MODELLING OF DOWNHOLE SEISMIC SOURCES II: AN ANALYSIS OF THE HEELAN/BREKHOVSKIKH RESULTS AND COMPARISON OF POINT SOURCE RADIATION TO RADIATION FROM BOREHOLES

by

J.A. Meredith, C.H. Cheng and M.N. Toksöz

Earth Resources Laboratory
Department of Earth, Atmospheric, and Planetary Sciences
Massachusetts Institute of Technology
Cambridge, MA 02139

ABSTRACT

The work of Heelan (1952, 1953a,b) was one of the first studies of wave propagation from a cylindrical boundary. Heelan attempted to model the radiation emanating from a cylindrical shot hole filled with dynamite. To do so he applied a constant stress to a finite length of an empty infinite cylindrical cavity embedded in an infinite elastic, homogeneous medium. The stresses he considered were axial, torsional, and radial stresses. The radial and axial stresses were required to be proportional to each other and of the same duration.

To date Heelan's work has been referenced in over 100 articles and 15 different journals including recent works (Paulsson, 1988) . His results have also been compared with results from the reciprocity theorem (White, 1953, 1960) and played an integral part of important books including those by Brekhovskikh (1960, 1980) and White (1965, 1983). His fundamental contributions were the description of shear wave lobes, the famous four-leaved rose, generated from a radial source in a borehole and that the radiation patterns for an axial source and a torsional source in a borehole have the same geometries as the point axial and torsional sources in infinite media.

Despite the importance of this work, Heelan's results have been criticized by Jordan (1962) who dismissed the work as mathematically unsound and Abo-Zena (1977) who devoted an appendix of his 1977 paper to criticizing Heelan's results. The main point of contention has been the use of contour analysis in his first paper (Heelan, 1953a).

Although Heelan's work did not include a fluid-filled borehole which is a crucial

omission for our purposes, his work may nonetheless be seen as a starting point for the modelling of downhole seismic sources. For instance, Lee and Balch (1982) developed radiation patterns for fluid boreholes which were simple extensions of Heelan's results. Additionally, one particular application of Heelan's theory is in the preliminary development of downhole seismic sources that often require dry holes until the electronics can be properly shielded. For that reason, an exhaustive examination of the mathematics and physics that went into Heelan's first paper was undertaken to determine if his formulation was correct.

The fundamental basis of Heelan's work was a variant of the Sommerfeld integral, an integral of cylindrical waves, in which he unfortunately did not specify the contour. To overcome this obstacle of an unknown contour a parallel method suggested by Brekhovskikh (1960, 1980) was implemented. Brekhovskikh used the Weyl integral, an integral over plane waves, to duplicate Heelan's results for the radial and torsional stresses. However he does no justification of the extensive algebra or analysis involved and does not include the effects of axial stress. Thus in this paper, we have completed and elucidated the work that Brekhovskikh initiated and moreover indirectly verified that Heelan's results were correct.

Additionally, we found that Abo-Zena's and Heelan's initial formulations were equivalent. The only difference was in a reversal of the separation of variables procedure necessary to replicate this work and also in Abo-Zena's use of the Laplace transform where Heelan used the Fourier transform. However, Abo-Zena's results do extend Heelan's by allowing the source function to vary over the distance in which it is applied. The far field results of Abo-Zena and Heelan are equivalent (White, 1983) only if a $\frac{1}{\mu}$ correction is applied to Abo-Zena's results.

The first half of this paper is very involved mathematically but much of the algebra is relegated to Appendix A. Having verified that Heelan's results were correct we then proceed to compare Heelan's results with well established point source representations known in the literature (White, 1983) and also with radiation patterns from point sources and stress sources in a fluid-filled borehole (Lee and Balch, 1982). These comparisons will help us isolate the propagation effects of the fluid and the geometrical effect of the borehole. One unique aspect to our approach will be the consideration of radiation from boreholes surrounded by varying lithologies instead of just the Poisson solid as is commonly done. The lithologies to be considered include a soft sediment (Pierre shale) and two more indurated sediments, Berea sandstone and Solenhofen limestone. By following this approach we show that the effect on the radiation magnitude can be substantial due to changes in lithology in addition to isolating the relative effects of the borehole and the fluid.

A DEFENSE AND ANALYSIS OF HEELAN'S RESULTS

Heelan published two papers (1953a,b) based on his thesis work (1952) which have been very influential in the description of downhole seismic sources. His first paper entitled "Radiation from a cylindrical source of finite length" showed the calculation of radiation from axisymmetric radial, axial and torsional stresses applied to a short length of an infinite cylindrical cavity embedded in an infinite elastic medium. Heelan's work was important because it was one of the first in geophysics to look at radiation from cylindrical objects. Heelan believed that applying stresses to a short length of an infinite cavity was a good model for modelling radiation from dynamite placed in cylindrical shot holes. However, the radius of his infinite cavity did not correspond exactly to the radius of the shothole but instead to the radial distance at which deformation becomes elastic following an explosion, Sharpe's equivalent cavity (Sharpe, 1942). His second paper was entitled "On the theory of head waves". This paper utilized integral expressions from the first paper to calculate head waves propagated along an interface due to spherical waves generated from a dynamite source. However, in this second paper Heelan only used the results for radial and torsional stresses since he surmised that explosions would not likely generate significant axial stresses. The results from Heelan's first paper on radiation will be the exclusive focus of this paper.

One major limitation of Heelan's work when discussed in the context of modelling downhole seismic sources is that the cavity Heelan used is empty. For most practical purposes the borehole will be under the water table so will be fluid-filled. An empty cavity implies that the wall of the cavity can be treated as a free surface and no continuity of displacement boundary conditions are allowed. Nonetheless, qualitatively at least, propagation from an empty borehole is a midpoint between propagation of point sources in infinite media and the propagation from a fluid-filled borehole so Heelan's results are important for our study here. Finally, there are occasions, for instance during seismic source development, that source excitation must be performed in dry holes (Paulsson, 1988).

Heelan's results have been cited in over 70 references and 13 different journals since 1953 including recent works (White, 1983; Paulsson, 1988). One of his major results was that compressional waves could be induced by an artificial source (dynamite) although at the time many researchers believed only P waves would be generated from such a source or that S waves observed from explosions were in fact of secondary origin. Another major contribution Heelan made was to calculate radiation patterns with simple geometric interpretations, the four-leaved rose for Sv radiation and the peanut-shaped pattern for P waves both from a radial source. Finally, his work has been used by others in further developments and research. For instance, the bulk of Heelan's analysis is replicated in Brekhovskikh's book "Waves in layered media"(1960, 1980). Additionally, there are many important references to Heelan's work by White

(1960, 1965, 1983) who used the reciprocity theorem and results from White's earlier paper (1953) to duplicate the radiation patterns from Heelan's paper. Thus White claimed confidence in the results from both techniques.

Almost all of the references of Heelan's work to date have only utilized the figures Heelan produced (Heelan, 1953a, Figures 2–4) without looking at the properties implied by the formulas used to produce the figures. Some of the implications are quite profound, especially the dependence of radiation amplitude on shear wave velocity.

Despite the importance of Heelan's work it has come under severe and justifiable criticism due to omissions in his papers and other matters. Another reason we feel Heelan's work has been criticized is due to the complexity of his algebra. The criticism comes in different forms, for instance, Jordan (1962) dismissed Heelan's work as mathematically unsound while Hazebroek (1966) pointed out that Heelan's analysis was improperly entitled since the cylinder was not closed. The most severe criticism has come from Abo-Zena who wrote a paper with a title similar to Heelan's first which was entitled "Radiation from a finite cylindrical explosive source" (Abo-Zena, 1977). In this paper, Abo-Zena devotes the appendix to criticism of Heelan's work.

White (1983) points out that the far field results of Abo-Zena (1977) are in fact equivalent to Heelan's results. Heelan's and Abo-Zena's patterns are equivalent except for a scaling factor in the denominator which is Lamè's parameter μ which does not affect the geometric shape of the radiation patterns but has a large effect on their amplitudes. We will demonstrate that the initial formulations of Abo-Zena and Heelan are equivalent and will show through verifying and extending Brekhovskikh's work that in fact Heelan's results are the correct results. However, despite the incorrectness of the scaling factor in Abo-Zena's work, it is in fact an extension of Heelan's work in that non-uniform stresses may be applied over the finite cylindrical cavity instead of the uniform stress applied over a cylindrical length required by Heelan.

One major problem in answering the criticisms of Heelan's work was that Heelan did not specify the complex contour C used in his analysis in either his papers (1953a,b) or thesis. Nor did he specify how the introduction of his source terms might affect his contour. It can be seen that Heelan's analysis is broken down into integrals of cylindrical waves, a Sommerfeld integral type problem, but the precise definition of the contour hampers proof of his results. There was no pictorial sketch and his verbal description of a loop was imprecise. Another difficulty leading to some of the criticisms was the complexity of the algebra Heelan produced and some of his unconventional though correct mathematical manipulations.

A more concrete treatment of the problem of radiation from cylindrical sources was initiated by Brekhovskikh (1960, 1980) referencing Heelan's work which clearly broke the wave propagation problem down into homogeneous and inhomogeneous plane waves using the Weyl integral. Heelan had used cylindrical waves, the Sommerfeld integral, instead of plane waves so to substantiate that the results of Brekhovskikh and Heelan are equivalent requires justification. Brekhovskikh's treatment is more straightforward and established than Heelan's and relies on the Weyl integral and contours very well known in electromagnetic theory (Stratton, 1941). Our contribution will be to follow up on the technique Brekhovskikh suggested and add important additional elucidation and analysis and confirm that Brekhovskikh's preliminary results are correct. Brekhovskikh did not calculate the effect of axial stress sources and that will be done here. This contribution is important in view of the criticism that Heelan's work has received and in our opinion independent corroboration through the techniques of Brekhovskikh is needed to answer the criticism.

Finally, a comparison will be made between the radiation patterns derived from Heelan/Brekhovskikh for a stress applied to the short length of an infinite elastic cavity and well known results for a point source applied in a particular direction in an infinite medium and a vertically incident force on a free surface as presented by Miller and Pursey (1953). This comparison of radiation behavior will allow separation of effects of the empty borehole from effects due to wave propagation in an infinite medium.

Heelan's Analysis

In Heelan's model, stresses are applied over a finite length of an empty infinite cylinder and on the boundary between the cylinder and infinite medium. The geometry is shown in Figure 1.

Heelan calculates the radiation for two uncoupled wave propagation problems with axisymmetric sources in axisymmetric media. The first problem he solves is the $P-Sv$ axisymmetric problem where the displacement potentials ϕ, ψ are calculated. But unlike the most common treatments for axisymmetric problems (Biot, 1952; White, 1965, 1983) Heelan does not recode ψ as was shown in Case 3 of Part I. Therefore his analysis corresponds to Case 3 for the $P-Sv$ problem and Case 4 for the Sh problem of Part I. Also, in a highly unusual manner Heelan uses $-\psi$ instead of ψ as his Sv displacement potential which affects the signs of his boundary conditions. Nonetheless due to symmetry considerations this does not affect the final results. It must be pointed out however that Pilant (1979, pg. 45) and others also used this convention and it is perfectly legitimate.

Because axisymmetry was assumed, there is no dependence on the θ component and no summation over θ. Therefore, Heelan writes his potentials ϕ, ψ, χ with the following integral transforms

$$\phi \quad = \quad Re \int_0^\infty e^{ikVt}\, dk \int_C f_o(\sigma, k) H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2 - k^2}}\, d\sigma \tag{1}$$

$$
\begin{aligned}
\psi &= Re \int_0^\infty e^{ikVt}\, dk \int_C g_o(\sigma,k) H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2-h^2}}\, d\sigma \\
\chi &= Re \int_0^\infty e^{ikVt}\, dk \int_C n_o(\sigma,k) H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2-h^2}}\, d\sigma
\end{aligned}
$$

In Heelan's notation, V and v are the compressional and shear wave velocities, k, h are wavenumbers and $kV = hv$, σ is an axial wavenumber, $\sqrt{\sigma^2 - k^2}$ and $\sqrt{\sigma^2 - h^2}$ are the radial wavenumbers, e^{ikVt} is the positive time dependence, and r is the radius of investigation. It is important to recognize that $kV = hv$ can be set equal to radial frequency ω. To check that this is permissible, we divide through by velocity and obtain the desired units for k, h, cycles over distance. This recognition of the kV integral as a frequency integral helps in comparing the analysis derived from the above equations with other author's developments. However, henceforth we will use the k, h symbolism to facilitate comparison with Heelan's and Brekhovskikh's results.

Heelan's notation caused us some initial difficulty because of the unusual treatment of the wavenumber term k. The wavenumber k is usually evaluated as an axial wavenumber that varies with the inverse of phase velocity (c) but in this development k is evaluated as a wavenumber that is fixed under the axial wavenumber (σ) integral. The usual definition of k is $\frac{\omega}{c}$ and in Heelan's paper k is evaluated as $\frac{\omega}{V}$ and similarly for h.

Another departure from conventions in describing sources from sonic well logging purposes was that the integrals of Eq. 1 represent a Hankel function of axial wavenumber multiplied by r instead of radial wavenumber multiplied by r, the common convention. This was due to a reversal of the final two steps in the separation of variables procedure Heelan used to obtain his wave functions. However, in separating his variables in this manner his results closely resemble the Sommerfeld integral which will be shown later.

In Part I , it was shown that the most common method to separate variables is to first separate in t yielding a time function $e^{i\omega t}$, (e^{ikVt}) then in z yielding a depth function $e^{ik_z z}$ ($e^{i\sigma z}$) and then finally in r yielding a modified Bessel or Hankel function of radial wavenumber times radius $H_0^{(1)}(lr), H_0^{(1)}(mr)$, where l and m equal $\sqrt{k_z^2 - \frac{\omega^2}{\alpha^2}}, \sqrt{k_z^2 - \frac{\omega^2}{\beta^2}}$ respectively or in Heelan's notation $\sqrt{\sigma^2 - k^2}, \sqrt{\sigma^2 - h^2}$.

Heelan's separation of variables strategy (Heelan did not specifically address this issue in his thesis or papers) was to reverse the final two steps and put the radial wavenumber under the exponent of z. To wit, we begin having separated the variables over frequency, but we ignore θ because of axisymmetry, rewriting Eq. I:47 (equation numbers from Part I will be designated I:)

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + k^2 + \frac{Z''}{Z} = 0 \tag{2}$$

The traditional method then further separates variables by bringing only the term $\frac{Z''}{Z}$ over to the right hand side and setting it equal to the axial wavenumber k_z^2 (σ^2). Heelan rearranges this separation of variables in a different manner as follows. He brings both the Z term and the wavenumber term over to the right hand side yielding

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2 - \frac{Z''}{Z} = \sigma^2 \tag{3}$$

which solutions in R are Bessel functions, modified Bessel functions or Hankel functions of order zero as explained in Part I. Heelan uses the radiation condition to limit his solutions to outgoing waves governed by the Hankel function $H_0^{(1)}(\sigma r)$.

Heelan's separation of variables then proceeds

$$\frac{Z''}{Z} - (\sigma^2 - k^2) = 0$$

Solving for Z yields the function

$$Z = e^{z\sqrt{\sigma^2 - k^2}} \tag{4}$$

An identical procedure is followed for the integrals dependent on h. Thus we can see that the equations and the boundary conditions that Heelan solves are comparable to the developments in Part I - the only difference is a rearrangement of the separation of variables procedure. To obtain the general solution we must superpose all potential solutions and thus integrate over all of the values of our parameters σ and ω. The displacement potential ϕ becomes

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\sigma, k) H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2 - k^2}} e^{ikVt} d\sigma dk \tag{5}$$

but we must allow complex values of the axial wavenumber σ so Heelan from conception evaluates Eq. 5 as a contour integral

$$\phi = \int_{-\infty}^{\infty}\int_{C} A(\sigma, k) H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2 - k^2}} e^{ikVt} d\sigma dk \tag{6}$$

Heelan specified that ϕ, ψ, χ are real and thus he is able to use a one sided Fourier transform operator $\mathrm{Re}\int_0^{\infty} e^{ikVt} dk$ as a shorthand notation (see the first few pages of Brekhovskikh (1960)). Since this has been the source of some criticism a proof of the validity of his operator is presented in an Appendix. Heelan writes his coefficient function as f_o, g_o, n_o and thus his integrals for the displacement potentials are (Eq. 1)(Heelan, 1953a, Eq. 5)

$$\begin{aligned} \phi &= Re\int_0^\infty e^{ikVt}\int_C f_o(\sigma,k)H_0^{(1)}(\sigma r)e^{i\sqrt{\sigma^2-k^2}}e^{ikVt}d\sigma\,dk \\ \psi &= Re\int_0^\infty e^{ikVt}\int_C g_o(\sigma,k)H_0^{(1)}(\sigma r)e^{i\sqrt{\sigma^2-h^2}}e^{ikVt}d\sigma\,dk \\ \chi &= Re\int_0^\infty e^{ikVt}\int_C h_o(\sigma,k)H_0^{(1)}(\sigma r)e^{i\sqrt{\sigma^2-h^2}}e^{ikVt}d\sigma\,dk \end{aligned} \tag{7}$$

We have used the potential symbol ψ instead of Θ which Heelan used. Heelan states in a footnote to his first paper (1953a) and his thesis (1952) that if $f_o = \frac{-\sigma}{2\sqrt{\sigma^2-k^2}}$ then the contour C can be deformed onto the real axis and the resulting integral is the classic Hertzian oscillator $\frac{e^{ikR}}{R}$, where $R = \sqrt{r^2+z^2}$. The resulting integral upon doing this substitution is the Sommerfeld integral as shown below.

We assume temporarily Heelan's supposition is correct and that the unknown contour C can be deformed onto the real axis after the substitution of $f_o = \frac{-\sigma}{2\sqrt{\sigma^2-k^2}}$ and obtain the integral

$$\phi = Re\int_0^\infty\int_{-\infty}^\infty \frac{-\sigma}{2\sqrt{\sigma^2-k^2}}H_0^{(1)}(\sigma r)e^{\sqrt{\sigma^2-k^2}}e^{ikVt}d\sigma dk \tag{8}$$

We temporarily set aside the integral with respect to k (our integral with respect to normalized frequency). We recall the following integral (Gradshteyn and Ryzhik, 1980, 4th edition, Eq. 6.616.3; Erdélyi et al., 1953, Eq. II-7.14.53)

$$\int_{-\infty}^\infty e^{itx}H_0^{(1)}(r\sqrt{\alpha^2-t^2})dt = -2i\frac{e^{i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}} \tag{9}$$

with arguments restricted to the domain

$$0 < \arg\sqrt{\alpha^2-t^2} < \pi, 0 < \arg\alpha < \pi \quad \text{r and x are real}$$

Putting Eq. 9 into Heelan's notation we obtain

$$\int_{-\infty}^\infty e^{itz}H_0^{(1)}(r\sqrt{k^2-t^2})d\sigma = -2i\frac{e^{ik\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} \tag{10}$$

Now we substitute $it = \sqrt{\sigma^2-k^2}$ and simplify

$$\frac{1}{2}\int_{-\infty}^\infty \frac{-\sigma}{\sqrt{\sigma^2-k^2}}H_0^{(1)}(\sigma r)e^{\sqrt{\sigma^2-k^2}z}d\sigma = \frac{e^{ikR}}{R} \tag{11}$$

Setting aside the integral over k we can see that Eq. 11 and Eq. 8 are in fact equivalent. The range limitations on Eq. 9 are satisfied if we assume k has a positive imaginary component which requires attenuation.

An important note is that the integral resulting from Heelan's supposition (Eq. 11 is in fact a Sommerfeld integral. Aki and Richards (1980, Eq. 6.7) identify the Sommerfeld integral as the following

$$\frac{e^{ikR}}{R} = \int_0^\infty \frac{\sigma J_0(\sigma r) e^{i\sqrt{\sigma^2-k^2}z}}{\sqrt{\sigma^2-k^2}} d\sigma \tag{12}$$

which they later rewrite (Aki and Richards, 1980, Eq. 6.15)

$$\frac{i}{2}\int_{-\infty}^{\infty} \frac{\sigma}{\sqrt{\sigma^2-k^2}} H_0^{(1)}(\sigma r) e^{i\sqrt{\sigma^2-k^2}z} d\sigma \tag{13}$$

both equations having been rewritten in Heelan's notation. A difference in phase i is attributed to Aki and Richards definition of radial wavenumber. The integrand is identified as a cylindrical wave governed by axial and radial wavenumbers. Comparison with Eq. 11 shows the equivalence. Thus Heelan's results represent the decomposition of a spherical wave into cylindrical plane waves.

Heelan next proceeds to match boundary conditions. His boundary conditions at the empty cavity are vanishing of normal stress and tangential stress for the P-Sv case and vanishing of azimuthal stress for the Sh case. Because the cylindrical boundary is a free surface there is no continuity in normal displacement boundary condition. As mentioned before, in an unusual manner Heelan equated his potential ψ to the negative of the potential ψ discussed in Chapter 2. Thus the stresses below are reversed in sign for the potential ψ from those presented in Part I. The stresses are

$$\begin{aligned} p_r &= \lambda \nabla^2 \phi + 2\mu \frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial r} - \frac{\partial^2 \psi}{\partial r \partial z}\right) \\ p_{rz} &= \mu \frac{\partial}{\partial r}\left(2\frac{\partial \phi}{\partial z} + \nabla^2 \psi - 2\frac{\partial^2 \psi}{\partial z^2}\right) \\ p_{r\theta} &= \mu\left(\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r}\frac{\partial \chi}{\partial r}\right) \end{aligned} \tag{14}$$

We add our stress formulae to our discontinuities in axial, radial, and torsional stress - our stress discontinuities being our sources and requiring the sums to vanish. Heelan mathematically formulates his sources with the following model. Stresses of constant magnitude P, Q, S are multiplied by a time varying amplitude factor $G(t)$ multiplied by a "boxcar" function in depth. The boxcar function is defined for a cylindrical length l centered at zero as $F(z) = 0 \;\; |z| > l, \;\; F(z) = 1 \;\; |z| <= l$. The boundary is at the radial distance $r = a$. We can equate our stresses to the boundary condition as follows

$$P\,G(t)F(z) + p_r = 0 \quad Q\,G(t)F(z) + p_{rz} = 0 \quad S\,G(t)F(z) + p_{r\theta} = 0 \tag{15}$$

This is a slight departure from Heelan's notation. Heelan uses the notation $P(t), Q(t)$ and $S(t)$ and specifies only in a parenthetical remark that they are proportional to each other. What this proportionality means is that $P(t)$ and $Q(t)$ have the same time function but different magnitudes. $S(t)$ for most cases of interest would have the same time function but because the P-Sv and Sh problems are uncoupled this is not a requirement. By using the common time function $G(t)$ we hope to clarify this important relationship.

Heelan makes use of the Fourier transform of the boxcar function which is commonly referred to as the "sinc" function

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin l\tau}{\tau} e^{iz\tau} d\tau$$

and transforms it to

$$F(z) = \frac{i}{\pi}\int_C \frac{\sigma \sinh(l\sqrt{\sigma^2 - k^2})}{\sigma^2 - k^2} e^{z\sqrt{\sigma^2 - k^2}} d\sigma \tag{16}$$

using the mapping $\sqrt{\sigma^2 - k^2} = i\tau$, $-\pi \leq \arg\,\tau \leq 0$. This shows the general integral transform for our boundary conditions; we Fourier transform $G(t)$ under k, and transform $F(z)$ under σ. Heelan next brings the boundary conditions (Eq. 15) under the integral and equates them. The results are algebraically complex and represent an intermediate step which are relegated to Appendix A of this paper. Heelan uses quite a few crafty substitutions which he did not divulge in his paper but are listed in the appendix. His algebraic results were verified to be correct.

Heelan then suggests that in the farfield we can treat n_o, f_o, g_o as being of small argument and therefore can be expanded in the parameters borehole radius a and the length of our finite cylinder l. An expansion in terms of small arguments is a common procedure in problems of this type. Although not stated in Heelan's work the expansions for small arguments he used were verified to be the following

$$\begin{aligned} \sinh x &\simeq ix \\ H_0^{(1)}(z) &\simeq \frac{2\ln(z)}{\pi} \\ H_1^{(1)}(z) &\simeq \frac{2}{i\pi z} \\ H_2^{(1)}(z) &\simeq \frac{4}{i\pi z^2} \end{aligned} \tag{17}$$

The Hankel function expansions may be seen in Abramowitz and Stegun (1964, Eq. 9.1.8,9.1.9). However, Heelan never uses the H_0 expansion because in his words "only the predominant terms in the expansion were kept". He uses the unstated justification that as z goes to zero the limit of $\frac{H_0^{(1)}}{H_1^{(1)}}$ is zero and thus $H_0^{(1)}$ terms are discarded. A similar argument was used by Abo-Zena (1977) with modified Bessel functions instead of Hankel functions.

Using these expansions for small arguments, Heelan develops the following for f_o, g_o, h_o

$$
\begin{aligned}
f_o &= P\frac{G(k)\Delta\sigma}{8\pi\mu\sqrt{\sigma^2-k^2}}(\frac{2\sigma^2}{h^2}+1-\frac{2v^2}{V^2})+Q\frac{G(k)A\sigma}{8\pi\mu h^2} \\
g_o &= P\frac{G(k)\Delta\sigma}{4\pi\mu h^2}+Q\frac{G(k)A\sigma}{8\pi\mu\sqrt{\sigma^2-h^2}} \\
n_o &= S\frac{G(k)\Delta\sigma}{8\pi\mu\sqrt{\sigma^2-h^2}}
\end{aligned}
\quad (18)
$$

where Δ equals $2\pi a^2 l$ the volume of the equivalent cavity and A equals $4\pi al$ the area of the equivalent cylindrical cavity. The extra factor of two in area and volume arise from the fact that the length of the cavity is $2l$.

Heelan uses Eq. 18 to evaluate radial, vertical and tangential displacements and places the results under the σ integral. The formula for the displacements can be seen from cases 3 and 4 in Part I, (Eq. I:32, Eq. I:36). However, again we remind you of the sign change on the term ψ.

$$
\begin{aligned}
U_r &= \frac{\partial\phi}{\partial r}-\frac{\partial^2\psi}{\partial r\partial z} \\
U_z &= \frac{\partial\phi}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) \\
U_\theta &= \frac{\partial\chi}{\partial r}
\end{aligned}
\quad (19)
$$

Heelan then substitutes the asymptotic expansions for large arguments for the Hankel functions into Eq. 19. At first glance this sequence of transformations may seem confusing. It's important to realize that Heelan uses the expansions for small arguments of the Hankel functions at the boundary $r = a$ (r is fixed) $H_\nu^{(1)}(\sigma a)$ and the asymptotic expansions in the far field where r is a variable going to infinity $H_\nu^{(1)}(\sigma r)$. The principal asymptotic expansions for large argument (Abramowitz and Stegun, 1964, 9.2.3) are

$$
H_\nu^{(1)}(\sigma r) = \sqrt{\frac{2}{\pi\sigma r}}e^{i(-\frac{\pi}{4}+\sigma r-\frac{\pi\nu}{2})} \quad (20)
$$

There is a misprint in Heelan's paper (1953a) just before Equation 9 where a factor of 2σ should be $\frac{\sigma}{2}$, the misprint was printed in its proper form in Heelan's thesis

(1952) and does not affect the final result. Upon substituting in the expansions for large arguments Heelan uses the method of steepest descent to evaluate the resulting displacement integrals and obtain the far field displacements. The result for U_r is

$$U_r = -\frac{2k\cos\phi}{R} f_o(\sigma_1)e^{-ikR} - \frac{2ih^2\cos^2\phi}{R} g_o(\sigma_2)e^{-ihR} \tag{21}$$

The value σ_1, σ_2 represent saddle points of the analysis $\sigma_1 = -k\sin\phi \quad \sigma_2 = -h\cos\phi$. $\tan\phi = \frac{r}{z}$ and $R = \sqrt{r^2+z^2}$.

Heelan then uses the expressions for f_o, g_o, n_o evaluated at σ_1, σ_2 and applies a one-sided Fourier transform operator to the above equation. In applying his operator Heelan utilizes the following relationships.

$$G(t-\frac{R}{V}) = Re\int_0^\infty e^{ikV(t-\frac{R}{V})}dk \tag{22}$$

$$\frac{d}{dt}[G(t-\frac{R}{V})] = Re\int_0^\infty ikVe^{ikV(t-\frac{R}{V})}dk \tag{23}$$

The difference between the two equations is the factor ikV. As seen in Eq. 21 a factor of k was introduced in the first term of the right hand side. This k in conjunction with the ikV factor in Eq. 23 produces the derivative of the stress applied over time.

Having completed that analysis, the resulting far field radial and vertical displacements for the P-Sv case and azimuthal displacements for the Sh case are then set up as follows.

$$\begin{aligned} U_r &= u_{rP} + u_{rSv} \\ U_z &= u_{zP} + u_{zSv} \\ U_\theta &= u_{\theta Sh} \end{aligned} \tag{24}$$

where Heelan uses the matrix representation for P of

$$\left|\begin{array}{c} U_{rP} \\ U_{zP} \end{array}\right| = \left| \frac{F_1(\phi)}{R}\frac{d}{dt}\{PG(t-\frac{R}{V})\} + \frac{G_1(\phi)}{R}SG(t-\frac{R}{v}) \right| \left|\begin{array}{c} \sin\phi \\ -\cos\phi \end{array}\right| = 0 \tag{25}$$

and for the Sv case

$$\left|\begin{array}{c} U_{rSv} \\ U_{zSv} \end{array}\right| = \left| \frac{F_2(\phi)}{R}\frac{d}{dt}\{PG(t-\frac{R}{v})\} + \frac{G_2(\phi)}{R}SG(t-\frac{R}{v}) \right| \left|\begin{array}{c} \cos\phi \\ \sin\phi \end{array}\right| = 0 \tag{26}$$

and for Sh

$$U_{\theta Sh} = \frac{K(\phi)}{R}\frac{d}{dt}\{SG(t-\frac{R}{v}\} \tag{27}$$

where the factors F_1, F_2, G_2, G_1, K_1 are Heelan's radiation pattern formulas. Heelan (1953a, eq. 12)

$$\begin{aligned}
F_1(\phi) &= \frac{\Delta(1-\frac{2v^2\cos^2\phi}{V^2})}{4\pi\mu V} &(28)\\
G_1(\phi) &= \frac{-Av^2\cos\phi}{4\pi\mu V^2} &(29)\\
F_2(\phi) &= \frac{\Delta\sin 2\phi}{4\pi\mu v} &(30)\\
G_2(\phi) &= \frac{A\sin\phi}{4\pi\mu} &(31)\\
K(\phi) &= \frac{\Delta\sin\phi}{4\pi\mu v} &(32)
\end{aligned}$$

Now the matrix formulation represented by Eq. 25 and Eq. 26 is slightly confusing. At first glance the middle matrix in these equations looks like a 1×2 matrix multiplying the rightmost 2×1 matrix. However the middle matrix instead is in fact a 1×1 matrix, a scaling factor. We will therefore multiply out these terms to remove the ambiguity and to facilitate comparison with Brekhovskikh's results to be shown later. Writing out the radial and vertical displacement components along with the contribution from U_r, U_z we have

$$\begin{aligned}
U_r &= \frac{F_1(\phi)\sin\phi}{R}\frac{d}{dt}\{PG(t-\frac{R}{V})\} + \frac{G_1(\phi)\sin\phi}{R}QG(t-\frac{R}{V}) &(33)\\
&\quad -\frac{F_1(\phi)\cos\phi}{R}\frac{d}{dt}\{PG(t-\frac{R}{V})\} - \frac{G_1(\phi)\cos\phi}{R}QG(t-\frac{R}{V})\\
U_z &= -\frac{F_1(\phi)\cos\phi}{R}\frac{d}{dt}\{PG(t-\frac{R}{V})\} - \frac{G_1(\phi)\cos\phi}{R}QG(t-\frac{R}{V})\\
&\quad +\frac{F_2(\phi)\sin\phi}{R}\frac{d}{dt}\{PG(t-\frac{R}{v})\} + \frac{G_2(\phi)\sin\phi}{R}QG(t-\frac{R}{v})
\end{aligned}$$

where the individual displacement components of Eq. 25 and Eq. 26 are

$$\begin{aligned}
U_{rP} &= \frac{F_1(\phi)\sin\phi}{R}\frac{d}{dt}\{PG(t-\frac{R}{V})\} + \frac{G_1(\phi)\sin\phi}{R}QG(t-\frac{R}{V})\\
U_{zP} &= -\frac{F_1(\phi)\cos\phi}{R}\frac{d}{dt}\{PG(t-\frac{R}{V})\} - \frac{G_1(\phi)\cos\phi}{R}QG(t-\frac{R}{V})\\
U_{rSv} &= \frac{F_2(\phi)\cos\phi}{R}\frac{d}{dt}\{PG(t-\frac{R}{v})\} + \frac{G_2(\phi)\cos\phi}{R}QG(t-\frac{R}{v})\\
U_{zSv} &= \frac{F_2(\phi)\sin\phi}{R}\frac{d}{dt}\{PG(t-\frac{R}{v})\} + \frac{G_2(\phi)\sin\phi}{R}QG(t-\frac{R}{v})
\end{aligned}$$

As can be seen from this system of equations, the P-wave components include contributions from both the dilatational potential ϕ and equivoluminal potential ψ due to the boundary conditions.

The behavior of Heelan's radiation patterns with variations in lithology are discussed later in this paper in comparison with well established point source representations and comparisons of related results for a fluid-filled borehole (Lee and Balch, 1982). Dimensionality analysis indicates that displacements calculated with these formulas, including the stress term and $\frac{1}{R}$ term yield displacements in terms of distance units which is the proper representation.

The geometric shape of these patterns can be viewed in a qualitative manner in Figure 2. It can be seen that F_1 which governs P wave propagation from a radial source is peanut-shaped, F_2 which governs Sv propagation from a radial source is a four-leaved rose, G_1 which governs P wave propagation from an axial source is circular, G_2 which governs Sv propagation from an axial source is also circular but inclined perpendicularly to G_1, and finally K is also circular but similar to G_1. G_1, G_2, K are qualitatively similar to what would be seen with a point source in an infinite medium.

Criticisms of Heelan's Results

Although Jordan (1962), Hazebroek (1966), and Abo-Zena (1977) have criticized Heelan's work, Abo-Zena's criticism is the most comprehensive, he devoted a whole appendix of his paper to criticism of Heelan's paper (1953a). Abo-Zena's criticisms, some of which are justifiable, will be answered here.

Abo-Zena has four principal criticisms. The first criticism is shared with Hazebroek and is primarily an issue of semantics. Abo-Zena (1977) and Hazebroek (1966) point out that Heelan's work is not for a cylinder of finite length since contributions from the ends of the cylinder are not considered. Thus Heelan's results are not for an isolated cavity but instead for a stress applied over a finite length of an infinite cavity which is true.

The second objection Abo-Zena has is due to a typographical error in Heelan's first paper that is not present in Heelan's thesis (1952). On page 687, Heelan describes the functionals $f_o(\sigma,k), g_o(\sigma,k), n_o(\sigma,k)$ with k being a wavenumber. On page 688 the lowercase k is mistakenly converted to a lower case r in the typesetting of the heading - the heading displaying $f_o(\sigma,r), g_o(\sigma,r), n_o(\sigma,r)$ and his thesis displaying the correct $f_o(\sigma,k)$, etc. (Heelan, 1952, pg. 19). Abo-Zena mistakenly interprets this heading as requiring Heelan to evaluate f_o, g_o, n_o as r goes to infinity thus yielding a point source approximation. In fact, the dependence is not on r as a variable but r at the boundary a which is a parameter. The far field expansion is done in the parameter a using an expansion for <u>small</u> arguments. An example of the expansion for small arguments is the approximation $\sin a = a$. Heelan does not evaluate his functionals in terms of r thus due to a misprint this particular criticism is unjustified.

A third criticism of Heelan's paper by Abo-Zena (1977) was related to Heelan's use of a particular type of Fourier transform. Abo-Zena recognized Heelan's operator as a Fourier transform but stated that the boundary conditions could not be equated under the integrand. It is shown in one of the appendices through the use of the Fourier integral theorem that Heelan's use of his Fourier transform was valid and just a short hand notation for the more conventional Fourier transform operator.

A final criticism of Abo-Zena concerns Heelan's contour integral analysis although Abo-Zena does not show what the unknown contour is either. The issue of the unspecified contour is the principal reason for this examination of Heelan's work. Abo-Zena states that the contour C from Heelan's papers cannot be equivalent for each integral and thus boundary conditions cannot be placed under the integral sign. However, Abo-Zena does not prove this assertion but it is justified since Heelan's contour was not specified. We believe that Heelan did in fact use the same contour but without a precise definition of what Heelan's contour was we cannot verify his steepest descent's analysis nor his final results. Heelan did not specify his contour any more concretely in his thesis (1952) than in his papers. For this reason a parallel development using the Weyl integral (Brekhovskikh, 1960, 1980; Stratton 1941) as suggested by Brekhovskikh (1960, 1980) will unambiguously demonstrate the correctness of Heelan's results.

Abo-Zena's Results

Abo-Zena (1977) also addressed the problem of wave propagation from a stress applied to a finite length of an infinite cylindrical cavity but approached the problem in a slightly different manner. To begin with Abo-Zena only considered an applied axisymmetric radial stress and neglected torsional and axial stresses. One important extension of Heelan's work that Abo-Zena made was to allow a longitudinally varying stress $F(t,z)$ to be applied at the boundary $r = a$ whereas Heelan's was required to be a constant along the length of the cylinder. Abo-Zena chose to use the common convention of recoding the potential ψ to ψ' as explained in Chapter 2. This yields reformulated boundary conditions but of course does not change the physics. Abo-Zena uses modified Bessel functions K_0 for ϕ and K_1 for ψ as in Chapter 2. Finally, Abo-Zena's work is different in that the integral transforms he used were first a Laplace transform on the time variable and then a Fourier transform over z.

Abo-Zena's transformed integrals over z are of the following form

$$\int_{-\infty}^{\infty} \sigma A(k,\sigma) K_0(r\gamma_L) e^{i\sigma Z} d\sigma \tag{34}$$

where $\gamma_L = \sqrt{\sigma^2 - k^2}$ slightly modified to conform with Heelan's notation introduced earlier. We can readily see that Eq. 34 is an integral over axial wavenumber and the modified Bessel function is evaluated as a function of radial wavenumber. Remember, that Heelan's integrals have Hankel functions, an analogous Bessel function, evaluated as a function of axial wavenumber instead of radial wavenumber. Thus Abo-Zena's result is also a type of Sommerfeld integral.

Abo-Zena then solves the same boundary conditions as Heelan, vanishing of tangential stress and vanishing of normal stress at the free surface by bringing them under the transforms. Abo-Zena also uses the method of steepest descent to yield his far field radial and tangential displacements. These equations are (Abo-Zena, 1977, Eq. 60, 61)

$$\begin{aligned} u_R &= -\frac{\Delta}{4\pi RlV}(1-\frac{2v^2}{V^2}\cos^2\phi)\int_{-l}^{l}\frac{\partial F(t_0,z)}{\partial t_0}|_{t-\frac{R}{V}}dt_0 \\ u_\phi &= -\frac{\Delta}{4\pi Rlv}\sin 2\phi\int_{-l}^{l}\frac{\partial F(t_0,z)}{\partial t_0}|_{t-\frac{R}{V}}dt_0 \end{aligned} \tag{35}$$

where these equations have been rewritten in Heelan's notation. An important difference in these equations is that they are magnitudes of spherical coordinates instead of cylindrical coordinates. If we assume that a factor l will emerge from the last integral in these two equations we can rewrite Abo-Zena's formulas in terms of the F_1, F_2 symbolism of Heelan as follows

$$\begin{aligned} F_1 &= -\frac{\Delta}{4\pi V}(1-2\frac{v^2}{V^2}\cos^2\phi) \\ F_2 &= -\frac{\Delta}{4\pi Rv}\sin 2\phi \end{aligned} \tag{36}$$

In comparing Eq. 36 with Eq. 28, Eq. 30 we can see that they are very similar. In fact, White (1983) labelled them equivalent, but as we can see a $\frac{1}{\mu}$ factor present in Heelan's results is not present in Abo-Zena's. This dependence on Lamé parameter μ does not affect the geometry of the radiation patterns but will affect their magnitude substantially. Dimensionality analysis shows that Abo-Zena's formulae will not yield function that have units of distance as displacements should. Additionally, in checking the algebra in Abo-Zena's paper we found that in the calculations from Eq. 30 to Eq. 54 (Abo-Zena, 1977) a factor of $\frac{1}{\mu}$ was inadvertently left off. We were unable to derive Eq. 55 of Abo-Zena. Dimensionality analysis also shows that Heelans' results were potentially correct but completion of the work initiated by Brekhovskikh is needed to clear up any final ambiguity.

Brekhovskikh's Treatment

Having shown that Heelan's (1952, 1953a) and Abo-Zena's (1977) results for radiation patterns differ its important to check Heelan's results carefully. Unfortunately, the contour Heelan used in his integrals can only be guessed at. Contours for Sommerfeld integral type problems can be extremely complex (Aki and Richards, 1980, Chapter 6) so random guessing is ill advised. Therefore, through work building on Heelan's results initiated by Brekhovskikh (1960) who expands the fields in plane waves instead of cylindrical waves, we will show in fact that Heelan's results are correct.

Because the two editions of Brekhovskikh's book (1960, 1980) have slightly different section numbers it's wise to note that the following analysis will exclusively reference his second edition (1980). Brekhovskikh's integral, fundamentally the Weyl integral, is well known from both electromagnetic and acoustic theory and similar developments are present in Stratton (1941). The steepest descent paths and contours are known and published so can be easily followed.

Some preliminary background material must be presented first related to the plane wave decomposition of a scalar field. In this background material it will be seen that the Hertzian oscillator in Brekhovskikh's treatment is introduced from conception.

Consider a spherical wave radiated at the origin and its decomposition into plane waves. Brekhovskikh (1980, Eq. 26.15) introduces the following form for the Hertzian oscillator

$$\frac{e^{ikR}}{R} = \frac{i}{2\pi}\int\int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y)}}{\sqrt{k^2 - k_x^2 - k_y^2}} dk_x dk_y \tag{37}$$

Eq. 37 is a decomposition in plane waves in the x-y plane where $z=0$ exclusively. It is necessary to downward and upward continue Eq. 37 to include all of z. To perform this continuation we integrate over z by introducing a third component to the exponential $ik_z z$ where $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ for $z > 0$ and $-k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ for $z < 0$. Thus we have the integrals (Brekhovskikh, 1980, Eq. 26.17; Aki and Richards, 1980, Eq. 6.4)

$$z \geq 0 \quad \frac{e^{ikR}}{R} = \frac{i}{2\pi}\int\int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y + k_z z)}}{k_z} dk_x dk_y dk_z \tag{38}$$

$$z \leq 0 \quad \frac{e^{ikR}}{R} = \frac{i}{2\pi}\int\int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y - k_z z)}}{k_z} dk_x dk_y dk_z$$

Eq. 38 represents the expansion of a spherical wave into plane waves with the wavenumber vector of the plane wave having components k_x, k_y, k_z. A difference in phase i between Aki and Richards and Brekhovskikh's treatment is attributable to Brekhovskikh's k_z equalling Aki and Richards ik_z.

From this point the treatment in Aki and Richards leads to expansion in cylindrical waves, the Sommerfeld integral, whereas Brekhovskikh and also Stratton (1941, pg. 573-578) maintain plane waves but express the normal coordinate vector k in terms of polar angles ϕ, θ. ϕ is not to be confused with the potential ϕ. Stratton terms this expansion the Weyl solution or Weyl integral (Stratton, 1941, Sec. 9.29). In order to reconstruct our spherical source or for that matter any source a finite distance away using a superposition of plane waves procedure we need to consider both inhomogeneous and homogeneous plane waves (Stratton, 1941). Homogeneous waves are waves for which planes of constant phase have constant amplitude whereas for inhomogeneous plane waves planes of constant phase have variable amplitudes. By allowing imaginary values for components of our wavenumber vector we will describe inhomogeneous waves.

We will now perform a transform of the coordinate systems under an integral. To do so we have to use the Jacobian of the transformation. For a general integral in coordinates (k_x, k_y, k_z) to be transformed to variables (k, ϕ, θ) we have

$$\iint f(k_x, k_y, k_z) dk_x dk_y dk_z = \iint f(k_x(k,\phi,\theta), k_y(k,\phi,\theta), k_z(k,\phi,\theta)) \left| \frac{\partial(k_x, k_y, k_z)}{\partial(k,\phi,\theta)} \right| dk d\phi d\theta \tag{39}$$

where the term $|\frac{\partial(k_x, k_y, k_z)}{\partial(k,\phi,\theta)}|$ is the absolute value of the Jacobian of the transformation.

The proper transformations $f(k_x(k,\phi,\theta),\ldots)$ etc. for the plane wave components are

$$k_x = k\sin\theta\cos\phi \quad k_y = k\sin\theta\sin\phi \quad k_z = k\cos\theta \tag{40}$$

and the absolute value of the Jacobian for our transformation equals $k^2 \sin\theta$. The geometric relationship between the polar angles and the wavenumber vector components can be seen from Figure 3 (Brekhovskikh, 1980, Figure 26.1).

The absolute value of the Jacobian for our transformation equals $k^2 \sin\theta$. Solutions for both homogeneous as well as inhomogeneous plane waves must be allowed and thus it is required to allow complex values for the angle θ. The range of θ is from zero to $\frac{\pi}{2} - i\infty$. As will be discussed later, the contour is deformable among these limits (Stratton, 1941). This leads to the following form for the integral

$$z \geq 0 \quad \frac{e^{ikR}}{R} = \frac{ik}{2\pi} \int_0^{\frac{\pi}{2}-i\infty} d\theta \int_0^{2\pi} \frac{e^{i(k_x x + k_y y + k_z z)}}{k_z} \sin\theta d\phi \tag{41}$$

$$z \leq 0 \quad \frac{e^{ikR}}{R} = \frac{ik}{2\pi} \int_0^{\frac{\pi}{2}-i\infty} d\theta \int_0^{2\pi} \frac{e^{i(k_x x + k_y y + k_z z)}}{k_z} \sin\theta d\phi$$

A particular contour can be seen in Figure 26.2 (Brekhovskikh, 1980).

In comparison, the substitutions leading to the Sommerfeld integral are

$$k_x = k\cos\phi \quad k_y = k\sin\phi \quad k_z = k_z \tag{42}$$

and the magnitude of the determinant of the Jacobian of the transformation equals k. This leads to precursors of the Sommerfeld integral (Aki and Richards, 1980, Eq. 6.6, rewritten to this notation)

$$z \geq 0 \quad \frac{e^{ikR}}{R} = \frac{1}{2\pi}\int_0^\infty dk \int_0^{2\pi} \frac{k}{k_z} e^{i(kr\cos\phi + k_z z)} d\phi \tag{43}$$

$$z \leq 0 \quad \frac{e^{ikR}}{R} = \frac{1}{2\pi}\int_0^\infty dk \int_0^{2\pi} \frac{k}{k_z} e^{i(kr\cos\phi - k_z z)} d\phi$$

The Sommerfeld integral (Eq. 12) is obtained from Eq. 43 using a suitably tranformed form of the Poisson's integral. The Poisson's integral form used is

$$J_0(kr) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikr\cos\theta} d\theta \tag{44}$$

which can be derived from Watson (1944, pg. 25, Eq. 2.3.2).

The integration over ϕ in Eq. 41 spans from 0 to 2π but the integration over θ includes homogeneous waves and ranges in the complex θ plane from 0 to $\frac{\pi}{2} - i\infty$. Now would be a good point to state that there is a misprint present in the second edition of Brekhovskikh's book (1980) but not in the first (Brekhovskikh, 1960) and it is not carried through in the analysis. This misprint is that the term $dk_x dk_y dk_z$ should read $\frac{dk_x dk_y}{k_z}$ just prior to Eq. 26.19 (Brekhovskikh, 1980).

We have the integral $\frac{e^{ikR}}{R}$ approximated as a superposition of plane waves, the Weyl integral Eq. 41. Our need is to approximate any scalar field in terms of these plane waves. To accomplish this we introduce a function $V(\theta)$ to multiply times our integrand to achieve any arbitrary scalar field. Brekhovskikh treats the function $V(\theta)$ as a reflection coefficient function but it can be thought of in more general terms. Upon so doing we have an arbitrary scalar field ϑ

$$\vartheta = \frac{ik}{2\pi}\int_0^{\frac{\pi}{2}-i\infty} d\theta \int_0^{2\pi} V(\theta) e^{i(k_x x + k_y y + k_z z)} \sin\theta d\phi \tag{45}$$

Expanded in terms of the full expansions for k_x, k_y, k_z (Eq. 40) we obtain (Brekhovskikh, 1980, Eq. 26.24)

$$\vartheta = \frac{ik}{2\pi}\int_0^{\frac{\pi}{2}-i\infty} d\theta \int_0^{2\pi} V(\theta) e^{ik((x\cos\phi + y\sin\phi)\sin\theta + z\cos\theta)} \sin\theta d\phi \tag{46}$$

and now performing a transformation to polar coordinates under the integral in terms of ϕ, $x = r\cos\phi_1, y = r\sin\phi_1$ we yield

$$\vartheta = \frac{ik}{2\pi}\int_0^{\frac{\pi}{2}-i\infty} d\theta \int_0^{2\pi} V(\theta) e^{ikr\sin\theta\cos(\phi-\phi_1) + ikz\cos\theta} \sin\theta d\phi \tag{47}$$

where we have used the transformation

$$r \cos\phi \cos\phi_1 + r \sin\phi \sin\phi_1 = r \cos(\phi - \phi_1) \tag{48}$$

and use the Poisson integral transformation (Eq. 44) to obtain

$$\vartheta = ik \int_0^{\frac{\pi}{2} - i\infty} V(\theta) J_0(kr \sin\theta) \sin\theta e^{ikz\cos\theta} d\theta \tag{49}$$

If $V(\theta)$ equals 1 Eq. 49 is another expression for a Hertzian oscillator.

We now use a Bessel function identity (Abramowitz and Stegun, 1964, Eqs. 9.1.3-4)

$$J_0(z) = \frac{1}{2}[H_0^{(1)}(z) + H_0^{(2)}(z)] \tag{50}$$

Upon substituting this value for J_0, splitting the integral into two integrals, using the formula $H_0^{(1)}(e^{-\pi i} z) = -H_0^{(1)}(z)$ (Abramowitz and Stegun, 1964, Eq. 9.1.6), and then reversing the sign of integration we obtain

$$\vartheta = ik \int_{-\frac{\pi}{2} + i\infty}^{\frac{\pi}{2} - i\infty} V(\theta) H_0^{(1)}(kr \sin\theta) \sin\theta e^{ikz\cos\theta} d\theta \tag{51}$$

Brekhovskikh calls the contour from $\frac{-\pi}{2} + i\infty$ to $\frac{\pi}{2} - i\infty$, "Γ_1", a notation we shall use subsequently. The contour is shown diagramatically in Figure 4.

The contour Γ_1 is a member of a family of contours originally described by Sommerfeld. An excellent discussion of the properties of this contour family is provided by Stratton (Pg. 367-368, 1941). The basic properties of the family are a form of radiation condition in that the contributions from the end points must vanish. The dominant behavior is under the exponential of the form $e^{i\varrho\cos\theta}$ where ϱ is some constant or a function that is constant under the particular variable of integration. We define θ in terms of real and imaginary parts, $\theta = \theta' + i\theta''$ so we can write

$$i\varrho\cos\theta = \varrho \sin\theta' \sinh\theta'' + i\varrho \cos\theta' \cosh\theta'' \tag{52}$$

Terms of the form $e^{-\varrho\infty}$ will vanish which require (θ', θ'') values of $(\frac{-\pi}{2}, \infty)$ and $(\frac{\pi}{2}, -\infty)$ respectively since $\sin(\frac{-\pi}{2})\sinh(\infty)$ and $\sin(\frac{\pi}{2})\sinh(-\infty)$ equal $\frac{-\infty}{2}$ $(-\infty)$. Thus our vanishing endpoint contributions coupled with our desire to span all values of imaginary wavenumber dictate that our path begin at $(\frac{-\pi}{2}, \infty)$ and terminate at $(\frac{\pi}{2}, \infty)$. In fact we can choose any pair of an infinite choice of endpoint pairs beginning and terminating at opposite imaginary infinities separated by π but our choice of the principal pair is usually sufficient. Within these endpoint limits, and of course strongly dependent on the singularities of the integrand, the contour is deformable throughout although a crossing of the real axis at 45 degrees at the origin often advantageously provides a saddle point for consideration. In the method of steepest descent used with the stationary phase approximation which we will apply later, we in fact shift the endpoints and the location of the saddlepoint to the right by an amount θ_0. For the particular member of the family provided by Γ_1 we can break down the integral into segments consisting of an integral over inhomogeneous waves, a segment integrating over the real θ axis consisting of homogeneous waves and another segment integrating over inhomogeneous waves. We again refer you to Figure 4.

Eq. 51 represents Eq. 26.27 from Brekhovskikh (1980). The background for Brekhovskikh's treatment is now complete and we reintroduce Heelan's integrals. Brekhovskikh uses Eq. 51 for the scalar fields ϕ, ψ, χ and writes the Heelan integrals as follows with f_o, g_o, n_o.

$$\begin{aligned} \phi &= Re \int_0^\infty e^{-ikVt} \int_{\Gamma_1} f_o(\theta, k) H_0^{(1)}(kr \sin\theta) e^{-ikz\cos\theta} \sin\theta d\theta dk \\ \psi &= Re \int_0^\infty e^{-ikVt} \int_{\Gamma_1} g_o(\theta, k) H_0^{(1)}(kr \sin\theta) e^{-i\kappa z\cos\gamma} \sin\theta d\theta dk \\ \chi &= Re \int_0^\infty e^{-ikVt} \int_{\Gamma_1} n_o(\theta, k) H_0^{(1)}(kr \sin\theta) e^{-i\kappa z\cos\gamma} \sin\theta d\theta dk \end{aligned} \tag{53}$$

Brekhovskikh uses a negative time dependence e^{-ikVt} and uses the notation b for v (shear wave velocity), c for V (compressional wave velocity), κ for h shear radial wavenumber. We will maintain Brekhovskikh's use of the negative time dependence but use Heelan's notations for velocities. Brekhovskikh as Heelan did specifies that $\kappa b = kc$ or in Heelan's notation $kV = hv$ and also that $k \sin\theta = \kappa \sin\gamma$ a condition arising from his separation of variables procedure. Brekhovskikh also uses the symbol h_o for the coefficient function instead of Heelan's n_o as used above.

It is not obvious that Brekhovskikh's treatment in terms of the Weyl integral will yield the same results as Heelan's. Moreover, Heelan's treatment can't be verifed because of the lack of knowledge about Heelan's contour. And finally, Brekhovskikh only considers the case for a radial stress and a torsional stress since he used Heelan's work as a basis for the calculation of head waves, the same treatment adopted in Heelan's second paper (1953b). Therefore the laborious procedure of duplicating Heelan's results using Brekhovskikh's integrals will be demonstrated here.

The boundary conditions are set up with the same mathematical model as before (Eq. 15)

$$PG(t)F(z) + p_r = 0 \quad SG(t)F(z) + p_{r\theta} = 0 \tag{54}$$

but Brekhovskikh did not consider Q, axial stress.

$$QG(t)F(z) + p_{rz} = 0$$

The Fourier transform of the boxcar function is the sinc function

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(lt)}{t} e^{izt} dt \tag{55}$$

We believe that Brekhovskikh makes the following change of variable, $t = -k\cos\theta$, in transforming the boxcar function to

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(lk\cos\theta)}{\cos\theta} e^{-ikz\cos\theta}\sin\theta d\theta \tag{56}$$

Brekhovskikh (1960, 1980, first few pages) explains his usage of the same Fourier transform operator that Heelan had used. As mentioned before, an appendix to this paper thoroughly explains the origins of this operator. In using this operator, Brekhovskikh leaves to an exercise that the following relationships are equivalent for $F(\nu)$ being real.

$$\begin{aligned} F(\nu) &= Re\int_{0}^{\infty} \Phi(\omega) e^{i\omega\nu} d\omega \\ \Phi(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} F(\nu) e^{-i\omega\nu} d\nu \\ F(\nu) &= \frac{1}{2}\int_{-\infty}^{\infty} \Phi(\omega) e^{i\omega\nu} d\omega \end{aligned} \tag{57}$$

Having completed our discussion of the Fourier transform operator we can bring our boundary conditions and sources under the integral as did Heelan. Our boundary conditions are vanishing of normal and tangential stress with our stresses displayed in Eq. 14.

In doing so we obtain a 2×2 system of equations for f_o, g_o (the P-Sv case). For convenience the argument of the Hankel function is not specified in the equation below but is $ka\sin\theta$ throughout. Additionally, a $\sin\theta$ term that cancels has been left off both sides of these equations

$$\begin{aligned} &-QG(k)\frac{\sin(lk\cos\theta)e^{-ikz\cos\theta}}{\pi\cos\theta} = \\ &\quad f_o(2i\mu k^2\sin\theta\cos\theta)H_1^{(1)}()e^{-ikz\cos\theta} \\ &\quad -g_o\left(2\mu k\sin\theta\kappa^2\cos^2\gamma - \mu k^3\sin\theta\frac{V^2}{v^2}\right)H_1^{(1)}()e^{-i\kappa z\cos\gamma} \\ &-PG(k)\frac{\sin(lk\cos\theta)e^{-ikz\cos\theta}}{\pi\cos\theta} = \\ &\quad f_o\left(-\lambda k^2 H_0^{(1)}() - 2\mu k^2\sin^2\theta\left[H_0^{(1)}() - \frac{H_1^{(1)}()}{ka\sin\theta}\right]\right)e^{-ikz\cos\theta} \end{aligned} \tag{58}$$

$$-g_o(2i\mu k^2 \sin^2\theta\kappa\cos\gamma)\left[H_0^{(1)}() - \frac{H_1^{(1)}()}{ka\sin\theta}\right] e^{-i\kappa z\cos\gamma}$$

and a simple algebraic equation for n_o

$$\mu n_o k^2 \sin^2\theta H_2^{(1)}(ka\sin\theta)e^{-i\kappa z\cos\gamma} = -SG(k)\frac{\sin(lk\cos\theta)e^{-ikz\cos\theta}}{\pi\cos\theta} \tag{59}$$

with a $\sin\theta$ term being left off of both sides in Eq. 58 and Eq. 59. We next solve for this system and expand for small arguments using Eq. 17, the details of which are left to Appendix A.

$$\begin{aligned} f_o &= iPG(k)\left(2(\frac{v^2}{V^2})\cos^2\theta - 1\right)(\frac{k\Delta}{8\pi\mu}) - QG(k)\frac{Av^2\cos\theta}{8\pi\mu V^2} \\ g_o &= PG(k)\cos\theta(\frac{v^2\Delta}{4\pi\mu V^2}) + QG(k)\frac{iAv^2\cos\theta}{8\pi\mu V^2\kappa\cos\gamma} \\ n_o &= SG(k)\frac{\Delta kv}{8\pi i\mu V}\frac{\cos\theta}{\cos\gamma} \end{aligned} \tag{60}$$

where Δ equals the volume of the cylindrical cavity. These equations may be profitably compared to Eq. 18 of Heelan's paper.

In Brekhovskikh's book, solutions were given for radial and torsional stresses only which were Eq. 60 (Brekhovskikh, 1980, Eq. 33.6)

$$\begin{aligned} f_o &= iPG(k)\left(2(\frac{v^2}{V^2})\cos^2\theta - 1\right)(\frac{k\Delta}{8\pi\mu}) \\ g_o &= PG(k)\cos\theta(\frac{v^2\Delta}{4\pi\mu V^2}) \\ n_o &= SG(k)\frac{\Delta kv\cos\theta}{4\mu i\pi V\cos\gamma} \end{aligned} \tag{61}$$

The equivalence becomes readily apparent if Q is set equal to zero in Eq. 60. There is however a discrepancy is in the denominator for n_o where we found a factor of 8 and Brekhovskikh a factor of 4. The factor of 8 we found is consistent with Heelan's results. We believe this difference is due to a typographical error in Brekhovskikh's book in that the error is corrected through the process of steepest descent analysis.

Method of steepest descent analysis

Once f_o, g_o, n_o are known for small arguments, displacements are calculated using Eq. 19. There will be two components of the radial displacement, one travelling at the P wave velocity and one at the S and similarly for the vertical component. We will designate these components as $U_{rP}, U_{rSv}, U_{zP}, U_{zSv}$. We consider the integrals below having already carried out differentiation with the $\frac{\partial}{\partial z}$ operator.

$$
\begin{aligned}
U_{rP} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}(\frac{iP\Delta}{8\pi\mu}kG(k)\left(2(\frac{v^2}{V^2})\cos^2\theta-1\right)-QG(k)\frac{Av^2\cos\theta}{8\pi\mu V^2}) \\
&\quad \frac{\partial}{\partial r}H_0^{(1)}(kr\sin\theta)e^{-ikz\cos\theta}\sin\theta d\theta dk \qquad (62)\\
U_{rSv} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}(\frac{iP\Delta v^2}{4\pi\mu V^2}\kappa\cos\gamma G(k)\cos\theta-\frac{QAv^2}{8\pi\mu V^2}G(k)\cos\theta) \\
&\quad \frac{\partial}{\partial r}H_0^{(1)}(kr\sin\theta)e^{-i\kappa z\cos\gamma}\sin\theta d\theta dk \\
U_{zP} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}(\frac{P\Delta}{8\pi\mu}k^2G(k)\cos\theta\left(2(\frac{v^2}{V^2})\cos^2\theta-1\right)+\frac{iQAv^2k\cos^2\theta}{8\pi\mu V^2}) \\
&\quad H_0^{(1)}(kr\sin\theta)e^{-ikz\cos\theta}\sin\theta d\theta dk \\
U_{zSv} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}(\frac{Pv^2\Delta}{4\pi\mu V^2}G(k)\cos\theta+\frac{iAQv^2\cos\theta}{8\pi\mu V^2\kappa\cos\gamma}G(k)) \\
&\quad \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}H_0^{(1)}(kr\sin\theta))e^{-i\kappa z\cos\gamma}\sin\theta d\theta dk \\
U_{\theta Sh} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}SG(k)\frac{\Delta v}{8\pi i\mu V}\frac{k\cos\theta}{\cos\gamma}\frac{\partial}{\partial r}H_0^{(1)}(kr\sin\theta)e^{-i\kappa z\cos\gamma}\sin\theta d\theta dk
\end{aligned}
$$

Remembering the U_{rSv} term is $-\frac{\partial^2\psi}{\partial r\partial z}$, the minus sign being particularly important.

We now perform a series of manipulations to get the integrals into the proper form for steepest descent analysis. We first apply the principal asymptotic expansion of Hankel functions for large arguments Eq. 20. (Abramowitz and Stegun, 1964). For instance

$$H_0^{(1)}(kr\sin\theta)\sim\sqrt{\frac{2}{\pi kr\sin\theta}}e^{i\frac{-3\pi}{4}}e^{ikr\sin\theta} \qquad (63)$$

Next we use the relations

$$-z=R\cos\theta_0 \qquad r=R\sin\theta_0 \qquad (64)$$

which can be seen from Figure 6 and a double angle formula to yield

$$
\begin{aligned}
U_{rP} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}-(\frac{iP\Delta}{4\pi\mu}kG(k)\left(2(\frac{v^2}{V^2})\cos^2\theta-1\right)-QG(k)\frac{Av^2\cos\theta}{4\pi\mu V^2}) \\
&\quad e^{\frac{-i3\pi}{4}}\sqrt{\frac{k}{2\pi r}}e^{ikR\cos(\theta-\theta_0)}\sin\theta\sqrt{\sin\theta}d\theta dk \qquad (65)\\
U_{rSv} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}-(\frac{iP\Delta v^2}{2\pi\mu V^2}G(k)\cos\theta\kappa\cos\gamma-\frac{QAv^2}{4\pi\mu V^2}G(k)\cos\theta)
\end{aligned}
$$

$$
\begin{aligned}
& e^{\frac{-i3\pi}{4}}\sqrt{\frac{k}{2\pi r}}e^{i\kappa R\cos(\gamma-\theta_0)}\sin\theta\sqrt{\sin\theta}d\theta dk \\
U_{zP} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}(\frac{P\Delta}{4\pi\mu}kG(k)\cos\theta\left(2(\frac{v^2}{V^2})\cos^2\theta-1\right)+\frac{iQAv^2\cos^2\theta}{4\pi\mu V^2}G(k)) \\
& e^{\frac{-i\pi}{4}}\sqrt{\frac{k}{2\pi r}}e^{ikR\cos(\theta-\theta_0)}\sqrt{\sin\theta}d\theta dk \\
U_{zSv} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}-(+\frac{Pv^2\Delta}{2\pi\mu V^2}G(k)\cos\theta+\frac{QiAv^2\cos\theta}{4\pi\mu V^2\kappa\cos\gamma}G(k)) \\
& e^{\frac{-i\pi}{4}}\sin^2\theta k\sqrt{\frac{k}{2\pi r}}e^{i\kappa R\cos(\gamma-\theta_0)}\sqrt{\sin\theta}d\theta dk \\
U_{\theta Sh} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}-SG(k)\frac{\Delta v}{4i\mu\pi V}\frac{\cos\theta}{\cos\gamma}e^{\frac{-i3\pi}{4}}\sqrt{\frac{k}{2\pi r}}e^{i\kappa R\cos(\gamma-\theta_0)}\sin\theta\sqrt{\sin\theta}d\theta dk
\end{aligned}
$$

Examining our equations we see that we have functions of θ in the integrand and under the exponent. We also have R which grows exponentially large as r goes to infinity. Since we wish to evaluate the radiation asymptotically for large values of r these integrals are perfect candidates for the method of steepest descent analysis.

We now apply the method of steepest descent analysis and the stationary phase approximation to evaluate the radiation in the far field. Good discussions about steepest descent analysis can be found in Aki and Richards and Brekhovskikh. We will use the notation of Brekhovskikh (1980). For the method of steepest descent we define two functions, one under the exponent $f(\theta)$ and one under the integrand $F(\theta)$ and also a parameter under the exponent ρ which dominates as our variable of integration goes to infinity. For the first integral (Eq. 65) we define

$$
\begin{aligned}
F(\theta) &= (\frac{iP\Delta}{8\pi\mu}kG(k)\left(2(\frac{v^2}{V^2})\cos^2\theta-1\right)-QG(k)\frac{Av^2\cos\theta}{8\pi\mu V^2})e^{i\frac{-3\pi}{4}}\sin\theta\sqrt{\sin\theta}\sqrt{\frac{k}{2\pi r}} \\
f(\theta) &= i\cos(\theta-\theta_0) \qquad (66) \\
\rho &= kR
\end{aligned}
$$

The strategy is to first find a saddle point. A saddle point is determined by solving the equation $\frac{\partial f}{\partial\theta}=0$ which for our purposes is on the real θ axis at $\theta=\theta_0$. From a saddle point the integral will rapidly decrease in two directions and rapidly increase in two directions. We transform $f(\theta)$ into a new function s which we will only integrate along the real axis from $-\infty$ to ∞. A property of the decomposition is that lines of steepest descent of the real part of s have constant values for the imaginary part of s and thus can be ignored. Since the increase and decrease will be symmetric we approximate it with hyperbolas of the form

$$f(\theta)=f(\theta_0)-s^2 \qquad (67)$$

To apply this technique to our problem we first set (Brekhovskikh, Eq. 28.4, 1980)

$$f(\theta) = i\cos(\theta - \theta_0) = f(\theta_0) - s^2 \tag{68}$$

or equivalently

$$\cos(\theta - \theta_0) = f(\theta_0) + is^2 \tag{69}$$

We take the real and imaginary parts of both sides of this equation first defining $\theta = \theta' + i\theta''$

$$\cos(\theta' + i\theta'' - \theta_0) = 1 + is^2 = \cos(\theta' - \theta_0)\cosh\theta'' - i\sin\theta'\sinh\theta'' \tag{70}$$

which yields for the saddle path

$$\begin{aligned} \cos(\theta' - \theta_0)\cosh\theta'' &= 1 \\ s^2 &= \sin(\theta' - \theta_0)\sinh\theta'' \end{aligned} \tag{71}$$

The endpoints of the saddle path are at $\theta = \frac{-\pi}{2} + \theta_0 + i\infty$ ($s = -\infty$) and $\theta = \frac{\pi}{2} - \theta_0 + i\infty$ ($s = \infty$). Furthermore the saddle path crosses the origin at θ_0 at an incidence of 45 degrees. We can see that this path is just a shift of θ_0 of a member of the contour family described by Stratton (1941). This path is presented graphically in Figure 5 and will be entitled Γ. It can be seen that in the deformation of Γ_1 to Γ we just stretched Γ_1 at both ends. We have a branch point at 0 due to terms of the form $\sqrt{\sin\theta}$. The branch cut emanating from zero can be cut along any direction. If we take the negative imaginary axis as our branch cut for instance we will not cross this branch cut in the deformation of Γ_1 into Γ. This same procedure is used for reflection problems and more general waveguide problems but a much more rigorous treatment of the singularities in the complex plane is required. We are fortunate here in only having one singularity which can be bypassed. So we can now rewrite our integral as

$$e^{\rho f(\theta_0)} \int_{-\infty}^{\infty} e^{-\rho s^2} \Phi(s) ds \tag{72}$$

Because $f(\theta)$ includes the imaginary i, the potential for rapid oscillation about the $\theta'' = 0$ axis far away from our saddle point exists. By and large this rapid oscillation will cancel itself in area and thus in the integration so we can use the stationary phase approximation. The stationary phase approximation can be applied to our integral which is for our particular case

$$\sqrt{\frac{2\pi}{kR|\cos''(0)|}} e^{\frac{-i\pi}{4}} e^{ikR\cos(0)} F(\theta_0) \tag{73}$$

after simplifying

$$\sqrt{\frac{2\pi}{kR}} e^{\frac{-i\pi}{4}} e^{ikR} F(\theta_0) \tag{74}$$

Formulating our first integral (Eq. 65) in this form we yield

$$U_{rP} = Re\int_0^\infty e^{-ikVt}(\frac{iP\Delta}{4\pi\mu}kG(k)\left(2(\frac{v^2}{V^2})\cos^2\theta_0 - 1\right) - QG(k)\frac{Av^2\cos\theta_0}{4\pi\mu V^2})$$
$$\sin\theta_0\sqrt{\frac{1}{r}}\frac{e^{ikR}}{\sqrt{R}}\sqrt{\sin\theta_0}dk \quad (75)$$

which we simplify by reintroducing our identity $r = R\sin\theta_0$ or for our purposes $\sqrt{r} = \sqrt{R\sin\theta_0}$, finally yielding

$$U_{rP} = Re\int_0^\infty (\frac{iP\Delta}{4\pi\mu}kG(k)\left(2(\frac{v^2}{V^2})\cos^2\theta_0 - 1\right) - QG(k)\frac{Av^2\cos\theta_0}{4\pi\mu V^2}) \quad (76)$$
$$\sin\theta e^{-ikVt}\frac{e^{ikR}}{R}dk$$

For carrying out the integration with respect to k we can place a time derivative operator under the integral in the following manner

$$-ikVRe\int_{-\infty}^{\infty} G(k)e^{-ikV(t-\frac{R}{V})}dk = \frac{d}{dt}Re\int_{-\infty}^{\infty} G(k)e^{-ikV(t-\frac{R}{V})}dk = \frac{d}{dt}G(t-\frac{R}{V}) \quad (77)$$

which yields for our resulting integral

$$U_{rP} = \frac{-P\Delta}{4\pi\mu V}\left(2(\frac{v^2}{V^2})\cos^2\theta_0 - 1\right)\sin\theta_0\frac{1}{R}\frac{d}{dt}G(t-\frac{R}{V}) - Q\frac{Av^2\cos\theta_0}{4\pi\mu V^2}\sin\theta\frac{1}{R}G(t-\frac{R}{V}) \quad (78)$$

Heelans' Eq. 13 for comparison is

$$U_{rP} = \frac{P\Delta}{4\pi\mu V}\left(1 - 2(\frac{v^2}{V^2})\cos^2\theta_0\right)\sin\theta_0\frac{1}{R}\frac{d}{dt}G(t-\frac{R}{V}) - QG(k)\frac{Av^2\cos\theta_0}{4\pi\mu V^2}\sin\theta\frac{1}{R}G(t-\frac{R}{V}) \quad (79)$$

and the equivalence is readily seen.

Although the procedure is cumbersome, the method of steepest descent with the stationary phase approximation substantially simplified our resulting integrands. Except for a factor of two that was divided into our development there is no change in the boundary condition equations in the integrand. This is a remarkably simple result from integral expressions as complicated as those in Eq. 62.

Proceeding with the same integration procedure with our other integrands we obtain results equivalent to Heelan's Eq. 13 (Eq. 33)

$$U_{rP} = -\frac{P\Delta}{4\pi\mu V}\left(2(\frac{v^2}{V^2})\cos^2\theta - 1\right)\sin\theta\frac{1}{R}\frac{d}{dt}G(t - \frac{R}{V}) - \frac{QAv^2}{4\pi\mu V^2}\cos\theta\sin\theta\frac{1}{R}G(t-\frac{R}{V})$$

$$
\begin{aligned}
U_{rSv} &= \frac{P\Delta}{4\pi\mu v R}\sin 2\theta\cos\theta\frac{1}{R}\frac{d}{dt}G(t-\frac{R}{v}) + \frac{QA}{4\pi\mu}\sin\theta\cos\theta\frac{1}{R}G(t-\frac{R}{v}) \qquad (80)\\
U_{zP} &= \frac{P\Delta}{4\pi\mu V}\left(2(\frac{v^2}{V^2})\cos^2\theta - 1\right)\cos\theta\frac{1}{R}\frac{d}{dt}G(t-\frac{R}{V}) + \frac{QAv^2}{4\pi\mu V^2}\cos^2\theta\frac{1}{R}G(t-\frac{R}{V})\\
U_{zSv} &= \frac{P\Delta}{4\pi\mu v}\sin 2\theta\sin\theta\frac{1}{R}\frac{d}{dt}G(t-\frac{R}{v}) + \frac{QA}{4\pi\mu}\sin^2\theta\frac{1}{R}G(t-\frac{R}{v})\\
U_{\theta Sh} &= \frac{S\Delta\sin\theta}{4\pi\mu v}\frac{1}{R}\frac{d}{dt}G(t-\frac{R}{v})
\end{aligned}
$$

One notational note is that we have dropped the subscript on θ_0 and that the symbol θ has been used for the polar angle instead of ϕ.

In performing the steepest descent analysis with the terms $e^{i\kappa(\gamma-\theta_0)}$ of Eq. 65 we used a change in variables of the integrals to γ and the relations

$$
k\sin\theta = \kappa\sin\gamma \quad k\cos\theta d\theta = \kappa\cos\gamma d\gamma \quad H_0^{(1)}(kr\sin\theta) = H_0^{(1)}(\kappa r\sin\gamma) \qquad (81)
$$

For reference integrals in γ were evaluated in the following form

$$
\begin{aligned}
U_{rSv} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}(-\frac{Pi\Delta}{2\pi\mu}G(k)\kappa\cos^2\gamma + \frac{QA}{4\pi\mu}G(k)\kappa\cos^2\gamma)\\
&\quad e^{\frac{-i3\pi}{4}}\sqrt{\frac{\kappa}{2\pi r}}e^{i\kappa R\cos(\gamma-\gamma_0)}\sin\gamma\sqrt{\sin\gamma}d\gamma dk\\
U_{zSv} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1}(-\frac{P\Delta}{2\pi\mu}G(k)\cos\gamma - \frac{iQA}{4\pi\mu\kappa}G(K))\\
&\quad ke^{\frac{-i\pi}{4}}\sin^2\theta\sqrt{\frac{\kappa}{2\pi r}}e^{i\kappa R\cos(\gamma-\gamma_0)}\sqrt{\sin\gamma}d\gamma dk\\
U_{\theta Sh} &= Re\int_0^\infty e^{-ikVt}\int_{\Gamma_1} -SG(k)\frac{\Delta}{4\pi i\mu}e^{\frac{-i3\pi}{4}}\sqrt{\frac{k}{2\pi r}}e^{i\kappa R\cos(\gamma-\gamma_0)}\sin\gamma\sqrt{\sin\gamma}d\theta dk
\end{aligned}
$$

Thus indirectly we have verified Heelan's results with the much more rigorous contour integration strategy initiated by Brekhovskikh but elucidated and expanded upon here. We will now shift emphasis and begin examining the consequences of these radiation pattern formulas. Specifically, a comparative study of radiation from an empty borehole, (the Heelan/Brekhovskikh results), a fluid-filled borehole and a point source in infinite media will be undertaken.

Comparison of Radiation Pattern Formulas

We now compare Heelan's radiation pattern formula with well known formulas for point sources in infinite media so that we can isolate the effect of stresses applied to the borehole. Additionally we will compare the formula of Heelan to those of Lee and Balch (1982) for a volume source in a fluid-filled borehole and a radial stress source applied to the wall of a fluid-filled borehole. This last comparison should allow us to differentiate the radiation behavior in an empty borehole versus a borehole containing a fluid, an important comparison both mathematically and physically.

To provide a range of parameters from which to test, we will evaluate the radiation pattern formula using density and velocity parameters from three different lithology types. Since sandstone, limestone and shale are the three most common sedimentary lithologies we chose a representative from each group. The velocities we have chosen to use are laboratory-determined velocites of the Solenhofen limestone from Bavaria (Press, 1986 after Hughes and Cross, 1951) and the Berea sandstone (Thomsen, 1986 after King, 1964) and in-situ determinations of properties of the Pierre shale (Thomsen, 1986 after White, 1982). The velocities and densities for these lithologies are presented in Table 1. All velocities are assumed to be isotropic velocities although it is known that Pierre shale especially is anisotropic. Additionally you will notice that the Berea sandstone has a very small Poisson's ratio of .16 compared to the Solenhofen's limestone .308, a fact which will help demonstrate effects due to Poisson's ratio.

This examination of radiation patterns for velocity and density parameters other than a Poisson solid is somewhat of a unique aspect to this work. When viewed in terms of lithology information the radiation pattern behavior can be quite surprising.

Radiation from a point source in an infinite medium

It is worthwhile to compare Heelan's results to established results for radiation from point sources that are not in boreholes. This will help us to separate the properties of the borehole from the properties of the infinite media.

The first treatment to be addressed will be the solution for radiation from a point force in the far field in an infinite elastic medium. This is a well established result but for reference we will use the formulas from White (Eq. 6-5, 1983) who presents

$$\begin{aligned} U_R &= \frac{\cos\theta}{4\pi\rho V^2 r} g(t-\frac{r}{V}) \\ U_\phi &= \frac{\sin\theta}{4\pi\rho v^2 r} g(t-\frac{r}{v}) \end{aligned} \tag{82}$$

This far field behavior is displayed in Figure 7 for a horizontally directed point force. The components U_R and U_ϕ are measured in spherical coordinates so that U_R measures the amplitude of the longitudinally polarized or P wave and U_ϕ represents the amplitude of the transverse or Sv wave. As can be seen from the equations, the independent factor governing these equations was the inverse velocity squared term. Excluding attenuation, these formulas show that since the shear wave velocity is always less than the compressional wave velocity the shear wave radiation will be greater. It can also be seen readily that U_r the P wave radiation is symmetric with respect to the axis longitudinally directed and U_ϕ to the axis orthogonal to the longitudinal axis.

Heelan's results — radiation from stresses on the wall of an empty borehole

Let Δ equal the volume of the source $2\pi a^2 l$. Let A equal $4\pi al$ area of vertical walls. An extra factor of two in volume and area is due to the length of the cavity being $2l$. V is compressional wave velocity, v is shear wave velocity. Heelan's radiation pattern equations are presented first in Heelan's notation and then simplified. For a radial source we have F_1

$$F_1(\phi) = \frac{\Delta}{4\pi\mu V}(1 - \frac{2v^2\cos^2\phi}{V^2}) \tag{83}$$

$$F_1(\phi) = \frac{a^2 l}{2\rho v^2 V}(1 - \frac{2v^2\cos^2\phi}{V^2})$$

Figure 8 shows the behavior of F_1. It is characterized by a peanut shape - the dimpling of the peanut shape along the axis of the borehole. This dimpling is due to reduced amplitude along this axis. The Pierre shale radiation pattern is very large because of its low shear wave velocity contributing to the inverse velocity squared factor in Eq. 84. Also for a radial source we have F_2

$$F_2(\phi) = \frac{\Delta}{4\pi\mu v}\sin 2\phi \tag{84}$$

$$F_2(\phi) = \frac{a^2 l}{2\rho v^3}\sin 2\phi$$

which is shown in Figure 9 and represents the classic four-leaved rose radiation pattern. Pierre shale, because of the inverse shear wave velocity cubed dependence, does not plot on the page.

For an axial source applied to the walls of an empty borehole we have G_1

$$G_1(\phi) = \frac{-Av^2}{4\pi\mu V^2}\cos\phi \tag{85}$$

$$G_1(\phi) = \frac{-al}{\rho V^2}\cos\phi$$

and G_2

$$G_2(\phi) = \frac{A}{4\pi\mu}\sin\phi \tag{86}$$

$$G_2(\phi) = \frac{al}{\rho v^2}\sin\phi$$

G_1 and G_2 are shown in Figure 10 and Figure 11 and are geomtrically what one would expect for a vertically directed point force. To visualize a vertically directed point force we turn White's patterns Figure 7 for a horizontally directed force 90 degrees on its axis.In fact, geometrically, the $\frac{1}{V^2}, \frac{1}{v^2}$ dependence displayed in Eq. 86 and Eq. 87, is exactly equivalent to the form for a point source. The only difference is a factor due to the volume of the axial source. Thus a major conclusion shall be that if only an axial stress is applied to a borehole wall, a vertical point source radiation is effectively achieved barring a scaling factor equal to half the displaced volume. Because the velocity dependence for an exclusively axial source is just inverse velocity squared instead of cubed the variations due to differences in shear wave velocities between the Pierre shale and Solenhofen limestone are considerably reduced.

For a torsional source we have K

$$K(\phi) = \frac{\Delta}{4\pi\mu v}\sin\phi \tag{87}$$

$$K(\phi) = \frac{a^2 l}{2\rho v^3}\sin\phi$$

which is shown in Figure 12. Again the inverse velocity cubed behavior is apparent so the Pierre shale radiation pattern again does not fit on the page.

We have to be careful when using these patterns. First off the F_1 factors multiply the derivatives of our stresses. Depending on the formulation of our stresses, this can radically raise or lower amplitudes. Secondly, an inverse radial dependence has been left off of these equations and should be taken into account.

There is a complicated relationship between the magnitudes of our radiation pattern formulas and the radial and vertical displacement components we wish to calculate, Eq. 33. The figures for F_1, F_2, G_1, G_2 only represent magnitudes of P and Sv components if we restrict ourselves to radial stress or axial stress at the exclusion of the other. In other words, if we apply only a radial stress F_1 and F_2 represent the magnitude of the P and Sv waves whereas if we apply only an axial stress G_1 and G_2 represent these same magnitudes. If we apply a combination of radial and axial stresses then we have to use the complication represented by Eq. 33. Finally, attenuation is not considered. With those caveats in mind we can nonetheless state that the effects of the dry borehole are negligible on the axial stresses and substantial on the radial and torsional stresses. The effects on the radial and torsional stresses are extremely dependent on the type of stress that is applied due to its derivative nature. Finally, because of inverse velocity dependences to the third and second power, soft sediments should show more radiation into the formation itself. This inverse velocity dependence is especially true with the shear wave velocity. The only term that is not affected by the shear wave velocity is the axial G_1.

Because soft sediments with low shear wave velocities transmit energy into the formation well the converse requires either that a) the high velocity materials dissipate more energy in the near field and/or b) high velocity sediments set up more energetic modes inside the borehole.

Radiation from a point source in a fluid-filled borehole

Lee and Balch (1982) solved equations identical to those presented in paper 1 of this series but with a different notation. They then proceeded to calculate the far field radiation pattern by using the expansions in small argument for the Hankel functions and discarding of terms that do not predominate in the far field similar to the technique Heelan used. Lee and Balch found surprisingly simple modifications to Heelan's results due to the presence of a fluid in the borehole. Lee and Balch's equations for a volume source in the middle of the borehole in terms of Heelan's parameters are for P and Sv radiation

$$F_{1vol}(\phi) = \frac{1}{(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \frac{v^2}{V^2}\cos^2\phi)} \frac{\rho_f}{4\pi\rho_2}(1 - \frac{2v^2\cos^2\phi}{V^2}) \tag{88}$$

$$F_{2vol}(\phi) = \frac{1}{(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \cos^2\phi)} \frac{\rho_f}{4\pi\rho v^3}\sin 2\phi \tag{89}$$

F_{1vol} differing from Eq. 28 by the factor

$$\frac{a^2 l\rho}{4\pi v^2(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \frac{v^2}{V_f^2}\cos^2\phi)} \tag{90}$$

and F_{2vol} differing from Eq. 30 by the factor

$$\frac{a^2 l\rho}{4\pi(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \cos^2\phi)} \tag{91}$$

You will notice an interesting result from Eq. 88 and Eq. 89 that for a volume source in the middle of a fluid-filled borehole there is no dependence on the radius of the borehole in the far field.

For a radial source applied to the borehole wall, no axial stress, Lee and Balch found the following radiation pattern formula for the P waves

$$F_{1r}(\phi) = \frac{(\frac{v^2}{V_f^2} - \frac{v^2}{V_f^2}\cos^2\phi)}{(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \frac{v^2}{V^2}\cos^2\phi)} \frac{a^2 l}{2\rho v^2 V}(1 - \frac{2v^2\cos^2\phi}{V^2}) \tag{92}$$

and for the Sv waves

$$F_{2r}(\phi) = \frac{(\frac{v^2}{V_f^2} - \cos^2\phi)}{(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \cos^2\phi)} \frac{a^2 l}{2\rho v^3}\sin 2\phi \tag{93}$$

F_{1r} differing from Eq. 28 by

$$\frac{(\frac{v^2}{V_f^2} - \frac{v^2}{V_f^2}\cos^2\phi)}{(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \frac{v^2}{V^2}\cos^2\phi)} \tag{94}$$

F_{2r} differing from Eq. 30 by

$$\frac{(\frac{v^2}{V_f^2} - \cos^2\phi)}{(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \cos^2\phi)} \tag{95}$$

from Heelan's results.

Figure 13 through Figure 16 show radiation patterns calculated using Lee and Balch's formula with the same three lithologies used to calculate the radiation for the point source and the radiation for a radial source in an empty borehole (Heelan's results). The differences between the $F1$ components for a volume source in a fluid-filled borehole Figure 13 and a radial source in a fluid-filled borehole Figure 14 is primarily in the Pierre shale. It can be seen that with the Pierre shale and a radial source more energy is radiated radially and less in a vertical direction causing a dimpling or an increase in the peanut-like shape of the radial source Figure 14. Especially strange behavior is seen with the Lee and Balch radiation patterns for the Sv wave radiation Figure 15 and Figure 16. The Lee and Balch equations have introduced new poles related to the tube wave velocity into the radiation patterns that have a dominant influence on the radiation patterns in soft low velocity sediments (Pierre Shale). In soft low velocity sediments the denominator term $(\frac{\rho_f}{\rho} + \frac{v^2}{V_f^2} - \frac{v^2}{V^2}\cos^2\phi)$ common to both F_{1vol} and F_{1r} goes to zero and thus the quotient goes to infinity at certain ϕ. For Pierre Shale surrounding the borehole, assuming the fluid velocity is 1500 $\frac{m}{sec}$ and the density of the fluid is $1\frac{g}{cm^3}$ we see an angle near 62 degrees from horizontal will cause the quotient to blow up.

Lee and Balch only showed radiation patterns for a Poisson solid so the treatment of the three different lithologies given here is different from their approach. We can thus echo Lee and Balch's conclusion that for soft sediments the effect of the fluid is substantial and also add that radiation pattern analyses for soft sediments have limitations.

Comparison of source representations

In viewing the radiation patterns presented thus far it is obvious that there is a substantial difference due to the following factors: the presence of the borehole, the presence of a fluid in the borehole and less difference due to volume source versus a radial source.

In comparing the fluid-filled versus the empty borehole results we can see that there is more relative peanut shape in the radial source in the fluid-filled borehole. This implies in a gross sense that the presence of the fluid is impeding or causing a drag against energy propagating in the vertical direction. The presence of a fluid in a soft sediment introduces poles which are not present in the dry case and thus negate radiation pattern analysis used for this purpose in soft sediments. Shear wave velocity has an important effect on radiation magnitude in both fluid-filled and empty boreholes. Poisson ratio variations have a much smaller effect on the amplitude radiation than shear wave effects.

CONCLUSIONS

Heelan's algebraic results have been directly verified and explained in this chapter in addition to the correction of typographical errors. The results of Heelan's contour integration have been indirectly verified through extensions of Brekhovskikh's work in this paper. Brekhovskikh's method uses a contour widely used in electromagnetics and acoustics whereas Heelan's contour is essentially unknown. Having indirectly verified Heelan's results are correct they nonetheless are surprising. A very strong dependence of radiation amplitude on shear wave velocity is seen in the far field results of Heelan.

Lee and Balch's results (1982) for studying source radiation which incorporate a fluid into the borehole have been compared to those of Heelan and the result for a point force. The presence of a fluid has an especially pronounced effect on the Sv wave radiation pattern as Lee and Balch have shown. Additionally, radiation pattern analysis from a fluid-filled borehole surrounded by a soft low velocity sediment breaks down at certain azimuths because of the presence of an additional pole. Shear wave velocity because of an inverse velocity squared and cubed dependence is the dominating factor on theoretically calculated radiation patterns. These conclusions point out the inherent strengths and weaknesses of radiation pattern analyses.

Velocities and Densities of Three Common Lithologies and Steel

Lithology/Property	V_p	V_s	σ	ρ
Solenhofen Limestone	5,970 (19,582)	2,880 (9,446)	.308	2.656
Berea Sandstone	4,206 (13,796)	2,664 (8,738)	.165	2.140
Pierre Shale	2,074 (6,803)	869 (2,850)	.394	2.25

Table 1: Velocities and densities of three common lithologies. Velocities in m/sec (ft/sec), densities in g/cm^3, σ is Poisson's ratio.

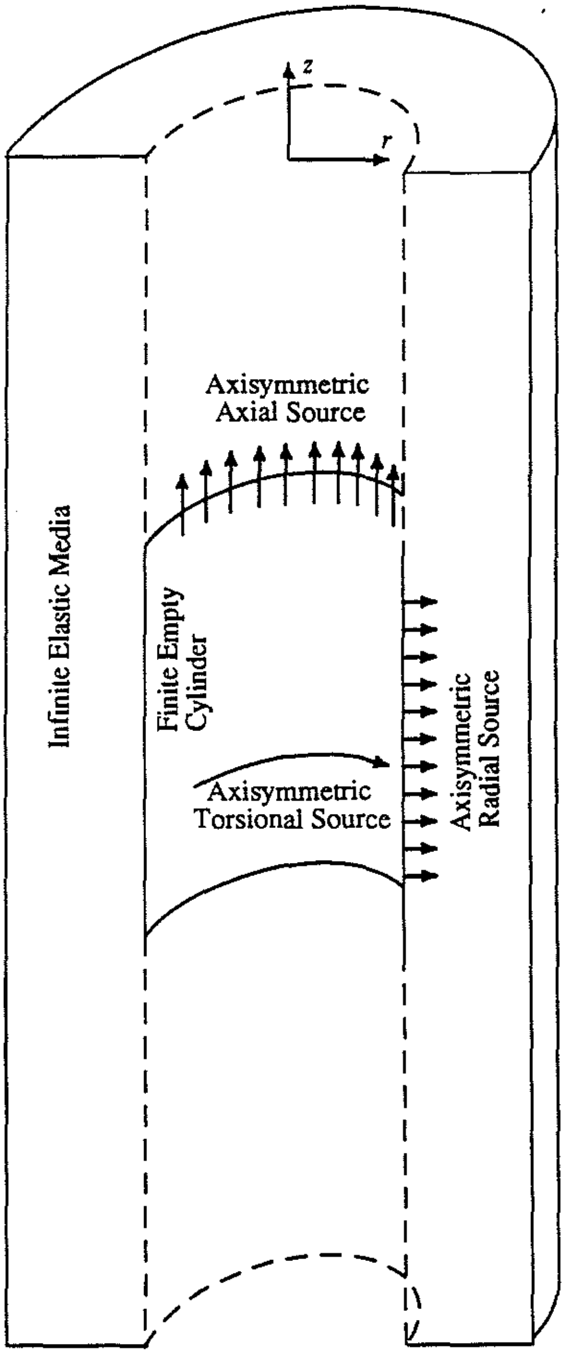

Figure 1: Geometry of Heelan's sources (1953a). An empty infinite cylinder is embedded in an infinite medium. Uniform axisymmetric axial and radial $P-Sv$ and torsional Sh sources are applied to a finite length of this cylinder at the cylinder-infinite medium boundary.

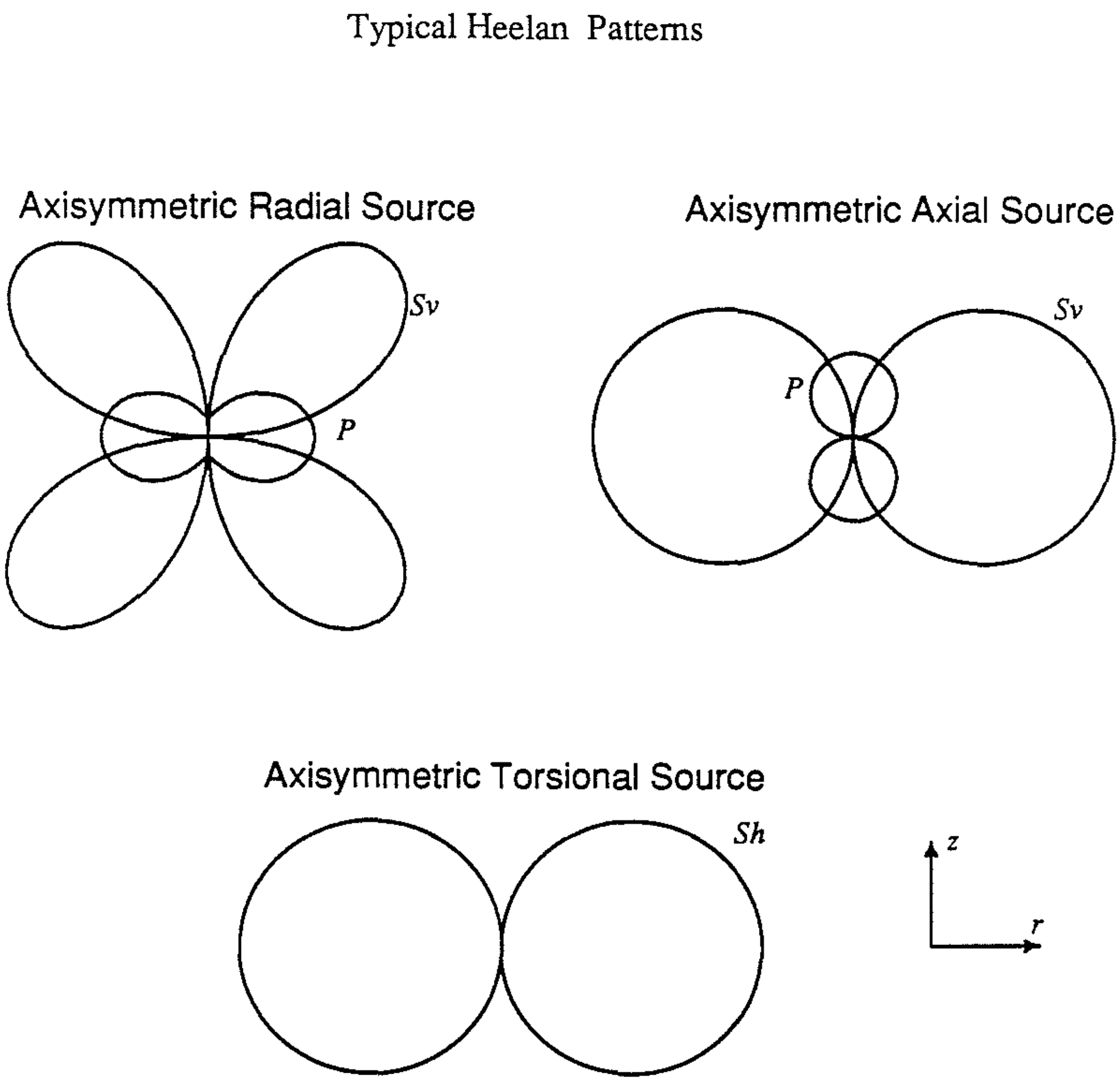

Figure 2: Pictorial description of Heelan's radiation patterns, not to scale, for axial, radial and torsional sources. Notice four leaved rose pattern with radial source Sv and peanut shape for radial source P.

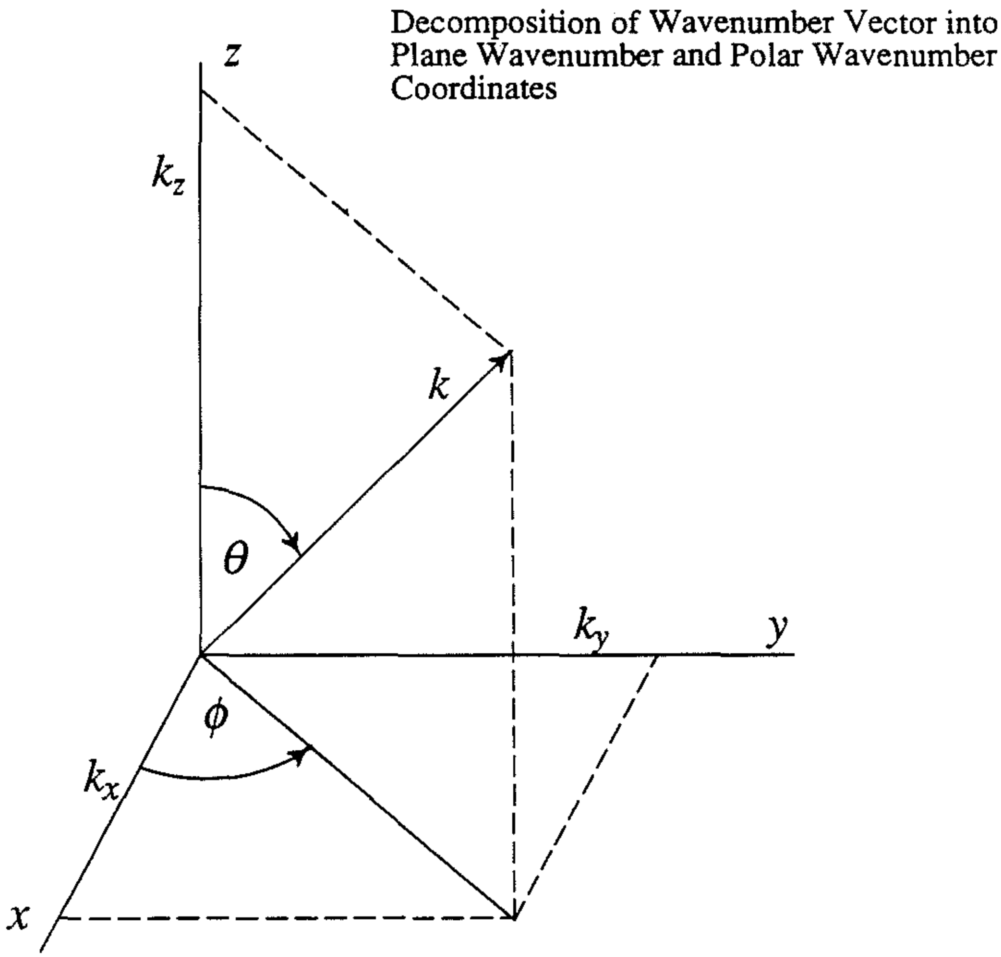

Figure 3: Decomposition of plane wave vector into polar angles (Brekhovskikh, 1960, Figure 26.1)

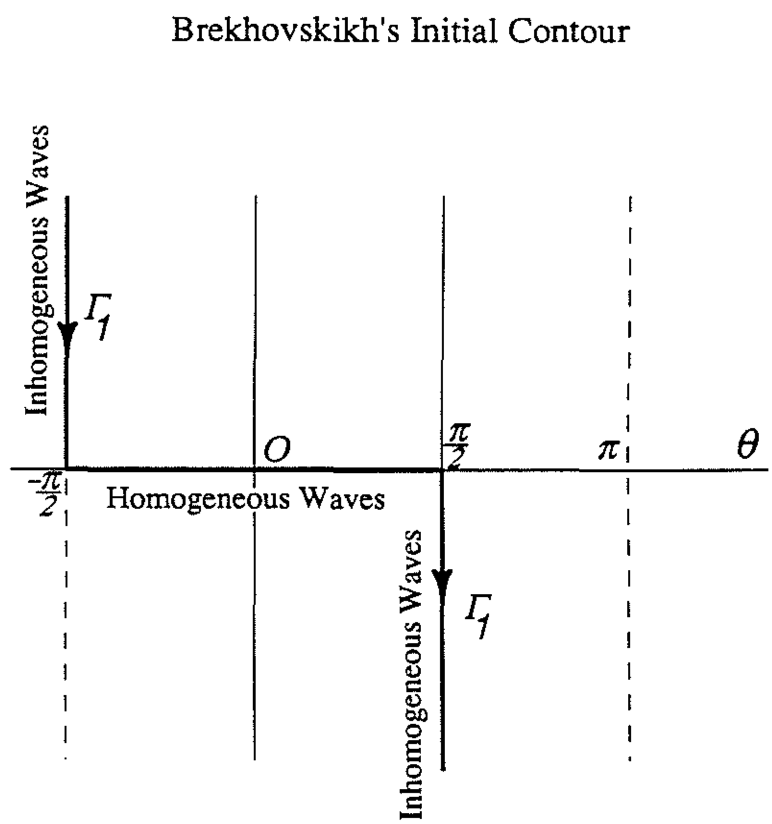

Figure 4: Contour Γ_1 used by Brekhovskikh (1960, 2nd Edition, Figure 28.1) for the Weyl integral. A member of a family of contours described by Stratton (1941, Figure 67, contour C_1). Includes all homogeneous and inhomogeneous plane waves and is conditionally deformable at will as long as it terminates at the endpoints $(\frac{-\pi}{2}, \infty)$ and $0, \frac{\pi}{2}$. Often advantageous to cross the origin to achieve a saddle point.

Brekhovskikh Steepest Descent Path

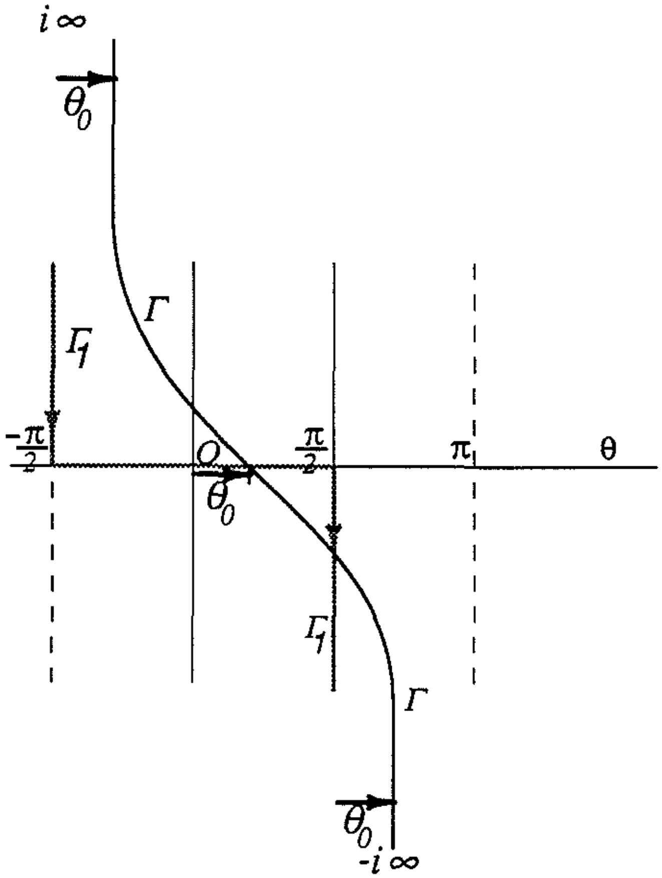

Figure 5: Deformed steepest descent path Γ incorporating stationary phase approximation used by Brekhovskikh (1960, 2nd Edition, Figure 28.1) along with original contour Γ_1.

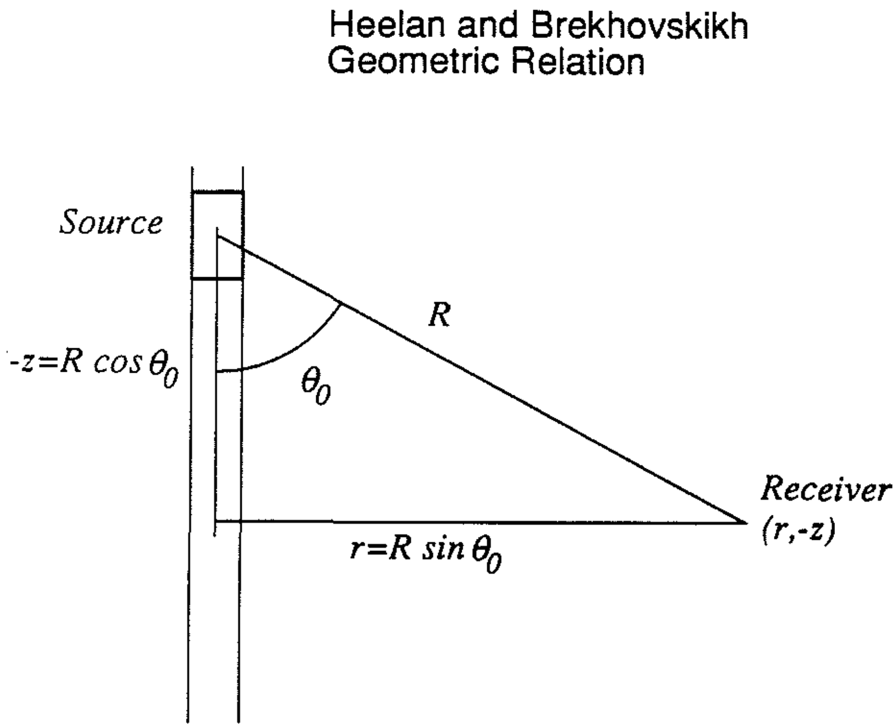

Figure 6: Geometry in the far field from Heelan and Brekhovskikh, The relationships between R, r, θ_0 are very important in the steepest descent analysis.

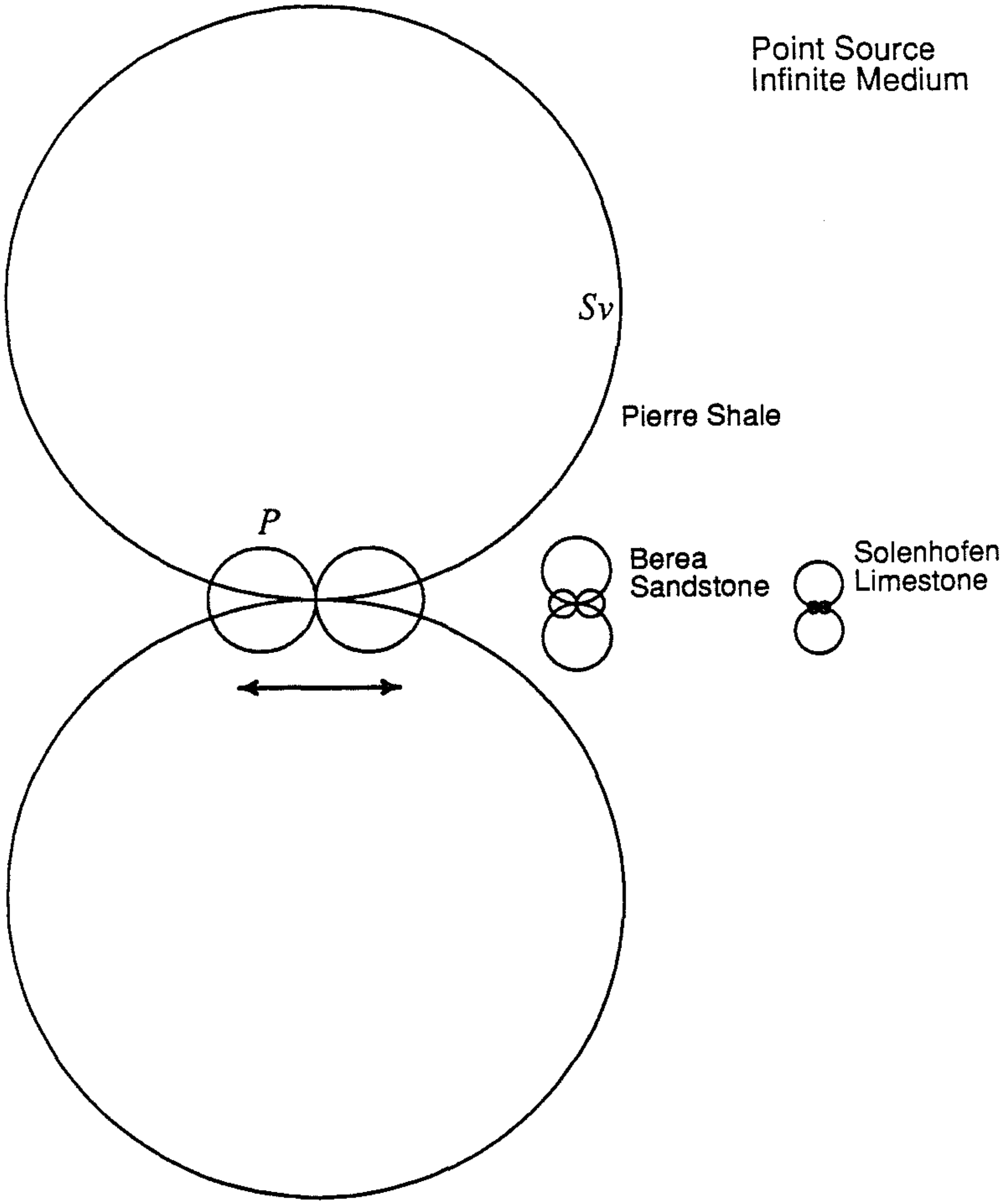

Figure 7: Radiation pattern for a point source in an infinite elastic medium in the far field from White (1965, Eq. 5-5, 1983, Eq. 6-5) for three different lithologies. Velocities and densities may be found in Table 1. Notice shape is very similar to that for an axial source of Heelan.

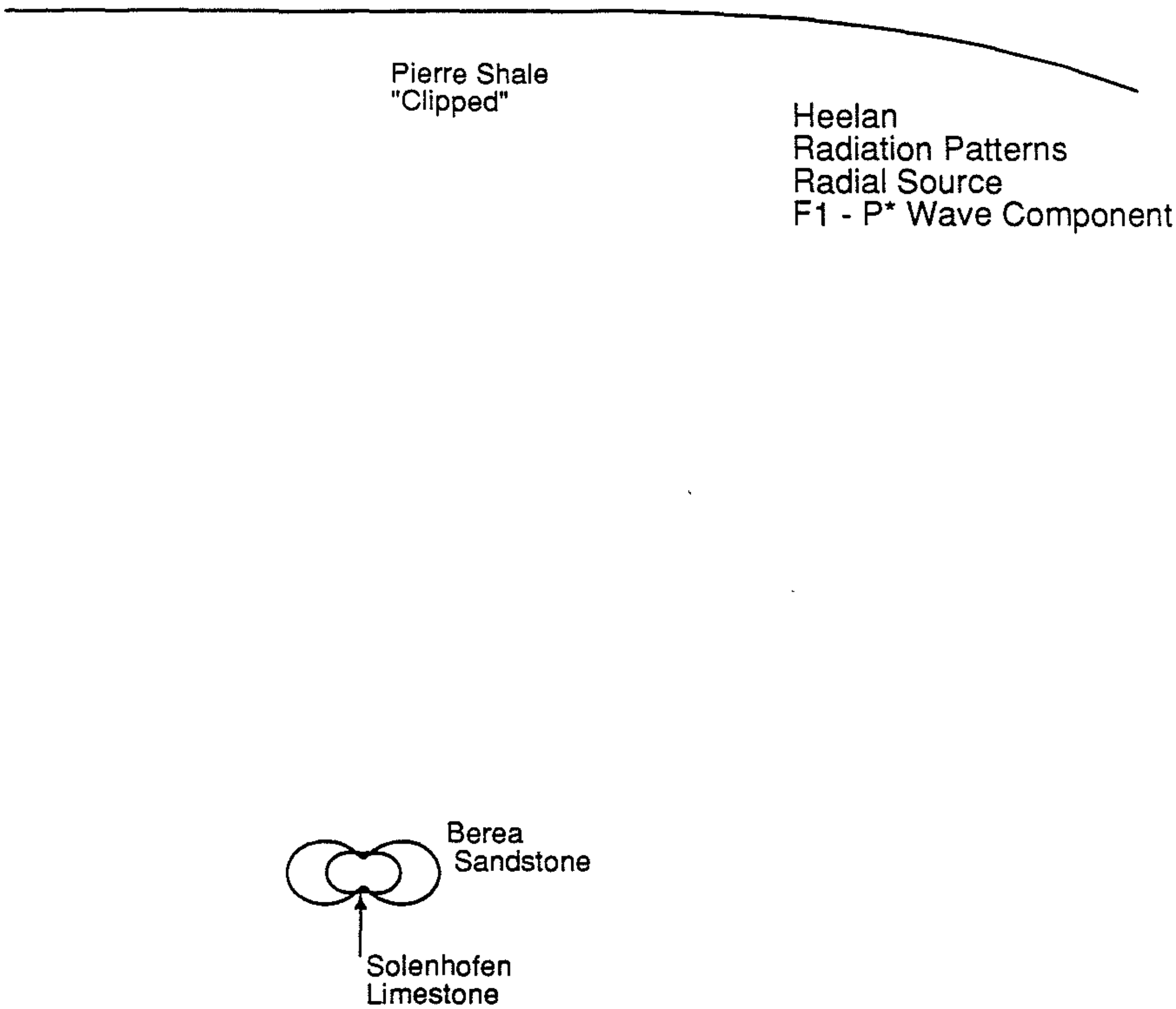

Figure 8: Magnitudes of F_1 ($P*$) for three different lithologies, Radial Stress - Empty Borehole. Notice extreme differences in amplitudes and peanut shape for P. Origin of peanut shape is reduced amplitude along the axis of the borehole. Pierre shale exceeds limits of page. *Can be considered magnitude of P if a radial source is applied exclusively. If axial source applied also, P is more complicated

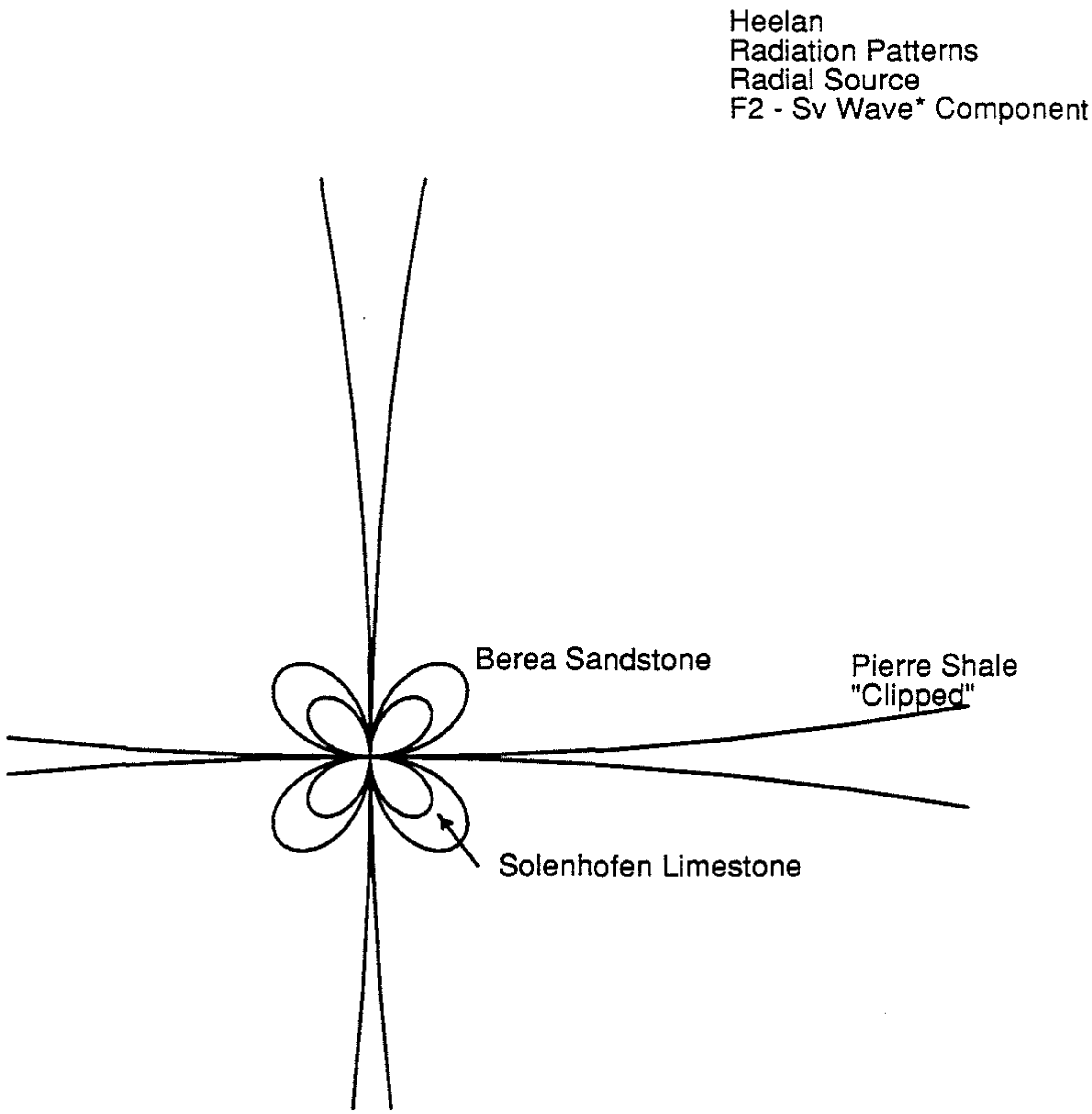

Figure 9: Magnitudes of F_2 (Sv)for three different lithologies, Radial Stress - Empty Borehole. Notice extreme differences in amplitudes. and four leaved rose pattern for Sv. Pierre Shale exceeds limits of page. *Can be considered magnitude of Sv if a radial source is only applied. If axial source applied also, Sv is more complicated

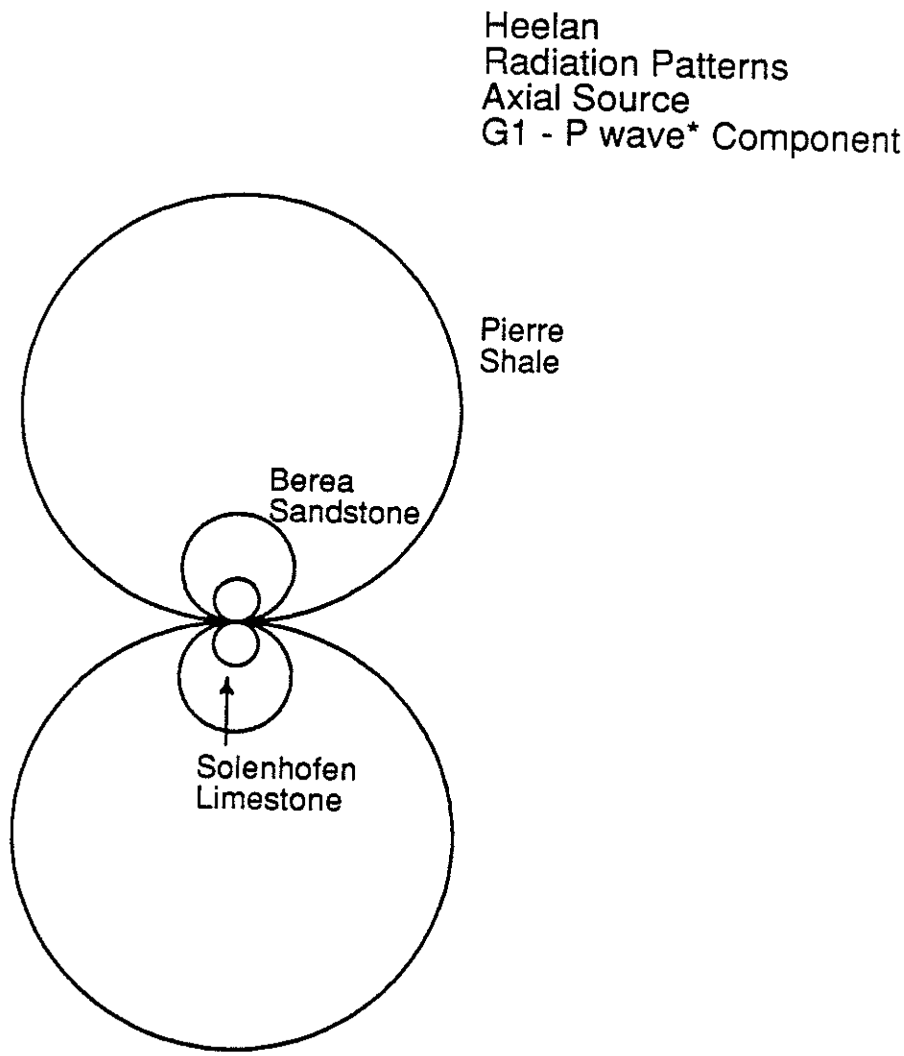

Figure 10: Magnitudes of G_1 (P) for three different lithologies, Axial stress - Empty Borehole. Notice differences in amplitudes but not as severe as for radial stress. Also notice circular geometry for patterns - the geometry unchanged from that of a point source. *Can be considered magnitude of P if an axial source is only applied. If radial source applied also, P is more complicated

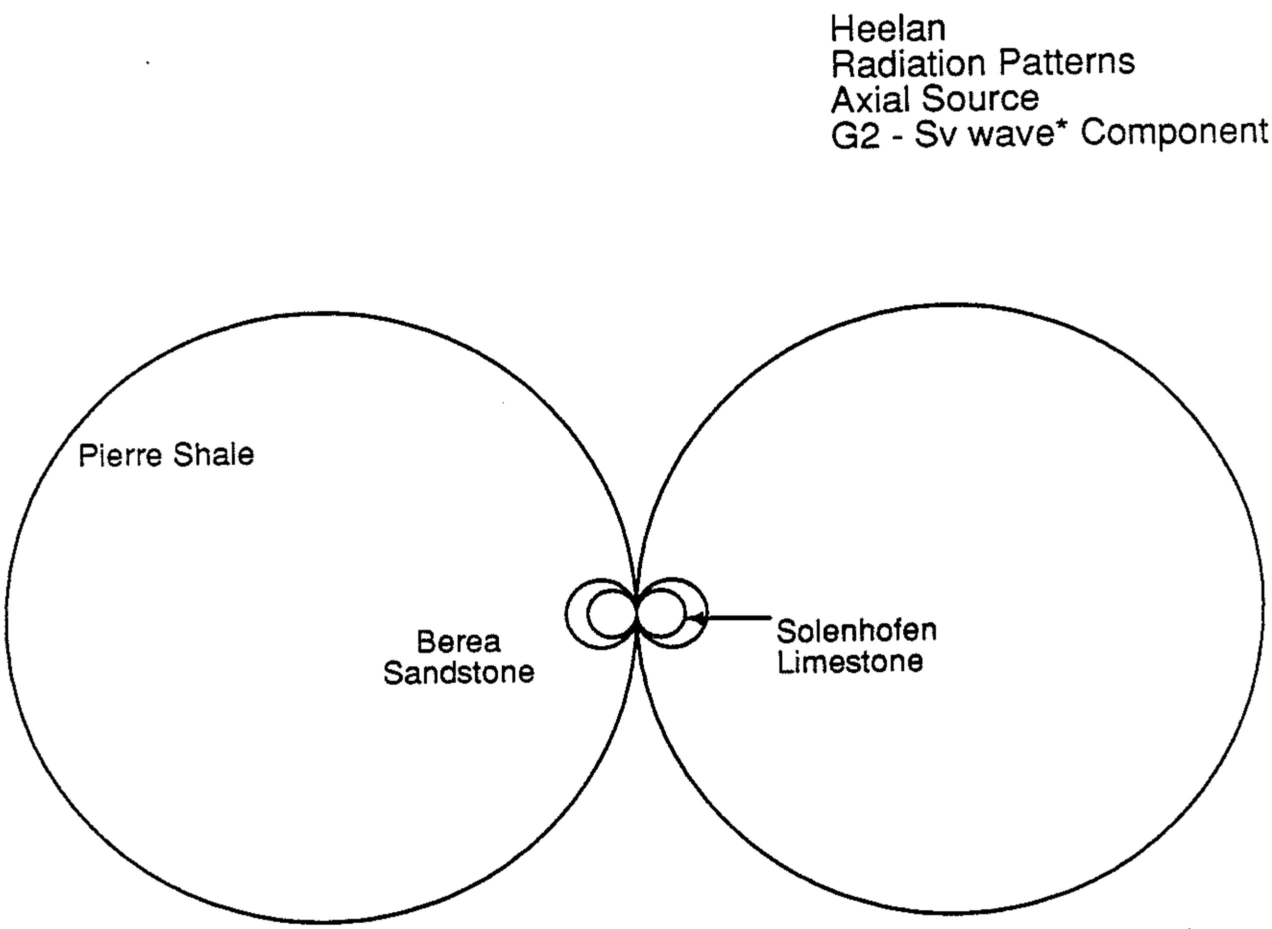

Figure 11: Heelan's Formula, Magnitudes of G_2 (Sv) for three different lithologies, Axial stress - Empty Borehole. Notice differences in amplitudes but not as severe as for radial stress. Also notice circular geometry for patterns - the geometry unchanged from that of a point source.

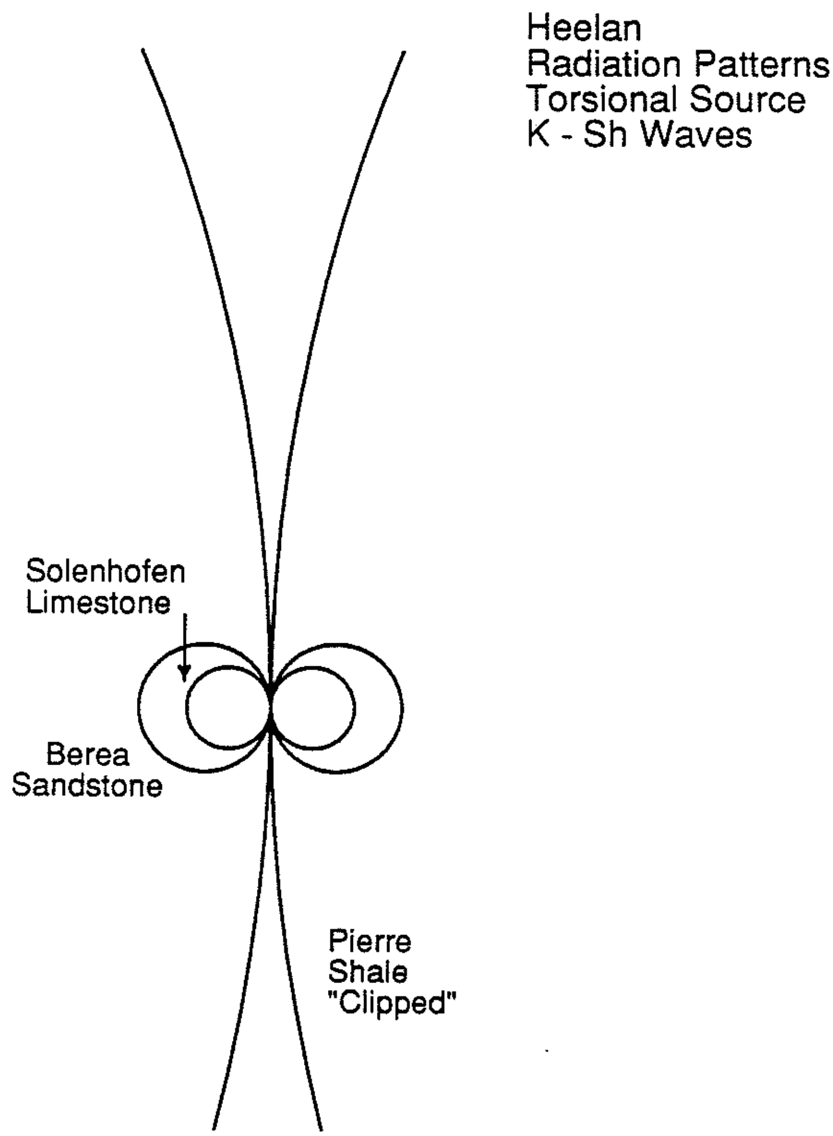

Figure 12: Heelan's Formula, Magnitudes of K (Sh) for three different lithologies, Torsional Stress - Empty Borehole. Notice differences in amplitudes but not as severe as case for radial stress. For this case geometry is same as point source representation. *Can be considered magnitude of Sv if an axial source is only applied. If radial source applied also, Sv is more complicated.

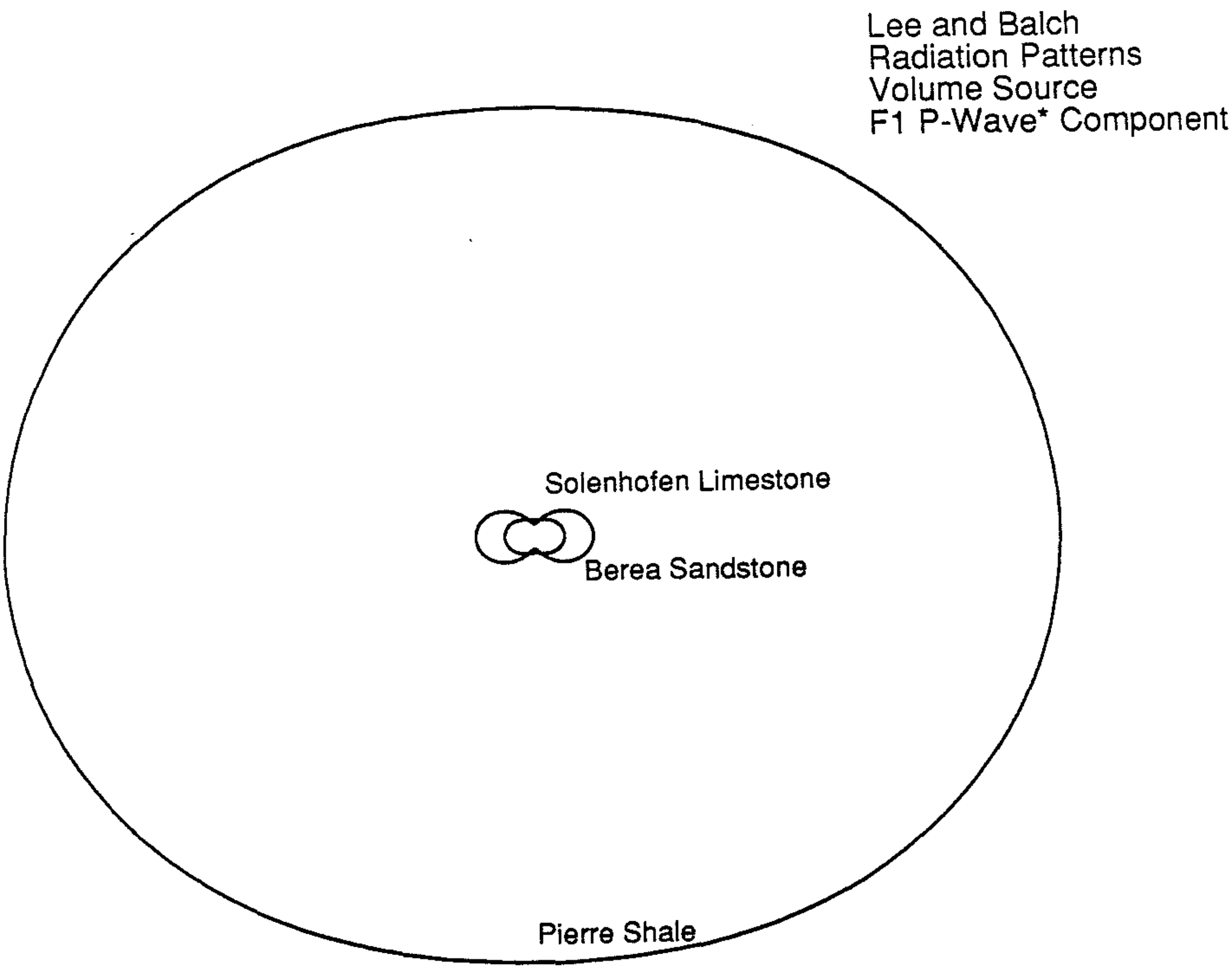

Figure 13: Lee and Balch (1982) Formula, Magnitudes of F_{1vol} (P) for three different lithologies, Volume Source - Fluid-Filled Borehole. Notice reduced difference between Pierre shale and other lithologies compared to Heelan's F_1.

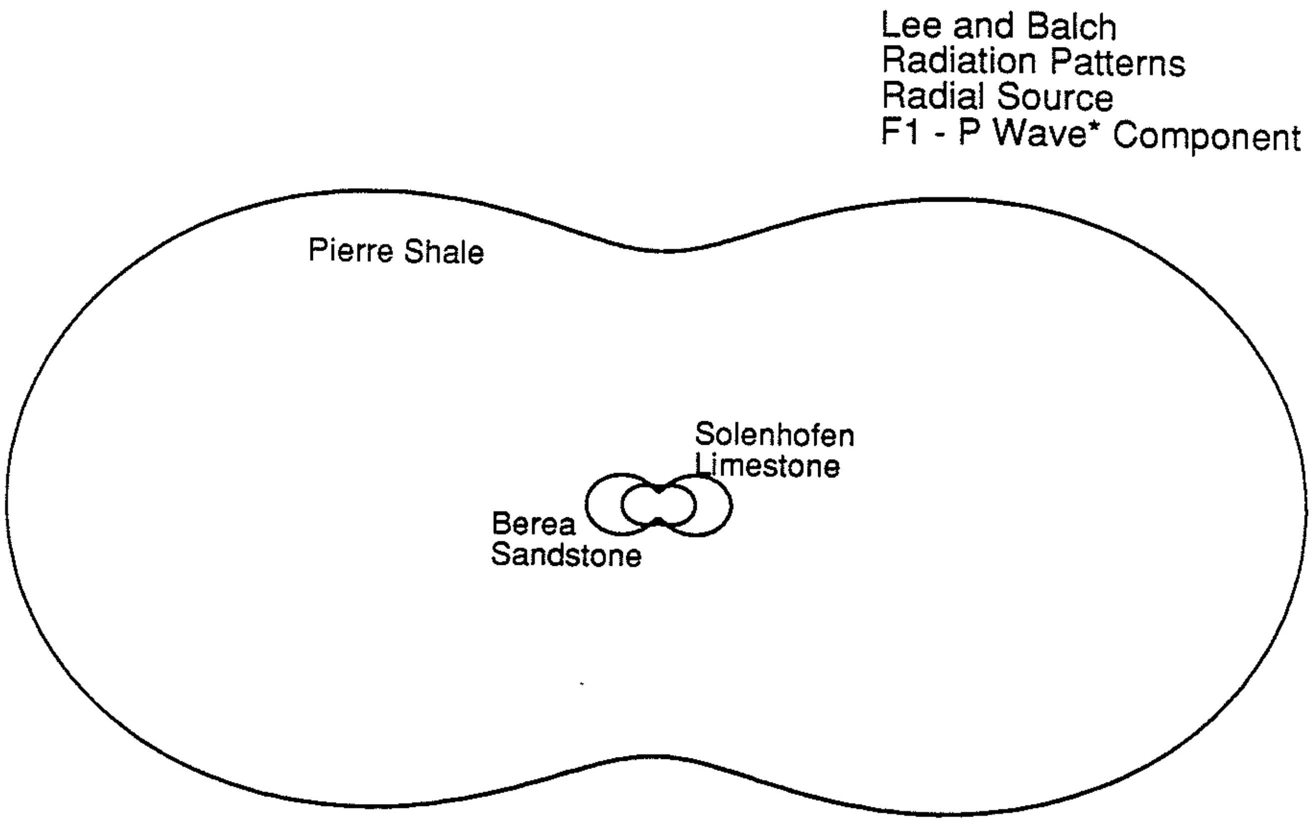

Figure 14: Lee and Balch (1982) Formula, Magnitudes of F_{1r} (P) for three different lithologies, Radial stress - Fluid-Filled Borehole. Notice reduced difference between Pierre shale and other lithologies compared to Heelan's F_1. *Can be considered magnitude of P if a radial source is applied exclusively. If axial source applied also, P is more complicated.

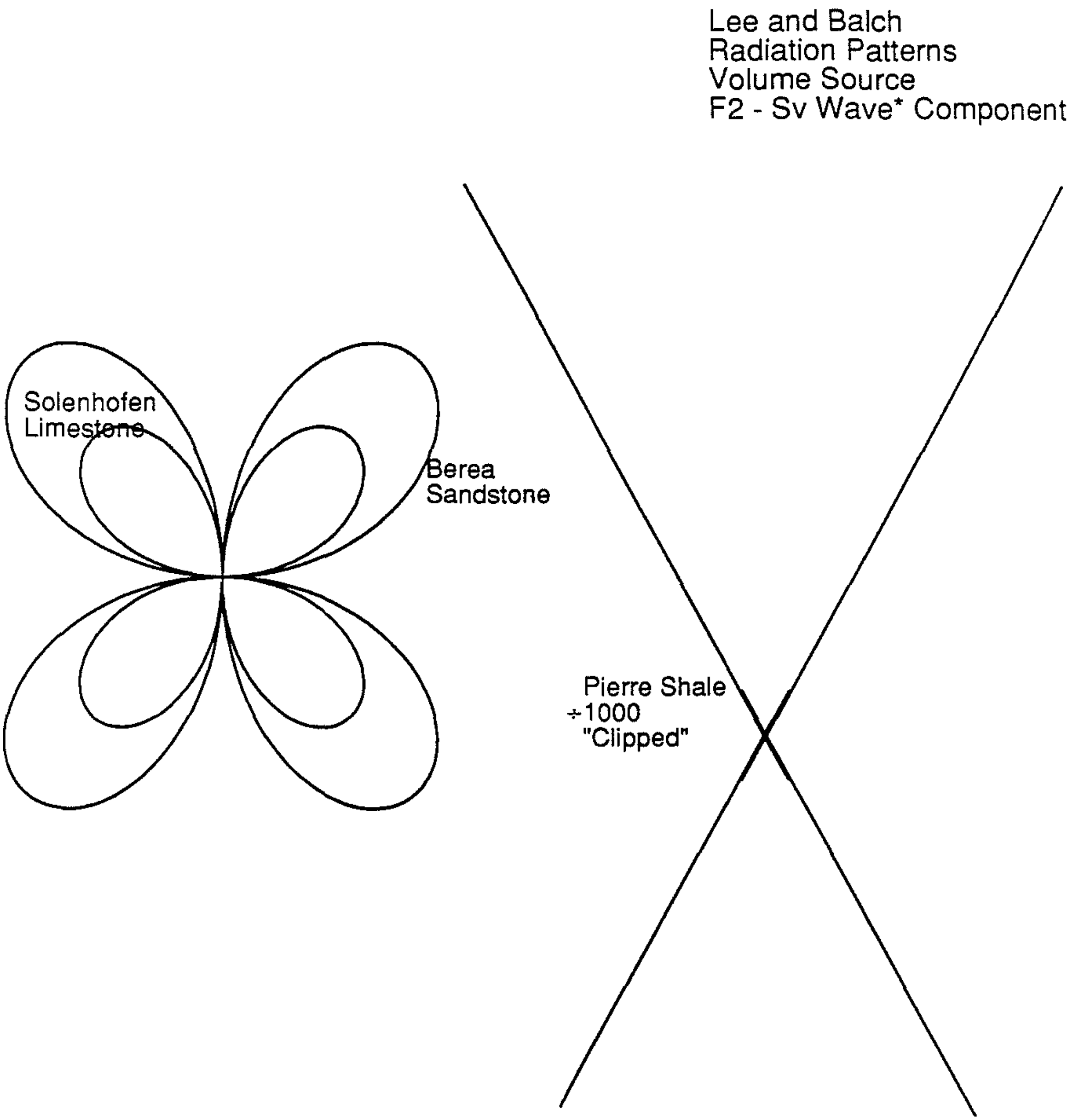

Figure 15: Lee and Balch (1982) Formula, Magnitudes of F_{2vol} (Sv) for three different lithologies, Volume Source - Fluid-Filled Borehole. Notice effect of tube wave pole causes radical behavior in Pierre shale radiation pattern. Hard lithologies compare favorably to F_2.

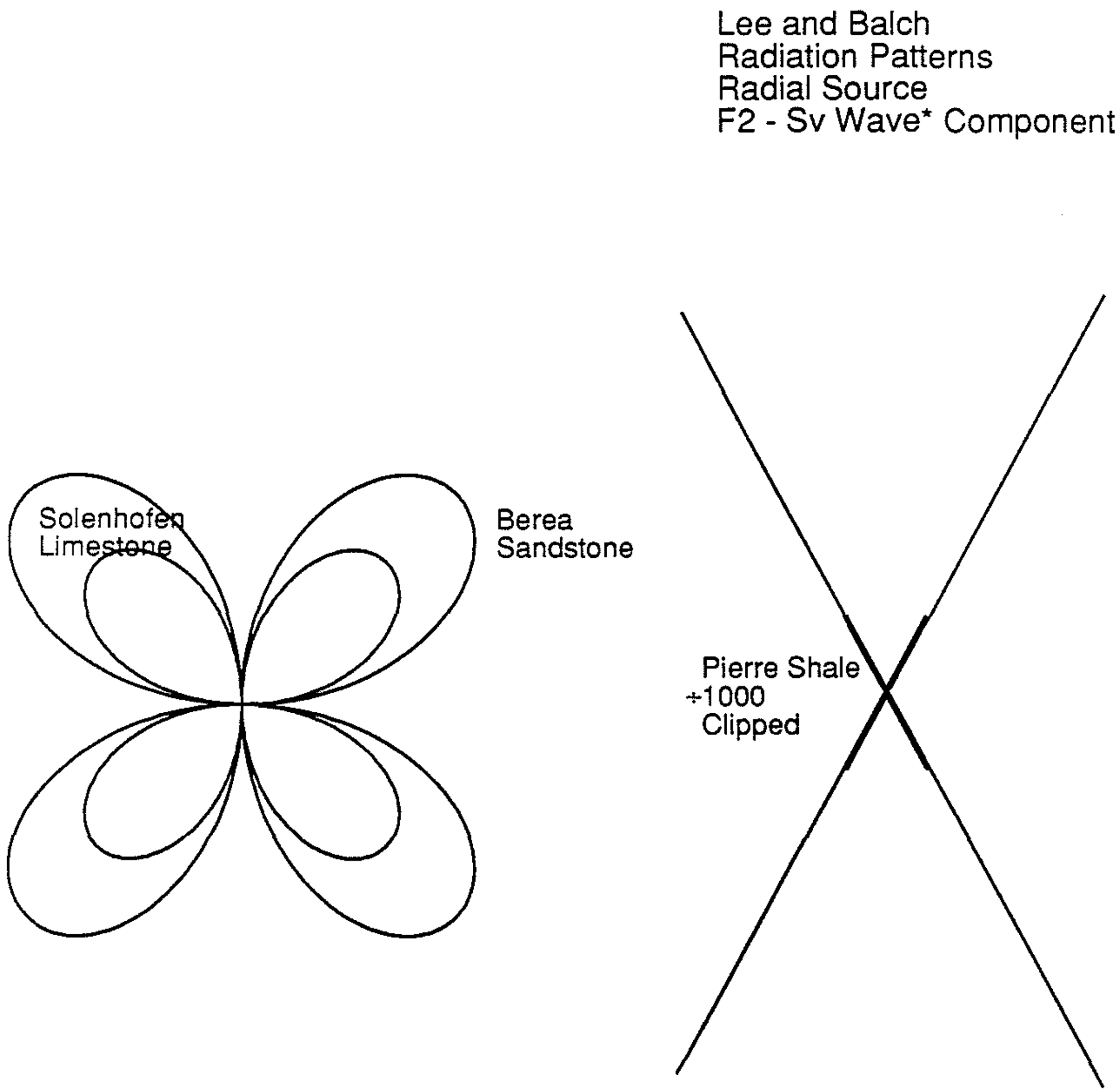

Figure 16: Lee and Balch (1982) Formula, Magnitudes of F_{2r} (Sv) for three different lithologies, Radial Source - Fluid-Filled Borehole. Notice effect of tube wave pole causes radical behavior in Pierre shale radiation pattern. Hard lithologies compare favorably to F_2. *Can be considered magnitude of Sv if a radial source is only applied. If axial source applied also, Sv is more complicated.

ACKNOWLEDGEMENTS

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Appendix A. HEELAN'S AND BREKHOVSKIKH'S ALGEBRA

The purpose of this appendix is to verify and fully disclose the algebra Heelan(1952, 1953a, 1953b) and Brekhovskikh (1960, 1980) used in their derivations and to elucidate some of the unstated procedures they used in solving their algebra which has caused difficulty for geophysicists trying to profitably analyze their work. This difficulty has caused some to dismiss this work as unsound. This appendix does not however address whether Heelan's contour integration is valid only that his and Brekhovskikh's algebra are correct.

One caveat is in order, since this appendix exclusively deals with algebraic manipulation it is equation oriented so please have a thorough understanding of the issues presented in chapter three before attempting to wade through it.

We have an empty borehole with a stress radiating into the surrounding formation. Since the empty borehole represents a free surface we do not have any continuity of displacement boundary conditions. Our possible boundary conditions are the vanishing of normal stress, azimuthal stress, and tangential stress. For the Sh case, the boundary condition is vanishing of azimuthal stress and for P-Sv the vanishing of axial (tangential) and normal (radial) stress. When stress from a source is added the total stress must still vanish. In Heelan's first paper (1953a), a radial and an axial stress were applied and in Heelan's second paper (1953b) and Brekhovskikh's work (1960, 1980) only a radial stress was applied.

We are fortunate in that the P-Sv and Sh problems are decoupled which leaves us with a 2×2 system of boundary condition equations and stresses for P-Sv and a simple equation to solve for Sh.

P-Sv case

We will begin with the most general case with having both radial and axial stresses applied. The second case where only a radial stress is constructed by setting the axial coefficient to 0. Our boundary conditions are vanishing of radial and axial stress. As mentioned earlier, Heelan uses an unusual convention in that his ψ equals the $-\psi$ of the derivations of Part I. This does not affect the physics of the problem though. These stresses are written (Eq. 14).

$$\begin{aligned} p_r &= \lambda\nabla^2\phi + 2\mu\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r} - \frac{\partial^2\psi}{\partial r\partial z}\right) \qquad \text{(A-1)} \\ p_{rz} &= \mu\frac{\partial}{\partial r}\left(2\frac{\partial\phi}{\partial z} + \nabla^2\psi - 2\frac{\partial^2\psi}{\partial z^2}\right) \\ p_{r\theta} &= \mu\left(\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r}\right) \end{aligned}$$

Where ϕ and ψ are defined to be Eq. 1

$$\begin{aligned} \phi &= Re \int_0^\infty e^{ikVt}\, dk \int_C f_o(\sigma,k) H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2-k^2}}\, d\sigma \\ \psi &= Re \int_0^\infty e^{ikVt}\, dk \int_C g_o(\sigma,k) H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2-h^2}}\, d\sigma \\ \chi &= Re \int_0^\infty e^{ikVt}\, dk \int_C n_o(\sigma,k) H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2-h^2}}\, d\sigma \end{aligned} \tag{A-2}$$

We add the stresses (Eq. A-1) plus our discontinuities in normal stress, $PG(k)$, and axial stress, $QG(k)$ to yield a sum of zero. The transformed values of $PG(k)$ and $QG(k)$ are the following from Eq. 16

$$\begin{aligned} PG(\sigma,k) &= PG(k)\frac{i\sigma \sinh(l\sqrt{\sigma^2-k^2})}{\pi(\sigma^2-k^2)} e^{z\sqrt{\sigma^2-k^2}} \\ QG(\sigma,k) &= QG(k)\frac{i\sigma \sinh(l\sqrt{\sigma^2-k^2})}{\pi(\sigma^2-k^2)} e^{z\sqrt{\sigma^2-k^2}} \end{aligned} \tag{A-3}$$

It's required to solve for the coefficient functions f_o and g_o. From our system of equations written below it is straightforward to see our Cramer's rule solution.

$$\begin{vmatrix} p_{rz1} & p_{rz2} \\ p_{r1} & p_{r2} \end{vmatrix} \begin{vmatrix} f_o \\ g_o \end{vmatrix} + \begin{vmatrix} QG(k) \\ PG(k) \end{vmatrix} = 0 \tag{A-4}$$

We have now equated the boundary conditions to the discontinuities in stress. Our Cramer's rule solutions for f_o and g_o are the following

$$\begin{aligned} f_o(\sigma,k) &= \frac{-QG(k)p_{r2} + PG(k)p_{rz2}}{p_{rz1}p_{r2} - p_{rz2}p_{r1}} \\ g_o(\sigma,k) &= \frac{-PG(k)p_{rz1} + QG(k)p_{r1}}{p_{rz1}p_{r2} - p_{rz2}p_{r1}} \end{aligned} \tag{A-5}$$

and for the case where no axial source stress is applied we set Q equal to zero and our boundary conditions remain equivalent. In this case, we solve for f_o, g_o yielding

$$\begin{aligned} f_o(\sigma,k) &= \frac{PG(k)p_{rz2}}{p_{rz1}p_{r2} - p_{rz2}p_{r1}} \\ g_o(\sigma,k) &= \frac{-PG(k)p_{rz1}}{p_{rz1}p_{r2} - p_{rz2}p_{r1}} \end{aligned} \tag{A-6}$$

We calculate our stresses using Eq. 14. Before doing so we remember Heelan's use of $-\psi$ for ψ and the relationship.

$$\nabla^2\phi = -k^2\phi \qquad \nabla^2\psi = -h^2\psi \tag{A-7}$$

We write our system of equations

$$
\begin{aligned}
-QG(k)\frac{i\sigma\sinh(l\sqrt{\sigma^2-k^2})e^{z\sqrt{\sigma^2-k^2}}}{\pi(\sigma^2-k^2)} &= \\
& f_o(-2\mu\sigma\sqrt{\sigma^2-k^2})H_1^{(1)}(\sigma a)e^{z\sqrt{\sigma^2-k^2}} \\
& +g_o(\mu\sigma(h^2+2\sigma^2-2k^2))H_1^{(1)}(\sigma a)e^{z\sqrt{\sigma^2-h^2}} \\
-PG(k)\frac{i\sigma\sinh(l\sqrt{\sigma^2-k^2})e^{z\sqrt{\sigma^2-k^2}}}{\pi(\sigma^2-k^2)} &= \\
& f_o\left((-\lambda k^2-2\mu\sigma^2)H_0^{(1)}(\sigma a)+2\mu\sigma\frac{H_1^{(1)}(\sigma a)}{a}\right)e^{z\sqrt{\sigma^2-k^2}} \\
& +g_o(2\mu\sigma^2\sqrt{\sigma^2-k^2})\left[H_0^{(1)}(\sigma a)-\frac{H_1^{(1)}(\sigma a)}{\sigma a}\right]e^{z\sqrt{\sigma^2-h^2}}
\end{aligned}
\tag{A-8}
$$

An algebraic difficulty arises because in Eq. A-8 we have potentials with two different exponential factors, $e^{z\sqrt{\sigma^2-k^2}}$ and $e^{z\sqrt{\sigma^2-h^2}}$.

To ease this algebraic burden, Heelan implements a very complex transformation. First Heelan substitutes a factor ϱ^2 equal to $(\sigma^2+h^2-k^2)$ into the ψ potential under the radical, although this crucial interim step was not defined in his papers and only very loosely defined in his thesis. In the final solution, for the $f_o(\sigma,k)$ potential, all ϱ terms cancel so there is no reverse tranformation. However we calculate the solution for $g_o(\varrho,k)$ and in order to transform it back to $g_o(\sigma,k)$ we set σ^2 equal to $(\sigma^2-h^2+k^2)$.

We thus rewrite our potentials temporarily setting aside the integrations as

$$
\begin{aligned}
\phi &= f_o(\sigma,k)H_0^{(1)}(\sigma r)e^{z\sqrt{\sigma^2-k^2}} \\
\psi &= g_o(\varrho,k)H_0^{(1)}(\varrho r)e^{z\sqrt{\sigma^2-k^2}}
\end{aligned}
\tag{A-9}
$$

and the change in ψ from Eq. A-2 is evident. So now we rewrite our Eq. A-8 in the following form.

$$
\begin{aligned}
-QG(k)\frac{i\sigma\sinh(l\sqrt{\sigma^2-k^2})}{\pi(\sigma^2-k^2)}e^{z\sqrt{\sigma^2-k^2}} &= \\
& f_o(-2\mu\sigma\sqrt{\sigma^2-k^2})H_1^{(1)}(\sigma a)e^{z\sqrt{\sigma^2-k^2}} \\
& +g_o(\mu\varrho(h^2+2\sigma^2-2k^2))H_1^{(1)}(\varrho a)e^{z\sqrt{\sigma^2-k^2}} \\
-PG(k)\frac{i\sigma\sinh(l\sqrt{\sigma^2-k^2})}{\pi(\sigma^2-k^2)}e^{z\sqrt{\sigma^2-k^2}} &= \\
& f_o\left((-\lambda k^2-2\mu\sigma^2)H_0^{(1)}(\sigma a)+2\mu\sigma\frac{H_1^{(1)}(\sigma a)}{a}\right)e^{z\sqrt{\sigma^2-k^2}}
\end{aligned}
\tag{A-10}
$$

$$+g_o(2\mu\varrho^2\sqrt{\sigma^2-k^2})\left[H_0^{(1)}(\varrho a)-\frac{H_1^{(1)}(\varrho a)}{\varrho a}\right]e^{z\sqrt{\sigma^2-k^2}}$$

Our denominator for the Cramer's rule solution is $p_{rz1}p_{r2}-p_{rz2}p_{r1}$ which after some simplification equals

$$p_{rz1}p_{r2}-p_{rz2}p_{r1}=$$

$$-4\sigma\mu^2\varrho^2(\sigma^2-k^2)H_1^{(1)}(\sigma a)H_0^{(1)}(\varrho a) \tag{A-11}$$

$$+\mu\varrho(h^2+2\sigma^2-2k^2)(\lambda k^2+2\mu\sigma^2)H_1^{(1)}(\varrho a)H_0^{(1)}(\sigma a) \tag{A-12}$$

$$-2\sigma\mu^2h^2\varrho\frac{H_1^{(1)}(\varrho a)H_1^{(1)}(\sigma a)}{a} \tag{A-13}$$

where we have discarded the exponential factors because of their cancellation througout the system of equations.

Heelan uses the expression "only predominant terms are kept in the expansion". What this means is that in the limit as z goes to zero, the ratio $\frac{H_0^{(1)}(z)}{H_1^{(1)}(z)}$ also goes to zero. Thus we can ignore factors in the denominator and numerator which have the Hankel function of order 0 (Eq. A-11, Eq. A-12). The only term that survives in the denominator is therefore Eq. A-13. Isolating Eq. A-13 we have for the denominator

$$-2\sigma\mu^2h^2\varrho\frac{H_1^{(1)}(\varrho a)H_1^{(1)}(\sigma a)}{a} \tag{A-14}$$

The numerator for the calculation of f_o is the following $(-QG(k)p_{r2}+PG(k)p_{rz2})$

$$-QG(k)2\varrho^2\mu\sqrt{\sigma^2-k^2}\left[H_0^{(1)}(\varrho a)-\frac{H_1^{(1)}(\varrho a)}{\varrho a}\right] \tag{A-15}$$

$$+PG(k)\mu\varrho(h^2+2\sigma^2-2k^2)H_1^{(1)}(\varrho a)$$

And the numerator for $g_o(\varrho,k)$ $(-PG(k)p_{rz1}+QG(k)p_{r1})$ the following

$$PG(k)2\sigma\mu\sqrt{\sigma^2-k^2}H_1^{(1)}(\sigma a)+QG(k)\left((-\lambda k^2-2\mu\sigma^2)H_0^{(1)}(\sigma a)+2\mu\sigma H_1^{(1)}(\sigma a)\right) \tag{A-16}$$

We evaluate f_o first. We substitute the values of $PG(k), QG(k)$ into Eq. A-15 and discard $H_0^{(1)}$ terms to obtain

$$QG(k)\frac{i\sigma\sinh(l\sqrt{\sigma^2-k^2})}{\pi(\sigma^2-k^2)}2\mu\varrho\sqrt{\sigma^2-k^2}\left[\frac{H_1^{(1)}(\varrho a)}{a}\right] \tag{A-17}$$

$$+PG(k)\frac{i\sigma\sinh(l\sqrt{\sigma^2-k^2})}{\pi(\sigma^2-k^2)}\mu\varrho(h^2+2\sigma^2-2k^2)H_1^{(1)}(\varrho a)$$

Dividing through by the denominator Eq. A-14 we obtain

$$
\begin{aligned}
&-QG(k)\frac{i\sinh(l\sqrt{\sigma^2-k^2})}{\pi h^2\mu(\sigma^2-k^2)}\sqrt{\sigma^2-k^2}\left[\frac{1}{H_1^{(1)}(\sigma a)}\right] \\
&-PG(k)\frac{i\sinh(l\sqrt{\sigma^2-k^2})}{2\pi\mu(\sigma^2-k^2)}(1+\frac{2\sigma^2}{h^2}-\frac{2k^2}{h^2})\left[\frac{a}{H_1^{(1)}(\sigma a)}\right]
\end{aligned}
\tag{A-18}
$$

Substituting in the following expansion for small arguments Eq. 17

$$
\begin{aligned}
\sinh x &\simeq x \\
H_1^{(1)}(z) &\simeq \frac{2}{i\pi z}
\end{aligned}
\tag{A-19}
$$

We finally obtain (Heelan, 1953a, Eq. 8a)

$$
f_o(\sigma,k)=\frac{PG(k)\sigma}{\sqrt{\sigma^2-k^2}}(1+\frac{2\sigma^2}{h^2}-\frac{2k^2}{h^2})\Delta\frac{1}{8\pi\mu}+QG(k)\frac{A\sigma}{8\pi\mu h^2}
\tag{A-20}
$$

where Δ equals the volume of the source $2\pi a^2 l$ and A equals the surface area of the finite length cylinder $4\pi a l$, where the factor of two arises due to the cavity length being $2l$.

And following the same procedures for $g_o(\varrho,k)$ we obtain

$$
-PG(k)\frac{ai\sigma\sinh(l\sqrt{\sigma^2-k^2})}{\pi\sqrt{\sigma^2-k^2}\varrho H_1^{(1)}(\varrho a)}-QG(k)\frac{i\sigma\sinh(l\sqrt{\sigma^2-k^2})}{\pi\mu h^2(\sigma^2-k^2)\varrho H_1^{(1)}(\varrho a)}
\tag{A-21}
$$

Applying our expansions for small arguments we obtain $g_o(\varrho,k)$

$$
PG(k)\frac{\pi a^2 l\sigma}{2\pi\mu h^2}+QG(k)\frac{\pi a l\sigma}{2\pi\mu h^2\sqrt{\sigma^2-k^2}}
\tag{A-22}
$$

Which we tranform to $g_o(\sigma,k)$ by replacing $\sqrt{\sigma^2-k^2}$ with $\sqrt{\sigma^2-h^2}$

$$
PG(k)\frac{\pi a^2 l\sigma}{\pi\mu h^2}+QG(k)\frac{al\sigma\pi}{2\pi\mu h^2\sqrt{\sigma^2-h^2}}
\tag{A-23}
$$

and finally yielding (Heelan, 1953a, Eq. 8b) analagous to Eq. A-20

$$
g_o(\sigma,k)=\frac{PG(k)\Delta\sigma}{4\pi\mu h^2}+\frac{QG(k)A\sigma}{8\pi\mu h^2\sqrt{\sigma^2-h^2}}
\tag{A-24}
$$

Sh **case**

The calculation of $n_o(k)$ is particularly simple because the Sh problem is uncoupled. No transformation is needed and just one equation is generated. The only boundary condition is to match our discontinuity in stress $S(k)$ with our continuity of azimuthal stress (Eq. 14) boundary condition. We thus write

$$-SG(k)\frac{i\sigma\sinh(l\sqrt{\sigma^2-h^2})}{\pi(\sigma^2-h^2)}e^{z\sqrt{\sigma^2-h^2}} = n_o(k)\mu\sigma^2 H_2^{(1)}(\sigma a)e^{z\sqrt{\sigma^2-h^2}} \qquad \text{(A-25)}$$

Since this Sh problem is uncoupled it is not necessary that $G(k)$ for the Sh problem be equivalent to $G(k)$ for the P-Sv problem though the physics of the problem would lead this to be the most obvious application. For small arguments we make the substitutions for sinh and the additional substitution

$$H_2^{(1)}(z) \simeq \frac{4}{i\pi z^2} \qquad \text{(A-26)}$$

to yield

$$n_o(k) = \frac{SG(k)\Delta\sigma}{8\pi\mu\sqrt{\sigma^2-h^2}} \qquad \text{(A-27)}$$

with the previous definition of Δ applying.

If we set Q equal to zero then we have the solutions for the cases given in Heelan's second paper (1953b) and Brekhovskikh's book (1960, 1980) where there is no axial stress applied.

Brekhovskikh's Algebra

P-Sv **case**

Brekhovskikh (1960, 1980) writes down Heelan's results for radial stress in his own notation (Eq. 33.4, 1980) but doesn't justify any of the algebraic steps. In light of the criticism Heelan's work we feel this is necessary. Although Heelan's algebra has been shown to be correct his choice of contour since it is unknown is not beyond reproach. The Weyl integral and the contour Brekhovskikh uses is well known and the algebra will be fully discussed here.

We begin with the same boundary conditions Eq. A-1 and we have our potentials in Brekhovskikh's notation setting aside the integrals with respect to k as

$$\begin{aligned} \phi &= f_o(k,\theta)H_0^{(1)}(kr\sin\theta)e^{-ikz\cos\theta} \\ \psi &= g_o(k,\theta)H_0^{(1)}(kr\sin\theta)e^{-i\kappa z\cos\gamma} \end{aligned} \qquad \text{(A-28)}$$

Our sources are expressed as follows

$$PG(k) = \frac{P}{\pi} \int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} \frac{\sin(lk\cos\theta)}{\cos\theta} e^{-ikz\cos\theta} \sin\theta d\theta \tag{A-29}$$

$$QG(k) = \frac{Q}{\pi} \int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} \frac{\sin(lk\cos\theta)}{\cos\theta} e^{-ikz\cos\theta} \sin\theta d\theta$$

$$SG(k) = \frac{S}{\pi} \int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} \frac{\sin(lk\cos\theta)}{\cos\theta} e^{-ikz\cos\theta} \sin\theta d\theta$$

where $kV = \kappa v, k\sin\theta = \kappa \sin\gamma$.

As with Heelan's work we make a substitution ϱ into ψ to obtain

$$\begin{aligned} \phi &= f_o(k,\theta) H_0^{(1)}(kr\sin\theta) e^{-ikz\cos\theta} \\ \psi &= g_o(\varrho,k,\theta) H_0^{(1)}(\varrho r) e^{-ikz\cos\theta} \end{aligned} \tag{A-30}$$

One will notice that derivatives with respect to z have changed sign from Heelan's to Brekhovskikh's treatment from being terms in $\sqrt{\sigma^2 - k^2}$ to $-ik\cos\theta$. We set up our reformulated boundary condition matrix with the substitution of the potentials from Eq. A-30 as

$$\begin{aligned} &-QG(k)\frac{\sin(lk\cos\theta)e^{-ikz\cos\theta}\sin\theta}{\pi\cos\theta} = \\ &\quad f_o(2i\mu k^2 \sin\theta\cos\theta) H_1^{(1)}(ka\sin\theta) e^{-ikz\cos\theta}\sin\theta \\ &\quad -g_o\left(2\mu\varrho k^2\cos^2\theta - \mu\varrho k^2\frac{V^2}{v^2}\right) H_1^{(1)}(\varrho a) e^{-ikz\cos\theta}\sin\theta \end{aligned} \tag{A-31}$$

$$\begin{aligned} &-PG(k)\frac{\sin(lk\cos\theta)e^{-ikz\cos\theta}\sin\theta}{\pi\cos\theta} = \\ &\quad f_o\left((-\lambda k^2 - 2\mu k^2\sin^2\theta) H_0^{(1)}(ka\sin\theta) + 2\mu k\sin\theta\frac{H_1^{(1)}(ka\sin\theta)}{a}\right) e^{-ikz\cos\theta}\sin\theta \\ &\quad -g_o(2i\mu k\cos\theta\varrho^2)\left[H_0^{(1)}(\varrho a) - \frac{H_1^{(1)}(\varrho a)}{\varrho a}\right] e^{-ikz\cos\theta}\sin\theta \end{aligned}$$

The $e^{-ikz\cos\theta}\sin\theta$ terms will be discarded henceforth.

Our denominator $p_{rz1}p_{r2} - p_{rz2}p_{r1}$ is after some simplification

$$\begin{aligned} &p_{rz1}p_{r2} - p_{rz2}p_{r1} = \\ &\quad (4\mu^2 k^3\cos^2\theta\sin\theta\varrho^2) H_0^{(1)}(\varrho a) H_1^{(1)}(ka\sin\theta) \qquad \text{(A-32)} \\ &\quad -(-\lambda k^2 - 2\mu k^2\sin^2\theta)(-2\mu\varrho k^2\cos^2\theta + \mu\varrho k^2\frac{V^2}{v^2}) H_1^{(1)}(\varrho a) H_0^{(1)}(ka\sin\theta) \qquad \text{(A-33)} \end{aligned}$$

$$\frac{-2\mu^2 k^3 \sin\theta V^2}{v^2 a} \varrho H_1^{(1)}(\varrho a) H_1^{(1)}(ka\sin\theta) \tag{A-34}$$

where $e^{-ikz\cos\theta}$ terms have been left off. We again find that for f_o our transformation in ϱ can be neglected and that our $H_0^{(1)}$ terms (Eq. A-32,Eq. A-33) can be discarded

So our denominator (Eq. A-34) can be written

$$-\frac{2\mu^2 k^3 \sin\theta V^2}{av^2} \varrho H_1^{(1)}(\varrho a) H_1^{(1)}(ka\sin\theta) \tag{A-35}$$

and for our numerators for the solution of f_o $(-QG(k)p_{r2} + PG(k)p_{rz2})$ we have

$$\begin{aligned} PG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}[-2\mu\varrho k^2\cos^2\theta + \mu k^2\frac{V^2}{v^2}\varrho]H_1^{(1)}(\varrho a)+ \\ -QG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}(\frac{2i\mu k\cos\theta\varrho}{a})H_1^{(1)}(\varrho a) \end{aligned} \tag{A-36}$$

and for our numerators for the solution of g_o $(-PG(k)p_{rz1} + QG(k)p_{r1})$ we have

$$\begin{aligned} -PG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}(2i\mu k^2\sin\theta\cos\theta)H_1^{(1)}(ka\sin\theta) \\ +QG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}(\frac{2\mu k\sin\theta}{a})H_1^{(1)}(ka\sin\theta) \end{aligned} \tag{A-37}$$

where again terms in H_0 have been neglected. As with the results in Heelan, we solve for f_o by dividing through with the denominator Eq. A-35 to obtain

$$PG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}(\frac{2v^2}{V^2}\cos^2\theta - 1)\frac{a}{2\mu k\sin\theta H_1^{(1)}(ka\sin\theta)} \tag{A-38}$$

$$+QG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}\frac{iv^2\cos\theta}{\mu V^2 k^2\sin\theta H_1^{(1)}(ka\sin\theta)} \tag{A-39}$$

Using the following expansions for small arguments

$$\begin{aligned} \sin x &\simeq x \\ H_1^{(1)}(z) &\simeq \frac{2}{i\pi z} \end{aligned} \tag{A-40}$$

we then obtain

$$f_o = +iPG(k)[(\frac{2v^2}{V^2})\cos^2\theta - 1](\frac{k\Delta}{8\pi\mu}) - QG(k)\frac{A\cos\theta}{8\pi\mu}\frac{v^2}{V^2} \tag{A-41}$$

where Δ, A have been previously defined as volume and length of the empty cavity of length $2l$. In Heelan's work the term $(\frac{2v^2}{V^2}\cos^2\theta - 1)$ is presented as $(1 - \frac{2v^2}{V^2}\cos^2\theta)$.

For g_o we divide through by the denominator to obtain

$$+PG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}\frac{iav^2\cos\theta}{\mu kV^2\varrho H_1^{(1)}(\varrho a)}$$
$$-QG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}\frac{v^2}{\mu V^2k^2\varrho H_1^{(1)}(\varrho a)} \tag{A-42}$$

We perform the same expansions for small argument to obtain

$$-PG(k)\frac{\pi a^2lv^2\cos\theta}{2\pi\mu V^2}$$
$$-QG(k)\frac{i\pi alv^2}{2\pi\mu V^2k} \tag{A-43}$$

Terms in ϱ have dropped out leaving

$$-PG(k)\frac{\Delta v^2\cos\theta}{4\pi\mu V^2}$$
$$-QG(k)\frac{iAv^2\cos\theta}{8\pi\mu V^2k\cos\theta} \tag{A-44}$$

Now making the inverse transformation for ϱ, $k\cos\theta=\kappa\cos\gamma$ we write for g_o

$$-PG(k)\frac{\Delta v^2\cos\theta}{4\pi\mu V^2}$$
$$-QG(k)\frac{iAv^2\cos\theta}{8\pi\mu V^2\kappa\cos\gamma} \tag{A-45}$$

The terms in $PG(k)$ were given by Brekhovskikh (Eq. 33.6, 1980) and the terms in $QG(k)$ are presented here for the first time. There is a sign change in our results for g_o which is as yet unresolved. Steepest descent analysis verifies that th achieve Heelan's results the sign used by Brekhovskikh is necessary. However, this is no cause for concern because of symmetry in both the horizontal and vertical axes

Sh case

The solution for Sh radiation, purely torsional motion is particularly simple - we write the equation without transformation and some simplification as

$$-SG(k)\frac{\sin(lk\cos\theta)}{\pi\cos\theta}e^{-i\kappa z\cos\gamma}=n_o\mu k^2\sin^2\theta H_2^{(1)}(ka\sin\theta)e^{-ikz\cos\theta} \tag{A-46}$$

where we have used the derivative operator for $H_1^{(1)}(z) = -H_2^{(1)}(z) + \frac{H_1^{(1)}(z)}{z}$. We now apply our expansions for small argument with another expansion, namely e^z can be expanded in the Taylor series and for small argument we only keep the first term z. Thus we write as in Brekhovskikh (1960, 1980, Eq. 33.6)

$$n_o = SG(k)(\frac{\cos\theta}{\cos\gamma})(\frac{vk\Delta}{8\pi i\mu V}) \tag{A-47}$$

The number 8 in the numerator of Eq. A-47 is given as the number 4 in Brekhovskikh's work but we believe this is a misprint. The number 8 abides by Heelan's development bearing in mind the different algebraic formulation. The number is self correcting in subsequent steepest descent analysis leading us to believe it is only typographical in nature.

Appendix B. TRANSFORMATION OF A FOURIER INTEGRAL OPERATOR

This short appendix will show the transformation of the Fourier integral operator used in Heelan's paper (1953a) and also used by other authors (Brekhovskikh, 1960, 1980; Pilant, 1978; Gilbert, 1964) to a well-recognized standard form. Issues of integrability, differentiability and continuity can be found in books on Fourier integrals (Wiener, 1933) and will not be addressed here. Heelan uses the operator $Re \int_0^\infty F(\omega)e^{ikVt}d\omega$. Keep in mind that for this analysis the time dependence, the sign of i, is irrelevant. Also since kV can be thought of in terms of frequency I will use the more common ω notation.

The Fourier integral theorem (Spiegel, Eq. 8.3, 1971; Bracewell, 1978) can be written in the following forms

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\omega(x-u)du d\omega \\ f(x) &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\omega(x-u)}du d\omega \\ f(x) &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} e^{i\omega x}\int_{-\infty}^{\infty} f(u)e^{-i\omega u)}du d\omega \end{aligned} \tag{B-1}$$

Let's consider the integral

$$\frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\omega(x-u)}du d\omega = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\omega(x-u)du d\omega + i\frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{-\infty}^{\infty} f(u)\sin\omega(x-u)a \tag{B-2}$$

Now for $f(u)$ defined real the two integrals on the right hand side will be real and imaginary respectively. If we take the real part of Eq. B-2 we obtain

$$\frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\omega(x-u)\,du\,d\omega \tag{B-3}$$

which is just equal to Eq. B-1. We can rewrite this as

$$\mathrm{Re}\frac{1}{\pi}\int_{\omega=0}^{\infty} F(\omega)e^{i\omega x}d\omega \tag{B-4}$$

where $F(omega)$ is defined to be

$$\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du \tag{B-5}$$

Thus we have proved that if $f(x)$ is real it can be represented by the following operator

$$\mathrm{Re}\frac{1}{\pi}\int_{\omega=0}^{\infty} F(\omega)e^{i\omega x}d\omega \tag{B-6}$$

showing Heelan's use of it was valid.