \, \varepsilon_{t-i} = a \sum_{i=0}^{\infty} \psi_i \, \varepsilon_{t-1-i} - b \sum_{i=0}^{\infty} \psi_i \, \varepsilon_{t-2-i} + \varepsilon_t - (a-b)\varepsilon_{t-1} \tag{A.19}$$

After equating coefficients of ε_{t-i}, we find that the coefficients have to satisfy the following conditions:

$$\psi_0 = 1, \text{ for } i = 0, \tag{A.20}$$
$$\psi_1 = a\psi_0 - (a-b), \text{ for } i = 1, \tag{A.21}$$
$$\psi_i = a\psi_{i-1} - b\psi_{i-2}, \text{ for } i \geq 2, \tag{A.22}$$

Assuming a solution of the kind

$$\psi_i = F_1 \alpha_1^i + F_2 \alpha_2^i, \tag{A.23}$$

we immediately derive that the solution has the form

$$\psi_i = F_1 \left(\frac{a + \sqrt{a^2 - 4b}}{2}\right)^i + F_2 \left(\frac{a - \sqrt{a^2 - 4b}}{2}\right)^i, \tag{A.24}$$

with equations (A.20) and (A.21) serving as the boundary conditions, which we use to find the constants F_1 and F_2. Resolving this system of two equations with two unknowns we derive that

$$F_1 = \frac{1}{2} + \frac{(2b-a)}{2\sqrt{a^2-4b}}, \quad F_2 = \frac{1}{2} - \frac{(2b-a)}{2\sqrt{a^2-4b}}, \tag{56}$$

as required.

Proof of Lemma 3:

This lemma simply summarises the conditions under which the formula (61) for the lag at which the first switch occurs is well-defined and therefore, such lag can be computed.

Proof of Proposition 3:

The proof of this proposition immediately follows from the infinite moving average representation (54), the definition of the h-step ahead forecast:

$$Y_T(h) \equiv E[Y_{T+h}|H_T] = \sum_{i=h}^{\infty} \psi_i \, \varepsilon_{T+h-i}, \tag{A.25}$$

and the i.i.d. assumption in relation to all ε_i.

Proof of Lemma 4:

The proof of this lemma simply follows from starting with equations (2) and (5), inserting the definitions of $\Delta P_{X,t}$ and $\Delta P_{Y,t}$ of (3), and collecting all components with $P_{X,t-i}$ and $P_{Y,t-i}$ together.

Proof of Proposition 4:

We start from equations (11) and (14), where we use formula (13) which is the definition of ϕ :

$$P_{i,t} = D_{i,t} + \phi N_{X,t}^{S}, \tag{11*}$$

$$P_{j,t} = D_{j,t} + \phi N_{Y,t}^{S}. \tag{14*}$$

We then aggregate across all equities in each style using formulas (4) and insert formulas (69) and (70) for the aggregate demand for equities in each two styles obtained in Lemma 4, which then leads us to formulas (71).