Sains Malaysiana 46(2)(2017): 335–347
http://dx.doi.org/10.17576/jsm-2017-4602-19

A Novel Collocation Method Based on Residual Error Analysis for Solving Integro-Differential Equations Using Hybrid Dickson and Taylor Polynomials

(Kaedah Novel Kolokasi Berdasarkan Analisis Sisa Ralat untuk Menyelesaikan Persamaan Integro-Pembezaan yang Menggunakan Hibrid Dickson dan Polinomial Taylor)

Ömür Kivanç Kürkçü*, Ersin Aslan & Mehmet Sezer

ABSTRACT

In this study, a novel matrix method based on collocation points is proposed to solve some linear and nonlinear integro-differential equations with variable coefficients under the mixed conditions. The solutions are obtained by means of Dickson and Taylor polynomials. The presented method transforms the equation and its conditions into matrix equations which comply with a system of linear algebraic equations with unknown Dickson coefficients, via collocation points in a finite interval. While solving the matrix equation, the Dickson coefficients and the polynomial approximation are obtained. Besides, the residual error analysis for our method is presented and illustrative examples are given to demonstrate the validity and applicability of the method.

Keywords: Collocation and matrix methods; Dickson and Taylor polynomials; integro-differential equations; nonlinear equations; pseudocode

ABSTRAK

Dalam kajian ini, kaedah matriks novel berdasarkan titik kolokasi adalah dicadangkan untuk menyelesaikan persamaan integro-pembezaan bagi sesetengah linear dan tak linear dengan pekali pemboleh ubah dalam keadaan bercampur-campur. Penyelesaian yang diperoleh dengan cara polinomial Dickson dan Taylor. Kaedah yang dibentangkan mengubah persamaan serta keadaannya ke dalam persamaan matriks yang bertepatan dengan sistem persamaan algebra linear dengan pekali Dickson tidak diketahui, melalui titik kolokasi dalam selang terhingga. Semasa menyelesaikan persamaan matriks ini, pekali Dickson dan penganggaran polinomial diperoleh. Selain itu, analisis sisa ralat bagi kaedah kami ini telah dikemukakan dan contoh ilustrasi diberi untuk menunjukkan kesahihan dan penerapan kaedah.

Kata kunci: Kolokasi dan kaedah matriks; polinomial Dickson dan Taylor; persamaan integro-pembezaan; persamaan tak linear; tatasusunan

Introduction

Integro-differential equations (IDEs) consist of differential and integral equations. These equations play an important role in the fields of applied mathematics and engineering, mechanics, physics, chemistry, potential theory, dynamics and ecology. These equations are also generally difficult to solve analytically; thereby, a numerical method is needed. In recent years, several numerical methods have been introduced such as the matrix and collocation methods based on Chebyshev (Akyüz-Daşcıoğlu 2006), Taylor (Sezer 1994), Legendre (Yalçınbaş et al. 2009) and Bessel (Yüzbaşı et al. 2011) polynomials, along with Adomian decomposition (Evans et al. 2005) and Wavelet moment (Babolian et al. 2007) methods.

Permutation and Dickson polynomials are widely used in mathematics, integer rings (Fernando 2013), finite fields (Bhargava et al. 1999), key cryptography (Wei et al. 2011), algebraic and number-theory (Stoll 2007). Dickson polynomials are denoted as $D_n(x, \alpha)$ and were introduced by Dickson (1896). These were later rediscovered by Brewer (1961). Dickson polynomials are defined as follows,

$$D_n(x,\alpha)=\sum_{p=0}^{\lfloor \frac{n}{2} \rfloor}\frac{n}{n-p}\binom{n-p}{p}(-\alpha)^p x^{(n-2p)};\ -\infty < x < \infty, \quad (1)$$

where the parameter $-\alpha$, $D_0(x,\alpha)=2$, $D_1(x,\alpha)$ and $n \geq 1$. Also, the Dickson polynomials $y = D_n(x,\alpha)$ satisfy the ordinary differential equations (Lidl et al. 1993)

$$(x^2 - 4\alpha)\, y'' + xy' - n^2 y = 0,\ n = 0,1,2,3,\ldots$$

and the recurrence relation (Lidl et al. 1993),

$$D_n(x, \alpha) = xD_{n-1}(x, \alpha) - \alpha D_{n-2}(x, \alpha),\ n \geq 2.$$

For further information about the Dickson polynomials see (Kürkçü et al. 2016 and therein references).

In this paper, the matrix relations between the Dickson polynomials and its expansions depend on the parameter-α with n and the novel method will be applied to mth-order linear and nonlinear integro-differential equations.

1. mth-order linear Fredholm integro-differential equation (FIDE)

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda_1 \int_a^b K_f(x,t)y(t)dt. \tag{2}$$

2. mth-order linear Volterra integro-differential equation (VIDE)

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda_2 \int_a^x K_v(x,t)y(t)dt. \tag{3}$$

3. mth-order linear Fredholm-Volterra integro-differential equation (FVIDE)

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda_1 \int_a^b K_f(x,t)y(t)dt + \lambda_2 \int_a^x K_v(x,t)y(t)dt. \tag{4a}$$

4. mth-order nonlinear Fredholm-Volterra integro-differential equation in the from

$$\begin{aligned}&\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + Z_1(x)y^2(x) + T_1(x)y^3(x) = g(x)\\ &+\lambda_1 \int_a^b K_f(x,t)y(t)dt + \lambda_2 \int_a^x K_v(x,t)y(t)dt\end{aligned}, \tag{4b}$$

under the mixed conditions

$$\sum_{k=0}^{m-1} \left[a_{jk}y^{(k)}(a) + b_{jk}y^{(k)}(b)\right] = \mu_j;\ j = 0,1,2,\ \ldots,\ m-1, \tag{5}$$

where $y(x)$ is an unknown function, the known functions $P_k(x)$, $Z_1(x)$, $T_1(x)$, $g(x)$, $K_f(x,t)$, $K_v(x,t)$ are described on $-\infty < a \leq x,\ t \leq b < \infty$ and a_{jk}, b_{jk}, λ_1, λ_2, μ_j are useful constants. Our purpose is to find an approximate solutions of (2), (3), (4a) and (4b). Hence, form of the solutions will be as follows (Kürkçü et al. 2016),

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} y_n D_n(x,\alpha),\ -\infty < a \leq x,\ t \leq b < \infty, \tag{6}$$

where y_n are unknown Dickson coefficients and N ($n \geq m$) is chosen as any positive integer. Also, Dickson polynomials $D_n(x, \alpha)$ were defined by (1). In order to obtain a solution in the form (6) of (2), (3), (4a) and (4b), we can use the collocation points,

$$x_i = a + \left(\frac{b-a}{N}\right)i,\ i = 0, 1, 2, \ldots, N, \tag{7}$$

where $a = x_0 < x_1 < \ldots < x_N = b$.

Fundamental Matrix Relations

In this and next sections, the whole relations will be based on (4a) and (4b). Let us write (4a) as the generalized integro-differential equation form,

$$D(x) = g(x) + F(x) + V(x), \tag{8}$$

where

$$D(x) = \sum_{k=0}^{m} P_k(x)y^{(k)}(x),\ F(x) = \lambda_1 \int_a^b K_f(x,t)y(t)dt$$
and $V(x) = \lambda_2 \int_a^x K_v(x,t)y(t)dt.$

$D(x)$, $F(x)$ and $V(x)$ are called as the differential, Fredholm and Volterra integral parts of (2), (3) and (4a), respectively. We transform these parts with mixed conditions (5) to matrix form. Here, if we establish the collocation points (5) in (8), then we have a system

$$D(x_i) = g(x_i) + F(x_i) + V(x_i), \tag{9}$$

where

$$D(x_i) = \sum_{k=0}^{m} P_k(x_i)y^{(k)}(x_i),\ F(x_i) = \lambda_1 \int_a^b K_f(x_i,t)y(t)dt$$
and $V(x_i) = \lambda_2 \int_a^x K_v(x_i,t)y(t)dt.$

Now we can transform the systems (9) into the matrix equations, respectively

$$\mathbf{D} = \mathbf{G}+\mathbf{F}+\mathbf{V}, \tag{10}$$

where

$$\mathbf{D} = \begin{bmatrix} D(x_0) \\ D(x_1) \\ \vdots \\ D(x_N) \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix},\ \mathbf{F} = \begin{bmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_N) \end{bmatrix} \text{and}$$

$$\mathbf{V} = \begin{bmatrix} V(x_0) \\ V(x_1) \\ \vdots \\ V(x_N) \end{bmatrix}.$$

Matrix Representation of Differential Part

Let us assume the function $y(x)$ and its derivatives have truncated Dickson series expansion of the form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} y_n D_n(x,\alpha),\ -\infty < a \leq x,\ t \leq b < \infty.$$

Hence, the solution is explained by (6) and its derivatives can be transformed to the matrix forms

$$[y(x)] = D(x,\alpha)\,Y \text{ and } [y^{(k)}(x)] = D^{(k)}(x,\alpha)\,Y, \tag{11}$$

such that

$$\boldsymbol{D}(x,\alpha) = [D_0(x,\alpha)\ D_1(x,\alpha)\ \ldots\ D_N(x,\alpha)],$$

$$\boldsymbol{D}^{(k)}(x,\alpha) = \begin{bmatrix} D_0^{(k)}(x,\alpha) & D_1^{(k)}(x,\alpha) & \cdots & D_N^{(k)}(x,\alpha) \end{bmatrix}$$

and the Dickson coefficients matrix

$$\boldsymbol{Y} = [y_0\ \ y_1\ \ \ldots\ \ y_N]^T.$$

On the other hand, we obtain the matrix $D(x,\alpha)$ by using the Dickson polynomial. The matrix is given for odd values of N

$$\underbrace{\begin{bmatrix} D_0(x,\alpha) \\ D_1(x,\alpha) \\ D_2(x,\alpha) \\ \vdots \\ D_{N-1}(x,\alpha) \\ D_N(x,\alpha) \end{bmatrix}}_{\boldsymbol{D}^T(x,\alpha)} = \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{N}{\left\lceil \frac{N}{2} \right\rceil}\binom{\left\lceil \frac{N}{2} \right\rceil}{\left\lfloor \frac{N}{2} \right\rfloor}(-\alpha)^{\left\lfloor \frac{N}{2} \right\rfloor} & 0 & \frac{N}{\left\lceil \frac{N}{2} \right\rceil+1}\binom{\left\lceil \frac{N}{2} \right\rceil+1}{\left\lfloor \frac{N}{2} \right\rfloor-1}(-\alpha)^{\left\lfloor \frac{N}{2} \right\rfloor-1} & \cdots & \frac{N}{N}\binom{N}{0}(-\alpha)^0 \end{bmatrix}}_{\boldsymbol{S}^T(\alpha)} \underbrace{\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ \vdots \\ x^{N-1} \\ x^N \end{bmatrix}}_{\boldsymbol{X}^T(x)}; \tag{12}$$

for even values of N

$$\underbrace{\begin{bmatrix} D_0(x,\alpha) \\ D_1(x,\alpha) \\ D_2(x,\alpha) \\ \vdots \\ D_{N-1}(x,\alpha) \\ D_N(x,\alpha) \end{bmatrix}}_{\boldsymbol{D}^T(x,\alpha)} = \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{N}{\frac{N}{2}}\binom{\frac{N}{2}}{\frac{N}{2}}(-\alpha)^{\frac{N}{2}} & 0 & \frac{N}{\frac{N}{2}+1}\binom{\frac{N}{2}+1}{\frac{N}{2}-1}(-\alpha)^{\frac{N}{2}-1} & 0 & \cdots & \frac{N}{N}\binom{N}{0}(-\alpha)^0 \end{bmatrix}}_{\boldsymbol{S}^T(\alpha)} \underbrace{\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ \vdots \\ x^{N-1} \\ x^N \end{bmatrix}}_{\boldsymbol{X}^T(x)}. \tag{13}$$

Hence, we write the matrix equation by using (12) and (13)

$$\boldsymbol{D}^{\boldsymbol{T}}(x,\alpha) = \boldsymbol{S}^{\boldsymbol{T}}(\alpha)\,\boldsymbol{X}^{\boldsymbol{T}}(x) \text{ or } \boldsymbol{D}(x,\alpha) = \boldsymbol{X}(x)\,\boldsymbol{S}(\alpha) \text{ and}$$

$$\boldsymbol{D}^{(k)}(x,\alpha) = \boldsymbol{X}^{(k)}(x)\,\boldsymbol{S}(\alpha). \tag{14}$$

Also, the following equations are obtained by using (11) and (14).

$$y(x) = \boldsymbol{X}(x)\,\boldsymbol{S}(\alpha)\,\boldsymbol{Y} \text{ and } y^{(k)}(x) = \boldsymbol{X}^{(k)}(x)\,\boldsymbol{S}(\alpha)\,\boldsymbol{Y}. \tag{15}$$

The relation (Kurt & Sezer 2008) between the matrix $X(x)$ and its derivative $X^{(k)}(x)$ is

$$\boldsymbol{X}^{(k)}(x) = \boldsymbol{X}(x)\,\boldsymbol{B}^k,\ (B^0\text{: Identity matrix }), \tag{16}$$

where

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From (15) and (16), we obtain $y^{(k)}(x) = \boldsymbol{X}(x)\,\boldsymbol{B}^k\boldsymbol{S}(\alpha)\,\boldsymbol{Y}$ and its representation $y^{(k)}(x_i) = \boldsymbol{X}(x_i)\,\boldsymbol{B}^k\boldsymbol{S}(\alpha)\boldsymbol{Y}$ with the collocation points. On the other hand, the matrix $\boldsymbol{D}$ corresponds to $D(x_i)$, $i = 0, 1, 2, \ldots, N$ can be formed as,

$$\boldsymbol{D} = \sum_{k=0}^{m} P_k y^{(k)} = \sum_{k=0}^{m} \boldsymbol{P}_k \boldsymbol{X}\boldsymbol{B}^k\boldsymbol{S}(\alpha)\boldsymbol{Y}, \tag{17}$$

where

$$\boldsymbol{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix},$$

$$\boldsymbol{X} = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \cdots & x_0^N \\ 1 & x_1 & \cdots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^N \end{bmatrix}.$$

Matrix Representation of Fredholm Integral Part

Now, we give the kernel function $K_f(x,t)$ for the Fredholm integral part $F(x)$ in the truncated Dickson and the Taylor series forms (Sezer 1996), respectively,

$$K_f(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{mn}^f D_m(x,\alpha) D_n(x,\alpha)$$

and

$$K_f(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{mn} x^m t^n,$$

where

$$k_{mn} = \frac{1}{m!n!}\frac{\partial^{m+n} K_f(0,0)}{\partial x^m\,\partial t^n},\quad m,n = 0,1,2,\ldots,N.$$

We can write the matrix forms of $K_t(x,t)$ for the Taylor and Dickson polynomials as

$$\left[K_t(x,t)\right] = \boldsymbol{X}(x)\boldsymbol{K}_t\boldsymbol{X}^T(t) \tag{18}$$

and

$$[K_f(x,t)] = \boldsymbol{D}(x,\alpha)\,\boldsymbol{K}_f\boldsymbol{D}^T(t,\alpha). \tag{19}$$

From the equality of the relations (18), (19) and by using the relation (14), we obtain the relation between the Dickson and Taylor coefficients of the kernel function $K_f(x,t)$:

$$\boldsymbol{X}(x)\boldsymbol{K}_t\boldsymbol{X}^T(t)=\boldsymbol{D}(x,\alpha)\boldsymbol{K}_f\boldsymbol{D}^T(t,\alpha)$$
$$=\boldsymbol{X}(x)\underbrace{\boldsymbol{S}(\alpha)\boldsymbol{K}_f\boldsymbol{S}^T(\alpha)}_{\boldsymbol{K}_t}\boldsymbol{X}^T(t),$$

$$\boldsymbol{K}_t=\boldsymbol{S}(\alpha)\boldsymbol{K}_f\boldsymbol{S}^T(\alpha)\Rightarrow\boldsymbol{K}_f=\left(\boldsymbol{S}(\alpha)\right)^{-1}\boldsymbol{K}_t\left(\boldsymbol{S}^T(\alpha)\right)^{-1}, \quad (20)$$

where

$\boldsymbol{K}_t=[k_{mn}]$, and $\boldsymbol{K}_f=\left[k_{mn}^f\right]$, $m,n=0,1,2,\ldots,N$.

By substituting the matrix forms (20) and (11) into the Fredholm integral part $F(x)$, we have the matrix equation

$$[F(x)]=\lambda_1\int_a^b\boldsymbol{D}(x,\alpha)\boldsymbol{K}_f\boldsymbol{D}^T(t,\alpha)\boldsymbol{D}(t,\alpha)\boldsymbol{Y}dt$$
$$=\lambda_1\boldsymbol{D}(x,\alpha)\boldsymbol{K}_f\underbrace{\left\{\int_a^b\boldsymbol{D}^T(t,\alpha)\boldsymbol{D}(t,\alpha)dt\right\}}_{\boldsymbol{Q}_f}\boldsymbol{Y}$$
$$=\lambda_1\boldsymbol{D}(x,\alpha)\boldsymbol{K}_f\boldsymbol{Q}_f\boldsymbol{Y}$$

where

$$\boldsymbol{Q}_f=\int_a^b\boldsymbol{S}^T(\alpha)\boldsymbol{X}^T(t)\boldsymbol{X}(t)\boldsymbol{S}(\alpha)dt$$
$$=\boldsymbol{S}^T(\alpha)\underbrace{\left\{\int_a^b\boldsymbol{X}^T(t)\boldsymbol{X}(t)dt\right\}}_{\boldsymbol{Q}_1}\boldsymbol{S}(\alpha)=\boldsymbol{S}^T(\alpha)\boldsymbol{Q}_1\boldsymbol{S}(\alpha)$$

$$\boldsymbol{Q}_1=[q_{mn}],\ q_{mn}=\frac{b^{m+n+1}-a^{m+n+1}}{m+n+1},\quad m,n=0,1,2,\ldots,N$$

Hence, we have the matrix connection of Fredholm integral part:

$$[F(x)]=\lambda_1\boldsymbol{D}(x,\alpha)\,\boldsymbol{K}_f\boldsymbol{Q}_f\boldsymbol{Y}.$$

If we utilize the collocation points $x=x_i$ ($i=0,1,2,\ldots,N$), then we obtain the system of the matrix equations

$$[F(x_i)]=\lambda_1\boldsymbol{D}(x_i,\alpha)\boldsymbol{K}_f\boldsymbol{Q}_f\boldsymbol{Y}\Rightarrow[F(x_i)]=\lambda_1\boldsymbol{X}(x_i)\boldsymbol{S}(\alpha)\boldsymbol{K}_f\boldsymbol{Q}_f\boldsymbol{Y}$$

or briefly, the matrix equation

$$\boldsymbol{F}=\lambda_1\boldsymbol{XS}(\alpha)\,\boldsymbol{K}_f\boldsymbol{Q}_f\boldsymbol{Y}. \quad (21)$$

Matrix Representation of Volterra Integral Part

Now we consider the kernel function $K_v(x,t)$ of the Volterra integral part $V(x)$ in (4a) and (4b) by using the similar procedure to previously discussed, we obtain the following results:

$$[V(x)]=\lambda_2\int_a^x\boldsymbol{X}(x)\boldsymbol{K}_t^v\boldsymbol{X}^T(t)\boldsymbol{X}(t)\boldsymbol{S}(\alpha)\boldsymbol{Y}dt$$
$$=\lambda_2\boldsymbol{X}(x)\boldsymbol{K}_t^v\underbrace{\left\{\int_a^x\boldsymbol{X}^T(t)\boldsymbol{X}(t)dt\right\}}_{\boldsymbol{Q}_v(x)}\boldsymbol{S}(\alpha)\boldsymbol{Y}$$
$$=\lambda_2\boldsymbol{X}(x)\boldsymbol{K}_t^v\boldsymbol{Q}_v(x)\boldsymbol{S}(\alpha)\boldsymbol{Y}$$

where

$$\boldsymbol{Q}_v(x)=[q_{kl}(x)],\ q_{kl}(x)=\frac{x^{k+l+1}-a^{k+l+1}}{k+l+1},\ k,\ l=0,1,2,\ldots,N;$$

and for $x=x_i$, ($i=0,1,2,\ldots,N$) the matrix system

$$[V(x_i)]=\lambda_2\boldsymbol{X}(x_i)\boldsymbol{K}_t^v\boldsymbol{Q}_v(x_i)\boldsymbol{S}(\alpha)\boldsymbol{Y}. \quad (22)$$

Consequently, the matrices system (22) is written in the matrix form

$$\boldsymbol{V}=\lambda_2\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{K}_v}\right)\left(\overline{\boldsymbol{Q}_v}\right)\boldsymbol{S}(\alpha)\boldsymbol{Y}, \quad (23)$$

where $\boldsymbol{K}_t^v=\left[k_{mn}^v\right]$, $k_{mn}^v=\dfrac{1}{m!n!}\dfrac{\partial^{m+n}K_v(0,0)}{\partial x^m\partial t^n}$; $m,\ n=0,1,2,\ldots,N$,

$$\overline{\boldsymbol{X}}=\begin{bmatrix}\boldsymbol{X}(x_0)&0&\cdots&0\\0&\boldsymbol{X}(x_1)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&0&\boldsymbol{X}(x_N)\end{bmatrix}_{(N+1)\times(N+1)^2},$$

$$\overline{\boldsymbol{K}_v}=\begin{bmatrix}\boldsymbol{K}_t^v&0&\cdots&0\\0&\boldsymbol{K}_t^v&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&0&\boldsymbol{K}_t^v\end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$\overline{\boldsymbol{Q}_v}=\begin{bmatrix}\boldsymbol{Q}_v(x_0)\\\boldsymbol{Q}_v(x_1)\\\vdots\\\boldsymbol{Q}_v(x_N)\end{bmatrix}_{(N+1)^2\times(N+1)}.$$

Matrix Representation of Nonlinear Parts

By using (7) and (15), we construct the matrix representation of nonlinear parts $Z_1(x)y^2(x)$ and $T_1y^3(x)$, respectively (Kürkçü et al. 2016),

$$Z_1(x_i)y^2(x_i)=\boldsymbol{Z}_1\boldsymbol{XS}(\alpha)\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{S}}(\alpha)\right)\overline{\boldsymbol{Y}}, \quad (24)$$

where

$$Z_1 = \mathrm{diag}\left[Z_1\left(x_i\right)\right]_{(N+1)\times(N+1)},\ \overline{\boldsymbol{S}}\left(\alpha\right) = \mathrm{diag}\left[\boldsymbol{S}\left(\alpha\right)\right]_{(N+1)^2\times(N+1)^2},$$
$$\overline{\boldsymbol{Y}} = \begin{bmatrix} y_0\boldsymbol{Y} & y_1\boldsymbol{Y} & \cdots & y_N\boldsymbol{Y} \end{bmatrix}^T_{(N+1)^2\times 1}.$$

Similarly,

$$T_1\left(x_i\right)y^3\left(x_i\right) = \boldsymbol{T}_1\boldsymbol{X}\boldsymbol{S}\left(\alpha\right)\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{S}}\left(\alpha\right)\right)\left(\overline{\overline{\boldsymbol{X}}}\right)\left(\overline{\overline{\boldsymbol{S}}}\left(\alpha\right)\right)\overline{\overline{\boldsymbol{Y}}}, \qquad (25)$$

where

$$\boldsymbol{T}_1 = \mathrm{diag}\left[T_1\left(x_i\right)\right]_{(N+1)\times(N+1)},\ \overline{\overline{\boldsymbol{X}}} = \mathrm{diag}\left[\overline{\boldsymbol{X}}\right]_{(N+1)^2\times(N+1)^3},$$

$$\overline{\overline{\boldsymbol{S}}}\left(\alpha\right) = \mathrm{diag}\left[\overline{\boldsymbol{S}}\left(\alpha\right)\right]_{(N+1)^3\times(N+1)^3} \text{ and}$$
$$\overline{\overline{\boldsymbol{Y}}} = \begin{bmatrix} y_0\overline{\boldsymbol{Y}} & y_1\overline{\boldsymbol{Y}} & \cdots & y_N\overline{\boldsymbol{Y}} \end{bmatrix}^T_{(N+1)^3\times 1}.$$

Matrix Representation of Mixed Conditions

We can find the corresponding matrix equations for the conditions (5), by using the relation (15),

$$\sum_{k=0}^{m-1}\left[a_{jk}\boldsymbol{X}\left(a\right)+b_{jk}\boldsymbol{X}\left(b\right)\right].\boldsymbol{B}^k\boldsymbol{S}\left(\alpha\right)\boldsymbol{Y} = \mu_j, j = 0,1,2,\ldots,\mathrm{m}-1, \qquad (26)$$

where

$$\boldsymbol{X}\left(a\right) = \begin{bmatrix} 1 & a^1 & \cdots & a^{N-1} & a^N \end{bmatrix},$$
$$\boldsymbol{X}\left(b\right) = \begin{bmatrix} 1 & b^1 & \cdots & b^{N-1} & b^N \end{bmatrix}.$$

Method of Solution

We now ready to build the fundamental matrix equation according to (4a). For this aim, we initially insert the matrix relations (17), (21) and (23) into (10) and then by simplifying, we obtain the fundamental matrix equation,

$$\left\{\boldsymbol{D}-\boldsymbol{F}-\boldsymbol{V}\right\}\boldsymbol{Y} =$$
$$\boldsymbol{G} \Rightarrow \underbrace{\left\{\sum_{k=0}^{m}\boldsymbol{P}_k\boldsymbol{X}\boldsymbol{B}^k\boldsymbol{S}\left(\alpha\right)-\lambda_1\boldsymbol{X}\boldsymbol{S}\left(\alpha\right)\boldsymbol{K}_f\boldsymbol{Q}_f-\lambda_2\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{K}}_v\right)\left(\overline{\boldsymbol{Q}}_v\right)\boldsymbol{S}\left(\alpha\right)\right\}}_{\boldsymbol{W}}\boldsymbol{Y} = \boldsymbol{G}, \qquad (27)$$

which corresponds to a system of $(N+1)$ algebraic equations for $(N + 1)$ unknown Dickson coefficients y_0, $y_1, \ldots, y_N$. Briefly, we can write (27) in the form:

$$\boldsymbol{W}\boldsymbol{Y} = \boldsymbol{G} \text{ or } [\boldsymbol{W}\,;\,\boldsymbol{G}], \qquad (28)$$

where

$$\boldsymbol{G} = [g(x_0)\quad g(x_1)\quad \ldots\quad g(x_{N-1})\quad g(x_N)]^T.$$

On the other hand, we can construct (26) for the conditions (5) , briefly as:

$$\boldsymbol{U}_j\boldsymbol{Y} = \mu_j \Rightarrow [\boldsymbol{U}_j\,;\,\mu_j],\ j = 0,1,2,\ldots, m-1, \qquad (29)$$

where

$$\boldsymbol{U}_j = \sum_{k=0}^{m-1}\left[a_{jk}\boldsymbol{X}\left(a\right)+b_{jk}\boldsymbol{X}\left(b\right)\right].$$
$$\boldsymbol{B}^k\boldsymbol{S}\left(\alpha\right)\boldsymbol{Y} \equiv \begin{bmatrix} u_{j0} & u_{j1} & u_{j2} & \cdots & u_{jN} \end{bmatrix}.$$

In order to obtain the solution of (4a) under the conditions (5), by changing the row matrices (29) by any m rows of the matrix (28), we get the augmented matrix

$$\left[\boldsymbol{W}^*\,;\,\boldsymbol{G}^*\right] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g\left(x_0\right) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g\left(x_1\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots; & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & g\left(x_{N-m}\right) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \mu_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \mu_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots; & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \mu_{m-1} \end{bmatrix}. \qquad (30)$$

If rankW^* =rank $[\boldsymbol{W}^*\,;\,\boldsymbol{G}^*] = N + 1$, then we can write $\boldsymbol{Y} = (\boldsymbol{W}^*)^{-1}\,\boldsymbol{G}^*$. Consequently, the Dickson coefficients y_k ($k = 0,1, \ldots, N$) are uniquely determined by (30). On the other hand, when det$(\boldsymbol{W}^*) = 0$, if rankW^*=rank $[\boldsymbol{W}^*\,;\,\boldsymbol{G}^*] < N + 1$, then we may find particular solutions. Else if rank≠rank $[\boldsymbol{W}^*\,;\,\boldsymbol{G}^*] < N + 1$, then it has no solution.

Furthermore, in order to solve (4b), we give the fundamental matrix equation by using (7), (17), (21) and (23)-(25).

$$\boldsymbol{W}\boldsymbol{Y}+\boldsymbol{Z}\overline{\boldsymbol{Y}}+\boldsymbol{T}\overline{\overline{\boldsymbol{Y}}} = \boldsymbol{G}, \qquad (31)$$

where $\boldsymbol{W} = [w_{ij}]$, $(i, j = 0,1, \ldots, N)$ represents the matrix form of the linear parts (as in (27)),

$$\boldsymbol{Z} = \left[z_{pq}\right] = \boldsymbol{Z}_1\boldsymbol{X}\boldsymbol{S}\left(\alpha\right)\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{S}}\left(\alpha\right)\right);$$

$p = 0, 1, \ldots, N + 1,$

$q = 0, 1, \ldots, (N+1)^2,$

$$\boldsymbol{T} = \left[t_{rs}\right] = \boldsymbol{T}_1\boldsymbol{X}\boldsymbol{S}\left(\alpha\right)\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{S}}\left(\alpha\right)\right)\left(\overline{\overline{\boldsymbol{X}}}\right)\left(\overline{\overline{\boldsymbol{S}}}\left(\alpha\right)\right);$$

$r = 0, 1, \ldots, N + 1,$

$s = 0, 1, \ldots, (N + 1)^3.$

Likewise, we obtain the following matrix equation by using (29) and (31):

$$\boldsymbol{W}^*\boldsymbol{Y}+\boldsymbol{Z}^*\overline{\boldsymbol{Y}}+\boldsymbol{T}^*\overline{\overline{\boldsymbol{Y}}}=\boldsymbol{G}^*\Rightarrow\left[\boldsymbol{W}^*;\ \boldsymbol{Z}^*;\ \boldsymbol{T}^*:\boldsymbol{G}^*\right]. \tag{32}$$

When the system (32) is solved, the unknown Dickson coefficients y_n are obtained. If they are substituted into (6), then we will get the Dickson polynomial solution via the method.

Residual Error Analysis

In this section, we will give an error analysis based on the residual function (Kürkçü et al. 2016) for the Dickson-Taylor collocation method. In addition, we will improve the Dickson polynomial solutions (6) by means of the residual error function. We can define the residual function of the Dickson-Taylor collocation method as:

$$R_N(x) = L[y_N(x)] - g(x), \tag{33}$$

where $L[y_N(x)]\cong g(x)$. The error function $e_N(x)$ can also be defined as:

$$e_N(x) = y(x) - y_N(x), \tag{34}$$

where $y(x)$ is the exact solution of the problem (4a). From (4a), (5), (33) and (34), we obtain the error equation (ODEs, FVIDEs, FIDEs or VIDEs):

$$L[e_N(x)] = L[y(x)] - L[y_N(x)] = -R_N(x),$$

with the homogeneous initial conditions

$$e_N^{(k)}(a)=0,$$

or briefly, the error problem is expressed as:

$$\left.\begin{aligned} L\left[e_N\left(x\right)\right] &= -R_N\left(t\right) \\ e_N^{(k)}(a) &= 0 \end{aligned}\right\}, \tag{35}$$

where the nonhomegeneous initial conditions (5) are reduced to homogeneous initial conditions

$$e_N^{(k)}(a)=0\,.$$

The error problem (35) can be solved by using the given procedure in Method of Solution Section. Then, we obtain the approximation

$$e_{N,M}(x)=\sum_{n=0}^{M}y_n^*D_n\left(x,\alpha\right),\ \left(M>N\right),$$

where $e_{N,M}(x)$ is the Dickson polynomial solution of the error problem obtained by using the residual error function. Consequently, the corrected Dickson polynomial solution $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$ is obtained by means of the polynomials $y_N(x)$ and $e_{N,M}(x)$. We also construct the error function $e_N(x) = y(x) - y_N(x)$, the estimated error function $e_{N,M}(x)$ and the corrected error function $E_{N,M}(x) = e_N(x) - e_{N,M}(x) = y(x) - y_{N,M}(x)$.

Note that this residual error analysis can not be used for the nonlinear (4b).

Numerical Examples

In this section, numerical examples are given to illustrate the efficiency and applicability of the method. The computations in the examples are calculated by using Mathematica 10 program. In Example 5.2, we calculate the values of the corrected Dickson polynomial solutions $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$, estimated error functions $e_{N,M}(x)$ and the corrected absolute error functions $|E_{N,M}(x)| = |y(x) - y_{N,M}(x)|$. Besides, we find a good approximation to exact solution of the nonlinear integro-differential equation in Example 5.4.

Example 5.1 (Akyüz-Daşcıoğlu et al. 2007; Yalçınbaş et al. 2009, 2000) First, let us consider the linear FIDE

$$y''\left(x\right)+xy'\left(x\right)-xy\left(x\right)=e^x-2\sin x+\int_{-1}^{1}\left(\sin x\right)e^{-t}y\left(t\right)dt$$
$$-1\le x,\, t\le 1,$$

with the initial conditions $y(0) = 1$ and $y'(0) = 1$. We suppose the problem has a Dickson polynomial solution,

$$y_N\left(x\right)=\sum_{n=0}^{3}y_nD_n\left(x,\alpha\right),$$

such that $N = 3$, $P_0(x) = -x$, $P_1(x) = x$, $P_2(x) = 1$, $K_f(x, t) = (\sin x)\, e^{-t}$, $-1 \le x, t \le 1$, $\lambda_1 = 1$, $\lambda_2 = 0$ and . For $g(x) = e^x - 2\sin(x)$. For $N = 3$, the collocation points are

$$x_i=a+\left(\frac{b-a}{N}\right)i,\ i=0,1,2,3\Rightarrow x_0=-1,$$
$$x_1=-\frac{1}{3},\ x_2=\frac{1}{3},\ x_3=1.$$

The fundamental matrix representation of the FIDE is

$$\underbrace{\left\{\left(\boldsymbol{P}_0\boldsymbol{X}\boldsymbol{B}^0+\boldsymbol{P}_1\boldsymbol{X}\boldsymbol{B}^1+\boldsymbol{P}_2\boldsymbol{X}\boldsymbol{B}^2\right)\boldsymbol{S}\left(\alpha\right)-\lambda_1\boldsymbol{X}\boldsymbol{S}\left(\alpha\right)\boldsymbol{K}_f\boldsymbol{Q}_f\right\}}_{\boldsymbol{W}}\boldsymbol{Y}=\boldsymbol{G}$$

or $[\boldsymbol{W}\,;\,\boldsymbol{G}]$,

where

$$\boldsymbol{P_0}=\begin{bmatrix}1&0&0&0\\0&\frac{1}{3}&0&0\\0&0&-\frac{1}{3}&0\\0&0&0&-1\end{bmatrix},\ \boldsymbol{P}_1=\begin{bmatrix}-1&0&0&0\\0&-\frac{1}{3}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&1\end{bmatrix},$$

$$\boldsymbol{P}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \boldsymbol{X}=\begin{bmatrix}1&-1&1&-1\\1&-\frac{1}{3}&\frac{1}{9}&\frac{1}{27}\\1&\frac{1}{3}&\frac{1}{9}&\frac{1}{27}\\1&1&1&1\end{bmatrix}.$$

$$\boldsymbol{Q}_f=\begin{bmatrix}8&0&-8\alpha+\frac{4}{3}&0\\0&\frac{2}{3}&0&-2\alpha+\frac{2}{5}\\-8\alpha+\frac{4}{3}&0&8\alpha^2-\frac{8\alpha}{3}+\frac{2}{5}&0\\0&-2\alpha+\frac{2}{5}&0&6\alpha^2-\frac{12\alpha}{5}+\frac{2}{7}\end{bmatrix},$$

$$\boldsymbol{G}=\begin{bmatrix}2.05082\\1.37092\\0.741223\\1.03534\end{bmatrix}.$$

For the given conditions, the fundamental matrices are acquired as, respectively,

$[\boldsymbol{U}_0\,;\boldsymbol{\mu}_0]=[2\quad 0\quad -2\alpha\quad 0\quad ;\quad 1]$ and

$[\boldsymbol{U}_1\,;\boldsymbol{\mu}_1]=[0\quad 1\quad 0\quad -3\alpha\quad ;\quad 1].$

The augmented matrix is

$$\left[\boldsymbol{W}^*\,;\boldsymbol{G}^*\right]=\begin{bmatrix}\frac{53}{9}&-\frac{47}{18}&-\frac{53\alpha}{9}+\frac{103}{18}&\frac{47\alpha}{6}-\frac{1307}{126}&;&2.05082\\\frac{533}{243}&-\frac{1663}{2430}&\frac{-5330\alpha+6179}{2430}&\frac{34923\alpha-38611}{17010}&;&1.37092\\2&0&-2\alpha&0&;&1\\0&1&0&-3\alpha&;&1\end{bmatrix}.$$

The solution of this system yields the Dickson coefficients matrix

$$\boldsymbol{Y}=[0.5+0.451521\alpha\quad 1+0.25052\alpha\quad 0.451521\quad 0.0835065]^T.$$

Hence, we get the approximate solution of the problem

$$y_3(x)=\sum_{n=0}^{3}y_nD_n(x,\alpha)=y_0D_0(x,\alpha)+y_1D_1(x,\alpha)+y_2D_2(x,\alpha)+y_3D_3(x,\alpha)\ ,$$

$$y_3(x)=1+x+0.451521x^2+0.0835065x^3.$$

The following approximate solutions have been given for $N = 6,7,8,9$ as respectively,

$y_6(x) = 1 + x + 0.5x^2 + 0.16675216123536205x^3 + 0.041778260722306185x^4 + 0.008146193641901422x^5+, 0.0009884800170341388x^6,$

$y_7(x) = 1 + x + 0.499999x^2 + 0.166664x^3 + 0.0416795x^4 + 0.0083539x^5 + 0.00137203x^6 + 0.000151515x^7,$

$y_8(x) = 1 + x + 0.5x^2 + 0.166666x^3 + 0.0416663x^4 + 0.00833489x^5 + 0.00139179x^6 + 0.000197236x^7 + 0.0000199175x^8,$

and

$y_9(x) = 1 + x + 0.5x^2 + 0.166667x^3 + 0.0416666x^4 + 0.00833326x^5 + 0.00138908x^6 + 0.000198799x^7 + 0.0000247112x^8 + 2.24328\times10^{-6}x^9.$

Also, the comparison of solutions with the exact solution $y(x) = e^x$ for Example 5.1 are shown in Table 1 and Figure 1.

In Figure 2, the interval [-1,1] cannot be changed. Because Fredholm integral is defined in this interval. If

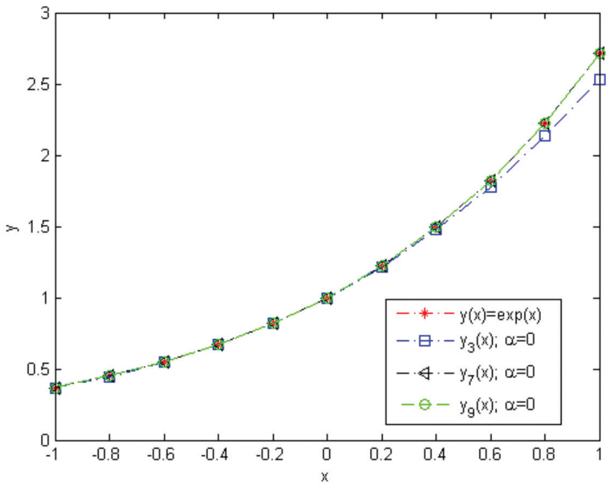

FIGURE 1. Comparison of the exact and the approximate solutions of Example 5.1 for $N = 3, 7, 9$

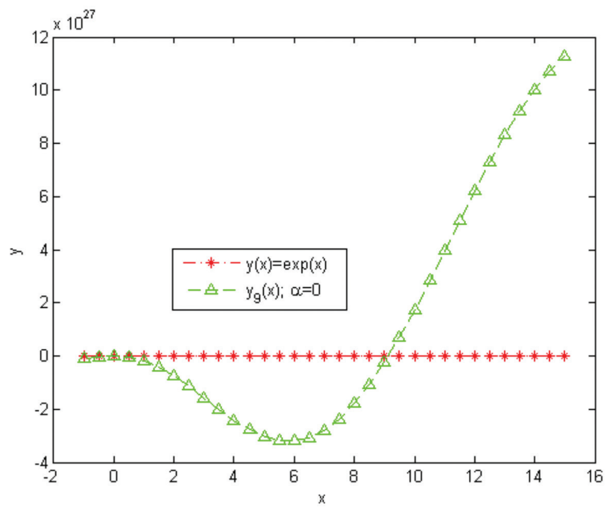

FIGURE 2. For the interval [-1,15] of Example 5.1

TABLE 1. Numerical results of Example 5.1

x_i	Exact Solution	Present method $N=3$	$N=6$	$N=7$	$N=8$	$N=9$
−1.0	0.36787944	0.36801433	0.36786839	0.36788077	0.36787945	0.36787945
−0.8	0.44932896	0.44621801	0.44932505	0.44932958	0.44932895	0.44932897
−0.6	0.54881164	0.54451010	0.54880867	0.54881192	0.54881163	0.54881164
−0.4	0.67032005	0.66689892	0.67031802	0.67032010	0.67032005	0.67032005
−0.2	0.81873075	0.81739278	0.81873028	0.81873073	0.81873075	0.81873075
0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.2	1.22140276	1.21872889	1.22140353	1.22140270	1.22140276	1.22140276
0.4	1.49182470	1.47758776	1.49182913	1.49182466	1.49182470	1.49182470
0.6	1.82211880	1.78058493	1.82211250	1.82211839	1.82211882	1.82211880
0.8	2.22554093	2.13172871	2.22541795	2.22553174	2.22554046	2.22554090
1.0	2.71828183	2.53502742	2.71766510	2.71821934	2.71827657	2.71828137
x_i	Legendre collocation method (Yalçınbaş et al. 2009)		Akyüz & Sezer Method (2007)		Yalcinbas & Sezer Method (2000)	
	$N=3$	$N=6$	$N=6$	$N=7$	$N=8$	$N=9$
−1.0	0.36801200	0.36795047	0.36804500	0.36787900	0.36805000	0.36787900
−0.8	0.44621670	0.44939678	0.44936100	0.44932900	0.44936300	0.44932800
−0.6	0.54450947	0.54887061	0.54881400	0.54881200	0.54881500	0.54881100
−0.4	0.66689869	0.67037064	0.67032000	0.67032000	0.67031900	0.67032000
−0.2	0.81739274	0.81877398	0.81873100	0.81873100	0.81873000	0.81873000
0	1.00000000	1.00003513	1.00000000	1.00000000	1.00000000	1.00000000
0.2	1.21872886	1.22143042	1.22140300	1.22140300	1.22140000	1.22140000
0.4	1.47758771	1.49184806	1.49182500	1.49182500	1.49182000	1.49182000
0.6	1.78058493	1.82212371	1.82211600	1.82211900	1.82211000	1.82211000
0.8	2.13172890	2.22542162	2.22550100	2.22554100	2.22549000	2.22554000
1.0	2.53502800	2.71766127	2.71806700	2.71828200	2.71805000	2.71828000

the interval is changed, the results will be unsuitable as seen from Figure 2 and its interval [-1,15].

Example 5.2 (Yüzbaşı et al. 2011) Second, let us consider the linear VIDE

$$y''(x)+xy'(x)-xy(x)=e^x+\frac{1}{2}x\cos x-\frac{1}{2}\int_0^x(\cos x)e^{-t}y(t)\,dt,$$

$$0\le x,\ t\le 1,$$

with the initial conditions $y(0)=1$ and $y'(0)=1$. Similarly, in order to find the Dickson polynomial solution, we initially take $N=3$ such that $P_0(x)=-x$, $P_1(x)=x$, $P_2(x)=1$, $g(x)=e^x+\frac{1}{2}x\cos x$, $\kappa_v(x,t)=(\cos x)e^{-t}$, $\lambda_1=0$, and $\lambda_2=-\frac{1}{2}$. For $N=3$, the collocation points are

$$x_0=0,\ x_1=\frac{1}{3},\ x_2=\frac{2}{3},\ x_3=1\,.$$

The matrix representation of the linear VIDE is

$$\underbrace{\left\{\left(\boldsymbol{P}_0\boldsymbol{X}\boldsymbol{B}^0+\boldsymbol{P}_1\boldsymbol{X}\boldsymbol{B}^1+\boldsymbol{P}_2\boldsymbol{X}\boldsymbol{B}^2\right)\boldsymbol{S}(\alpha)-\lambda_2\left(\overline{\boldsymbol{X}}\right)\left(\boldsymbol{K}_v\right)\left(\overline{\boldsymbol{Q}}_v\right)\boldsymbol{S}(\alpha)\right\}}_{\boldsymbol{W}}\boldsymbol{Y}=\boldsymbol{G}$$

or $[\boldsymbol{W};\boldsymbol{G}]$,

where $\boldsymbol{B}$ and $\boldsymbol{S}(\alpha)$ matrices are the same as in Example 5.1;

$$\boldsymbol{P_0}=\begin{bmatrix}0&0&0&0\\0&-\frac{1}{3}&0&0\\0&0&-\frac{2}{3}&0\\0&0&0&-1\end{bmatrix},\ \boldsymbol{P}_1=\begin{bmatrix}0&0&0&0\\0&\frac{1}{3}&0&0\\0&0&\frac{2}{3}&0\\0&0&0&1\end{bmatrix},$$

$$\boldsymbol{P_2}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \boldsymbol{X}=\begin{bmatrix}1&0&0&0\\1&\frac{1}{3}&\frac{1}{9}&\frac{1}{27}\\1&\frac{2}{3}&\frac{4}{9}&\frac{8}{27}\\1&1&1&1\end{bmatrix},$$

$$\overline{\boldsymbol{X}}=\begin{bmatrix}1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&\frac{1}{3}&\frac{1}{9}&\frac{1}{27}&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&\frac{2}{3}&\frac{4}{9}&\frac{8}{27}&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\end{bmatrix},$$

$$\boldsymbol{K}_t^v = \begin{bmatrix} 1 & -1 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{12} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{\boldsymbol{K}}_v = \begin{bmatrix} \left[\boldsymbol{K}_t^v\right]_{4\times4} & [0]_{4\times4} & [0]_{4\times4} & [0]_{4\times4} \\ [0]_{4\times4} & \left[\boldsymbol{K}_t^v\right]_{4\times4} & [0]_{4\times4} & [0]_{4\times4} \\ [0]_{4\times4} & [0]_{4\times4} & \left[\boldsymbol{K}_t^v\right]_{4\times4} & [0]_{4\times4} \\ [0]_{4\times4} & [0]_{4\times4} & [0]_{4\times4} & \left[\boldsymbol{K}_t^v\right]_{4\times4} \end{bmatrix}_{16\times16},$$

$$\overline{\boldsymbol{Q}_v} = \begin{bmatrix} \boldsymbol{Q}_v(0) \\ \boldsymbol{Q}_v\left(\frac{1}{3}\right) \\ \boldsymbol{Q}_v\left(\frac{2}{3}\right) \\ \boldsymbol{Q}_v(1) \end{bmatrix}_{16\times4}, \quad \boldsymbol{G} = \begin{bmatrix} 1. \\ 1.55311 \\ 2.20970 \\ 2.98843 \end{bmatrix}.$$

The condition matrices are obtained as

$$[\boldsymbol{U}_0 ; \boldsymbol{\mu}_0] = [2 \quad 0 \quad -2\alpha \quad 0 \quad ; \quad 1] \text{ and}$$

$$[\boldsymbol{U}_1 ; \boldsymbol{\mu}_1] = [0 \quad 1 \quad 0 \quad -3\alpha \quad ; \quad 1].$$

Thereby, the augmented matrix for Example 5.2 is

$$\left[\boldsymbol{W}^* ; \boldsymbol{G}^*\right] = \begin{bmatrix} 0 & 0 & 2 & 0 & ; & 1 \\ -\frac{13961}{34992} & \frac{255397}{1049760} & \frac{646508}{295245}+\frac{13961\alpha}{34992} & -\frac{127(-1093508+380079\alpha)}{66134880} & ; & 1.55311 \\ 2 & 0 & -2\alpha & 0 & ; & 1 \\ 0 & 1 & 0 & -3\alpha & ; & 1 \end{bmatrix}.$$

By solving the system, we obtain the Dickson coefficients matrix

$$\boldsymbol{Y} = [0.5+0.5\alpha \quad 1+0.592084222416703\alpha \quad 0.5 \quad 0.1973614074722343]^T$$

and the approximate solution of linear VIDE

$$y_3(x) = 1+x+0.5x^2 + 0.1973614074722343x^3.$$

In similar way, we obtain the solution of the problem for $N = 7$,

$$y_7(x) = 1+x+0.55x^2 + 0.1666689439997x^3 + 0.0416485846103x^4 + 0.0083947557404x^5 + 0.0012837229956x^6 + 0.0002855234008x^7.$$

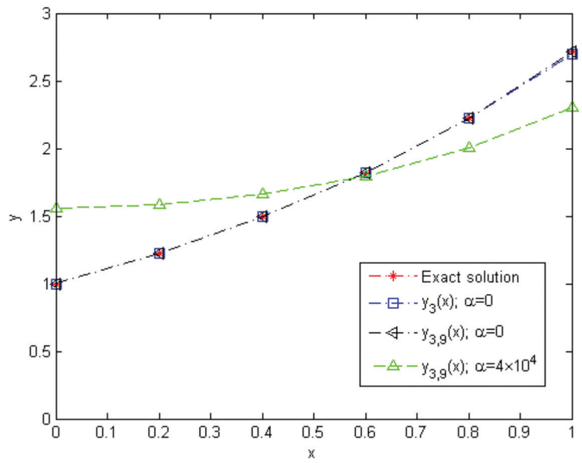

FIGURE 3. Comparison of the exact, approximate and the corrected Dickson polynomial solutions according to the parameter-α for Example 5.2

TABLE 2. Numerical results of Example 5.2

x_i	Exact solution $y(x_i)=e^{x_i}$	Present method		Bessel polynomial method (Yüzbaşı et al. 2011)	
		$N=3$	$N=7$	$N=3$	$N=7$
0	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000
0.2	1.2214027581602	1.2215788912598	1.2214027614222	1.2215788912598	1.2214027614222
0.4	1.4918246976413	1.4926311300782	1.4918247044117	1.4926311300782	1.4918247044117
0.6	1.8221188003905	1.8226300640140	1.8221188108838	1.8226300640140	1.8221188108838
0.8	2.2255409284925	2.2210490406258	2.2255409520233	2.2210490406257	2.2255409520234
1.0	2.7182818284590	2.6973614074722	2.7182815307467	2.6973614074720	2.7182815307470

TABLE 3. Numerical results of the exact and the approximate solutions for N =3 and M =5,9 of Example 5.2

x_i	Exact solution $y(x_i)=e^{x_i}$	Present method $y_3(x_i)$	Corrected Dickson polynomial solution $y_{3,5}(x_i)$; $\alpha=0$	$y_{3,5}(x_i)$; $\alpha=10$	$y_{3,9}(x_i)$; $\alpha=0$
0	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000
0.2	1.2214027581602	1.2215788912598	1.2214039392159	1.2214039392159	1.2214027581651
0.4	1.4918246976413	1.4926311300782	1.4918273197988	1.4918273197988	1.4918246976517
0.6	1.8221188003905	1.8226300640140	1.8221232119132	1.8221232119132	1.8221188004062
0.8	2.2255409284925	2.2210490406258	2.2255392749173	2.2255392749173	2.2255409285146
1.0	2.7182818284590	2.6973614074722	2.7181514509198	2.7181514509198	2.7182818277320

Table 2 indicates the comparison of solutions with the exact solution $y(x) = e^x$.

Now, we calculate the corrected Dickson polynomial solutions for $N = 3$ and $M = 5,9$. In Table 3 and Figure 3, we compare the exact solution and the approximate solutions for N=3 and M=5,9.

$y_{3,5}$ = $1+x+0.5x^2 + 0.03977088752456758x^4 + 0.01129410303351437x^5 + (0.1670864603617452 - (1.387778780781445e - 17)\alpha)x^3 - (1.387778780781445e - 17)\alpha^2 + (2.775557561562891e - 17)x\alpha^2$.

$y_{3,9}$ = $1+0.0000224849x^8 + (4.07868e - 6)x^9 + (0.000200566 + (6.77626e - 21)\alpha)x^7 + (0.00138771 + (2.71051e - 20)\alpha)x^6 + (3.46945e - 18)\alpha^3 + (2.168e - 19)\alpha^4 + (0.00833372 + (4.33681e - 19)\alpha - (5.42101e - 20)\alpha 2)x^5 + (0.0416666 + (1.04083e - 17)\alpha + (4.33681e - 19)\alpha^2)x^4 + (0.5 - (1.38778e - 17)\alpha^2 - (9.75782e - 19)\alpha^3)x^2 + (0.166667 + (3.46945e - 17)\alpha - (8.67362e - 19)\alpha^2 + (2.1684e - 19(\alpha^3)x^3 + x(1 + (4.16334e - 17)\alpha - (4.16334e - 17)\alpha^2 - (3.79471e - 19)\alpha^4)$.

Similarly, we calculate the corrected Dickson polynomial solutions for N=7 and M=9. The comparisons are given in Table 4. Then, the comparison of the corrected absolute errors are given in Tables 5 and 6.

As seen from Tables 5 and 6, the corrected absolute errors are close to zero. So, when the values of M increase, the accuracy of solution increases. However, when the values of parameter-α increase, the tolerance increases.

Example 5.3 (Akyüz-Daşcıoğlu 2006) Let us consider the linear FVIDE

$$xy''(x) - xy'(x) + 2y(x) = \frac{x^4}{12} - \frac{x^3}{6} - \frac{x^2}{2} - \frac{13x}{6} + \frac{17}{12} + \int_0^1 (x+t)y(t)\,dt + \int_0^x (x-t)y(t)\,dt,$$

$0 \le x,\ t \le 1$,

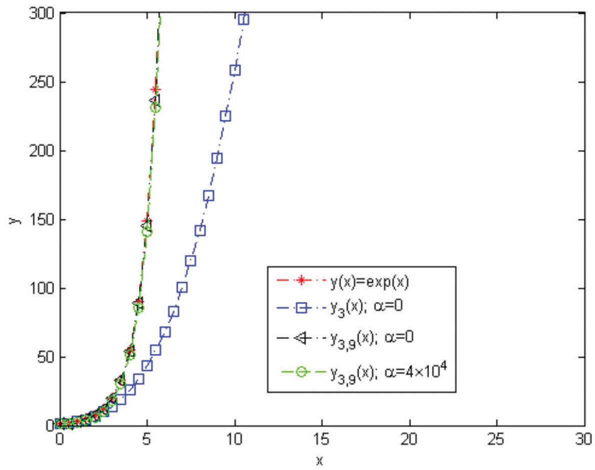

FIGURE 4. For the interval [0,30] of Example 5.2

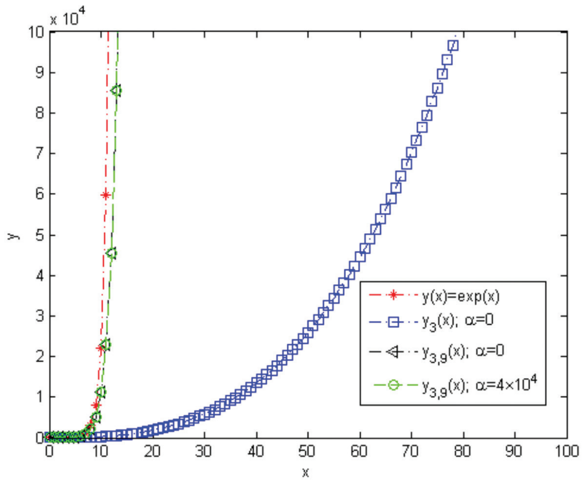

FIGURE 5. For the interval [0,100] of Example 5.2

with the conditions $y(0) = 1$, $y'(0) - 2y(1) + 2y(0) = 1$. In order to solve the above problem, we take $N = 5$. Hence, the matrix representation of linear FVIDE is

$$\underbrace{\left\{\left(\boldsymbol{P}_0\boldsymbol{XB}^0 + \boldsymbol{P}_1\boldsymbol{XB}^1 + \boldsymbol{P}_2\boldsymbol{XB}^2\right)\boldsymbol{S}(\alpha) - \lambda_1\boldsymbol{XS}(\alpha)\boldsymbol{K}_f\boldsymbol{Q}_f - \lambda_2\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{K}_v}\right)\left(\overline{\boldsymbol{Q}_v}\right)\boldsymbol{S}(\alpha)\right\}}_{\boldsymbol{W}}\boldsymbol{Y} = \boldsymbol{G}$$

or $[\boldsymbol{W}\,;\,\boldsymbol{G}]$

TABLE 4. Numerical results of the exact and the approximate solutions for N=7 and M=9 of Example 5.2

x_i	Exact solution $y(x_i) = e^{x_i}$	Present method $y_7(x_i)$	Corrected Dickson polynomial solution $y_{7,9}(x_i)$; $\alpha = 0$	$y_{7,9}(x_i)$; $\alpha = 10$	$y_{7,9}(x_i)$; $\alpha = 100$
0	1.0000000000000	1.0000000000000	1.000000000000000	1.0000000000000000	1.0000000000000000
0.2	1.2214027581602	1.2215788912598	1.221402764679255	1.2214027646792536	1.2214027646591990
0.4	1.4918246976413	1.4926311300782	1.491824711171731	1.4918247111717295	1.4918247111529113
0.6	1.8221188003905	1.8226300640140	1.822118821361273	1.8221188213612722	1.8221188213437278
0.8	2.2255409284925	2.2210490406258	2.225540975531958	2.2255409755319570	2.2255409755157194

TABLE 5. Numerical results of the corrected absolute errors for N=3, M=5,9 of Example 5.2

x_i	Absolute errors	Corrected absolute errors			
	$\lvert e_3(x_i)\rvert = \lvert y(x_i)-y_3(x_i)\rvert$	$\lvert E_{3,5}(x_i)\rvert$; $\alpha=0$	$\lvert E_{3,5}(x_i)\rvert$; $\alpha=10$	$\lvert E_{3,9}(x_i)\rvert$; $\alpha=0$	$\lvert E_{3,9}(x_i)\rvert$; $\alpha=10$
0	0	0	$1.33227e-15$	0	$5.77316e-15$
0.2	$1.76133e-04$	$1.18106e-06$	$1.18106e-06$	$4.9396e-12$	$4.9436e-12$
0.4	$8.06432e-04$	$2.62216e-06$	$2.62216e-06$	$1.04274e-11$	$1.04297e-11$
0.6	$5.11264e-04$	$4.41152e-06$	$4.41152e-06$	$1.57181e-11$	$1.57185e-11$
0.8	$4.49189e-03$	$1.65358e-06$	$1.65358e-06$	$2.21236e-11$	$2.21219e-11$
1.0	$2.09204e-02$	$1.30378e-04$	$1.30378e-04$	$7.27014e-10$	$7.27018e-10$

TABLE 6. Numerical results of the corrected absolute errors for N=7, M=9 of Example 5.2

x_i	Absolute errors	Corrected absolute errors			
	$\lvert e_7(x_i)\rvert = \lvert y(x_i)-y_7(x_i)\rvert$	$\lvert E_{7,9}(x_i)\rvert$; $\alpha=0$	$\lvert E_{7,9}(x_i)\rvert$; $\alpha=10$	$\lvert E_{7,9}(x_i)\rvert$; $\alpha=100$	$\lvert E_{7,9}(x_i)\rvert$; $\alpha=10^4$
0	0	0	$1.77636e-15$	$2.125e-11$	$2.16797e-03$
0.2	$3.26201e-09$	$6.51909e-09$	$6.51908e-09$	$6.49903e-09$	$2.05943e-03$
0.4	$6.77044e-09$	$1.35305e-08$	$1.35305e-08$	$1.35116e-08$	$1.95085e-03$
0.6	$1.04932e-08$	$2.09708e-08$	$2.09708e-08$	$2.09532e-08$	$1.84224e-03$
0.8	$2.35308e-08$	$4.70395e-08$	$4.70395e-08$	$4.70233e-08$	$1.73357e-03$
1.0	$2.97712e-07$	$5.94698e-07$	$5.94698e-07$	$5.94712e-07$	$1.62554e-03$

where $P_0(x)=2,\ P_1(x)=-x,\ P_2(x)=x,\ g(x)=\frac{x^4}{12}-\frac{x^3}{6}-\frac{x^2}{2}-\frac{13x}{6}+\frac{17}{12},\ K_f(x,t)=(x+t),\ K_v(x,t)=(x-t)$ and $\lambda_1=\lambda_2=1$. Also, the collocation points are

$$x_0=0,\ x_1=\frac{1}{5},\ x_2=\frac{2}{5},\ x_3=\frac{3}{5},\ x_4=\frac{4}{5},\ x_5=1.$$

We obtain the augmented matrix as

$$\left[\boldsymbol{W}^*;\boldsymbol{G}^*\right]=\begin{bmatrix} 3 & -\frac{1}{3} & -\frac{1}{4}-3\alpha & -\frac{1}{5}+\alpha & -\frac{1}{6}+\alpha+3\alpha^2 & -\frac{1}{7}+\alpha-\frac{5\alpha^2}{3} & ; & \frac{17}{12} \\ \frac{64}{25} & -\frac{88}{375} & \frac{52}{625}-\frac{64\alpha}{25} & -\frac{563}{31250}+\frac{88\alpha}{125} & \frac{16\left(-556-1625\alpha+12500\alpha^2\right)}{78125} & -\frac{238138}{1640625}+\frac{563\alpha}{6250}-\frac{88\alpha^2}{75} & ; & \frac{1804}{1875} \\ \frac{51}{25} & -\frac{18}{125} & \frac{3109}{7500}-\frac{51\alpha}{25} & \frac{3\left(6203+4500\alpha\right)}{31250} & \frac{73437}{156250}-\frac{3109\alpha}{1875}+\frac{51\alpha^2}{25} & \frac{445786}{1640625}-\frac{18609\alpha}{6250}-\frac{18\alpha^2}{25} & ; & \frac{3461}{7500} \\ \frac{36}{25} & -\frac{26}{375} & -\frac{6}{625}\left(-77+150\alpha\right) & \frac{49691+6500\alpha}{31250} & \frac{479198}{234375}-\frac{1848\alpha}{625}+\frac{36\alpha^2}{25} & \frac{1156748}{546875}-\frac{49691\alpha}{6250}-\frac{26\alpha^2}{75} & ; & -\frac{166}{1875} \\ 2 & 0 & -2\alpha & 0 & 2\alpha^2 & 0 & ; & 1 \\ 0 & -1 & -2 & -2+3\alpha & -2+8\alpha & -2+10\alpha-5\alpha^2 & ; & 1 \end{bmatrix}$$

and the Dickson coefficients matrix

$$\boldsymbol{Y}=\begin{bmatrix} \frac{1}{2}-\alpha & 1 & -1 & 0 & 0 & 0 \end{bmatrix}^T.$$

Thereby, we get the solution

$$y(x) = -x^2 + x + 1,$$

which is the exact solution.

Example 5.4 Finally, let us consider the nonlinear Volterra integro-differential equation

$$y''(x) - 2y(x) + y^2(x) = g(x) + \int_0^x \frac{1}{\cos(t)} y(t)\,dt,\ 0 \le x, t \le 1$$

with the conditions $y(0) = 1$ and $y'(0) = 0$. The exact solution of the equation is $y(x) = \cos(x)$. Here $P_0(x) = -2$, $P_2(x) = 1$, $\lambda_2 = 1$ and $g(x) = \cos^2(x) - 3\cos(x) - x$. We now construct the fundamental matrix equation from (32).

$$\left\{\boldsymbol{P}_0\boldsymbol{X}\boldsymbol{B}^0 + \boldsymbol{P}_2\boldsymbol{X}\boldsymbol{B}^2 - \lambda_2\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{K}}_v\right)\left(\overline{\boldsymbol{Q}}_v\right)\right\}\boldsymbol{S}(\alpha)\boldsymbol{Y} + \boldsymbol{Z}_1\boldsymbol{X}\boldsymbol{S}(\alpha)\left(\overline{\boldsymbol{X}}\right)\overline{\boldsymbol{S}}(\alpha)\overline{\boldsymbol{Y}} = \boldsymbol{G}.$$

When this system is solved, we obtain the Dickson polynomial solutions by applying N=3 and some different values of the parameter-α.

$$y_3(x)\,\big|_{\alpha=0} = 1 - 0.5x^2 - 0.00521369x^3,$$

$$y_3(x)\,\big|_{\alpha=-0.5} = 1 - 1.27329\times10^{-15}x - 0.5745505x^2 - 0.0140165x^3,$$

$$y_3(x)\,\big|_{\alpha=0.4} = 1 + 2.82326\times10^{-16}x - 0.46197x^2 - 0.00168x^3,$$

$$y_3(x)\,\big|_{\alpha=0.9} = 1 - 1.04084\times10^{-17}x - 0.42714x^2 + 0.00103x^3.$$

As seen from Figure 6, we achieved consistent aproximate solutions by using the present method. If the parameter-α is choosen in [-0.5, 0.9], the results of Example 5.4 will be close to the exact solution. For the best approximation, the parameter-α is choosen as α = 0.4. In addition, except for this interval, the results will be connected to complex or null space. Therefore, the parameter-α should be choosen in this interval. Also, as seen from Figures 4, 5 and 7, when the interval is expanded, the results have been deviated a little from the exact solutions, but the good approximations have been obtained by the present method.

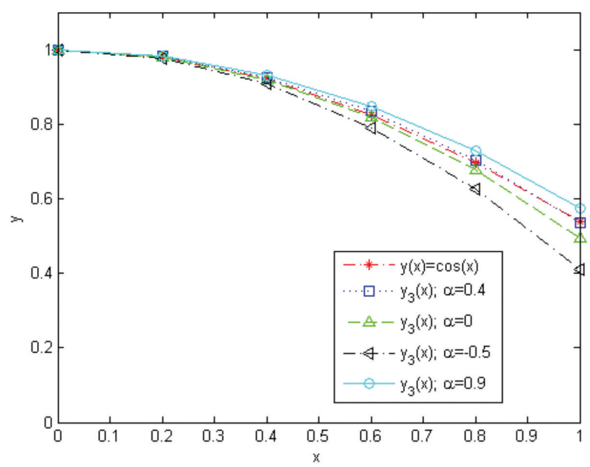

FIGURE 6. Comparison of the exact and the approximate solutions of Example 5.4 for N = 3 with α = -0.5, 0, 0.4, 0.9

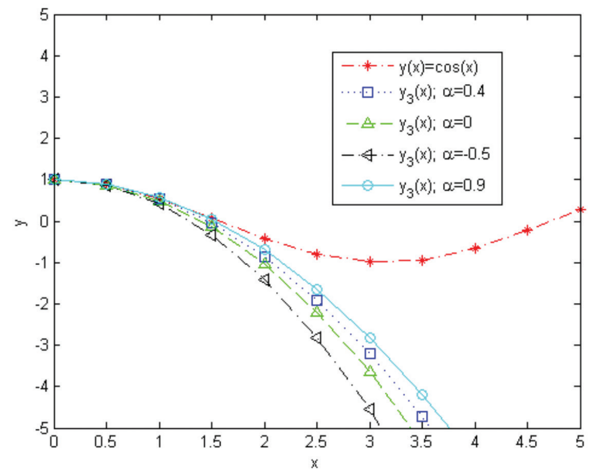

FIGURE 7. For the interval [0,5] of Example 5.4

ALGORITHM

In this section, the Pseudocode has been given for calculation of (4a). This can also be applied to (2) and (3).

Step 1

a. Input the number of truncated Dickson polynomial solution $N \in \mathbb{N}$ such that $N \ge m$ (6).
b. Determine a, b, λ_1, λ_2, $P_0(x)$, …, $P_k(x)$, (k = 0, 1, …, m), $K_f(x, t)$, $K_v(x, t)$, $g(x)$ and mixed conditions.
c. The mixed conditions put in (5).
d. According to N (N is even or odd), set $\boldsymbol{S}(\alpha)$.

Step 2 Set the collocation points x_i, i = 0, 1, …, N. There are $x_0 = a$ and $x_N = b$.

Step 3

a. Construct the matrices $\boldsymbol{P}_k$ $(k = 0, \ldots, m)$, $\boldsymbol{B}$, $\boldsymbol{X}$, $\boldsymbol{K}_f$, $\boldsymbol{Q}_f$, $\overline{\boldsymbol{X}}$, $\overline{\boldsymbol{K}}_v$, and $\overline{\boldsymbol{Q}}_v$ from (27).
b. Compute $\boldsymbol{W}$ and $\boldsymbol{G}$ matrices.
c. Construct the conditional (m-1)-rows matrices from (29).

Step 4 Construct the augmented $[\boldsymbol{W}^*; \boldsymbol{G}^*]$ matrix from (30).

Step 5 If rank $\boldsymbol{W}^*$ = rank $[\boldsymbol{W}^*; \boldsymbol{G}^*] = N + 1$, then solve the system by using Gaussian elimination method (or to solve the $\boldsymbol{Y} = (\boldsymbol{W}^*)^{-1}\,\boldsymbol{G}^*$).

Step 6 Substituting all elements of the Dickson coefficients matrix solution, respectively, into (6). Finally, this will be our solution.

Conclusion

High-order linear and nonlinear Fredholm-Volterra integro-differential equations (FVIDEs) are usually difficult to solve as analytically. Therefore, it is necessary to use approximate methods. For these purposes, the present method has been given to find consistent approximate solutions. One of the remarkable advantages of the present method, the Dickson coefficients obviously find with the aid of the computer programs. At the same time, the presence of parameter-α is required to use computer program along with the present method for the accuracy of solutions. The results related with examples have been shown in Tables 1-6 and Figures 1-7. As seen from tables and figures, when the value of N is increased, the numerical results improve. On the other hand, if the interval $a \leq x, t \leq b$ is taken, the width intervals as [0,30], [0,100]…, it is seen that the approximations are not good. We have also improved the approximate solutions by using the residual error analysis. The present method can be developed for the systems of differential, integral and integro-differential equations. But some modifications are required.

References

Akyüz-Daşcıoğlu, A. & Sezer, M. 2007. A Taylor polynomial approach for solving high-order linear Fredholm integro-differential equations in the most general form. *Int. J. Comput. Math.* 84(4): 527-539.

Akyüz-Daşcıoğlu, A. 2006. A Chebyshev polynomial approach for linear Fredholm-Volterra integro-differential equations in the most general form. *Appl. Math. Comput.* 181: 103-112.

Babolian, E., Lotfi, T. & Paripour, M. 2007. Wavelet moment method for solving Fredholm integral equations of the first kind. *Appl. Math. Comput.* 186: 1467-1471.

Bhargava, M. & Zieve, M.E. 1999. Factorizing Dickson polynomials over finite fields. *Finite Fields Appl.* 5: 103-111.

Brewer, B.W. 1961. On certain character sums. *Transactions of the American Mathematical Society* 99(2): 241-245.

Dickson, L.E. 1896. The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group I-II. *Annals of Mathematics* 11(1/6): 65-120.

Evans, D.J. & Raslan, K.R. 2005. The Adomian decomposition method for solving delay differential equation. *Int. J. Comput. Math.* 82: 49-54.

Fernando, N. 2013. A study of permutation polynomials over finite fields, (Graduate Theses and Dissertations), http://scholarcommons.usf.edu/etd/4484. *University of South Florida*.

Kürkçü, Ö.K., Ersin, A. & Sezer, M. 2016. A numerical approach with error estimation to solve general integro-differential-difference equations using Dickson polynomials. *Appl. Math. Comput.* 276: 324-339.

Karamete, A. & Sezer, M. 2002. A Taylor collocation method for the solution of linear integro-differential equations. *Int. J. Comput. Math.* 79(9): 987-1000.

Kurt, N. & Sezer, M. 2008. Polynomial solution of high-order linear Fredholm integro-differential equations with constant coefficients. *J. Franklin Inst.* 345: 839-850.

Lidl, R., Mullen, G.L. & Turnwald, G. 1993. Dickson polynomials, Pitman monographs and surveys in pure and applied mathematics. *Longman Scientific and Technical*. Harlow, Essex. p. 65.

Sezer, M. 1996. A method for approximate solution of the second order linear differential equations in terms of Taylor polynomials. *Int. J. Math. Educ. Sci. Technol.* 27(6): 821-834.

Sezer, M. 1994. Taylor polynomial solutions of Volterra integral equations. *Int. J. Math. Educ. Sci. Technol.* 25(5): 625-633.

Stoll, T. 2007. Complete decomposition of Dickson-type recursive polynomials and a related Diophantine equation. *Formal Power Series and Algebraic Combinatorics*, Tianjin, China.

Wei, P., Liao, X. & Wong, K-W. 2011. Key exchange based on Dickson polynomials over finite field with 2^m. *Journal of Computers* 6(12): 2546-2551.

Yalçınbaş, S., Sezer, M. & Sorkun, H. 2009. Legendre polynomial solutions of high-order linear Fredholm integro-differential equations. *Appl. Math. Comput.* 210: 334-349.

Yalçınbaş, S. & Sezer, M. 2000. The approximate solution of high-order linear Volterra-Fredholm integro-differential equations in terms of Taylor polynomials. *Appl. Math. Comput.* 112: 291-308.

Yüzbaşı, Ş., Şahin, N. & Sezer, M. 2011. Bessel polynomial solutions of high-order linear Volterra integro-differential equations. *Computers and Mathematics with Applications* 62: 1940-1956.

Ömür Kıvanç Kürkçü*
Department of Mathematics
Faculty of Science
Celal Bayar University, Manisa 45140
Turkey

Ersin Aslan
Turgutlu Vocational Training School
Celal Bayar University, Manisa
Turkey

Mehmet Sezer
Department of Mathematics
Faculty of Science
Celal Bayar University, Manisa 45140
Turkey

*Corresponding author; email: omurkivanc@outlook.com

Received: 1 May 2015
Accepted: 18 June 2016