Conservation in Signal Processing Systems

by

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Abstract

Conservation principles have played a key role in the development and analysis of many existing engineering systems and algorithms. In electrical network theory for example, many of the useful theorems regarding the stability, robustness, and variational properties of circuits can be derived in terms of Tellegen's theorem, which states that a wide range of quantities, including power, are conserved. Conservation principles also lay the groundwork for a number of results related to control theory, algorithms for optimization, and efficient filter implementations, suggesting potential opportunity in developing a cohesive signal processing framework within which to view these principles. This thesis makes progress toward that goal, providing a unified treatment of a class of conservation principles that occur in signal processing systems. The main contributions in the thesis can be broadly categorized as pertaining to a mathematical formulation of a class of conservation principles, the synthesis and identification of these principles in signal processing systems, a variational interpretation of these principles, and the use of these principles in designing and gaining insight into various algorithms. In illustrating the use of the framework, examples related to linear and nonlinear signal-flow graph analysis, robust filter architectures, and algorithms for distributed control are provided.

Thesis Supervisor: Alan V. Oppenheim
Title: Ford Professor of Engineering

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Chapter 1

Introduction

Conjugate effort and flow variables are deeply connected to our understanding of physical systems. Also referred to as "effort" and "flow" variables or "across" and "through" variables, conjugate variables represent physical quantities that when multiplied together indicate the amount of power consumed or generated by a given system. In physical systems that are assembled as a lossless interconnection of physical subsystems, the total power consumed or produced by the interconnection is zero, i.e. power is conserved. A lossless physical interconnection of K subsystems, each with conjugate effort and flow variables denoted e_k and f_k respectively, therefore has a conservation law that may be written as

$$e_1 f_1 + \cdots + e_K f_K = 0. \tag{1.1}$$

In such physical systems, Eq. 1.1 holds independent of whether the interconnected subsystems are linear or nonlinear, time-invariant or time-varying, or deterministic or stochastic. As such, the use of Eq. 1.1 in the derivation of useful mathematical theorems about physical systems often implies not only that the theorems apply very broadly, but also that the application of linear or nonlinear transformations may be used as a tool in the corresponding derivations. And furthermore, such transformations may be used to modify existing theorems in arriving at additional related results.

One may look to electrical networks to find a very broad class of such theorems originating from equations of the form of Eq. 1.1. In this class of physical systems, Eq. 1.1 is embodied by Tellegen's Theorem, [34] and a comprehensive summary of many of the accompanying theorems, which address among other things stability, sensitivity, and variational principles in electrical networks, is found in [31]. Eq. 1.1 also forms a cornerstone of the bond graph methodology, applied widely in the analysis, design and control of mechanical, thermal, hydraulic, electrical, and other physical systems. [30,35] The bond graph framework has also been applied in the analysis of social and economic systems as well. [6]

In contrast to physical systems, many current signal processing architectures, including general-purpose computers and digital signal processors, implement algorithms in a way that is often far-removed from the physics underlying their implementation. One advantage to this is that a wide range of signal processing algorithms can be realized that might otherwise be difficult or impossible to implement directly in discrete physical devices, including for example transform-based coding, cepstral processing, and adaptive filtering. However, the high degree of generality facilitated by these types of architectures comes with the expense of losing some of the powerful analytic tools traditionally applied in the design and analysis of the restricted set of systems that is allowed physically, and derivations of many of these tools stem from equations of the form of Eq. 1.1.

A common strategy to overcome this essentially involves designing signal processing algorithms that mimic the equations or sets of equations describing a specific physical system or class of physical systems. Any signal processing algorithm that can be put in the form of the equations is then regarded as being of a special class, to which a wide range of theorems often apply. For example, the class of signal processing systems consisting of two subsystems interconnected to form a feedback loop is a canonical representation into which it is often desirable to place control systems, and about which many useful results are known. And in the early work by Zames [43,44] describing open-loop conditions for closed-loop stability in this class of systems, the equivalent electrical network is often referenced.

Another class of signal processing algorithms developed in this spirit is the wave-digital class of structures, which are based on the equations describing physical microwave filters, and which have exceptional stability and robustness properties, even in the presence of parameter perturbations. [18] These properties were originally proven by drawing analogies to reference physical microwave systems, which are known to have similar characteristics. [17] The stability properties of other signal processing structures, such as lattice filters, have likewise been determined by manipulating them to fit the form of the equations describing wave-digital filters.

This strategy has also been used in the field of optimization. The network-based optimization algorithm developed by Dennis to solve the multi-commodity network flow problem was derived by designing a reference electrical network, with the network "content" being equivalent to the cost function in the original optimization. [14, 15] Chua also discussed the use of nonreciprocal elements such as operational amplifiers in realizing the idealized components in Dennis' formulation, in addition to those required in a broader range of nonlinear problems. [8,9] In Dennis' work, the question of finding an optimal set of primal and dual decision variables, shown by Dennis to be equivalent to voltages and currents in the network, also involved ensuring that the network would indeed reach steady state, i.e. it involved ensuring stability of the network. Theorems regarding the stationarity of network content and the stability of electrical networks can be derived by starting with Tellegen's Theorem, which as was previously mentioned takes the form of Eq. 1.1.

Indeed conservation principles are at work in a wide class of useful systems and algorithms, and this suggests potential opportunity in developing a cohesive signal processing framework within which to view them. This thesis makes progress toward that goal, providing a unified treatment of a class of conservation principles related to signal processing algorithms, and enriching and providing new connections between these principles and key fields of application. The main contributions in the thesis can be broadly categorized as pertaining to:

- the mathematical formulation of a class of conservation principles,

- the synthesis and identification of these principles in signal processing systems,
- a variational interpretation of these principles, and
- the use of these principles in designing and gaining insight into specific algorithms.

Specifically, in Chapter 2 we review various forms of system representation that will be useful in discussing conservation, and we present a theorem pertinent to translating between them. There are a variety of conservation principles in the literature that, in an appropriate basis, are reminiscent of Eq. 1.1, and we establish a framework in Chapter 3 for placing these on equal footing. Also in Chapter 3 we use the theory of Lie groups to address the question of what vector spaces constraining the variables in the left-hand side of Eq. 1.1 result in the right-hand side of Eq. 1.1 evaluating to zero. Chapter 4 further interprets this result within the context of signal-flow graphs and electrical network theory, providing graph-based techniques for synthesizing conservative interconnections and identifying conservation in pre-specified interconnections.

As is the case with electrical networks, a conservative interconnection can in many cases be viewed as operating at a stationary point of a functional, and in Chapter 5 we present a multidimensional stationarity principle that generalizes the variational principles previously established in electrical network theory to a broader class of systems commonly encountered in signal processing algorithms. Also in Chapter 5 we use the tools of optimization theory and convex analysis to gain further insight into the meaning of these principles, and we discuss their time dynamics, pertinent to algorithms where time is a meaningful quantity. Chapter 6 illustrates with examples the application of the principles established in Chapters 2 through 5.

Chapter 2

System representations and manipulations

In physical systems, conservation pertains to constraints in a system, rather than which system variables, if any, are considered inputs or outputs. However, many signal processing systems are specified using an input-output representation. In this chapter we discuss the relationship between these and other system representations that will be used in the remainder of the thesis. The chapter begins by reviewing the behavioral representation of Willems [39], input-output representations including linear and nonlinear signal-flow graphs, and the related topic of image representations. A theorem related to performing manipulations between these representations is presented, and in the process we present a theorem for system inversion that generalizes the flow graph reversal theorems of Mason and Kung [24, 25] to linear and nonlinear systems represented as a general interconnection of maps.

2.1 Behavioral representations

The basic idea underlying the behavioral representation, a complete treatment of which can be found in [39], is that of viewing systems not as maps from sets of input variables to output variables, but rather as constraints between variables, some of which may be system inputs; others, system outputs; and still others for which the designation of input or output might be ambiguous. The convention is that "the behavior" of a system refers to the entire collection of sets of system variables that are consistent with the constraints imposed by the system.

Given a system R that represents constraints between a total of K variables $x_1, \ldots, x_K$, its behavior may be written formally as the set $\mathcal{S}$ of those length-K vectors of system variables that are permitted by the constraints imposed by R. The variables x_k may in general be arbitrary mathematical objects, and in this thesis we will mainly be concerned with variables that represent some type of signal or scalar quantities.

An interconnection of systems is addressed in a straightforward way from the behavioral viewpoint. In particular, the behavior resulting from an interconnection of any two systems, interconnected via variable sharing, is the intersection of the behaviors of the uncoupled systems. For example, given two systems R and R$'$ each having a total of K variables $x_1, \ldots, x_K$ and $x'_1, \ldots, x'_K$ and having respective behaviors $\mathcal{S}$ and $\mathcal{S}'$, the interconnected system obtained by sharing variables as

$$\begin{aligned} x_1 &= x'_1 \\ &\vdots \\ x_K &= x'_K \end{aligned} \tag{2.1}$$

has the behavior $\mathcal{S}_i$ specified by

$$\mathcal{S}_i = \mathcal{S} \cap \mathcal{S}'. \tag{2.2}$$

2.2 Input-output representations

The basic idea in input-output representations is to specify the components of a system using functional relationships, i.e. using a function or functions of the form $M : \mathcal{C} \rightarrow \mathcal{D}$ that map every element of an input set $\mathcal{C}$ to a unique element of the output set $\mathcal{D}$. Given an input element $c \in \mathcal{C}$, the corresponding output element $d \in \mathcal{D}$

is related to c as

$$d = M(c). \tag{2.3}$$

The convention in this thesis will be that whenever the term "function" is used, it will refer to a relationship where each element in an input set is mapped to a unique output element. From a behavioral viewpoint, the function in Eq. 2.3 has a behavior $\mathcal{S}$ that may be written as

$$\mathcal{S} = \left\{ \begin{bmatrix} c \\ M(c) \end{bmatrix} : c \in \mathcal{C} \right\}, \tag{2.4}$$

i.e. its behavior is the set of pairs of variables c and d that are consistent with Eq. 2.3.

2.2.1 Linear and nonlinear signal-flow graphs

There are several common forms of system representation within the more general class of input-output representations, and a particularly pervasive subclass is that of linear and nonlinear signal-flow graphs. In this form of representation, signal processing systems are described by a collection of nodes and associated node variables, connected using branch functions that may generally be linear or nonlinear. The value of a node variable is the sum of the output variables from the incident branch functions that are directed toward the node, in addition to possible contribution from an external input, and the node variable is used as an input to the incident branch functions that are directed away from the node.

In continuous- and discrete-time signal-flow graphs where the instantaneous values of the node variables are real scalars, a given node variable w_k in a signal-flow graph containing P nodes may be related to the branch variables v_{jk} as

$$w_k = \sum_{j=1}^{N_k} v_{jk} \ (+x_k), \quad k = 1, \ldots, P, \tag{2.5}$$

where v_{jk} represents the output value of the branch that connects node j to node k, with the total number of such branches directed toward node k denoted N_k, and where x_k is a potential external input to the node. A given branch variable may accordingly be written as

$$v_{jk} = M_{jk}(w_j), \tag{2.6}$$

where $M_{jk} : \mathbb{R} \to \mathbb{R}$ is the branch function that maps from the value of the variable at node j to the contribution of the branch to the variable at node k. [28] [29]

A pertinent question is that of whether a signal-flow graph represented as an interconnection of functions implements an overall functional relationship, and the examples depicted in Fig. 2-1 illustrate that this is generally not the case. Referring to this figure, the input and output variables in systems (a)-(c) satisfy the respective equations $d_a = 2c_a$, $d_b = c_b/2$ and $c_c = 0$. As such, the relationships between the input and output variables in systems (a) and (b) are functions. For system (c), the output variable d_c may take on any value as long as the input variable c_c is zero, and we say that the system cannot be realized as a function from c_c to d_c.

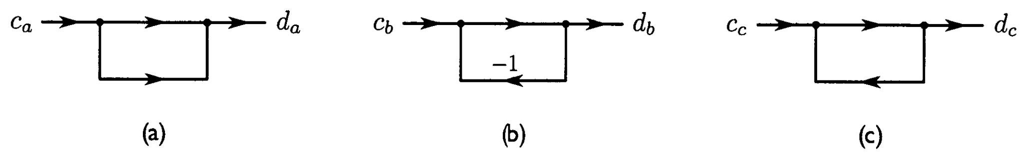


Figure 2-1: (a) A signal-flow graph that is a function. (b) A signal-flow graph that is a function and contains a closed loop. (c) A signal flow graph that is not a function.

The issue of whether a signal-flow graph is a map will be especially relevant in systems that are implemented using a technology that necessitates the functional dependency of outputs on inputs, as with digital signal processors and general-purpose computers. The issue will be less critical in implementations that make use of, e.g., analog and continuous-time technology, although the question of whether a system is a map will still in this domain provide insight into whether an observed output value is unique.

Systems (b) and (c) in Fig. 2-1 are also examples of signal-flow graphs that contain closed loops, i.e. loops containing no storage elements, and a natural question is that of what bearing this has, if any, on whether a signal-flow graph implements an overall functional relationship. In discrete-time systems, the existence of delay-free loops, a subclass of closed loops, is related to whether the overall signal-flow graph implements a function. As was shown in [12], a discrete-time signal-flow graph having causal branch functions that contains no delay-free loops is known to be a function itself, since it is computable. However, as is illustrated by systems (b) and (c) in Fig. 2-1, the existence of a delay-free loop does not imply anything in general about whether a system is a function.

2.2.2 Interconnective systems

We also call attention to a class of input-output representations where the behaviors of subsystems are separated from the relationships that couple them together, as has been done in, e.g., [4,37,38]. From this perspective, a system is viewed as having two parts: constitutive relations, e.g. a set of systems that are uncoupled from one another, and an interconnecting system to which the subsystems and the overall system input and output are connected. The variables that are shared by the interconnecting system and the constitutive relations are referred to as the interconnection terminal variables, and each such variable may either be an input to or output from the interconnection. The designation of whether each interconnection terminal variable is an interconnection input or an interconnection output will be referred to as the input-output configuration. We will refer to this form of system representation as an **interconnective** representation.

In an interconnective representation, the constitutive relations and the interconnecting system may all be possibly nonlinear and time-varying systems that are allowed to have memory. The key point of the representation is to emphasize the distinction between the many independent constitutive subsystems, which are individually connected to a common interconnecting subsystem. Many of the results in Chapters 3 and 4 will pertain to the interconnecting component of an overall system in an interconnective representation, and as such will have the convenient property that they will not depend on the specific behaviors of the constitutive relations, facilitating their application in a variety of systems. Example interconnective representations for a generic system and for the feedback system in Fig. 3-1 are respectively illustrated in Figs. 2-2 and 2-3.

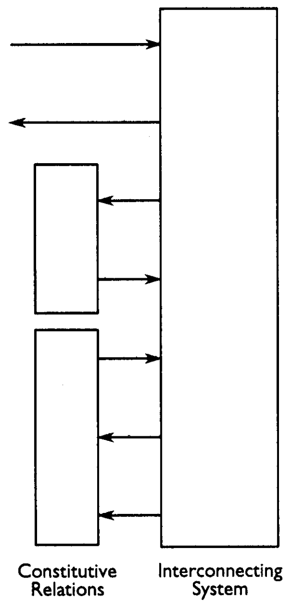


Figure 2-2: An interconnective representation of a generic signal processing system.

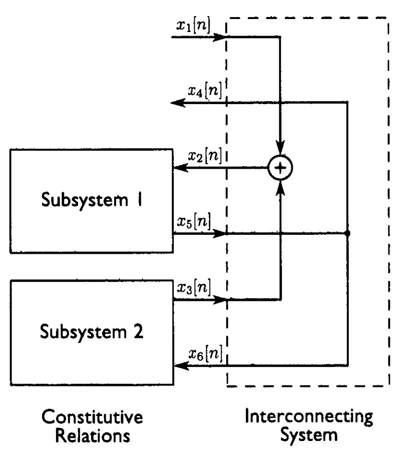


Figure 2-3: An interconnective representation of the feedback system in Fig. 3-1.

As was previously mentioned, the behavior of an interconnection of systems is the intersection of the behaviors of the individual systems, and the interconnected system in Fig. 2-3 illustrates this. Referring to this figure, if we represent Subsystem 1 and

Subsystem 2 using functions M_1 and M_2 for which

$$x_5[n] = M_1(x_2[n]) \tag{2.7}$$

and

$$x_3[n] = M_2(x_6[n]), \tag{2.8}$$

then the behavior $\mathcal{S}_r$ of the uncoupled constitutive relations may be written as

$$\mathcal{S}_r = \left\{ \begin{bmatrix} x_1[n] \\ x_4[n] \\ x_2[n] \\ M_1(x_2[n]) \\ M_2(x_6[n]) \\ x_6[n] \end{bmatrix} : \begin{bmatrix} x_1[n] \\ x_4[n] \\ x_2[n] \\ x_6[n] \end{bmatrix} \in \mathcal{C}^4 \right\}, \tag{2.9}$$

where $\mathcal{C}^4 = \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ is used to denote the set of allowable signals over which the relationships in the system are defined. The behavior $\mathcal{S}_c$ of the interconnecting system is likewise written as

$$\mathcal{S}_c = \left\{ \begin{bmatrix} x_1[n] \\ x_5[n] \\ x_1[n] + x_3[n] \\ x_5[n] \\ x_3[n] \\ x_5[n] \end{bmatrix} : \begin{bmatrix} x_1[n] \\ x_5[n] \\ x_3[n] \end{bmatrix} \in \mathcal{C}^3 \right\}, \tag{2.10}$$

and the interconnected behavior $\mathcal{S}_i$ is the set of all signals consistent with both behaviors, i.e.

$$\mathcal{S}_i = \mathcal{S}_r \cap \mathcal{S}_c. \tag{2.11}$$

2.3 Image representations

In moving between input-output and behavioral representations, it will be useful to refer to systems for which all of the terminal variables x_k are viewed as outputs that are driven by a set of hidden internal variables ϕ_k. This type of system representation is referred to as an image representation, [2,39] reflective of the fact that the behavior of the system is the image of a function M that relates the internal variables to the terminal variables as

$$M\left(\begin{bmatrix}\phi_1\\ \vdots \\ \phi_J\end{bmatrix}\right) = \begin{bmatrix}x_1\\ \vdots \\ x_K\end{bmatrix}, \tag{2.12}$$

with the behavior of the system being written in the case where ϕ_k are real as

$$\mathcal{S} = \left\{ M\left(\begin{bmatrix}\phi_1\\ \vdots \\ \phi_J\end{bmatrix}\right) : \begin{bmatrix}\phi_1\\ \vdots \\ \phi_J\end{bmatrix} \in \mathbb{R}^J \right\}. \tag{2.13}$$

As an example illustrating this, an image representation for the interconnecting component of the system in Fig. 2-3 is depicted in Fig. 2-4. Referring to this figure, the form of the expression for its behavior in Eq. 2.10 is reflected in the structure of the system. It will often be the case that an expression for the behavior of an input-output system will be suggestive of an image representation.

Image representations will also be useful in realizing input-output representations of systems, given a pre-specified behavior. The general strategy in doing this, depicted in Fig. 2-5, will be to begin with a behavior that is specified in terms of a function from a set of hidden input variables to the set of output terminal variables. The task will then be to perform system manipulations on the image representation to arrive at a behaviorally-equivalent system where the hidden variables are instead outputs. At this point there will be no dependence on the hidden variables, and the resulting system may be regarded as implementing a functional relationship between the terminal variables. Section 2.4 discusses the specifics of a class of system manipulations

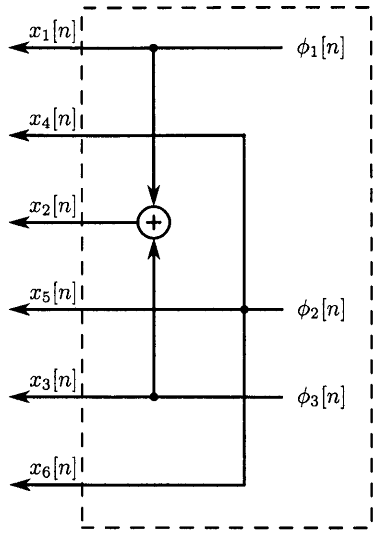


Figure 2-4: Image representation for the interconnecting component of the system in Fig. 2-3.

that will be useful in doing this.

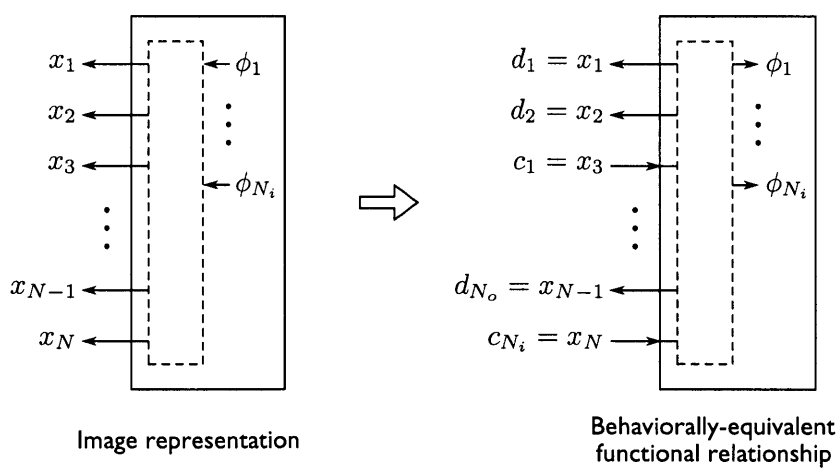


Figure 2-5: The general strategy behind obtaining a functional relationship from an image representation of a system.

2.4 Manipulations between representations

In viewing signal processing systems from the perspectives of the previously-mentioned representations, it will often be the case that the representation in which a useful result is most directly stated will be different from the domain in which it is implemented. As an example of this, in Chapter 3 we will discuss conservation from a behavioral perspective, and in Chapter 4 these principles will be related to signal-flow graph representations. This section establishes some tools for translating between these domains.

2.4.1 Behaviorally-equivalent, multiple-input, multiple-output systems

Given a pre-specified behavior, there may in general be a number of different functions or interconnections of functions to which the behavior corresponds. As a straightforward example, consider a map that is invertible in the sense that it is both a one-to-one and onto mapping from the set of variables in its domain to the set of variables in its codomain. It is straightforward to show that the inverse map is behaviorally-equivalent to the forward map, with the input and output variables exchanged. Written formally, function $M : \mathcal{C} \to \mathcal{D}$ has as its behavior the set of allowable (c, d) variable pairs given by Eq. 2.4. If M is invertible, the behavior of the inverse function $M^{-1} : \mathcal{D} \to \mathcal{C}$ is in turn given by

$$\mathcal{S}' = \left\{ \begin{bmatrix} M^{-1}(d) \\ d \end{bmatrix} : d \in \mathcal{D} \right\}. \tag{2.14}$$

As M is a one-to-one and onto correspondence between the sets $\mathcal{C}$ and $\mathcal{D}$, we may equivalently write

$$\begin{aligned} \mathcal{S}' &= \left\{ \begin{bmatrix} M^{-1}(M(c)) \\ M(c) \end{bmatrix} : c \in \mathcal{C} \right\} & (2.15) \\ &= \left\{ \begin{bmatrix} c \\ M(c) \end{bmatrix} : c \in \mathcal{C} \right\} & (2.16) \\ &= \mathcal{S}, & (2.17) \end{aligned}$$

and we say that M and M^{-1} are behaviorally equivalent. Behavioral equivalence of inverse systems lays the groundwork for a number of theorems regarding the inversion of linear and nonlinear systems, discussed in greater detail in [4]. In this thesis, there will not be a particular emphasis on inversion, although essentially any of the following results can be applied to that problem by drawing upon the behavioral equivalence property of inverse systems, e.g. Eqns. 2.15-2.17.

We have seen that for a single-input, single-output system, a behaviorally-equivalent function with the input and output configurations interchanged is an inverse. For systems with many inputs and outputs, the concept of obtaining behaviorally-equivalent systems will be useful in this thesis as well. We note that for the multiple-input, multiple output case, behavioral equivalence implies inversion only in the case where all of the input and output configurations are interchanged, and we will in general be interested in behaviorally-equivalent systems where some subset of configurations are interchanged.

Motivated by these considerations, the following theorem provides a necessary and sufficient condition under which the configuration for an input-output pair of terminal variables in a two-input, two-output function may be reversed, such that the resulting system is itself a valid map. As the domains and codomains of the input and output variables are allowed to be arbitrary and accordingly may themselves be sets of vectors or n-tuples of variables, the theorem is immediately applicable to general multiple-input, multiple output systems as well.

Theorem 2.1. *This theorem pertains to a two-input, two-output system, written as two functions* $M_1 : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}_1$ *and* $M_2 : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}_2$ *that each operate on a pair of variables* (c_1, c_2)*, with* $c_1 \in \mathcal{C}_1$ *and* $c_2 \in \mathcal{C}_2$*, such that* $M_1(c_1, c_2) = d_1$ *and* $M_2(c_1, c_2) = d_2$*, where* $d_1 \in \mathcal{D}_1$ *and* $d_2 \in \mathcal{D}_2$*. Then a behaviorally-equivalent pair of functions* $M_1' : \mathcal{D}_1 \times \mathcal{C}_2 \to \mathcal{C}_1$ *and* $M_2' : \mathcal{D}_1 \times \mathcal{C}_2 \to \mathcal{D}_2$ *exists, if and only if each of the functions* $M_1^{(c_2)} : \mathcal{C}_1 \to \mathcal{D}_1$*, defined as*

$$M_1^{(c_2)}(c_1) \equiv M_1(c_1, c_2), \tag{2.18}$$

is an invertible function for all $c_2 \in \mathcal{C}_2$*. Writing the behavior of the original pair of functions as*

$$\mathcal{B} = \left\{(c_1, c_2, M_1(c_1, c_2), M_2(c_1, c_2)) : c_1 \in \mathcal{C}_1, c_2 \in \mathcal{C}_2\right\}, \tag{2.19}$$

and writing the behavior of the primed pair of functions as

$$\mathcal{B}' = \left\{(M_1'(d_1, c_2), c_2, d_1, M_2'(d_1, c_2)) : d_1 \in \mathcal{D}_1, c_2 \in \mathcal{C}_2\right\}, \tag{2.20}$$

the specific notion of behavioral equivalence is that $\mathcal{B} = \mathcal{B}'$*. A summary of the result in this theorem is illustrated in Fig. 2-6.*

Proof. We first show that invertibility of $M_1^{(c_2)}$ for all $c_2 \in \mathcal{C}_2$ implies that a pair of primed functions exists that are behaviorally equivalent to the original pair. In doing so we explicitly define M_1' using the inverse of $M_1^{(c_2)}$, i.e.

$$M_1'(d_1, c_2) \equiv {M_1^{(c_2)}}^{-1}(d_1), \tag{2.21}$$

and we define M_2' in terms of M_2 and M_1' as

$$M_2'(d_1, c_2) \equiv M_2\left(M_1'(d_1, c_2), c_2\right), \tag{2.22}$$

with M_1' defined as in Eq. 2.21. The behavior $\mathcal{B}'$ of the primed system is accordingly

$$\mathcal{B}' = \left\{ \left(M_1^{(c_2)^{-1}}(d_1), c_2, d_1, M_2\left(M_1^{(c_2)^{-1}}(d_1), c_2 \right) \right) : d_1 \in \mathcal{D}_1, c_2 \in \mathcal{C}_2 \right\}. \tag{2.23}$$

As we have assumed that both of $M_1^{(c_2)}$ and $M_1^{(c_2)^{-1}}$ are invertible functions for all $c_2 \in \mathcal{C}_2$, we may perform the substitution $c_1 = M_1^{(c_2)^{-1}}(d_1)$ and write

$$\begin{aligned} \mathcal{B}' &= \left\{ \left(c_1, c_2, M_1^{(c_2)}(c_1), M_2(c_1, c_2) \right) : M_1^{(c_2)}(c_1) \in \mathcal{D}_1, c_2 \in \mathcal{C}_2 \right\} && (2.24) \\ &= \left\{ \left(c_1, c_2, M_1(c_1, c_2), M_2(c_1, c_2) \right) : c_1 \in \mathcal{C}_1, c_2 \in \mathcal{C}_2 \right\}, && (2.25) \end{aligned}$$

resulting in $\mathcal{B}' = \mathcal{B}$.

In showing that the existence of a behaviorally-equivalent pair of primed functions implies that $M_1^{(c_2)}$ is invertible for all $c_2 \in \mathcal{C}_2$, we proceed by proving the contrapositive statement, "if $M_1^{(c_2)}$ is not invertible for all $c_2 \in \mathcal{C}_2$, then there does not exist a pair of primed functions that is behaviorally equivalent to the original pair." Let $\hat{c}_2 \in \mathcal{C}_2$ denote a value corresponding to a function $M_1^{(\hat{c}_2)}$ that is not invertible. Then for this function $M_1^{(\hat{c}_2)}$ there exist at least two distinct input values that correspond to the same output value, i.e. there exist values $c_1' \in \mathcal{C}_1$ and $c_1'' \in \mathcal{C}_1$, $c_1' \neq c_1''$, such that

$$M_1^{(\hat{c}_2)}(c_1') = M_1^{(\hat{c}_2)}(c_1''), \tag{2.26}$$

or equivalently, such that

$$M_1(c_1', \hat{c}_2) = M_1(c_1'', \hat{c}_2) \tag{2.27}$$

The corresponding output value is denoted $\hat{d}_1 = M_1(c_1', \hat{c}_2) = M_1(c_1'', \hat{c}_2)$. We now have some information about two of the elements in the behavior of the original system, i.e. these elements are

$$\left(c_1', \hat{c}_2, \hat{d}_1, M_2(c_1', \hat{c}_2) \right) \in \mathcal{B} \tag{2.28}$$

and

$$\left(c_1'', \hat{c}_2, \hat{d}_1, M_2(c_1'', \hat{c}_2) \right) \in \mathcal{B}. \tag{2.29}$$

The pertinent question is whether behaviorally-equivalent primed functions exist, i.e. we are interested finding functions whose behavior, written as in Eq. 2.20, has as two of its elements the left-hand sides of Eqns. 2.28 and 2.29. However as $c_1' \neq c_1''$, no satisfactory function M_1' can exist. □

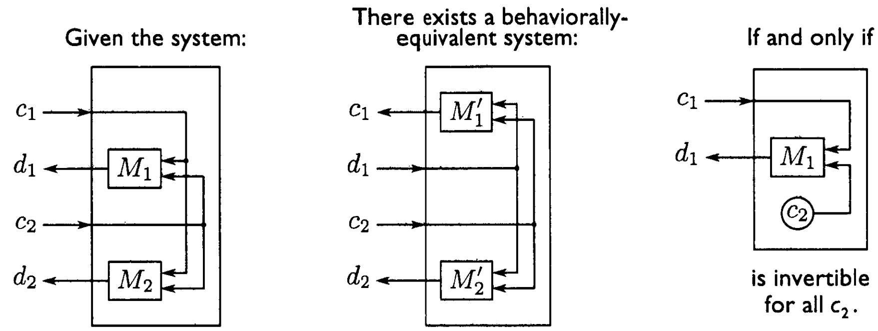


Figure 2-6: Illustration of Thm. 2.1.

For a system that is a multiple-input, multiple output linear map from a vector of N_i real input scalars to a vector of N_o real output scalars, the map may be represented in terms of a gain matrix $G : \mathbb{R}^{N_i} \longrightarrow \mathbb{R}^{N_o}$ as

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N_o} \end{bmatrix} = G \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N_i} \end{bmatrix}, \tag{2.30}$$

where each of the scalar coefficients c_k and d_k are real-valued. The behavior of the system may accordingly be written in the form of Eq. 2.4 as

$$\mathcal{B} = \left\{ \Big[c_1, c_2, \ldots, c_{N_i}, (G\mathbf{c})_1, (G\mathbf{c})_2, \ldots, (G\mathbf{c})_{N_o} \Big]^{tr} : \mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{N_i} \end{bmatrix}^{tr} \in \mathbb{R}^{N_i} \right\}, \tag{2.31}$$

with $(G\mathbf{c})_k$ indicating the value of entry k in the vector $(G\mathbf{c})$.[1] Writing the set $\mathcal{B}$ in

[1]In this thesis, boldface variables will specifically be used to denote column vectors in $\mathbb{R}^N$. Vectors in abstract vector spaces will generally be written as usual using italicized variables, i.e. we will write $\mathbf{x} \in \mathbb{R}^N$ and $x \in V$.

terms of the range of a block matrix as

$$\mathcal{B} = \text{range}\left(\begin{bmatrix} I_{N_i} \\ G \end{bmatrix}\right), \tag{2.32}$$

we see that the behavior of a linear map of the form of Eq. 2.30 is a vector space. The number of linearly-independent columns of the matrix in the right-hand side of Eq. 2.32 is the dimension of the vector space, and as such the vector space $\mathcal{B}$ has dimension N_i.

The following corollary illustrates the application of Thm. 2.1 to multiple-input, multiple output linear, maps of the form of Eq. 2.30. It is applicable to systems that can be represented from an input-output perspective as a matrix multiplication, as is the case with, e.g., linear, memoryless interconnections.

Corollary 2.1. *This corollary pertains to an N_i-input, N_o-output linear, memoryless system that accepts N_i real-valued scalars and produces N_o real-valued scalars, i.e. the system may be represented as a matrix multiplication of the form of Eq. 2.30, where $G : \mathbb{R}^{N_i} \longrightarrow \mathbb{R}^{N_o}$ is a real-valued matrix. Then a behaviorally-equivalent matrix G' exists for which*

$$\begin{bmatrix} c_1 \\ d_2 \\ \vdots \\ d_{N_o} \end{bmatrix} = G' \begin{bmatrix} d_1 \\ c_2 \\ \vdots \\ c_{N_i} \end{bmatrix}, \tag{2.33}$$

if and only if the gain from c_1 to d_1 through G is nonzero, i.e. if and only if $G_{1,1} \neq 0$. Writing the behavior of G as in Eq. 2.31 and the behavior of G' as

$$\mathcal{B}' = \left\{ \Big[(G'\mathbf{c})_1, c_2, \ldots, c_{N_i}, d_1, (G'\mathbf{c})_2, \ldots, (G'\mathbf{c})_{N_o}\Big]^{tr} : \mathbf{c} = \begin{bmatrix} d_1 & c_2 & \cdots & c_{N_i} \end{bmatrix}^{tr} \in \mathbb{R}^{N_i} \right\}, \tag{2.34}$$

the specific notion of behavioral equivalence is that $\mathcal{G} = \mathcal{G}'$.

Proof. We proceed by applying Thm. 2.1 to the operation of matrix multiplication by G as specified by Eq. 2.30. Referring to the notation in Thm. 2.1, the domains and codomains of the maps are $\mathcal{C}_1 = \mathbb{R}$, $\mathcal{C}_2 = \mathbb{R}^{N_i-1}$, $\mathcal{D}_1 = \mathbb{R}$, and $\mathcal{D}_2 = \mathbb{R}^{N_o-1}$. The associated maps $M_1 : \mathbb{R} \times \mathbb{R}^{N_i-1} \to \mathbb{R}$ and $M_2 : \mathbb{R} \times \mathbb{R}^{N_i-1} \to \mathbb{R}^{N_o-1}$ may accordingly be written in terms of G as

$$M_1\left(c_1, \begin{bmatrix} c_2 \\ \vdots \\ c_{N_i} \end{bmatrix}\right) = \left(G \begin{bmatrix} c_1 \\ \vdots \\ c_{N_i} \end{bmatrix}\right)_1 \tag{2.35}$$

and

$$M_2\left(c_1, \begin{bmatrix} c_2 \\ \vdots \\ c_{N_i} \end{bmatrix}\right) = \begin{bmatrix} \left(G \begin{bmatrix} c_1 \\ \vdots \\ c_{N_i} \end{bmatrix}\right)_2 \\ \vdots \\ \left(G \begin{bmatrix} c_1 \\ \vdots \\ c_{N_i} \end{bmatrix}\right)_{N_o} \end{bmatrix}. \tag{2.36}$$

The map M_1 may equivalently be written as a sum in terms of the elements of G, i.e.

$$M_1\left(c_1, \begin{bmatrix} c_2 \\ \vdots \\ c_{N_i} \end{bmatrix}\right) = G_{1,1}c_1 + G_{1,2}c_2 + \cdots + G_{1,N_i}c_{N_i}, \tag{2.37}$$

which is an invertible map for all $[c_2, \ldots, c_{N_i}]^{tr} \in \mathbb{R}^{N_1-1}$, if and only if $G_{1,1} \neq 0$. We therefore apply Thm. 2.1 and claim that a behaviorally-equivalent system exists. In showing that the system is linear, we select M_1' and M_2' as was done in the proof for Thm. 2.1, i.e.

$$M_1'\left(d_1, \begin{bmatrix} c_2 \\ \vdots \\ c_{N_i} \end{bmatrix}\right) = (d_1 - G_{1,2}c_2 - \cdots - G_{1,N_i}c_{N_i})/G_{1,1} \tag{2.38}$$

and

$$M_2'\left(d_1, \begin{bmatrix} c_2 \\ \vdots \\ c_{N_i} \end{bmatrix}\right) = M_2\left((d_1 - G_{1,2}c_2 - \cdots - G_{1,N_i}c_{N_i})/G_{1,1}, \begin{bmatrix} c_2 \\ \vdots \\ c_{N_i} \end{bmatrix}\right). \quad (2.39)$$

By inspection, the map M_1' is linear. As a composition of linear maps is itself linear, M_2' is linear as well. $\square$

2.4.2 Behaviorally-equivalent interconnections of functions

Thm. 2.1 provides a necessary and sufficient condition for behavioral equivalence, and a pertinent question is that of how such a behaviorally-equivalent system might be obtained from an original system. When we have a system represented in an interconnective form as in, e.g. Fig. 2-2, a convenient way of doing this will often be to perform behaviorally-equivalent modifications to the interconnecting system, with variable sharing between the original constitutive relations and modified interconnection resulting in an overall system that is behaviorally equivalent.

Given a pre-specified system represented as an interconnection of functions and a pair of terminal variables whose input-output configurations we wish to exchange, a convenient way of obtaining a behaviorally-equivalent system will specifically be to identify a functional path from the unmodified input to the unmodified output, and refer to this as the interconnecting system. Then the inputs and outputs to the interconnecting system are the overall input and output whose configurations we wish to exchange, in addition to the internal inputs and outputs connected to the functional path. The desired system can likewise be obtained by creating a behaviorally-equivalent path where the overall input and output configurations have been exchanged and the internal input and output configurations remain unmodified, using, e.g. the straightforward rules depicted in Fig. 2-7.

An example of the use of these rules in inverting the nonlinear system as discussed in [4, 7] is illustrated in Fig. 2-8. Referring to this figure, the elements along the outlined path in Fig. 2-8(a) are replaced with the behaviorally-equivalent elements depicted in Fig. 2-7, resulting in the inverse system in Fig. 2-8(b).

(a)

(b)

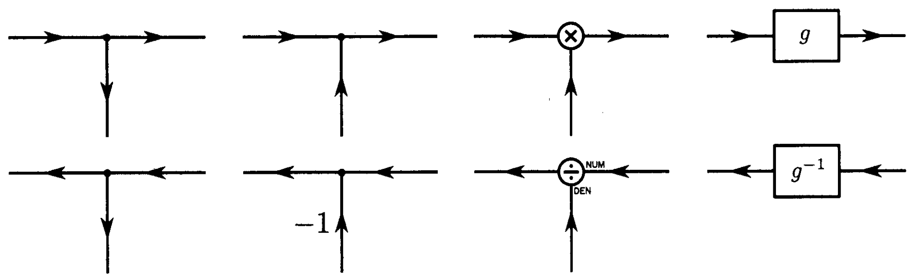


Figure 2-7: (a) Elements along a functionally-dependent path from an input to an output whose configurations are to be exchanged. (b) Behaviorally-equivalent elements that reverse the path.

(a)

(b)

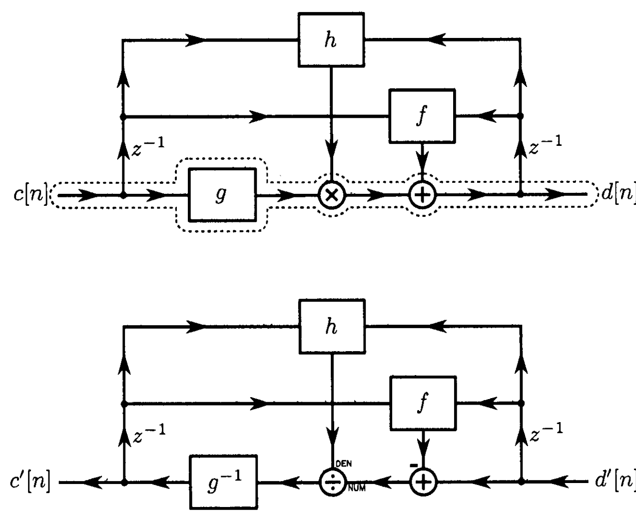


Figure 2-8: (a) Nonlinear system illustrating the functional path, or interconnecting system, from $c[n]$ to $d[n]$ that is used in exchanging the input-output configurations of these variables. (b) Behaviorally-equivalent system obtained by performing path reversal in the interconnecting system.

2.5 Partial taxonomy of 2-input, 2-output linear systems

On a number of occasions related to viewing conservation principles under a change of basis, we will be interested in implementing a linear transformation of the behavior of a set of variables in a larger system. A specific sub-class of these transformations that we will commonly encounter will be those corresponding to linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. As the pertinent variables in the original system may be represented in a number of possible input-output configurations, applying an appropriate transformation generally involves realizing the pertinent behavior in a system that has a compatible input-output configuration.

Toward these ends, Fig. 2-9 depicts a partial taxonomy of behaviorally-equivalent linear signal-flow graphs that implement the linear transformation

$$x_3 = ax_1 + bx_2 \tag{2.40}$$

$$x_4 = cx_1 + dx_2. \tag{2.41}$$

Referring to this figure, the signal-flow graphs were generated by beginning with various implementations for the transformation specified in Eqns. 2.40-2.41, taking x_1 and x_2 as inputs and x_3 and x_4 as outputs, and performing path reversal to realize the depicted systems. The bent arrows in the figure indicate these manipulations. Still referring to this figure, interconnections along upper-right, lower-left diagonals have equivalent bottom branch configurations, and interconnections along upper-left, lower-right diagonals have equivalent upper branch configurations.

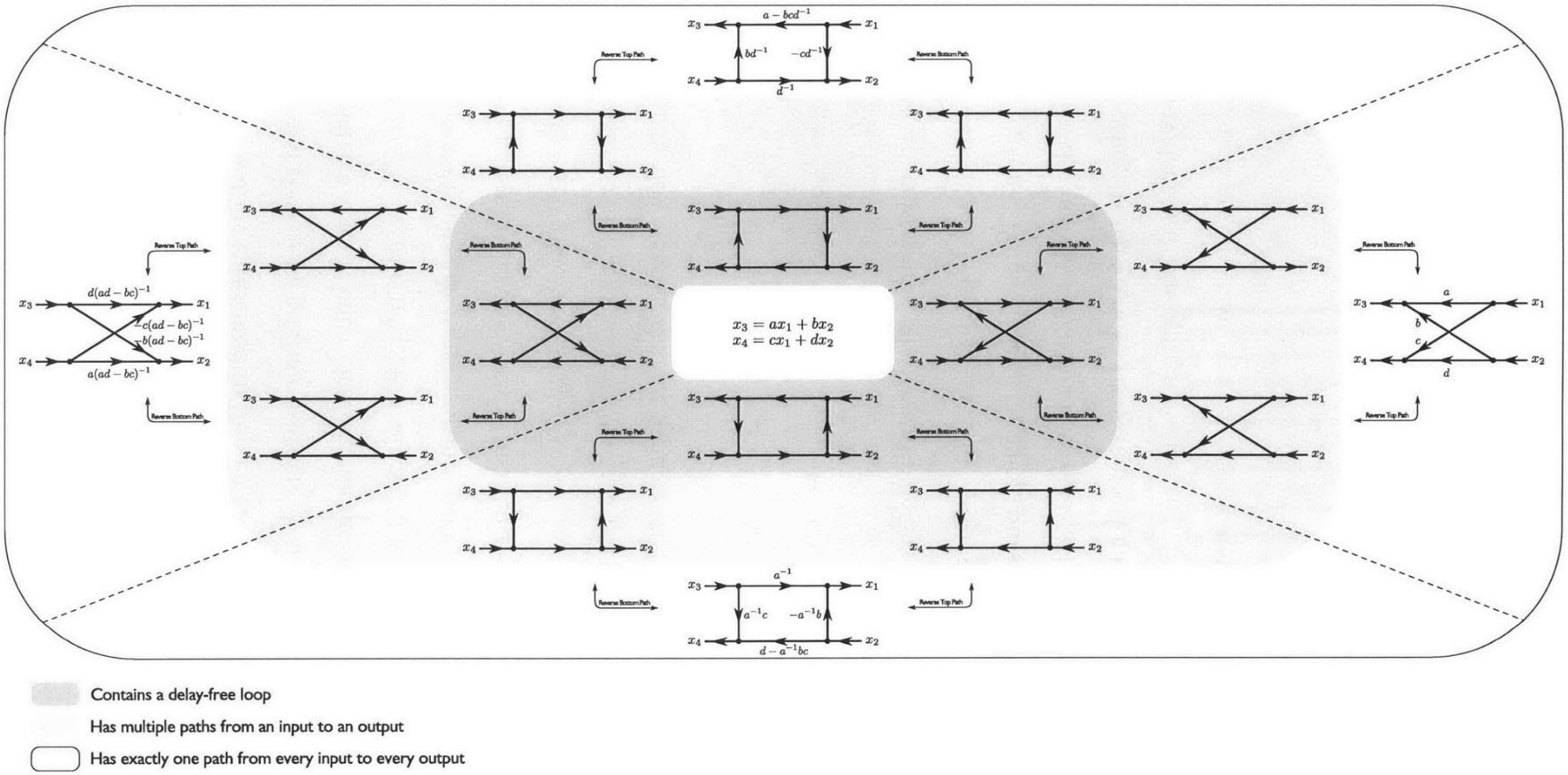


Figure 2-9: Partial taxonomy of behaviorally-equivalent 2-input, 2-output, linear, memoryless interconnections. The white region contains interconnections in four input-output configurations, and the bent arrows indicate manipulations that can be made by reversing the upper and lower input-output paths. Branch gains for the interconnections in the two gray regions may be obtained using path reversal and are omitted here for clarity. Interconnections along upper-right, lower-left diagonals have equivalent bottom branch configurations, and interconnections along upper-left, lower-right diagonals have equivalent upper branch configurations.

Chapter 3

Conservation framework

As was previously mentioned, we are concerned in this thesis with conservation laws reminiscent of Eq. 1.1, with the general motivating problems being

- the design of signal processing algorithms for which a conservation law of the form of Eq. 1.1 is obeyed,
- the identification of conservation laws of the form of Eq. 1.1 in existing signal processing algorithms, and
- the role of these conservation laws in obtaining new and useful results.

Toward these ends, we focus in this chapter on gaining further insight into the fundamental principles of conservation laws that take the form of Eq. 1.1.

In particular, we explore the question of what properties the left-hand side of Eq. 1.1 has, in addition to that of what causes the right-hand side of Eq. 1.1 evaluate to zero, laying much of the groundwork needed to address the remaining issues regarding the synthesis, identification, and use of conservation in signal processing algorithms. The details uncovered in doing so will in turn form a foundation for the remainder of the thesis. As the principles developed in this chapter will apply in a number of essentially unrelated applications, they will be viewed as a unifying framework within which to discuss conservation in signal processing systems.

A common theme in the remainder of the thesis will be that conservation is a property of a linear interconnecting system, and the results in this chapter form a very general foundation for applications such as this. We begin the chapter by formalizing the pertinent notion of conservation, introducing what we will refer to as an organized variable space (OVS) and illustrating its use by describing known conservation principles in existing classes of signal processing algorithms. In cases where conservation is a result of variables lying in a vector space, we draw a distinction between whether an equation of the form of Eq. 1.1 corresponds to pairwise orthogonality of vectors or to orthogonality of vector subspaces, and we present a theorem establishing conditions on which this distinction may equivalently be based. We conclude the chapter by showing that the set of all conservative vector spaces forms a smooth manifold, in the process writing the generating set of matrices for the Lie group that can be used to move between them.

3.1 Organized variable spaces

In physical systems where conservation laws of the form of Eq. 1.1 hold, the corresponding conjugate variables may represent two of a wide range of different quantities. In these systems, a natural way to define conjugate variables in turn often involves identifying variables that can generically be thought of as efforts and flows. In signal processing systems, however, the system variables may be unitless or may have no particular physical meaning. Although an effort-flow classification may still be effective in certain cases, e.g. for signal processing systems that simulate electrical networks or that move along continuous trajectories as in [21], it ultimately has the potential to lead to misguided or ambiguous concepts. For example, describing a quantity as a flow implies that something has been differentiated with respect to time, a notion that may require further clarification within the context of a discrete-time system. The concept of conjugate variables in this thesis therefore explicitly does not make use of this type of distinction.

A natural question, then, is that of how we might expect to identify candidate variables that may potentially result in a conservation principle of the form of Eq. 1.1. This thesis takes the viewpoint that the critical issue is not what the variables represent, but rather the way that they are organized in giving rise to an expression akin to the left-hand side of Eq. 1.1, and that the behavior of the underlying signal processing system is what leads to the right-hand side of the equation evaluating to zero. These ideas are formalized in this thesis using an idea that we refer to as an **organized variable space** (OVS).

3.1.1 The correspondence map

The central idea behind the OVS is to organize a collection of system variables, and to do so using the tools of linear algebra. The motivation behind the use of linear algebra is to allow conjugate variables to be defined as linear combinations of system variables, a property that will allow conservation in systems such as wave-digital filters to be placed on equal footing with conservation in, e.g., electrical networks. The interpretation will be that the values of the variables in a signal processing system can be thought of as coefficients in a basis expansion of a vector that lies in a finite-dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$, defined over the real numbers, and that a quadratic form of a specific class can be used to map these coefficients to a real number. If the underlying signal processing system constrains its system variables so that the quadratic form evaluates to 0, the OVS will be said to be conservative for the behavior of the system.

A good reference for the basic principles in the theory of quadratic forms is [27], and an attempt will be made in this thesis to formulate the key ideas in a way that does not require such a reference. One reason is that as the theory of quadratic forms is a rich topic in its own right, some of the accepted terminology in that field coincides with familiar concepts in inner product spaces. For example, an "orthogonal decomposition of a vector space" in the theory of quadratic forms does not generally have the usual inner-product space interpretation. Our approach will be to begin with an inner product space and use the inner product, in addition to a linear map, to define a quadratic form. This is indeed reminiscent of the usual progression in the theory of quadratic forms, where a bilinear form is first defined and is then used to create a quadratic form. However, the approach here in explicitly defining an inner product will allow us to use the properties of inner products after the quadratic form has been defined, and to relate these back to the structure of the quadratic form in useful ways.

We will specifically be concerned with an even-dimensional inner product space $(V, \langle .,. \rangle)$, where $2L = \dim V \geq 2$, in addition to an associated quadratic form $Q : V \to \mathbb{R}$ that is defined in terms of the inner product as

$$Q(x) = \langle Cx, x \rangle, \tag{3.1}$$

where $C : V \to V$ is a linear map that will be assumed to be self-adjoint[1] in this definition without loss of generality, i.e. $C^* = C$. The key restriction on C is that it will be required to be invertible, with a total of L positive and L negative eigenvalues. The map C in this definition will be referred to as a **correspondence map** because in mapping V onto itself, it implicitly specifies a correspondence between any two vectors $x, x' \in V$ for which $Cx = x'$.

It is straightforward to verify that the quadratic form in the left-hand side of

[1]In this thesis, the adjoint of a linear map $M : V \to V$ on an inner product space $(V, \langle .,. \rangle)$ will be denoted M^*, i.e. $\langle Mx, x' \rangle = \langle x, M^* x' \rangle$, $\forall x, x' \in V$.

Eq. 1.1 has a valid correspondence map by writing it in the following way:

$$
\begin{aligned}
Q\left(\begin{bmatrix} e_1 \\ \vdots \\ e_K \\ f_1 \\ \vdots \\ f_K \end{bmatrix}\right) &= e_1 f_1 + \cdots + e_K f_K \\
&= \left\langle \underbrace{\begin{bmatrix} & & & \frac{1}{2} & & \\ & & & & \ddots & \\ & & & & & \frac{1}{2} \\ \frac{1}{2} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{2} & & & \end{bmatrix}}_{C} \begin{bmatrix} e_1 \\ \vdots \\ e_K \\ f_1 \\ \vdots \\ f_K \end{bmatrix}, \begin{bmatrix} e_1 \\ \vdots \\ e_K \\ f_1 \\ \vdots \\ f_K \end{bmatrix} \right\rangle,
\end{aligned} \tag{3.2}
$$

with $\langle ., . \rangle$ denoting the standard inner product on $\mathbb{R}^{2K}$. In this equation, the matrix C can be diagonalized as

$$C = S^{tr} \Lambda S, \tag{3.3}$$

with

$$
S = \begin{bmatrix} \frac{\sqrt{2}}{2} & & & \frac{\sqrt{2}}{2} & & \\ & \ddots & & & \ddots & \\ & & \frac{\sqrt{2}}{2} & & & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & & & -\frac{\sqrt{2}}{2} & & \\ & \ddots & & & \ddots & \\ & & \frac{\sqrt{2}}{2} & & & -\frac{\sqrt{2}}{2} \end{bmatrix} \tag{3.4}
$$

and

$$\lambda = \begin{bmatrix} \frac{1}{2} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{2} & & & \\ & & & -\frac{1}{2} & & \\ & & & & \ddots & \\ & & & & & -\frac{1}{2} \end{bmatrix}, \tag{3.5}$$

i.e. it is a map from $\mathbb{R}^{2K}$ to $\mathbb{R}^{2K}$ that has a total of K positive and K negative eigenvalues.

In the language of quadratic forms, the previously-mentioned conditions on the eigenvalues of the correspondence map C used in defining $Q(x)$ is equivalent to saying that $Q(x)$ is regular, with signature (L, L), again where $2L = \dim V$. In this thesis, we will refer to a regular quadratic form defined on a $2J$-dimensional vector space that has signature (J, J) as being **balanced**.

3.1.2 Partition decompositions

A technique that will be used to further describe the structure of a quadratic form $Q(x)$ defined as in Eq. 3.1 will be to refer to certain direct-sum decompositions of V that have special properties with respect to $Q(x)$. The first such decomposition that we will call attention to will be referred to as a **partition decomposition** of V. A partition decomposition will specifically be defined as a direct-sum decomposition of V whose decomposition subspaces linearly separate $Q(x)$ into balanced quadratic forms acting on the subspaces. Written formally, a partition decomposition will refer to a set of vector subspaces $\{V_1, \ldots, V_K\}$ ($V_k \neq \{0\},\ k = 1, \ldots, K$) for which

$$V = V_1 \oplus \cdots \oplus V_K \tag{3.6}$$

and for which there exist balanced quadratic forms $Q_k : V_k \to \mathbb{R},\ k = 1, \ldots, K$, such that $Q(x)$ can be written as

$$Q\left(x_1 + \cdots + x_K\right) = Q_1\left(x_1\right) + \cdots + Q_K\left(x_K\right), \quad x_k \in V_k,\ k = 1, \ldots, K. \tag{3.7}$$

The specific sense in which the quadratic forms $Q_k(x_k)$ are balanced is that they can be defined in terms of individual correspondence maps $C_k : V_k \to V_k$ as

$$Q_k(x_k) = \langle C_k x_k, x_k \rangle, \quad x_k \in V_k, \tag{3.8}$$

where each C_k is a linear, invertible map that has an equal number of positive and negative eigenvalues, and where the inner product $\langle ., . \rangle$ as defined on V also serves as the inner product on the subspace V_k.

Note that as the quadratic forms $Q_k(x_k)$ are balanced, the subspaces V_k in a partition decomposition will be even-dimensional. We will typically label a set of vector subspaces forming a partition decomposition as $\mathcal{D}_p = \{V_1, \ldots, V_K\}$, and the subspaces V_k in a partition decomposition will be referred to as **partition subspaces**.

A special name will be given to a partition decomposition where each subspace has dimension 2. In this case the decomposition will have a total of $L = \dim V/2$ subspaces, which is the maximum number of partition subspaces allowed in a partition decomposition of a $2L$-dimensional vector space V. This type of partition decomposition will accordingly be referred to as a **maximal-$\mathcal{D}_p$ decomposition**.

3.1.3 Conjugate decompositions

We have emphasized the viewpoint that the left-hand side of Eq. 1.1 can be thought of as a quadratic form acting on a vector in $\mathbb{R}^{2K}$. An alternative perspective, and one that is widely used in describing power conservation principles including those in electrical network theory, is to instead view the left-hand side of Eq. 1.1 as an inner product taken between two vectors, each of which is in a smaller-dimensional space $\mathbb{R}^K$. Formalizing the relationship between these two interpretations may be fairly straightforward in the case of Eq. 1.1, where the form of the left-hand side of the equation is naturally suggestive of an inner product. However in the more general case where a balanced quadratic form $Q(\cdot)$ has been defined over an abstract, even-dimensional inner product space $(V, \langle .,. \rangle)$, the relationship between $Q(\cdot)$ and its interpretation as an inner product has the potential to be more elusive. In facilitating our understanding of this relationship, we will make use of a direct-sum decomposition of V that we will refer to as a **conjugate decomposition**.

A conjugate decomposition of an even-dimensional inner product space $(V, \langle .,. \rangle)$ having a balanced quadratic form $Q : V \rightarrow \mathbb{R}$ describes how to decompose a vector in V so that $Q(\cdot)$ acts like an inner product on the decomposed elements. It is specifically defined as a set of two vector subspaces $\{V_A, V_B\}$ of equal dimension that decompose V as

$$V = V_A \oplus V_B, \tag{3.9}$$

such that elements in the subspaces can be mapped to the arguments of an inner product in a smaller-dimensional inner product space $(U, \langle .,. \rangle_U)$ in a way that $Q(\cdot)$, acting on a vector in V, is equivalent to the inner product $\langle .,. \rangle_U$ acting on the mapped components taken from V_A and V_B. The formal condition on the subspaces V_A and V_B in the decomposition will be that given these subspaces, there exists an inner product space $(U, \langle .,. \rangle_U)$ over the real numbers, as well as invertible, linear maps $M_A : V_A \rightarrow U$ and $M_B : V_B \rightarrow U$, for which

$$Q(x_A + x_B) = \langle M_A x_A, M_B x_B \rangle_U, \quad x_A \in V_A, \; x_B \in V_B. \tag{3.10}$$

We will typically label a set of vector subspaces forming a conjugate decomposition as $\mathcal{D}_c = \{V_A, V_B\}$, and the subspaces V_A and V_B will be referred to as **conjugate subspaces**.

We have been careful to provide a definition for a conjugate decomposition in a way that depends on the existence of an inner product space $(U, \langle .,. \rangle_U)$ and on the existence of maps $M_A : V_A \rightarrow U$ and $M_B : V_B \rightarrow U$, as opposed to requiring that they be specified explicitly. The reason for this is that, given one set of maps and an inner product space that are known to satisfy Eq. 3.10, linear transformations can be used to obtain other suitable combinations of maps and inner product spaces, i.e. for a given conjugate decomposition they will not be unique. Nonetheless, it will be useful to give names to a particular inner product space and pair of maps that satisfy Eq. 3.10 for a pre-specified conjugate decomposition of V. We will specifically refer to an appropriate inner product space $(U, \langle .,. \rangle_U)$ as a **comparison space** and the mappings M_A and M_B will be referred to as **conjugate mappings**.

3.1.4 OVS definition

With the previously-mentioned terms having been established, we are prepared to write formal definitions for two key elements in the conservation framework. The first will be referred to as an **organization** of an inner product space, and the second, which will consist of an organization in addition to the collection of elements composing an inner product space, will be referred to as an **organized variable space**.

An organization $\mathcal{O}$ of an inner product space $(V, \langle .,. \rangle)$ will be defined as a 3-tuple containing a correspondence map, a partition decomposition, and a conjugate decomposition, with the additional requirement that each partition subspace is decomposed by the conjugate subspaces and vice-versa vice-versa. We proceed by writing the definitions formally.

Definition 3.1. *An organization of an even-dimensional inner product space* $(V, \langle .,. \rangle)$ *over the real numbers is defined as a 3-tuple*

$$\mathcal{O} = (C, \mathcal{D}_p, \mathcal{D}_c) \tag{3.11}$$

whose elements are

C*: a correspondence map for a balanced quadratic form*

$\mathcal{D}_p$*: an associated partition decomposition, and*

$\mathcal{D}_c$*: an associated conjugate decomposition,* *with the partition and conjugate decompositions satisfying*

$$V_A = (V_A \cap V_1) \oplus \cdots \oplus (V_A \cap V_K), \tag{3.12}$$

$$V_B = (V_B \cap V_1) \oplus \cdots \oplus (V_B \cap V_K), \tag{3.13}$$

and

$$V_k = (V_k \cap V_A) \oplus (V_k \cap V_B), \quad k = 1, \ldots, K. \tag{3.14}$$

In Eqns. 3.12-3.14, K denotes the number of subspaces in the partition decomposition $\mathcal{D}_p$.

Definition 3.2. *An organized variable space (OVS) is defined as an even-dimensional inner product space $(V, \langle .,. \rangle)$ over the real numbers, in addition to an organization of the space. An OVS is written*

$$\mathfrak{U} = (V, \langle .,. \rangle, \mathcal{O}), \tag{3.15}$$

with $\mathcal{O}$ being an organization of $(V, \langle .,. \rangle)$.

A set of vectors $\mathcal{S} \subset V$ for which the quadratic form $Q(x)$ associated with an OVS $\mathfrak{U}$ is known to evaluate to 0, i.e. for which

$$Q(x) = 0, \quad x \in \mathcal{S}, \tag{3.16}$$

will be referred to as a **conservative set** for $\mathfrak{U}$. We will also say that $\mathfrak{U}$ is **conservative over $\mathcal{S}$.** Note that although the set $\mathcal{S}$ in Eq. 3.16 is required to be a subset of V, it need not be a vector space.

3.2 Examples of organized variable spaces

Before going further, we present some examples of OVSs defined over various known signal processing systems that are conservative over their respective behaviors. The examples are specifically chosen to illustrate the use of the language of OVSs and to provide insight into the situations where an OVS can be applied. This appears to be the first occasion in which some of these systems have been placed on equal footing in this sense, and a path for future research might include applying useful known theorems regarding one type of system to other systems, using the OVS as a vehicle for translating between the domains.

In this section and in the various examples throughout this thesis, we will often encounter signal processing systems that are composed of an interconnection of subsystems, and where the constraints imposed by the interconnection will be sufficient to result in conservation. As the sharing of variables between the interconnecting system and any subsystems will only further restrict the associated conservative set, we will discuss conservation in these examples with the subsystems generally remaining unspecified. This underscores the breadth of systems to which conservation can be applied, which as is the case with electrical networks includes systems that are nonlinear, time-varying and stochastic.

3.2.1 Electrical networks

The power conservation principle for an electrical network containing a total of K elements, with associated currents i_k and voltages v_k, is written as

$$v_1 i_1 + \cdots + v_K i_K = 0. \tag{3.17}$$

In the OVS language, we interpret the voltage and current variables as being coefficients in a basis expansion of a vector in $\mathbb{R}^{2K}$ such that

$$[v_1, \ldots, v_K, i_1, \ldots, i_K]^{tr} \in \mathbb{R}^{2K}. \tag{3.18}$$

The corresponding OVS is written

$$\mathfrak{U} = (\mathbb{R}^{2K}, \langle .,. \rangle, \mathcal{O}), \tag{3.19}$$

with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^{2K}$, and with the elements of the organization $\mathcal{O} = (C, \mathcal{D}_p, \mathcal{D}_c)$ being

$$\begin{aligned}
C &= \frac{1}{2}\begin{bmatrix} 0_K & I_K \\ I_K & 0_K \end{bmatrix} &(3.20)\\
\mathcal{D}_p &= \left\{\operatorname{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(K)}\right), \ldots, \operatorname{span}\left(\mathbf{e}^{(K+1)}, \mathbf{e}^{(2K)}\right)\right\} &(3.21)\\
\mathcal{D}_c &= \left\{\operatorname{span}\left(\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(K)}\right), \operatorname{span}\left(\mathbf{e}^{(K+1)}, \ldots, \mathbf{e}^{(2K)}\right)\right\}, &(3.22)
\end{aligned}$$

where C is a matrix that swaps and scales the first and second halves of a vector, and where $\mathbf{e}^{(k)}$ denotes the length-$2K$ column vector containing zeros in all of its entries, with the exception of the kth, which has value one. The associated quadratic form is

$$Q(\mathbf{x}) = \langle C\mathbf{x}, \mathbf{x} \rangle, \quad \mathbf{x} \in \mathbb{R}^{2K} \tag{3.23}$$

and may equivalently be written using the standard inner product $\langle .,. \rangle_{\mathbb{R}^K}$ on the comparison space $\mathbb{R}^K$ as

$$Q(\mathbf{x}) = \langle M_A\mathbf{x}, M_B\mathbf{x} \rangle_{\mathbb{R}^K}, \quad \mathbf{x} \in \mathbb{R}^{2K}, \tag{3.24}$$

with example conjugate mappings being specified as matrices from $\mathbb{R}^{2K}$ to $\mathbb{R}^K$ as

$$M_A = \begin{bmatrix} I_K & 0_K \end{bmatrix} \tag{3.25}$$

and

$$M_B = \begin{bmatrix} 0_K & I_K \end{bmatrix}. \tag{3.26}$$

It is straightforward to verify that M_A and M_B invertibly map vectors in the respective conjugate subspaces to $\mathbb{R}^K$.[2]

It is a result of Tellegen's theorem that $\mathfrak{U}$ is conservative over the behavior of the

[2]This is an appropriate place to emphasize a benefit of having developed the organized variable space in a coordinate-free setting. Working in $\mathbb{R}^N$, for example, we would have been confronted with matrices such as those in Eqns. 3.25-3.26, which, being matrices that do not have right inverses, might have obscured their interpretation as invertible linear maps from the conjugate subspaces to the comparison space.

interconnection, i.e. that the Kirchoff laws for the network imply

$$Q(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{S}, \tag{3.27}$$

where $\mathcal{S}$ denotes the set of vectors that satisfy these laws. It should be noted that while Tellegen's theorem for electrical networks implies that the quadratic form evaluates to zero, the theorem is actually a statement of conservation in a broader sense that relates to orthogonality of vector spaces. The spirit of this more general form is embodied in what we refer to as strong conservation, which is covered later in this chapter in Section 3.4.

3.2.2 Feedback systems

In the language established by Willems regarding dissipative systems, [37, 38] the interconnection structure in a feedback system in the form of the system in Fig. 3-1 is referred to as being neutral.[3] We illustrate that for this system, neutrality coincides with OVS conservation.

Referring again to Fig. 3-1, the assumption is that the interconnecting structure is linear, memoryless and time-invariant, and that the subsystems can generally be nonlinear, time-varying and stochastic. Using the OVS language, we interpret the instantaneous values of the interconnection variables, at time $n = n_0$ as being coefficients in a basis expansion of a vector in $\mathbb{R}^6$ such that

$$[x_1[n_0], x_2[n_0], x_3[n_0], x_4[n_0], x_5[n_0], x_6[n_0]]^{tr} \in \mathbb{R}^6. \tag{3.28}$$

The corresponding OVS is written

$$\mathfrak{U} = (\mathbb{R}^6, \langle ., . \rangle, \mathcal{O}), \tag{3.29}$$

with $\langle ., . \rangle$ denoting the standard inner product on $\mathbb{R}^6$, and with the elements of the

[3]The interested reader is also pointed toward [41] and [42], which discuss the concepts of losslessness and dissipation as they pertain to electrical network theory.

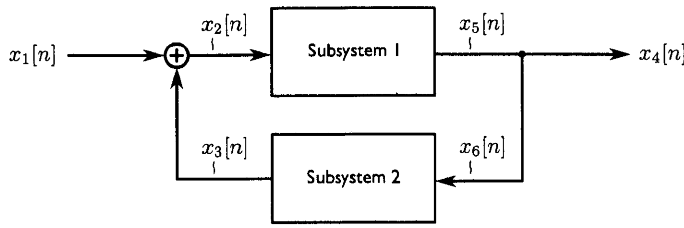


Figure 3-1: An interconnected system with feedback.

organization $\mathcal{O} = (C, \mathcal{D}_p, \mathcal{D}_c)$ being

$$C = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{3.30}$$

$$\mathcal{D}_p = \left\{\text{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(4)}\right), \text{span}\left(\mathbf{e}^{(2)}, \mathbf{e}^{(5)}\right), \text{span}\left(\mathbf{e}^{(3)}, \mathbf{e}^{(6)}\right)\right\} \tag{3.31}$$

$$\mathcal{D}_c = \left\{\text{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}\right), \text{span}\left(\mathbf{e}^{(4)}, \mathbf{e}^{(5)}, \mathbf{e}^{(6)}\right)\right\}. \tag{3.32}$$

It is straightforward to verify that C is balanced, i.e. that it has 3 positive eigenvalues and 3 negative eigenvalues. The associated quadratic form is

$$Q(\mathbf{x}) = \langle C\mathbf{x}, \mathbf{x}\rangle, \quad \mathbf{x} \in \mathbb{R}^{2K} \tag{3.33}$$

and may equivalently be written using the standard inner product $\langle ., . \rangle_{\mathbb{R}^3}$ on the comparison space $\mathbb{R}^3$ as

$$Q(\mathbf{x}) = \langle M_A\mathbf{x}, M_B\mathbf{x}\rangle_{\mathbb{R}^K}, \quad \mathbf{x} \in \mathbb{R}^{2K} \tag{3.34}$$

with example conjugate mappings being

$$M_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{3.35}$$

and

$$M_B = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.36}$$

The quadratic form may accordingly be written as

$$Q(\mathbf{x}) = -x_1[n_0]x_4[n_0] + x_2[n_0]x_5[n_0] - x_3[n_0]x_6[n_0]. \tag{3.37}$$

We claim that this OVS $\mathfrak{U}$ is conservative over the behavior of the interconnection, i.e. that

$$Q(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{S}, \tag{3.38}$$

where $\mathcal{S}$ is the set of vectors permitted by the interconnection structure. This can be verified by writing the interconnection equations,

$$x_2[n_0] = x_1[n_0] + x_3[n_0] \tag{3.39}$$

$$x_4[n_0] = x_5[n_0] = x_6[n_0], \tag{3.40}$$

and substituting them into Eq. 3.37 to obtain

$$\begin{aligned} Q(\mathbf{x}) &= -x_1[n_0]x_5[n_0] + (x_1[n_0] + x_3[n_0])\, x_5[n_0] - x_3[n_0]x_5[n_0] \\ &= 0, \quad \mathbf{x} \in \mathcal{S}. \end{aligned} \tag{3.41}$$

This conservation principle, in conjunction with appropriate conditions on the subsystems, forms the basis for the theorems regarding open-loop conditions for closed loop stability that are presented in, e.g., [37,38,43,44]. In [43], transformations on the system in Fig. 3-1 are also used to turn positivity conditions for system stability into conic conditions. In the language of OVSs, this is equivalent to saying that there are multiple organizations that result in conservation over the interconnection behavior.

In exploring this perspective further, we refer to a related OVS $\mathfrak{U}^{(\sigma)}$ that is written in terms of a real-valued scalar parameter σ and that is defined over the inner product space $(\mathbb{R}^6, \langle .,. \rangle)$, in addition to an organization $\mathcal{O}^{(\sigma)}$ written as

$$\mathcal{O}^{(\sigma)} = \left(C^{(\sigma)}, \mathcal{D}_p, \mathcal{D}_c^{(\sigma)}\right). \tag{3.42}$$

The elements of the organization are

$$C^{(\sigma)} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2\sigma & 0 \\ 0 & 0 & -1 & 0 & 0 & 2\sigma \end{bmatrix} \tag{3.43}$$

$$\mathcal{D}_p = \left\{\operatorname{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(4)}\right), \operatorname{span}\left(\mathbf{e}^{(2)}, \mathbf{e}^{(5)}\right), \operatorname{span}\left(\mathbf{e}^{(3)}, \mathbf{e}^{(6)}\right)\right\} \tag{3.44}$$

$$\mathcal{D}_c^{(\sigma)} = \left\{\operatorname{span}\left(\mathbf{e}^{(1)}, (\mathbf{e}^{(2)} - \sigma\mathbf{e}^{(5)}), (\mathbf{e}^{(3)} - \sigma\mathbf{e}^{(6)})\right), \operatorname{span}\left(\mathbf{e}^{(4)}, \mathbf{e}^{(5)}, \mathbf{e}^{(6)}\right)\right\}, \tag{3.45}$$

and the quadratic form can be written in terms of the system variables as

$$Q^{(\sigma)}(\mathbf{x}) = -x_1[n_0]x_4[n_0] + (x_2[n_0] - \sigma x_5[n_0])x_5[n_0] - (x_3[n_0] - \sigma x_6[n_0])x_6[n_0]. \tag{3.46}$$

Mapping the conjugate subspaces to the comparison space $\mathbb{R}^3$, the quadratic form can be written using the standard inner product $\langle .,. \rangle_{\mathbb{R}^3}$ as

$$Q^{(\sigma)}(\mathbf{x}) = \langle M_A^{(\sigma)}\mathbf{x}, M_B\mathbf{x}\rangle_{\mathbb{R}^3}, \quad \mathbf{x} \in \mathbb{R}^6, \tag{3.47}$$

with example conjugate mappings being

$$M_A^{(\sigma)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\sigma & 0 \\ 0 & 0 & -1 & 0 & 0 & \sigma \end{bmatrix} \tag{3.48}$$

and

$$M_B = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.49}$$

It is straightforward to verify that the OVS $\mathfrak{U}^{(\sigma)}$ is conservative over the behavior of the interconnection for any finite value of σ by substituting the interconnection equations 3.39-3.40 into Eq. 3.46, i.e. we have

$$-x_1[n_0]x_4[n_0]+(x_2[n_0]-\sigma x_5[n_0])x_5[n_0]-(x_3[n_0]-\sigma x_6[n_0])x_6[n_0] = 0, \quad \mathbf{x} \in \mathcal{S}, \tag{3.50}$$

where $\mathcal{S}$ denotes the set of vectors permitted by the interconnection structure.

We have not directly addressed the question of whether the matrix $C^{(\sigma)}$, defined in Eq. 3.43, is balanced. The line of reasoning that we will use in answering this, and which we will use throughout the thesis in similar situations, utilizes a theorem referred to in the theory of quadratic forms as Sylvester's law of inertia. The theorem states that given a real, symmetric matrix $R : \mathbb{R}^N \to \mathbb{R}^N$, performing a change of coordinates as

$$\widehat{R} = S^{tr} R S, \tag{3.51}$$

with $S : \mathbb{R}^N \to \mathbb{R}^N$ being an invertible matrix, will result in a matrix $\widehat{R} : \mathbb{R}^N \to \mathbb{R}^N$ that has the same signature as that of R. As this is equivalent to saying that $\widehat{R}$ will have the same number of positive and negative eigenvalues as R, we conclude that performing an invertible, linear transformation on the correspondence map for a balanced quadratic form will result in a new map that is balanced, and as such will also be a valid correspondence map.

That the matrix $C^{(\sigma)}$ defined in Eq. 3.43 is balanced can accordingly be verified by noting that it is related to the balanced correspondence map C in Eq. 3.30 according to

$$C^{(\sigma)} = \left(S^{(\sigma)}\right)^{tr} C S^{(\sigma)}, \tag{3.52}$$

with $S^{(\sigma)}$ being defined in terms of the conjugate mappings $M_A^{(\sigma)}$ and M_B in Eqns. 3.48-3.49 as

$$S^{(\sigma)} = \begin{bmatrix} M_A^{(\sigma)} \\ M_B \end{bmatrix}. \tag{3.53}$$

3.2.3 Wave-digital interconnections

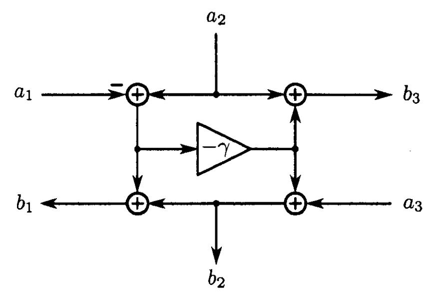
The wave-digital class of signal processing structures, discussed in, e.g., [18], have among their many desirable properties stability characteristics that are exceptionally robust to parameter perturbations. Structures within this class are composed of a specific set of subsystems and interconnections that are analogous to the physical components used in the design and implementation of microwave filters. An overall wave-digital structure is assembled by sharing variables between so-called elements and interconnecting structures, done in such a way that every element shares its variables with an interconnecting structure, as opposed sharing directly with another element.

A good reference regarding conservation in wave-digital filters is [17], in which the conserved quantity is referred to as pseudopower. In wave-digital filters, a pseudopower is defined for each interconnection port, and the specific sense of conservation is that the sum of the port pseudopowers, taken over all ports in the system, evaluates to zero.

The approach in [17] was to write the conservation principle before having defined any specific wave-digital interconnections. As such, the conservation principle in [17] can be thought of as a condition in determining whether an interconnection structure is admissible within the wave-digital framework. In describing the conservation principle using the OVS language, we instead choose to consider in detail the two commonly-used wave-digital interconnections that are depicted in Fig. 3-2. In

Section 6.3 we will discuss wave-digital filters with greater generality as we use the OVS framework in conjunction with intervening results to provide a straightforward method for generating the set of all linear wave-digital interconnections.

(a)



(b)

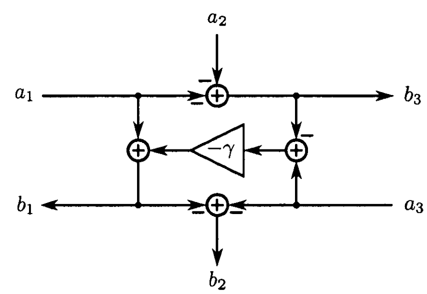


Figure 3-2: (a) Parallel and (b) series wave-digital interconnections.

Again referring to Fig. 3-2, we define OVSs in terms of the inner product space $(\mathbb{R}^6, \langle .,. \rangle)$ as

$$\mathfrak{U}^{(a)} = \left(\mathbb{R}^6, \langle .,. \rangle, \mathcal{O}^{(a)}\right) \tag{3.54}$$

and

$$\mathfrak{U}^{(b)} = \left(\mathbb{R}^6, \langle .,. \rangle, \mathcal{O}^{(b)}\right) \tag{3.55}$$

that respectively correspond to the so-called parallel and series interconnections in Fig. 3-2(a) and Fig. 3-2(b), and with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^6$. Referring to either structure, the interconnection variables will be interpreted as coefficients in a basis expansion of a vector in $\mathbb{R}^6$, in the sense that

$$[a_1, a_2, a_3, b_1, b_2, b_3] \in \mathbb{R}^6. \tag{3.56}$$

The associated elements of the organizations $\mathcal{O}^{(a)} = (C^{(a)}, \mathcal{D}_p, \mathcal{D}_c)$ and $\mathcal{O}^{(b)} = (C^{(b)}, \mathcal{D}_p, \mathcal{D}_c)$ are

$$
\begin{aligned}
C^{(a)} &= \frac{1}{2}\begin{bmatrix} \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\gamma) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & -(1-\gamma) & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} && (3.57) \\
C^{(b)} &= \frac{1}{2}\begin{bmatrix} (1-\gamma) & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma(1-\gamma) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma(1-\gamma) \end{bmatrix} && (3.58) \\
\mathcal{D}_p &= \left\{\operatorname{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(4)}\right), \operatorname{span}\left(\mathbf{e}^{(2)}, \mathbf{e}^{(5)}\right), \operatorname{span}\left(\mathbf{e}^{(3)}, \mathbf{e}^{(6)}\right)\right\} && (3.59) \\
\mathcal{D}_c &= \left\{\operatorname{span}\left(\mathbf{e}^{(1)} + \mathbf{e}^{(4)}, \mathbf{e}^{(2)} + \mathbf{e}^{(5)}, \mathbf{e}^{(3)} + \mathbf{e}^{(6)}\right), \right. \\
&\quad \left. \operatorname{span}\left(\mathbf{e}^{(1)} - \mathbf{e}^{(4)}, \mathbf{e}^{(2)} - \mathbf{e}^{(5)}, \mathbf{e}^{(3)} - \mathbf{e}^{(6)}\right)\right\}, && (3.60)
\end{aligned}
$$

with the partition decomposition $\mathcal{D}_p$ and conjugate decomposition $\mathcal{D}_c$ being common to both $\mathcal{O}^{(a)}$ and $\mathcal{O}^{(b)}$. The respective quadratic forms associated with $\mathfrak{U}^{(a)}$ and $\mathfrak{U}^{(a)}$ can be written as

$$Q^{(a)}(\mathbf{x}) = \gamma(a_1^2 - b_1^2) + (1-\gamma)(a_2^2 - b_2^2) + (a_3^2 - b_3^2) \tag{3.61}$$

and

$$Q^{(b)}(\mathbf{x}) = (1-\gamma)(a_1^2 - b_1^2) + \gamma(a_2^2 - b_2^2) + \gamma(1-\gamma)(a_3^2 - b_3^2). \tag{3.62}$$

$Q^{(a)}(\mathbf{x})$ and $Q^{(a)}(\mathbf{x})$ can also be formulated in terms of the standard inner product

$\langle .,. \rangle_{\mathbb{R}^3}$ on the comparison space $\mathbb{R}^3$ as

$$Q^{(a)}(\mathbf{x}) = \langle M_A^{(a)}\mathbf{x}, M_B^{(a)} \rangle_{\mathbb{R}^3} \tag{3.63}$$

and

$$Q^{(b)}(\mathbf{x}) = \langle M_A^{(b)}\mathbf{x}, M_B^{(b)} \rangle_{\mathbb{R}^3}, \tag{3.64}$$

with example sets of conjugate mappings for the two OVSs being

$$M_A^{(a)} = \begin{bmatrix} \gamma & 0 & 0 & \gamma & 0 & 0 \\ 0 & (1-\gamma) & 0 & 0 & (1-\gamma) & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{3.65}$$

$$M_B^{(a)} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{3.66}$$

and

$$M_A^{(b)} = \begin{bmatrix} (1-\gamma) & 0 & 0 & (1-\gamma) & 0 & 0 \\ 0 & \gamma & 0 & 0 & \gamma & 0 \\ 0 & 0 & (1-\gamma) & 0 & 0 & (1-\gamma) \end{bmatrix} \tag{3.67}$$

$$M_B^{(b)} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & \gamma & 0 & 0 & -\gamma \end{bmatrix}. \tag{3.68}$$

It is straightforward to verify by substituting the interconnection equations into the corresponding quadratic forms that $\mathfrak{U}^{(a)}$ is conservative over the behavior of the interconnection depicted in Fig. 3-2(a) and that $\mathfrak{U}^{(b)}$ is conservative over the behavior of the interconnection depicted in Fig. 3-2(b), i.e. that

$$Q^{(a)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{S}^{(a)} \tag{3.69}$$

and

$$Q^{(b)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{S}^{(b)}, \tag{3.70}$$

where $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ denote the behaviors of the respective interconnections.

3.2.4 Lattice filters

The FIR and IIR classes of lattice structures are used in a wide range of signal processing applications including adaptive filtering, speech modification and speech coding. Among their attractive qualities is the fact that they have causally stable and invertible responses, even in the presence of heavily-quantized coefficients as long as the coefficients have magnitude less than 1. [29]

A lattice structure is assembled by connecting the linear, memoryless interconnections in Fig. 3-3 to intermediate two-port causal systems that are generally allowed to contain memory. The interconnection in Fig. 3-3(a) is typical of FIR lattice structures, and the interconnection in Fig. 3-3(b) is typical of IIR lattice structures. Referring to this pair of figures, the variables a, b, c, d denote the instantaneous values of the interconnection inputs and outputs. The behavior of the the FIR interconnection is identical to that of the IIR interconnection as can be verified using path reversal, and as such we proceed by describing a single OVS that will be shown to be conservative over the behavior of either.

The OVS will specifically be defined over the inner product space $(\mathbb{R}^4, \langle .,. \rangle)$, as

$$\mathfrak{U} = (\mathbb{R}^4, \langle .,. \rangle, \mathcal{O}), \tag{3.71}$$

with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^4$. The interconnection variables will be interpreted as coefficients in a basis expansion of a vector in $\mathbb{R}^4$ such that

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4, \tag{3.72}$$

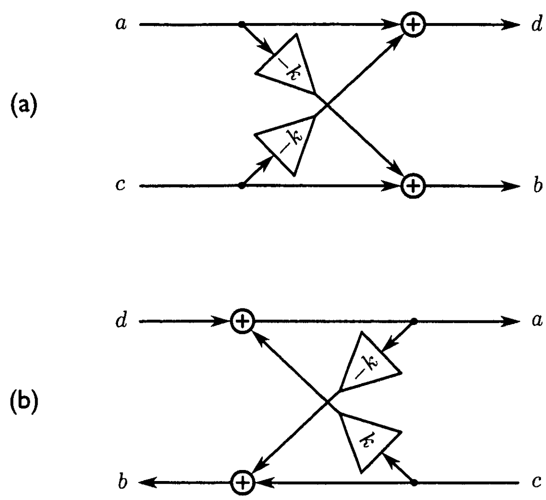


Figure 3-3: (a) FIR and (b) IIR lattice interconnections.

and the elements of the organization $\mathcal{O} = (C, \mathcal{D}_p, \mathcal{D}_c)$ will be defined as

$$C = \frac{1}{2}\begin{bmatrix} (k^2-1) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -(k^2-1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.73}$$

$$\mathcal{D}_p = \left\{\operatorname{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(3)}\right), \operatorname{span}\left(\mathbf{e}^{(2)}, \mathbf{e}^{(4)}\right)\right\} \tag{3.74}$$

$$\mathcal{D}_c = \left\{\operatorname{span}\left(\mathbf{e}^{(1)} + \mathbf{e}^{(3)}, \mathbf{e}^{(2)} + \mathbf{e}^{(4)}\right), \operatorname{span}\left(\mathbf{e}^{(1)} - \mathbf{e}^{(3)}, \mathbf{e}^{(2)} - \mathbf{e}^{(4)}\right)\right\}. \tag{3.75}$$

The quadratic form $Q(\cdot)$ associated with the correspondence map C is accordingly written as

$$Q(\mathbf{x}) = (k^2-1)(a^2-c^2) - (b^2-d^2), \tag{3.76}$$

and may be represented using the standard inner product $\langle .,. \rangle_{\mathbb{R}^2}$ on the comparison space $\mathbb{R}^2$ as

$$Q(\mathbf{x}) = \langle M_A\mathbf{x}, M_B\mathbf{x} \rangle_{\mathbb{R}^2}, \tag{3.77}$$

with example conjugate mappings being

$$M_A = \begin{bmatrix} (k^2-1) & 0 & (k^2-1) & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \tag{3.78}$$

and

$$M_B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}. \tag{3.79}$$

It is a straightforward exercise to verify that $\mathfrak{U}$ is conservative over the behavior of the interconnection, i.e. that

$$Q(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{S}, \tag{3.80}$$

where $\mathcal{S}$ denotes the interconnection behavior. This can be done, for example, by writing equations for either interconnection structure in Fig. 3-3 and combining them with Eq. 3.76.

3.3 Transformations on $Q(x)$ and conservative sets

Section 3.2 illustrated how the OVS describes conservation and what the OVS can be used to describe. Beyond being a language for describing a class of conservation principles, the OVS will find its perhaps most compelling applications in the design and analysis of signal processing systems, and a primary mechanism in facilitating this will be the application of invertible, linear transformations.

The focus will in particular be on invertible transformations for several reasons, some of which will become more clear as we proceed with the discussion. A key motivation to mention upfront is that this will avoid the use of transformations that map elements of a behavior to 0, resulting in conservation by way of introducing an ambiguity. I.e. if noninvertible transformations were allowed, it would be possible to make any set a conservative set by mapping all of its elements to the zero element, resulting in an OVS that would provide little insight into the behavior of the underlying system.

3.3.1 Relationships between transformations of OVS elements

We will mainly be concerned with performing transformations on sets, conservative or otherwise, in addition to performing transformations on the quadratic form associated with an OVS. In many applications, transformations applied to the conservative set will correspond to a modification of system behavior, and transformations applied to the quadratic form will result in a modification of the way that conservation is described.

From a mathematical perspective, a conservative set $\mathcal{S}$ and quadratic form $Q(x)$ are related by the equation $Q(x) = 0,\ x \in \mathcal{S}$, and performing a transformation on one accordingly corresponds to a transformation on the other. For an OVS $\mathfrak{U} = (V, \langle .,. \rangle, (C, \mathcal{D}_p, \mathcal{D}_c))$ and a linear transformation $T : V \to V$, this relationship may be written formally as an equivalence between two statements, i.e.

$$Q(Tx) = 0,\ x \in \mathcal{S} \quad \Leftrightarrow \quad Q(x) = 0,\ x \in T(\mathcal{S}), \tag{3.81}$$

where the notation $T(\mathcal{S})$ is used to indicate the set that results from applying T to every element of $\mathcal{S}$, i.e.

$$T(\mathcal{S}) = \{Tx : x \in \mathcal{S}\}. \tag{3.82}$$

When applying a transformation to a quadratic form $Q(x)$, it is straightforward to show that the resulting functional

$$Q'(x) = Q(Tx) \tag{3.83}$$

will also be a valid quadratic form, e.g.

$$Q(Tx) = \langle CTx, Tx \rangle = \langle T^* CTx, x \rangle = Q'(x), \tag{3.84}$$

although it is not generally the case that a valid partition decomposition $\mathcal{D}_p$ and conjugate decomposition $\mathcal{D}_c$ for $Q(x)$ will also be valid partition decomposition and conjugate decomposition for $Q'(x)$. It will therefore be customary to transform these elements in conjunction with $Q(x)$ as

$$\begin{aligned} \mathcal{D}'_p &= \{V'_1, \ldots, V'_K\} \\ &= \left\{T^{-1}(V_1), \ldots, T^{-1}(V_K)\right\} \end{aligned} \tag{3.85}$$

and

$$\begin{aligned} \mathcal{D}'_c &= \{V'_A, V'_B\} \\ &= \left\{T^{-1}(V_A), T^{-1}(V_B)\right\}, \end{aligned} \tag{3.86}$$

where the notation $T^{-1}(V_k)$ represents an application of T^{-1} to every element of V_k, consistent with convention in Eq. 3.82. We will likewise transform any conjugate mappings $M_A : V_A \longrightarrow U$ and $M_B : V_B \longrightarrow U$ associated with $Q(x)$ as

$$M'_A = M_A T \tag{3.87}$$

and

$$M'_B = M_B T. \tag{3.88}$$

It can be verified that $\mathcal{D}_p$, $\mathcal{D}_c$, M_A, and M_B being valid for $Q(x)$ implies that $\mathcal{D}'_p$, $\mathcal{D}'_c$, M'_A, and M'_B, as respectively defined in Eqns. 3.85, 3.86, 3.87, and 3.88, are valid for $Q'(x)$, as defined in Eq. 3.83. The general strategy in doing this involves beginning with the defining equations for partition decompositions, conjugate decompositions, and conjugate mappings, and performing a change of variables. In particular, we demonstrate that the transformed partition decomposition $\mathcal{D}'_p$ is valid for $Q'(x)$ by beginning with Eq. 3.7 and performing a change of variables as

$$\begin{aligned}
Q\left(Tx^{(1)}+\cdots+Tx^{(K)}\right) &= Q_1\left(Tx^{(1)}\right)+\cdots+Q_K\left(Tx^{(K)}\right), \quad Tx^{(k)}\in V_k,\ k=1,\ldots,K \\
Q\left(T\left(x^{(1)}+\cdots+x^{(K)}\right)\right) &= Q_1\left(Tx^{(1)}\right)+\cdots+Q_K\left(Tx^{(K)}\right), \quad x^{(k)}\in T^{-1}(V_k),\ k=1,\ldots,K \\
Q'\left(x^{(1)}+\cdots+x^{(K)}\right) &= Q'_1\left(x^{(1)}\right)+\cdots+Q'_K\left(x^{(K)}\right), \quad x^{(k)}\in V'_k,\ k=1,\ldots,K,
\end{aligned}$$

with the quadratic forms $Q'_k : V'_k \longrightarrow \mathbb{R}$ that operate on the transformed partition subspaces being defined as

$$Q'_k(x) = Q_k(Tx), \quad x\in V'_k,\ k=1,\ldots,K. \tag{3.89}$$

The transformed conjugate decomposition $\mathcal{D}'_c$ and any conjugate mappings M'_A and M'_B can likewise be shown to be valid for $Q'(x)$ by beginning with Eq. 3.10 and performing the following manipulations:

$$\begin{aligned}
Q\left(Tx^{(A)}+Tx^{(B)}\right) &= \left\langle M_ATx^{(A)}, M_BTx^{(B)}\right\rangle_U, \quad Tx^{(A)}\in V_A,\ Tx^{(B)}\in V_B \\
Q\left(T\left(x^{(A)}+x^{(B)}\right)\right) &= \left\langle M_ATx^{(A)}, M_BTx^{(B)}\right\rangle_U, \quad x^{(A)}\in T^{-1}(V_A),\ x^{(B)}\in T^{-1}(V_B) \\
Q'\left(x^{(A)}+x^{(B)}\right) &= \left\langle M'_Ax^{(A)}, M'_Bx^{(B)}\right\rangle_U, \quad x^{(A)}\in V'_A,\ x^{(B)}\in V'_B.
\end{aligned}$$

3.3.2 Canonical conjugate bases

A convenient property of an arbitrary OVS is that the correspondence map C for the quadratic form $Q(x)$ performs a swapping operation on the first and second halves of length-$2L$ column vectors in $\mathbb{R}^{2L}$ when represented in an appropriate basis, i.e. the two halves of the vector play the roles of conjugate variables in an expression of the form of Eq. 1.1. In particular, given a $2L$-dimensional OVS $\mathfrak{U} = (V, \langle .,. \rangle, (C, \mathcal{D}_p, \mathcal{D}_c))$ and a basis $\mathcal{B} = \{v_1, \ldots, v_{2L}\}$ of V, where every element $x \in V$ maps to a unique column vector of coefficients $\mathbf{x} \in \mathbb{R}^{2L}$ as

$$x = \mathbf{x}_1 v_1 + \cdots + \mathbf{x}_{2L} v_{2L}, \quad x \in V, \tag{3.90}$$

the basis $\mathcal{B}$ is said to be a **canonical conjugate basis** if the representation of C as a matrix in this basis takes the form

$$C^{(\mathcal{B})} = \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix}. \tag{3.91}$$

It is fairly straightforward to select such a basis using, for example, the following steps:

(1) Pick a basis $\mathcal{B}'$ for V for which the transpose of a matrix written in the basis coincides with the adjoint of the associated linear map.[4] Denote C, represented as a matrix in this basis, as $C^{(\mathcal{B}')}$.

(2) Perform an eigen decomposition of $C^{(\mathcal{B}')}$ as

$$C^{(\mathcal{B}')} = R^{tr} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2L} \end{bmatrix} R, \tag{3.92}$$

with $R^{tr} = R^{-1}$ and with $\lambda_1, \ldots, \lambda_{2L}$ being ordered from largest to smallest.[5] As a consequence of the requirement that C is balanced, $\lambda_1, \ldots, \lambda_L$ will be strictly positive and $\lambda_{L+1}, \ldots, \lambda_{2L}$ will be strictly negative.

(3) Define the change of basis matrix S as

$$S = \frac{\sqrt{2}}{2} \begin{bmatrix} I_L & I_L \\ I_L & -I_L \end{bmatrix} \begin{bmatrix} \sqrt{|\lambda_1|} & & 0 \\ & \ddots & \\ 0 & & \sqrt{|\lambda_{2L}|} \end{bmatrix} R. \tag{3.93}$$

[4]Such a basis will always exist.

[5]As C is self-adjoint, $C^{(\mathcal{B}')}$ is symmetric, and consequently such a decomposition will always exist.

It is straightforward to verify that the matrix $C^{(\mathcal{B}')}$ is written in terms of S as

$$C^{(\mathcal{B}')} = S^{tr} \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} S, \tag{3.94}$$

and consequently the basis $\mathcal{B}$ that is obtained by applying S to the elements of $\mathcal{B}'$ is a canonical conjugate basis, i.e.

$$C^{(\mathcal{B})} = \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix}. \tag{3.95}$$

The key motivation behind representing correspondence maps in a canonical conjugate basis is that it results in a quadratic form resembling the standard expression involved in power conservation and other similar laws. Specifically, given an arbitrary $2L$-dimensional OVS $\mathfrak{U} = (V, \langle.,.\rangle, (C, \mathcal{D}_p, \mathcal{D}_c))$ with a canonical conjugate basis $\mathcal{B}$, the associated quadratic form may be represented in this basis as a map from $\mathbb{R}^{2L}$ to $\mathbb{R}$ as

$$Q^{(\mathcal{B})}\left([a_1, \ldots, a_L, b_1, \ldots, b_L]\right) = 2a_1 b_1 + \cdots + 2a_L b_L. \tag{3.96}$$

From this, we may also define a comparison space $U = \mathbb{R}^L$ and associated conjugate mappings

$$M_A^{(\mathcal{B})} = \sqrt{2} \begin{bmatrix} I_L & 0_L \end{bmatrix} \tag{3.97}$$

and

$$M_B^{(\mathcal{B})} = \sqrt{2} \begin{bmatrix} 0_L & I_L \end{bmatrix} \tag{3.98}$$

in the basis.

Another important consequence of the fact that an arbitrary OVS has a canonical conjugate basis is that it allows many of the results that will be developed in the remainder of the thesis to not be critically dependent on the specific OVS to which they apply. For example, Section 3.4 will discuss the synthesis of conservative sets that are vector subspaces, and will do so by working with a canonical conjugate basis. Likewise, Chapter 5 will discuss the variational properties of certain conservative

OVSs, and the development will proceed by writing theorems regarding the particular comparison space $\mathbb{R}^L$ without loss of generality.

3.3.3 $\mathcal{D}_p$-invariant transformations

It will also be useful to call attention to the particular case where an OVS is defined as $\mathfrak{U} = \left(\mathbb{R}^{2L}, \langle .,. \rangle, (C, \{V_1, \dots, V_K\}, \{V_A, V_B\})\right)$, with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^{2L}$, where

$$
\begin{aligned}
V_A &= \operatorname{span}\left(\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(L)}\right), && (3.99)\\
V_B &= \operatorname{span}\left(\mathbf{e}^{(L+1)}, \dots, \mathbf{e}^{(2L)}\right), && (3.100)
\end{aligned}
$$

and where the partition subspaces span subsets of the vectors $\mathbf{e}^{(k)} \in \mathbb{R}^{2L}$ as

$$
\begin{aligned}
V_1 &= \operatorname{span}\left(\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(\dim V_1)}\right) && (3.101)\\
V_2 &= \operatorname{span}\left(\mathbf{e}^{(\dim V_1+1)}, \dots, \mathbf{e}^{(\dim V_1+\dim V_2)}\right) && (3.102)\\
&\vdots\\
V_K &= \operatorname{span}\left(\mathbf{e}^{(2L-\dim V_K)}, \dots \mathbf{e}^{2L}\right). && (3.103)
\end{aligned}
$$

In this situation, which was commonly encountered in the examples in Section 3.2, the correspondence map C is written as a $2L \times 2L$ matrix that takes the form

$$
C = \begin{bmatrix}
E_1 & & & F_1 & & \\
& \ddots & & & \ddots & \\
& & E_K & & & F_K \\
G_1 & & & H_1 & & \\
& \ddots & & & \ddots & \\
& & G_K & & & H_K
\end{bmatrix}, \tag{3.104}
$$

with each sub-matrix E_k, F_k, G_k, H_k that corresponds to the associated partition subspace V_k being a $(\dim V_k/2) \times (\dim V_k/2)$ matrix.

For this OVS, the associated quadratic form may be written as

$$
\begin{aligned}
Q(\mathbf{x}) &= \left[\mathbf{x}_1, \ldots, \mathbf{x}_{\dim V_1/2}\right] C_1 \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{\dim V_1/2} \end{bmatrix} \\
&\quad + \cdots + \left[\mathbf{x}_{2L-\dim V_K}, \ldots, \mathbf{x}_{2L}\right] C_K \begin{bmatrix} \mathbf{x}_{2L-\dim V_K} \\ \vdots \\ \mathbf{x}_{2L} \end{bmatrix}, \qquad (3.105)
\end{aligned}
$$

with each matrix C_k being the $\dim V_k \times \dim V_k$ matrix defined as

$$
C_k = \begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix}, \quad k = 1, \ldots K. \tag{3.106}
$$

Each matrix C_k will be balanced, and we can accordingly write a matrix S_k for each as in Eq. 3.93, resulting in

$$
C_k = S_k^{tr} \begin{bmatrix} 0_{\dim V_k/2} & I_{\dim V_k/2} \\ I_{\dim V_k/2} & 0_{\dim V_k/2} \end{bmatrix} S_k. \tag{3.107}
$$

Partitioning each matrix S_k into four sub-matrices of equal size as

$$
S_k = \begin{bmatrix} E_k^{(S)} & F_k^{(S)} \\ G_k^{(S)} & H_k^{(S)} \end{bmatrix}, \tag{3.108}
$$

we may construct a $2L \times 2L$ transforming matrix

$$
S = \begin{bmatrix}
E_1^{(S)} & & & F_1^{(S)} & & \\
& \ddots & & & \ddots & \\
& & E_K^{(S)} & & & F_K^{(S)} \\
G_1^{(S)} & & & H_1^{(S)} & & \\
& \ddots & & & \ddots & \\
& & G_K^{(S)} & & & H_K^{(S)}
\end{bmatrix} \tag{3.109}
$$

that can be used to represent the correspondence map in a canonical conjugate form while leaving the partition subspaces unchanged.

Written formally, we define the transformed quadratic form as

$$
\begin{aligned}
Q'(\mathbf{x}) &= Q(S\mathbf{x}) && (3.110)\\
&= \langle CS\mathbf{x}, S\mathbf{x}\rangle && (3.111)\\
&= \left\langle \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} \mathbf{x}, \mathbf{x} \right\rangle, && (3.112)
\end{aligned}
$$

with Eq. 3.112 following from the adjoint theorem and Eq. 3.107. We note that the transformed partition subspaces remain unchanged,

$$V_k = S^{-1}(V_k), \quad k = 1, \ldots, K, \tag{3.113}$$

as S is an invertible transformation that has the partition subspaces as invariant subspaces, i.e. S is $\mathcal{D}_p$-invariant. The conjugate subspaces are transformed as

$$V'_A = S^{-1}(V_A) \tag{3.114}$$

and

$$V'_B = S^{-1}(V_B), \tag{3.115}$$

and will in general be different from the original conjugate subspaces. Likewise, any conjugate mappings $M_A : \mathbb{R}^{2L} \rightarrow \mathbb{R}^L$ and $M_B : \mathbb{R}^{2L} \rightarrow \mathbb{R}^L$ will be transformed, resulting in

$$M'_A = M_A S \tag{3.116}$$

and

$$M'_B = M_B S. \tag{3.117}$$

As in a canonical conjugate basis the conjugate mappings may be represented as

$$M'_A = \sqrt{2} \begin{bmatrix} I_L & 0_L \end{bmatrix} \tag{3.118}$$

and

$$M_B' = \sqrt{2} \begin{bmatrix} 0_L & I_L \end{bmatrix}, \tag{3.119}$$

knowledge of the transformation S used in changing to this basis allows us to determine valid conjugate mappings for the original OVS $\mathfrak{U}$ by writing

$$M_A = \sqrt{2} \begin{bmatrix} I_L & 0_L \end{bmatrix} S^{-1} \tag{3.120}$$

and

$$M_B = \sqrt{2} \begin{bmatrix} 0_L & I_L \end{bmatrix} S^{-1}. \tag{3.121}$$

If we are given a conservative set $\mathcal{S}$ for the OVS $\mathfrak{U}$, it may be transformed as

$$\mathcal{S}' = S^{-1}(\mathcal{S}) \tag{3.122}$$

to obtain a conservative set $\mathcal{S}'$ for the transformed OVS

$$\mathfrak{U}' = \left(\mathbb{R}^{2L}, \langle .,. \rangle, (\{V_1, \dots, V_K\}, \{V_A', V_B'\}), \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} \right). \tag{3.123}$$

It is an illustrative and fairly straightforward exercise to find a $\mathcal{D}_p$-invariant transformation that transforms to a canonical conjugate basis for one of the examples in Section 3.2.

3.3.4 $Q(x)$-invariant transformations

Given an OVS $\mathfrak{U} = (V, \langle .,. \rangle, (C, \mathcal{D}_p, \mathcal{D}_c))$, another important consideration will be that of determining the set of transformations that leave the quadratic form $Q(x)$ unchanged. This will, in particular, facilitate the discussion of conservative vector spaces in Subsection 3.4.2. Written formally, we are interested in the set $\mathcal{G}_Q$ of transformations $T : V \to V$ that satisfy

$$Q'(x) = Q(Tx) = \langle CTx, Tx \rangle = \langle Cx, x \rangle = Q(x), \quad T \in \mathcal{G}_Q, \; x \in V. \tag{3.124}$$

As $Q(x)$ is a balanced quadratic form, i.e. as it is nondegenerate, the theory of quadratic forms states that the set $\mathcal{G}_Q$ is a group, with the group law being composition of maps. We verify that the group axioms are satisfied:

(1) **Closure.** Given transformations T and T' for which $Q(Tx) = Q(x),\ x \in V$ and $Q(T'x) = Q(x),\ x \in V$, we have that $Q(TT'x) = Q(T'x) = Q(x),\ x \in V$.

(2) **Associativity.** Given transformations $T, T', T'' \in \mathcal{G}_Q$, associativity of linear maps implies that $T(T'T'') = (TT')T''$.

(3) **Identity element.** The identity element is the identity map I, which is shown to formally satisfy Eq. 3.124 by writing $Q(Ix) = Q(x)\ x \in V$, and which indeed satisfies $TI = IT = T$ for any $T \in \mathcal{G}_Q$.

(4) **Inverse element.** We first show that every element $T \in \mathcal{G}_Q$ is invertible. From Eq. 3.124, we have that each $T \in \mathcal{G}_Q$ satisfies

$$T^*CT = C, \tag{3.125}$$

where C is an invertible map by virtue of being a correspondence map. The transformation T must consequently be invertible for Eq. 3.125 to hold. It remains to be shown that $T \in \mathcal{G}_Q$ implies $T^{-1} \in \mathcal{G}_Q$. Multiplying both sides of Eq. 3.125 by T^{-1*} on the left and by T^{-1} on the right results in

$$C = T^{-1*}CT^{-1}, \tag{3.126}$$

and we conclude that $T \in \mathcal{G}_Q$ implies $T^{-1} \in \mathcal{G}_Q$.

The elements of the group $\mathcal{G}_Q$ can be generated by choosing a canonical conjugate basis $\mathcal{B}$ and multiplying matrices that leave the correspondence map, represented as a matrix in this basis, unchanged. The group of such matrices forms a matrix Lie group, and the following theorem explicitly lists the 1-dimensional subgroups of matrices that generate the group.

Theorem 3.1. *The group $\mathcal{G}_Q^{(\mathcal{B})}$ of $2L \times 2L$ matrices T that satisfy*

$$T^{tr} \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} T = \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} \tag{3.127}$$

is a matrix Lie group that can be generated by multiplying matrices from the following subgroups:

(1)

$$T_1^{[q;t)} = \begin{array}{rc} & \begin{array}{ccccccccccc} & & & \overset{q}{\downarrow} & & & & \overset{q+L}{\downarrow} & & & \end{array} \\ \begin{array}{r} \\ \\ \\ q \rightarrow \\ \\ \\ \\ q+L \rightarrow \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& e^{t} &&&&&&& \\ &&&& 1 &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& 1 &&&& \\ &&&&&&& e^{-t} &&& \\ &&&&&&&& 1 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{pmatrix} \end{array},$$

$$1 \leq q \leq L, \;\; t \in \mathbb{R},$$

(2)

$$T_2^{(q,r;t)} = \begin{array}{cc} & \begin{array}{ccccc} & & q & r+L & \\ & & \downarrow & \downarrow & \end{array} \\ \begin{array}{r} \\ r \rightarrow \\ \\ q+L \rightarrow \\ \\ \end{array} & \begin{pmatrix} 1 & & & & \\ & \ddots & t & & \\ & & & & \\ & & & -t & \ddots \\ & & & & 1 \end{pmatrix} \end{array},$$

$$1 \leq q \leq L,\ 1 \leq r \leq L,\ q \neq r,\ t \in \mathbb{R},$$

(3)

$$T_3^{(q,r;t)} = \begin{array}{cc} & \begin{array}{ccccc} & r & q & & \\ & \downarrow & \downarrow & & \end{array} \\ \begin{array}{r} \\ \\ \\ r+L \rightarrow \\ q+L \rightarrow \\ \\ \end{array} & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & & & \\ & & t & & \\ & -t & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array},$$

$$1 \leq q \leq L,\ 1 \leq r \leq L,\ q \neq r,\ t \in \mathbb{R},$$

(4)

$$T_4^{[q,r;t)} = \begin{array}{r@{}c}
 & \begin{matrix} & & \overset{r+L}{\downarrow} & \overset{q+L}{\downarrow} & & \end{matrix} \\
\begin{matrix} \\ r \to \\ q \to \\ \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & & \\ & \ddots & & t & & \\ & & -t & & & \\ & & & & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}
\end{array},$$

$$1 \leq q \leq L,\ 1 \leq r \leq L,\ q \neq r,\ t \in \mathbb{R},$$

(5)

$$T_5^{[q]} = \begin{array}{r@{}c}
 & \begin{matrix} & & & \overset{q}{\downarrow} & & & & \overset{q+L}{\downarrow} & & & \end{matrix} \\
\begin{matrix} \\ \\ \\ q \to \\ \\ \\ \\ q+L \to \\ \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & -1 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & & & & & -1 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}
\end{array},$$

$$1 \leq q \leq L,$$

(6)

$$
T_6^{[q]} = \begin{array}{r c}
 & \begin{array}{ccccccccccc} & & & \overset{q}{\downarrow} & & & & \overset{q+L}{\downarrow} & & & \end{array} \\
\begin{array}{r} \\ \\ \\ q \rightarrow \\ \\ \\ \\ q+L \rightarrow \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & 0 & & & & 1 & & & \\
 & & & & 1 & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & 1 & & & & \\
 & & & 1 & & & & 0 & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{pmatrix}
\end{array},
$$

$1 \leq q \leq L.$

The group $\mathcal{G}_Q^{(\mathcal{B})}$ is, in particular, isomorphic to the matrix indefinite orthogonal group $O(L, L)$, i.e. the group of invertible $2L \times 2L$ matrices preserving a quadratic form whose matrix is diagonal, with the first L diagonal elements being 1 and the last L diagonal elements being -1. Consequently, $\mathcal{G}_Q^{(\mathcal{B})}$ has four connected components as does $\mathcal{G}_Q$, and $T_5^{[p]}$ and $T_6^{[p']}$ can be used to move between them for any fixed p and p'.

Proof. As the group $\mathcal{G}_Q$ is isomorphic to the so-called indefinite orthogonal group $O(L, L)$, the proof of this theorem amounts to the standard exercise of using the Lie algebra for the connected component of $\mathcal{G}_Q$ that contains the identity element to write the one-parameter generating subgroups, in addition to writing transformations for moving discontinuously between the four connected components of $\mathcal{G}_Q$. Appendix A goes through this argument in detail. $\square$

3.4 Conservation over vector spaces

In the examples in Section 3.2, OVSs were defined over signal processing systems for which the associated quadratic form $Q(x)$ evaluated to 0 for all vectors x permitted by a linear interconnection, i.e. each OVS was conservative over a set $\mathcal{S} = W$ that was a vector space. The first key issue that we explore in this section is that of how conservation over a vector space is viewed from the perspective of the comparison space U. As was previously mentioned, conservation implies that the inner product $\langle .,. \rangle_U$ on the comparison space will evaluate to 0, and if this corresponds to vector space orthogonality on the comparison space, the OVS will receive the special designation of being strongly conservative. We conclude the section by providing a method for generating the manifold of all conservative vector spaces for a pre-specified OVS using the set of transformations in Thm. 3.1.

Another reason for focusing on conservation over vector spaces has to do with the previously-mentioned emphasis on signal processing systems represented as a linear interconnection of subsystems that are allowed to be time-varying, nonlinear or stochastic. Conservation over vector spaces is therefore an appealing focus of study because of its broad applicability, in addition to its natural amenability to analysis.

3.4.1 Strong conservation

In discussing conservation over vector spaces, we will specifically refer to an OVS denoted $\mathfrak{U} = (V, \langle .,. \rangle, (C, \mathcal{D}_p, \{V_A, V_B\}))$, along with a comparison space U having an associated abstract inner product $\langle .,. \rangle_U$, as well as conjugate mappings M_A and M_B from the conjugate subspaces V_A and V_B to U. Given a vector subspace $W \subset V$ over which $\mathfrak{U}$ is conservative, we may use Eq. 3.10 to write the conservation principle in terms of the conjugate mappings as

$$\langle M_A x_A, M_B x_B \rangle_U = 0, \quad x_A + x_B \in W,\ x_A \in V_A,\ x_B \in V_B. \tag{3.128}$$

The key issue is that Eq. 3.128 may either represent pairwise orthogonality between the vectors $M_A x_A$ and $M_B x_B$ for each $x_A + x_B \in W$, or it may be representative of orthogonality between vector spaces. The former will be referred to as weak conservation and the latter will be referred to as strong conservation. Fig. 3-4 depicts an example of this distinction for the case where $V = \mathbb{R}^4$, $U = \mathbb{R}^2$, where $\langle .,. \rangle_{\mathbb{R}^2}$ is the standard inner product on $\mathbb{R}^2$, and where the conjugate mappings are

$$M_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.129}$$

and

$$M_B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.130}$$

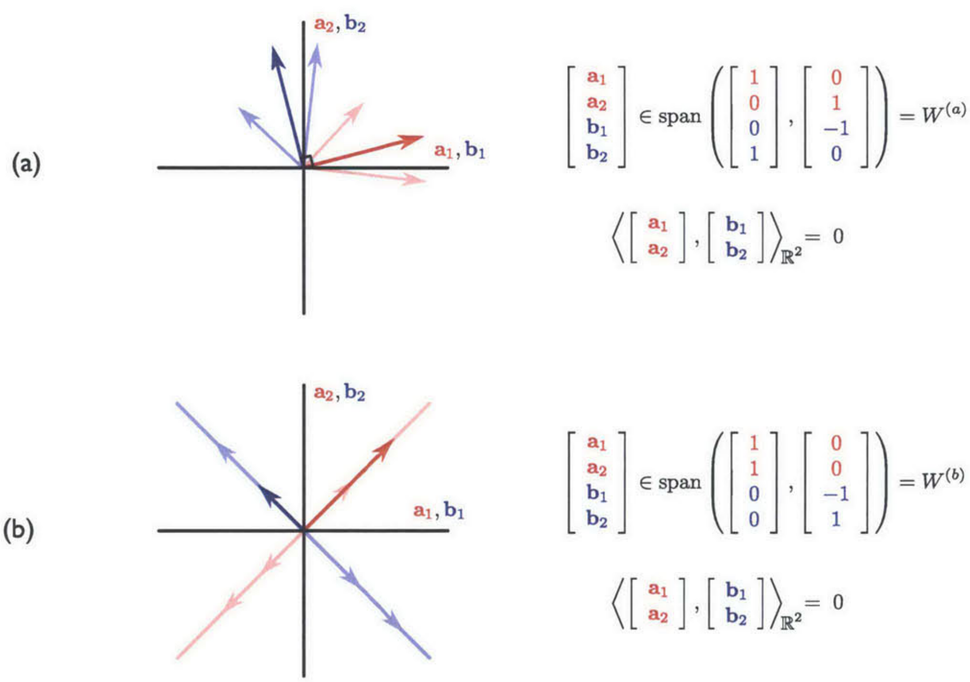


Figure 3-4: Conservation over subspaces of $\mathbb{R}^4$, as viewed from the perspective of the comparison space $\mathbb{R}^2$. (a) Pairwise orthogonality, corresponding to weak conservation. (b) Orthogonal subspaces, corresponding to strong conservation.

In providing a formal definition for strong conservation, we will refer to vector subspaces W_A and W_B that represent the respective sets of vectors x_A and x_B that result from decomposing each vector in W into the components in the conjugate subspaces V_A and V_B. In doing this, it will be convenient to make use of oblique projection operators

$$P_A : V \rightarrow V \tag{3.131}$$

and

$$P_B : V \rightarrow V, \tag{3.132}$$

i.e. linear maps for which $P_A^2 = P_A$ and $P_B^2 = P_B$, that additionally satisfy the following relationships:

$$\begin{aligned} \mathrm{range}(P_A) &= V_A & (3.133) \\ \mathrm{range}(P_B) &= V_B & (3.134) \\ P_A + P_B &= I, & (3.135) \end{aligned}$$

where I is the identity operator on V.[6] The operators P_A and P_B can be used to uniquely decompose an arbitrary vector $x \in V$ into its respective components $x_A \in V_A$ and $x_B \in V_B$ as

$$\begin{aligned} x &= P_A x + P_B x & (3.136) \\ &= x_A + x_B, & (3.137) \end{aligned}$$

with the components x_A and x_B being written as

$$x_A = P_A x \tag{3.138}$$

and

$$x_B = P_B x. \tag{3.139}$$

[6]It is straightforward to verify that a pair of oblique projections P_A and P_B satisfying Eqns. 3.133-3.135 always exists. E.g. define $P_A = P_A^2$ as satisfying Eq. 3.133 and also $\mathrm{null}(P_A) = V_B$, consistent with the property of oblique projections that $\mathrm{range}(P_A) \oplus \mathrm{null}(P_A) = V$. Then Eq. 3.134 follows from Eq. 3.135, and $P_A = P_A^2$ implies $P_B = P_B^2$.

The subspaces W_A and W_B are formally defined in terms of P_A and P_B as

$$W_A = \{P_A x : x \in W\} \tag{3.140}$$

and

$$W_B = \{P_B x : x \in W\} . \tag{3.141}$$

Given a pair of conjugate mappings M_A and M_B, we will also refer to the vector subspaces of the comparison space U that are obtained by respectively mapping W_A and W_B to U through M_A and M_B:

$$M_A(W_A) = \{M_A x : x \in W_A\} \tag{3.142}$$

and

$$M_B(W_B) = \{M_B x : x \in W_B\} . \tag{3.143}$$

With this notation established, we present the following theorem that formally establishes the definition of strong conservation.

Theorem 3.2. *Given a $2L$-dimensional OVS that is conservative over a vector subspace $W \subset V$, in addition to arbitrary conjugate mappings M_A and M_B that map to an abstract comparison space $(U, \langle .,. \rangle_U)$, the following are equivalent:*

(1) $W_A \subseteq W$

(2) $W_B \subseteq W$

(3) $W_A \oplus W_B = W$

(4) $\dim W_A + \dim W_B = \dim W$

(5) $M_A(W_A)$ *and* $M_B(W_B)$ *are orthogonal vector spaces under* $\langle .,. \rangle_U$.

An OVS that is conservative over a vector subspace W and satisfies (1)-(5) will be referred to as **strongly conservative**, *and a conservative OVS that is not strongly conservative will be referred to as* **weakly conservative**.

Proof. We proceed by proving the equivalence of (1)-(5) in the following order:

(a) (1) $\Leftrightarrow$ (2)

(b) (1) and (2) $\Leftrightarrow$ (3)

(c) (3) $\Leftrightarrow$ (4)

(d) (3) $\Leftrightarrow$ (5).

(a): (1) $\Rightarrow$ (2).

$$\begin{aligned} W_A \subseteq W &\Rightarrow \{P_A x : x \in W\} \subseteq W && (3.144)\\ &\Rightarrow \left\{ \underbrace{x - P_A x}_{P_B x} : x \in W \right\} \subseteq W && (3.145)\\ &\Rightarrow W_B \subseteq W. && (3.146) \end{aligned}$$

(a): (1) $\Leftarrow$ (2).

$$\begin{aligned} W_B \subseteq W &\Rightarrow \{P_B x : x \in W\} \subseteq W && (3.147)\\ &\Rightarrow \left\{ \underbrace{x - P_B x}_{P_A x} : x \in W \right\} \subseteq W && (3.148)\\ &\Rightarrow W_A \subseteq W. && (3.149) \end{aligned}$$

(b): (1) and (2) $\Leftrightarrow$ (3). We note that any time we write $W_A + W_B$, $W_A \oplus W_B$ may equivalently be written, as $W_A \subseteq V_A$, $W_B \subseteq V_B$, and $V_A \oplus V_B = V$. In proceeding with the proof, the following line of reasoning can be used to show that $W_A \oplus W_B \supseteq W$:

$$\begin{aligned} W_A + W_B &= \{x_A + x_B : x_A \in W_A,\ x_B \in W_B\} && (3.150)\\ &= \{P_A x + P_B x' : x, x' \in W\} && (3.151)\\ &\supseteq \{P_A x + P_B x' : x, x' \in W,\ x = x'\} && (3.152)\\ &= \left\{ \underbrace{(P_A + P_B)}_{I} x : x \in W \right\} && (3.153)\\ &= W. && (3.154) \end{aligned}$$

Therefore, $W_A + W_B = W$ if and only if $W_A + W_B \subseteq W$, which as is indicated by Eq. 3.150 will occur if and only if $W_A \subseteq W$ and $W_B \subseteq W$.

(c): (3) $\Rightarrow$ (4). It is a fundamental result in linear algebra that

$$\dim(W_A + W_B) = \dim W_A + \dim W_B - \dim W_A \cap W_B. \tag{3.155}$$

As $W_A \subseteq V_A$, $W_B \subseteq V_B$ and $V_A \oplus V_B = V$, we have

$$W_A \cap W_B = \{0\}, \tag{3.156}$$

resulting in

$$\dim W_A + \dim W_B = \dim(W_A + W_B) = \dim W. \tag{3.157}$$

(c): (3) $\Leftarrow$ (4). Using Eqns. 3.155 and 3.156 in conjunction with statement (4), we write

$$\dim W = \dim W_A + \dim W_B = \dim(W_A + W_B). \tag{3.158}$$

It was shown in Eqns. 3.150-3.154 that $W \subseteq W_A + W_B$, and using Eq. 3.158 we conclude that $W = W_A + W_B$.

(d): (3) $\Leftrightarrow$ (5). With $\perp_U$ used to denote orthogonality of vector subspaces of U under $\langle .,. \rangle_U$, we write

$$\begin{aligned}
M_A(W_A) \perp_U M_B(W_B) \quad &\Leftrightarrow \quad \langle y_A, y_B \rangle_U = 0, \quad y_A \in M_A(W_A),\ y_B \in M_B(W_B) &(3.159)\\
&\Leftrightarrow \quad \langle M_A x_A, M_B x_B \rangle_U = 0, \quad x_A \in W_A,\ x_B \in W_B &(3.160)\\
&\Leftrightarrow \quad Q(x_A + x_B) = 0, \quad x_A \in W_A,\ x_B \in W_B &(3.161)\\
&\Leftrightarrow \quad Q(x_A + x_B) = 0, \quad x_A + x_B \in W_A + W_B,\\
&\qquad\qquad x_A \in W_A,\ x_B \in W_B. &(3.162)
\end{aligned}$$

The theorem pertains to conservation over the subspace W, stated formally as

$$Q(x_A + x_B) = 0, \quad x_A + x_B \in W,\ x_A \in W_A,\ x_B \in W_B, \tag{3.163}$$

and we conclude that Eq. 3.162 is equivalent to Eq. 3.163 if and only if $W = W_A + W_B$. $\square$

Illustrating the use of Thm. 3.2 with the examples depicted in Fig. 3-4, we write the OVS $\mathfrak{U} = (\mathbb{R}^4, \langle .,. \rangle, (C, \mathcal{D}_p, \mathcal{D}_c))$, with the elements of the organization being

$$
\begin{aligned}
C &= \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} && (3.164) \\
\mathcal{D}_p &= \{V_A, V_B\} \\
&= \left\{\text{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(3)}\right), \text{span}\left(\mathbf{e}^{(2)}, \mathbf{e}^{(4)}\right)\right\} && (3.165) \\
\mathcal{D}_c &= \left\{\text{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}\right), \text{span}\left(\mathbf{e}^{(3)}, \mathbf{e}^{(4)}\right)\right\}, && (3.166)
\end{aligned}
$$

and we define the projection operators $P_A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ and $P_B : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ as

$$
P_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.167}
$$

and

$$
P_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.168}
$$

We also select $U = (\mathbb{R}^2, \langle .,. \rangle_{\mathbb{R}^2})$ as the comparison space, with the conjugate mappings being written as in Eqns. 3.129-3.130. In these definitions, $\langle .,. \rangle$ is used to denote the standard inner product on $\mathbb{R}^4$, and $\langle .,. \rangle_{\mathbb{R}^2}$ is used to denote the standard inner product on $\mathbb{R}^2$.

Referring to Fig. 3-4, the subspace $W^{(a)}$ in Fig. 3-4(a) is weakly conservative, and the subspace $W^{(b)}$ in Fig. 3-4(b) is strongly conservative. This can be seen by checking for any of the equivalent conditions in Thm. 3.2. For example, defining the subspaces

$$W_A^{(a)} = P_A(W^{(a)}) \tag{3.169}$$

$$W_B^{(a)} = P_B(W^{(a)}) \tag{3.170}$$

$$W_A^{(b)} = P_A(W^{(b)}) \tag{3.171}$$

$$W_B^{(b)} = P_B(W^{(b)}), \tag{3.172}$$

it is straightforward to show that

$$\underbrace{\dim W}_{2} \neq \underbrace{\dim W_A^{(a)}}_{2} + \underbrace{\dim W_B^{(a)}}_{2}, \tag{3.173}$$

as is indicative of weak conservation, and that

$$\underbrace{\dim W}_{2} = \underbrace{\dim W_A^{(b)}}_{1} + \underbrace{\dim W_B^{(b)}}_{1}, \tag{3.174}$$

as is indicative of strong conservation.

3.4.2 The manifold of conservative vector spaces

A question that will be especially pertinent in designing conservative signal processing algorithms is that of given a pre-specified OVS, what vector subspaces are conservative, and the answer relates to the $Q(x)$-invariant transformations that were discussed in Subsection 3.3.4. In particular, an arbitrary conservative vector space can be generated by beginning with an arbitrary conservative vector space of the same dimension, and applying transformations from the group preserving the quadratic form $Q(x)$. As this group was shown to be a Lie group, there is a smoothness to the set of such transformations, and we say that the set of conservative vector spaces forms a smooth manifold.

We further illustrate with an example that the set of conservative vector spaces for a pre-specified OVS is not a vector space itself, as can be seen by defining the following 1-partition OVS:

$$\mathfrak{U} = \left(\mathbb{R}^2, \langle .,. \rangle, \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \left\{\mathbb{R}^2\right\}, \left\{\text{span}\left(\mathbf{e}^{(1)}\right), \text{span}\left(\mathbf{e}^{(2)}\right)\right\} \right)\right), \tag{3.175}$$

with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^2$. The associated quadratic form is written as

$$Q\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = 2x_1x_2. \tag{3.176}$$

By inspection, the entire set $\mathcal{S}$ of vectors x for which $Q(x) = 0$ is $\mathcal{S} = \text{span}\left(\mathbf{e}^{(1)}\right) \cup \text{span}\left(\mathbf{e}^{(2)}\right)$, which is not a vector space but rather a union of vector spaces.

This statement regarding the synthesis of conservative vector spaces is formalized in the following theorem, which paraphrases a standard result in the theory of quadratic forms using the terminology of OVSs.

Theorem 3.3. *Given an OVS $\mathfrak{U} = (V, \langle .,. \rangle, (C, \mathcal{D}_p, \mathcal{D}_c)$, the manifold of J-dimensional vector subspaces $W \subset V$ over which $\mathfrak{U}$ is conservative may be obtained by beginning with an arbitrary J-dimensional conservative subspace $W' \subset V$ and applying the group of transformations in $\mathcal{G}_Q$ preserving the associated quadratic form $Q(x) = \langle Cx, x \rangle$.*

Proof. We proceed by showing that given any two J-dimensional conservative subspaces $W \subset V$ and $W' \subset V$, there exists a transformation $T \in \mathcal{G}_Q$ for which

$$W' = \{Tx : x \in W\}. \tag{3.177}$$

The subspaces W and W' are of the same dimension, and consequently there exists a linear map that invertibly maps W to W'. We refer to this map as $T' : W \to W'$. As W and W' are conservative, the map T' is an isometry between the spaces, i.e.

$$Q(x) = Q(T'x) = 0, \quad \forall x \in W. \tag{3.178}$$

As discussed in [27], the extension theorem of Witt states that given an isometry between any two subspaces of a quadratic space, there exists an extension of the isometry to an isometry on the whole space. Written formally, given the subspaces W and W', there exists an extension $T : V \to V$ of T' that satisfies

$$Tx = T'x, \quad x \in W \tag{3.179}$$

and

$$Q(Tx) = Q(x), \quad x \in V. \tag{3.180}$$

Therefore for any such subspaces W and W', there exists a transformation T that is in the group $\mathcal{G}_Q$ and satisfies Eq. 3.177.

It follows that, given a conservative vector space W of dimension J, the set of transformations used to realize all conservative vector spaces W' of dimension J is contained in $\mathcal{G}_Q$. In showing that $\mathcal{G}_Q$ is contained in the set of all such transformations, we observe that every $T \in \mathcal{G}_Q$ is invertible, and consequently every vector space $W' = \{Tx : x \in W\}$ is a conservative space and has the same dimension as W. $\square$

A related question is that of what the maximum dimension of a conservative space can be. In the language of quadratic forms, a subspace $W \subset V$for which $Q(x) = 0, \ x \in W$ is referred to as a totally isotropic subspace, and the maximum allowable dimension of a totally isotropic subspace of V is referred to as the isotropy index. In general, a given quadratic form that is non-degenerate with signature (p, q) has an isotropy index that is the minimum of p and q.

As we are concerned with balanced quadratic forms, i.e. those whose signature is (L, L) for a $2L$-dimensional OVS, we conclude that the maximum dimension of a conservative vector space for a $2L$-dimensional OVS is L. An L-dimensional vector space over which a $2L$-dimensional OVS is conservative will accordingly be referred to as a **maximal conservative** vector space.

Chapter 4

Conservative interconnecting systems

In Chapter 3 we discussed conservation from the perspective of abstract vector spaces, and in this chapter the ideas are interpreted within the context of linear interconnection structures. The theme will be to use the results in Chapter 2 to apply the principles in Chapter 3 to systems in an input-output representation. In doing so, there will be a focus on linear signal-flow graphs, and many of the results will be equally applicable to interconnections represented as multiple-input, multiple-output linear systems.

The chapter begins by using the theorems in Chapter 3 related to the synthesis of conservative vector spaces to develop techniques for generating conservative linear interconnections, also connecting these results to electrical network-based transformations. The structure of the Lie group $\mathcal{G}_Q$ will also be interpreted within this context, facilitating discussion of signal-flow graph conditions for strong and weak conservation. These conditions will be useful in arriving at strategies for identifying conservation in pre-specified linear interconnections, illustrating their application within the context of a speed control system for a chain of vehicles.

4.1 Image representations of conservative interconnections

In Chapter 3, the tools of group theory were used as a foundation for discussing conservation over vector spaces. It was shown in Thm. 3.3 that given an OVS, any conservative J-dimensional vector space could be generated by beginning with an arbitrary J-dimensional conservative vector space and applying transformations from the group preserving the associated quadratic form. The group was in turn decomposed into subgroups of transformations in Thm. 3.1. In this section, these subgroups will be used to gain insight into the specific structure of the behavior of conservative vector spaces. The approach will be to view the families of transformations as manipulations of image representations that are realized as linear signal-flow graphs, with an electrical analog being provided for each. This perspective will allow the families of transformations to be interpreted using familiar signal processing and electrical network principles, providing insight into the structure of the group and the relationship between the families of transformations, and laying the groundwork for the development of signal-flow graph theorems pertaining to strongly- and weakly-conservative interconnections. As an arbitrary conservative vector space is a subspace of a maximal conservative vector space, we will focus our attention on maximal conservative spaces.

Given a pre-specified $2L$-dimensional OVS $\mathfrak{U} = (V, \langle ., . \rangle, \mathcal{O})$, with $\mathcal{O} = (C, \mathcal{D}_p, \mathcal{D}_c)$, we may formally write the process in Thm 3.3 in terms of $2L \times 2L$ matrices corresponding to a canonical conjugate basis $\mathcal{B}$ for $\mathfrak{U}$ as

$$W = \text{range}\left(R_{N_t} R_{N_t-1} \cdots R_1 R_0\right), \tag{4.1}$$

where R_0 is a (singular) matrix whose range is a J-dimensional vector space over which $\mathfrak{U}$ is conservative, and where each of $R_1, \ldots, R_{N_t}$ is one of the (invertible) matrices listed in Thm. 3.1. The conservative vector subspace W_k obtained after applying k

such transformations, $k = 0, \ldots, N_t$, is accordingly written as

$$W_k = \text{range}\left(R_k \cdots R_1 R_0\right), \tag{4.2}$$

with W_0 being the initial conservative space, W_1 being the conservative space obtained after applying R_1, W_2 being the conservative space obtained after applying R_2, and so forth.

Signal-flow graph representations of the matrices in Thm. 3.1, can likewise be cascaded in a manner that is one-to-one with the cascade of matrices in Eq. 4.1. These representations, in addition to associated electrical network representations, are depicted in Fig. 4-1. Referring to this figure, the signal-flow graphs are direct implementations of the matrices in Thm. 3.1, and the electrical network representations are multi-port systems that implement the transformations under the requirement that the port condition is satisfied, i.e. under the requirement that the current entering any port is equal to the current leaving the port.

Still referring to Fig. 4-1, the variables $a_1, \ldots, a_L$ and $a'_1, \ldots, a'_L$ represent the entries of a vector $x \in \mathbb{R}^{2L}$ in the conjugate subspace V_A, and the variables $b_1, \ldots, b_L$ and $b'_1, \ldots, b'_L$ represent the entries that are in V_B. The systems preserve the associated quadratic form, i.e. if the set of unprimed variables lies in a conservative space, then the set of primed variables lies in a conservative space as well. As such, in a canonical conjugate basis any conservative space may be obtained by beginning with a system whose behavior is an arbitrary conservative space and cascading the appropriate systems in a process that is one-to-one with the use of the cascade of matrices in Eq. 4.1.

In particular, any conservative space may be obtained in this way by beginning first with a signal-flow graph-based image representation for an arbitrary conservative space W_0, corresponding to R_0 in Eq. 4.1, and cascading the appropriate transformation signal-flow graphs, corresponding to $R_1, \ldots, R_{N_t}$ in this equation. This is specifically done by connecting the output variables $a'_1, \ldots, a'_L, b'_1, \ldots, b'_L$ in the initial image representation to the corresponding input variables $a_1, \ldots, a_L, b_1, \ldots, b_L$ in the next signal-flow graph, connecting the output variables in that signal-flow graph to the input variables in the next, and so on. The resulting signal-flow graph will in turn be an image representation of a conservative space W_{N_t} that is of the same dimension as the initial conservative space W_0.

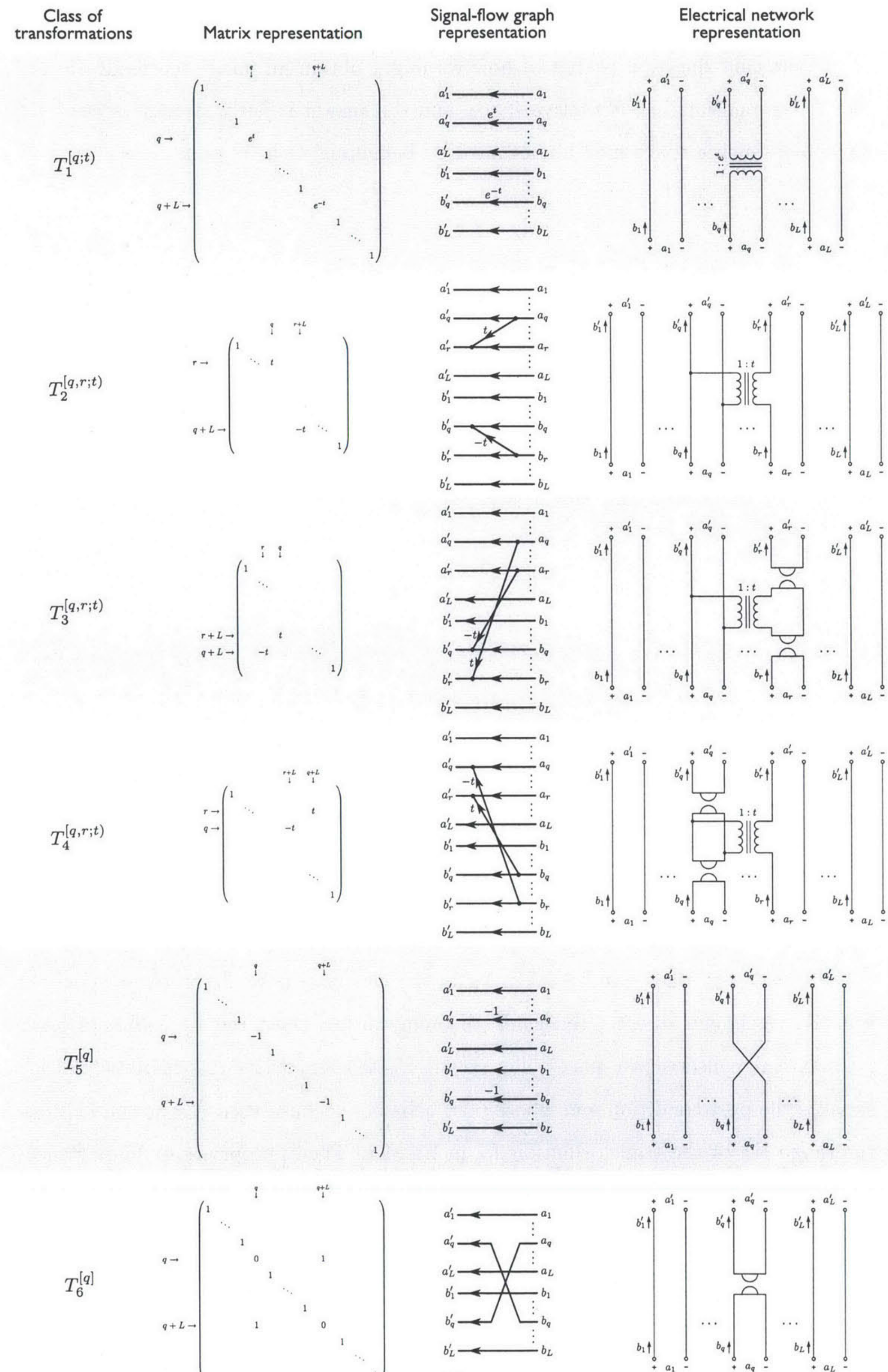


Figure 4-1: Classes of one-parameter subgroups of $\mathcal{G}_Q^{(\mathcal{B})}$ and corresponding representations as matrices, signal-flow graphs and multi-port electrical networks.

A pertinent question is that of how we might obtain an image representation of initial J-dimensional conservative space, and the answer is fairly straightforward. In particular, such a space may be obtained by beginning with the constraints

$$a_1 = \cdots = a_L = 0 \tag{4.3}$$

and

$$b_1 = \cdots = b_L = 0, \tag{4.4}$$

and then for each a_k, b_k pair, $k = 1, \ldots, J$, removing this constraint from exactly one of a_k and b_k. The resulting vector space will thus be J-dimensional and conservative, as the product of each pair will be zero, i.e.

$$a_k b_k = 0, \quad k = 1, \ldots, L, \tag{4.5}$$

and a total of J variables will be unconstrained real numbers. An image representation that is generated in this manner can then be manipulated to obtain an input-output representation of a conservative interconnection, using the path reversal technique discussed in Chapter 2. An example depicting this process is illustrated in Fig. 4-2. Referring to Fig. 4-2, it is straightforward to verify that the condition in Cor. 2.1 required to perform path reversal is met, as the gain from the input to the output of each of the paths to be reversed is 1.

A similar approach can be taken using the electrical network representations in Fig. 4-1, although due to Tellegen's theorem we are restricted to L-dimensional, i.e. maximal, conservative spaces. The strategy is to begin with an initial network that satisfies the port condition and whose port behavior is conservative, representative of the range W_0 of the transformation R_0 in Eq. 4.1. The appropriate multi-port networks in Fig. 4-1, each implementing one of the transformations R_k for $k = 1, \ldots, N_t$, can then be cascaded with this system, resulting in an electrical network whose port behavior is that of any L-dimensional conservative vector space. The initial conservative space, whose behavior is W_0, can be generated by either shorting or leaving

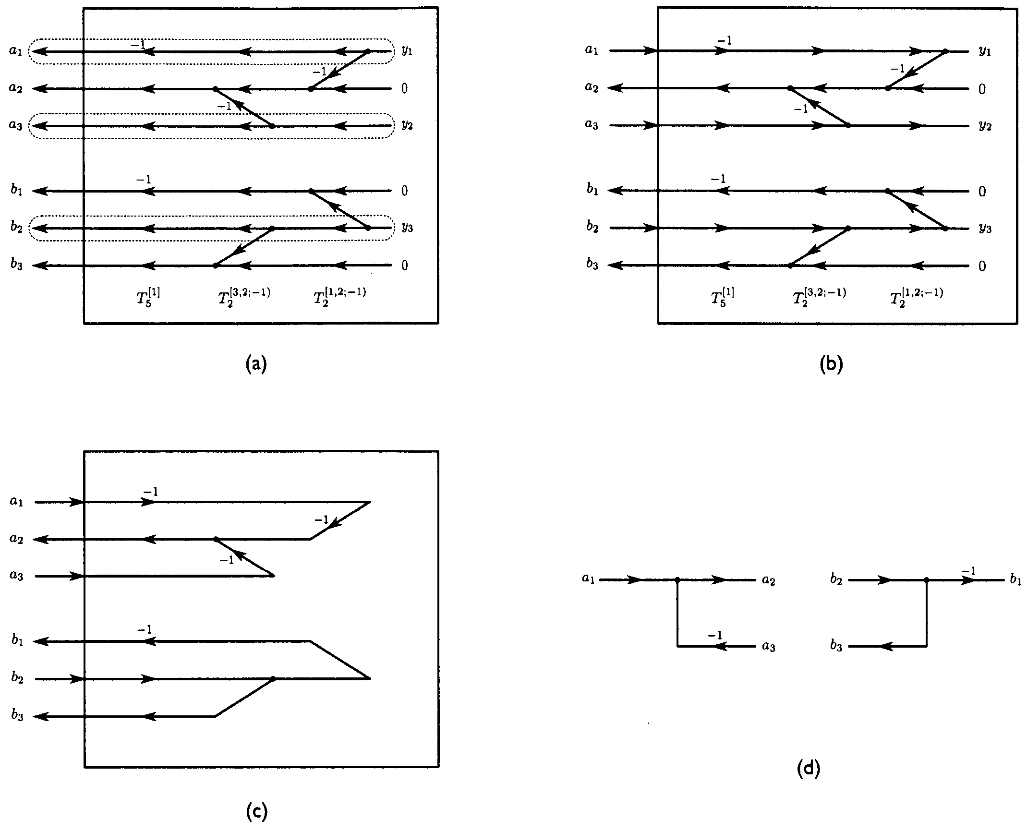


Figure 4-2: (a) Image representation for a conservative interconnection generated by cascading the transforming systems in Fig. 4-1. (b) Behaviorally-equivalent interconnection in input-output form, obtained using path reversal. (c) Simplified input-output representation. (d) Resulting structure after rearranging the layout of the graph.

open the unprimed ports on the transformation corresponding to R_1. In particular, shorting a particular port k implies that the port voltage a_k is zero and that the port current b_k is unconstrained. Likewise, leaving a particular port k open implies that the port voltage a_k is unconstrained and that the port current b_k is zero. Both operations satisfy the port condition and have a conservative initial behavior, resulting in a cascade of electrical networks whose final port behavior is conservative. Fig. 4-3 illustrates an example where this process is performed using the same transformations and initial conservative space as in Fig. 4-2, resulting in an electrical network corresponding to three elements in series.

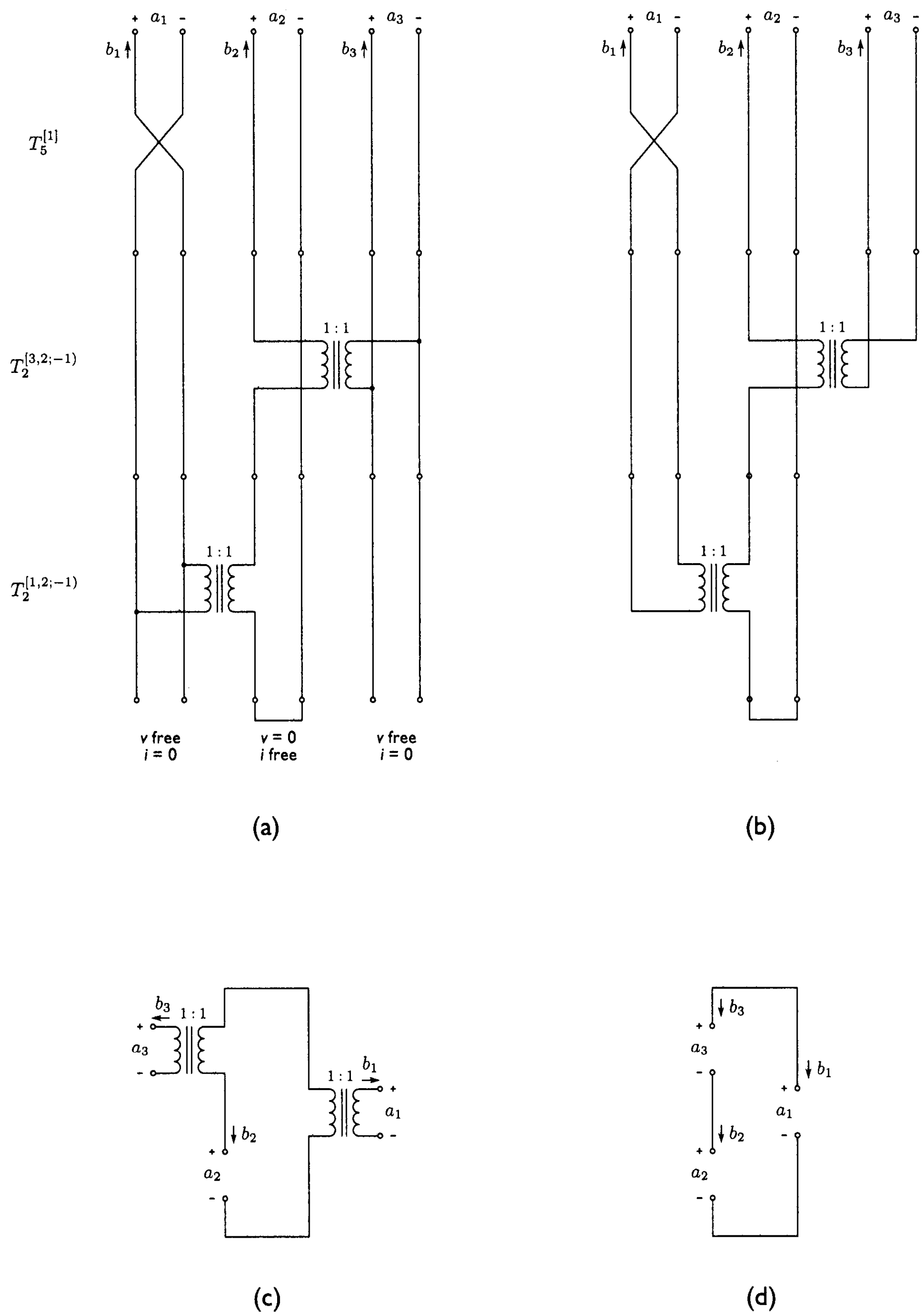


Figure 4-3: (a) Port-conservative electrical network generated by cascading the transforming networks in Fig. 4-1. (b) Simplified representation. (c) Resulting network after rearranging its layout. (d) Network obtained after removing the 1 : 1 transformers.

4.2 Comments on the structure of $\mathcal{G}_Q$

The representations of the families of transformations in Fig. 4-1 may be used to gain insight into the structure of the group of transformations that preserve the quadratic form associated with a given OVS, denoted previously as $\mathcal{G}_Q$. We have seen that the transformations in Fig. 4-1 preserve the quadratic form written in a canonical conjugate basis in particular, and that for an arbitrary OVS of the same dimension, a change of basis can be used to obtain a corresponding quadratic form-preserving transformation. It was also shown in the proof for Thm. 3.1 that the group of transformations that preserves the quadratic form for an arbitrary OVS of dimension $2L$ is, to within a change of basis, identical to the so-called indefinite orthogonal group with signature (L, L), denoted $O(L, L)$. As such, the group $\mathcal{G}_Q$ was said to be isomorphic to $O(L, L)$, and the statement of being isomorphic was written $\mathcal{G}_Q \cong O(L, L)$. In this section, we use the signal-flow graph and electrical network representations of these transformations to gain insight into the structure of $\mathcal{G}_Q$ and its relationship to the structure of $O(L, L)$. It will be shown that some of the families of transformations, in particular $T_3^{[q,r;t)}$, $T_4^{[q,r;t)}$ and $T_6^{[q,r;t)}$, do not always preserve the strength of conservation, while the others do. This is in turn related to the issue of which of the four connected components of $\mathcal{G}_Q$ a given transformation lies.

4.2.1 Isomorphisms with $O(L, L)$, $SO(L, L)$ and $SO^+(L, L)$

It was shown in the proof for Thm. 3.1 that the group $\mathcal{G}_Q$ is isomorphic to $O(L, L)$, and as $O(L, L)$ is known to have four connected components, $\mathcal{G}_Q$ was shown to have four connected components as well. The group $O(L, L)$ is specifically the group of transformations that preserves the quadratic form

$$Q\left([x_1, \ldots, x_{2L}]^{tr}\right) = x_1^2 + \cdots + x_L^2 - x_{L+1}^2 - \cdots - x_{2L}^2, \tag{4.6}$$

and the four connected components of $O(L, L)$ relate to whether a given transformation reverses the orientation of the subspaces

$$V_+ = \mathrm{span}(e_1, \ldots, e_L) \tag{4.7}$$

and

$$V_- = \mathrm{span}(e_{L+1}, \ldots, e_{2L}). \tag{4.8}$$

There are two subgroups of $O(L, L)$ that commonly receive special attention. One is the so-called special orthogonal group, denoted $SO(L, L)$, which consists of two of the components of $O(L, L)$. These components are specifically the one that reverses the orientation of both subspaces V_+ and V_- and the one that contains the identity element. The latter component is a subgroup itself and is denoted $SO^+(L, L)$. It was shown in the proof of Thm. 3.1 that to within a change of basis, each of the transformations in the family $T_5^{[q]}$ reverses the orientations of both V_+ and V_-, each of the transformations in the family $T_6^{[q]}$ reverses the orientation of one of V_+ and V_-, and that the other families of transformations $T_1^{[q;t)}, \ldots, T_4^{[q,r;t)}$ reverse the orientation of neither subspace. As such, a group of transformations that is isomorphic to the orthogonal group $O(L, L)$ may be obtained by beginning with the identity element and applying transformations from $T_1^{[q;t)}, \ldots, T_6^{[q]}$. If an even number of transformations in the family $T_6^{[q]}$ is used, the resulting transformation will be in a group that is isomorphic to the special orthogonal group $SO(L, L)$. Likewise, if an even number of transformations in $T_6^{[q]}$ is used and an even number of transformations in $T_5^{[q]}$ is used, the resulting transformation will be in a group that is isomorphic to $SO^+(L, L)$. The relationship between these groups is illustrated in Fig. 4-4.

4.2.2 The families $T_1^{[q;t)}$, $T_2^{[q,r;t)}$ and $T_5^{[q]}$ generate all strongly-conservative vector spaces

As is indicated in Fig. 4-4, the transformations in families $T_1^{[q;t)}$, $T_2^{[q,r;t)}$ and $T_5^{[q]}$ preserve strong conservation, i.e. given a space W that is strongly conservative, the

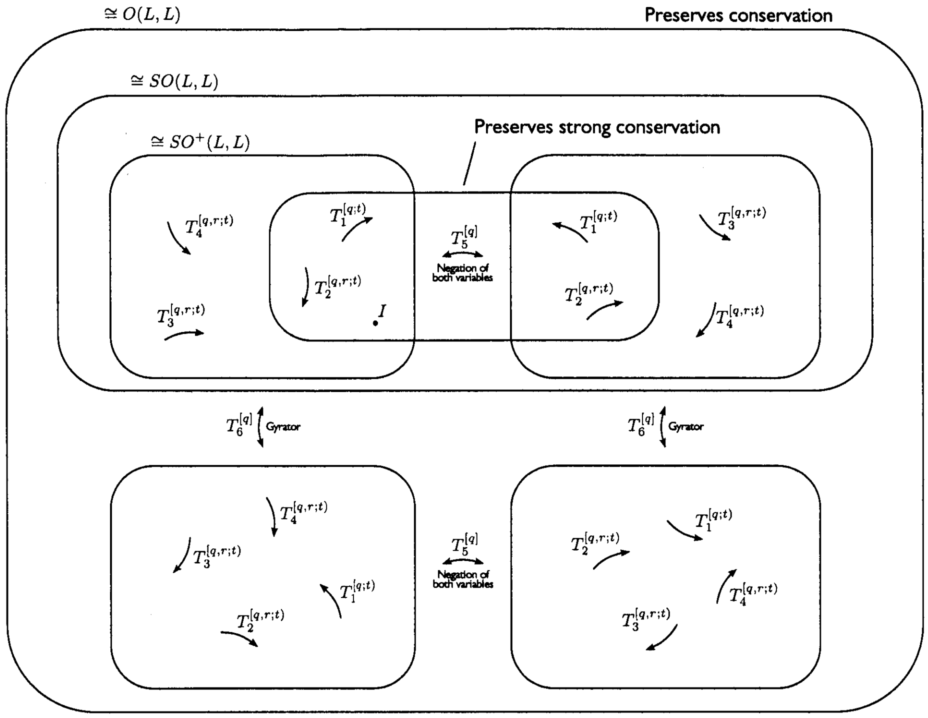


Figure 4-4: The structure of the group $\mathcal{G}_Q$ in terms of the families $T_1^{[q;t]}, \ldots, T_6^{[q]}$.

transformed space $W' = \{Tx : x \in W\}$ is strongly conservative if T is generated using the transformations in these families. This can be readily seen by noting that these transformations preserve the dimension of the behaviors of the variables making up the conjugate subspaces. As such, the if the necessary and sufficient condition for strong conservation that is mentioned in Thm. 3.2, and which pertains to the dimensions of these spaces, is satisfied before transformation, then it will be satisfied after transformation as well.

It can also be shown that, beginning with a strongly-conservative space where in a given pair of conjugate variables one is unconstrained and the other is set to zero, transformations from the families $T_1^{[q;t)}$, $T_2^{[q,r;t)}$ and $T_5^{[q]}$ can be used to generate all vector spaces that are strongly-conservative. This is equivalent to showing that the process may be used to generate an arbitrary vector space for the behavior of one of the conjugate spaces, as the behavior of the other space will be the (unique) orthogonal complement. This can be seen by viewing the effect of the transformations on the conjugate subspace $V_A = \mathbb{R}^L$ and making the observation that an arbitrary vector subspace W_A of V_A can be generated by beginning with a diagonal projection matrix T_0 whose range is the dimension of W_A, and multiplying an appropriate invertible matrix T to obtain a resulting matrix whose range is W_A. As the matrix T is invertible, it admits an LUP decomposition, i.e. there exists a lower-triangular matrix L, an upper-triangular matrix U, and a permutation matrix P for which $LUP = T$, and accordingly we may write range$(LUPT_0) = W_A$. For any diagonal projection matrix T_0 and permutation matrix P, the range of the matrix PT_0 is the span of some subset of the vectors $e_1, \ldots, e_L$, which can be generated by taking a given conjugate variable and either leaving it unconstrained or setting it to zero. This is exactly the effect that our technique for generating an initial strongly-conservative space has on the conjugate subspace V_A. The matrix LU can be generated by multiplying the elementary row multiplication and row addition matrices, which is precisely how the matrices in the families $T_1^{[q;t)}$, $T_2^{[q,r;t)}$ and $T_5^{[q]}$ operate on the conjugate subspace V_A as well. As such, our technique can create an arbitrary vector space $W_A \subset V_A$, and we conclude that it can be used to create an arbitrary strongly-conservative space.

We may use these observations to conclude that Tellegen's theorem, which as was previously mentioned is a statement of strong conservation for electrical networks, applies to a broader class of interconnections than those that are a result of the Kirchoff laws. In particular, the electrical network representations of $T_1^{[q;t)}$, $T_2^{[q,r;t)}$ and $T_5^{[q]}$ in Fig. 4-1, when connected to an initial set of ports where each is either shorted or open, generate the broadest class of electrical interconnections to which a statement of strong conservation, such as Tellegen's theorem, can apply. These interconnections involve ideal transformers, underscoring potential reasons why they are not a primary focus in the study of electrical networks.

4.2.3 The families $T_1^{[q;t)}$, $T_2^{[q,r;t)}$, $T_5^{[q]}$, and $T_6^{[q]}$ generate all conservative vector spaces

It was shown in Thm. 3.1 that the families $T_1^{[q;t)}, \ldots, T_6^{[q]}$ can be used to generate $\mathcal{G}_Q$, the group of transformations that preserves the C-induced quadratic form Q. As was previously mentioned, this was based on the observation that given a $2L$-dimensional OVS, $\mathcal{G}_Q$ is isomorphic to the indefinite orthogonal group $O(L,L)$, which has four connected components. As such, the transformations $T_1^{[q;t)}, \ldots, T_4^{[q,r;t)}$ generate a group that is isomorphic to the component containing the identity element, denoted $SO^+(L,L)$, and the theory of Lie groups tells us that two additional transformations are required to move between $SO^+(L,L)$ and the other three components making up $O(L,L)$. We saw that two such transformations can be obtained by selecting any transformation in the family $T_5^{[q]}$ and selecting any transformation in the family $T_6^{[q]}$, as for any fixed q_1, q_2 between 1 and L, $T_5^{[q_1]}$ reverses the orientation of both of the subspaces V_+ and V_-, and $T_6^{[q_2]}$ reverses the orientation of one of the subspaces V_+ and V_-, with V_+ and V_- being defined as in Eqns. 4.7-4.8. As such, the transformations in families $T_1^{[q;t)}, \ldots, T_6^{[q]}$ allow for the use of all transformations in $T_5^{[q]}$ and $T_6^{[q]}$, and families $T_1^{[q;t)}, \ldots, T_6^{[q]}$ can accordingly be said to generate $\mathcal{G}_Q$ redundantly.

Using identities relating the transformations in families $T_3^{[q,r;t)}$ and $T_4^{[q,r;t)}$ to those in $T_2^{[q,r;t)}$ and $T_6^{[q]}$, we conclude that the redundancy in families $T_1^{[q;t)}, \ldots, T_6^{[q]}$ can be reduced by eliminating $T_3^{[q,r;t)}$ and $T_4^{[q,r;t)}$. The identities that we use are specifically

$$T_3^{[q,r;t)} = T_6^{[r]} T_2^{[q,r;t)} T_6^{[r]} \tag{4.9}$$

and

$$T_4^{[q,r;t)} = T_6^{[q]} T_2^{[q,r;t)} T_6^{[q]}. \tag{4.10}$$

Eqns. 4.9-4.10 can be readily derived by performing signal-flow graph manipulations on their representations in Fig. 4-1. Fig. 4-5 depicts these identities as viewed from the perspective of signal-flow graphs.

The generating set consisting of $T_1^{[q;t)}$, $T_2^{[q,r;t)}$, $T_5^{[q]}$, and $T_6^{[q]}$, in addition to other

(a)

(b)

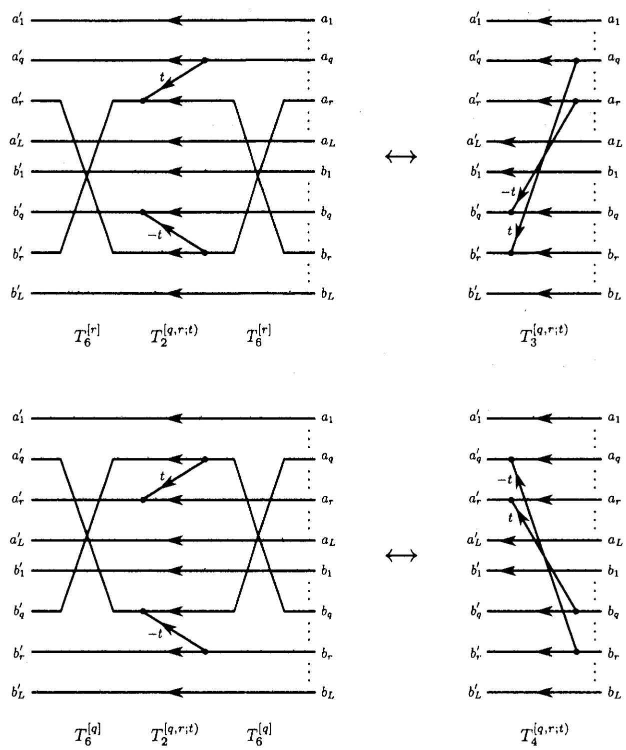


Figure 4-5: Signal-flow graph representations of the identities in Eqns. 4.9-4.10, which relate families $T_3^{[q,r;t]}$ and $T_4^{[q,r;t]}$ to families $T_2^{[q,r;t]}$ and $T_6^{[q]}$.

relevant sets of subgroups of $\mathcal{G}_Q$, are listed in Table 4.1. Referring to this table, a key point is that the previously-mentioned generating set differs from the set preserving strong conservation only in that the family $T_6^{[q]}$ is omitted from the latter. In Section 4.4, this fact will be used as the basis of a technique for strengthening the sense of conservation in conservative systems that are known to be weakly-conservative.

Set of subgroups of $\mathcal{G}_Q$ ($\cong O(L,L)$):	$T_1^{[q;t]}$	$T_2^{[q,r;t]}$	$T_3^{[q,r;t]}$	$T_4^{[q,r;t]}$	$T_5^{[q]}$	$T_6^{[q]}$
Generates $\mathcal{G}_Q$:	•	•	•	•	•[1]	•[1]
Generates subgroup $\cong SO(L,L)$:	•	•	•	•	•	
Generates subgroup $\cong SO^+(L,L)$:	•	•	•	•		
Generates $\mathcal{G}_Q$:	•	•			•[2]	•
Preserves strong conservation:	•	•			•	

[1]No more than one transformation in each of $T_5^{[q]}$ and $T_6^{[q]}$ is required.
[2]No more than one transformation in $T_5^{[q]}$ is required.

Table 4.1: Some relevant sets of subgroups of $\mathcal{G}_Q$. A dot indicates that the subgroup corresponding to the column is in the set corresponding to the row.

4.3 Generating matched conservative interconnecting systems

In Section 4.1 it was shown that a signal-flow graph for a maximal conservative interconnection with $2L$ terminal variables could be obtained by beginning with a conservative image representation involving L exogenous variables, applying transformations from the group preserving the C-induced quadratic form for an appropriately-defined OVS, and using path reversal to bring the image representation to the form of a linear map, represented as a signal-flow graph with an associated gain matrix G. In this section, we address the question of how to design conservative linear interconnections from the perspective of designing, from the outset, an appropriate interconnection gain matrix G. As such, we illustrate that the interconnection gain matrix for a conservative interconnection takes on a special form, and that strong conservation imposes specific additional structure on G. Moreover, all conservative and strongly-conservative behaviors will be shown to be obtainable by designing gain matrices with the appropriate respective form. This will facilitate the design of strongly- and weakly-conservative signal-flow graph interconnections, where the conditions on G will be related to flow graph transposition, negation of certain branches, and separability of the graph into independent sub-graphs interconnecting each of the conjugate spaces. In particular, separability of the signal-flow graph will be related to conservation strength, and will be shown to be a mechanism by which conservation principles applicable to "two distinct networks having the same topology" arise.

We will specifically focus discussion on conservation in linear interconnecting systems where the correspondence map for the OVS is represented in its canonical conjugate basis. As was shown in Chapter 3, an arbitrary correspondence map can be represented in this way. We will be referring to an OVS $\mathfrak{U}$ that is defined as

$$\mathfrak{U} = \left(\mathbb{R}^{2L}, \langle .,. \rangle, \mathcal{O}\right), \tag{4.11}$$

with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^{2L}$, and with the conjugate subspaces and the correspondence map in the organization $\mathcal{O} = (C, \mathcal{D}_p, \{V_A, V_B\})$ being written as

$$\begin{aligned} C &= \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} && (4.12) \\ V_A &= \text{span}(e_1, \ldots, e_L) && (4.13) \\ V_B &= \text{span}(e_{L+1}, \ldots, e_{2L}), && (4.14) \end{aligned}$$

where I_L and 0_L are respectively the $L \times L$ identity and zero matrices. The associated quadratic form is accordingly

$$Q\left([a_1, \ldots, a_L, b_1, \ldots, b_L]^{tr}\right) = 2a_1b_1 + \ldots 2a_Lb_L. \qquad (4.15)$$

The variables $a_1, \ldots, a_L$ denote the interconnection terminal variables in the conjugate subspace V_A and the variables $b_1, \ldots, b_L$ denote the interconnection terminal variables in the conjugate subspace V_B. Conservation of $\mathfrak{U}$ over a vector space W corresponds to the conservation law

$$2a_1b_1 + \ldots 2a_Lb_L = 0, \quad [a_1 \ldots, a_L, b_1, \ldots, b_L]^{tr} \in W. \qquad (4.16)$$

In discussing linear interconnections having conservative behaviors, we will specifically be interested in interconnections that are maps, and which have a total of $2L$ interconnection terminal variables $a_1, \ldots, a_L, b_1, \ldots, b_L$. A natural subclass consists of those interconnections having L inputs and L output, and for which every conjugate pair of variables a_k and b_k has one input and one output. We refer to this class of interconnections as being **input-output matched**. One reason for specifying to this class is that it is common in many existing systems for which we have identified conservation to have an ensemble of two-variable constitutive relations that are maps, each of which is connected to a pair of conjugate variables in the interconnection, and this necessitates the use of one input and one output variable for each pair of conjugate variables.

We will define an L-dimensional linear, matched input-output interconnection as one that contains a total of $2L$ interconnection terminal variables, denoted using the notation in Eq. 4.15 as $a_1, \dots, a_L, b_1, \dots, b_L$, and for which the following statements hold for a given conjugate pair of variables a_k and b_k, $k = 1, \dots, L$:

$$a_k \text{ is an interconnection input} \quad \Leftrightarrow \quad b_k \text{ is an interconnection output} \tag{4.17}$$

$$a_k \text{ is an interconnection output} \quad \Leftrightarrow \quad b_k \text{ is an interconnection input.} \tag{4.18}$$

Such an interconnection has a total of L inputs and consequently has a linear behavior with L degrees of freedom, i.e. its behavior is an L-dimensional vector space. We will denote the interconnection terminal variables in the conjugate subspace V_A using the vector $\mathbf{a} = [a_1, \dots, a_L]^{tr}$ and the interconnection terminal variables in the conjugate subspace V_B as using $\mathbf{b} = [b_1, \dots, b_L]^{tr}$.

In determining the behavior of the linear interconnecting system, we establish an indexing convention for the input and output interconnection terminal variables, equating interconnection input variables c_k, $k = 1, \dots, L$ and interconnection output variables d_k, $k = 1, \dots, L$ to the conjugate variables a_k and b_k. In equating these variables, the subscript k of c_k or d_k will refer to which of the L pairs of conjugate variables a_k or b_k that the variable c_k or d_k is equated. Using this notation, the rules in Eqns. 4.17-4.18 describing an input-output matched interconnection may be written as

$$a_k = c_k \quad \Leftrightarrow \quad b_k = d_k \tag{4.19}$$

$$a_k = d_k \quad \Leftrightarrow \quad b_k = c_k, \tag{4.20}$$

with exactly one of Eqns. 4.19-4.20 being held. We introduce a permutation matrix P that encodes which of Eqns. 4.19-4.20 holds for a given index k, i.e. P encodes the relationship between the variables $a_1, \dots, a_L$, $b_1, \dots, b_L$, $c_1, \dots, c_L$, and $d_1, \dots, d_L$ as

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = P \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}, \tag{4.21}$$

with the vectors $\mathbf{c} = [c_1, \ldots, c_L]^{tr}$ and $\mathbf{d} = [d_1, \ldots, d_L]^{tr}$ respectively denoting the interconnection input and output variables. The interconnection gain matrix G is in turn related to $\mathbf{c}$ and $\mathbf{d}$ according to

$$\mathbf{d} = G\mathbf{c}, \tag{4.22}$$

and we have

$$\begin{aligned} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} &= P \begin{bmatrix} \mathbf{c} \\ G\mathbf{c} \end{bmatrix} && (4.23) \\ &= P \begin{bmatrix} I_L \\ G \end{bmatrix} \mathbf{c}. && (4.24) \end{aligned}$$

The structure that satisfying exactly one of Eqns. 4.19-4.20 for each index k imposes on the matrix P can be readily seen by viewing P as being composed of four $L \times L$ matrix blocks, i.e.

$$P = \begin{bmatrix} P^{(ac)} & P^{(ad)} \\ P^{(bc)} & P^{(bd)} \end{bmatrix}. \tag{4.25}$$

Satisfying exactly one of Eqns. 4.19-4.20 implies that each of the matrices $P^{(ac)}$, $P^{(ad)}$, $P^{(bc)}$, and $P^{(bd)}$ will be diagonal, with zero- and one-valued entries. The structure that is imposed by Eqns. 4.19-4.20 on the diagonal elements of the blocks of P is specifically that exactly one of the following holds for each $k = 1, \ldots, L$:

$$P^{(ac)}_{k,k} = P^{(bd)}_{k,k} = 1 \quad \text{and} \quad P^{(ad)}_{k,k} = P^{(bc)}_{k,k} = 0 \tag{4.26}$$

$$P^{(ac)}_{k,k} = P^{(bd)}_{k,k} = 0 \quad \text{and} \quad P^{(ad)}_{k,k} = P^{(bc)}_{k,k} = 1, \tag{4.27}$$

and we observe from these equations and the previously-mentioned structure of P

that the blocks satisfy

$$P^{(bd)} = P^{(ac)} \tag{4.28}$$

$$P^{(ad)} = I_L - P^{(ac)} \tag{4.29}$$

$$P^{(bc)} = I_L - P^{(ac)} \tag{4.30}$$

and are individually symmetric and idempotent.

4.3.1 A condition for conservation

With the structure of the correspondence between conjugate variables and interconnection inputs and outputs in place, it is straightforward to determine conditions on the interconnection gain matrix G that result in conservation of the OVS $\mathfrak{U}$, as defined in Eq. 4.11. We are specifically interested in conditions on G under which the quadratic form associated with this OVS evaluates to zero, resulting in a conservation law of the form of Eq. 4.16. Eqns. 4.21-4.22 may accordingly be used to relate G to the variables a_k and b_k in Eq. 4.16.

Before writing this relationship formally, we make the observation that the permutation matrix P in Eq. 4.21 is in the group $\mathcal{G}_Q$ preserving the quadratic form in Eq. 4.15 associated with the OVS $\mathfrak{U}$. This can readily be seen by writing the expression for the transformed quadratic form explicitly:

$$
\begin{aligned}
Q(P\mathbf{x}) &= \langle P^{tr}CP\mathbf{x}, \mathbf{x}\rangle && (4.31)\\
&= \left\langle \begin{bmatrix} P^{(ac)} & P^{(ad)} \\ P^{(bc)} & P^{(bd)} \end{bmatrix}^{tr} \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} \begin{bmatrix} P^{(ac)} & P^{(ad)} \\ P^{(bc)} & P^{(bd)} \end{bmatrix} \mathbf{x}, \mathbf{x} \right\rangle && (4.32)\\
&= \left\langle \begin{bmatrix} P^{(ac)} & P^{(ad)} \\ P^{(bc)} & P^{(bd)} \end{bmatrix}^{tr} \begin{bmatrix} P^{(bc)} & P^{(bd)} \\ P^{(ac)} & P^{(ad)} \end{bmatrix} \mathbf{x}, \mathbf{x} \right\rangle && (4.33)\\
&= \left\langle \begin{bmatrix} P^{(ac)} & I_L - P^{(ac)} \\ I_L - P^{(ac)} & P^{(ac)} \end{bmatrix} \begin{bmatrix} I_L - P^{(ac)} & P^{(ac)} \\ P^{(ac)} & I_L - P^{(ac)} \end{bmatrix} \mathbf{x}, \mathbf{x} \right\rangle && (4.34)\\
&= \left\langle \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} \mathbf{x}, \mathbf{x} \right\rangle && (4.35)\\
&= Q(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{2L}, && (4.36)
\end{aligned}
$$

with Eq. 4.32 following from the definition of P in Eq. 4.25 and with Eq. 4.34 following from Eqns. 4.28-4.30. We therefore conclude that the following relationship holds between the quadratic form associated with $\mathfrak{U}$ and the interconnection input and output variables:

$$Q\left(\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}\right) = 2\langle \mathbf{a}, \mathbf{b}\rangle_{\mathbb{R}^L} = 2\langle \mathbf{c}, \mathbf{d}\rangle_{\mathbb{R}^L}, \tag{4.37}$$

with $\langle .,. \rangle_{\mathbb{R}^L}$ denoting the standard inner product on $\mathbb{R}^L$.

In establishing conditions for conservation, we are specifically interested in the Eq. 4.37 evaluating to zero. Substituting Eq. 4.22 in Eq. 4.37, we wish to find conditions on G for which the following equation is satisfied:

$$\langle \mathbf{c}, G\mathbf{c}\rangle_{\mathbb{R}^L} = 0, \ \forall \mathbf{c} \in \mathbb{R}^L. \tag{4.38}$$

Decomposing the matrix G into its symmetric and skew-symmetric components as

$$G = \frac{1}{2}(G + G^{tr}) + \frac{1}{2}(G - G^{tr}) \tag{4.39}$$

and substituting this into the expression $\langle \mathbf{c}, G\mathbf{c}\rangle_{\mathbb{R}^L}$ in Eq. 4.38, we obtain

$$
\begin{aligned}
\left\langle \mathbf{c}, \left(\frac{1}{2}(G+G^{tr}) + \frac{1}{2}(G-G^{tr})\right)\mathbf{c}\right\rangle_{\mathbb{R}^L} &= \left\langle \mathbf{c}, \frac{1}{2}(G+G^{tr})\mathbf{c}\right\rangle_{\mathbb{R}^L} \\
&\quad + \left\langle \mathbf{c}, \frac{1}{2}(G-G^{tr})\mathbf{c}\right\rangle_{\mathbb{R}^L} \\
&= \left\langle \mathbf{c}, \frac{1}{2}(G+G^{tr})\mathbf{c}\right\rangle_{\mathbb{R}^L}. \qquad (4.40)
\end{aligned}
$$

We are interested in conditions on G for which $\langle \mathbf{c}, G\mathbf{c}\rangle_{\mathbb{R}^L} = 0$ for all vectors $\mathbf{c}$, written formally in terms of Eq. 4.40 as

$$
\left\langle \mathbf{c}, \frac{1}{2}(G+G^{tr})\mathbf{c}\right\rangle_{\mathbb{R}^L} = 0, \quad \mathbf{c} \in \mathbb{R}^L, \qquad (4.41)
$$

and accordingly Eq. 4.41 is satisfied if and only if the symmetric component of G is zero. This is equivalent to the condition that the matrix G is skew symmetric, i.e. that

$$
G = -G^{tr}. \qquad (4.42)
$$

Fig. 4-6 depicts a maximal conservative interconnection, represented in terms of its interconnection gain matrix G.

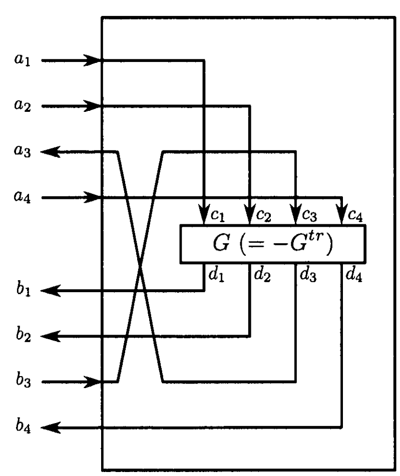


Figure 4-6: A maximal conservative interconnection.

4.3.2 A condition for strong conservation

In Thm. 3.2 it was shown that a conservative vector space W is strongly-conservative if and only if the dimensions of the behaviors of the conjugate spaces sum to the dimension of W, and from this we write a condition for strong conservation involving the interconnection gain matrix G. The approach is specifically to constrain the dimensions of the behaviors of the conjugate spaces to sum to L by creating an interconnecting system that is composed of two linear maps: one that has a behavior of dimension L_A with a total of L_A inputs and couples the variables in V_A, denoted Interconnection A, and one that has a behavior of dimension $L_B = L - L_A$ with a total of L_B inputs that couples the variables in V_B, denoted Interconnection B. Interconnection A will accordingly have a total of L_B outputs and Interconnection will have a total of L_A outputs.

We will denote the respective gain matrices for Interconnection A and Interconnection B as $G^{(A)} : \mathbb{R}^{L_A} \to \mathbb{R}^{L_B}$ and $G^{(B)} : \mathbb{R}^{L_B} \to \mathbb{R}^{L_A}$. The specific relationship between the vectors of input variables $\mathbf{c}^{(A)} \in \mathbb{R}^{L_A}$ and output variables $\mathbf{d}^{(A)} \in \mathbb{R}^{L_B}$ for Interconnection A is given by

$$\mathbf{d}^{(A)} = G^{(A)}\mathbf{c}^{(A)}, \tag{4.43}$$

and the relationship between the vectors of input variables $\mathbf{c}^{(B)} \in \mathbb{R}^{L_B}$ and output variables $\mathbf{d}^{(B)} \in \mathbb{R}^{L_A}$ for Interconnection B is accordingly

$$\mathbf{d}^{(B)} = G^{(B)}\mathbf{c}^{(B)}. \tag{4.44}$$

In determining conditions on $G^{(A)}$ and $G^{(B)}$ that will result in conservation, we first write expressions relating the vectors of conjugate variables $\mathbf{a}$ and $\mathbf{b}$ to the vectors $\mathbf{c}^{(A)}$, $\mathbf{d}^{(A)}$, $\mathbf{d}^{(B)}$, and $\mathbf{c}^{(B)}$, and in doing so, we must specify that the vectors of conjugate variables $\mathbf{a} \in \mathbb{R}^L$ and $\mathbf{b} \in \mathbb{R}^L$ are in a matched input-output configuration. This relationship was stated formally in Eqns. 4.17-4.18.

We proceed by establishing an indexing convention for the elements of the input and output vectors $\mathbf{c}^{(A)}$, $\mathbf{d}^{(A)}$, $\mathbf{d}^{(B)}$, and $\mathbf{c}^{(B)}$, and we use permutation matrices to specify the correspondence between these indices and the indices of the conjugate vectors $\mathbf{a}$ and $\mathbf{b}$. The convention is that the index of a given input variable for Interconnection A is identical to that of the corresponding output variable for Interconnection B, and vice-versa. Written formally, we implement the rules in Eqns. 4.17-4.18 by requiring that exactly one of the following holds for each pair of conjugate variables a_k and b_k, $k = 1, \ldots, L$:

$$a_k = c_p^{(A)} \quad \Leftrightarrow \quad b_k = d_p^{(B)} \tag{4.45}$$

$$a_k = d_p^{(A)} \quad \Leftrightarrow \quad b_k = c_p^{(B)}, \tag{4.46}$$

where for each k, the value of p is in the range $p = 1, \ldots, L_A$ if Eq. 4.45 holds and is in the range $p = 1, \ldots, L_B$ is Eq. 4.46 holds, in such a way that each conjugate variable pair a_k-b_k is connected to one input-output pair $c_p^{(A)}$-$d_p^{(B)}$ or $c_p^{(B)}$-$d_p^{(A)}$. In other words, the pth input to Interconnection A will be matched to the pth output from Interconnection B, and vice-versa, and the index p for this pair will not necessarily be the index k for the associated pair of conjugate variables.

As the strategy in generating linear interconnections with strongly-conservative behaviors is to specify two interconnections, one coupling the variables in the vector $\mathbf{a}$ and the other coupling the variables in the vector $\mathbf{b}$, we will use two permutation matrices to encode the correspondence between the vectors of input and output variables and the vectors of conjugate variables. In particular, the permutation matrix denoted $P^{(A)} : \mathbb{R}^L \to \mathbb{R}^L$ will be used to encode the correspondence between $\mathbf{a}$, $\mathbf{c}^{(A)}$ and $\mathbf{d}^{(A)}$ as

$$\mathbf{a} = P^{(A)} \begin{bmatrix} \mathbf{c}^{(A)} \\ \mathbf{d}^{(A)} \end{bmatrix}, \tag{4.47}$$

and the permutation matrix denoted $P^{(B)} : \mathbb{R}^L \to \mathbb{R}^L$ will be used to encode the correspondence between $\mathbf{b}$, $\mathbf{c}^{(B)}$ and $\mathbf{d}^{(B)}$ as

$$\mathbf{b} = P^{(B)} \begin{bmatrix} \mathbf{d}^{(B)} \\ \mathbf{c}^{(B)} \end{bmatrix}. \tag{4.48}$$

Note that in the vector in the right-hand side of Eq. 4.48, the order of the input and output sub-vectors has been reversed with respect to those in the vector in Eq. 4.47. As every element of $\mathbf{c}^{(A)}$ is matched to an element of $\mathbf{d}^{(B)}$ and every element of $\mathbf{c}^{(B)}$ is matched to an element of $\mathbf{d}^{(A)}$, this convention facilitates discussion of the relationship between $P^{(A)}$ and $P^{(B)}$ that must be satisfied so that the variables in Interconnection A and Interconnection B are in a matched input-output configuration, i.e. so that exactly one of Eqns. 4.45-4.46 holds. These equations, in particular, specify that the permutation matrices for Interconnection A and Interconnection B are equal, i.e. that

$$P^{(A)} = P^{(B)}. \tag{4.49}$$

This relationship follows directly from Eqns. 4.45-4.46, which stated another way, specify that the entry of $\mathbf{a}$ to which a given entry of $\mathbf{c}^{(A)}$ maps is the same entry of $\mathbf{b}$ to which the corresponding entry of $\mathbf{d}^{(B)}$ maps, and accordingly that the the entry of $\mathbf{b}$ to which a given entry of $\mathbf{c}^{(B)}$ maps is the same entry of $\mathbf{a}$ to which the corresponding entry of $\mathbf{d}^{(A)}$ maps. As the mapping is the same for each interconnection, the corresponding permutation matrices encoding the mapping are identical.

With the relationship between the input and output variables and conjugate variables established, we proceed by determining conditions on the respective gain matrices $G^{(A)}$ and $G^{(B)}$ for Interconnections A and B that will result in conservation. As we have explicitly constrained the dimensions of the behaviors of these interconnections to be vector spaces whose dimensions sum to L, i.e. the dimension of the behavior of the overall set of terminal variables, these conditions will specifically result in strong conservation. Following the form of Eq. 4.37, we write the associated quadratic from as

$$
\begin{aligned}
Q\left(\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}\right) &= 2\langle \mathbf{a}, \mathbf{b} \rangle_L \\
&= 2\left\langle P^{(A)}\begin{bmatrix} \mathbf{c}^{(A)} \\ G^{(A)}\mathbf{c}^{(A)} \end{bmatrix}, P^{(B)}\begin{bmatrix} G^{(B)}\mathbf{c}^{(B)} \\ \mathbf{c}^{(B)} \end{bmatrix}\right\rangle_{\mathbb{R}^L}. \qquad (4.50)
\end{aligned}
$$

The right-hand side of Eq. 4.50 is obtained by performing the substitutions in Eqns. 4.47-4.48, followed by those in Eqns. 4.43-4.44. Eq. 4.50 may be further simplified by using the identity $P^{(A)} = P^{(B)}$ and noting that the transpose of a permutation matrix is its inverse, resulting in

$$
\begin{aligned}
Q\left(\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}\right) &= 2\left\langle \begin{bmatrix} \mathbf{c}^{(A)} \\ G^{(A)}\mathbf{c}^{(A)} \end{bmatrix}, P^{(A)^{tr}}P^{(B)}\begin{bmatrix} G^{(B)}\mathbf{c}^{(B)} \\ \mathbf{c}^{(B)} \end{bmatrix}\right\rangle_{\mathbb{R}^L} && (4.51) \\
&= 2\left\langle \mathbf{c}^{(A)}, G^{(B)}\mathbf{c}^{(B)} \right\rangle_{\mathbb{R}^{L_A}} + 2\left\langle G^{(A)}\mathbf{c}^{(A)}, \mathbf{c}^{(B)} \right\rangle_{\mathbb{R}^{L_B}} && (4.52) \\
&= 2\left\langle \mathbf{c}^{(A)}, G^{(B)}\mathbf{c}^{(B)} \right\rangle_{\mathbb{R}^{L_A}} + 2\left\langle \mathbf{c}^{(A)}, G^{(A)^{tr}}\mathbf{c}^{(B)} \right\rangle_{\mathbb{R}^{L_A}} && (4.53) \\
&= 2\left\langle \mathbf{c}^{(A)}, \left(G^{(B)} + G^{(A)^{tr}}\right)\mathbf{c}^{(B)} \right\rangle_{\mathbb{R}^{L_A}}. && (4.54)
\end{aligned}
$$

Here $\langle .,. \rangle_{\mathbb{R}^{L_A}}$ and $\langle .,. \rangle_{\mathbb{R}^{L_B}}$ respectively denote the standard inner products on $\mathbb{R}^{L_A}$ and $\mathbb{R}^{L_B}$, and Eq. 4.53 in particular is obtained by taking the adjoint of $G^{(A)}$, which coincides with its transpose when using these inner products. Setting the quadratic form to zero for all values of the input vectors $\mathbf{c}^{(A)}$ and $\mathbf{c}^{(B)}$, we obtain

$$
\left\langle \mathbf{c}^{(A)}, \left(G^{(B)} + G^{(A)^{tr}}\right)\mathbf{c}^{(B)} \right\rangle_{\mathbb{R}^{L_A}} = 0, \ \forall \mathbf{c}^{(A)} \in \mathbb{R}^{L_A}, \mathbf{c}^{(B)} \in \mathbb{R}^{L_B}, \qquad (4.55)
$$

which is satisfied if and only if $G^{(B)} + G^{(A)^{tr}} = 0$, i.e. if and only if

$$
G^{(B)} = -G^{(A)^{tr}}. \qquad (4.56)
$$

We have thus shown that by beginning with an arbitrary linear interconnection map coupling the variables in one conjugate subspace and creating an interconnection map for the other conjugate subspace whose gain matrix is the negative transpose of that of the first, we will obtain an interconnection whose behavior is strongly conservative. An interconnection that was created using this strategy is depicted in Fig. 4-7.

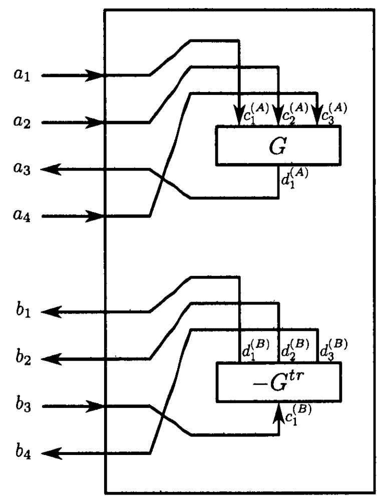


Figure 4-7: A maximal, strongly-conservative interconnection.

An important consequence of developing this technique is that if the interconnection map for one of the conjugate subspaces is represented as a linear, memoryless signal-flow graph, the signal-flow graph for the other conjugate subspace may be created in a straightforward way. Specifically, the second signal-flow graph is obtained from the first by taking the signal-flow graph transpose and negating all of either the input or output branches. That this process creates a pair of linear signal-flow graphs whose gain matrices satisfy Eq. 4.56 is a direct consequence of the transposition theorem, discussed in detail in, e.g. [28]. In particular, the matrix transpose in the right-hand side of Eq. 4.56 corresponds to taking the signal-flow graph transpose, and the negation in the right-hand side of this equation corresponds to the negation of the inputs or of the outputs in the resulting graph.

4.4 Identifying maximal-$\mathcal{D}_p$ conservation in matched interconnecting systems

In the previous section, techniques were developed for creating conservative, matched input-output interconnections, and the general strategy was to begin by formulating conditions for imposing strong and weak conservation on interconnection gain matrices. The sense of conservation specifically pertained to an OVS whose correspondence map was written as a matrix in a canonical conjugate basis, i.e. an OVS for which the conservation law was written as a standard inner product, and invertible transformation interconnections were applied to variables in the partition subspaces to obtain interconnections having other conservation laws. In this section, we pose the related question of, given a linear, input-output matched interconnection and pre-specified partition subspaces, how to determine what conservation laws, if any, might be obeyed.

As was discussed in Chapter 3, a partition-invariant change of basis matrix may in general be selected for transforming an arbitrary OVS into a canonical conjugate representation. As such, the strategy will be to begin with a pre-specified interconnection gain matrix G and collection of partition subspaces $\mathcal{D}_p$, and apply transformations to each of the partition subspaces in an attempt to obtain an interconnection whose gain matrix G' satisfies $G' = -G'^{tr}$, i.e. whose behavior is conservative. If this is successful, the resulting transformations can be used to define an OVS that is conservative over the behavior of the original interconnection G. We will focus discussion on matched input-output interconnections having N interconnection terminal variables, with the OVS being defined over the inner product space $\left(\mathbb{R}^N, \langle .,. \rangle\right)$ and having a total of $N/2 = L$ partition subspaces, with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^N$. In this sense, the discussion will apply to an arbitrary maximal-$\mathcal{D}_p$ OVS, and its partition decomposition may be written as

$$\mathcal{D}_p = \{V_1, \ldots, V_L\}, \tag{4.57}$$

with

$$V_k = \text{span}\left(\mathbf{e}^{(k)}, \mathbf{e}^{(k+L)}\right), \ k = 1, \ldots, L. \tag{4.58}$$

The conjugate decomposition and correspondence map will initially be unspecified.

Defining the behavior W of the interconnection as

$$W = \text{range}\left(\begin{bmatrix} I_L \\ G \end{bmatrix}\right), \tag{4.59}$$

with I_L denoting the $L \times L$ identity matrix, we are interested in a transformation $T : \mathbb{R}^N \to \mathbb{R}^N$ for which the following hold:

(1) T is invertible.

(2) The subspaces $V_1, \ldots, V_L$, defined in Eq. 4.58, are invariant subspaces of T.

(3) The transformed subspace W', defined as

$$W' = \{Tx : x \in W\}, \tag{4.60}$$

is conservative, in the sense that the OVS defined by Eqns. 4.11-4.12 and Eqns. 4.57-4.58 is conservative over W'.

Looking into these requirements further, (2) implies that T can be written as

$$T = \begin{bmatrix} T_{1,1}^{(1)} & & & T_{1,2}^{(1)} & & \\ & \ddots & & & \ddots & \\ & & T_{1,1}^{(L)} & & & T_{1,2}^{(L)} \\ T_{2,1}^{(1)} & & & T_{2,2}^{(1)} & & \\ & \ddots & & & \ddots & \\ & & T_{2,1}^{(L)} & & & T_{2,2}^{(L)} \end{bmatrix}, \tag{4.61}$$

with each entry $T_{1,1}^{(k)}, T_{1,2}^{(k)}, T_{2,1}^{(k)}, T_{2,2}^{(k)}$ $(k = 1, \ldots, L)$ being a real scalar composing a matrix $T^{(k)}$ that denotes the action of T on subspace V_k, i.e.

$$T^{(k)} = \begin{bmatrix} T^{(k)}_{1,1} & T^{(k)}_{1,2} \\ T^{(k)}_{2,1} & T^{(k)}_{2,2} \end{bmatrix}. \tag{4.62}$$

As it is required that T is invertible, the transformation on each of the subspaces V_k must be invertible as well, and so each matrix $T^{(k)}$ must be invertible. Based upon the results in Section 4.3, requirement (3) is equivalent to the requirement that W' is the behavior of some matched input-output interconnection having a skew-symmetric gain matrix, i.e.

$$W' = \text{range}\left(\begin{bmatrix} I_L \\ G' \end{bmatrix}\right), \tag{4.63}$$

for some $L \times L$ matrix G' satisfying

$$G' = -G'^{tr}. \tag{4.64}$$

A primary issue addressed in this section will be that of, given an interconnection behavior W, how to obtain an invertible matrix in the form of Eq. 4.61 that results in a conservative vector space W', i.e. that results in an interconnection having a gain matrix satisfying Eq. 4.64. If such a transformation exists, it will then be used to define a conjugate decomposition $\mathcal{D}'_p = \{V_A, V_B\}$ and correspondence map C', the remaining ingredients needed to define an OVS that is conservative over W. In this case we will say that the interconnection is conservative, with a conservation law that is defined by the quadratic form associated with C'. The strategy for obtaining such a matrix T will in turn be used to show that, with the exception of a degenerate case, all 2-input, 2-output linear interconnections are conservative, and that for these interconnections a closed-form expression for T can be written.

As was the case with the interconnecting system in the example in Subsection 3.2.2, multiple conservation laws will generally exist for a pre-specified interconnection structure, and a secondary issue addressed in this section will be that of how to obtain multiple such laws. It will be shown that given an OVS that is conservative for the interconnection, another OVS having the same partition decomposition and potentially with a different conservation law may be obtained using transformations related to the behavior of the interconnection. An OVS with a having a distinct conservation law may be obtained by parameterizing the set of such transforming matrices using a modified Iwasawa decomposition.

4.4.1 Partition transformations for identifying and strengthening conservation

As was previously mentioned, we are interested in applying invertible transformations to the partition subspaces so as to obtain an interconnection behavior that is conservative. The emphasis will be on linear, matched input-output interconnections, and as we have previously written a straightforward condition for conservation in terms of the interconnection gain matrix for this class of interconnections, the strategy will be to use a graph-based approach. In particular, the search for transformations resulting in a conservative space will be equivalent to the task of choosing parameters in an appropriate transformation graph that, when coupled with the original interconnection, results in a new interconnection whose graph matrix satisfies $G = -G^{tr}$, i.e. that is skew-symmetric.

The condition that the transforming system has the partition subspaces as its invariant subspaces is equivalent to the requirement that the transforming systems couple variables in a given partition subspace only to variables in the same subspace. As such, we are interested in a total of L 2-input, 2-output systems whose behaviors realize the transformations $T^{(k)}$, defined in Eq. 4.62, i.e. we are interested connecting a system from Fig. 2-9 to each pair of variables c_k, d_k in the original interconnection, and choosing the variables in the transforming system so that the coupled interconnection has a gain matrix that is skew-symmetric.

We proceed by determining how the parameters in the transforming system affect the gain matrix when coupled to the original interconnection. Denoting the gain matrix for the coupled interconnection as G', we will write an equation relating G'

to G when one such 2-input, 2-output transforming system is coupled to the interconnection. As the overall transforming system may be composed of L such systems, this relationship can be applied a total of L times to determine the gain matrix for the final coupled interconnection. The relationship between an original interconnection and the modified interconnection obtained by coupling a transforming system is depicted in Fig. 4-8.

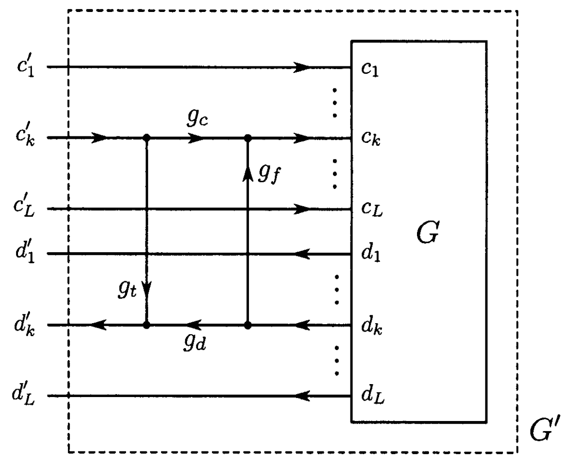


Figure 4-8: A linear interconnection coupled to a 2-input, 2-output transforming system.

The transforming system in Fig. 4-8 consists of four branch gains. We will refer to the gains g_c and g_d as the input and output gains, respectively, as they modify the gains in the paths from c'_k to c_k and d_k to d'_k. The gain g_t will be referred to as the crosstalk gain, as it allows modification of the crosstalk from c'_k to d'_k, and we will refer to g_f as the feedback gain. Denoting the respective vectors of interconnection inputs and outputs corresponding to G as $\mathbf{c}$ and $\mathbf{d}$ and denoting the vectors of interconnection inputs and outputs corresponding to G' as $\mathbf{c}'$ and $\mathbf{d}'$, we write the relationships between the interconnection terminal variables and gain matrices as

$$\mathbf{d} = G\mathbf{c} \tag{4.65}$$

and

$$\mathbf{d}' = G'\mathbf{c}'. \tag{4.66}$$

The relationships between the vectors $\mathbf{c}$, $\mathbf{d}$, $\mathbf{c}'$, and $\mathbf{d}'$ in Fig. 4-8 can be written formally as

$$\mathbf{c} = (I_L + (1 - g_c)P_k)\,\mathbf{c}' + g_f P_k \mathbf{d} \tag{4.67}$$

and

$$\mathbf{d}' = (I_L + (1 - g_d)P_k)\,\mathbf{d} + g_t P_k \mathbf{c}', \tag{4.68}$$

where P_k is an $L \times L$ diagonal matrix with zeros in all of its diagonal entries except the kth, which has value 1. By performing straightforward algebraic manipulations on Eqns. 4.67-4.68, the relationship between the input and output in the coupled interconnection can be written as

$$\mathbf{d}' = \left[g_t P_k + (I_L + (1 - g_d)P_k)\, G\, (I - g_f P_k G)^{-1} (I + (1 - g_c)P_k)\right] \mathbf{c}'. \tag{4.69}$$

Eq. 4.69 takes the form of Eq. 4.66, and we conclude that the gain matrix for the coupled interconnection is

$$G' = g_t P_k + (I_L + (1 - g_d)P_k)\, G\, (I - g_f P_k G)^{-1} (I + (1 - g_c)P_k)\,. \tag{4.70}$$

Looking further into the form of Eq. 4.70, the innermost term can be manipulated using the matrix inversion lemma to obtain

$$\begin{aligned} G\,(I - g_f P_k G)^{-1} &= G + g_f G P_k\,(I_L - g_f P_k G)^{-1} P_k G \\ &= G + g_f \left((I_L - g_f P_k G)^{-1}\right)_{k,k} G P_k G \end{aligned} \tag{4.71}$$

i.e. the innermost term, which is affected by the feedback gain g_f, corresponds to the addition of a scaled rank-1 matrix to G that consists of the outer product between G's kth row and kth column. We accordingly substitute Eq. 4.71 into Eq. 4.70 to obtain

$$G' = g_t P_k + (I_L + (1 - g_d)P_k) \underbrace{\left(G + g_f \left(\left(I_L - g_f P_k G\right)^{-1}\right)_{k,k} G P_k G\right)}_{\widehat{G}} (I + (1 - g_c)P_k)\,, \tag{4.72}$$

denoting the innermost term, i.e. the right-hand side of Eq. 4.71, as $\widehat{G}$.

The form of Eq. 4.72 allows us to make the following observations relating the branch gains in Fig. 4-8 to their effect in modifying the gain matrix.

(1) A nonzero feedback gain g_f corresponds to adding to G a matrix to that is proportional to GP_kG, resulting in the matrix $\widehat{G}$ that is indicated in Eq. 4.72.

(2) The input gain g_c scales column k of $\widehat{G}$ by $1 - g_c$.

(3) The output gain g_d scales row k of $\widehat{G}$ by $1 - g_d$.

(4) A nonzero crosstalk gain g_t adds a constant to the kth diagonal entry of $\widehat{G}$.

4.4.2 A strategy for identifying transformations

With the relationship between the branch gains and the corresponding modifications to G now in place, we develop a strategy for identifying conservation in a class of linear interconnections. We will focus on transforming systems that take the form of Fig. 4-8, with the feedback gain g_f being zero. This will allow us to specialize the set of modifications on G to those that result in column scalings, row scalings, and addition of terms along the matrix diagonal.

Another important reason for specializing to this class is that it will result in a straightforward condition for invertibility of the transforming system. Setting the feedback gain to $g_f = 0$, we write the relationship between the the variables c_k, d_k, c'_k, and d'_k for the transforming system in Fig. 4-8 as

$$\begin{bmatrix} c'_k \\ d'_k \end{bmatrix} = \underbrace{\begin{bmatrix} g_c^{-1} & 0 \\ g_t & g_d \end{bmatrix}}_{T^{(k)}} \begin{bmatrix} c_k \\ d_k \end{bmatrix}. \tag{4.73}$$

The relationship between the original interconnection behavior W and the transformed interconnection behavior W' is accordingly given by Eq. 4.60, with the elements composing T in Eq. 4.60 being defined in terms of 2×2 matrices $T^{(1)}, \ldots, T^{(L)}$ taking the form of Eq. 4.62. For the system in Fig. 4-8, we have in particular that $T^{(i)} = I_L$ for $i \neq k$ and that $T^{(k)}$ is defined as in Eq. 4.73. As we are interested in a transformation T that is invertible, which is equivalent to each of $T^{(1)}, \ldots, T^{(L)}$ being invertible, we require of $T^{(k)}$ that its determinant is well-defined and nonzero, i.e. that the input and output gains g_c and g_d are both nonzero.

With this established, it is straightforward to write a strategy for obtaining an OVS over which the interconnection behavior is conservative:

(1) Beginning with the gain matrix G for the original interconnection, perform a sequence of row and column scalings in an attempt to obtain a matrix $\widetilde{G}$ that is skew-symmetric with the exception of its diagonal elements, i.e. that satisfies

$$\widetilde{G} + \widetilde{G}^{tr} = D, \tag{4.74}$$

where D is a diagonal $L \times L$ matrix.

(2) Apply a sequence of transformations of the form of Fig. 4-8, with the input gains and output gains chosen to encode the manipulations in Step (1), and with the crosstalk gains chosen so as to cancel crosstalk in the interconnection, i.e. with $g_t^{(k)}$ in each being the negative of the kth diagonal element of $\widetilde{G}$.

(3) Write the 2×2 matrices $T^{(k)}$ composing the overall matrix T that corresponds to this transformation. The matrix T can then be used to define the following

OVS, which will be conservative over W:

$$\begin{aligned}
\mathfrak{U} &= \left(\mathbb{R}^{2L}, \langle .,. \rangle, \mathcal{O}\right) & (4.75)\\
\mathcal{O} &= \left(C, \mathcal{D}_p, \mathcal{D}_c\right) & (4.76)\\
C &= T^{tr} \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} T & (4.77)\\
\mathcal{D}_p &= \left\{\operatorname{span}\left(\mathbf{e}^{(1)}, \mathbf{e}^{(L+1)}\right), \operatorname{span}\left(\mathbf{e}^{(2)}, \mathbf{e}^{(L+2)}\right), \dots, \operatorname{span}\left(\mathbf{e}^{(L)}, \mathbf{e}^{(2L)}\right)\right\} & (4.78)\\
\mathcal{D}_c &= \left\{\operatorname{span}\left(\left(T^{tr}\right)_1, \dots, \left(T^{tr}\right)_L\right), \operatorname{span}\left(\left(T^{tr}\right)_{L+1}, \dots, \left(T^{tr}\right)_{2L}\right)\right\}, & (4.79)
\end{aligned}$$

with $(T^{tr})_k$ being the transpose of row k of T and with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^{2L}$.

That the OVS defined in Step (3) is a valid OVS can be seen by noting that partition subspaces for the correspondence map

$$\widehat{C} = \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} \tag{4.80}$$

can be written as

$$\widehat{V}_A = \operatorname{span}\left(\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(L)}\right) \tag{4.81}$$

and

$$\widehat{V}_B = \operatorname{span}\left(\mathbf{e}^{(L+1)}, \dots, \mathbf{e}^{(2L)}\right). \tag{4.82}$$

Then referring to Eq. 4.77, the row space of the first L rows of T is mapped by T to $\widehat{V}_A$ and the row space of the last L rows of T is mapped by T to $\widehat{V}_B$. These spaces are written formally as the spans of the transposes of rows of T in Eq. 4.79. As T is a partition-invariant transformation, the partition decomposition in Eq. 4.78, which is valid for $\widehat{C}$, is also valid for C.

The goal in performing the row and column scalings in Step (1) was to result in a matrix $\widetilde{G}$ satisfying Eq. 4.74, and the question still remains of what failing to be able to do this implies, if anything, about the existence of an OVS that is conservative over W. This issue is discussed further in Subsection 4.4.5, where we conclude that specializing to the class of invertible transforming systems having zero feedback gain does, indeed, allow for the possibility of overlooking a transformation that might otherwise result in obtaining a conservative space W'.[1]

Even with the restriction that the feedback gain in these transformations is zero, the strategy can be used to conclude that every 2-input, 2-output linear interconnection having a gain matrix written as

$$G = \begin{bmatrix} f & g \\ h & i \end{bmatrix} \tag{4.83}$$

is conservative as long as both g and h are nonzero, i.e. as long as the input c_1 affects the output d_2 and the input c_2 affects the output d_1. In this case, scaling the first column by g and scaling the second column by $-h$ results in a matrix $\widetilde{G}$ that is written as

$$\widetilde{G} = \begin{bmatrix} fg & -gh \\ gh & -hi \end{bmatrix}, \tag{4.84}$$

satisfying the requirement in Eq. 4.74 in Step (1). The corresponding transforming systems in Step (2) are in turn those whose input and output gains are $g_c^{(1)} = g$, $g_c^{(2)} = -h$, $g_d^{(1)} = 1$, and $g_d^{(2)} = 1$, and whose crosstalk gains are $g_t^{(1)} = -fg$ and $g_t^{(2)} = hi$, canceling the diagonal entries of $\widetilde{G}$. Applying the transforming systems results in an interconnection whose gain matrix G' is

$$G' = \begin{bmatrix} 0 & -gh \\ gh & 0 \end{bmatrix}. \tag{4.85}$$

Fig. 4-9 illustrates the process of beginning with a 2-input, 2-output linear interconnection and applying appropriate transforming systems so that the resulting interconnection has a skew-symmetric gain matrix, i.e. so that it is conservative. In

[1]It is not enough to conclude that such transformations can be overlooked simply because the number of available parameters in the matrix has been reduced from four to three. We will see in Subsection 4.4.5 that the set of all such transformations can indeed be described by a three-parameter group, although this group is not the group of invertible lower triangular matrices, i.e. it is not those of the form of the matrix in Eq. 4.73.

particular, Fig. 4-9(a) depicts a linear interconnection whose gain matrix G is as specified as in Eq. 4.83, Fig. 4-9(b) depicts the use of the previously-mentioned transforming systems, and Fig. 4-9(c) shows the system in Fig. 4-9(b) after performing signal flow-graph simplifications that reveal the elements in the final interconnection gain matrix G' specified in Eq. 4.84. The transformed system, depicted in Fig. 4-9(c), is conservative for an OVS in a canonical conjugate basis, and the original system, depicted in Fig. 4-9(a), is conservative for the transformed OVS that is defined as

$$\begin{aligned}
\mathfrak{U}' &= \left(\mathbb{R}^4, \langle .,. \rangle, \mathcal{O}'\right) && (4.86)\\
\mathcal{O}' &= \left(C', \mathcal{D}_p', \mathcal{D}_c'\right) && (4.87)\\
C' &= \begin{bmatrix} g^{-1} & 0 & -f & 0 \\ 0 & -h^{-1} & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} g^{-1} & 0 & 0 & 0 \\ 0 & -h^{-1} & 0 & 0 \\ -f & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2fg^{-1} & 0 & g^{-1} & 0 \\ 0 & 2h^{-1}i & 0 & -h^{-1} \\ g^{-1} & 0 & 0 & 0 \\ 0 & -h^{-1} & 0 & 0 \end{bmatrix} && (4.88)\\
\mathcal{D}_p' &= \left\{\operatorname{span}\left(e_1, e_3\right), \operatorname{span}\left(e_2, e_4\right)\right\} && (4.89)\\
\mathcal{D}_c' &= \left\{ \operatorname{span}\left(\begin{bmatrix} g^{-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -h^{-1} \\ 0 \\ 0 \end{bmatrix} \right), \operatorname{span}\left(\begin{bmatrix} -f \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -i \\ 0 \\ 1 \end{bmatrix} \right) \right\} \\
&= \left\{\operatorname{span}(e_1, e_2), \operatorname{span}(e_3 - fe_1, e_4 - ie_2)\right\}, && (4.90)
\end{aligned}$$

with $\langle .,. \rangle$ denoting the standard inner product on $\mathbb{R}^4$. As the transformed interconnection in the example in Fig. 4-9(c) is conjugate-separable, the transformed OVS $\mathfrak{U}'$ for the interconnection in Fig. 4-9(a) is strongly conservative over its behavior, and the associated conservation law is

$$g^{-1}c_1(d_1 - fc_1) - h^{-1}c_2(d_2 - ic_2) = 0. \tag{4.91}$$

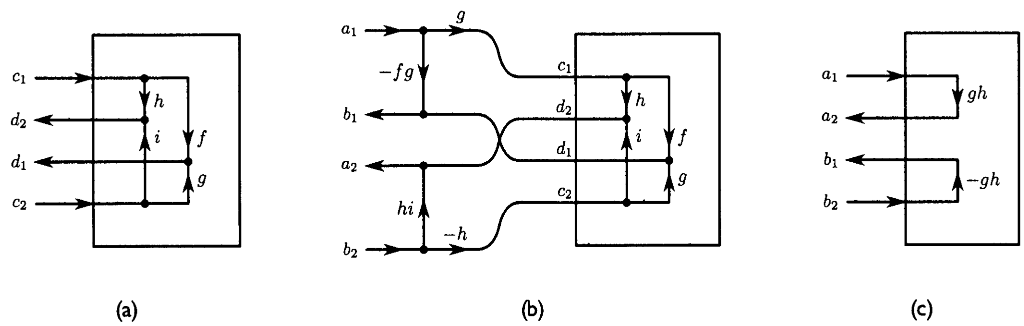


Figure 4-9: (a) A general 2-input, 2-output, linear interconnection. (b) Interconnection with transforming systems applied to partition subspaces. (c) Transformed interconnection after performing flow graph simplifications, revealing that the system in (a) is strongly-conservative under this transformation.

4.4.3 A strategy for strengthening weak conservation

While the strategy in Subsection 4.4.2 provides a mechanized way to identify whether a linear interconnection has a conservation law, it does not directly address the question of whether an interconnection is strongly conservative in particular. Indeed, the system in Fig. 4-9 happens to be strongly conservative under the transformation that resulted from the use of the strategy, but as the strategy is based upon the condition that was developed in Subsection 4.3.1 pertinent to the existence of conservation, as opposed to its strength, this will not always be the case.

In this subsection we address the issue of obtaining a strongly-conservative OVS for a linear interconnection by providing an algorithm for performing transformations on a weakly-conservative OVS in an attempt to obtain an OVS that is strongly conservative, i.e. in an attempt to strengthen it. We are again interested in transformations that preserve the partition decomposition of the OVS, and as such we require that the transformations have the partition subspaces as invariant subspaces. We are therefore interested in invertible transformations that:

(1) are composed of 2×2 matrices, as in Eqns. 4.61-4.62,

(2) preserve the fact that the OVS is conservative over the behavior of the interconnection, and

(3) do not preserve conservation strength.

As will be discussed in Subsection 4.4.5, the set of transformations referred to in requirement (2) will in general depend on the behavior of the interconnection. With the goal being to present a straightforward algorithm that can be applied to an arbitrary interconnection, we interpret (2) as requiring that the transformation preserves conservation over an arbitrary behavior. Put another way, we require that such a transformation be in the group of transformations $\mathcal{G}_Q$ preserving the quadratic form associated with C.

We are interested in transformations in $\mathcal{G}_Q$ that have the partition subspaces as invariant subspaces and that do not preserve the strength of conservation. Referring to the transformations listed in Fig. 4-1, those that meet the invariant subspace requirement are in classes $T_1^{[q;t)}$, $T_5^{[q]}$ and $T_6^{[q]}$. Of those classes, the transformations composing $T_6^{[q]}$, which we refer to as the gyrator transformations, are the only ones that do not preserve the strength of conservation, as is indicated in Table 4.1. With this in mind, we write the following algorithm that attempts to strengthen weakly-conservative interconnections:

(1) Apply gyrator transformations to some subset of the partition subspaces.

(2) Check whether the interconnection is strongly conservative, e.g. by checking whether the dimensions of the behaviors of the conjugate subspaces sum to L, or equivalently by checking whether the interconnection is conjugate-separable.

(3) If the interconnection is not strongly conservative, apply gyrator transformations to a different subset of the partition subspaces and go to step (2).

The process is depicted in Fig. 4-10 for a 2-input, 2-output linear interconnection. The original interconnection is depicted in Fig. 4-10(a) and the transformed interconnection, transformed using the technique in Subsection 4.4.2, is depicted in Fig. 4-10(b). As is apparent in the simplified signal-flow graph depicted in Fig. 4-10(c), the interconnection is not conjugate-separable, and as such the interconnection in Fig. 4-10(a) is weakly conservative under this transformation. Fig. 4-10(d) depicts the application of a gyrator transformation to the partition subspace V_1, resulting in a simplified interconnection in Fig. 4-10(e) that is conjugate separable and as such is strongly-conservative for a 4-dimensional OVS in a canonical conjugate basis, under the transformation defined by

$$a_1 = d_1 - fc_1 \tag{4.92}$$

$$b_1 = g^{-1}c_1 \tag{4.93}$$

$$a_2 = -h^{-1}c_2 \tag{4.94}$$

$$b_2 = d_2 - ic_2. \tag{4.95}$$

The conservation law is accordingly

$$a_1b_1 + a_2b_2 = (d_1 - fc_1)g^{-1}c_1 - h^{-1}c_2(d_2 - ic_2) = 0. \tag{4.96}$$

4.4.4 Identifying conservation in a bilateral vehicle speed control system

As an example illustrating the identification of conservation laws in an existing system, we consider a distributed system for controlling the speed of a chain of vehicles. The system is bilateral, i.e. the speed of each vehicle in the chain is a function of the distances between the vehicles immediately leading and following it. The system relating the speeds of the vehicles, discussed in detail in [20], is depicted in Fig. 4-11. Referring to this figure, the systems $H_1(s)$ and $H_2(s)$ in the chain will remain unspecified, and as such we will initially be interested in obtaining a conservation

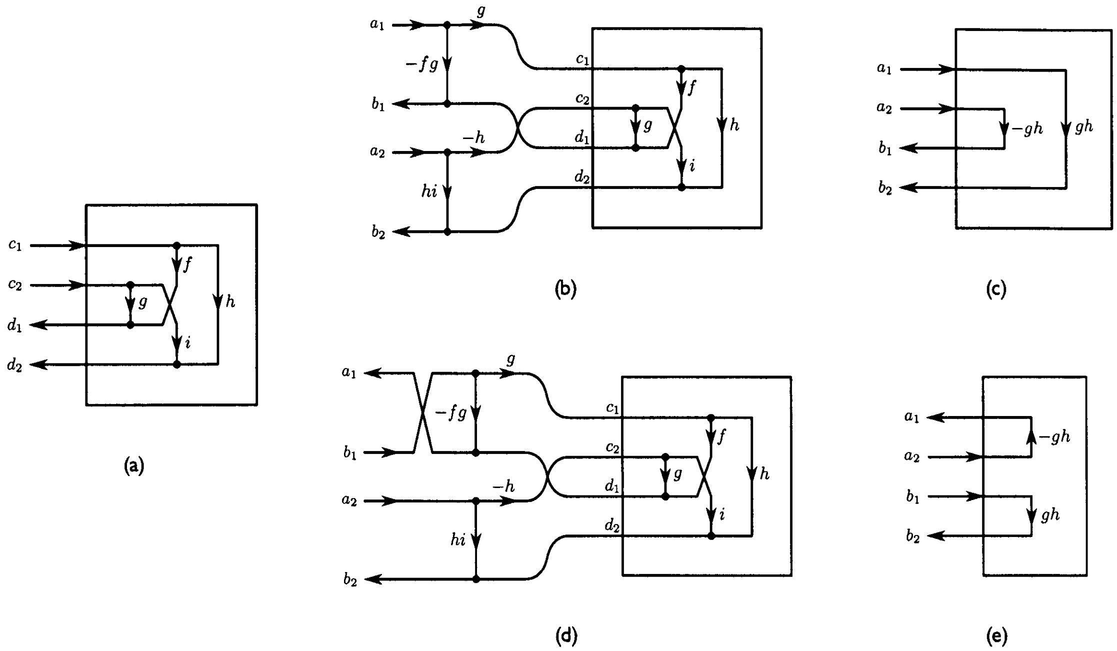


Figure 4-10: (a) A general 2-input, 2-output, linear interconnection. (b) Interconnection with transforming systems as specified by the technique in Subsection 4.4.2 applied to partition subspaces. (c) Transformed interconnection in (b) after performing flow graph simplifications, revealing that the system in (a) is weakly-conservative under this transformation. (d) Transformed interconnection having a gyrator transformation applied to the first partition subspace. (e) Transformed interconnection in (d) after performing flow graph simplifications, revealing that under that transformation, the system in (a) is strongly-conservative.

law involving the instantaneous values of the interconnection variables. Although the chain of systems is infinite, it is composed of a series of identical summing junctions, one of which is indicated by the dotted box in this figure. The strategy will accordingly be to determine a conservation law for a single such junction, as a summation of the conservation laws for an arbitrary number of junctions in the chain will itself be a conservation law.

We proceed by writing the gain matrix G for the interconnection in the dotted box in Fig. 4-11:

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{4.97}$$

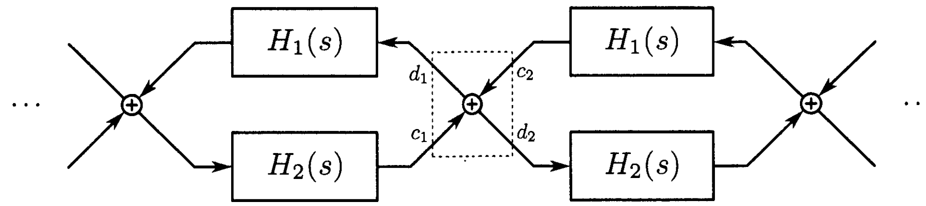


Figure 4-11: Block diagram relating vehicle speeds for a chain of vehicles under bilateral speed control.

The relationship between the interconnection inputs and outputs is

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = G \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{4.98}$$

and the behavior W of the interconnection is accordingly

$$W = \text{range}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\right), \tag{4.99}$$

in the sense that the vector $\mathbf{x}$ of interconnection terminal variables will always be an element of W, i.e.

$$\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{bmatrix} \in W. \tag{4.100}$$

The goal is to find an OVS $\mathfrak{U} = (\mathbb{R}^4, \langle .,. \rangle, \mathcal{O})$ that is conservative over W and that has an orientation $\mathcal{O} = (\mathcal{D}_c, \mathcal{D}_p, C)$ where the partition subspaces in $\mathcal{D}_p$ are

$$V_1 = \text{span}(e_1, e_3) \tag{4.101}$$

and

$$V_2 = \text{span}(e_2, e_4). \tag{4.102}$$

Noting that the matrix G in Eq. 4.97 satisfies the condition that g and h are both nonzero when written as in Eq. 4.83, it is possible to use the previously-established result pertaining to conservation in 2-input, 2-output linear interconnections to obtain the desired OVS. However, it is instructive to proceed by performing the appropriate manipulations on G, bringing it into a skew-symmetric form.

The manipulations will specifically be written in terms of how they relate to the corresponding gains in the transforming systems as depicted in Fig. 4-8, and there will be two such transforming systems, one for each partition subspace. We begin with the input and output gains being $g_c^{(1)} = g_d^{(1)} = g_c^{(2)} = g_d^{(2)} = 1$ and with the crosstalk gains being $g_t^{(1)} = g_t^{(2)} = 0$, i.e. we will begin with the original, untransformed interconnection, and we indicate the sequence of changes in the transforming system gain terms that result in the corresponding changes to the interconnection gain matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{g_c^{(2)}=-1} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \xrightarrow{g_t^{(1)}=-1} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \xrightarrow{g_t^{(2)}=1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.103}$$

The original interconnection, the transforming systems indicated by the manipulations in Eq. 4.103, and the simplified transformed interconnection are depicted in Fig. 4-12. As is apparent in Fig. 4-12(b), the gyrator transformation is used the partition subspace V_1 so that the transformed interconnection is conjugate-separable, and we conclude that with the conjugate decomposition and correspondence map being defined as

$$\begin{aligned} \mathcal{D}_c &= \left\{ \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right), \operatorname{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right) \right\} && (4.104) \\ &= \left\{ \operatorname{span}(e_1, e_2), \operatorname{span}(e_3 - e_1, e_4 - e_2) \right\} && (4.105) \end{aligned}$$

and

$$
\begin{aligned}
C &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},
\end{aligned}
\tag{4.106}
$$

the OVS $\mathfrak{U}$ is strongly conservative over the behavior W of the original interconnection. We can in turn write the following conservation law for the interconnection:

$$c_1(d_1 - c_1) - c_2(d_2 - c_2) = 0. \tag{4.107}$$

Using this example as a springboard for discussion, an alternative system for vehicle control that is based upon the principles in this thesis is discussed in Section 6.5.

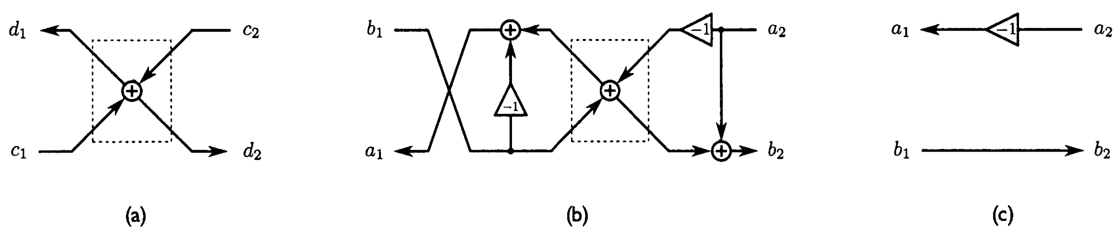


Figure 4-12: (a) Interconnection structure for the system in Fig. 4-11. (b) Interconnection with transforming systems applied to partition subspaces. (c) Transformed interconnection after performing flow graph simplifications, revealing that the system in (a) is strongly-conservative under this transformation.

4.4.5 Obtaining all conservation laws for a conservative behavior

As was illustrated with the feedback network in Subsection 3.2.2, a linear interconnection will in general have multiple conservation laws. The strategy in Subsection 4.4.2 can be used to obtain one such law for a pre-specified interconnection, and in this subsection we address the question of obtaining multiple conservation laws from an initial law by applying transformations to the partition subspaces. We specifically are interested in transformations that are invertible and that have the partition subspaces as invariant subspaces. As this class of transformations can be composed of matrices $T^{(k)}$ as in Eqns. 4.61-4.62, we are interested in 2×2 matrices $T^{(k)}$ that:

(1) preserve the fact that the OVS is conservative over the behavior of the interconnection and

(2) result in distinct conservation laws.

In satisfying requirement (2) it will be useful to distinguish between transformations that modify the OVS but do not affect the quadratic form, as with the gyrator transformation, and those that do modify the quadratic form. Toward these ends, we will make use of what will be referred to as a modified Iwasawa decomposition of each invertible 2×2 matrix $T^{(k)}$ composing the overall transformation as in Eqns. 4.61-4.62. We specifically define a modified Iwasawa decomposition of an arbitrary 2×2 matrix $T^{(k)}$ as

$$T^{(k)} = (-1)^n \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^r \begin{bmatrix} t & 0 \\ 0 & \frac{1}{t} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \eta, \quad \begin{array}{c} n \in \{0,1\} \\ r \in \{0,1\} \\ t > 0 \\ 0 \leq \phi < \pi \\ \eta > 0 \end{array}, \tag{4.108}$$

with n and r being binary variables that respectively select whether the expression is negated and whether an initial gyrator transformation is used, and with x being a real number. It can be shown that an arbitrary 2×2 invertible transformation can be uniquely decomposed into the form of Eq. 4.108, and that the sign and magnitude of the determinant of $T^{(k)}$ will respectively be $(-1)^r$ and η^2.[2]

As is apparent in Eq. 4.108, an important consequence of the modified Iwasawa decomposition is that it arranges an invertible transformation so that the components that lie in $\mathcal{G}_Q$, namely $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} t & 0 \\ 0 & \frac{1}{t} \end{bmatrix}$, are separated out from those that do not. Elements of $\mathcal{G}_Q$ do not have an effect on the quadratic form, and accordingly we conclude that the set of 2×2 matrices that result in distinct conservation laws can be parameterized by the variables x, ϕ and η in Eq. 4.108, as an identical quadratic form will result given an arbitrary choice of n, r and t.

It is also an illustrative exercise to verify, using the modified Iwasawa decomposition, that the limited set of transformations that were used in the strategy for identifying conservation in Subsection 4.4.2 indeed have the potential to fail to identify conservation laws. In particular, the lower-triangular form of the transformation in Eq. 4.73 restricts the parameter ϕ in its modified Iwasawa decomposition to be $\phi = \pi/2$, allowing for the potential to overlook a wide range of conservation laws. With this in mind, it is all the more remarkable that the reduced set of transformation in Subsection 4.4.2 can be used to identify conservation in arbitrary 2-input, 2-output interconnections, with the exception of those in a degenerate class.

In satisfying requirement (1), we use the fact that requirement (2) means that we need not consider transformations that result in other conservative behaviors, as these are precisely the transformations that preserve the quadratic form, i.e. that leave the conservation law unchanged. As such, we are interested in transformations that preserve the fact that the OVS is conservative by leaving the behavior of the interconnection W unchanged. Put another way, we are interested in transformations that have W as an invariant subspace.

[2] The decomposition in Eq. 4.108 is referred to as a modified Iwasawa decomposition because of its resemblance to the form of the Iwasawa decomposition for an invertible 2×2 matrix, $T^{(k)} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & \frac{1}{t} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$. Indeed the uniqueness of our modified decomposition in Eq. 4.108 follows in a straightforward way from the fact that an Iwasawa decomposition is unique, as is discussed in, e.g., [11].

We conclude that requirements (1) and (2) can be re-written as the following requirements:

(1) The overall matrix T must have the interconnection behavior W as an invariant subspace.

(2) The matrices $T^{(k)}$ composing T must be of the form

$$T^{(k)} = \begin{bmatrix} 1 & x_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi_k & -\sin\phi_k \\ \sin\phi_k & \cos\phi_k \end{bmatrix} \eta_k, \quad \begin{array}{c} 0 \leq \phi_k < \pi \\ \eta_k > 0 \end{array} . \tag{4.109}$$

The specific details of satisfying these requirements will in general relate to the relationship between W and the partition subspaces.

Chapter 5

Variational principles of strongly-conservative spaces

A convenient aspect of many physical systems is that they operate at extremal points of variational problems. Variational principles indeed form much of the theoretical foundation of classical mechanics, embodied by what is referred to as the principle of stationary, or least, action. As its name suggests, the principle states that the trajectory of a mechanical system will lie at a stationary point of a quantity that is obtained by integrating the corresponding Lagrangian with respect to time, referred to as the action. [1]

Electrical networks in steady state operate according to a similar principle: the vector of currents $\mathbf{i}$ lies at a stationary point of the total content $Q(\mathbf{i})$, a scalar quantity that is obtained by summing the individual contents $Q^{(k)}(\mathbf{i}_k)$ of the elements in the network, each of which involves integrating the corresponding voltage. As is the case with classical mechanics, where an alternative formulation of the variational principle may be obtained in terms of a dual function, i.e. the Hamiltonian, the vector of voltages $\mathbf{v}$ in a steady-state electrical network lies at a stationary point of a dual scalar quantity, referred to as the total co-content $R(\mathbf{v})$. As with content, the total co-content can be obtained by summing the individual contents $R^{(k)}(\mathbf{v}_k)$ of the network elements, each of which involves integrating the corresponding current.[1][2]

The focus of this chapter will be on formulating similar variational principles for systems that have a strongly-conservative OVS. The development will be done in the spirit of electrical network theory as opposed to that of classical mechanics, in the sense that the variational principles will pertain to continuous, differentiable functions in conjunction with a previously-established conservation law. This is in contrast to the commonly-followed sequence in classical mechanics, where the principle of stationary action is stated first, and where conservation laws are subsequently derived from it by performing continuous transformations, i.e. using Noether's theorem. [1]

In a number of applications, the use of the variational principles in this chapter will represent a natural progression for applying the framework that begins with using the techniques developed in Chapters 3 and 4 for creating and identifying conservation. However we emphasize that the results in Chapters 3 and 4 are intended to stand on their own right. Indeed in electrical network theory, there are a number of useful results that are based on Tellegen's theorem and do not require the concepts of content and co-content, even though in using these principles a significant number of additional results can be proven.

As is the case with electrical networks, the discussion will focus on an inner product that is taken between vectors of conjugate variables. The material in this chapter will accordingly be pertinent to the comparison space of the OVS, which as a consequence of strong conservation will naturally contain a pair of orthogonal vector spaces related to the conservative set. As was mentioned in Chapter 3, a fundamental property of the OVS is that an arbitrary $2L$-dimensional OVS always has conjugate mappings to the comparison space $\mathbb{R}^L$, such that the standard inner product on $\mathbb{R}^L$ coincides with the quadratic form for the OVS. The results in this chapter will likewise be formulated without loss of generality in $\mathbb{R}^L$ and using the standard inner product,

[1] The standard variables used in denoting content and co-content in the literature are respectively $G(\cdot)$ and $J(\cdot)$. To avoid a conflict in notation, we will refer to these quantities using $Q(\cdot)$ and $R(\cdot)$.

[2] Consistent with the previously-established convention, a boldface variable will denote a column vector, and a subscript k will be used in denoting the scalar value in its kth entry.

denoted in this chapter as $\langle .,. \rangle$.[3]

Drawing upon previously-defined concepts of content and co-content, we begin the chapter by developing more general notions of content and co-content that seem not to appear in the literature, and we prove their stationarity. The remainder of the chapter involves connecting these concepts with existing results, using the tools of optimization theory to facilitate their interpretation as minimization and maximization, and using the tools of stability theory to show that they can serve as potential Lyapunov functions when dynamics are involved.

5.1 OVS content and co-content

There have been several definitions, refinements and generalizations of the terms content and co-content in the electrical network theory literature. Millar [26] defined the content $Q^{(k)}(i)$ and co-content $R^{(k)}(v)$ for a nonlinear resistor having an invertible v-i characteristic as

$$Q^{(k)}(i) = \int^{i} v(\tau)d\tau \tag{5.1}$$

and

$$R^{(k)}(v) = \int^{v} i(\tau)d\tau, \tag{5.2}$$

with $v(i)$ and $i(v)$ indicating the functional relationships between current and voltage, and with the lower limits of integration being specified on a case-by-case basis. [31] Also in [26], generalized definitions of content and co-content applicable to a broader class of dynamic and time-varying elements were defined in terms of voltage and current trajectories $v(t)$ and $i(t)$ as

$$Q^{(k)}(t) = \int^{t} v(\tau)i'(\tau)d\tau \tag{5.3}$$

[3]It has been the convention previously to write the inner product for a comparison space U as $\langle .,. \rangle_U$. As all inner products that appear in this chapter will be taken on the comparison space, we remove the subscript for notational clarity and write $\langle .,. \rangle$.

and

$$R^{(k)}(t) = \int^{t} i(\tau)v'(\tau)d\tau, \tag{5.4}$$

with $v'(t)$ and $i'(t)$ denoting the first derivatives of the functions $v(t)$ and $i(t)$ with respect to time. Chua [8] provided a definition of content and co-content for memoryless elements having a parameterizable v-i characteristic essentially by interpreting $v(\cdot)$ and $i(\cdot)$ in Eqns. 5.3-5.4 not as voltage and current trajectories through time but rather as functions $v(y)$ and $i(y)$ describing the v-i characteristic for a specific element in terms of an independent parameter y. Based upon another result in [8], co-content has been defined for a multi-port, voltage-controlled element, i.e. for an element where the vector of port currents $\mathbf{i}$ was taken to be a function of the port voltages $\mathbf{v}$. The expression was written in [9] using a path integral as

$$R^{(k)}(\mathbf{v}) = \int_{\mathbf{0}}^{\mathbf{v}} \mathbf{i}(\mathbf{v}) \cdot d\mathbf{v}. \tag{5.5}$$

In defining suitable notions of OVS content and co-content, we wish to address vector-valued relationships as in Eq. 5.5, but we also aim to do so in a way that does not require an *a priori* specification of which conjugate variables are functions of others. We accordingly draw on the positive aspects of these definitions to formulate concepts of content and co-content that are parametric and that involve vector-valued functions of an independent, vector-valued parameter. This type of generalization does not seem to exist currently in the literature.

5.1.1 Definition

We will be working in the vector space $\mathbb{R}^L$, with $\langle .,. \rangle$ denoting the standard inner product on the space. Conservation will specifically involve two subspaces $A \subseteq \mathbb{R}^L$ and $B \subseteq \mathbb{R}^L$ that are orthogonal, i.e. that meet the formal requirement

$$\langle \mathbf{a}, \mathbf{b} \rangle = 0, \quad \mathbf{a} \in A, \ \mathbf{b} \in B. \tag{5.6}$$

We will be referring to variations of the vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to a vector $\mathbf{y} \in \mathbb{R}^M$, where $\mathbf{a}$ and $\mathbf{b}$ are varied in such a way that they remain in the respective subspaces A and B.

The strategy in doing this will be to utilize two vector-valued functions,

$$\mathbf{f} : \mathbb{R}^M \rightarrow \mathbb{R}^L \tag{5.7}$$

and

$$\mathbf{g} : \mathbb{R}^M \rightarrow \mathbb{R}^L, \tag{5.8}$$

that are written in terms of a vector-valued independent variable $\mathbf{y}$ as $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$, and to focus on small variations around any point $\mathbf{y}^\star$ for which $\mathbf{f}(\mathbf{y}^\star) \in A$ and $\mathbf{g}(\mathbf{y}^\star) \in B$. The assumption implicit in doing this will be that the functions are smooth in the vicinity of any such point, i.e. that the Jacobians of $\mathbf{f}$ and $\mathbf{g}$ exist at $\mathbf{y}^\star$. Using the notation $\mathbf{f}_k(\mathbf{y})$ and $\mathbf{g}_k(\mathbf{y})$ to denote the functionals $\mathbf{f} : \mathbb{R}^M \rightarrow \mathbb{R}$ and $\mathbf{g} : \mathbb{R}^M \rightarrow \mathbb{R}$ that respectively map $\mathbf{y}$ to the kth entry of $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$, the Jacobians of $\mathbf{f}$ and $\mathbf{g}$, evaluated at a point $\mathbf{y}^\star$, will be written as

$$J_{\mathbf{f}}(\mathbf{y}^\star) = \begin{bmatrix} \frac{\partial \mathbf{f}_1}{\partial \mathbf{y}_1} & \cdots & \frac{\partial \mathbf{f}_1}{\partial \mathbf{y}_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{f}_L}{\partial \mathbf{y}_1} & \cdots & \frac{\partial \mathbf{f}_L}{\partial \mathbf{y}_M} \end{bmatrix}_{\mathbf{y}=\mathbf{y}^\star} \tag{5.9}$$

$$= \begin{bmatrix} (\nabla \mathbf{f}_1(\mathbf{y}^\star))^{tr} \\ \vdots \\ (\nabla \mathbf{f}_L(\mathbf{y}^\star))^{tr} \end{bmatrix} \tag{5.10}$$

and

$$J_{\mathbf{g}}(\mathbf{y}^{\star}) = \left[\begin{array}{ccc} \frac{\partial \mathbf{g}_1}{\partial \mathbf{y}_1} & \cdots & \frac{\partial \mathbf{g}_1}{\partial \mathbf{y}_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{g}_L}{\partial \mathbf{y}_1} & \cdots & \frac{\partial \mathbf{g}_L}{\partial \mathbf{y}_M} \end{array}\right]_{\mathbf{y}=\mathbf{y}^{\star}} \tag{5.11}$$

$$= \left[\begin{array}{c} (\nabla \mathbf{g}_1(\mathbf{y}^{\star}))^{tr} \\ \vdots \\ (\nabla \mathbf{g}_L(\mathbf{y}^{\star}))^{tr} \end{array}\right]. \tag{5.12}$$

Our formal definitions of the functions for total content

$$Q : \mathbb{R}^M \rightarrow \mathbb{R} \tag{5.13}$$

and co-content

$$R : \mathbb{R}^M \rightarrow \mathbb{R} \tag{5.14}$$

will be written as a sum of quantities pertaining to the entries of a vector in $\mathbb{R}^L$ that are referred to as the individual contents

$$Q^{(k)} : \mathbb{R}^M \rightarrow \mathbb{R} \tag{5.15}$$

and co-contents

$$R^{(k)} : \mathbb{R}^M \rightarrow \mathbb{R}, \tag{5.16}$$

$k = 1, \ldots, L$. We will specifically define the individual contents and co-contents as any such functions that satisfy the following relationships:

$$\nabla Q^{(k)}(\mathbf{y}) = \mathbf{g}_k(\mathbf{y}) \nabla \mathbf{f}_k(\mathbf{y}) \tag{5.17}$$

$$\nabla R^{(k)}(\mathbf{y}) = \mathbf{f}_k(\mathbf{y}) \nabla \mathbf{g}_k(\mathbf{y}) \tag{5.18}$$

$$Q^{(k)}(\mathbf{y}) + R^{(k)}(\mathbf{y}) = \mathbf{f}_k(\mathbf{y}) \mathbf{g}_k(\mathbf{y}), \tag{5.19}$$

with the total content and co-content being written formally as

$$Q(\mathbf{y}) = \sum_{k=1}^{L} Q^{(k)}(\mathbf{y}) \tag{5.20}$$

and

$$R(\mathbf{y}) = \sum_{k=1}^{L} R^{(k)}(\mathbf{y}). \tag{5.21}$$

Note that there is some redundancy in Eqns. 5.17-5.19. In particular, exactly one of Eqns. 5.17 and 5.18 may be eliminated without affecting the definition. This can be seen by substituting Eq. 5.19 into Eq. 5.17, resulting in

$$\nabla\left(\mathbf{f}_k(\mathbf{y})\mathbf{g}_k(\mathbf{y}) - R^{(k)}(\mathbf{y})\right) = \mathbf{g}_k(\mathbf{y})\nabla\mathbf{f}_k(\mathbf{y}). \tag{5.22}$$

Using linearity of the gradient, Eq. 5.22 can be rearranged as

$$\begin{aligned}\nabla R^{(k)}(\mathbf{y}) &= \nabla\left(\mathbf{f}_k(\mathbf{y})\mathbf{g}_k(\mathbf{y})\right) - \mathbf{g}_k(\mathbf{y})\nabla\mathbf{f}_k(\mathbf{y}) && (5.23)\\ &= \mathbf{f}_k(\mathbf{y})\nabla\mathbf{g}_k(\mathbf{y}), && (5.24)\end{aligned}$$

where Eq. 5.24 follows from the product rule for gradients. We conclude that Eqns. 5.17 and 5.19 imply Eq. 5.18, and by a similar line of reasoning, Eqns. 5.18 and 5.19 imply Eq. 5.17. The reason for writing all three equations is to emphasize the symmetry in the definitions.

The total content $\mathbf{f}(\mathbf{y})$ and co-content $\mathbf{g}(\mathbf{y})$ are not required to evaluate to vectors in the orthogonal subspaces A and B, and any values of $\mathbf{y}$ for which they do are precisely those points around which a variational principle may be stated. In doing this we will consider small variations of $\mathbf{y}$ for which $\mathbf{f}(\mathbf{y})$ remains in A to first order as well as small variations of $\mathbf{y}$ for which $\mathbf{g}(\mathbf{y})$ remains in B to first order. Related to this, the directional derivative will be of use, which is commonly defined in vector calculus in any of the following equivalent forms:

$$\begin{aligned} D_{\mathbf{u}}\mathbf{f}(\mathbf{y}) &= \lim_{\delta \to 0} \frac{\mathbf{f}(\mathbf{y}+\delta\mathbf{u}) - \mathbf{f}(\mathbf{y})}{\delta} && (5.25) \\ &= \begin{bmatrix} D_{\mathbf{u}}\mathbf{f}_1(\mathbf{y}) \\ \vdots \\ D_{\mathbf{u}}\mathbf{f}_L(\mathbf{y}) \end{bmatrix} && (5.26) \\ &= \begin{bmatrix} (\nabla \mathbf{f}_1(\mathbf{y}))^{tr}\,\mathbf{u} \\ \vdots \\ (\nabla \mathbf{f}_L(\mathbf{y}))^{tr}\,\mathbf{u} \end{bmatrix} && (5.27) \\ &= J_{\mathbf{f}}(\mathbf{y})\mathbf{u}. && (5.28) \end{aligned}$$

The interpretation is that $D_{\mathbf{u}}\mathbf{f}(\mathbf{y})$ is the rate at which the function $\mathbf{f}$ changes at the point $\mathbf{y}$ in the direction $\mathbf{u}$. [36] The variational principle is stated in the following theorem.

Theorem 5.1 (Points of orthogonality have stationary OVS content and co-content). *This theorem pertains to two orthogonal subspaces $A \subseteq \mathbb{R}^L$ and $B \subseteq \mathbb{R}^L$ and two functions $\mathbf{f} : \mathbb{R}^M \to \mathbb{R}^L$ and $\mathbf{g} : \mathbb{R}^M \to \mathbb{R}^L$, as well as a point $\mathbf{y}^\star$ for which $\mathbf{f}(\mathbf{y}^\star) \in A$ and $\mathbf{g}(\mathbf{y}^\star) \in B$, and for which $J_{\mathbf{f}}(\mathbf{y}^\star)$ and $J_{\mathbf{g}}(\mathbf{y}^\star)$ exist. At any such point, the total content Q is stationary with respect to small variations taken in any direction $\mathbf{u}^{(Q)}$ for which $D_{\mathbf{u}^{(Q)}}\mathbf{f}(\mathbf{y}^\star) \in A$. Likewise, the total co-content R is stationary with respect to small variations taken in any direction $\mathbf{u}^{(R)}$ for which $D_{\mathbf{u}^{(R)}}\mathbf{g}(\mathbf{y}^\star) \in B$. Furthermore, $Q(\mathbf{y}^\star) = -R(\mathbf{y}^\star)$ at any such point.*

Proof. We proceed by evaluating the directional derivatives of the total content and co-content in the respective directions $\mathbf{u}^{(Q)}$ and and $\mathbf{u}^{(R)}$, and showing that they evaluate to zero, i.e. we demonstrate that these quantities are stationary with respect to the allowed variations in $\mathbf{y}$.

By linearity of the directional derivative, the directional derivative of the total content $D_{\mathbf{u}^{(Q)}}Q(\mathbf{y}^\star)$ is the sum of the directional derivatives of the individual contents, written as

$$D_{\mathbf{u}^{(Q)}} Q(\mathbf{y}^\star) = \sum_{k=1}^{L} D_{\mathbf{u}^{(Q)}} Q^{(k)}(\mathbf{y}^\star). \tag{5.29}$$

Writing the directional derivatives in terms of the gradients of the individual contents $Q^{(k)}$ and substituting in the definition of $Q^{(k)}$ from Eq. 5.17, we obtain

$$\begin{aligned} D_{\mathbf{u}^{(Q)}} Q(\mathbf{y}^\star) &= \mathbf{u}^{(Q)^{tr}} \sum_{k=1}^{L} \nabla Q^{(k)}(\mathbf{y}^\star) && (5.30) \\ &= \mathbf{u}^{(Q)^{tr}} \sum_{k=1}^{L} \mathbf{g}_k(\mathbf{y}^\star) \nabla \mathbf{f}_k(\mathbf{y}^\star). && (5.31) \end{aligned}$$

The summation over k can be written as a matrix multiplication involving a matrix whose columns are the gradients of the functionals $\mathbf{f}_k$, which when left-multiplied by $\mathbf{u}^{(Q)^{tr}}$ evaluates to the directional derivative of $\mathbf{f}$, resulting in

$$\begin{aligned} D_{\mathbf{u}^{(Q)}} Q(\mathbf{y}^\star) &= \mathbf{u}^{(Q)^{tr}} \left[\begin{array}{ccc} \nabla \mathbf{f}_1(\mathbf{y}^\star) & \cdots & \nabla \mathbf{f}_L(\mathbf{y}^\star) \end{array} \right] \mathbf{g}(\mathbf{y}^\star) && (5.32) \\ &= \left[\begin{array}{ccc} D_{\mathbf{u}^{(Q)}} \mathbf{f}_1(\mathbf{y}^\star) & \cdots & D_{\mathbf{u}^{(Q)}} \mathbf{f}_L(\mathbf{y}^\star) \end{array} \right] \mathbf{g}(\mathbf{y}^\star) && (5.33) \\ &= \left(D_{\mathbf{u}^{(Q)}} \mathbf{f}(\mathbf{y}^\star) \right)^{tr} \mathbf{g}(\mathbf{y}^\star). && (5.34) \end{aligned}$$

As $D_{\mathbf{u}^{(Q)}} \mathbf{f}(\mathbf{y}^\star) \in A$, $\mathbf{g}(\mathbf{y}^\star) \in B$ and $A \perp B$, we have

$$D_{\mathbf{u}^{(Q)}} Q(\mathbf{y}^\star) = 0. \tag{5.35}$$

Following the same line of reasoning, the directional derivative of R likewise evaluates to zero in the direction $\mathbf{u}^{(R)}$:

$$\begin{aligned}
D_{\mathbf{u}^{(R)}} R(\mathbf{y}^\star) &= \sum_{k=1}^{L} D_{\mathbf{u}^{(R)}} R^{(k)}(\mathbf{y}^\star) && (5.36)\\
&= \mathbf{u}^{(R)^{tr}} \sum_{k=1}^{L} \nabla R^{(k)}(\mathbf{y}^\star) && (5.37)\\
&= \mathbf{u}^{(R)^{tr}} \sum_{k=1}^{L} \mathbf{f}_k(\mathbf{y}^\star) \nabla \mathbf{g}_k(\mathbf{y}^\star) && (5.38)\\
&= \mathbf{u}^{(R)^{tr}} \left[\begin{array}{ccc} \nabla \mathbf{g}_1(\mathbf{y}^\star) & \cdots & \nabla \mathbf{g}_L(\mathbf{y}^\star) \end{array} \right] \mathbf{f}(\mathbf{y}^\star) && (5.39)\\
&= \left[\begin{array}{ccc} D_{\mathbf{u}^{(R)}} \mathbf{g}_1(\mathbf{y}^\star) & \cdots & D_{\mathbf{u}^{(R)}} \mathbf{g}_L(\mathbf{y}^\star) \end{array} \right] \mathbf{g}(\mathbf{y}^\star) && (5.40)\\
&= \left(D_{\mathbf{u}^{(R)}} \mathbf{f}(\mathbf{y}^\star)\right)^{tr} \mathbf{g}(\mathbf{y}^\star) && (5.41)\\
&= 0, && (5.42)
\end{aligned}$$

again where Eq. 5.42 follows from orthogonality of the subspaces A and B.

It remains to be shown that $Q(\mathbf{y}^\star) = -R(\mathbf{y}^\star)$. This can be seen by writing the sum of the total content and co-content in terms of the individual contents and co-contents as

$$\begin{aligned}
Q(\mathbf{y}^\star) + R(\mathbf{y}^\star) &= \sum_{k=1}^{L} Q^{(k)}(\mathbf{y}^\star) + R^{(k)}(\mathbf{y}^\star) && (5.43)\\
&= \sum_{k=1}^{L} \mathbf{f}_k(\mathbf{y}^\star) \mathbf{g}_k(\mathbf{y}^\star) && (5.44)\\
&= 0, && (5.45)
\end{aligned}$$

where Eq. 5.44 is obtained by performing the substitution in Eq. 5.19, and Eq. 5.45 follows from orthogonality of the subspaces A and B. $\square$

There are many potential interpretations for the meanings of our definitions of content and co-content as they pertain to conservative signal processing systems. For $M = 1$, the individual contents and co-contents may represent time trajectories in a continuous-time system. A common situation with $M = L$ arises when a conservative interconnection is coupled to a memoryless nonlinearity whose image representation is described by the functions $\mathbf{f}$ and $\mathbf{g}$, with $\mathbf{y}$ being a vector of the exogenous variables ϕ_k in the representation. Nonlinearities that can be represented as a functional relationship may likewise be described by setting $\mathbf{f}(\mathbf{y}) = \mathbf{y}$ or $\mathbf{g}(\mathbf{y}) = \mathbf{y}$. Still another situation is where a variable, conservative interconnection is coupled to fixed subsystems. In this case, variations in the OVS defined over the interconnection might result in transformations of the conjugate mappings to the comparison space $\mathbb{R}^L$ where the variational principles are applied, and from the perspective of this space the variations would resemble those associated with a fixed conservative interconnection coupled to varying subsystems. In Chapter 6, we will discuss these interpretations in greater detail within the context of specific applications.

5.1.2 Relationship to integral definitions

Existing notions of content and co-content have been formulated as integrals, taken either with respect to a single variable or along a multidimensional path. In this subsection, we show that such definitions are particular instances of our notions of OVS content and co-content as defined in Eqns. 5.13-5.21. In doing so, we begin by taking the path integrals of Eqns. 5.17 and 5.18, in addition to the sum of these equations.

As Eqns. 5.17-5.18 are gradient fields, their path integrals are path invariant. It is therefore sufficient to specify an initial and a final point of integration, denoted $\mathbf{y}^{(0)}$ and $\mathbf{y}$, respectively. Integrating both sides of Eq. 5.17 results in

$$\begin{aligned} \int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \nabla Q^{(k)}(\mathbf{u}) \cdot d\mathbf{u} &= \int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \mathbf{g}_k(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}) \cdot d\mathbf{u} && (5.46) \\ Q^{(k)}(\mathbf{y}) - Q^{(k)}\left(\mathbf{y}^{(0)}\right) &= \int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \mathbf{g}_k(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}) \cdot d\mathbf{u}, && (5.47) \end{aligned}$$

with Eq. 5.47 following from the gradient theorem. Integrating both sides of Eq. 5.18, we likewise obtain

$$\int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \nabla R^{(k)}(\mathbf{u}) \cdot d\mathbf{u} = \int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \mathbf{f}_k(\mathbf{u}) \nabla \mathbf{g}_k(\mathbf{u}) \cdot d\mathbf{u} \tag{5.48}$$

$$R^{(k)}(\mathbf{y}) - R^{(k)}\left(\mathbf{y}^{(0)}\right) = \int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \mathbf{f}_k(\mathbf{u}) \nabla \mathbf{g}_k(\mathbf{u}) \cdot d\mathbf{u}. \tag{5.49}$$

In integrating the sum of Eqns. 5.17 and 5.18, we first use the product rule for gradients to write

$$\mathbf{g}_k(\mathbf{y}) \nabla \mathbf{f}_k(\mathbf{y}) + \mathbf{f}_k(\mathbf{y}) \nabla \mathbf{g}_k(\mathbf{y}) = \nabla \left(\mathbf{f}_k(\mathbf{y}) \mathbf{g}_k(\mathbf{y})\right), \tag{5.50}$$

and we accordingly write the integral of Eq. 5.50 as

$$\int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \nabla Q^{(k)}(\mathbf{u}) \cdot d\mathbf{u} + \int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \nabla R^{(k)}(\mathbf{u}) \cdot d\mathbf{u} = \int_{\mathbf{y}^{(0)}}^{\mathbf{y}} \nabla \left(\mathbf{f}_k(\mathbf{u}) \mathbf{g}_k(\mathbf{u})\right) \cdot d\mathbf{u} \tag{5.51}$$

$$Q^{(k)}(\mathbf{y}) - Q^{(k)}\left(\mathbf{y}^{(0)}\right) + R^{(k)}(\mathbf{y}) - R^{(k)}\left(\mathbf{y}^{(0)}\right) = \mathbf{f}_k(\mathbf{y}) \mathbf{g}_k(\mathbf{y}) - \mathbf{f}_k\left(\mathbf{y}^{(0)}\right) \mathbf{g}_k\left(\mathbf{y}^{(0)}\right). \tag{5.52}$$

The first key observation in connecting the notions of OVS content and co-content to previous definitions is that, as it would seem, the definitions in [8,9,26,31] implicitly specify functions $\mathbf{f}^{(k)}$ and $\mathbf{g}^{(k)}$, as well as a point $\mathbf{y}^{(0)}$ for which

$$Q^{(k)}\left(\mathbf{y}^{(0)}\right) + R^{(k)}\left(\mathbf{y}^{(0)}\right) = \mathbf{f}_k\left(\mathbf{y}^{(0)}\right) \mathbf{g}_k\left(\mathbf{y}^{(0)}\right), \quad k = 1, \ldots, L, \tag{5.53}$$

i.e. for which Eqns. 5.47, 5.49 and 5.52 take the form of Eqns. 5.17-5.19. Assuming that Eq. 5.53 holds, the lower limits of integration may be dropped for notational convenience, and we have the individual contents and co-contents written in integral form as

$$Q^{(k)}(\mathbf{y}) = \int^{\mathbf{y}} \mathbf{g}_k(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}) \cdot d\mathbf{u} \tag{5.54}$$

and

$$R^{(k)}(\mathbf{y}) = \int^{\mathbf{y}} \mathbf{f}_k(\mathbf{u}) \nabla \mathbf{g}_k(\mathbf{u}) \cdot d\mathbf{u}. \tag{5.55}$$

If each functional $\mathbf{f}_k(\mathbf{y})$ and $\mathbf{g}_k(\mathbf{y})$ is a function only of the kth entry of $\mathbf{y}$, Eqns. 5.54-5.55 can equivalently be written in terms of $f^{(k)} : \mathbb{R} \to \mathbb{R}$ and $g^{(k)} : \mathbb{R} \to \mathbb{R}$

as

$$Q^{(k)}(\mathbf{y}_k) = \int^{\mathbf{y}_k} g^{(k)}(u) f^{(k)\prime}(u) du \tag{5.56}$$

and

$$R^{(k)}(\mathbf{y}_k) = \int^{\mathbf{y}_k} f^{(k)}(u) g^{(k)\prime}(u) du, \tag{5.57}$$

with $f^{(k)\prime}(u)$ and $g^{(k)\prime}(u)$ respectively denoting the first derivatives of $f^{(k)}$ and $g^{(k)}$. Eqns. 5.56-5.57 take the form of the expressions for parametric individual content and co-content that were defined in [8].

In addressing the case where the value of $\mathbf{g}$ is a function of the value of $\mathbf{f}$, it is useful to set $M = L$ and define

$$\mathbf{f}(\mathbf{y}) = \mathbf{y}, \tag{5.58}$$

with the gradients of the functionals $\mathbf{f}_k$ being

$$\nabla \mathbf{f}_k(\mathbf{y}) = \mathbf{e}^{(k)}. \tag{5.59}$$

In Eq. 5.59, $\mathbf{e}^{(k)}$ is used to denote a column vector with zeros in all entries except for the kth, which has value 1. We likewise write the total content in terms of the expression for individual content in Eq. 5.54 as

$$\begin{aligned} Q(\mathbf{y}) &= \sum_{k=1}^{L} \int^{\mathbf{y}} \mathbf{g}_k(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}) \cdot d\mathbf{u} && (5.60) \\ &= \sum_{k=1}^{L} \int^{\mathbf{y}} \mathbf{g}_k(\mathbf{u}) \mathbf{e}^{(k)} \cdot d\mathbf{u} && (5.61) \\ &= \int_0^1 \sum_{k=1}^{L} \mathbf{g}_k(\mathbf{u}(\tau)) \mathbf{e}^{(k)^{tr}} \mathbf{u}'(\tau) d\tau, && (5.62) \end{aligned}$$

where Eq. 5.62 was obtained using the vector calculus definition of path integration, stated in terms of a parameterized path $\mathbf{u} : \mathbb{R} \rightarrow \mathbb{R}^M$ for which $\mathbf{u}(0) = \mathbf{y}^{(0)}$ and $\mathbf{u}(1) = \mathbf{y}$, and exchanging the summation and integration. As

$$\sum_{k=1}^{L} \mathbf{g}_k(\mathbf{u}(\tau)) \mathbf{e}^{(k)^{tr}} = \mathbf{g}^{tr}(\mathbf{u}(\tau)), \tag{5.63}$$

Eq. 5.62 can be written as

$$Q(\mathbf{y}) = \int_0^1 \mathbf{g}^{tr}(\mathbf{u}(\tau))\mathbf{u}'(\tau)d\tau \tag{5.64}$$

$$= \int^{\mathbf{y}} \mathbf{g}(\mathbf{u}) \cdot d\mathbf{u}. \tag{5.65}$$

Likewise if $\mathbf{f}$ is a function of the value of $\mathbf{g}$, we have

$$R(\mathbf{y}) = \int^{y} \mathbf{f}(\mathbf{u}) \cdot d\mathbf{u}. \tag{5.66}$$

Eqns. 5.65 and 5.66 take the form of the definitions of multidimensional content and co-content, as written in, e.g., [8, 9].

We have seen thus far that integrating the equations for OVS content and co-content results in familiar expressions for these quantities, and we wish to emphasize further that the utility of OVS content and co-content lies beyond their formulations simply as differential expressions for integral quantities. As was previously mentioned, the main point of defining OVS content and co-content in the way that we have done is specifically to avoid a priori specification of inputs or outputs through the use of a parametric representation, and to do so in a multidimensional setting. The implications of this with respect to existing definitions is perhaps best seen by writing the definitions of total OVS content and co-content in integral form.

Summing over the integral expression for the individual contents as written in Eq. 5.54, we obtain

$$Q(\mathbf{y}) = \sum_{k=1}^{L} \int^{\mathbf{y}} \mathbf{g}_k(\mathbf{u})\nabla\mathbf{f}_k(\mathbf{u}) \cdot d\mathbf{u} \tag{5.67}$$

$$= \int_0^1 \sum_{k=1}^{L} \mathbf{g}_k(\mathbf{u}(\tau)) \left(\nabla\mathbf{f}_k(\mathbf{u}(\tau))\right)^{tr} \mathbf{u}'(\tau)d\tau, \tag{5.68}$$

with $\mathbf{u} : \mathbb{R} \to \mathbb{R}^M$ representing a path for which $\mathbf{u}(0) = \mathbf{y}^{(0)}$ and $\mathbf{u}(1) = \mathbf{y}$. Using the identity written in Eqns. 5.26-5.27, we perform the substitution

$$\left(\nabla \mathbf{f}_k(\mathbf{u}(\tau))\right)^{tr} \mathbf{u}'(\tau) = D_{\mathbf{u}'(\tau)} \mathbf{f}_k(\mathbf{u}(\tau)), \tag{5.69}$$

resulting in

$$Q(\mathbf{y}) = \int_0^1 \left\langle D_{\mathbf{u}'(\tau)} \mathbf{f}\left(\mathbf{u}(\tau)\right), \mathbf{g}\left(\mathbf{u}(\tau)\right) \right\rangle d\tau. \tag{5.70}$$

An intuitive interpretation for total OVS content is therefore that it represents an integral along a path through the vector field $\mathbf{g}$, taken with respect to changes in the vector field $\mathbf{f}$ along the path. The total OVS co-content is likewise written in integral form as

$$R(\mathbf{y}) = \int_0^1 \left\langle \mathbf{f}\left(\mathbf{u}(\tau)\right), D_{\mathbf{u}'(\tau)} \mathbf{g}\left(\mathbf{u}(\tau)\right) \right\rangle d\tau \tag{5.71}$$

and has a complementary interpretation.

5.1.3 Composing $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ as functions on subvectors

As was previously mentioned in Eqns. 5.20-5.21, total content and co-content are formulated as the respective sums of the individual contents and co-contents. In this sense, the total content and co-content are linearly separable, and we further emphasize in this subsection that if the functions $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ are decomposable into functions on subvectors, the total content and co-content are likewise decomposable in a conformal way.

This stems from the observation that if the functional $\mathbf{f}^{(k)}(\mathbf{y})$ depends only on certain entries in $\mathbf{y}$, then the individual content $Q^{(k)}(\mathbf{y})$ depends only on those entries as well. Likewise, if $\mathbf{g}^{(k)}(\mathbf{y})$ depends only on certain entries of $\mathbf{y}$, then $R^{(k)}(\mathbf{y})$ depends only on those same entries. This can be seen by reviewing the definition in Eq. 5.17, which results in the following implication:

$$\frac{\partial \mathbf{f}_k(\mathbf{y})}{\partial \mathbf{y}_i} = 0 \;\Rightarrow\; \frac{\partial Q^{(k)}(\mathbf{y})}{\partial \mathbf{y}_i} = 0. \tag{5.72}$$

Eq. 5.18 accordingly results in

$$\frac{\partial \mathbf{g}_k(\mathbf{y})}{\partial \mathbf{y}_i} = 0 \Rightarrow \frac{\partial R^{(k)}(\mathbf{y})}{\partial \mathbf{y}_i} = 0. \tag{5.73}$$

A consequence of Eq. 5.72 is that if the function $\mathbf{f}(\mathbf{y})$ is defined in terms of a total of J functions $\mathbf{f}^{(j)} : \mathbb{R}^{M^{(j)}} \to \mathbb{R}^{L^{(j)}}$, $j = 1, \ldots, J$, as

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}\left(\begin{bmatrix} \mathbf{y}^{(1)} \\ \vdots \\ \mathbf{y}^{(J)} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{f}^{(1)}\left(\mathbf{y}^{(1)}\right) \\ \vdots \\ \mathbf{f}^{(L)}\left(\mathbf{y}^{(L)}\right) \end{bmatrix}, \tag{5.74}$$

with $\mathbf{y}^{(j)}$ denoting a subvector of $\mathbf{y}$, then each function $\mathbf{f}^{(j)}\left(\mathbf{y}^{(j)}\right)$ has a well-defined content $Q^{(j)} : \mathbb{R}^{M^{(j)}} \to \mathbb{R}$, written $Q^{(j)}\left(\mathbf{y}^{(j)}\right)$. Likewise, Eq. 5.73 implies that if we have a function $\mathbf{g}(\mathbf{y})$ defined in terms of a total of J functions $\mathbf{g}^{(j)} : \mathbb{R}^{M^{(j)}} \to \mathbb{R}^{L^{(j)}}$, $j = 1, \ldots, J$, as

$$\mathbf{g}(\mathbf{y}) = \mathbf{g}\left(\begin{bmatrix} \mathbf{y}^{(1)} \\ \vdots \\ \mathbf{y}^{(J)} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{g}^{(1)}\left(\mathbf{y}^{(1)}\right) \\ \vdots \\ \mathbf{g}^{(L)}\left(\mathbf{y}^{(L)}\right) \end{bmatrix}, \tag{5.75}$$

then each function $\mathbf{g}^{(j)}\left(\mathbf{y}^{(j)}\right)$ has a well-defined co-content $R^{(j)} : \mathbb{R}^{M^{(j)}} \to \mathbb{R}$, written $R^{(j)}\left(\mathbf{y}^{(j)}\right)$. The total OVS content and co-content may accordingly be written as

$$Q(\mathbf{y}) = \sum_{j=1}^{J} Q^{(j)}\left(\mathbf{y}^{(j)}\right) \tag{5.76}$$

and

$$R(\mathbf{y}) = \sum_{j=1}^{J} R^{(j)}\left(\mathbf{y}^{(j)}\right), \tag{5.77}$$

i.e. the terms in the sums involve a decomposition of the vector $\mathbf{y}$ that is conformal with the decompositions of $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ in Eqns. 5.74-5.75.

5.1.4 Re-parameterizing $\mathbf{f}(\mathbf{y})$-$\mathbf{g}(\mathbf{y})$, $\mathbf{f}(\mathbf{y})$-$Q(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$-$R(\mathbf{y})$ contours

We have defined OVS content and co-content, as well as the pertinent functions $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$, in terms of a vector-valued parameter $\mathbf{y}$. A relevant question in doing this is that of what relationships, if any, are affected by a re-parameterization of $\mathbf{y}$. The specific sense of re-parameterization that we address will be a replacement of the variable $\mathbf{y}$ with a function $\mathbf{h} : \mathbb{R}^M \longrightarrow \mathbb{R}^M$, resulting in re-parameterized functions $\widehat{\mathbf{f}}(\mathbf{y})$ and $\widehat{\mathbf{g}}(\mathbf{y})$ that are written formally as

$$\widehat{\mathbf{f}}(\mathbf{y}) = \mathbf{f}\left(\mathbf{h}(\mathbf{y})\right), \tag{5.78}$$

and

$$\widehat{\mathbf{g}}(\mathbf{y}) = \mathbf{g}\left(\mathbf{h}(\mathbf{y})\right). \tag{5.79}$$

In discussing this, we will assume that $\mathbf{f}(\mathbf{y})$, $\mathbf{g}(\mathbf{y})$ and $\mathbf{h}(\mathbf{y})$ are continuous and everywhere differentiable, and we will additionally require that the image of $\mathbf{h}(\mathbf{y})$ is $\mathbb{R}^M$. Written formally, we require that the Jacobians $J_{\mathbf{f}}(\mathbf{y})$, $J_{\mathbf{g}}(\mathbf{y})$ and $J_{\mathbf{h}}(\mathbf{y})$ exist at all points $\mathbf{y} \in \mathbb{R}^M$, and that

$$\left\{\mathbf{h}(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^M\right\} = \mathbb{R}^M. \tag{5.80}$$

It is straightforward to demonstrate that the relationship between $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ is identical to the relationship between $\widehat{\mathbf{f}}(\mathbf{y})$ and $\widehat{\mathbf{g}}(\mathbf{y})$. In particular, it follows from Eq. 5.80 that the surface traced out by $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ when evaluated at all points $\mathbf{y} \in \mathbb{R}^M$ is identical to the surface traced out by $\widehat{\mathbf{f}}(\mathbf{y})$ and $\widehat{\mathbf{g}}(\mathbf{y})$, as substituting $\mathbf{h}(\mathbf{y})$ in place of $\mathbf{y}$ does not restrict the set of all points for which $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ are evaluated.

We claim that the relationship between $\mathbf{f}(\mathbf{y})$ and $Q(\mathbf{y})$ also remains unaffected, as does the relationship between $\mathbf{g}(\mathbf{y})$ and $R(\mathbf{y})$. As $Q(\mathbf{y})$ and $R(\mathbf{y})$ are defined in terms of $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$, this must be verified with greater care. The approach in doing so is to define the re-parameterized individual contents $\widehat{Q}^{(k)}(\mathbf{y})$ and co-contents $\widehat{R}^{(k)}(\mathbf{y})$ as

$$\widehat{Q}^{(k)}(\mathbf{y}) = Q^{(k)}\left(\mathbf{h}(\mathbf{y})\right) \tag{5.81}$$

and

$$\widehat{R}^{(k)}(\mathbf{y}) = R^{(k)}\left(\mathbf{h}(\mathbf{y})\right), \tag{5.82}$$

$k = 1, \ldots, L$, with the re-parameterized total content and co-content in turn being

$$\widehat{Q}(\mathbf{y}) = \sum_{k=1}^{L} \widehat{Q}^{(k)}(\mathbf{y}) \tag{5.83}$$

and

$$\widehat{R}(\mathbf{y}) = \sum_{k=1}^{L} \widehat{R}^{(k)}(\mathbf{y}), \tag{5.84}$$

and to demonstrate that $\widehat{Q}^{(k)}(\mathbf{y})$ and $\widehat{R}^{(k)}(\mathbf{y})$ are valid individual contents and co-contents for the re-parameterized functions $\widehat{\mathbf{f}}(\mathbf{y})$ and $\widehat{\mathbf{g}}(\mathbf{y})$.

Taking the gradient of both sides of Eq. 5.81 results in

$$\begin{aligned} \nabla\widehat{Q}^{(k)}(\mathbf{y}) &= \nabla Q^{(k)}\left(\mathbf{h}(\mathbf{y})\right) & (5.85)\\ &= J_{\mathbf{h}}^{tr}(\mathbf{y})\left(\nabla Q^{(k)}(\mathbf{y})\big|_{\mathbf{h}(\mathbf{y})}\right), & (5.86) \end{aligned}$$

where Eq. 5.86 follows from the multidimensional chain rule. Substituting in the expression for $\nabla Q^{(k)}(\mathbf{y})$ in Eq. 5.17 results in

$$\begin{aligned} \nabla\widehat{Q}^{(k)}(\mathbf{y}) &= J_{\mathbf{h}}^{tr}(\mathbf{y})\left(\nabla\mathbf{f}_k(\mathbf{y})\big|_{\mathbf{h}(\mathbf{y})}\right)\mathbf{g}_k\left(\mathbf{h}(\mathbf{y})\right) & (5.87)\\ &= \mathbf{g}_k\left(\mathbf{h}(\mathbf{y})\right)\nabla\mathbf{f}_k\left(\mathbf{h}(\mathbf{y})\right) & (5.88)\\ &= \widehat{\mathbf{g}}_k(\mathbf{y})\nabla\widehat{\mathbf{f}}_k(\mathbf{y}), & (5.89) \end{aligned}$$

where Eq. 5.88 again follows from the multidimensional chain rule, i.e. from the equation $\nabla\mathbf{f}_k\left(\mathbf{h}(\mathbf{y})\right) = J_{\mathbf{h}}^{tr}(\mathbf{y})\left(\nabla\mathbf{f}_k(\mathbf{y})\big|_{\mathbf{h}(\mathbf{y})}\right)$, and where Eq. 5.89 follows from substituting Eqns. 5.78-5.79 into Eq. 5.88. We conclude that re-parameterizing the expression for an individual content as $Q^{(k)}\left(\mathbf{h}(\mathbf{y})\right)$ is equivalent to forming an individual content from the re-parameterized functionals $\mathbf{f}_k\left(\mathbf{h}(\mathbf{y})\right)$ and $\mathbf{g}_k\left(\mathbf{h}(\mathbf{y})\right)$.

The same holds for the individual co-contents, as can be verified by substituting

$\mathbf{h}(\mathbf{y})$ for $\mathbf{y}$ in Eq. 5.19, resulting in

$$R^{(k)}(\mathbf{h}(\mathbf{y})) = \mathbf{f}_k(\mathbf{h}(\mathbf{y}))\,\mathbf{g}_k(\mathbf{h}(\mathbf{y})) - Q^{(k)}(\mathbf{h}(\mathbf{y})) \tag{5.90}$$

$$\widehat{R}^{(k)}(\mathbf{y}) = \widehat{\mathbf{f}}_k(\mathbf{y})\widehat{\mathbf{g}}_k(\mathbf{y}) - \widehat{Q}^{(k)}(\mathbf{y}), \tag{5.91}$$

i.e. the re-parameterized co-content satisfies Eq. 5.19. As was previously mentioned, Eqns. 5.17 and 5.19 imply Eq. 5.18, and we conclude that $Q(\mathbf{h}(\mathbf{y}))$ and $R(\mathbf{h}(\mathbf{y}))$ are a valid content and co-content corresponding to $\mathbf{f}(\mathbf{h}(\mathbf{y}))$ and $\mathbf{g}(\mathbf{h}(\mathbf{y}))$. From Eq. 5.80, the image of $\mathbf{h}(\mathbf{y})$ is $\mathbb{R}^M$, and we conclude that the $\mathbf{f}(\mathbf{y})$-$Q(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$-$R(\mathbf{y})$ contours are invariant to re-parameterization by $\mathbf{h}(\mathbf{y})$.

5.1.5 Some example contours

It is worth pointing out that that even in the case where $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$, as well the corresponding content $Q(\mathbf{y})$ and co-content $R(\mathbf{y})$, are everywhere differentiable, a fairly broad class of contours can result. An example of this is depicted in Figs. 5-1, 5-2 and 5-3, which illustrate the relevant contours for the case where individual content and co-content corresponding to the 1-dimensional functions

$$\mathbf{f}_k(\mathbf{y}) = \begin{cases} (\mathbf{y}_k - 1)^2, & \mathbf{y}_k \geq 1 \\ 0, & -1 \leq \mathbf{y}_k < 1 \\ -(\mathbf{y}_k + 1)^2, & \mathbf{y}_k < -1 \end{cases} \tag{5.92}$$

and

$$\mathbf{g}_k(\mathbf{y}) = \begin{cases} 1, & \mathbf{y}_k \geq 1 \\ \sin\left(\frac{\pi}{2}\mathbf{y}_k\right), & -1 \leq \mathbf{y}_k < 1 \\ -1, & \mathbf{y}_k < -1 \end{cases} \tag{5.93}$$

have been computed as

$$Q^{(k)}(\mathbf{y}) = \begin{cases} (\mathbf{y}_k - 1)^2, & \mathbf{y}_k \geq 1 \\ 0, & -1 \leq \mathbf{y}_k < 1 \\ (\mathbf{y}_k + 1)^2, & \mathbf{y}_k < -1 \end{cases} \tag{5.94}$$

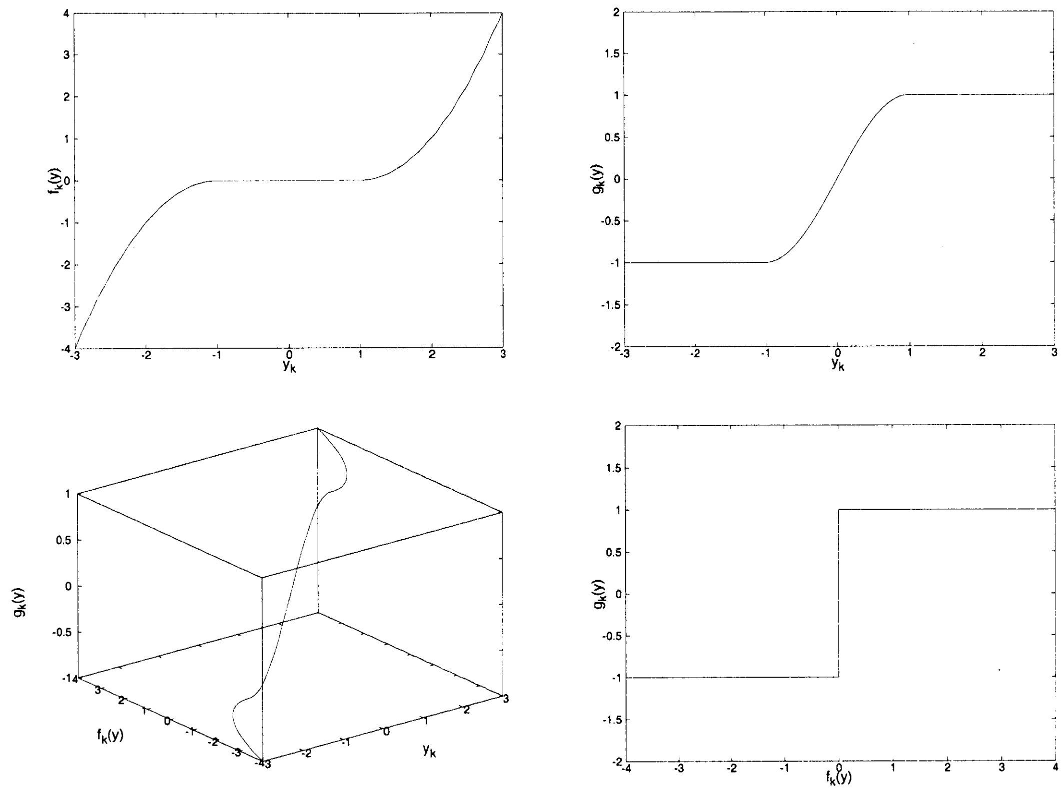


Figure 5-1: Plots comparing $\mathbf{f}_k(\mathbf{y_k})$, $\mathbf{g}_k(\mathbf{y_k})$ and $\mathbf{y_k}$, as pertaining to Eqns. 5.92-5.95.

and

$$R^{(k)}(\mathbf{y}) = 0. \tag{5.95}$$

In particular Fig. 5-1 illustrates that the $\mathbf{f}_k(\mathbf{y})$-$\mathbf{g}_k(\mathbf{y})$ contour is not representable as a function, Fig. 5-2 depicts a $\mathbf{f}_k(\mathbf{y})$-$Q^{(k)}(\mathbf{y})$ contour that is not differentiable, and Fig. 5-3 illustrates that the $\mathbf{g}_k(\mathbf{y})$-$R^{(k)}(\mathbf{y})$ contour has compact support.

It is illustrative to see the progression from contours that represent functional relationships to those that do not. In particular, invertibility of the $\mathbf{f}_k(\mathbf{y}_k)$-$\mathbf{g}_k(\mathbf{y}_k)$ contour appears to be related to convexity of the $\mathbf{f}_k(\mathbf{y}_k)$-$Q^{(k)}(\mathbf{y}_k)$ contour, and in turn to whether the $\mathbf{g}_k(\mathbf{y}_k)$-$R^{(k)}(\mathbf{y}_k)$ represents a functional relationship, as is depicted by the ensemble of contours in Fig. 5-4. Referring to this figure, we emphasize that the irregularities in certain of the $\mathbf{g}_k(\mathbf{y}_k)$-$R^{(k)}(\mathbf{y}_k)$ contours are not artifacts due to

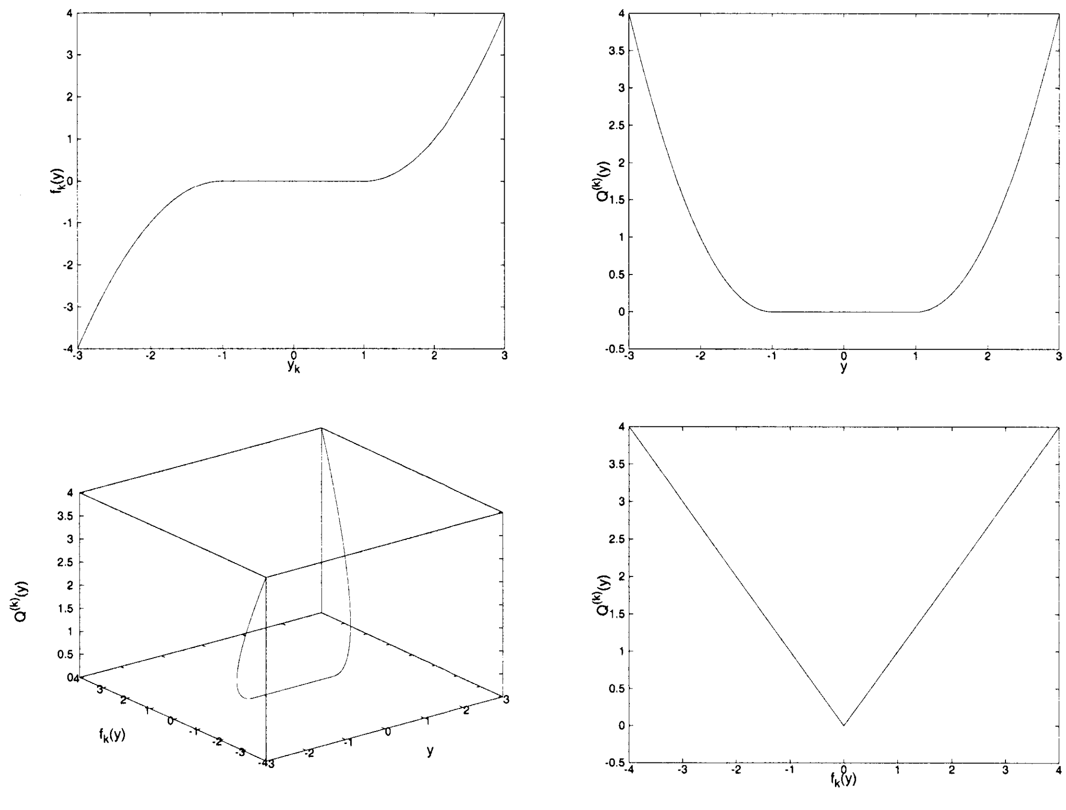


Figure 5-2: Plots comparing $\mathbf{f}_k(\mathbf{y_k})$, $Q^{(k)}(\mathbf{y_k})$ and $\mathbf{y_k}$, as pertaining to Eqns. 5.92-5.95.

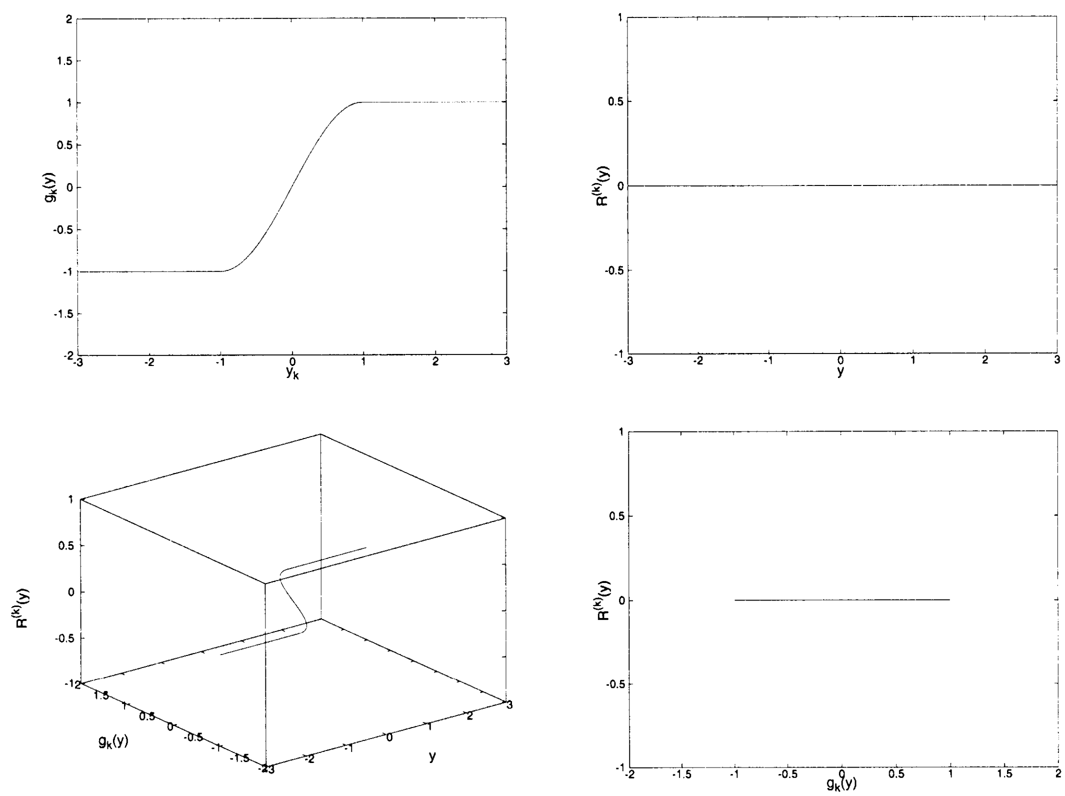


Figure 5-3: Plots comparing $\mathbf{g}_k(\mathbf{y_k})$, $R^{(k)}(\mathbf{y_k})$ and $\mathbf{y_k}$, as pertaining to Eqns. 5.92-5.95.

the plotting routines used, but rather are a result of the contour smoothly changing direction and turning back on itself as $\mathbf{y}_k$ increases.

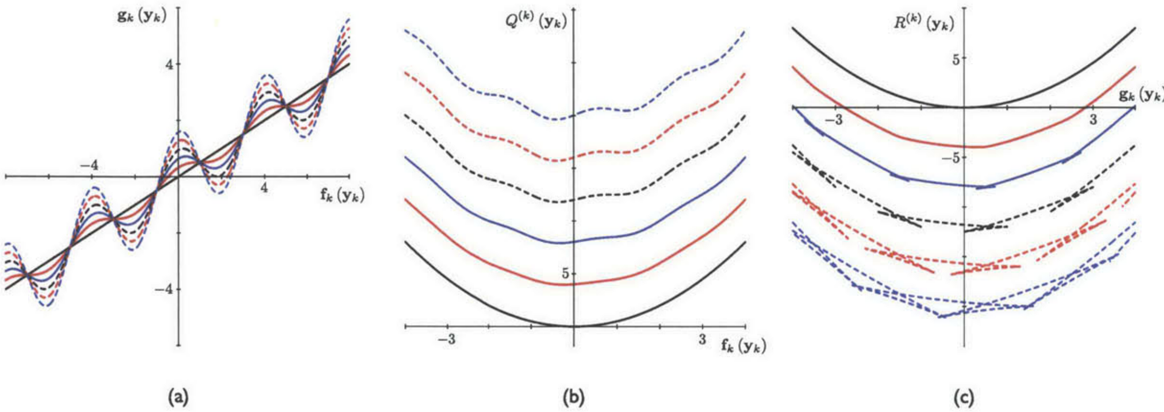


Figure 5-4: (a) Example contours progressing from invertible to noninvertible $\mathbf{f}(\mathbf{y})$-$\mathbf{g}(\mathbf{y})$ relationships. (b) Corresponding $\mathbf{f}(\mathbf{y})$-$Q^{(k)}(\mathbf{y})$ contours. (c) Corresponding $\mathbf{g}(\mathbf{y})$-$R^{(k)}(\mathbf{y})$ contours.

5.1.6 Functionally-related $\mathbf{f}(\mathbf{y})$-$Q(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$-$R(\mathbf{y})$ contours

We have established the concepts of OVS content and co-content by specifying a set of conditions in Eqns. 5.17-5.19 that must be obeyed. In the case where the individual contents $Q^{(k)}(\mathbf{y})$ are differentiable and functionally dependent on $\mathbf{f}_k(\mathbf{y})$ or the co-contents $R^{(k)}(\mathbf{y})$ are differentiable and functionally dependent on $\mathbf{g}_k(\mathbf{y})$, it is possible to use the defining equations 5.17-5.19 to begin with a pre-specified $Q^{(k)}(\mathbf{y})$ or $R^{(k)}(\mathbf{y})$ and write functionals $\mathbf{f}_k(\mathbf{y})$ and $\mathbf{g}_k(\mathbf{y})$ for which $Q^{(k)}(\mathbf{y})$ or $R^{(k)}(\mathbf{y})$ are well-defined.

The particular sense in which we demonstrate this is to begin with a sum of individual contents, denoted

$$Q^{(k,\dots,k+\ell)}(\mathbf{y}) = \sum_{i=k}^{\ell} Q^{(i)}(\mathbf{y}), \tag{5.96}$$

with each individual content $Q^{(i)}(\mathbf{y})$ in the sum being differentiable and individually depending only on the entries of $\mathbf{y}$ in the range $\mathbf{y}_k, \ldots, \mathbf{y}_{k+\ell}$, and with the equation

$$\mathbf{f}_i(\mathbf{y}) = \mathbf{y}_i, \quad i = k, \ldots, k+\ell \tag{5.97}$$

formally establishing the functional dependency of $Q^{(k,\ldots,k+\ell)}(\mathbf{y})$ on $\mathbf{f}_k(\mathbf{y}), \ldots, \mathbf{f}_{k+\ell}(\mathbf{y})$. Then combining Eqns. 5.17 and 5.97 results in

$$\mathbf{g}_i(\mathbf{y})\mathbf{e}^{(i)} = \nabla Q^{(i)}(\mathbf{y}), \quad i = k, \ldots, k+\ell \tag{5.98}$$

and substituting this into Eq. 5.96 results in

$$\nabla Q^{(k,\ldots,k+\ell)}(\mathbf{y}) = \begin{bmatrix} \vdots \\ 0 \\ \mathbf{g}_k(\mathbf{y}) \\ \vdots \\ \mathbf{g}_{k+\ell}(\mathbf{y}) \\ 0 \\ \vdots \end{bmatrix}. \tag{5.99}$$

We conclude that given a differentiable content $Q^{(k,\ldots,k+\ell)}(\mathbf{y})$ that is functionally dependent on $\mathbf{f}_k, \ldots, \mathbf{f}_{k+\ell}$, we may obtain valid functions $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ using the condition in Eq. 5.97, in addition to

$$\mathbf{g}_i(\mathbf{y}) = \frac{\partial Q^{(k,\ldots,k+\ell)}(\mathbf{y})}{\partial \mathbf{y}_i}, \quad i = k, \ldots, k+\ell. \tag{5.100}$$

Furthermore, as $Q^{(k,\ldots,k+\ell)}(\mathbf{y})$ depends only on $\mathbf{y}_k, \ldots, \mathbf{y}_{k+\ell}$, each functional $\mathbf{g}_i(\mathbf{y})$ in Eq. 5.100 will depend only on these values as well, and we say that $\mathbf{g}_i(\mathbf{y})$, $i = k, \ldots, k+\ell$ is a function of $\mathbf{f}_i(\mathbf{y})$, $i = k, \ldots, k+\ell$.

Likewise, given a sum of co-contents defined as

$$R^{(k,\ldots,k+\ell)}(\mathbf{y}) = \sum_{i=k}^{\ell} R^{(i)}(\mathbf{y}), \tag{5.101}$$

with each individual co-content $R^{(i)}(\mathbf{y})$ in the sum being differentiable and individually depending only on the entries of $\mathbf{y}$ in the range $\mathbf{y}_k, \ldots, \mathbf{y}_{k+\ell}$, and with the functional dependence of each $R^{(i)}(\mathbf{y})$ on $\mathbf{g}_i(\mathbf{y})$ being written formally as

$$\mathbf{g}_i(\mathbf{y}) = \mathbf{y}_i, \quad i = k, \ldots, k+\ell \tag{5.102}$$

then a function $\mathbf{f}(\mathbf{y})$ corresponding to a well-defined co-content $R^{(k,\ldots,k+\ell)}(\mathbf{y})$ may be obtained by satisfying

$$\mathbf{f}_i(\mathbf{y}) = \frac{\partial R^{(k,\ldots,k+\ell)}(\mathbf{y})}{\partial \mathbf{y}_i}, \quad i = k, \ldots, k+\ell. \tag{5.103}$$

Furthermore, each functional $\mathbf{f}_i(\mathbf{y})$ in Eq. 5.103 will depend only on $\mathbf{y}_k, \ldots, \mathbf{y}_{k+\ell}$, and we say that $\mathbf{f}_i(\mathbf{y}),\ i = k, \ldots, k+\ell$ is a function of $\mathbf{g}_i(\mathbf{y}),\ i = k, \ldots, k+\ell$.

We have just demonstrated that a differentiable $Q(\mathbf{y})$ being functionally dependent on $\mathbf{f}(\mathbf{y})$ results in a function $\mathbf{g}(\mathbf{y})$ that is functionally dependent on $\mathbf{f}(\mathbf{y})$, and likewise that a differentiable $\widehat{R}(\mathbf{y})$ being functionally dependent on $\widehat{\mathbf{g}}(\mathbf{y})$ results in a function $\widehat{\mathbf{f}}(\mathbf{y})$ that is functionally dependent on $\widehat{\mathbf{g}}(\mathbf{y})$. A pertinent question is that of how these are related to the co-content $R(\mathbf{y})$ corresponding to $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$, as well as to the content $\widehat{Q}(\mathbf{y})$ corresponding to $\widehat{\mathbf{f}}(\mathbf{y})$ and $\widehat{\mathbf{g}}(\mathbf{y})$. Indeed, these quantities are straightforward to define using Eq. 5.19 as

$$R^{(i)}(\mathbf{y}) = \mathbf{y}_i \mathbf{g}_i(\mathbf{y}) - Q^{(i)}(\mathbf{y}), \quad i = k, \ldots, k+\ell \tag{5.104}$$

and

$$\widehat{Q}^{(i)}(\mathbf{y}) = \widehat{\mathbf{f}}_i(\mathbf{y})\mathbf{y}_i - \widehat{R}^{(i)}(\mathbf{y}), \quad i = k, \ldots, k+\ell. \tag{5.105}$$

However, the $(\mathbf{g}_k(\mathbf{y}), \ldots, \mathbf{g}_{k+\ell}(\mathbf{y}))$-$R^{(k,\ldots,k+\ell)}(\mathbf{y})$ and $(\mathbf{f}_k(\mathbf{y}), \ldots, \mathbf{f}_{k+\ell}(\mathbf{y}))$-$Q^{(k,\ldots,k+\ell)}(\mathbf{y})$ surfaces will not generally exhibit a functional relationship, as was seen with the line contours in Fig. 5-4.

A multidimensional example illustrating this issue is depicted in Figs. 5-5, 5-6 and 5-7. In particular, Fig. 5-5 depicts surfaces for which $Q^{(k,k+1)}(\mathbf{y})$ is a function of $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$, $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$ is a function of $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$, and for which $R^{(k,k+1)}(\mathbf{y})$ happens also to be a function of $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$. In Fig. 5-6, dimples are incrementally added to the original $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$-$Q^{(k,k+1)}(\mathbf{y})$ surface, resulting in a final $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$-$R^{(k,k+1)}(\mathbf{y})$ surface that no longer exhibits a functional relationship. Fig. 5-7 depicts the final surface in greater detail, along with the associated functional relationship from $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$ to $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$.

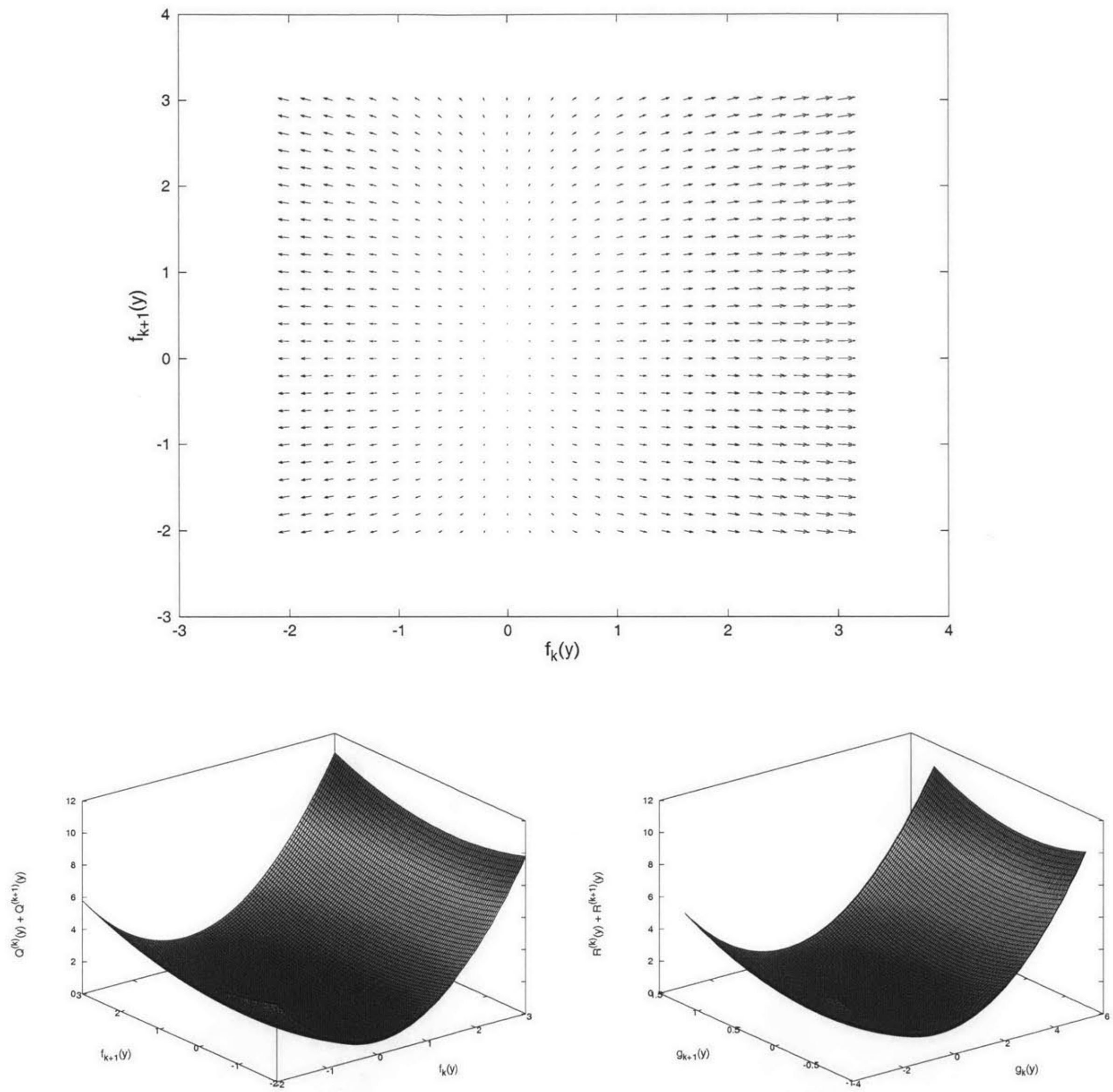


Figure 5-5: Example surfaces for which $\mathbf{Q}^{(k,k+1)}(\mathbf{y})$ and $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$ are functions of $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$. Top: vector field representing the two dimensional function from $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$ to $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$. Bottom left: surface representing the function from $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$ to $\mathbf{Q}^{(k,k+1)}(\mathbf{y})$. Bottom right: parametric $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$-$\mathbf{R}^{(k,k+1)}(\mathbf{y})$ surface, obtained using, e.g., Eq. 5.104.

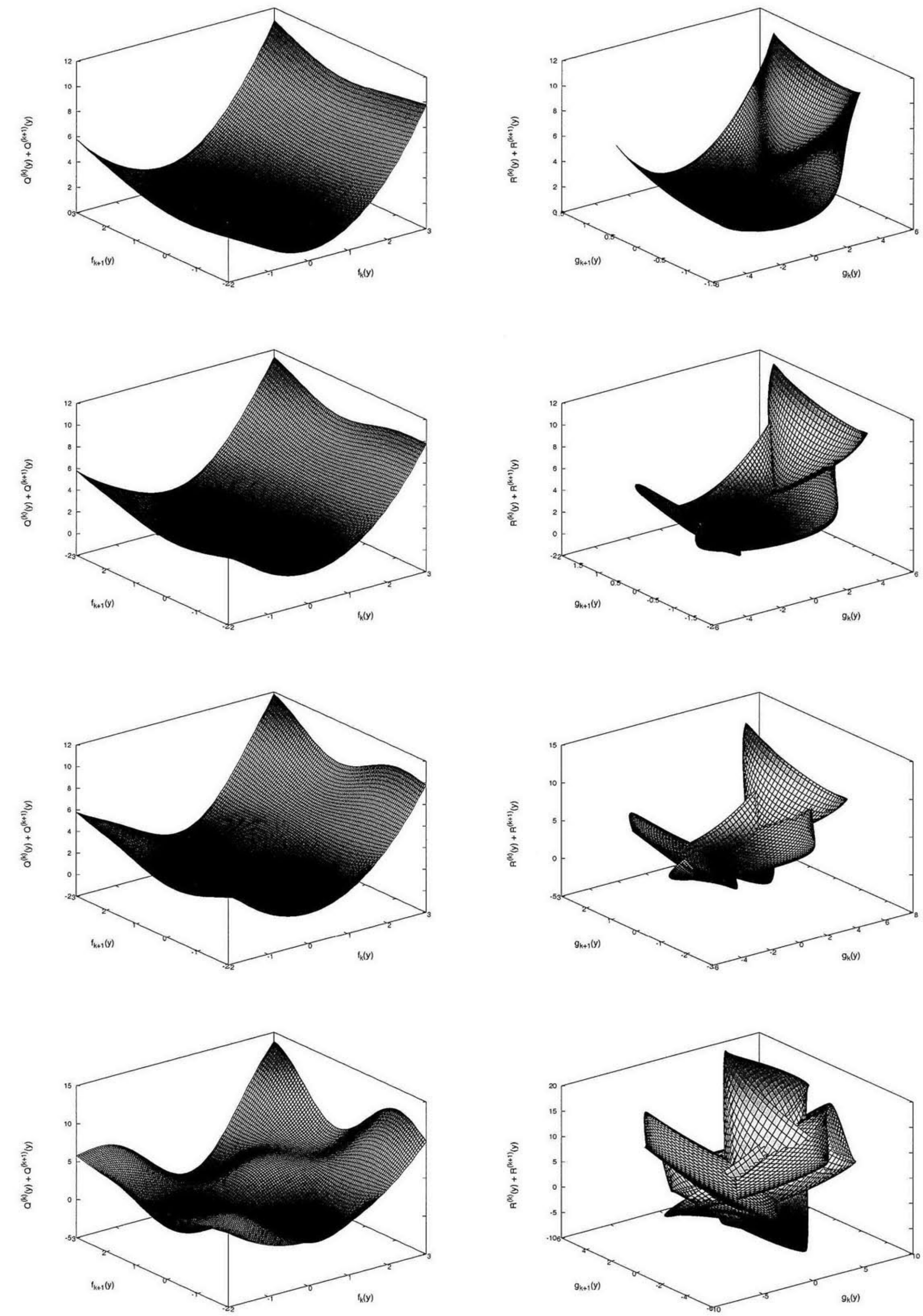


Figure 5-6: Left column: surfaces for which $\mathbf{Q}^{(k,k+1)}(\mathbf{y})$ is a function of $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$. Right column: corresponding parametric $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$-$\mathbf{R}^{(k,k+1)}(\mathbf{y})$ surfaces, obtained using, e.g., Eq. 5.104.

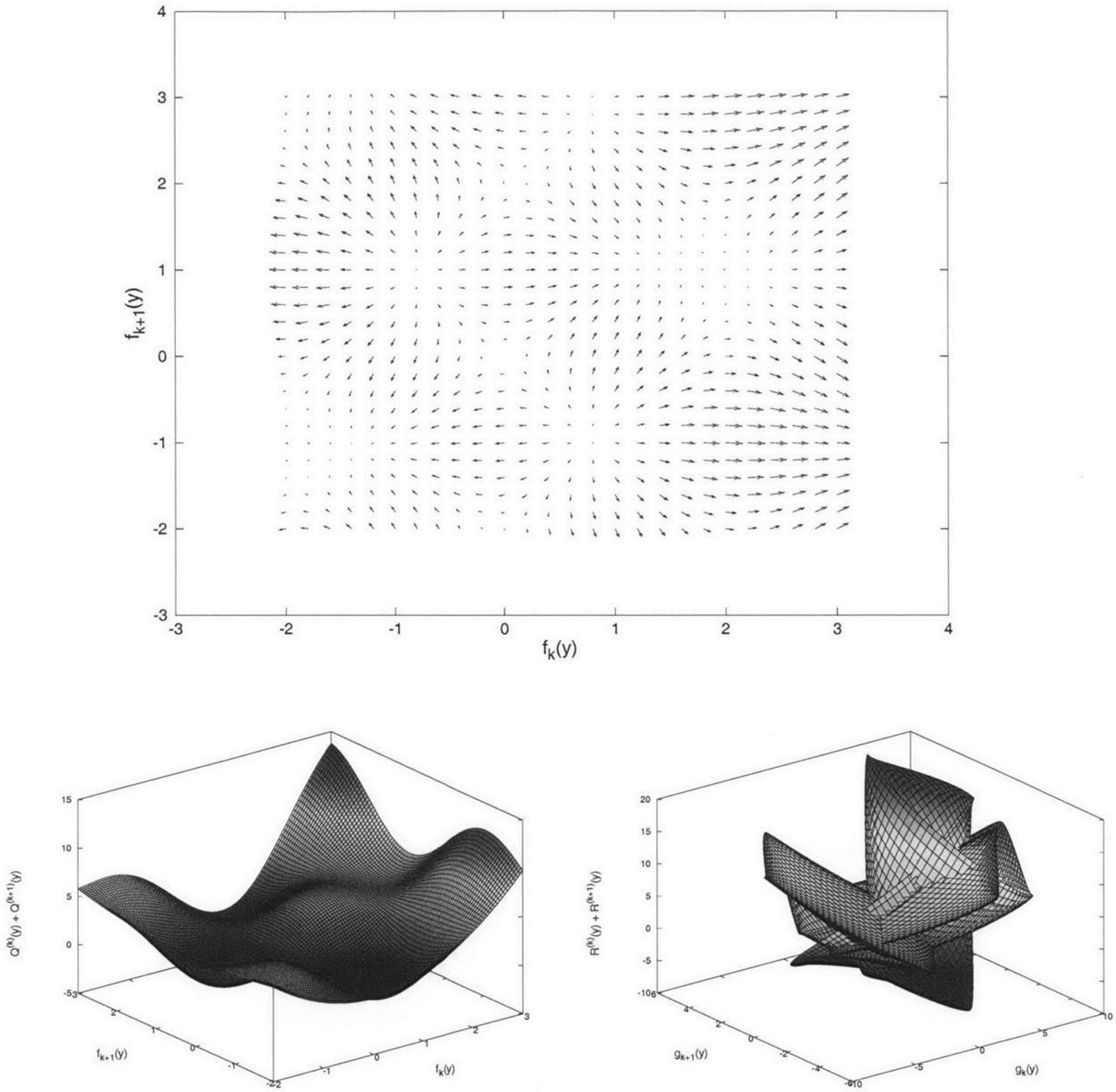


Figure 5-7: Example surfaces corresponding to the bottom row in Fig. 5-6. Top: vector field representing the two dimensional function from $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$ to $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$. Bottom left: surface representing the function from $(\mathbf{f}_k(\mathbf{y}), \mathbf{f}_{k+1}(\mathbf{y}))$ to $\mathbf{Q}^{(k,k+1)}(\mathbf{y})$. Bottom right: parametric $(\mathbf{g}_k(\mathbf{y}), \mathbf{g}_{k+1}(\mathbf{y}))$-$\mathbf{R}^{(k,k+1)}(\mathbf{y})$ surface, obtained using, e.g., Eq. 5.104.

5.2 Connections with optimization theory

In Thm. 5.1, stationarity of OVS content $Q(\mathbf{y})$ and co-content $R(\mathbf{y})$ was established for any point $\mathbf{y}^\star$ where the functions $\mathbf{f}(\mathbf{y}^\star)$ and $\mathbf{g}(\mathbf{y}^\star)$ lied in orthogonal vector subspaces A and B. Paraphrasing the theorem, any small movement $\delta\mathbf{u}^{(Q)}$ for which $\mathbf{f}\left(\mathbf{y}^\star + \delta\mathbf{u}^{(Q)}\right)$ remained in A was shown to be a point of zero slope for $Q(\mathbf{y})$, and any small movement $\delta\mathbf{u}^{(R)}$ for which $\mathbf{g}\left(\mathbf{y}^\star + \delta\mathbf{u}^{(R)}\right)$ remained in B was shown to be a point of zero slope for $R(\mathbf{y})$.

In this sense, Thm. 5.1 related $\mathbf{f}(\mathbf{y})$ to $Q(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ to $R(\mathbf{y})$, and it is this pair of relationships that bears a resemblance to dual cost functions in certain constrained optimization problems. As the theorem does not involve minimization, maximization, or any notion of convexity or concavity, we take a moment to emphasize the occasions where the variational principle coincides with common classes of optimization problems. The intent of this section is to draw on a the rich body of work in the field of optimization theory to gain further insight into the meaning behind content and co-content in these situations, utilizing a few specific examples as a prelude to future research. We will focus attention to functions $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ that are continuous and everywhere differentiable, although Thm. 5.1 does not require these properties at every point $\mathbf{y}$.

We proceed by making the following observation pertaining to the use of Thm. 5.1 in cases where it is known that every stationary point of $Q(\mathbf{y})$ for which $\mathbf{f}(\mathbf{y}) \in A$ is a global minimum, and where every stationary point of $R(\mathbf{y})$ for which $\mathbf{g}(\mathbf{y}) \in B$ is also a global minimum. Then in these cases we have that any vector $\mathbf{y}^\star$ satisfying $\mathbf{f}(\mathbf{y}^\star) \in A$ and $\mathbf{g}(\mathbf{y}^\star) \in B$ is a solution to both of the following optimization problems:

$$\begin{aligned} \min_{\mathbf{y}\in\mathbb{R}^M} \quad & Q(\mathbf{y}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{y}) \in A \end{aligned} \tag{5.106}$$

and

$$\begin{aligned} \max_{\mathbf{y}\in\mathbb{R}^M} \quad & -R(\mathbf{y}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{y}) \in B. \end{aligned} \tag{5.107}$$

Furthermore for any such vector $\mathbf{y}^\star$, Thm. 5.1 implies that the optimal cost for (5.106) is equal to the optimal cost for (5.107).

Indeed the form of these problems, in addition to the equivalence of their cost functions, is reminiscent of the form of dual problems in optimization theory. Two major distinctions, however, are that the two problems are coupled together through the variable $\mathbf{y}$, and that there is no guarantee a priori that the functions $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ will generally be dual in any traditional sense. Motivated by this, Figs. 5-8 through 5-11 depict some example $\mathbf{f}(\mathbf{y})$-$\mathbf{g}(\mathbf{y})$ contours that result in standard, uncoupled dual optimization problems that are dual in the sense of monotropic optimization. [32] The assumption is that the functions $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ used in generating the contours are everywhere differentiable, even though many of these contours have edges that are not smooth. An example where smooth functions were used in generating non-smooth contours was depicted in Figs. 5-1 through 5-3.

The uncoupling of the problems specifically occurs by taking advantage of the fact that neither of $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ is required to be invertible. For example, in the contours in Fig. 5-8 depicting primal equality constraints and dual linear cost, the associated variable $\mathbf{y}$ does not play a role in the primal problem and as such has a dual cost contribution that is uncoupled.

5.3 Dynamics of OVS content and co-content

As was initially discussed in [26] and emphasized in [19, 40], an electrical network containing 2-terminal resistors, voltage sources, and linear capacitors has "shrinking co-content," i.e. the sum of the individual co-contents of the memoryless components is nonincreasing with respect to time. This observation has served as a foundation for

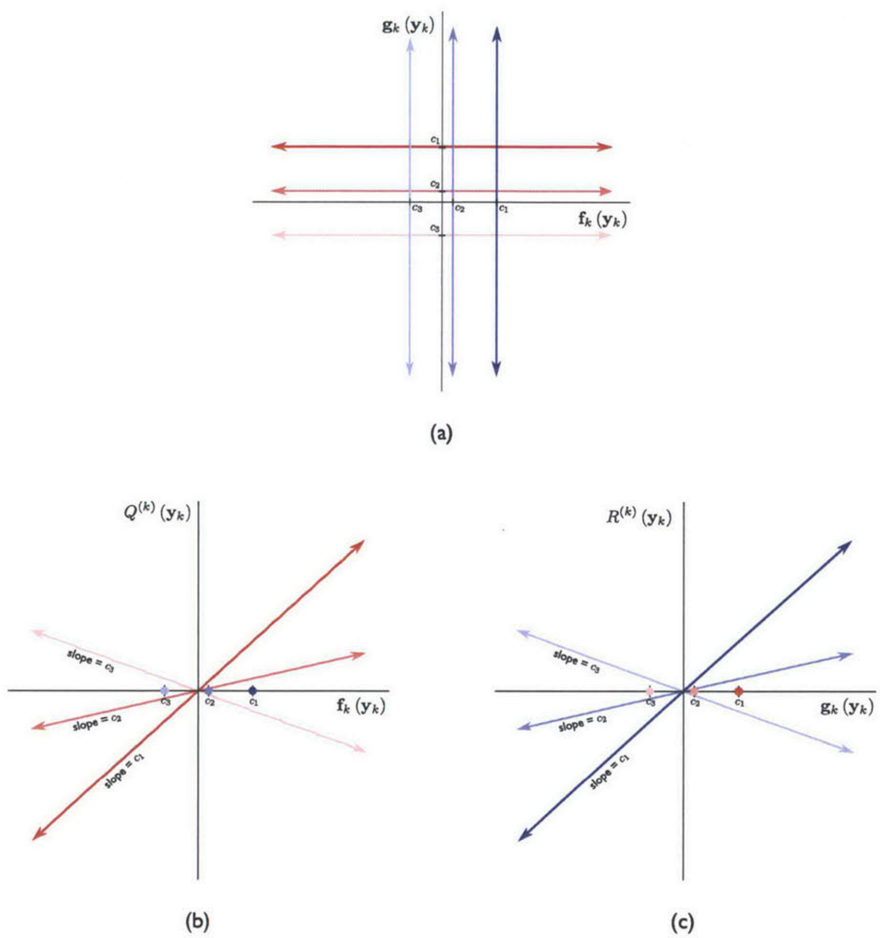


Figure 5-8: $\mathbf{f}(\mathbf{y})$-$\mathbf{g}(\mathbf{y})$ contours corresponding to primal equality constraints and dual linear cost terms; and vice-versa.

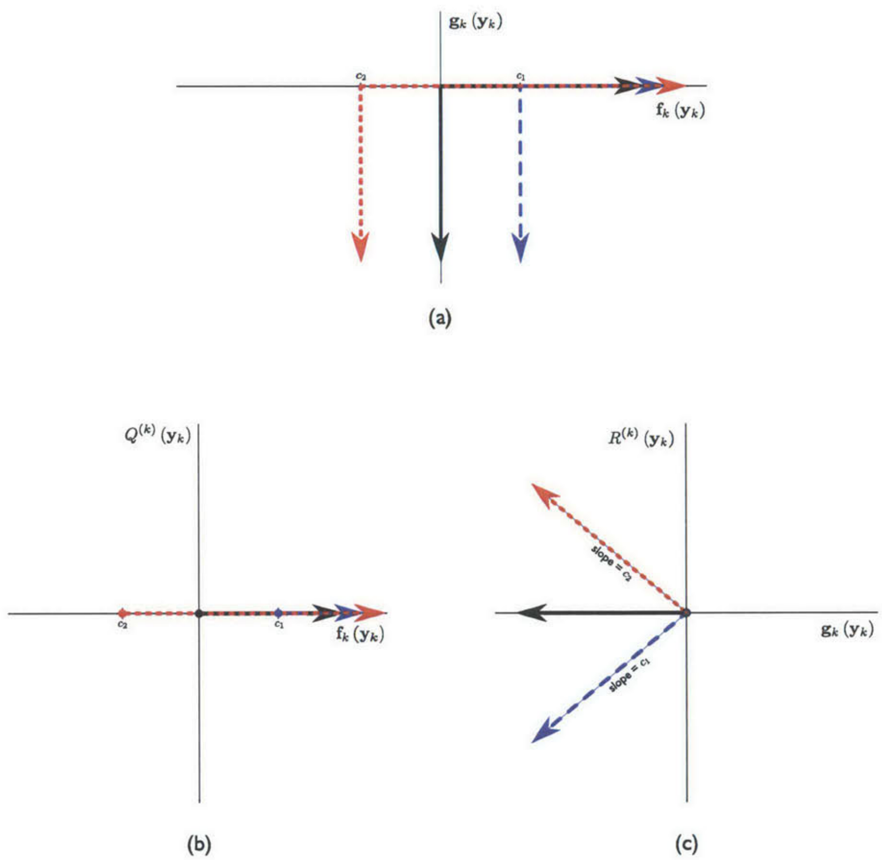


Figure 5-9: $\mathbf{f}(\mathbf{y})$-$\mathbf{g}(\mathbf{y})$ contours corresponding to primal inequality constraints and dual inequality constraints with linear cost.

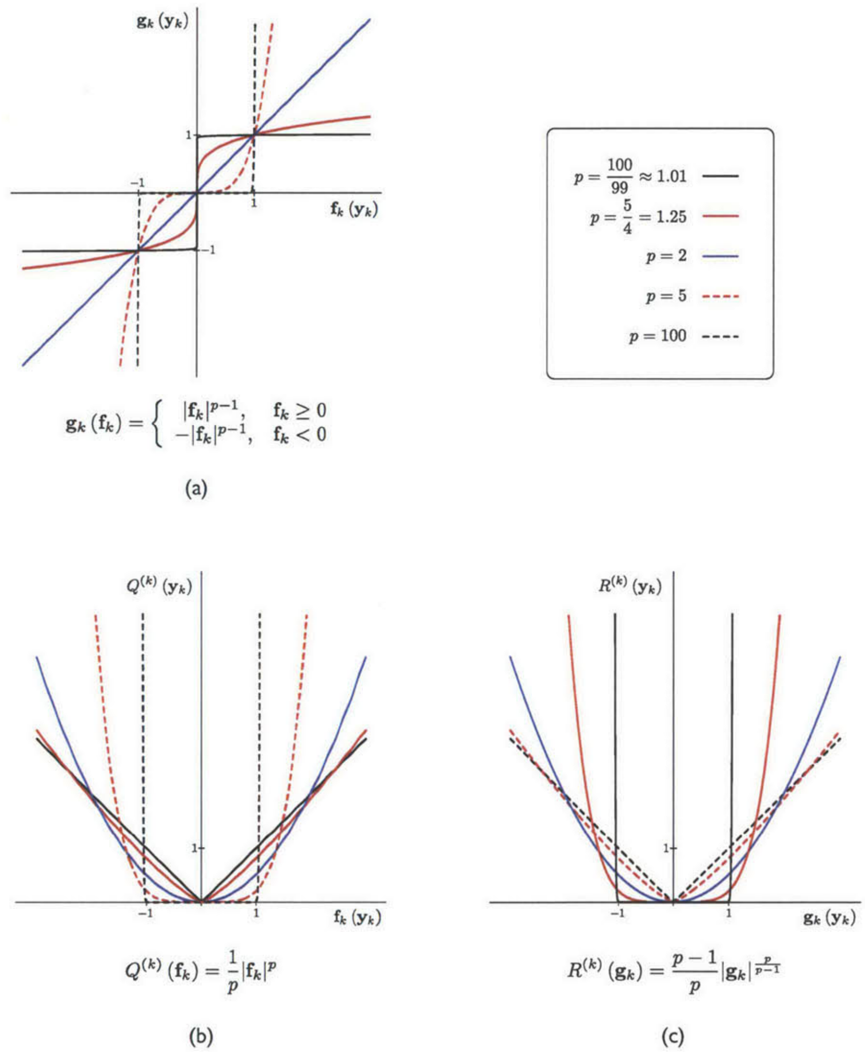


Figure 5-10: $\mathbf{f}(\mathbf{y})$-$\mathbf{g}(\mathbf{y})$ contours corresponding to primal and dual scaled power law cost terms, useful for example in constructing convex p-norms.

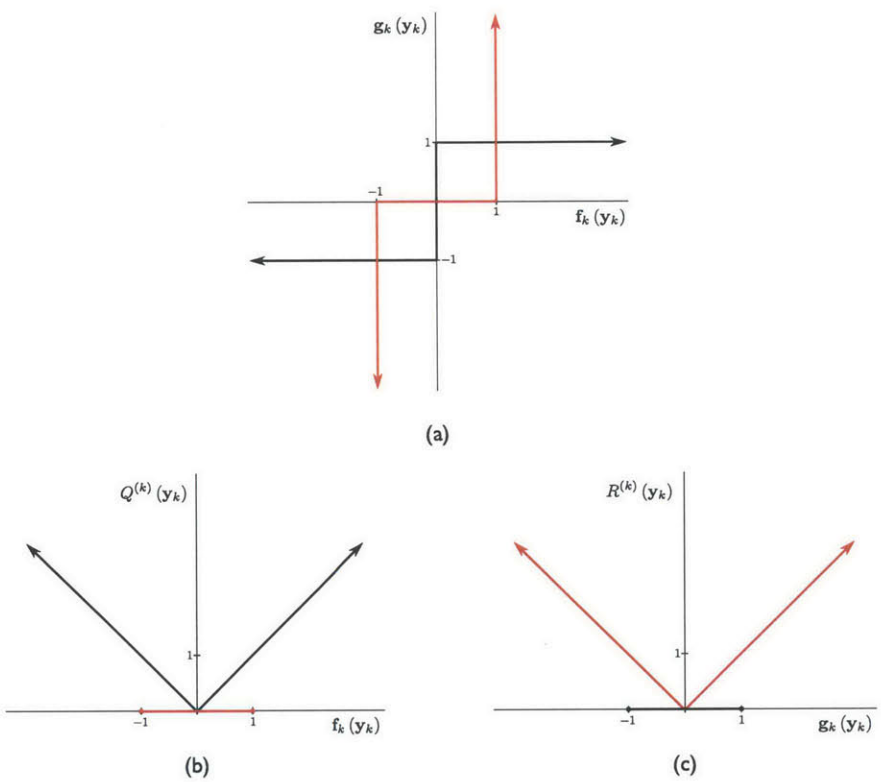


Figure 5-11: $\mathbf{f}(\mathbf{y})$-$\mathbf{g}(\mathbf{y})$ contours corresponding to primal absolute value cost terms and dual inequality constraints and vice-versa.

various nonlinear electrical network designs aimed at performing linear and nonlinear optimization and image processing, e.g. [9,19,22,40]. In these applications, in addition to others too numerous to cite, co-content not only acted as a cost function, but also played the role of a Lyapunov function in facilitating stability analysis of the systems.

In developing OVS content and co-content, we have been careful to provide formulations for which a concept of time is not required. However in designing conservative systems having dynamic behavior, it may be of interest to use content and co-content in describing their time evolution. Within this context, we present a theorem resembling the shrinking co-content principle in [40] that embodies the spirit of that result within the context of OVS content and co-content.

Pertinent to this, we are concerned with functions

$$\mathbf{f} : \mathbb{R}^M \to \mathbb{R}^L \tag{5.108}$$

and

$$\mathbf{g} : \mathbb{R}^M \to \mathbb{R}^L \tag{5.109}$$

that are continuous and everywhere differentiable, and that have valid associated total content

$$Q : \mathbb{R}^M \to \mathbb{R} \tag{5.110}$$

and co-content

$$R : \mathbb{R}^M \to \mathbb{R}, \tag{5.111}$$

i.e. that satisfy Eqns. 5.17-5.19.

Theorem 5.2 (Principle of shrinking OVS content). *This theorem pertains to the functions in Eqns. 5.108-5.111, in addition to two orthogonal vector spaces $A \subseteq \mathbb{R}^L$ and $B \subseteq \mathbb{R}^L$, as well as two augmented vector spaces $A_{aug} \subseteq \mathbb{R}^{K+L}$ and $B_{aug} \subseteq \mathbb{R}^{K+L}$ that satisfy the following properties:*

(1) A_{aug} is orthogonal to B_{aug}, under the standard inner product on $\mathbb{R}^{K+L}$.

(2) For an arbitrary vector $\mathbf{x} \in \mathbb{R}^L$*,*

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{x} \end{bmatrix} \in A_{aug} \Leftrightarrow \mathbf{w} \in A. \tag{5.112}$$

(3)

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{0}^{(K)} \end{bmatrix} \in B_{aug} \Leftrightarrow \mathbf{w} \in B, \tag{5.113}$$

with $\mathbf{0}^{(K)}$ *denoting the length-*K *column vector of zeros.*

Then for any differentiable trajectory $\mathbf{y} : \mathbb{R} \to \mathbb{R}^L$ *that satisfies*

$$\begin{bmatrix} \mathbf{f}(\mathbf{y}(t)) \\ \mathbf{x}(t) \end{bmatrix} \in A_{aug} \tag{5.114}$$

and

$$\begin{bmatrix} \mathbf{g}(\mathbf{y}(t)) \\ \mathbf{x}'(t) \end{bmatrix} \in B_{aug} \tag{5.115}$$

for some differentiable $\mathbf{x} : \mathbb{R} \to \mathbb{R}^K$*, the total content* $Q(\mathbf{y}(t))$ *is nonincreasing with respect to* t*, i.e.*

$$\frac{dQ(\mathbf{y}(t))}{dt} \leq 0. \tag{5.116}$$

Given a fixed value of $t = t^\star$*, Eq. 5.116 is satisfied with equality if and only if* $\mathbf{f}(\mathbf{y}(t^\star)) \in A$ *and* $\mathbf{g}(\mathbf{y}(t^\star)) \in B$*, i.e. if and only if the conditions required for applying Thm. 5.1 hold.*

Exchanging the roles of $\mathbf{f}$ *and* $\mathbf{g}$*,* A *and* B*, and* A_{aug} *and* B_{aug}*, a dual theorem pertinent to shrinking co-content* $R(\mathbf{y}(t))$ *can be stated as well.*

Proof. We begin by noting that Eq. 5.114 implies that

$$\begin{bmatrix} \frac{d\mathbf{f}(\mathbf{y}(t))}{dt} \\ \frac{d\mathbf{x}(t)}{dt} \end{bmatrix} \in A_{aug}. \tag{5.117}$$

As A_{aug} and B_{aug} are orthogonal vector subspaces, the following may be written using

Eqns. 5.115 and 5.117:

$$\sum_{\ell=1}^{L} \frac{d\mathbf{f}_\ell(\mathbf{y}(t))}{dt} \mathbf{g}_\ell(\mathbf{y}(t)) + \sum_{k=1}^{K} \left(\frac{d\mathbf{x}_k(t)}{dt} \right)^2 = 0. \tag{5.118}$$

Applying the multidimensional chain rule results in

$$\sum_{\ell=1}^{L} \mathbf{g}_\ell(\mathbf{y}(t)) \frac{d\mathbf{f}_\ell(\mathbf{y}(t))}{dt} = \sum_{\ell=1}^{L} \mathbf{g}_\ell(\mathbf{y}(t)) \left(\nabla \mathbf{f}_\ell(\mathbf{y}(t))\right)^{tr} \mathbf{y}'(t), \tag{5.119}$$

and substituting in the expression for individual content in Eq. 5.17, we obtain

$$\begin{aligned} \sum_{\ell=1}^{L} \mathbf{g}_\ell(\mathbf{y}(t)) \frac{d\mathbf{f}_\ell(\mathbf{y}(t))}{dt} &= \sum_{\ell=1}^{L} \left(\nabla Q^{(\ell)}(\mathbf{y}(t))\right)^{tr} \mathbf{y}'(t) && (5.120) \\ &= \sum_{\ell=1}^{L} \frac{dQ^{(\ell)}(\mathbf{y}(t))}{dt}, && (5.121) \end{aligned}$$

with Eq. 5.121 again following from the multidimensional chain rule. The right-hand side of this equation is equal to the total content, and we have

$$\sum_{\ell=1}^{L} \mathbf{g}_\ell(\mathbf{y}(t)) \frac{d\mathbf{f}_\ell(\mathbf{y}(t))}{dt} = \frac{dQ(\mathbf{y}(t))}{dt}. \tag{5.122}$$

Substituting Eq. 5.122 into Eq. 5.118 results in

$$\frac{dQ(\mathbf{y}(t))}{dt} + \sum_{k=1}^{K} \left(\frac{d\mathbf{x}_k(t)}{dt} \right)^2 = 0, \tag{5.123}$$

and we conclude that

$$\frac{dQ(\mathbf{y}(t))}{dt} \leq 0. \tag{5.124}$$

With this in place, the conditions for which

$$\frac{dQ(\mathbf{y}(t^\star))}{dt} = 0 \tag{5.125}$$

follow in a straightforward way from requirements (1) and (2) in the statement of the theorem. In particular,

$$\frac{dQ(\mathbf{y}(t^\star))}{dt} = 0 \;\Leftrightarrow\; \begin{bmatrix} \mathbf{g}(\mathbf{y}(t^\star)) \\ \mathbf{0}^{(K)} \end{bmatrix} \in B_{aug} \;\Leftrightarrow\; \mathbf{g}(\mathbf{y}(t^\star)) \in B, \tag{5.126}$$

and $\mathbf{f}(\mathbf{y}(t^\star)) \in A$ by construction. □

Some example subsystems indicating the use Thm. 5.2 are depicted in Fig. 5-12. Referring to this figure, the interpretations of the functions $\mathbf{f}(\mathbf{y})$ and $\mathbf{g}(\mathbf{y})$ is that they are nonlinearities composing an image representation of subsystems. For the systems in Fig. 5-12(a) and (c), the orthogonal vector subspaces A and B are realized using conjugate signal-flow graphs with the convention that inputs to Interconnection B are negated with respect to outputs from Interconnection A. The systems in Fig. 5-12(b) and (d) have interconnecting structures that implement the augmented vector subspaces A_{aug} and B_{aug}. Under the interpretation that the integrator boxes are time integrals, the respective total content and co-content for these systems, a subsystem of which is depicted in each of Figs. 5-12(b) and (d), is nonincreasing with respect to time.

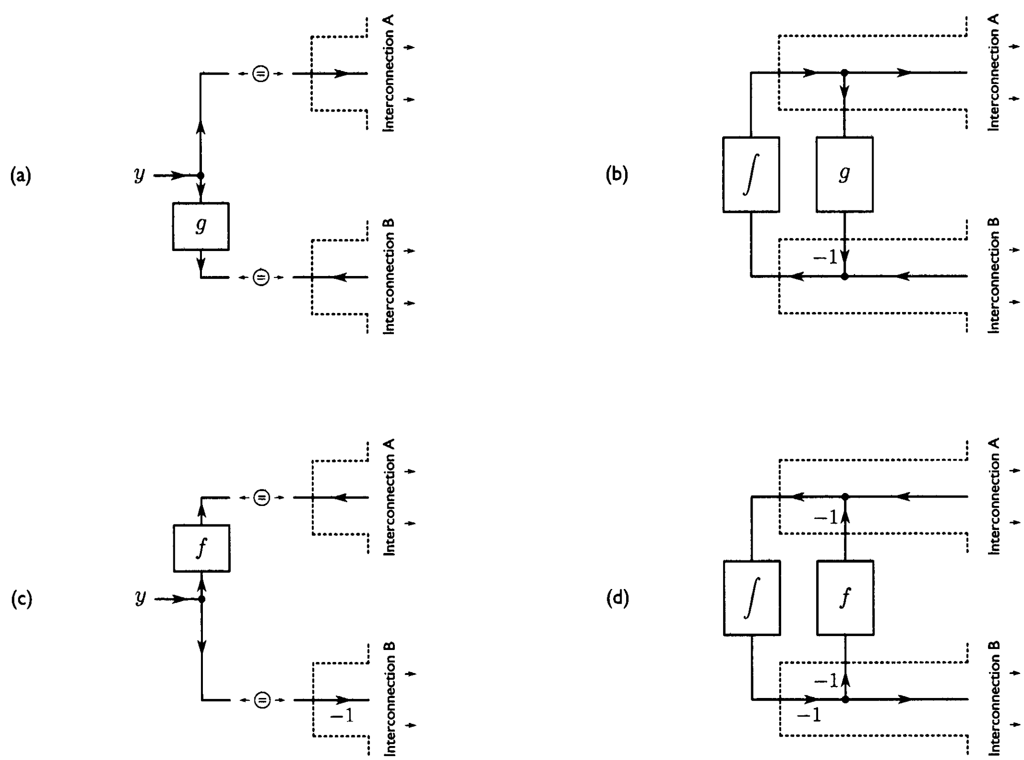


Figure 5-12: (a) Conservative subsystem for which $dQ(\mathbf{y}(t^\star))/dt = 0$. (b) Subsystem with $dQ(\mathbf{y}(t^\star))/dt \leq 0$ and that becomes (a) in steady state. (c) Conservative subsystem for which $dR(\mathbf{y}(t^\star))/dt = 0$. (d) Subsystem with $dR(\mathbf{y}(t^\star))/dt \leq 0$ and that becomes (c) in steady-state.

Chapter 6

Examples and conclusions

This chapter contains a collection of examples, with the intent being to illustrate the use of various aspects of the framework presented in this thesis. In doing this, an emphasis will be placed on providing insight into potential ways in which the framework can be applied. With this in mind, certain of the applications may be viewed as illustrative examples, while others, such as the system for traffic density control in Section 6.5, represent a way of solving a problem that appears to be new.

A key goal of the thesis has been to unify various signal processing systems within a cohesive framework. From this perspective, the examples in this chapter serve also as concluding remarks for the thesis, providing additional context for the elements of the framework and suggesting potential in future applications.

6.1 Inversion of feedback-based compensation systems

The example in this section illustrates the use of the results in Chapter 2 pertaining to system inversion. The specific context is that we are given a system containing an invertible nonlinearity that has been approximately compensated for by closing a feedback loop around it. From the output of this compensated system, we wish to design a system that provides an exact inverse, i.e. that can be used to obtain the input signal to the compensated system.

Fig. 6-1 illustrates the formulation of the problem. As was discussed in, e.g. [5], it is sometimes possible to compensate for a memoryless nonlinear system f represented as in Fig. 6-1(a) by projecting the associated error forward in time to future samples using, e.g. a compensating system that takes the form of Fig. 6-1(b). The function g in Fig. 6-1(b), sometimes not written explicitly in this class of compensating systems, is a memoryless nonlinearity that may for example represent quantization in the processing or damping in feeding back the error. Under certain conditions on f, the spectrum of $d[n]$ due to the error will tend to have a highpass response, and if the input signal $c[n]$ is sufficiently oversampled so that most of its energy falls in a low frequency band for which the contribution from the error is relatively minor, a simple lowpass system as in Fig. 6-1(c) can be used for reconstruction.[1]

Using the results in Chapter 2 related to system inversion, path reversal can be performed on the path from $c[n]$ to $d[n]$ in Fig. 6-1(b), resulting in the behaviorally-equivalent nonlinear reconstruction system in Fig. 6-1(d). The sense of behavioral equivalence is specifically that every $c[n]$-$d[n]$ signal pair consistent with the system in Fig. 6-1(b) is a $c'[n]$-$d'[n]$ signal pair consistent with the system in Fig. 6-1(d), i.e. they are inverses. As the system in Fig. 6-1(d) is a nonlinear feedforward system, it is guaranteed to be stable.

Figs. 6-1, 6-3 and 6-4 illustrate the use of the systems in Fig. 6-1 in compensating for a nonlinearity and performing approximate, lowpass; and exact, nonlinear reconstruction. The signal $c[n]$ is a trumpet solo recorded over background accompaniment at 44.1 kHz, oversampled by a factor of 16 using the function `resample` in GNU Octave, which in doing so uses a length-1160 FIR, Kaiser interpolation filter having 60dB stopband rejection, as specified by the heuristic method in [29]. For this example, the nonlinear functions f and g depicted in Fig. 6-2 are used. The function g represents quantization in the feedback loop, and the function f is the undesired nonlinearity in the system. In systems such as power amplifiers, the function f may

[1] Specific conditions on f for which the associated error in the compensating system has a highpass response is not the focus here. The assumption is simply that the compensating and reconstruction systems in Fig. 6-1(b)-(c) are useful within the context of whatever specific application is at hand.

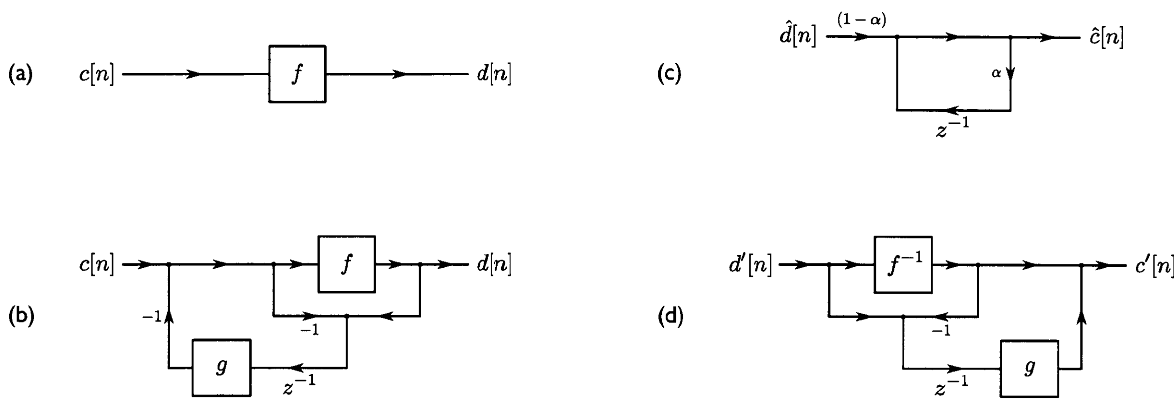


Figure 6-1: (a) Original system. (b) Compensating system. (b) Lowpass approximate reconstruction system. (c) Nonlinear exact reconstruction system.

often be more benign than the one depicted in Fig. 6-2. The function f was chosen in this example to illustrate the effectiveness of the technique with regard to systems having a fairly severe nonlinearity.

The original signal $c[n]$ and distortion error signal $f(c[n]) - c[n]$ pertaining to this example are depicted in Fig. 6-3. Using the 2-norm as an indication of closeness, we have for this example that

$$||f(c[n]) - c[n]|| \approx 37.8. \tag{6.1}$$

Performing approximate lowpass reconstruction using the system in Fig. 6-1(c) with $\alpha = 0.6$, which was experimentally determined to correspond to the minimum 2-norm reconstruction error, resulted in

$$||\hat{c}[n] - c[n]|| \approx 11.7. \tag{6.2}$$

Performing exact reconstruction using the system in Fig. 6-1(d) resulted in an error signal with a 2-norm that essentially reflected the numerical error in GNU Octave:

$$||c'[n] - c[n]|| \approx 1.27 \times 10^{-14}. \tag{6.3}$$

The error signals for the lowpass reconstruction method and the nonlinear reconstruc-

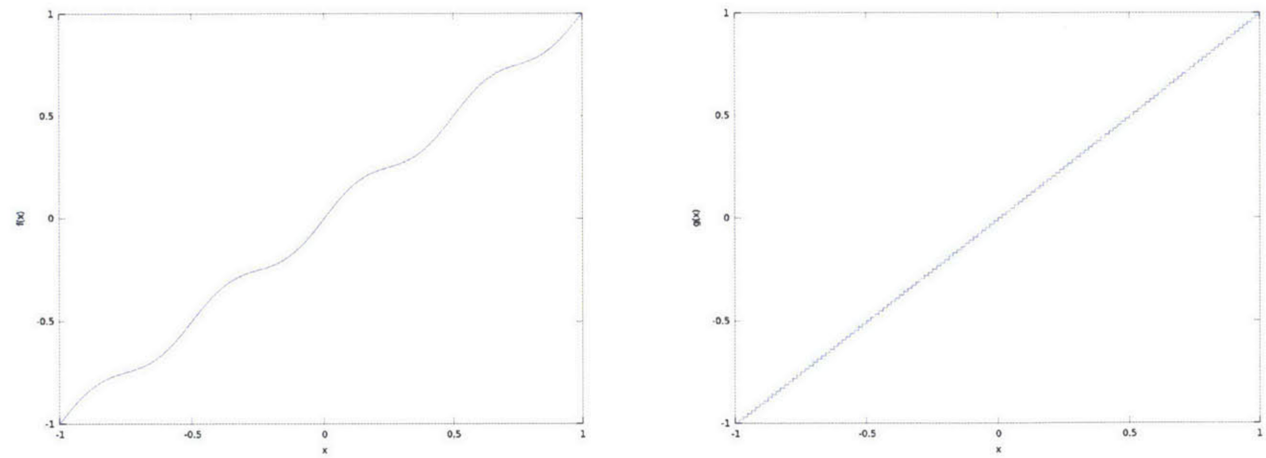


Figure 6-2: Nonlinear characteristics for the systems f and g in Fig. 6-1.

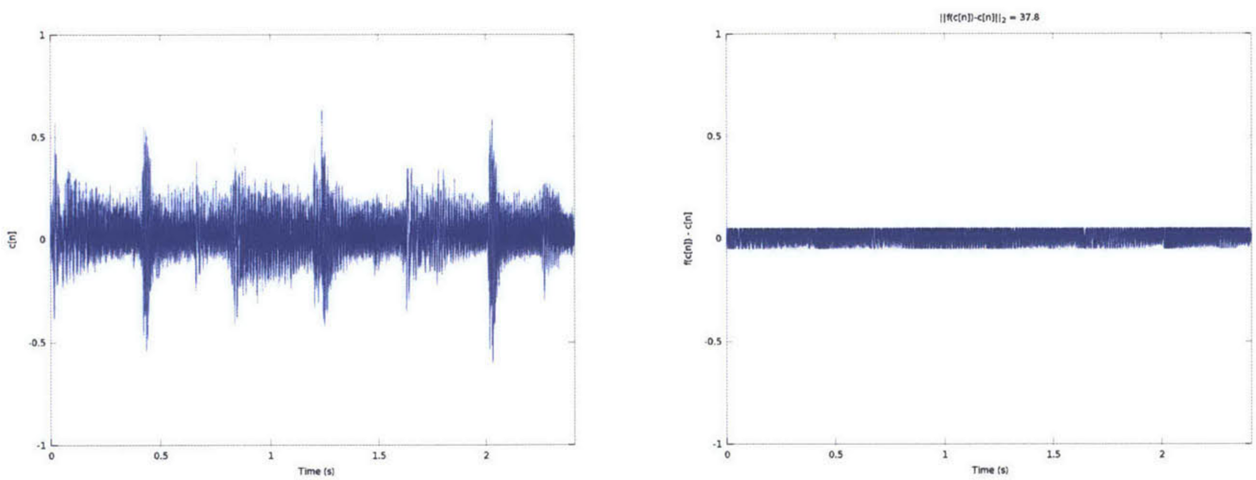


Figure 6-3: (left) Original signal $c[n]$. (right) Distortion error signal $f(c[n]) - c[n]$.

tion method obtained using the results in Chapter 2 are depicted in Fig. 6-4.

6.2 A generalized Tellegen's theorem for signal-flow graphs

The results in Chapter 4 pertaining to the creation of conservative, linear signal-flow graph interconnections can be used to generalize an existing theorem for signal-flow graph nodes that resembles Tellegen's theorem for electrical networks. This theorem,

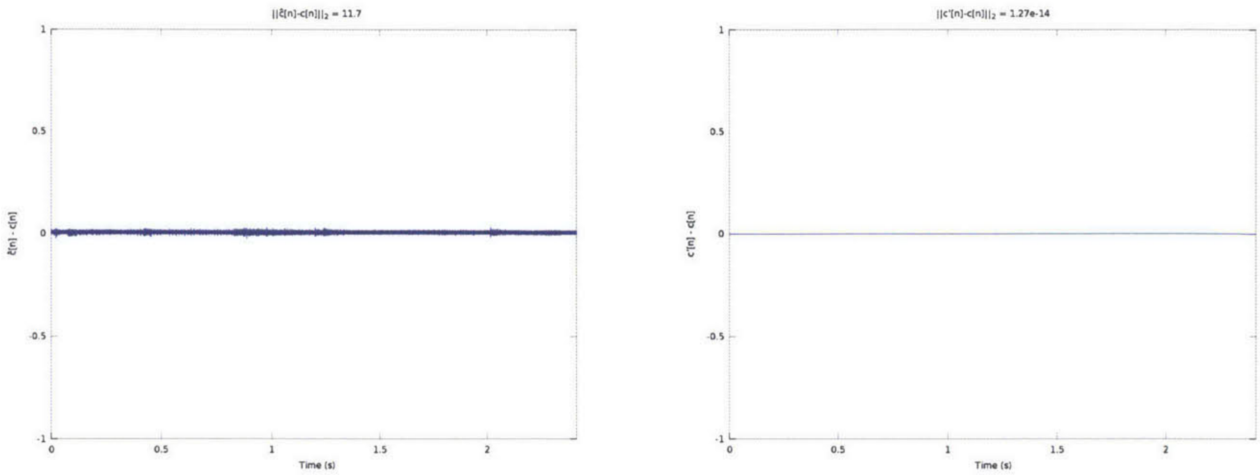


Figure 6-4: (left) Lowpass reconstruction error signal $\hat{c}[n] - c[n]$. (right) Nonlinear reconstruction error signal $c'[n] - c[n]$.

which is commonly referred to as "Tellegen's theorem for signal-flow graphs," has been used in deriving various signal-flow graph theorems, including those related to transposition and calculation of parameter sensitivities. [3, 10, 13, 16, 28, 33] The theorem states that for two topologically-equivalent signal-flow graphs, with P denoting the number of network nodes and $M^{(k)}$ denoting the number of inputs to a particular node k, the following equation is satisfied:

$$\sum_{k=1}^{P}\sum_{j=1}^{M^{(k)}}\left(w'^{(k)}v_j^{(k)} - w^{(k)}v_j'^{(k)}\right) = 0. \tag{6.4}$$

In Eq. 6.4, $w^{(k)}$ and $v_j^{(k)}$ respectively denote the node variables and node inputs in the first network, and $w'^{(k)}$ and $v_j'^{(k)}$ respectively denote the node variables and node inputs in the second network. The convention is specifically that the variables associated with a single node k are related according to

$$w^{(k)} = \sum_{j=1}^{M^{(k)}} v_j^{(k)} \tag{6.5}$$

and

$$w'^{(k)} = \sum_{j=1}^{M^{(k)}} v_j'^{(k)} \tag{6.6}$$

in the first and second networks, respectively, and that the node inputs may generally be connected to branches or to variables that are external to the system. As Eqns. 6.5-6.6 and the proof of Eq. 6.4 in, e.g., [28] do not involve specification of the branch functions or subsystems, Tellegen's theorem for signal-flow graphs can be applied to a wide variety of networks including those that have nonlinear and time-varying branches.

Using the results pertaining to conditions for conservation in Chapter 4, we derive a generalization of the theorem applicable to signal-flow graphs where the nodes are allowed to have inputs and outputs with non-unity gains. As with the original theorem, the branches that connect the nodes will be allowed to be arbitrary. There will again be two networks, although this time the networks will not be topologically equivalent but rather topologically complementary, i.e. the number of outputs from a particular node in the first network will equal the number of inputs to the corresponding node in the second network, and vice-versa. In particular, the relationship between a corresponding pair of generalized nodes will be that they are transposes, with the associated equations being

$$w_j^{(k)} = d_j^{(k)} \sum_{i=1}^{M^{(k)}} c_i^{(k)} v_i^{(k)}, \quad j = 1, \ldots, N^{(k)} \tag{6.7}$$

and

$$w_i'^{(k)} = c_i^{(k)} \sum_{j=1}^{N^{(k)}} d_j^{(k)} v_j'^{(k)}, \quad i = 1, \ldots, M^{(k)}. \tag{6.8}$$

In Eqns. 6.7-6.8, $M^{(k)}$ denotes the number of inputs to a specific node k in the first network, which equals the number of outputs from node k in the second, and $N^{(k)}$ denotes the number of outputs from a specific node k in the first network, which equals number of inputs to node k in the second. The variables $w_j^{(k)}$ and $w_i'^{(k)}$ represent node outputs and the variables $v_i^{(k)}$ and $v_j'^{(k)}$ represent node inputs in the first and second networks, respectively. In the first network, a given generalized node k has input gains specified by $c_i^{(k)}$ and output gains specified by $d_j^{(k)}$, and in the second network the roles of these variables are exchanged. Fig. 6-5 depicts the relationships between these variables for a pair of nodes in the two networks.

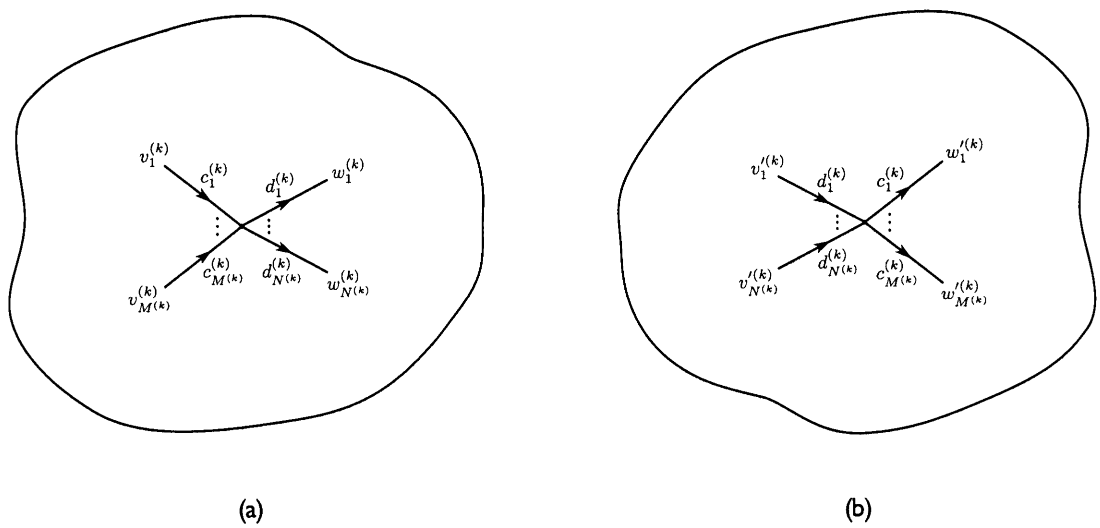


Figure 6-5: A pair of nodes in the first (a) and second (b) networks as respectively described by Eqns. 6.7 and 6.8.

Theorem 6.1 (Generalized Tellegen's theorem for signal-flow graphs). *Consider a pair of topologically-complementary signal-flow graphs described by Eqns. 6.7-6.8 and depicted in Fig. 6-5, and that contain a total of P generalized nodes in each. For any set of values taken on by the node inputs and node outputs in the two networks,*

$$\sum_{k=1}^{P}\left(\sum_{i=1}^{M^{(k)}} w_i'^{(k)} v_i^{(k)} - \sum_{j=1}^{N^{(k)}} w_j^{(k)} v_j'^{(k)}\right) = 0. \tag{6.9}$$

Proof. We proceed by demonstrating that the innermost expression in Eq. 6.9 evaluates to zero, in the sense that

$$\sum_{i=1}^{M^{(k)}} w_i'^{(k)} v_i^{(k)} - \sum_{j=1}^{N^{(k)}} w_j^{(k)} v_j'^{(k)} = 0, \quad k = 1, \ldots, P. \tag{6.10}$$

We do this by defining an OVS that corresponds to the specific pair of nodes k pertinent to Eq. 6.10 and that is conservative over the behavior of the nodes, as specified in Eqns. 6.7-6.8.

The OVS is defined as $\mathfrak{U} = \left(\mathbb{R}^{2M^{(k)}+2N^{(k)}}, \langle.,.\rangle, \mathcal{O}\right)$, with $\langle.,.\rangle$ denoting the standard inner product on $\mathbb{R}^{2M^{(k)}+2N^{(k)}}$. The interpretation is that the node inputs and node outputs are coefficients in a basis expansion of a vector in $\mathbb{R}^{2M^{(k)}+2N^{(k)}}$, in the sense that

$$\left[v_1^{(k)},\ldots,v_{M^{(k)}}^{(k)},w_1^{(k)},\ldots,w_{N^{(k)}}^{(k)},w_1^{\prime(k)},\ldots,w_{M^{(k)}}^{\prime(k)},v_1^{\prime(k)},\ldots,v_{N^{(k)}}^{\prime(k)}\right]^{tr} \in \mathbb{R}^{2M^{(k)}+2N^{(k)}}. \tag{6.11}$$

In specifying the elements of the organization $\mathcal{O} = (C, \mathcal{D}_p, \mathcal{D}_c)$, we define $K^{(k)} = M^{(k)} + N^{(k)}$ and write

$$\begin{aligned}
C &= \frac{1}{2}\begin{bmatrix} I_{K^{(k)}} & 0_{K^{(k)}} \\ 0_{K^{(k)}} & I_{K^{(k)}} \end{bmatrix} && (6.12)\\
\mathcal{D}_p &= \left\{\mathrm{span}\left(\mathbf{e}^{(1)},\mathbf{e}^{\left(K^{(k)}\right)}\right),\ldots,\mathrm{span}\left(\mathbf{e}^{\left(K^{(k)}+1\right)},\mathbf{e}^{\left(2K^{(k)}\right)}\right)\right\} && (6.13)\\
\mathcal{D}_c &= \left\{\mathrm{span}\left(\mathbf{e}^{(1)},\ldots,\mathbf{e}^{\left(K^{(k)}\right)}\right),\mathrm{span}\left(\mathbf{e}^{\left(K^{(k)}+1\right)},\ldots,\mathbf{e}^{\left(2K^{(k)}\right)}\right)\right\}. && (6.14)
\end{aligned}$$

The associated quadratic form $Q(\mathbf{x}) = \langle C\mathbf{x}, \mathbf{x}\rangle$ is written as

$$Q(\mathbf{x}) = \sum_{i=1}^{M^{(k)}} w_i^{\prime(k)} v_i^{(k)} - \sum_{j=1}^{N^{(k)}} w_j^{(k)} v_j^{\prime(k)}. \tag{6.15}$$

Substituting in Eqns. 6.5-6.6, we obtain

$$\begin{aligned}
Q(\mathbf{x}) &= \sum_{i=1}^{M^{(k)}}\sum_{j=1}^{N^{(k)}} c_i^{(k)} d_j^{(k)} v_j^{\prime(k)} v_i^{(k)} - \sum_{j=1}^{N^{(k)}}\sum_{i=1}^{M^{(k)}} d_j^{(k)} c_i^{(k)} v_i^{(k)} v_j^{\prime(k)}\\
&= 0, && (6.16)
\end{aligned}$$

i.e. $\mathfrak{U}$ is conservative over the behavior of a pair of nodes in the networks. Eq. 6.10 in turn holds, as does Eq. 6.9.

We can alternatively prove this theorem using the condition for conservation in

Chapter 4. Specifically, negating the inputs to each generalized node in Fig. 6-5(b), we observe that the gain matrix for any such node is the negative transpose of the gain matrix for the corresponding node in Fig. 6-5(a), and the two nodes form a conservative interconnection in a canonical conjugate basis with the $v_j'^{(k)}$ being negated, i.e. Eq. 6.10 holds. $\square$

As is depicted in Fig. 6-6, Tellegen's theorem for signal flow graphs can be seen as a special case of Thm. 6.1 by setting $N^{(k)} = M^{(k)}$ and $c_j^{(k)} = d_i^{(k)} = 1$ in Eqns. 6.7-6.8, and by bringing out one of the node outputs in each node pair to obtain a corresponding pair of conventional signal-flow graph nodes.

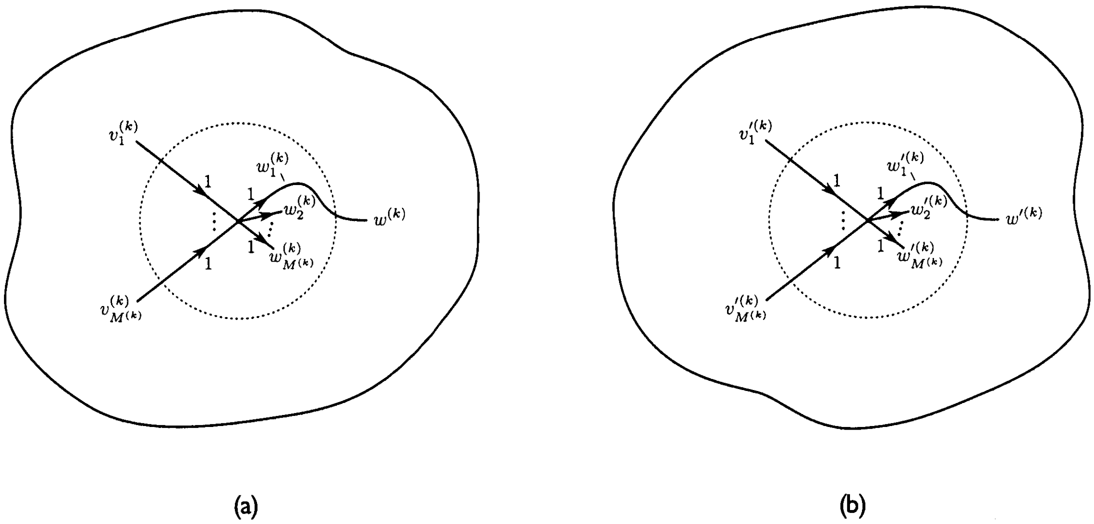


Figure 6-6: (a) First and (b) second network pertaining to Tellegen's theorem for signal-flow graphs, as a special case of Thm. 6.1.

6.3 The set of lossless wave-digital building blocks

This example uses the results in Chapter 4 regarding the creation of conservative, linear interconnections, in combination with the results in Chapter 3 pertaining to canonical conjugate bases and the results in Chapter 2 pertaining to linear transforming flow-graphs, to prescribe a method for creating an arbitrary element from the set of lossless wave-digital building blocks. The conservation principle for the wave-digital class of structures, which are known to have exceptional stability and robustness properties, was stated in [17]. Using the notation in [17], it was written formally as

$$\sum_{k=1}^{K} p_k = 0, \tag{6.17}$$

with p_k denoting the so-called instantaneous pseudopower absorbed by a particular wave-digital building block. For an n-port building block, the absorbed pseudopower was specifically written as

$$p = \sum_{v=1}^{n} (a_v^2 - b_v^2) g_v, \tag{6.18}$$

with g_v denoting the admittance at a particular port v. In designing the lossless portion of wave-digital structures, the frequently-used strategy has been to refer to a table of commonly-used of building blocks, as in [18]. Two such blocks were discussed in the example in Subsection 3.2.3.

A problem with designing wave-digital structures using a limited collection of lossless building blocks is the inherent possibility of overlooking a wide range of interconnection behaviors and associated filter topologies. We address this by using the techniques in Chapter 4 to formulate a technique for obtaining an arbitrary linear wave-digital building block that is lossless for a pre-specified set of port admittances.

The strategy, which is depicted in Figs. 6-7 and 6-8, follows in a straightforward way from the condition for strong conservation in Chapter 4. In particular, the general approach is to begin with a conjugate-separable pair of linear interconnections where the gain matrix for one is arbitrary and the gain matrix for the other is its negative transpose. This pair of interconnections is strongly conservative in a canonical conjugate basis, and by applying appropriate two-input, two-output transforming systems having branch gains that are specified in terms of the desired port admittances, an interconnecting system having the desired behavior can be obtained. The specific transforming systems corresponding to a desired port impedance g_k are depicted in Fig. 6-7(a) for the two two possible input-output configurations that may be encountered. These structures, which were obtained from the partial taxonomy of

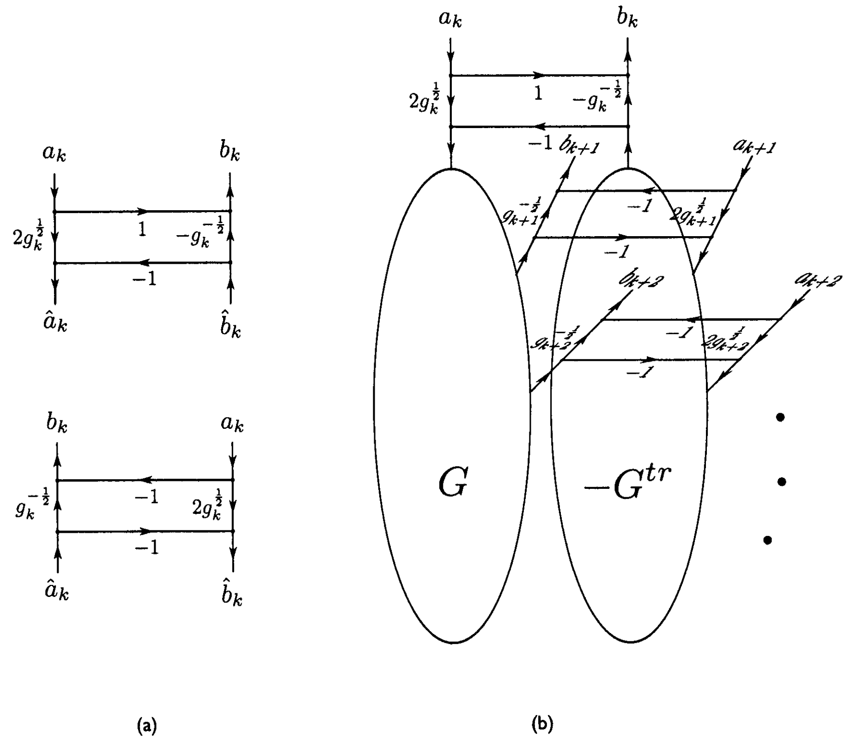


Figure 6-7: (a) Systems for transforming a conservation law in a canonical conjugate basis to a wave-digital conservation law. (b) Transforming systems coupled to a strongly-conservative interconnection.

transforming systems in Fig. 2-9, are joined to the terminal variables of the conjugate-separable graphs as depicted in Fig. 6-7(b).

An issue in using the approach in Fig. 6-7 is that the technique will generally result in delay-free loops, but as a consequence of the relationship between conservation and the gain matrices for conjugate signal-flow graph interconnections, the loop can always be factored out. In particular, Fig. 6-8 illustrates a method for eliminating these. Referring to this figure, the process involves replacing the delay-free loop with a multiple-input, multiple-output system that is computable. This is done by writing an equation relating the variables in the loop and solving for the variables that are outputs. In particular, the structure of the transforming systems in Fig. 6-7(a) and the relationship between the internal gains in the conjugate-separable interconnecting structures implies that any such loop can be replaced with a four-input, four-output linear system implementing

$$\begin{bmatrix} g^{(A)} \\ h^{(A)} \\ g^{(B)} \\ h^{(B)} \end{bmatrix} = \frac{1}{1+(G_{k,\ell})^2} \begin{bmatrix} -G_{k,\ell} & 1 & -1 & G_{k,\ell} \\ 1 & G_{k,\ell} & -G_{k,\ell} & (G_{k,\ell})^2 \\ -1 & -G_{k,\ell} & G_{k,\ell} & 1 \\ G_{k,\ell} & (G_{k,\ell})^2 & 1 & -G_{k,\ell} \end{bmatrix} \begin{bmatrix} i^{(A)} \\ f^{(A)} \\ i^{(B)} \\ f^{(B)} \end{bmatrix}, \tag{6.19}$$

with the naming convention for the variables being as depicted in Fig. 6-8(a). It is furthermore noted that this technique will not introduce additional delay free loops, as its only affect on G and $-G^{tr}$ is to set an entry in each to zero.

It is an illustrative exercise to begin with an interconnection implementing

$$\hat{a}_1 = \hat{a}_2 \tag{6.20}$$

$$\hat{b}_1 = -\hat{b}_2, \tag{6.21}$$

which is strongly conservative under the interpretation that the variables $\hat{a}_k$ and $\hat{b}_k$ represent coefficients of vectors in conjugate subspaces for an OVS, and then to apply the technique in this section to obtain a two-input, two-output wave-digital interconnection having arbitrary port impedances.

6.4 Linearly-constrained p-norm minimization

This example illustrates the use of the techniques in Chapters 4-5 in writing an algorithm for linearly-constrained, convex$p$-norm minimization. We are specifically interested in solving

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^K} \quad & \frac{1}{p}||\mathbf{x}||_p^p \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}, \end{aligned} \tag{6.22}$$

with A being an $M \times K$ matrix, and with $p > 1$.

In doing this, we define functions $\mathbf{f} : \mathbb{R}^{K+M} \to \mathbb{R}^{K+M}$ and $\mathbf{g} : \mathbb{R}^{K+M} \to \mathbb{R}^{K+M}$,

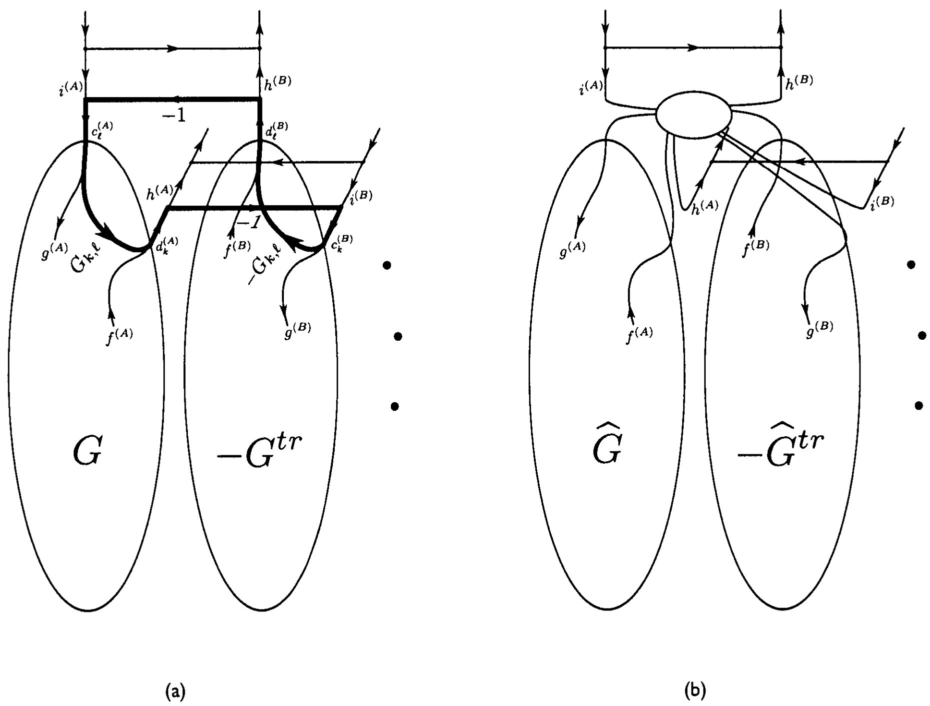


Figure 6-8: (a) Identification of a delay-free loop in the system in Fig. 6-7(b). (b) Replacement of the delay free loop with a computable subsystem.

and use the relationships in Fig. 5-10 and Thm. 5.1 to write the desired conditions for content and co-content stationarity:

$$
\begin{aligned}
\mathbf{f}_k(\mathbf{y}) &= \begin{cases} |\mathbf{y}_k|^{\frac{1}{p-1}}, & \mathbf{y}_k \geq 0 \\ -|\mathbf{y}_k|^{\frac{1}{p-1}}, & \mathbf{y}_k < 0 \end{cases}, \quad k = 1, \ldots, K && (6.23) \\
\mathbf{f}_{k+K}(\mathbf{y}) &= \mathbf{b}_k, \quad k = 1, \ldots, M && (6.24) \\
\mathbf{g}(\mathbf{y}) &= \mathbf{y} && (6.25) \\
V_A &= \text{range}\left(\begin{bmatrix} I_K \\ A \end{bmatrix} \right) \subseteq \mathbb{R}^{K+M} && (6.26) \\
V_B &= V_A^{\perp} \subseteq \mathbb{R}^{K+M} && (6.27) \\
\mathbf{f}(\mathbf{y}) &\in V_A && (6.28) \\
\mathbf{g}(\mathbf{y}) &\in V_B. && (6.29)
\end{aligned}
$$

A system depicting Eqns. 6.23-6.29 is illustrated in Fig. 6-9(a). Referring to this figure, the conservative interconnecting systems have been created using the technique in Chapter 4 for creating strongly-conservative systems. Making the substitution in Fig. 5-12(c)-(d), we obtain a dynamic system whose co-content decreases with respect to time until it reaches a minimum. As the co-content corresponds to the negative of the dual cost and we are dealing with a convex optimization problem, this primal cost, i.e. content, will be minimized by this system as well.

Illustrating the dynamics of this system with an example, we select

$$
A = \begin{bmatrix} -1.6 & -0.8 & 2 & -0.8 & -0.4 \\ 1.3 & -0.1 & -1 & 1 & 0.5 \end{bmatrix} \tag{6.30}
$$

and

$$
\mathbf{b} = \begin{bmatrix} 1.5 \\ 1.4 \end{bmatrix}. \tag{6.31}
$$

Fig. 6-10 depicts coefficient trajectories for various values of p that were obtained by performing a discrete-time simulation of the system in Fig. 6-9(b), with A and $\mathbf{x}$ being defined as in Eqns. 6.30-6.31. In computing the discrete-time simulation, the

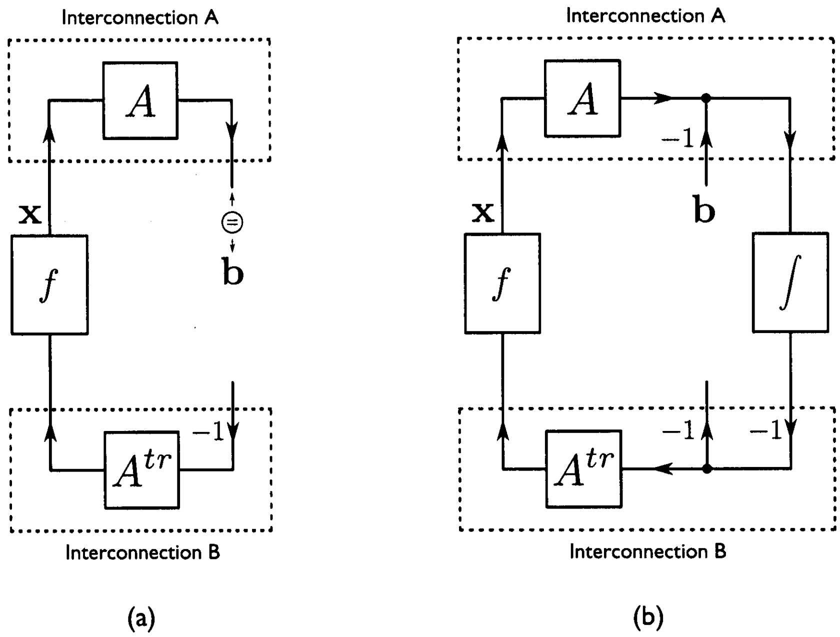


Figure 6-9: (a) Desired conservative system. (b) Co-content-minimizing system that becomes (a) in steady state.

approach was to fix the constant of integration and finely discretize the time axis, subsequently approximating the integrator with a first-order accumulator followed by a single-sample delay to avoid delay-free loops. The net result was a system where the integrator was replaced with a causal system of the form

$$H(z) = \varepsilon \frac{z^{-1}}{1 - z^{-1}}, \tag{6.32}$$

with increasingly fine time discretization corresponding to smaller values of ε. For the plots in Fig. 6-10, $\varepsilon = 0.005$ was chosen.

6.5 A distributed system for vehicle density control

In this example, the line of reasoning that was used in the example in Section 6.4 is applied to the problem of designing a system for controlling a chain of N vehicles

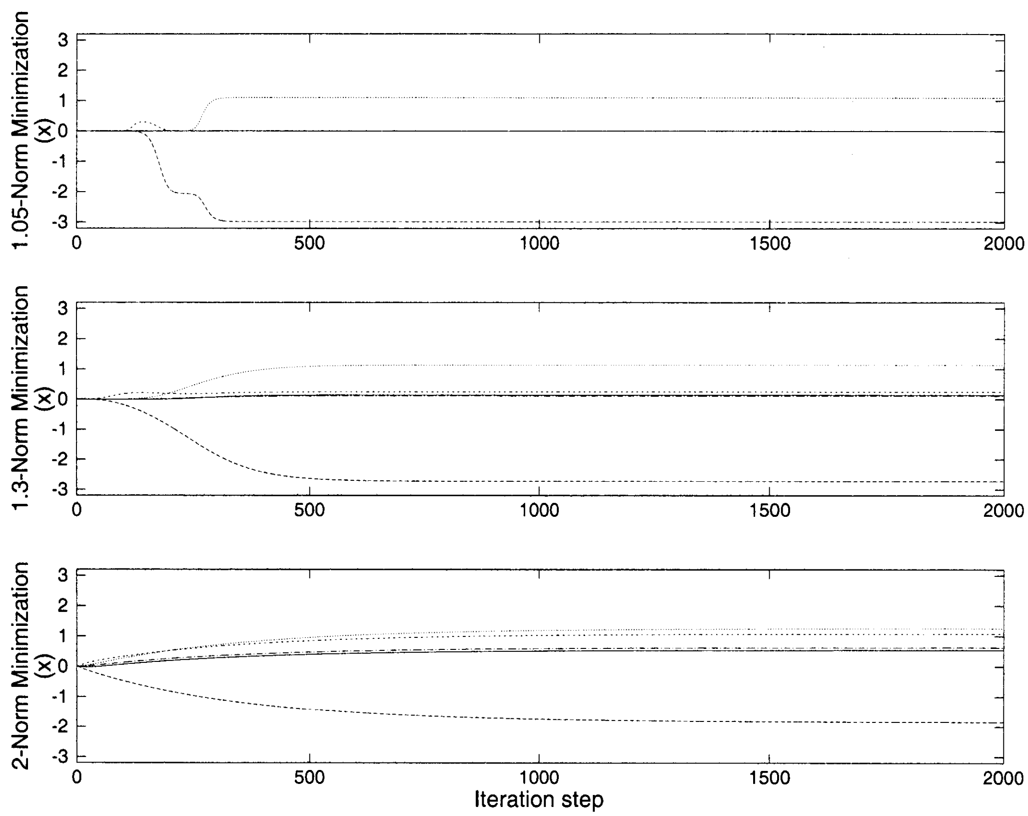


Figure 6-10: Trajectories of $\mathbf{x}$ as minimum-$||\mathbf{x}||_p$ solution is computed. Top panel: $p = 1.05$, middle panel: $p = 1.3$, bottom panel: $p = 2$.

following behind a leading vehicle. The key goal is for each to maintain a target following distance, pre-selected to be both safe and to result in a desired traffic density, and for any perturbations in the vehicle positions to be dealt with gracefully while avoiding collisions with others in the chain. If a chain of vehicles is assembled where each operates under a car-following based system for adaptive cruise control, growing oscillatory behavior usually results. [20] The goal here is to create a distributed system that behaves as desired while avoiding these kind of oscillations.

This example continues the theme of discussion in Subsection 4.4.4 about identifying a conservation principle in a distributed vehicle control system. However, the sequence of development in this section is to begin with a problem statement and, using the framework in this thesis, arrive at a conservative distributed system that offers a solution that is complementary to the approach in [20].

With a translational offset removed, the positions of the vehicles in the chain will be denoted $x_0, \ldots, x_{N-1}$, and the distance between the $(k-1)$st and kth vehicles will be denoted d_k. The convention will be that the position of the leading vehicle is fixed to $x_0 = 0$, and that the position axis moving backwards away from the leading vehicle, through the chain of vehicles that are hopefully following it, will be increasing. A negative relative velocity of a specific vehicle will accordingly bring it closer to the lead, and a positive relative velocity will take it further from the lead.

The overall system will be designed to minimize a sum of distance penalties $Q(d_k)$, where the penalty function Q is selected to take a minimum value at the target following distance. The minimization involved in doing this is written formally as

$$\begin{aligned} \min_{x_0,\ldots,x_9,d_1,\ldots,d_9} \quad & \sum_{k=1}^{N-1} Q(d_k) \\ \text{s.t.} \quad & x_0 = 0 && (6.33) \\ & d_k = x_k - x_{k-1}, \quad k = 1, \ldots, N-1. && (6.34) \end{aligned}$$

Using the general approach in Section 6.4, a content-minimizing system can be designed for finding a local minimum of the optimization problem formulated in (6.33). The resulting system for $N = 10$ is depicted in Fig. 6-11. Referring to this figure,

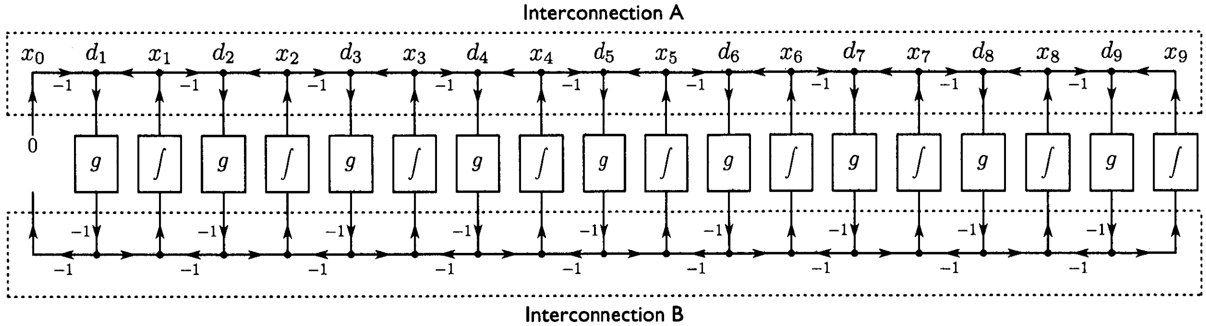


Figure 6-11: Interconnected system of vehicles designed to minimize the distance penalty in (6.33), for $N = 10$. The variables available to a particular vehicle k are d_k, x_k and d_{k+1} for $k = 1, \ldots, 8$, and d_k and x_k for $k = 9$.

Interconnection A was obtained by writing a signal-flow graph for implementing the linear equality constraints in (6.33), Interconnection B was obtained by taking its negative transpose to create a strongly-conservative interconnection, and the subsystems implementing the memoryless, nonlinear function g were selected to have Q as their individual contents by using

$$g(x) = \frac{dQ(x)}{dx}. \tag{6.35}$$

Integrators were appropriately interconnected to result in a total content function whose contribution from the memoryless elements decreases with time until reaching a local minimum, serving also as a Lyapunov function for the system.

Still referring to Fig. 6-11, the signal-flow graph representations of Interconnections A and B indicate that the system is well-suited to a distributed implementation. In particular, the variables used in controlling the position of a particular vehicle k are those that are available locally: d_k and d_{k+1} for $k = 1, \ldots, 8$, and d_k for $k = 9$. This suggests an implementation that is obtained by implementing a straightforward algorithm in each vehicle k for controlling the setpoint of its cruise control system:

(1) Measure the distance d_k to the leading vehicle.

(2) If a trailing vehicle exists, measure the distance d_{k+1} to the trailing vehicle. Otherwise set $d_{k+1} = 0$.

(3) Compute the setpoint v_k for the cruise control system in vehicle k according to

$$v_k = \frac{dx_k}{dt} = -(g(d_{k+1}) - g(d_k)) \tag{6.36}$$

and repeat.

An interactive simulation of the system in Fig. 6-11 was written in the language Processing. As with the example in Section 6.4, the integrators were approximated using discrete-time systems as

$$H(z) = \sigma \frac{z^{-1}}{1 - z^{-1}}, \tag{6.37}$$

with $\sigma = 0.1$ for this example, and with one sample period of the simulation being computed every video frame. In the simulation, the user has the option to choose from three penalty functions: a symmetric penalty function, with g and Q respectively being

$$g(d) = g_s(d) = 3(d-1) \tag{6.38}$$

$$Q(d) = Q_s(d) = \frac{3}{2}(d-1)^2, \tag{6.39}$$

an asymmetric penalty function, with g and Q respectively being

$$g(d) = g_a(d) = \begin{cases} \frac{1}{10}(d-1), & d > 1 \\ 4(d-1), & d \leq 1 \end{cases} \tag{6.40}$$

$$Q(d) = Q_a(d) = \begin{cases} \frac{1}{20}(d-1)^2, & d > 1 \\ 2(d-1)^2, & d \leq 1 \end{cases}, \tag{6.41}$$

and a nonconvex penalty function, with g and Q respectively being

$$g(d) \;=\; g_n(d) \;=\; \begin{cases} 3(d-1), & d > \frac{7}{8} \\ -3\left(d-\frac{3}{4}\right), & \frac{5}{8} < d \le \frac{7}{8} \\ 3\left(d-\frac{1}{2}\right), & d \le \frac{5}{8} \end{cases} \tag{6.42}$$

$$Q(d) \;=\; Q_n(d) \;=\; \begin{cases} \frac{3}{2}(d-1)^2, & d > \frac{7}{8} \\ -\frac{3}{2}\left(d-\frac{3}{4}\right)^2, & \frac{5}{8} < d \le \frac{7}{8} \\ \frac{3}{2}\left(d-\frac{1}{2}\right)^2, & d \le \frac{5}{8} \end{cases} . \tag{6.43}$$

Screen captures from the simulation are depicted in Fig. 6-12. The horizontal line of orange rectangles represents the positions of vehicles in the chain, and clicking on any one of the blue sliders perturbs a vehicle position by overriding the output value of the associated integrator. Subsystems in the inset system diagram indicate the instantaneous values of the associated products in the conservation law by glowing red if the product is positive and blue if the product is negative. As a consequence of conservation, the presence of a red glowing block implies that there must also be a blue glowing block, and vice-versa.

Two typical steady-state configurations are depicted in Fig. 6-12(a) and Fig. 6-12(b), which respectively are representative of the two convex and one nonconvex penalty functions. Figs. 6-12(c)-(e) depict the system approximately 30 frames after discontinuously setting $x_5 = -1$ while using each of the three penalty functions, with the initial state for each being as depicted in Fig. 6-12(a). In Fig. 6-12(c), the use of the symmetric penalty function Q_s results in collision avoidance with the trailing vehicles quickly catching up. In Fig. 6-12(d), the use of the asymmetric penalty function Q_a results in collision avoidance with the trailing vehicles more gradually reaching the target following distance. In Fig. 6-12(e), the nonconvex penalty function causes some of the vehicles to follow more closely than others, corresponding to the two local minima of Q_n. These examples suggest potential in further exploring the use of other penalty functions as well.

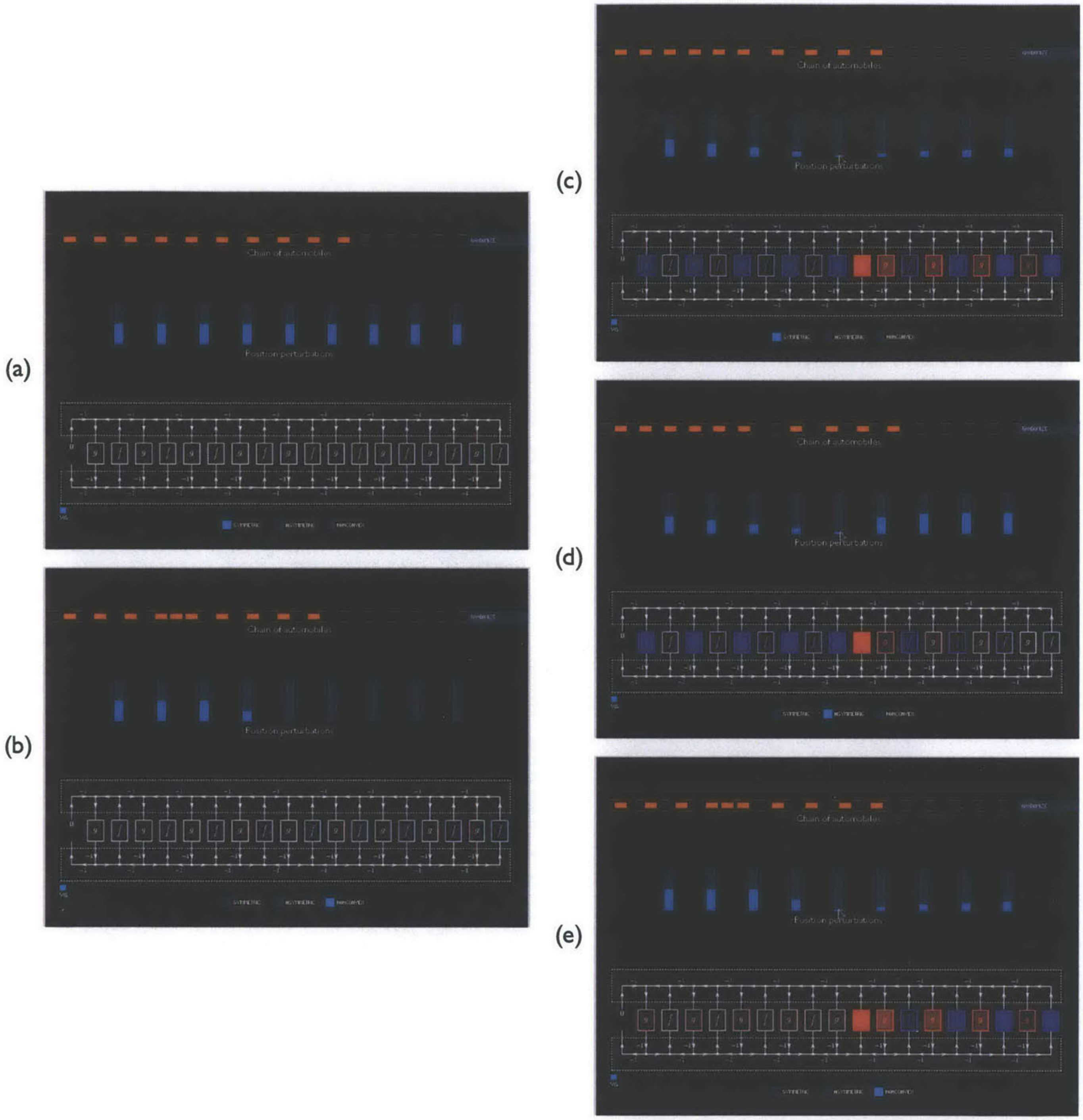


Figure 6-12: Screen captures from a simulation of the vehicle control system in Fig. 6-11. (a) Steady-state configuration typical of using the symmetric and asymmetric penalty functions Q_s and Q_a respectively listed in Eqns. 6.39 and 6.41. (b) Steady-state configuration typical of using the nonconvex penalty function Q_n in Eq. 6.43. (c) Configuration approximately 30 frames after discontinuously setting $x_5 = -1$ when using the symmetric penalty function Q_s. (d) Configuration approximately 30 frames after discontinuously setting $x_5 = -1$ when using the asymmetric penalty function Q_a. (e) Configuration approximately 30 frames after discontinuously setting $x_5 = -1$ when using the nonconvex penalty function Q_n. Subsystems in the inset system diagrams are colored red if the associated product in the conservation law is positive at that time, and blue if the associated product is negative at that time.

Appendix A

Proof of Thm. 3.1

We begin by writing the identity

$$\begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I_L & I_L \\ I_L & -I_L \end{bmatrix} \begin{bmatrix} I_L & 0_L \\ 0_L & -I_L \end{bmatrix} \begin{bmatrix} I_L & I_L \\ I_L & -I_L \end{bmatrix} \tag{A.1}$$

and observing that this implies that $\mathcal{G}_Q^{(\mathcal{B})}$ is isomorphic to the so-called indefinite orthogonal group, denoted $O(L, L)$. The group $O(L, L)$ is specifically a Lie group that consists of the set of invertible $2L \times 2L$ matrices T for which

$$T^{tr} \begin{bmatrix} I_L & 0_L \\ 0_L & -I_L \end{bmatrix} T = \begin{bmatrix} I_L & 0_L \\ 0_L & -I_L \end{bmatrix}. \tag{A.2}$$

The group $O(L, L)$ is known to have four connected components, and as such $\mathcal{G}_Q^{(\mathcal{B})}$ has four connected components as well. Referring to $O(L, L)$, the component in which a given matrix lies indicates which of the subspaces

$$V_+ = \text{span}\left(\mathbf{e}^{(1)}, \ldots \mathbf{e}^{(L)}\right) \tag{A.3}$$

and

$$V_- = \text{span}\left(\mathbf{e}^{(L+1)}, \ldots \mathbf{e}^{(2L)}\right) \tag{A.4}$$

has its orientation reversed by the matrix.

The two components where neither or both orientations are reversed, i.e. the two components whose elements have a positive determinant, is commonly referred to as the special orthogonal group $SO(L, L) \subset O(L, L)$. The subgroup of $SO(L, L)$ that consists of a single connected component containing transformations that reverse neither orientation is commonly denoted $SO^{+}(L, L)$. The set of one-parameter continuous subgroups for generating $SO^{+}(L, L)$ can be obtained in the usual way using the exponential map. [23]

The approach in proving that the families of transformations in Thm. 3.1 generate $\mathcal{G}_Q^{(\mathcal{B})}$ is thus to show that $T_1^{[q;t)}$, $T_2^{[q,r;t)}$, $T_3^{[q,r;t)}$, and $T_4^{[q,r;t)}$ are those that are obtained from the Lie algebra for $\mathcal{G}_Q^{(\mathcal{B})}$ and accordingly generate the connected component of $\mathcal{G}_Q^{(\mathcal{B})}$ that contains the identity element. The final step then involves showing that an arbitrary element of $T_5^{[q]}$ and an arbitrary element of $T_6^{[q]}$ can be used to move between the four components of $\mathcal{G}_Q^{(\mathcal{B})}$, generating the entire group.

In obtaining the one parameter subgroups for generating the connected component containing the identity element, we make the substitution

$$T = e^{Ut}, \tag{A.5}$$

where U is a $2L \times 2L$ matrix and t is a real parameter, and we write Eq. 3.127 as

$$e^{U^{tr}t} \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} e^{Ut} = \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix}, \tag{A.6}$$

which by performing left and right matrix multiplications results in

$$\begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} e^{U^{tr}t} \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} = e^{-Ut}. \tag{A.7}$$

Using the identity $Y^{-1}e^{U}Y = e^{Y^{-1}UY}$, we obtain

$$e^{\left(\begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} U^{tr}t \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix}\right)} = e^{-Ut}. \tag{A.8}$$

Writing the matrix U in terms of four $L \times L$ matrices E, F, G, and H as

$$U = \begin{bmatrix} E & F \\ G & H \end{bmatrix}, \tag{A.9}$$

the condition required of U for satisfying Eq. 3.127 is

$$\begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix}^{tr} \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} = - \begin{bmatrix} E & F \\ G & H \end{bmatrix}. \tag{A.10}$$

Performing further simplifications, we obtain

$$\begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} \begin{bmatrix} E^{tr} & G^{tr} \\ F^{tr} & H^{tr} \end{bmatrix} \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix} = - \begin{bmatrix} E & F \\ G & H \end{bmatrix}, \tag{A.11}$$

resulting in

$$\begin{bmatrix} H^{tr} & F^{tr} \\ G^{tr} & E^{tr} \end{bmatrix} = - \begin{bmatrix} E & F \\ G & H \end{bmatrix}. \tag{A.12}$$

The set of matrices U satisfying the condition on the sub-matrices in Eq. A.12 therefore form a vector space of dimension $(2L^2 - L)$, which is also the dimension of the associated Lie algebra.

We will use $D^{[r,q]}$ to denote the $L \times L$ matrix containing all zeros, with the exception of the entry in row r and column q, which has value 1. Then it is a straightforward matter to verify that the family $T_1^{[q;t)}$ is generated by substituting

$$U = \begin{bmatrix} D^{[q,q]} & 0_L \\ 0_L & -D^{[q,q]} \end{bmatrix} \tag{A.13}$$

into Eq. A.5; the family $T_2^{[q,r;t)}$ is generated by substituting

$$U = \begin{bmatrix} D^{[r,q]} & 0_L \\ 0_L & -D^{[q,r]} \end{bmatrix} \tag{A.14}$$

into Eq. A.5; the family $T_3^{[q,r;t)}$ is generated by substituting

$$U = \begin{bmatrix} 0_L & 0_L \\ D^{[r,q]} - D^{[q,r]} & 0_L \end{bmatrix} \tag{A.15}$$

into Eq. A.5; and the family $T_4^{[q,r;t)}$ is generated by substituting

$$U = \begin{bmatrix} 0_L & D^{[r,q]} - D^{[q,r]} \\ 0_L & 0_L \end{bmatrix} \tag{A.16}$$

into Eq. A.5. The matrix U in each substitution satisfies Eq. A.12, and as there are a total of $2L^2 - L$ such distinct matrices, they can be used to generate the connected component of $\mathcal{G}_Q^{(\mathcal{B})}$ that contains the identity element.

It remains to be shown that an arbitrary element of $T_5^{[q]}$ and an arbitrary element of $T_6^{[q]}$ can be used to move between the four components of $\mathcal{G}_Q^{(\mathcal{B})}$, generating the entire group. In doing this, we perform similarity transformation to relate these to the transformations that move between the connected components of $O(L, L)$. In particular, we write $T_5^{[q]}$ as

$$\begin{aligned} T_5^{[q]} &= \begin{bmatrix} I_L - 2D^{[q,q]} & 0_L \\ 0_L & I_L - 2D^{[q,q]} \end{bmatrix} && \text{(A.17)} \\ &= \frac{1}{2} \begin{bmatrix} I_L & I_L \\ I_L & -I_L \end{bmatrix} \begin{bmatrix} I_L - 2D^{[q,q]} & 0_L \\ 0_L & I_L - 2D^{[q,q]} \end{bmatrix} \begin{bmatrix} I_L & I_L \\ I_L & -I_L \end{bmatrix}, && \text{(A.18)} \end{aligned}$$

and we write $T_6^{[q]}$ as

$$\begin{aligned} T_6^{[q]} &= \begin{bmatrix} I_L - D^{[q,q]} & D^{[q,q]} \\ D^{[q,q]} & I_L - D^{[q,q]} \end{bmatrix} && \text{(A.19)} \\ &= \frac{1}{2} \begin{bmatrix} I_L & I_L \\ I_L & -I_L \end{bmatrix} \begin{bmatrix} I_L & 0_L \\ 0_L & I_L - 2D^{[q,q]} \end{bmatrix} \begin{bmatrix} I_L & I_L \\ I_L & -I_L \end{bmatrix}. && \text{(A.20)} \end{aligned}$$

From the form of the middle matrix in Eq. A.18, we conclude that $T_5^{[q]}$ maps to a transformation in $O(L,L)$ that reverses the orientation of both V_+ and V_-, and the form of the middle matrix in Eq. A.20 likewise tells us that $T_6^{[q]}$ maps to a transformation in $O(L,L)$ that reverses the orientation of V_- but not V_+. Combinations of these can therefore be used to move between the four connected components of $O(L,L)$, and as such, combinations of $T_5^{[q]}$ and $T_6^{[q]}$ can be used to move between the four connected components of $\mathcal{G}_Q^{(\mathcal{B})}$, completing the proof.

Appendix B

Glossary of terms

Balanced quadratic form: A quadratic form whose associated matrix is invertible, with the number of positive and negative eigenvalues being equal.

Comparison space: A vector space on which the quadratic form associated with an OVS acts as an inner product.

Canonical conjugate basis: A basis in which the correspondence map for a $2L$-dimensional OVS is written as

$$C = \begin{bmatrix} 0_L & I_L \\ I_L & 0_L \end{bmatrix}.$$

In a canonical conjugate basis, the associated quadratic form written $\mathbf{x}^{tr} C \mathbf{x}$ takes the form of the standard inner product on $\mathbb{R}^L$.

Conjugate decomposition $\mathcal{D}_c$: A direct-sum decomposition of the vector space $V = V_A \oplus V_B$ used in defining an OVS that designates the components of vectors that map to a comparison space.

Conservative set: A set of vectors over which an OVS is conservative, i.e. for which the associated quadratic form evaluates to zero.

Correspondence map C: A linear, invertible, self-adjoint map indicating a correspondence between elements of the vector space used in defining an OVS. The associated quadratic form is written in terms of a correspondence map as $Q(x) = \langle Cx, x \rangle$.

Input-output matched interconnecting system: An interconnecting system where for a given pair of conjugate variables, exactly one is an input and one is an output.

Maximal-$\mathcal{D}_p$ decomposition: A partition decomposition having the maximum number of elements permitted by the structure of the OVS. For an OVS defined using a $2L$-dimensional vector space, a maximal-$\mathcal{D}_p$ decomposition will have a total of L subspaces, with each being a 2-dimensional subspace.

Organization $\mathcal{O}$: A correspondence map, partition decomposition, and conjugate decomposition used in defining an OVS.

Organized variable space (OVS) $\mathfrak{U}$: An inner product space in addition to an organization of the space.

Partition decomposition $\mathcal{D}_p$: A direct-sum decomposition of the vector space $V = V_1 \oplus \cdots \oplus V_K$ used in defining an OVS that indicates the subspaces over which the associated quadratic form is linearly-separable.

Strongly-conservative set: A conservative set that is a vector subspace, and that results in conservation being viewed in a comparison space as orthogonality between vector subspaces.

Weakly-conservative set: A conservative set that is a vector subspace and that is not strongly conservative. Weak conservation is viewed in a comparison subspace as pairwise orthogonality, as opposed to orthogonality between vector subspaces.

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Epilogue

Over the course of the graduate program here at MIT, I had the good fortune of hearing Amar Bose give various lectures at Bose Corporation regarding his view on research. A common point in some of these talks was that the logical progression of a technical document can sometimes be very different from the chronological order in which the contributing research might have been done. As this thesis was supervised in Al Oppenheim's unique and characteristic style of an "intellectual adventure," I believe that Amar Bose's comment is especially relevant. The intent of this epilogue is to offer a glimpse into the way in which the thesis evolved chronologically, with the goal being to give the reader a view of the document from a somewhat different perspective that is complementary to the written sequence.

In embarking on the research for this thesis, the intellectual adventure began with a simple question from Al: "What can thermodynamics inspire about signal processing?" My initial approach in attempting to answer this mostly involved trying to muster as much creativity as possible in thinking about what existing signal processing systems might implicitly be behaving according to the laws of thermodynamics. I made a decision early on to steer away from the commonly-discussed connections with information entropy by opting to think about deterministic systems.

Also around this time I was thinking about dual circuits, bond graphs, and the relationship between positive-real and minimum-phase systems. There was no deep connection that I was trying to find between these other than that they all had convenient properties and happened to be physical systems. It seemed intriguing that the straightforward mechanical process of obtaining a dual circuit resulted in a network with an inverted impedance, and I thought that perhaps the technique could be used to invert certain linear or nonlinear signal-flow graphs that happened to simulate circuits. (The eventual results in Chapter 2 regarding the inversion of nonlinear signal-flow graphs ended up having little to do with electrical networks.)

It also seemed remarkable, at this point before having really dived into Jan Willems' work in dissipative system theory, that an interconnection of stable (positive-real) electrical elements resulted in an overall stable system. I remember wondering why the signal-flow graphs that can be drawn for simulating electrical networks seem very different from the wave-digital structures, even though those structures also have similar properties.

About a year after having begun the Ph.D. research, I was sitting in on a class that John Wyatt was teaching, pertaining to functional analysis and linear algebra for signal processing. In one the lectures, he presented a proof of Tellegen's Theorem from a linear algebra perspective, viewing it as a statement of orthogonality between vector subspaces. Al had introduced me to Tellegen's Theorem for electrical networks as I was working on my master's thesis, since in that work I had made use of the Tellegen-like theorem that exists for signal-flow graphs. Viewing energy conservation in terms of orthogonal vector spaces resonated with me and Al, and I began to think more about using orthogonality as a means of identifying thermodynamic laws in signal processing algorithms.

From May 2009 to May 2012, there were four committee meetings and a year of intense writing. In the first meeting I argued that if signal processing algorithms can be constructed to obey the laws of thermodynamics in some sense, they ought to have convenient properties. The rationale was that many physical systems happened to have convenient properties, and that these systems also happened to obey the laws of thermodynamics. As examples, I had a few slides with signal flow graphs that essentially simulated the voltage and current equations in electrical networks.

A few days before the second committee meeting, I came to the conclusion that if the thesis were going to discuss thermodynamics, and if conservation of energy corresponded to the linear algebra concept of orthogonality, then the tools of linear algebra could probably be used in picking out variables from a signal processing system in a way that maps to an inner product that evaluates to zero. The main idea was that there ought to be some structure to these maps, and although I had no idea how it might be useful, that it could perhaps at least be of academic interest. I spent probably more time than I needed to drawing a very busy figure graphically depicting vector space decompositions and linear maps, which ended up resembling a snowman standing next to a giant flower.

Although the idea of organizing variables using linear algebra was not mature enough to make an impact in the second meeting, a number of things had fallen into place by the third meeting, which also represented a shift of focus away from thermodynamics and toward conservation. There was an emphasis on viewing conservation as being related to the linear interconnecting component of a signal processing algorithm, including an early version of a result in Chapter 4 about creating what would later be referred to as strongly-conservative interconnections. The idea of organizing variables to form an inner product had not yet been cleanly separated from conditions under which the inner product might evaluate to zero, but there was reference to the possibility that Lie groups could be used to determine all conservative vector spaces. There were also preliminary versions of the stationary content and co-content theorems that would later appear in Chapter 5, shown to illustrate ways in which conservation could potentially be useful.

One influence that had nudged me down the path toward Lie groups was a cultural interest in Lorentz transformations that various members of the Digital Signal Processing Group had adopted at the time, and which had been nurtured by Al after reading Einstein's book on special relativity. The stationary content and co-content examples were inspired by Jack Dennis' Ph.D. thesis, which my academic advisor Sanjoy Mitter had pointed me toward some time before.

By the fourth committee meeting, the formal math behind the concept of the organized variable space had materialized, although it would subsequently go through a revision during the writing. More work had been done on the use of group theory in creating conservative vector spaces, and a slide distinguished between strong and weak conservation within this context. There was also a result about system inversion that had solidified while writing a paper for the 2011 IEEE DSP Workshop, and which would form the basis for the necessary and sufficient condition for behavioral equivalence in Chapter 2.

The framework significantly solidified during the process of writing. The writing of the thesis initially began with an early draft of what would become Chapter 3. This version mostly consisted of theorems, corollaries and lemmas provided without much additional description, put down on paper as a way of trying to organize everything without having the material in other chapters to provide context. Then a preliminary draft of Chapter 5 was written. Chapters 3 and 5 would eventually be re-written, almost in their entirety. In writing Chapter 4, the text grew to the point that the chapter split into two, with the first part eventually becoming Chapter 2. Chapter 6 provided a natural place to collect examples that I had been using in thinking about how the framework might be applied, and some of these developed fairly late in the process. For example, the bilateral vehicle density control example was motivated by a discussion with Berthold Horn that took place more than a semester before the thesis defense date, but the particular solution in the thesis came together about a week and a half before the defense.

I hope that this epilogue has offered an alternative perspective into the thesis, illustrating that the sequence of preparing a technical document can sometimes be very different from the sequence of presentation.