# DYNAMIC AND STATIC GREEN'S FUNCTIONS IN TRANSVERSELY ISOTROPIC ELASTIC MEDIA

**by**

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## ABSTRACT

Concise and numerically feasible dynamic and static Green's functions are obtained in dyadic form by solving the wave equation and the equilibrium equation with general source distribution in transversely isotropic (TI) media. The wave and equilibrium equations are solved by using an extended version of the Kupradze method originally developed for isotropic media. The dynamic Green's function is expressed through three scalar quantities characterizing the propagation of $SH$ and $P$-$SV$ waves in a transversely isotropic medium. The 2-D inverse Laplacian operator contained in previous Green's function expressions is eliminated without limiting to special cases and geometries. The final dyadic form is similar to that of the isotropic dyadic Green's function, and therefore lends itself to easy analytical and numerical manipulations. The static Green's function has the same dyadic form as the dynamic function except that the three scalars must be redefined. From the dynamic Green's function, displacements due to vertical, horizontal, and explosive sources are explicitly given. The displacements of the explosive source show that an explosive source in a TI medium excites not only the *quasi-P* wave, but also the *quasi-SV* wave. The singular properties of the Green's functions are also addressed through their surface integrals in the limit of coinciding receiver and source. The singular contribution is shown to be $-1/2$ when the static stress Green's function is integrated over a half elliptical surface.

## INTRODUCTION

Wave propagation from various seismic sources placed inside a fluid-filled borehole embedded in a layered transversely isotropic medium is of great interest and importance to geophysicists dealing with crosshole, vertical seismic profiling, and acoustic logging data. In simulating wave propagation in this geometry, ordinary numerical techniques, such as the finite difference method and the finite element method, encounter computational difficulties because of the significant scale difference between the borehole diameter and the formation extent. A technique (Bouchon, 1992; Dong et al., 1992) perfectly suited to this kind of geometry is the boundary element method (BEM). It is a semi-analytical method because the only discretization occurs at the borehole boundary and the propagation of waves is realized through the use of the dynamic Green's function. This technique also requires the static Green's function to regularize the boundary surface integral when the source and the receiver coincide. These essential requirements of the Green's functions in the BEM technique motivate this work.

Although plane wave propagation in TI media has been studied by many workers (e.g. Fedorov, 1968; Crampin, 1985; among others), literature on the static and dynamic Green's functions of the TI medium are at most scarce. Among the existing ones, most of them provided the solution in component and numerical forms. Pan and Chou (1976) presented explicit solutions of the equilibrium equation in terms of displacement and stress components for vertical and horizontal forces. In their solution procedure, three displacement potentials and an assumed solution form with unknown coefficients were used. Buchwald (1959) solved the wave equation for three strains: $\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$, $\frac{\partial u_z}{\partial z}$ and $\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)$. The far-field approximation of these strains was given using a stationary phase approximation. Other workers (White, 1984; White et al., 1984; Mandal and Toksöz, 1990) employed numerical method to study the radiated waveforms of line source, vertical and horizontal point forces, and explosion source. Kazi-Aoual et al. (1988) devised an algorithm using the Kupradze method (Kupradze, 1979) for calculating the dynamic Green's function. In Kazi-Aoual et al. (1988), the dynamic Green's function is expressed as the cofactor matrix of a symmetrical matrix of differential operators operating on a single scalar. The scalar is represented by the Hankel transform. Ben-Menahem and Sena (1990) and Sena (1992) extended the work of Buchwald and obtained the dynamic Green's tensor in the form of the Hankel transform by recovering the displacement vector from the three strains. This extension is significant because a fairly simple dynamic Green's tensor is given in terms of dyadic notation. However, due to the presence of a 2-D inverse Laplacian operator in the expression, this Green's function does not lend itself easily to numerical and analytical manipulations.

In this paper, we present a unified treatment of the dynamic and the static Green's functions and show that they can be conveniently expressed by a single dyadic form, with different meanings of the symbols for each case, of course. Unlike the previous studies, we solve the equilibrium and wave equations with a general source distribution by using an extended version of the Kupradze method. This not only simplifies the previous derivations but also enhances their rigorousness. More importantly, the final solution does not contain the 2-D inverse Laplacian operator and is valid for the source at an arbitrary location, which is critical for the BEM technique. We then apply the dynamic Green's function to obtain the displacements for the vertical, horizontal and explosive point sources. Finally, the static Green's function is used to compute the singular contribution when integrating the dynamic stress function over a boundary surface in the limit of the source coinciding with the receiver. These results are directly applicable to BEM simulation of wave radiation and scattering.

## THE DYNAMIC GREEN'S FUNCTION

In a Cartesian coordinate system $(x, y, z)$ with unit vector $(\hat{x}, \hat{y}, \hat{z})$, let $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{F} = (F_x, F_y, F_z)$, respectively, be the displacement vector and external body force of a transversely isotropic medium characterized by the five independent elastic stiffness constants, $c_{11}$, $c_{13}$, $c_{33}$, $c_{44}$, and $c_{66}$. The frequency domain wave equation in terms of the displacement components for a transversely isotropic medium can be written in the following form

$$\begin{aligned} c_{11}\frac{\partial^2 u_x}{\partial x^2} + c_{66}\frac{\partial^2 u_x}{\partial y^2} + c_{44}\frac{\partial^2 u_x}{\partial z^2} + (c_{11} - c_{66})\frac{\partial^2 u_y}{\partial x \partial y} + (c_{13} + c_{44})\frac{\partial^2 u_z}{\partial x \partial z} + \rho\omega^2 u_x &= -F_x, \\ c_{11}\frac{\partial^2 u_y}{\partial y^2} + c_{66}\frac{\partial^2 u_y}{\partial x^2} + c_{44}\frac{\partial^2 u_y}{\partial z^2} + (c_{11} - c_{66})\frac{\partial^2 u_x}{\partial x \partial y} + (c_{13} + c_{44})\frac{\partial^2 u_z}{\partial y \partial z} + \rho\omega^2 u_y &= -F_y, \\ c_{33}\frac{\partial^2 u_z}{\partial z^2} + c_{44}\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right) + (c_{13} + c_{44})\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \rho\omega^2 u_z &= -F_z. \end{aligned} \tag{1}$$

In the above equation, the relation $c_{12} = c_{11} - 2c_{66}$ is used. These equations can be easily compared with those in White (1983), where Love's notation for the elastic constants is used. Grouping the first two equations together in terms of transverse displacement, $\mathbf{u}_t = u_x\hat{x} + u_y\hat{y}$, and rewriting the third equation, we obtain

$$c_{66}\nabla_t^2\mathbf{u}_t + c_{44}\frac{\partial^2 \mathbf{u}_t}{\partial z^2} + (c_{11} - c_{66})\nabla_t\nabla_t \cdot \mathbf{u}_t + (c_{13} + c_{44})\frac{\partial}{\partial z}\nabla_t u_z + \rho\omega^2\mathbf{u}_t = -\mathbf{F}_t, \tag{2}$$

$$c_{44}\nabla_t^2 u_z + c_{33}\frac{\partial^2 u_z}{\partial z^2} + (c_{13} + c_{44})\frac{\partial}{\partial z}\nabla_t \cdot \mathbf{u}_t + \rho\omega^2 u_z = -F_z, \tag{3}$$

where, $\mathbf{F}_t = F_x\hat{x} + F_y\hat{y}$. Similar to solving the elastic wave equation for the isotropic case, where the curl and divergence are taken on both sides of the equation, Eqs. (2) and (3) can be solved by taking the transverse curl and the transverse divergence on both sides.

We first take the transverse curl, defined as $\nabla_t \times = [\nabla - \frac{\partial}{\partial z}\hat{z}]\times$, of Eq. (2) to obtain

$$c_{66}\nabla_t^2\nabla_t \times \mathbf{u}_t + c_{44}\frac{\partial^2}{\partial z^2}\nabla_t \times \mathbf{u}_t + \rho\omega^2\nabla_t \times \mathbf{u}_t = -\nabla_t \times \mathbf{F}_t. \qquad (4)$$

The gradient terms disappeared because $\nabla_t \times \nabla_t u = 0$. By virtue of Green's superposition theorem, the solution of this equation in terms of the transverse curl of $\mathbf{u}_t$ is

$$\nabla_t \times \mathbf{u}_t = \int_V g(\mathbf{x}, \mathbf{x}')\, \nabla_t' \times \mathbf{F}_t'\, d\mathbf{x}', \qquad (5)$$

where $\mathbf{x} \doteq (x, y, z)$ is the receiver location, and $\mathbf{x}' \doteq (x', y'z')$ is the source location. $g(\mathbf{x}, \mathbf{x}')$ is the Green's function of the scalar wave equation

$$c_{66}\nabla_t^2 g + c_{44}\frac{\partial^2 g}{\partial z^2} + \rho\omega^2 g = -\delta(\mathbf{x} - \mathbf{x}'). \qquad (6)$$

This function is readily obtained following transformation $s = \sqrt{c_{66}/c_{44}}\, z$ and $\delta(az) = 1/a\delta(z)$. If $e^{-i\omega t}$ dependence is assumed for the wavefield, this Green's function is

$$g(\mathbf{x}, \mathbf{x}') = \frac{e^{ik_0 R}}{4\pi\sqrt{c_{44}c_{66}}R}, \qquad (7)$$

where, $R = \sqrt{(x - x')^2 + (y - y')^2 + c_{66}/c_{44}(z - z')^2}$ is the distance from the source to the receiver and $k_0 = \omega/\sqrt{c_{66}/\rho}$ is the wave number. In the isotropic limit, $c_{66} = c_{44} = \mu$, and $g$ reduces to the scalar shear wave Green's function.

Taking the transverse divergence of eq. (2) and the $z$ derivative of eq. (3), we obtain two coupled equations,

$$\left(c_{11}\nabla_t^2 + c_{44}\frac{\partial^2}{\partial z^2}\right)\nabla_t \cdot \mathbf{u}_t + (c_{13} + c_{44})\nabla_t^2\frac{\partial u_z}{\partial z} + \rho\omega^2\nabla_t \cdot \mathbf{u}_t = -\nabla_t \cdot \mathbf{F}_t, \qquad (8)$$

$$\left(c_{44}\nabla_t^2 + c_{33}\frac{\partial^2}{\partial z^2}\right)\frac{\partial u_z}{\partial z} + (c_{13} + c_{44})\frac{\partial^2}{\partial z^2}\nabla_t \cdot \mathbf{u}_t + \rho\omega^2\frac{\partial u_z}{\partial z} = -\frac{\partial F_z}{\partial z}. \qquad (9)$$

These equations can be solved using an extended version of the Kupradze method (Kupradze, 1963) outlined for an isotropic medium. First, we rewrite the coupled equation in a matrix form

$$\begin{bmatrix} L_z & (c_{13} + c_{44})\nabla_t^2 \\ (c_{13} + c_{44})\frac{\partial^2}{\partial z^2} & L_t \end{bmatrix} \begin{bmatrix} \nabla \cdot \mathbf{u}_t \\ \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} -\nabla_t \cdot \mathbf{F}_t \\ -\frac{\partial}{\partial z}F_z \end{bmatrix}, \qquad (10)$$

where

$$L_t = c_{44}\nabla_t^2 + c_{33}\frac{\partial^2}{\partial z^2} + \rho\omega^2, \qquad (11)$$

$$L_z = c_{11}\nabla_t^2 + c_{44}\frac{\partial^2}{\partial z^2} + \rho\omega^2. \qquad (12)$$

In the Kupradze method, the unknowns of the system are expressed in terms of the cofactor matrix of the original symmetrical matrix operating on a single scalar. The system is greatly simplified because the product of a symmetrical matrix and its cofactor matrix results in an identity matrix scaled by the determinant of the original matrix. This method no longer applies in our case due to the loss of symmetry of the matrix in Eq. (10). Kazi-Aoual et al. (1988) can still apply the Kupradze method because they solve Eq. (1), which is symmetric when written in matrix form. Instead of the cofactor matrix, the adjoint of the original matrix must be used for the nonsymmetrical system. The adjoint of a matrix is defined as the transpose of its cofactor matrix. The product of a matrix with its adjoint is the identity matrix scaled by its determinant. Following this method, we assume

$$\begin{bmatrix} \nabla_t \cdot \mathbf{u}_t \\ \frac{\partial u_z}{\partial z} \end{bmatrix} = \int_V \begin{bmatrix} L_t & -(c_{13}+c_{44})\nabla_t^2 \\ -(c_{13}+c_{44})\frac{\partial^2}{\partial z^2} & L_z \end{bmatrix} \phi(\mathbf{x},\mathbf{x}') \begin{bmatrix} \nabla_t' \cdot \mathbf{F}_t' \\ \frac{\partial}{\partial z'} F_z' \end{bmatrix} d\mathbf{x}'. \quad (13)$$

Substituting Eq. (13) into Eq. (10), we obtain

$$\left[L_t L_z - (c_{13}+c_{44})^2 \frac{\partial^2}{\partial z^2}\nabla_t^2\right]\phi(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'). \quad (14)$$

The scalar Eq. (14) can be solved using the Fourier transform method. Defining the 3-D spatial Fourier transform as follows,

$$\mathbf{FT}\{f(\mathbf{x})\} = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx f(\mathbf{x}) e^{-i[k_x(x-x')+k_y(y-y')+k_z(z-z')]}, \quad (15)$$

and applying it to the above equation, we have

$$\phi(k_x,k_y,k_z,\omega) = \frac{-1}{(c_{11}k^2+c_{44}k_z^2-\rho\omega^2)(c_{44}k^2+c_{33}k_z^2-\rho\omega^2)-(c_{13}+c_{44})^2k^2k_z^2}, \quad (16)$$

where, $k^2 = k_x^2 + k_y^2$ is the transverse or horizontal wavenumber. To return to the spatial coordinates, one takes the inverse transform first, then changes the rectangular space $(x-x', y-y', z-z')$ and wavenumber $(k_x, k_y, k_z)$ domain into cylindrical coordinates $(D, \theta_D, z)$ and $(k, \theta_k, k_z)$. Integration over $\theta_k$ produces a zeroth order Bessel function of the first kind, i.e., $2\pi J_0(kD) = \int_0^{2\pi} d\theta_k e^{ikD\cos(\theta_k-\theta_D)}$. The final result is

$$\phi(\mathbf{x},\mathbf{x}',\omega) = \frac{-1}{(2\pi)^2 c_{33}c_{44}} \int_0^{\infty} kJ_0(kD)dk \int_{-\infty}^{\infty} dk_z \frac{e^{ik_z(z-z')}}{(k_z^2-\nu_a^2)(k_z^2-\nu_b^2)}. \quad (17)$$

In this equation, $D = \sqrt{(x-x')^2+(y-y')^2}$, and $\nu_a^2$ and $\nu_b^2$ represent the two roots of the denominator in Eq. (16), corresponding to *quasi-P* and *quasi-SV* waves, respectively. After regrouping terms, the denominator is

$$\begin{aligned} c_{33}c_{44}k_z^4 &+ [(c_{11}c_{33}-c_{13}^2-2c_{13}c_{44})k^2-(c_{33}+c_{44})\rho\omega^2]k_z^2 \\ &+ (c_{44}k^2-\rho\omega^2)(c_{11}k^2-\rho\omega^2) = 0, \end{aligned} \quad (18)$$

whose roots are

$$\nu_b^2 = \frac{(c_{33}+c_{44})\rho\omega^2 - (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})k^2 + \sqrt{Ak^4+Bk^2+C}}{2c_{33}c_{44}}, \quad (19)$$

$$\nu_a^2 = \frac{(c_{33}+c_{44})\rho\omega^2 - (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})k^2 - \sqrt{Ak^4+Bk^2+C}}{2c_{33}c_{44}}, \quad (20)$$

where,

$$\begin{aligned} A &= (c_{11}c_{33} - c_{13}^2)[c_{11}c_{33} - (c_{13}+2c_{44})^2], \\ B &= -2\rho\omega^2(c_{33}-c_{44})[c_{11}c_{33} - (c_{13}+2c_{44})^2], \\ &\quad +4\rho\omega^2 c_{44}(c_{13}+c_{44})(c_{13}+2c_{44}-c_{33}) \\ C &= (c_{33}-c_{44})^2\rho^2\omega^4. \end{aligned} \quad (21)$$

To calculate the $k_z$ integral properly, one should notice that the integrand has four poles at $k_z = \pm\nu_a$ and $k_z = \pm\nu_b$. Moreover, for a real $\omega$, these four poles lie on the real $k_z$ axis, rendering the integral undefined. However, if a complex $\omega$ is assumed, these poles are off the real axis, and the integral is well-defined. If we assume $Im[\nu_a] > 0$ and $Im[\nu_b] > 0$, then for $z - z' > 0$, we have to close the contour in the upper half of the $k_z$ plane. By Cauchy's theorem, the real axis integration is equivalent to $2\pi$ times the residue of poles at $k_z = \nu_a$ and $k_z = \nu_b$. Similarly, for $z - z' < 0$, the integral is equal to the pole contribution at $-\nu_a$ and $-\nu_b$. The combined result valid for all $z - z'$ is

$$\phi(\mathbf{x},\mathbf{x}',\omega) = \frac{-i}{4\pi c_{33}c_{44}} \int_0^\infty \frac{1}{\nu_b^2 - \nu_a^2} \left( \frac{e^{i\nu_b|z-z'|}}{\nu_b} - \frac{e^{i\nu_a|z-z'|}}{\nu_a} \right) kJ_0(kD)dk. \quad (22)$$

Once $g$ and $\phi$ are determined, we obtained the transverse curl, the transverse divergence, and the $z$ derivative of the displacement vector, $\mathbf{u}$. These quantities are

$$\nabla_t \times \mathbf{u}_t = \int_V g(\mathbf{x},\mathbf{x}')\nabla_t' \times \mathbf{F}_t' d\mathbf{x}', \quad (23)$$

$$\nabla_t \cdot \mathbf{u}_t = \int_V L_t\phi(\mathbf{x},\mathbf{x}')\nabla_t' \cdot \mathbf{F}_t' d\mathbf{x}' - (c_{13}+c_{44}) \int_V \nabla_t^2\phi(\mathbf{x},\mathbf{x}') \frac{\partial F_z'}{\partial z'} d\mathbf{x}', \quad (24)$$

$$\frac{\partial u_z}{\partial z} = \int_V L_z\phi(\mathbf{x},\mathbf{x}') \frac{\partial F_z'}{\partial z'} d\mathbf{x}' - (c_{13}+c_{44}) \int_V \frac{\partial^2\phi(\mathbf{x},\mathbf{x}')}{\partial z^2} \nabla_t' \cdot \mathbf{F}_t' d\mathbf{x}'. \quad (25)$$

Now, a few vector identities and integration by parts can be used to recover the total displacement vector. Using vector identity $g\nabla_t' \times \mathbf{f}' = \nabla_t' \times (g\mathbf{f}') - \nabla_t' g \times \mathbf{f}'$ in Eq. (23) and noticing that $\nabla_t' g = -\nabla_t g$, we have

$$\begin{aligned} \nabla_t \times \mathbf{u}_t &= \int_V \{\nabla_t' \times [g(\mathbf{x},\mathbf{x}')\mathbf{F}_t'] + \nabla_t g(\mathbf{x},\mathbf{x}') \times \mathbf{F}_t'\} d\mathbf{x}' \\ &= \int_V \nabla_t g(\mathbf{x},\mathbf{x}') \times \mathbf{F}_t' d\mathbf{x}'. \end{aligned} \quad (26)$$

In (26), since the integrand of the first integral is in a differential form, the integral can be evaluated at the boundary surface of the volume. This results in zero because $\mathbf{F}_t$ is a body force and not supported at the boundary surface. Applying $\phi\nabla_t \cdot \mathbf{f} = \nabla_t \cdot (\phi \mathbf{f}) - \nabla_t\phi \cdot \mathbf{f}$ and Green's theorem to the first integral of (24) by noticing that the surface integral is zero again because $\mathbf{F}_t$ is not supported on the surface ($\mathbf{F}$ is a volume source), and using integration by parts to the second integral, we obtain

$$\nabla_t \cdot \mathbf{u}_t = \int_V \nabla_t L_t \phi(\mathbf{x}, \mathbf{x}') \cdot \mathbf{F}_t'\, d\mathbf{x}' - (c_{13} + c_{44}) \int_V \nabla_t^2 \frac{\partial \phi(\mathbf{x}, \mathbf{x}')}{\partial z} \hat{z} \cdot \mathbf{F}'\, d\mathbf{x}'. \tag{27}$$

Similarly, for (25), we have

$$\frac{\partial u_z}{\partial z} = \int_V \frac{\partial}{\partial z} L_z \phi(\mathbf{x}, \mathbf{x}') \hat{z} \cdot \mathbf{F}'\, d\mathbf{x}' - (c_{13} + c_{44}) \int_V \frac{\partial^2}{\partial z^2} \nabla_t \phi(\mathbf{x}, \mathbf{x}') \cdot \mathbf{F}_t'\, d\mathbf{x}'. \tag{28}$$

Using the identity $\nabla_t^2 \mathbf{u}_t = \nabla_t \nabla_t \cdot \mathbf{u}_t - \nabla_t \times \nabla_t \times \mathbf{u}_t$, the total displacement vector can be recovered as

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_t + u_z \hat{z} \\
&= \frac{1}{\nabla_t^2} \left[\nabla_t \nabla_t \cdot \mathbf{u}_t - \nabla_t \times \nabla_t \times \mathbf{u}_t\right] + \hat{z} \int \frac{\partial u_z}{\partial z} dz \\
&= \int_V \frac{\nabla_t \nabla_t}{\nabla_t^2} L_t \phi(\mathbf{x}, \mathbf{x}') \cdot \mathbf{F}'\, d\mathbf{x}' - (c_{13} + c_{44}) \int_V \nabla_t \frac{\partial}{\partial z} \phi \hat{z} \cdot \mathbf{F}'\, d\mathbf{x}' \\
&\quad - \int_V \frac{1}{\nabla_t^2} \nabla_t \times (\nabla_t g(\mathbf{x}, \mathbf{x}') \times \mathbf{F}_t') d\mathbf{x}' \\
&\quad + \int_V L_z \phi(\mathbf{x}, \mathbf{x}') \hat{z}\hat{z} \cdot \mathbf{F}'\, d\mathbf{x}' - (c_{13} + c_{44}) \int_V \frac{\partial}{\partial z} \hat{z} \nabla_t \phi \cdot \mathbf{F}'\, d\mathbf{x}' \\
&= \int_V \left[g\bar{\mathbf{I}} + \hat{z}\hat{z}(L_z\phi - g) - (c_{13} + c_{44}) \frac{\partial}{\partial z} (\nabla_t \hat{z} + \hat{z}\nabla_t)\phi + \frac{\nabla_t \nabla_t}{\nabla_t^2} (L_t \phi - g)\right] \cdot \mathbf{F}' d\mathbf{x}'.
\end{aligned} \tag{29}$$

In the above derivation, the following identities have been used:

$$\begin{aligned}
\nabla_t \times (\nabla_t g \times \mathbf{F}_t') &= \nabla_t \nabla_t g \cdot \mathbf{F}_t' - \nabla_t^2 g \mathbf{F}_t', \\
\bar{\mathbf{I}} &= \bar{\mathbf{I}}_t + \hat{z}\hat{z}, \\
\mathbf{F}_t &= \bar{\mathbf{I}}_t \cdot \mathbf{F}, \\
\frac{1}{\nabla_t^2} \nabla_t^2 &= 1.
\end{aligned}$$

The last equation in the above says that the displacement field can be determined for any kind of source by convolving the source with a certain function then integrating over the source volume. This is exactly the statement of Green's superposition theorem, and this certain function (the integrand) is just the Green's function for the wave equation. Thus, The dynamic Green's function (tensor), denoted by $\overline{\mathbf{G}}$ and expressed in dyadic form, is

$$\overline{\mathbf{G}} = g\bar{\mathbf{I}} + \hat{z}\hat{z}(L_z \phi - g) - (c_{13} + c_{44}) \frac{\partial}{\partial z} (\nabla_t \hat{z} + \hat{z}\nabla_t)\phi + \frac{\nabla_t \nabla_t}{\nabla_t^2} (L_t \phi - g). \tag{30}$$

The meaning of $L_t$, $L_z$, $g$ and $\phi$ in the above equation are defined in Eqs. (11), (12), (7) and (22), respectively. The inverse Laplacian operator, $\frac{1}{\nabla_t^2}$, does not have simple form except for fields or sources with $z$-axis symmetry. With the $z$-axis symmetry, this operator (Ben-Menahem and Sena, 1990; Sena, 1992) is

$$\nabla_t^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \longleftrightarrow \frac{1}{\nabla_t^2}u = \int \frac{dr}{r}\int rudr. \tag{31}$$

Even in this special case, the inverse Laplacian incurs integration with respect to Bessel functions which are not easily obtained. Fortunately, as it is shown later, this inverse Laplacian operator can in fact be replaced by integration over $z$ in the general cases.

Without further simplification, the isotropic limit can be obtained with the assistance of the two scalar wave equations. In the isotropic limit, $c_{11} = c_{33} = \lambda + 2\mu$, $c_{13} = \lambda$ and $c_{44} = c_{66} = \mu$,

$$A = 0,\ \ B = 0,\ \ C = (\lambda+\mu)^2\rho^2\omega^4,\ \ \nu_b^2 = \frac{\rho\omega^2}{\mu} - k^2,\ \ \nu_a^2 = \frac{\rho\omega^2}{\lambda+2\mu} - k^2. \tag{32}$$

Using the Sommerfeld representation for a point source, $\phi$ is simplified to

$$\phi = \frac{-1}{(\lambda+\mu)\rho\omega^2}(g_\beta - g_\alpha) = \frac{-1}{(\lambda+\mu)\rho\omega^2}\left(\frac{e^{ik_\beta R}}{4\pi R} - \frac{e^{ik_\alpha R}}{4\pi R}\right), \tag{33}$$

where $g_\beta$ and $g_\alpha$ are the scalar Green's function of the scalar wave equations

$$\nabla^2 g_\beta + \frac{\rho\omega^2}{\mu} g_\beta = -\delta(\mathbf{x}-\mathbf{x}'),\quad \nabla^2 g_\alpha + \frac{\rho\omega^2}{\lambda+2\mu} g_\alpha = -\delta(\mathbf{x}-\mathbf{x}'). \tag{34}$$

These two equations can be used to simplify $L_t\phi$ and $L_z\phi$ at any point, including the source point. We then have

$$L_z\phi = \frac{g_\beta}{\mu} - \frac{1}{\rho\omega^2}\frac{\partial^2(g_\alpha - g_\beta)}{\partial z^2},\quad L_t\phi = \frac{g_\beta}{\mu} - \frac{1}{\rho\omega^2}\nabla_t^2(g_\alpha - g_\beta). \tag{35}$$

With these results and $g = g_\beta/\mu$ from (7), the Green's function becomes

$$\overline{\mathbf{G}} = \frac{1}{\mu}g_\beta\overline{\mathbf{I}} + \frac{1}{\rho\omega^2}\left[\left(\frac{\partial}{\partial z}\nabla_t\hat{z} + \frac{\partial}{\partial z}\hat{z}\nabla_t\right) + \hat{z}\hat{z}\frac{\partial^2}{\partial z^2} + \nabla_t\nabla_t\right](g_\beta - g_\alpha) \tag{36}$$

$$= \frac{1}{\rho\omega^2}\left[k_\beta^2 g_\beta\overline{\mathbf{I}} + \nabla\nabla(g_\beta - g_\alpha)\right], \tag{37}$$

which is exactly the dynamic Green's function for the isotropic elastic medium (Kupradze, 1963; Ben-Menahem and Singh, 1981).

## THE STATIC GREEN'S FUNCTION

To determine the singular behavior of the dynamic Green's function when a field point approaches the source point, the static case must be considered because it is required to regularize the surface integrals of the dynamic Green's function (Kupradze, 1963). Following the procedure of the previous section and set the frequency to zero ($\omega = 0$), one finds that the static Green's function has the same form as Eq. (30) except that $L_t$, $L_z$, $g$ and $\phi$ must be redefined. Operators $L_t$ and $L_z$ stay the same as in Eqs. (11) and (12) with $\omega = 0$, while more work is required in order to obtain $g$ and $\phi$. In the static limit, Eqs. (6) and (14) become

$$c_{66}\nabla_t^2 g + c_{44}\frac{\partial^2 g}{\partial z^2} = -\delta(\mathbf{x}-\mathbf{x}') \quad (38)$$

$$\left[c_{33}c_{44}\frac{\partial^4}{\partial z^4} + (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})\nabla_t^2\frac{\partial^2}{\partial z^2} + c_{11}c_{44}\nabla_t^4\right]\phi = -\delta(\mathbf{x}-\mathbf{x}'). \quad (39)$$

The solution of the first equation is

$$g = \frac{\nu_g}{4\pi c_{66}}\frac{1}{R_g}; \quad R_g = \sqrt{(x-x')^2 + (y-y')^2 + \nu_g^2(z-z')^2}, \quad (40)$$

where, $\nu_g = \sqrt{c_{66}/c_{44}}$. The second equation can be factorized into

$$c_{33}c_{44}\nu_a^2\nu_b^2\left(\nabla_t^2 + \frac{1}{\nu_a^2}\frac{\partial^2}{\partial z^2}\right)\left(\nabla_t^2 + \frac{1}{\nu_b^2}\frac{\partial^2}{\partial z^2}\right)\phi = -\delta(\mathbf{x}-\mathbf{x}'), \quad (41)$$

where, $\nu_a^2$ and $\nu_b^2$ are the negative counterpart of the solutions of the equation

$$c_{33}c_{44}\nu^4 + (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})\nu^2 + c_{11}c_{44} = 0, \quad (42)$$

i.e.,

$$\nu_a^2 = \frac{(c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}) + \sqrt{(c_{11}c_{33} - c_{13}^2)[c_{11}c_{33} - (c_{13} + 2c_{44})^2]}}{2c_{33}c_{44}}, \quad (43)$$

$$\nu_b^2 = \frac{(c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}) - \sqrt{(c_{11}c_{33} - c_{13}^2)[c_{11}c_{33} - (c_{13} + 2c_{44})^2]}}{2c_{33}c_{44}}. \quad (44)$$

In order to be a solution of Eq. (41), $\phi$ must satisfy

$$\left(\nabla_t^2 + \frac{1}{\nu_b^2}\frac{\partial^2}{\partial z^2}\right)\phi = \frac{\nu_a}{4\pi c_{33}c_{44}\nu_a^2\nu_b^2}\frac{1}{R_a}, \quad (45)$$

$$\left(\nabla_t^2 + \frac{1}{\nu_a^2}\frac{\partial^2}{\partial z^2}\right)\phi = \frac{\nu_b}{4\pi c_{33}c_{44}\nu_a^2\nu_b^2}\frac{1}{R_b}, \quad (46)$$

where,

$$R_a = \sqrt{(x-x')^2+(y-y')^2+\nu_a^2(z-z')^2},$$
$$R_b = \sqrt{(x-x')^2+(y-y')^2+\nu_b^2(z-z')^2}.$$

In arriving at the above equations, we employed the transformations $s_a = \nu_a(z-z')$, $s_b = \nu_b(z-z')$, and $\delta(s/c) = c\delta(s)$, and the Poisson's equation

$$\nabla^2 \frac{1}{4\pi R} = -\delta(\mathbf{x}-\mathbf{x}'), \tag{47}$$

where, $R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$. From Eqs. (45) and (46), we obtain

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{1}{4\pi c_{33}c_{44}(\nu_b^2-\nu_a^2)}\left[\frac{\nu_b}{R_b}-\frac{\nu_a}{R_a}\right], \tag{48}$$

$$\nabla_t^2 \phi = \frac{1}{4\pi c_{33}c_{44}(\nu_b^2-\nu_a^2)}\left[\frac{1}{\nu_a R_a}-\frac{1}{\nu_b R_b}\right]. \tag{49}$$

Assuming $\nu_a$ and $\nu_b > 0$ (or $Re[\nu_a]$ and $Re[\nu_b] > 0$), integration over $z$ yields

$$\frac{\partial \phi}{\partial z} = \frac{\mathrm{sgn}(z-z')}{4\pi c_{33}c_{44}(\nu_b^2-\nu_a^2)}\{\ln[R_b+\nu_b|z-z'|] - \ln[R_a+\nu_a|z-z'|]\}, \tag{50}$$

and

$$\phi = \frac{1}{4\pi c_{33}c_{44}(\nu_b^2-\nu_a^2)}\{\ |z-z'|\ln[R_b+\nu_b|z-z'|] - \tfrac{R_b}{\nu_b} - |z-z'|\ln[R_a+\nu_a|z-z'|] + \tfrac{R_a}{\nu_a}\ \}. \tag{51}$$

Except for the absolute value, this expression is the same as the assumed solution form in Pan and Chou (1976, Eq. 19). The absolute value of $z-z'$ is necessary because for $\nu_a$ or $\nu_b > 0$, and $z-z' = -R_a/\nu_a$ or $-R_b/\nu_b$, Eq. (51) yields a finite solution, instead of the infinity when the absolute value sign is absent.

The isotropic limit cannot be obtained from the above expression for $\phi$. This is because at the limit, $\nu_a = \nu_b = 1$, and Eq. (41) reduces to

$$\mu(\lambda+2\mu)\left(\nabla_t^2+\frac{\partial^2}{\partial z^2}\right)^2 \phi = -\delta(\mathbf{x}-\mathbf{x}'). \tag{52}$$

Using the identity $\nabla^2 R = \frac{2}{R}$, the solution for this equation is

$$\phi = \frac{1}{8\pi c_{33}c_{44}} R. \tag{53}$$

Substitution of this $\phi$ and $g$ into (30) yields the isotropic static Green's function (Love, 1944), i.e.,

$$\overline{\mathbf{G}} = \frac{1}{4\pi\mu R}\overline{\mathbf{I}} - \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\nabla\nabla R.$$

## CALCULATION OF $\frac{\nabla_t \nabla_t}{\nabla_t^2}(L_t\phi - g)$

The last term in the dynamic Green's function (Eq. 30) is a simple but very abstract expression. Its meaning is not easily defined in general. Even for special cases, numerical calculation of the Green's function renders integration of Bessel functions with respect to spatial coordinates. This, along with the integration over wave numbers, presents many numerical difficulties. Moreover, this operator prevents further analytical manipulation of the Green's function. An alternative form, therefore, is necessary. In the following, we show that a simple and numerically feasible expression is indeed available.

Before we proceed, let's understand why $\frac{1}{\nabla_t^2}$ disappears in the isotropic case (Eq. 37). As seen from equation (35), $L_t\phi$ cancels out $g$ and leaves $-\frac{1}{\rho\omega^2}\nabla_t^2(g_\alpha - g_\beta)$. This cancels out the inverse Laplacian. To obtain (35), the two independent scalar wave equations (34) for the $P$ and $S$ waves are used. In the case of transverse isotropy, we no longer have two separate scalar wave equations for the *quasi-P* and *quasi-S* waves. Instead, we have a fourth order scalar equation (Eq. 14) for the $P - SV$ waves and a second order equation (Eq. 6) for the $SH$ wave. Equation (14) indicates the inevitable involvement of operator $L_z$ in the calculation. This suggests that we first compute $L_z(L_t\phi - g)$ rather that $(L_t\phi - g)$ alone.

From equations (14) and (6), a simple manipulation yields

$$L_z(L_t\phi - g) = \nabla_t^2 \left[ (c_{13} + c_{44})^2 \frac{\partial^2 \phi}{\partial z^2} - (c_{11} - c_{66}) g \right]. \tag{54}$$

Because $L_z$ is a linear operator, the above result suggests that $L_t\phi - g = \nabla_t^2 \psi$, where $\psi$ is an unknown function to be determined. Thus, we obtain

$$\frac{\nabla_t \nabla_t}{\nabla_t^2}(L_t\phi - g) = \nabla_t \nabla_t \, \psi, \tag{55}$$

and

$$L_z \psi = (c_{13} + c_{44})^2 \frac{\partial^2 \phi}{\partial z^2} - (c_{11} - c_{66}) g. \tag{56}$$

Now that we have got rid of the inverse Laplacian, what is left to do is to determine $\psi$ by solving the inhomogeneous equation (56), where the differential operator $L_z$ is defined by equation (12).

The right hand side of equation (56) has the form of Hankel transform because

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{i}{4\pi c_{33} c_{44}} \int_0^\infty \frac{1}{\nu_b^2 - \nu_a^2} \left( \nu_b \, e^{i\nu_b |z-z'|} - \nu_a \, e^{i\nu_a |z-z'|} \right) k J_0(kD) dk \tag{57}$$

$$g = \frac{i}{4\pi c_{44}} \int_0^\infty \frac{e^{i\nu_c |z-z'|}}{\nu_c} k J_0(kD) dk, \tag{58}$$

where, $\nu_a$ and $\nu_b$ are defined in Eqs. (19) and (20), and $\nu_c = \sqrt{(\rho\omega^2 - c_{66}k^2)/c_{44}}$. This suggests that the solution $\psi$ should also be in the form of Hankel transform. i.e.,

$$\psi = \frac{i}{4\pi}\int_0^\infty f(z,z')\, kJ_0(kD)dk. \tag{59}$$

Substituting this solution form and Eqs. (57) and (58) into equation (56), we obtain the following ordinary differential equation for $f(z,z')$

$$\begin{aligned} \frac{d^2 f(z,z')}{dz^2} + \nu_z^2 f(z,z') &= p(z,z') \\ &= \frac{S_1}{\nu_b^2 - \nu_a^2}\left[\nu_b e^{i\nu_b|z-z'|} - \nu_a e^{i\nu_a|z-z'|}\right] - \frac{S_2}{\nu_c}e^{i\nu_c|z-z'|}. \end{aligned} \tag{60}$$

In arriving at the above equation, the following definitions and identities were used,

$$\nu_z^2 = \frac{c_{11}}{c_{44}}\left(\frac{\rho\omega^2}{c_{11}} - k^2\right), \quad S_1 = \frac{(c_{13}+c_{44})^2}{c_{33}c_{44}^2}, \quad S_2 = \frac{c_{11}-c_{66}}{c_{44}^2}, \tag{61}$$

$$\begin{aligned} \nabla_t^2 J_0(kD) &= \nabla_t^2 \sum_{m=0}^\infty \epsilon_m J_m(kr_0)J_m(kr)\cos m(\theta-\theta_0) \\ &= \sum_{m=0}^\infty \epsilon_m J_m(kr_0)\nabla_t^2[J_m(kr)\cos m(\theta-\theta_0)] \\ &= \sum_{m=0}^\infty \epsilon_m J_m(kr_0)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J_m(kr)}{\partial r}\right) - \frac{m^2}{r^2}J_m(kr)\right]\cos m(\theta-\theta_0)] \\ &= \sum_{m=0}^\infty \epsilon_m J_m(kr_0)[\,-k^2 J_m(kr)\,]\cos m(\theta-\theta_0)] \\ &= -k^2 J_0(kD). \end{aligned} \tag{62}$$

For the second identity, the addition theorem (Watson, 1944) of Bessel's functions was used.

Equation (60) can be solved with the aid of the Green's function for this ordinary differential equation. This Green's function, denoted by $q(z,z'')$, and satisfying the continuity condition of $q$ and discontinuity condition of $\frac{\partial q}{\partial z}$ at the source level $z = z''$, is

$$q(z,z'') = \frac{e^{i\nu_z|z-z''|}}{2i\nu_z}. \tag{63}$$

Using Green theorem on Eq. (60), we obtain

$$f(z,z') = \int_{-\infty}^{\infty} q(z,z'')\, p(z'',z')dz''. \tag{64}$$

When $p(z'', z')$ (defined in (60)) is substituted into the above equation, there are three integrals of the type $\int_{-\infty}^{\infty} e^{i\nu_1|z-z''|}e^{i\nu_2|z''-z'|}dz''$. This type of integral is readily computed by dividing the integral into three sub-domain integrals: $\int_{-\infty}^{\infty} = \int_{-\infty}^{z'} + \int_{z'}^{z} + \int_{z}^{\infty}$ for $z > z'$ and $\int_{-\infty}^{\infty} = \int_{-\infty}^{z} + \int_{z}^{z'} + \int_{z'}^{\infty}$ for $z < z'$. The final result is

$$\int_{-\infty}^{\infty} e^{i\nu_1|z-z''|}e^{i\nu_2|z''-z'|}dz'' = \frac{2i}{\nu_1^2 - \nu_2^2}\left[\nu_1 e^{i\nu_2|z-z'|} - \nu_2 e^{i\nu_1|z-z'|}\right]. \tag{65}$$

Using this result in (65), we obtain

$$\begin{aligned} f(z, z') = & \quad \frac{1}{\nu_z}\left[\frac{S_2}{\nu_z^2 - \nu_c^2} - \frac{S_1\nu_z^2}{(\nu_z^2 - \nu_a^2)(\nu_z^2 - \nu_b^2)}\right] e^{i\nu_z|z-z'|} \\ & - \frac{S_1\nu_a}{(\nu_b^2 - \nu_a^2)(\nu_z^2 - \nu_a^2)} e^{i\nu_a|z-z'|} \\ & + \frac{S_1\nu_b}{(\nu_b^2 - \nu_a^2)(\nu_z^2 - \nu_b^2)} e^{i\nu_b|z-z'|} - \frac{S_2}{\nu_c(\nu_z^2 - \nu_c^2)} e^{i\nu_c|z-z'|}. \end{aligned} \tag{66}$$

The above seems to suggest four types of propagating waves. However, a closer examination of the first term of $f(z, z')$ shows that it vanishes altogether. This result agrees with the physics that only three kinds of waves exist in a TI medium, $\nu_a$ part for the *quasi*-P wave, $\nu_b$ part for the *quasi*-SV wave, and $\nu_c$ part for the SH wave. Thus, the final result is

$$\begin{aligned} f(z, z') = & - \frac{S_1\nu_a}{(\nu_b^2 - \nu_a^2)(\nu_z^2 - \nu_a^2)} e^{i\nu_a|z-z'|} \\ & + \frac{S_1\nu_b}{(\nu_b^2 - \nu_a^2)(\nu_z^2 - \nu_b^2)} e^{i\nu_b|z-z'|} - \frac{S_2}{\nu_c(\nu_z^2 - \nu_c^2)} e^{i\nu_c|z-z'|}. \end{aligned} \tag{67}$$

Substituting this result back into (59) and using (55), we obtain a simple form for the originally complicated term. This simplified expression can be implemented easily on the computer.

In the isotropic limit,

$$\begin{aligned} & S_1 = \frac{(\lambda + \mu)^2}{\mu^2(\lambda + 2\mu)}; \quad S_2 = \frac{\lambda + \mu}{\mu^2}; \quad \nu_b^2 - \nu_a^2 = \rho\omega^2 \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}; \\ & \nu_c^2 = \nu_b^2 = \rho\omega^2/\mu - k^2; \quad \nu_z^2 - \nu_a^2 = \frac{\lambda + \mu}{\mu}\nu_a^2; \\ & \nu_z^2 - \nu_b^2 = \nu_z^2 - \nu_c^2 = -\frac{\lambda + \mu}{\mu}k^2, \end{aligned}$$

then

$$f(z, z') = \frac{1}{\rho\omega^2}\left(\frac{e^{i\nu_b|z-z'|}}{\nu_b} - \frac{e^{i\nu_a|z-z'|}}{\nu_a}\right), \tag{68}$$

and, using the Sommerfeld integral, we have

$$\psi = \frac{i}{4\pi} \int_0^\infty f(z, z')\, kJ_0(kD) dk = \frac{g_\beta - g_\alpha}{\rho\omega^2}, \tag{69}$$

which agrees with the result of equation (36).

For the static case, equations (48) and (49) yield

$$L_t\phi = \frac{1}{4\pi c_{33} c_{44} (\nu_b^2 - \nu_a^2)} \left\{ \frac{c_{44} - c_{33}\nu_a^2}{\nu_a} \frac{1}{R_a} - \frac{c_{44} - c_{33}\nu_b^2}{\nu_b} \frac{1}{R_b} \right\}. \tag{70}$$

$\nu_a^2$ and $\nu_b^2$ are now defined by equations (43) and (44). $1/R_a$, $1/R_b$ satisfy

$$\left[ \nabla_t^2 + \frac{1}{\nu_i^2} \partial_z^2 \right] \frac{1}{R_i} = -\frac{4\pi}{\nu_i} \delta(\mathbf{x} - \mathbf{x}'), \tag{71}$$

where, subscript $i$ represents $a$, $b$, or $g$. A manipulation of Eq. (71), with the fact that operators $\frac{\partial^2}{\partial z^2}$ and $\frac{1}{\nabla_t^2}$ commute in cylindrical coordinates, yields

$$\begin{aligned} \frac{1}{\nabla_t^2} \frac{1}{R_i} &= -\int \frac{\nu_i^2}{R_i} d^2 z - \frac{4\pi\nu_i}{\nabla_t^2} \int \delta(\mathbf{x} - \mathbf{x}') d^2 z \\ &= -\nu_i |z - z'| \ln[R_i + \nu_i |z - z'|] + R_i - \frac{4\pi\nu_i}{\nabla_t^2} \int \delta(\mathbf{x} - \mathbf{x}') d^2 z. \end{aligned} \tag{72}$$

In calculating $\frac{1}{\nabla_t^2}(L_t\phi - g)$, the second term in (72) drops out due to cancellation, and we obtain

$$\begin{aligned} \psi &= \frac{1}{\nabla_t^2}(L_t\phi - g) \\ &= \frac{c_{44} - c_{33}\nu_b^2}{4\pi c_{33} c_{44} (\nu_b^2 - \nu_a^2)\nu_b} \left\{ \nu_b |z - z'| \ln[R_b + \nu_b |z - z'|] - R_b \right\} \\ &\quad - \frac{c_{44} - c_{33}\nu_a^2}{4\pi c_{33} c_{44} (\nu_b^2 - \nu_a^2)\nu_a} \left\{ \nu_a |z - z'| \ln[R_a + \nu_a |z - z'|] - R_a \right\} \\ &\quad + \frac{\nu_g}{4\pi c_{66}} \left\{ \nu_g |z - z'| \ln[R_g + \nu_g |z - z'|] - R_g \right\}. \end{aligned} \tag{73}$$

It is interesting to note from the above calculations that the inverse Laplacian is essentially removed by integration over $z$. The end result of $z$ integration is basically to introduce amplitude weighting for different waves. The computations are based on operator manipulation and therefore the results are valid for any source geometry.

## DISPLACEMENT FOR SOME FUNDAMENTAL SOURCES

Before we proceed to calculate the displacements, we summarize the results of the previous sections. The dynamic and static Green's function in transversely isotropic media can be expressed in the following single dyadic form

$$\overline{\mathbf{G}} = g\overline{\mathbf{I}}_t + \hat{z}\hat{z}L_z\phi - (c_{13}+c_{44})(\nabla_t\hat{z}+\hat{z}\nabla_t)\frac{\partial\phi}{\partial z} + \nabla_t\nabla_t\psi. \tag{74}$$

For the dynamic case, $L_z$, $g$, $\phi$, and $\psi$ are defined in equations (12), (7) or (58), (22), and (59) plus (67), respectively. For the static case, these symbols are defined in equations (12) with $\omega = 0$, (40), (51), and (73).

Practically, we now have all the tools necessary to solve wave propagation and scattering problems in a TI medium. As the basic applications of Green's functions, we calculate the displacements produced by vertical, horizontal and explosive point sources.

### Vertical Point Force

For a vertical point force (parallel to the symmetry axis) at the origin, $\mathbf{F}(\mathbf{x}) = \hat{z}\delta(\mathbf{x})$. Using the Green theorem, we obtain

$$\mathbf{u} = \int_V \overline{\mathbf{G}}(\mathbf{x},\mathbf{x}')\cdot\mathbf{F}(\mathbf{x}')d\mathbf{x}' = \hat{z}L_z\phi - (c_{13}+c_{44})\nabla_t\frac{\partial\phi}{\partial z}. \tag{75}$$

The displacement vector does not depend on $g$, indicating that a point force along the symmetry axis does not excite $SH$ waves. Writing out in components, we have

$$U_r = \frac{\operatorname{sgn}(z)}{4\pi}\int_0^\infty S_{ab}k^2J_1(kr)\left[e^{i\nu_b|z|} - e^{i\nu_a|z|}\right]dk, \tag{76}$$

$$U_z = \frac{i}{4\pi}\int_0^\infty kJ_0(kr)\left[S_b e^{i\nu_b|z|} - S_a e^{i\nu_a|z|}\right]dk, \tag{77}$$

where,

$$S_{ab} = \frac{c_{13}+c_{44}}{c_{33}c_{44}(\nu_b^2-\nu_a^2)}, \quad S_a = \frac{\nu_a^2-\nu_z^2}{c_{33}(\nu_b^2-\nu_a^2)\nu_a}, \quad S_b = \frac{\nu_b^2-\nu_z^2}{c_{33}(\nu_b^2-\nu_a^2)\nu_b}.$$

### Horizontal Point Force

For a point force at the origin and directed along an isotropic plane (say along $\hat{x}$), $\mathbf{F}(\mathbf{x}) = \hat{x}\delta(\mathbf{x})$. The displacement vector is

$$\mathbf{u} = g\hat{x} - (c_{13}+c_{44})\frac{\partial^2\phi}{\partial x\partial z}\hat{z} + \nabla_t\frac{\partial\psi}{\partial x}. \tag{78}$$

Now $g$ is included in the final expression, indicating that a horizontal point force excites all three waves: $SH$, *quasi-P*, and *quasi-SV*. In their components, the displacements are

$$U_r = \frac{i\cos\varphi}{4\pi}\int_0^\infty kJ_1(kr)\left[T_a e^{i\nu_a|z|} - T_b e^{i\nu_b|z|}\right]dk + \frac{i\cos\varphi}{4\pi}\int_0^\infty f(z,0)\,k^2\frac{J_1(kr)}{r}dk, \quad (79)$$

$$U_\varphi = \frac{i\sin\varphi}{4\pi c_{44}}\int_0^\infty kJ_1(kr)\frac{e^{i\nu_c|z|}}{\nu_c}dk + \frac{i\sin\varphi}{4\pi}\int_0^\infty f(z,0)\,k^2\frac{J_1(kr)}{r}dk, \quad (80)$$

$$U_z = \frac{\mathrm{sgn}(z)}{4\pi}\cos\varphi\int_0^\infty S_{ab}k^2 J_0(kr)\left[e^{i\nu_a|z|} - e^{i\nu_b|z|}\right]dk. \quad (81)$$

The azimuthal angle $\varphi$ is measured from the $x$-axis in the $x-y$ plane. $S_{ab}$ is the same as defined before, and $T_a$ and $T_b$ are

$$T_a = \frac{S_1\nu_a k^2}{(\nu_b^2-\nu_a^2)(\nu_z^2-\nu_a^2)}, \quad T_b = \frac{S_1\nu_b k^2}{(\nu_b^2-\nu_a^2)(\nu_z^2-\nu_b^2)}.$$

The second terms in $U_r$ and $U_\varphi$ represent the near-field part of the wave field. In the far-field, only the first terms contribute. However, in the BEM modeling of downhole sources, these near terms are crucial in satisfying the boundary conditions. The above simple forms allow easy computation of the near-field.

## Explosive Point Source

For an explosive point source at the origin, the displacement vector is obtained by taking the divergence of the Green's function with respect to the source coordinates. Since $\nabla = -\nabla'$ for functions $g$, $\phi$, and $\psi$, we have

$$\mathbf{u} = -\hat{z}\frac{\partial}{\partial z}(L_z\phi) - \nabla_t(L_t\phi) + (c_{13}+c_{44})(\nabla_t^2\hat{z} + \frac{\partial}{\partial z}\nabla_t)\frac{\partial\phi}{\partial z}. \quad (82)$$

In the above, $L_t\phi = \nabla_t^2\psi + g$ is used. The curl of $\mathbf{u}$ is

$$\nabla\times\mathbf{u} = \nabla_t\times\hat{z}\frac{\partial}{\partial z}\left[(c_{13}+2c_{44}-c_{11})\nabla_t^2 + (c_{33}-c_{13}-2c_{44})\frac{\partial^2}{\partial z^2}\right]\phi. \quad (83)$$

For an explosive source in a transversely isotropic medium, the curl of the displacement field is not zero. This implies that an explosive source excites not only the *quasi-P* but also the *quasi-SV* waves. In the isotropic limit, $c_{11} = c_{33} = c_{13} + 2c_{44}$, the curl of the displacement field due to an explosive source is zero, indicating that only the compressional wave exists. The two displacement components are

$$U_r = \frac{i}{4\pi}\int_0^\infty \left[(S_{ab}\nu_a + T_a)e^{i\nu_a|z|} - (S_{ab}\nu_b + T_b)e^{i\nu_b|z|}\right]k^2 J_1(kr)dk, \quad (84)$$

$$U_z = \frac{\mathrm{sgn}(z)}{4\pi}\int_0^\infty \left[(S_b\nu_b - S_{ab}k^2)e^{i\nu_b|z|} - (S_a\nu_a - S_{ab}k^2)e^{i\nu_a|z|}\right]kJ_0(kr)dk. \quad (85)$$

In the isotropic limit, $S_{ab}\nu_b + T_b = 0$ and $S_b\nu_b - S_{ab}k^2 = 0$. $SV$ wave contribution to the displacements vanishes. The displacements reduce to the gradient of $\phi$.

## Radiation Patterns in Two Media

We now evaluate the above displacement integrals in two particular TI media. The first medium (Mesaverde sandstone - Ben-Menahem and Sena, 1990) has a density of 2870 $kg/m^3$ and the following elastic constants (in $10^9 Pa$): $c_{11} = 50$, $c_{33} = 45$, $c_{13} = -8.6$, $c_{44} = 24.6$, and $c_{66} = 26.6$. The parameters of the second medium (plexiglas-aluminum - White, 1984) are: $\rho$=1950 $kg/m^3$, $c_{11} = 51.8$, $c_{33} = 21.4$, $c_{44} = 3.65$, and $c_{66} = 14.1$. The first medium is only slightly anisotropic, while the second medium is extremely anisotropic. The displacements are calculated at receivers placed circularly around the sources. For the case of horizontal force, the receiver array makes a $45^o$ azimuthal angle with the $x - z$ plane. The radius of the receiver array is 75 $m$ for the first medium and 30 $m$ for the second. The calculated displacements are then rotated to the spherical coordinates. The resulting radial, tangential, and azimuthal components are plotted in a way to show both the amplitudes and phase fronts.

Figure 1, Figures 2 and 3, and Figure 4, respectively, show the displacements due to a vertical, horizontal ($\hat{x}$), and explosive point source in the first medium. The phase fronts of the $P$, $SV$, and $SH$ waves are almost circular. The $P$ and $SH$ (azimuthal component in Figure 3) waves travel a little faster in the horizontal direction than in the vertical direction. The $SV$ wave travels faster in the $45^o$ direction. That the $P$ and $SV$ waves sustain large amplitudes in tgeh wide angle range (Figure 1) suggests large lobes in the radiation pattern. Figure 4 shows the existence of $SV$ waves for an explosion in the medium. If examined carefully, this $SV$ wave exhibits two lobes in each quadrant. A similar pattern can also be seen for the vertical force (Figure 1). However, this phenomenon is absent for the case of horizontal force.

Figures 5–8 show the displacements in the second medium. One immediate observation is the triplication of $SV$ waves. Although the magnitude of one branch is significantly smaller than the other two, three branches of $SV$ wave are clearly seen on Figure 6, where the point force is in the isotropic plane. If the source is a point force along the symmetry axis or explosion, one branch of the triplication disappears. The vertical force result agrees with the calculation of White (1984) for a line source approximated by a borehole along the symmetry axis. White (1984) also demonstrated the difference in energy fall-off for the *quasi-P* and *quasi-SV* waves. Amplitude decays as $-1$ power of distance for the $P$ wave, and as $-0.8$ for the $SV$ waves near the triplication. Our results support this observation.

## SURFACE INTEGRATION OF THE GREEN'S FUNCTIONS

In the boundary element method, integration of the Green's function over boundary surfaces is a necessary calculation. However, the Green's function is singular when receiver and source coincide. The displacement Green's function has first order singularity which is removable when integrated over the surface. On the other hand, the stress Green's function has a second order singularity and the surface integral is not defined. Then, its principal value has to be used. The contribution of this second order singularity to the integral can be evaluated analytically. When the receiver and the source approach each other, the surface integral of the dynamic Green's function can be regularized using the static Green's function (Kupradze, 1963). The singularity integral of the dynamic Green's function is reduced to the integration of the static Green's function over a half elliptical surface around the source point. The limit as the axes of the elliptical surface go to zero is the singular contribution.

Using the static Green's function, we first calculate the displacement and stress field for the vertical point force. We obtain

$$\begin{aligned} u_x &= -(c_{13}+c_{44})\frac{\partial^2\phi}{\partial x\partial z} \\ &= \frac{(c_{13}+c_{44})\mathrm{sgn}(z-z')}{4\pi c_{33}c_{44}(\nu_b^2-\nu_a^2)}\left\{\frac{x-x'}{R_b[R_b+\nu_b|z-z'|]}-\frac{x-x'}{R_a[R_a+\nu_a|z-z'|]}\right\} \quad (86) \\ u_y &= -(c_{13}+c_{44})\frac{\partial^2\phi}{\partial y\partial z} \\ &= \frac{(c_{13}+c_{44})\mathrm{sgn}(z-z')}{4\pi c_{33}c_{44}(\nu_b^2-\nu_a^2)}\left\{\frac{y-y'}{R_b[R_b+\nu_b|z-z'|]}-\frac{y-y'}{R_a[R_a+\nu_a|z-z'|]}\right\} \quad (87) \\ u_z &= L_z\phi. \quad (88) \end{aligned}$$

Notice that the displacements have first order singularity only when $\mathbf{x}\to\mathbf{x}'$. The stress along the $\hat{z}$ direction on the surface whose normal is $\hat{x}$, $\tau_{xz}$ , is

$$\begin{aligned} \tau_{xz} &= c_{44}\left[\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right] \\ &= \frac{-1}{4\pi c_{33}(\nu_b^2-\nu_a^2)}\left\{\frac{c_{11}+c_{13}\nu_a^2}{\nu_a}\frac{x-x'}{R_a^3}-\frac{c_{11}+c_{13}\nu_b^2}{\nu_b}\frac{x-x'}{R_b^3}\right\}. \quad (89) \end{aligned}$$

The stress has second order singularity when the receiver point coincides with the source point ($R_a=0$ and $R_b=0$). If this stress is integrated over a surface that includes the source, a finite value results. This value can be obtained by integrating $\tau_{xz}$ over all possible *dydz* surrounding the source. This integration can be replaced by integration over a half ellipsoidal surface around the source point. The ellipsoidal surface is defined by

$$x-x'=R\sin\theta\cos\varphi,\ \ y-y'=R\sin\theta\sin\varphi,\ \ z-z'=\frac{1}{\nu}R\cos\theta. \quad (90)$$

Using $s = \nu z$, the ellipsoidal surface can be transformed into a spherical surface in the $(x, y, s)$ system. Then, surface mapping between $dydz$ and a differential ellipsoidal surface is

$$dydz \sin\theta \cos\varphi = \frac{1}{\nu} dyds \sin\theta \cos\varphi = \frac{1}{\nu} R^2 \sin\theta d\theta d\varphi, \tag{91}$$

as shown in Figure 9, where, $dz = \frac{1}{\nu} ds$, $ds = Rd\theta/\sin\theta$ and $dy = R\sin\theta d\varphi/\cos\varphi$. Then,

$$\begin{aligned} \int \tau_{xz} dydz &= \frac{-1}{4\pi c_{33}(\nu_b^2 - \nu_a^2)} \left( \frac{c_{11} + c_{13}\nu_a^2}{\nu_a^2} - \frac{c_{11} + c_{13}\nu_b^2}{\nu_b^2} \right) \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{\pi} \sin\theta d\theta \\ &= \frac{-c_{11}}{2c_{33}\nu_a^2\nu_b^2} = -\frac{1}{2}. \end{aligned} \tag{92}$$

For a point force in the $\hat{x}$ direction, we have

$$\begin{aligned} u_x &= g + \frac{\partial^2}{\partial x^2} \frac{1}{\nabla_t^2} (L_t \phi - g) \\ &= \frac{S_a}{\nu_a} \left( \frac{1}{D_a} - \frac{(x - x')^2}{D_a^2 R_a} \right) - \frac{S_b}{\nu_b} \left( \frac{1}{D_b} - \frac{(x - x')^2}{D_b^2 R_b} \right) + \frac{\nu_g}{4\pi c_{66}} \left( \frac{1}{D_g} - \frac{(y - y')^2}{D_g^2 R_g} \right) \end{aligned} \tag{93}$$

$$\begin{aligned} u_y &= \frac{\partial^2}{\partial x \partial y} \frac{1}{\nabla_t^2} (L_t \phi - g) \\ &= \frac{S_b}{\nu_b} \frac{(x - x')(y - y')}{D_b^2 R_b} - \frac{S_a}{\nu_a} \frac{(x - x')(y - y')}{D_a^2 R_a} + \frac{\nu_g}{4\pi c_{66}} \frac{(x - x')(y - y')}{D_g^2 R_g} \end{aligned} \tag{94}$$

$$\begin{aligned} u_z &= -(c_{13} + c_{44}) \frac{\partial^2 \phi}{\partial x \partial z} \\ &= -S_c \left( \frac{x - x'}{D_b R_b} - \frac{x - x'}{D_a R_a} \right) \operatorname{sgn}(z - z'). \end{aligned} \tag{95}$$

In the above equations, $D_a = R_a + \nu_a|z - z'|$ and $D_b = R_b + \nu_b|z - z'|$. The scalars $S_a$, $S_b$ and $S_c$ are

$$S_a = \frac{c_{44} - c_{33}\nu_a^2}{4\pi c_{33}c_{44}(\nu_b^2 - \nu_a^2)}, \quad S_b = \frac{c_{44} - c_{33}\nu_b^2}{4\pi c_{33}c_{44}(\nu_b^2 - \nu_a^2)}, \quad S_c = \frac{c_{13} + c_{44}}{4\pi c_{33}c_{44}(\nu_b^2 - \nu_a^2)}. \tag{96}$$

Then, the normal stress, $\tau_{xx}$, is

$$\begin{aligned} \tau_{xx} &= c_{11} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) - 2c_{66} \frac{\partial u_y}{\partial y} + c_{13} \frac{\partial u_z}{\partial z} \\ &= -\frac{c_{11} S_a}{\nu_a} \frac{x - x'}{R_a^3} + \frac{2c_{66} S_a}{\nu_a} \left( \frac{x - x'}{R_a D_a^2} - \frac{(x - x')(y - y')^2}{D_a^2 R_a^3} - \frac{2(x - x')(y - y')^2}{R_a^2 D_a^3} \right) \end{aligned}$$

$$+\frac{c_{11}S_b}{\nu_b}\frac{x-x'}{R_b^3}-\frac{2c_{66}S_b}{\nu_b}\left(\frac{x-x'}{R_bD_b^2}-\frac{(x-x')(y-y')^2}{D_b^2R_b^3}-\frac{2(x-x')(y-y')^2}{R_b^2D_b^3}\right)$$
$$-\frac{\nu_g}{2\pi}\left(\frac{x-x'}{R_gD_g^2}-\frac{(x-x')(y-y')^2}{D_g^2R_g^3}-\frac{2(x-x')(y-y')^2}{R_g^2D_g^3}\right)$$
$$+c_{13}S_c\left(\frac{\nu_b(x-x')}{R_b^3}-\frac{\nu_a(x-x')}{R_a^3}\right). \quad (97)$$

Similarly, integrating $\tau_{xx}$ over a small half ellipsoidal surface results in $-1/2$. This is so because integration of the second, the fourth, the fifth and the last term is zero, as shown in Appendix A, while integration of the first and third term is the same as in $\tau_{xz}$ for the vertical force.

## CONCLUSIONS

The dynamic and static Green's functions have been obtained by solving the wave and equilibrium equations with general sources in a transversely isotropic elastic medium. The use of an extended Kupradze method makes possible a simplified, yet rigorous derivation. The two Green's functions are shown to have a single dyadic form expressed through three scalars: $g$ for the $SH$ wave, $\phi$ for the $P-SV$ waves, and $\psi$ for $P-SV-SH$ waves. In deriving these functions, the 2-D inverse Laplacian operator is removed to obtain simplified and numerically feasible expressions for the Green's functions. The final result is valid for arbitrary sources at arbitrary locations. This is particularly important to BEM implementation of wave propagation and scattering problems. The dynamic Green's function is applied to obtain simple analytical expression for the displacements produced by three point sources. Evaluations of these displacements show agreement with previous numerical studies. The singular contribution, when integrating stresses over a half elliptical surface at the limit of the receiver coinciding with the source, is shown to be negative one-half of the applied force.

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## APPENDIX A: PROOF OF ZERO INTEGRALS

In this appendix, we prove that integrating the second term in equation (97) over a half elliptical surface results in zero. According to Eqs. (90) and (91), we have

$$\begin{aligned}
I &= \int dy\, dz \left(\frac{x-x'}{R_a D_a^2} - \frac{(x-x')(y-y')^2}{D_a^2 R_a^3} - \frac{2(x-x')(y-y')^2}{R_a^2 D_a^3}\right) \\
&= \frac{1}{\nu_a}\int_0^\pi \sin\theta d\theta \int_{-\pi/2}^{\pi/2} d\varphi \left(\frac{R_a^2}{D_a^2} - \frac{(y-y')^2}{D_a^2} - \frac{2R_a(y-y')^2}{D_a^3}\right) \\
&= \frac{1}{\nu_a}\int_0^\pi d\theta \int_{-\pi/2}^{\pi/2} d\varphi \left(\frac{\sin\theta}{(1+|\cos\theta|)^2} - \frac{\sin^3\theta\sin^2\varphi}{(1+|\cos\theta|)^2} - \frac{2\sin^3\theta\sin^2\varphi}{(1+|\cos\theta|)^3}\right) \\
&= \frac{\pi}{\nu_a}\int_0^\pi d\theta \left(\frac{\sin\theta}{(1+|\cos\theta|)^2} - \frac{\sin^3\theta}{2(1+|\cos\theta|)^2} - \frac{\sin^3\theta}{(1+|\cos\theta|)^3}\right).
\end{aligned}$$

If we divide the $\theta$ integral into two regions, 0 to $\pi/2$ and $\pi/2$ to $\pi$, and combine the first and the third term, we obtain

$$\begin{aligned}
I &= \frac{\pi}{\nu_a}\int_0^{\pi/2} \frac{\sin\theta\ \cos\theta}{(1+\cos\theta)^2} d\theta - \frac{\pi}{\nu_a}\int_{\pi/2}^{\pi} \frac{\sin\theta\ \cos\theta}{(1-\cos\theta)^2} d\theta \\
&\quad -\frac{\pi}{2\nu_a}\int_0^{\pi/2} \frac{\sin^3\theta}{(1+\cos\theta)^2} d\theta - \frac{\pi}{2\nu_a}\int_{\pi/2}^{\pi} \frac{\sin^3\theta}{(1-\cos\theta)^2} d\theta \\
&= \frac{2\pi}{\nu_a}\int_0^{\pi/2} \frac{\sin\theta\cos\theta}{(1+\cos\theta)^2} d\theta - \frac{\pi}{\nu_a}\int_0^{\pi/2} \frac{\sin^3\theta}{(1+\cos\theta)^2} d\theta \\
&\overset{\theta=2t}{=} -\frac{4\pi}{\nu_a}\int_0^{\pi/4} \frac{\sin^3 t}{\cos t} dt + \frac{2\pi}{\nu_a}\int_0^{\pi/4} \left(\frac{\sin t}{\cos t} - \frac{\sin^3 t}{\cos^3 t}\right) dt \\
&= -\frac{4\pi}{\nu_a}[\cos^2 t/2 - \ln\cos t]_0^{\pi/4} + \frac{2\pi}{\nu_a}[-\ln\cos t]_0^{\pi/4} \\
&\quad -\frac{2\pi}{\nu_a}[\frac{1}{2\cos^2 t} + \ln\cos t]_0^{\pi/4} \\
&= 0.
\end{aligned}$$

The same results are obtained for the fourth and fifth terms in equation (97).

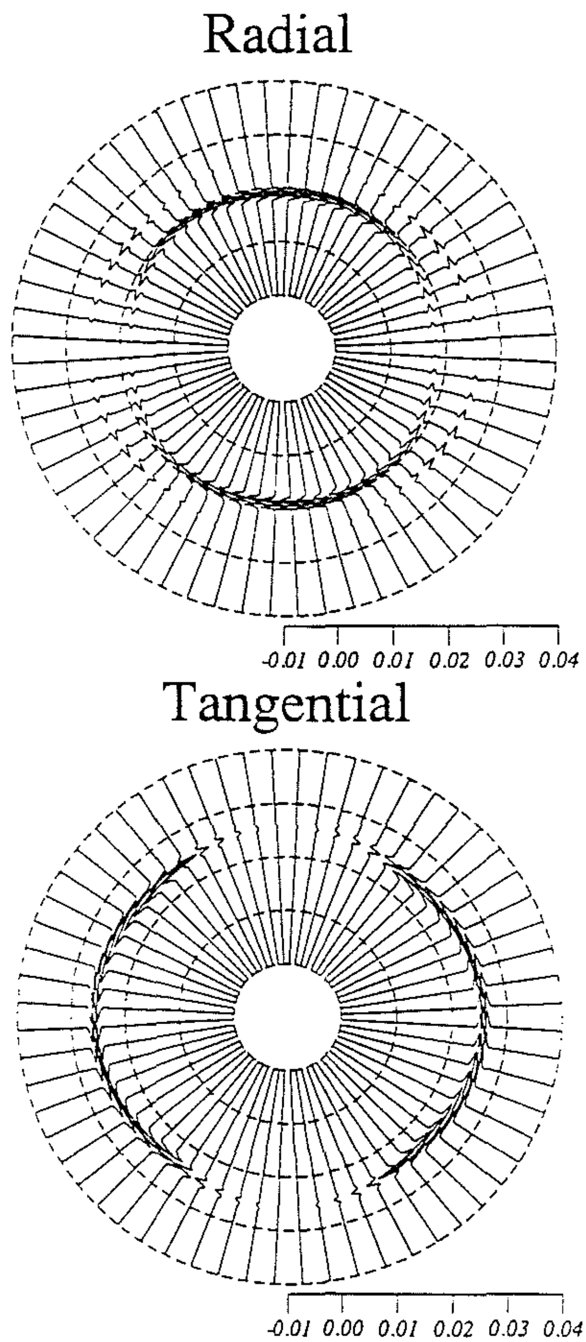


Figure 1: The radial and tangential components of the displacement produced by a point force along the symmetry axis (vertical) in a slightly anisotropic medium: Mesaverde sandstone. The wave amplitudes are normalized to the same scale. Receivers are 75 $m$ away and time is in seconds.

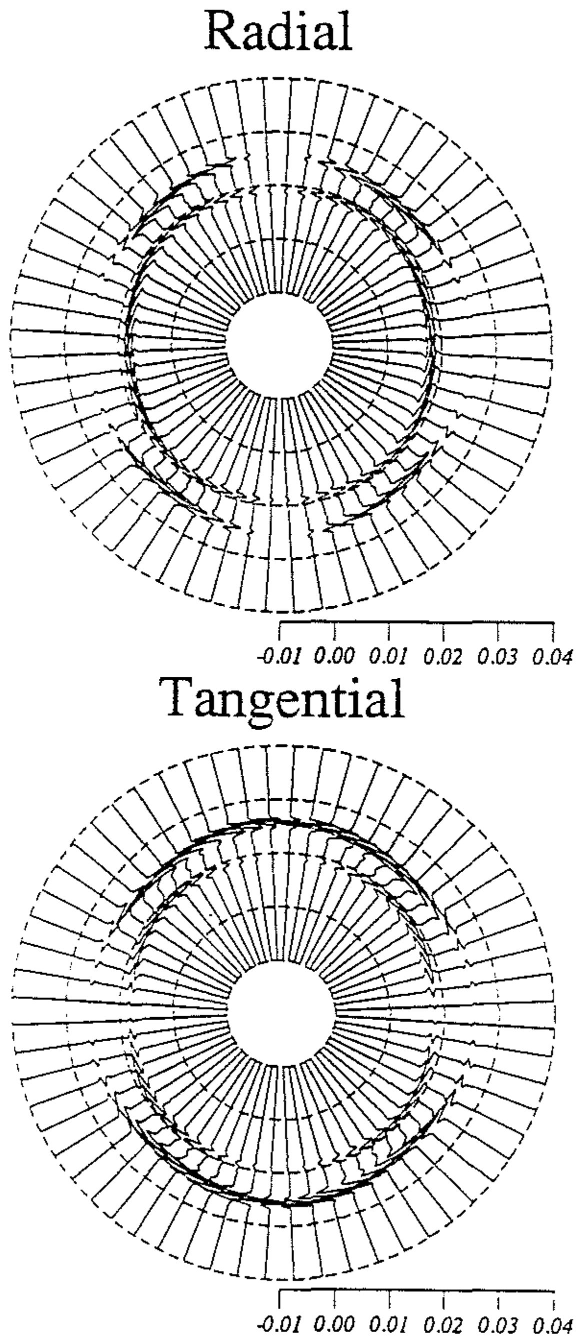


Figure 2: The radial and tangential components of the displacement produced by a point force along the isotropic plane (horizontal) in Mesaverde sandstone. Receivers are 75 $m$ away at $45^o$ azimuth angle from the force direction.

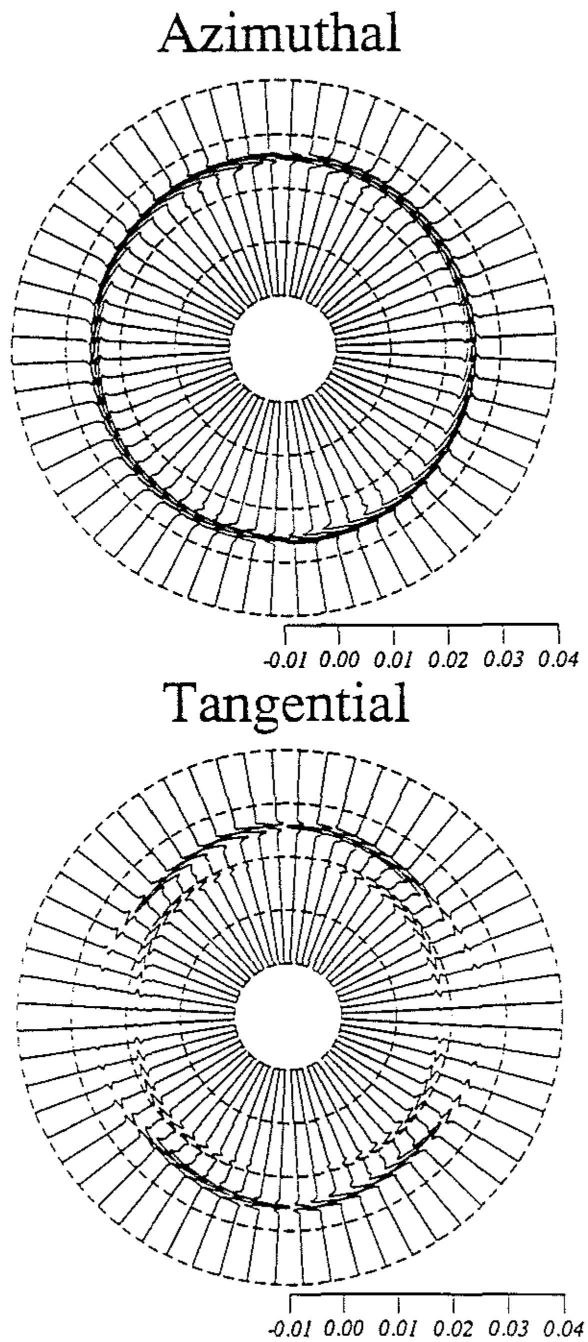


Figure 3: The azimuthal (*SH* wave) and tangential components of the displacement produced by a point force along the isotropic plane (horizontal) in Mesaverde sandstone. Same receiver position as Figure 2.

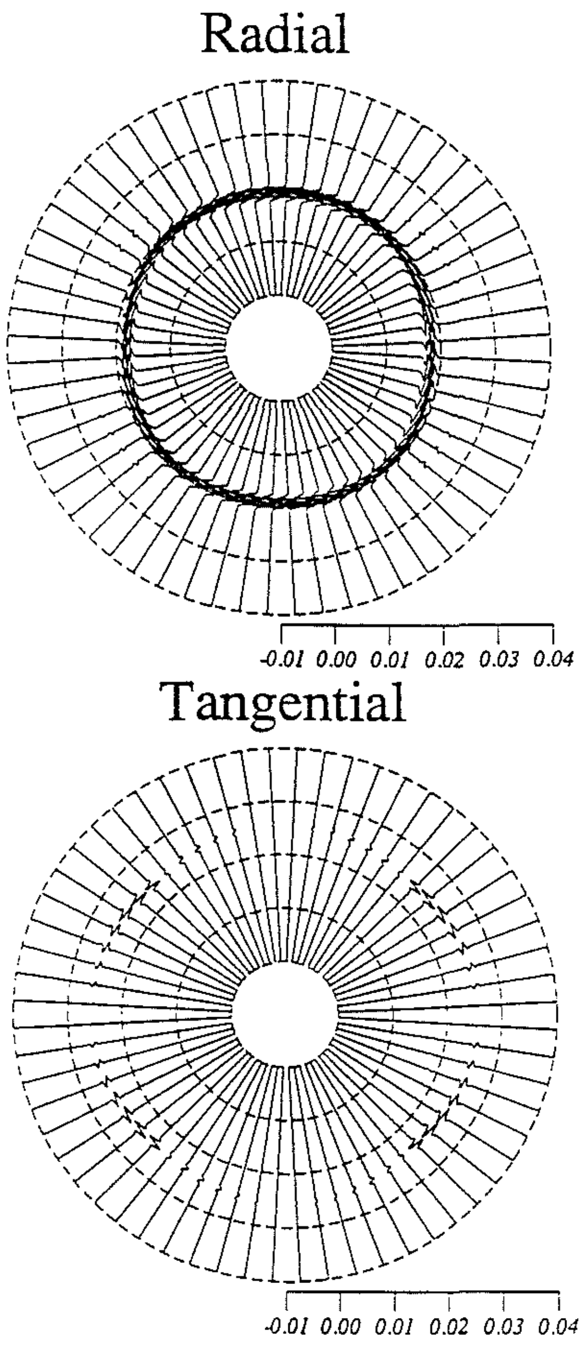


Figure 4: The radial and tangential components of the displacement produced by an explosion in Mesaverde sandstone.

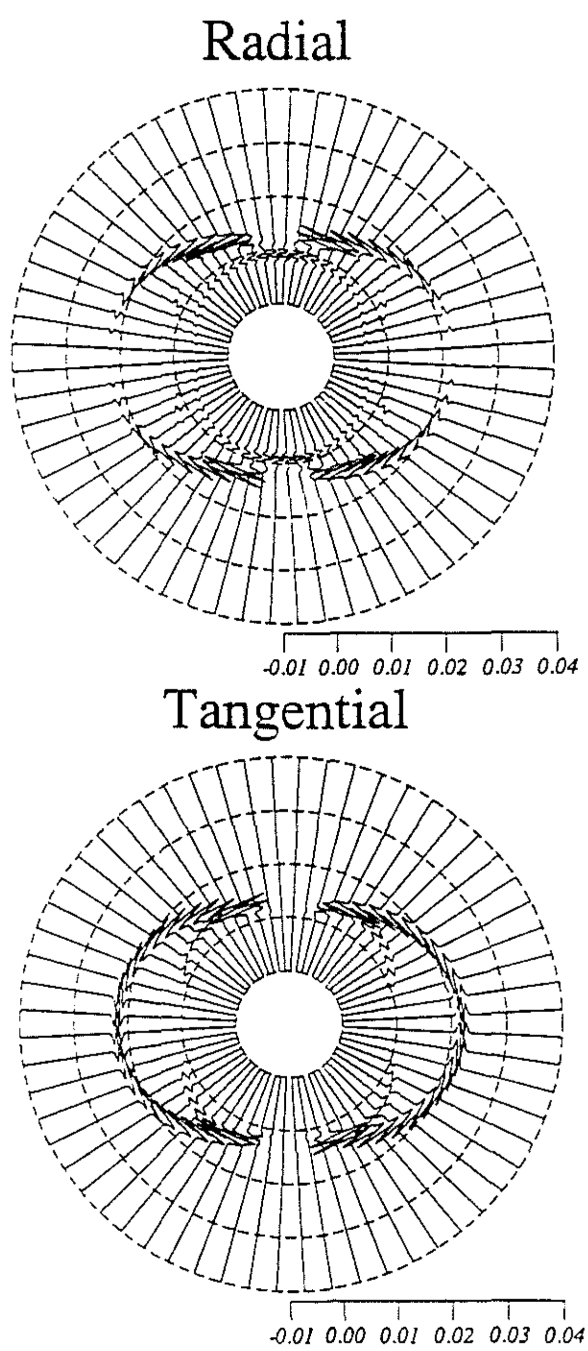


Figure 5: The radial and tangential components of the displacement produced by a point force along the symmetry axis (vertical) in a highly anisotropic medium: plexiglas-aluminum. Receivers are 30 $m$ away from the source.

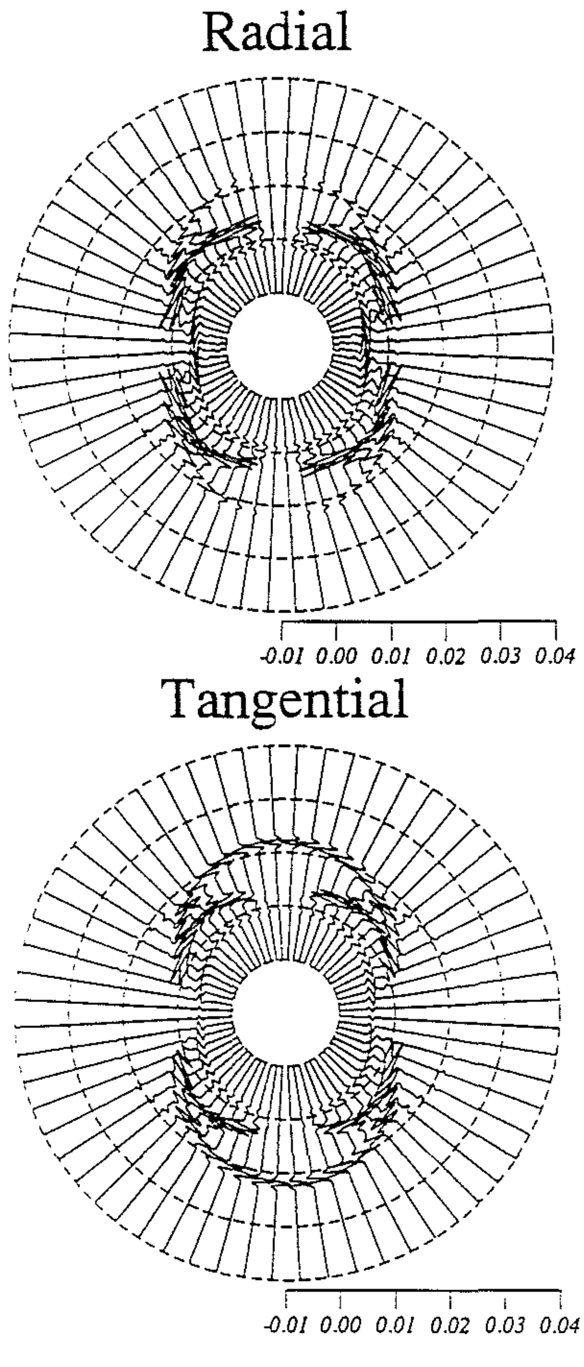


Figure 6: The radial and tangential components of the displacement produced by a point force along the isotropic plane (horizontal) in plexiglas-aluminum. Receivers are 30 $m$ away at $45^o$ azimuth angle from the force direction.

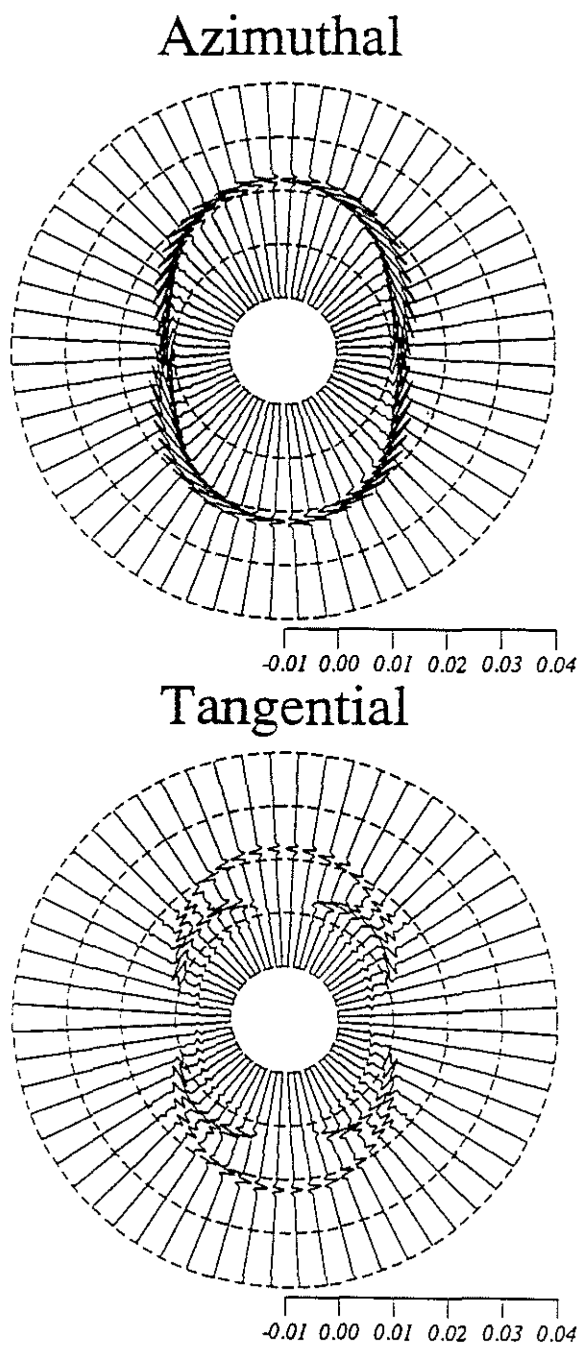


Figure 7: The azimuthal ($SH$ wave) and tangential components of the displacement produced by a point force along the isotropic plane (horizontal) in plexiglas-aluminum. Same receiver position as Figure 6.

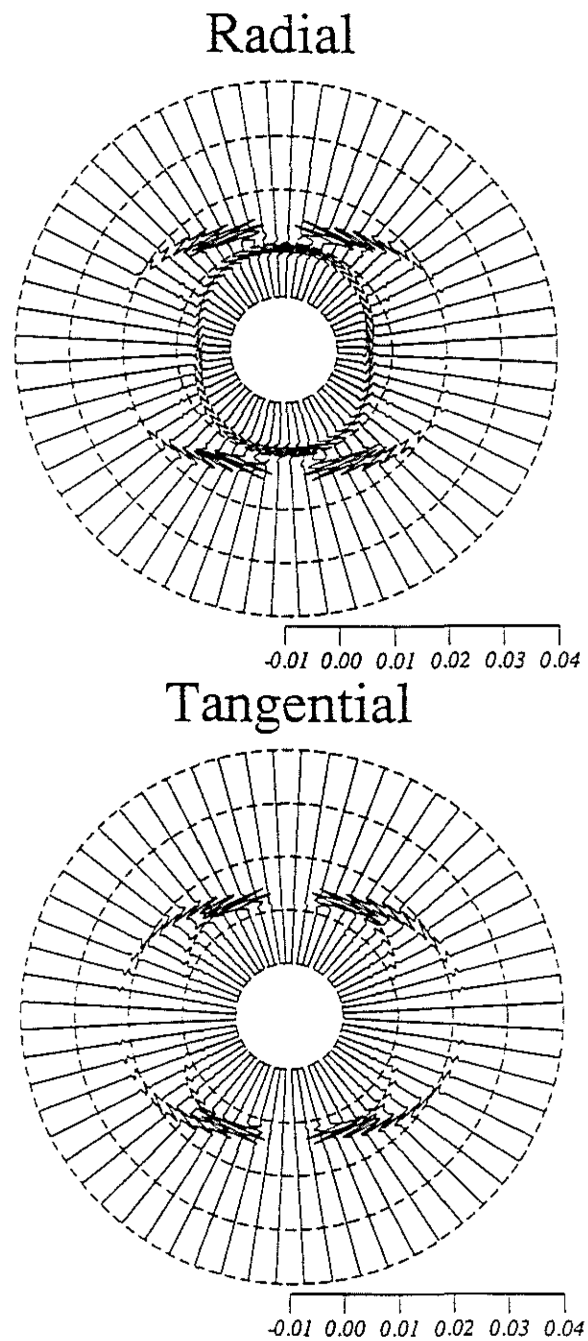


Figure 8: The radial and tangential components of the displacement produced by an explosion in plexiglas-aluminum.

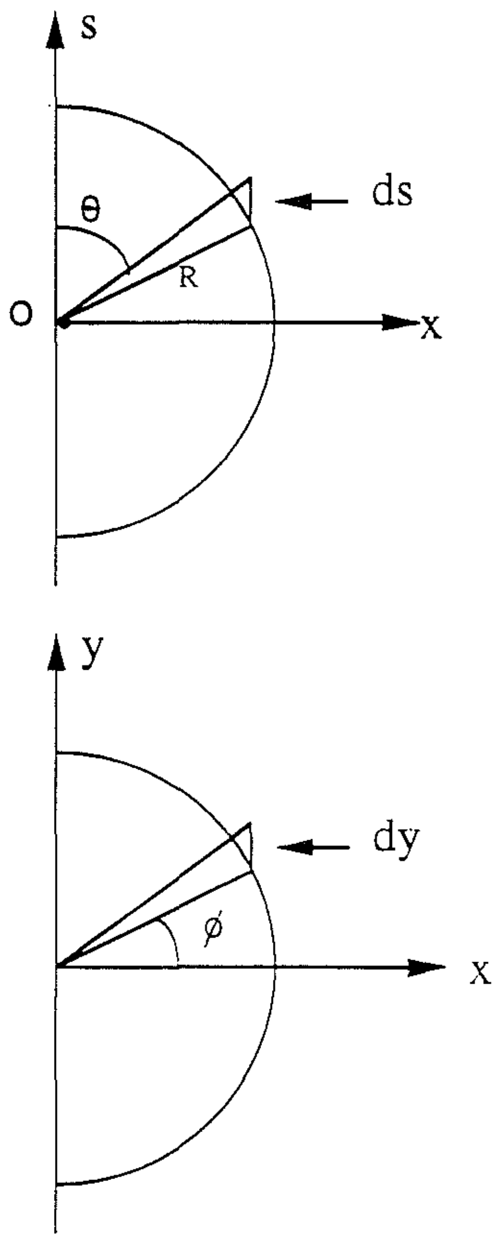


Figure 9: The geometry of the differential surfaces used in evaluating the singular contribution of surface integration of Green's functions.