

PARABOLIC INDUCTION AND RESTRICTION FUNCTORS FOR RATIONAL CHEREDNIK ALGEBRAS

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1. INTRODUCTION

Parabolic induction and restriction functors play an important role in the representation theory of finite and affine Hecke algebras. This makes it desirable to generalize them to the setting of double affine Hecke algebras, or Cherednik algebras. However, a naive attempt to do so fails: the definition of parabolic induction and restriction functors for finite and affine Hecke algebras uses the fact that the Hecke algebra attached to a parabolic subgroup can be embedded into the Hecke algebra attached to the whole group, which is not the case in the double affine setting.

One of the main goals of this paper is to circumvent this difficulty in the case of rational Cherednik algebras. The price to pay is that our functors depend on an additional parameter, which is a point b of the reflection representation whose stabilizer is the parabolic subgroup at hand. The functors for different values of b are isomorphic, but not canonically, and there is nontrivial monodromy with respect to b .

More specifically, let W be a finite group acting faithfully on a finite dimensional complex vector space \mathfrak{h} . Let c be a conjugation invariant function on the set S of reflections in W , and $H_c(W, \mathfrak{h})$ the corresponding rational Cherednik algebra. Let $\mathcal{O}_c(W, \mathfrak{h})_0$ be the category of $H_c(W, \mathfrak{h})$ -modules which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and locally nilpotent under the action of \mathfrak{h} . Let $W' \subset W$ be a parabolic subgroup, $\mathfrak{h}' = \mathfrak{h}/\mathfrak{h}^{W'}$, and c' be the restriction of c to the set of reflections in W' . Then we define the parabolic induction and restriction functors

$$\mathrm{Res}_b : \mathcal{O}_c(W, \mathfrak{h})_0 \rightarrow \mathcal{O}_c(W', \mathfrak{h}')_0, \quad \mathrm{Ind}_b : \mathcal{O}_c(W', \mathfrak{h}')_0 \rightarrow \mathcal{O}_c(W, \mathfrak{h})_0, \quad b \in \mathfrak{h}_{\mathrm{reg}}^{W'}.$$

We show that these functors are exact, and the second one is right adjoint to the first one. We also compute some of their values, and study their dependence on b ; this dependence is characterized in terms of local systems with nontrivial monodromy. In particular, we show that in the case $W' = 1$, the functor Res_b (where b is a variable) is the same as the KZ functor of [GGOR].

As a by-product, we show that the category $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ of “Whittaker” modules over $H_c(W, \mathfrak{h})$ (i.e. the category of $H_c(W, \mathfrak{h})$ -modules, finitely generated over $\mathbb{C}[\mathfrak{h}]$, on which $\mathbb{C}[\mathfrak{h}^*]^W$ acts with generalized eigenvalue $\lambda \in \mathfrak{h}^*$) is equivalent to $\mathcal{O}_c(W_\lambda, \mathfrak{h})_0$.

Next, we give some applications of the parabolic induction and restriction functors. First, we give a simple proof of the Gordon-Stafford theorem [GS], which characterizes the values of c (for $W = S_n, \mathfrak{h} = \mathbb{C}^{n-1}$) for which the rational Cherednik algebra is Morita equivalent to its spherical subalgebra. In particular, we remove the condition $c \notin 1/2 + \mathbb{Z}$, which was expected to be unnecessary. Also, we determine some values of c for Coxeter groups for which there exist finite dimensional representations of the rational Cherednik algebra, and find the number of such irreducible representations. Finally, we find all the irreducible aspherical representations in category \mathcal{O} of the rational Cherednik algebra for $W = S_n$. They turn out to coincide with representations for $c \in (-1, 0)$ which are killed by the Knizhnik-Zamolodchikov functor, and their number for each $c = -r/m$ ($2 \leq m \leq n$, $0 < r < m$, $(r, m) = 1$) is equal to the number of non- m -regular partitions of n . This confirms a conjecture of A. Okounkov and the first author. Also, this result implies that the spherical Cherednik algebra $A_c(S_n)$ is simple if $-1 < c < 0$, and allows us to strengthen the main result of [BFG] about localization functors for Cherednik algebras in positive characteristic.

At the end of the paper we include an appendix by the second author, in which the techniques of this paper are applied to the study of reducibility of the polynomial representation of the trigonometric Cherednik algebra.

Remark. We note that the analogs of our parabolic induction and restriction functors in the representation theory of semisimple Lie algebras are the translation functors between the regular and singular category \mathcal{O} ; see [MS].

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2. RATIONAL CHEREDNIK ALGEBRAS

2.1. Definition of rational Cherednik algebras. Let \mathfrak{h} be a finite dimensional vector space over \mathbb{C} , and $W \subset GL(\mathfrak{h})$ a finite subgroup. A reflection in W is an element $s \neq 1$ such that $\text{rk}(s - 1) = 1$. Denote by S the set of reflections in W . Let $c : S \rightarrow \mathbb{C}$ be a W -invariant function. For $s \in S$, let $\alpha_s \in \mathfrak{h}^*$ be a generator of $\text{Im}(s|_{\mathfrak{h}^*} - 1)$, and $\alpha_s^\vee \in \mathfrak{h}$ be the generator of $\text{Im}(s|_{\mathfrak{h}} - 1)$, such that $(\alpha_s, \alpha_s^\vee) = 2$.

Definition 2.1. (see e.g. [EG, E1]) The rational Cherednik algebra $H_c(W, \mathfrak{h})$ is the quotient of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the ideal generated by the relations

$$[x, x'] = 0, [y, y'] = 0, [y, x] = (y, x) - \sum_{s \in S} c_s(y, \alpha_s)(x, \alpha_s^\vee)s,$$

$x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$.

Remark. In [EG, E1], rational Cherednik algebras are defined for (complex) reflection groups W , but this assumption plays no essential role in the theory, and the same definition can be used for any finite group. In fact, this is a rather trivial generalization, since any W acting on \mathfrak{h} contains a canonical normal subgroup W_{ref} generated by the complex reflections in W , and one has $H_c(W, \mathfrak{h}) = \mathbb{C}[W] \otimes_{\mathbb{C}[W_{\text{ref}}]} H_c(W_{\text{ref}}, \mathfrak{h})$, with natural multiplication.

An important role in the representation theory of rational Cherednik algebras is played by the element

$$(1) \quad \mathbf{h} = \sum_i x_i y_i + \frac{\dim \mathfrak{h}}{2} - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} s,$$

where y_i is a basis of \mathfrak{h} , x_i the dual basis of \mathfrak{h}^* , and λ_s is the nontrivial eigenvalue of s in \mathfrak{h}^* . Its usefulness comes from the fact that it satisfies the identities

$$(2) \quad [\mathbf{h}, x_i] = x_i, [\mathbf{h}, y_i] = -y_i.$$

2.2. A geometric approach to rational Cherednik algebras. In [E2], a geometric point of view on rational Cherednik algebras is suggested, in the spirit of the theory of D-modules; this point of view will be useful in the present paper. Namely, in [E2], the algebra $H_c(W, \mathfrak{h})$ is sheafified over \mathfrak{h}/W (as a usual $\mathcal{O}_{\mathfrak{h}/W}$ -module). This yields a quasicohherent sheaf of algebras, $H_{c,W,\mathfrak{h}}$, such that for any affine open subset $U \subset \mathfrak{h}/W$, the algebra of sections $H_{c,W,H}(U)$ is $\mathbb{C}[U] \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c(W, \mathfrak{h})$.

One of the main ideas of [E2] (see [E2], Section 2.9) is that the same sheaf can be defined more geometrically as follows. Let \tilde{U} be the preimage of U in \mathfrak{h} . Then the algebra $H_{c,W,\mathfrak{h}}(U)$ is the algebra of linear operators on $\mathcal{O}(\tilde{U})$ generated by $\mathcal{O}(\tilde{U})$, the group W , and Dunkl-Opdam operators

$$\partial_a + \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(a)}{\alpha_s} (s - 1),$$

where $a \in \mathfrak{h}$.

2.3. The category $\mathcal{O}_c(W, \mathfrak{h})$. The algebra $H_c(W, \mathfrak{h})$ contains commutative subalgebras $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$. We define the category $\mathcal{O}_c(W, \mathfrak{h})$ to be the category of $H_c(W, \mathfrak{h})$ -modules which are finitely generated over $\mathbb{C}[\mathfrak{h}] = S\mathfrak{h}^*$ and locally finite under the action of \mathfrak{h} . We have a decomposition

$$\mathcal{O}_c(W, \mathfrak{h}) = \bigoplus_{\lambda \in \mathfrak{h}^*/W} \mathcal{O}_c(W, \mathfrak{h})_\lambda,$$

where $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ is the full subcategory of those objects of $\mathcal{O}_c(W, \mathfrak{h})$ on which the algebra $\mathbb{C}[\mathfrak{h}^*]^W$ acts with generalized eigenvalue λ . For convenience, below we will use the notation $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ for $\lambda \in \mathfrak{h}^*$, rather than \mathfrak{h}^*/W .

We note that we have a canonical equivalence of categories ${}^1\zeta : \mathcal{O}_c(W, \mathfrak{h})_\lambda \rightarrow \mathcal{O}_c(W, \mathfrak{h}/\mathfrak{h}^W)_\lambda$, defined by the formula

$$\zeta(M) = \{v \in M : yv = \lambda(y)v, y \in \mathfrak{h}^W\}.$$

This implies that the category $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ depends only on the restriction of λ to the W -invariant complement of \mathfrak{h}^W in \mathfrak{h} .

The most interesting case is $\lambda = 0$. The category $\mathcal{O}_c(W, \mathfrak{h})_0$ is the category of $H_c(W, \mathfrak{h})$ -modules which are finitely generated under $\mathbb{C}[\mathfrak{h}]$ and locally nilpotent under the action of \mathfrak{h} . This is what is usually called category \mathcal{O} ; it is discussed in detail in [GGOR]. It is easy to see using equation (2) that the element \mathbf{h} acts locally finitely in any $M \in \mathcal{O}_c(W, \mathfrak{h})_0$, with finite dimensional generalized eigenspaces, and real parts of eigenvalues bounded below.

The most important objects in the category $\mathcal{O}_c(W, \mathfrak{h})_0$ are the standard modules $M_c(W, \mathfrak{h}, \tau) = \text{Ind}_{W \otimes \mathbb{C}[\mathfrak{h}^*]}^{H_c(W, \mathfrak{h})} \tau$, where τ is an irreducible representation of W with the zero action of \mathfrak{h} , and their irreducible quotients $L_c(W, \mathfrak{h}, \tau)$.

It is easy to show that the category $\mathcal{O}_c(W, \mathfrak{h})_0$ contains all finite dimensional $H_c(W, \mathfrak{h})$ -modules.

Remark 2.2. We note that the category $\mathcal{O}_c(W, \mathfrak{h})_0$ is analogous to category \mathcal{O} for semisimple Lie algebras, while the category $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ is analogous to the category of Whittaker modules.

2.4. Completion of rational Cherednik algebras at zero and Jacquet functors. Jacquet functors for rational Cherednik algebras were defined by Ginzburg, [Gi]. Let us recall their construction.

For any $b \in \mathfrak{h}$ we can define the completion $\widehat{H}_c(W, \mathfrak{h})_b$ to be the algebra of sections of the sheaf $H_{c, W, \mathfrak{h}}$ on the formal neighborhood of the image of b in \mathfrak{h}/W . Namely, $\widehat{H}_c(W, \mathfrak{h})_b$ is generated by regular functions on the formal neighborhood of the W -orbit of b , the group W and Dunkl-Opdam operators.

The algebra $\widehat{H}_c(W, \mathfrak{h})_b$ inherits from $H_c(W, \mathfrak{h})$ the natural filtration F^\bullet by order of differential operators, and each of the spaces $F^n \widehat{H}_c(W, \mathfrak{h})_b$ has a projective limit topology; the whole algebra is then equipped with the topology of the nested union (or inductive limit).

Consider the completion of the rational Cherednik algebra at zero, $\widehat{H}_c(W, \mathfrak{h})_0$. It naturally contains the algebra $\mathbb{C}[[\mathfrak{h}]]$. Define the category $\widehat{\mathcal{O}}_c(W, \mathfrak{h})$ of representations of $\widehat{H}_c(W, \mathfrak{h})_0$ which are finitely generated over $\mathbb{C}[[\mathfrak{h}]]$.

¹It is obvious that $H_{c_1}(W_1, \mathfrak{h}_1) \otimes H_{c_2}(W_2, \mathfrak{h}_2) = H_{c_1, c_2}(W_1 \times W_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$, and this isomorphism defines an equivalence of categories

$$\mathcal{O}_{c_1, c_2}(W_1 \times W_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2)_{\lambda_1, \lambda_2} \rightarrow \mathcal{O}_{c_1}(W_1, \mathfrak{h}_1)_{\lambda_1} \otimes \mathcal{O}_{c_2}(W_2, \mathfrak{h}_2)_{\lambda_2}.$$

In particular, if we take $W_1 = W$, $W_2 = 1$, $\mathfrak{h}_1 = \mathfrak{h}/\mathfrak{h}^W$, $\mathfrak{h}_2 = \mathfrak{h}^W$, this equivalence specializes to the equivalence ζ . If W acts trivially on \mathfrak{h} , then ζ identifies the category of D -modules on \mathfrak{h} with locally nilpotent action of $y - \lambda(y)$ with the category of vector spaces, which, upon taking Fourier transforms, is an instance of Kashiwara's lemma.

We have a completion functor $\widehat{\cdot} : \mathcal{O}_c(W, \mathfrak{h}) \rightarrow \widehat{\mathcal{O}}_c(W, \mathfrak{h})$, defined by

$$\widehat{M} = \widehat{H}_c(W, \mathfrak{h})_0 \otimes_{H_c(W, \mathfrak{h})} M = \mathbb{C}[[\mathfrak{h}]] \otimes_{\mathbb{C}[\mathfrak{h}]} M.$$

Also, for $N \in \widehat{\mathcal{O}}_c(W, \mathfrak{h})$, let $E(N)$ be the space spanned by generalized eigenvectors of \mathfrak{h} in N . Then it is easy to see that $E(N) \in \mathcal{O}_c(W, \mathfrak{h})_0$.

The following theorem is standard in the theory of Jacquet functors.

Theorem 2.3. *The restriction of the completion functor $\widehat{\cdot}$ to $\mathcal{O}_c(W, \mathfrak{h})_0$ is an equivalence of categories $\mathcal{O}_c(W, \mathfrak{h})_0 \rightarrow \widehat{\mathcal{O}}_c(W, \mathfrak{h})$. The inverse equivalence is given by the functor E .*

Proof. The proof is standard, but we give it for reader's convenience.

It is clear that $M \subset \widehat{M}$, so $M \subset E(\widehat{M})$ (as M is spanned by generalized eigenvectors of \mathfrak{h}). Let us demonstrate the opposite inclusion. Pick generators m_1, \dots, m_r of M which are generalized eigenvectors of \mathfrak{h} with eigenvalues μ_1, \dots, μ_r . Let $0 \neq v \in E(\widehat{M})$. Then $v = \sum_i f_i m_i$, where $f_i \in \mathbb{C}[[\mathfrak{h}]]$. Assume that $(\mathfrak{h} - \mu)^N v = 0$ for some N . Then $v = \sum_i f_i^{(\mu - \mu_i)} m_i$, where for $f \in \mathbb{C}[[\mathfrak{h}]]$ we denote by $f^{(d)}$ the degree d part of f . Thus $v \in M$, so $M = E(\widehat{M})$.

It remains to show that $\widehat{E(N)} = N$, i.e. that N is the closure of $E(N)$. In other words, letting \mathfrak{m} denote the maximal ideal in $\mathbb{C}[[\mathfrak{h}]]$, we need to show that the natural map $E(N) \rightarrow N/\mathfrak{m}^j N$ is surjective for every j .

To do so, note that \mathfrak{h} preserves the descending filtration of N by subspaces $\mathfrak{m}^j N$. On the other hand, the successive quotients of these subspaces, $\mathfrak{m}^j N / \mathfrak{m}^{j+1} N$, are finite dimensional, which implies that \mathfrak{h} acts locally finitely on their direct sum $\text{gr} N$, and moreover each generalized eigenspace is finite dimensional. Now for each $\beta \in \mathbb{C}$ denote by $N_{j, \beta}$ the generalized β -eigenspace of \mathfrak{h} in $N/\mathfrak{m}^j N$. We have surjective homomorphisms $N_{j+1, \beta} \rightarrow N_{j, \beta}$, and for large enough j they are isomorphisms. This implies that the map $E(N) \rightarrow N/\mathfrak{m}^j N$ is surjective for every j , as desired. \square

Example. Suppose that $c = 0$. Then Theorem 2.3 specializes to the well known fact that the category of W -equivariant local systems on \mathfrak{h} with a locally nilpotent action of partial differentiations is equivalent to the category of all W -equivariant local systems on the formal neighborhood of zero in \mathfrak{h} . In fact, both categories in this case are equivalent to the category of finite dimensional representations of W .

We can now define the composition functor $\mathcal{J} : \mathcal{O}_c(W, \mathfrak{h}) \rightarrow \mathcal{O}_c(W, \mathfrak{h})_0$, by the formula $\mathcal{J}(M) = E(\widehat{M})$. The functor \mathcal{J} is called the Jacquet functor ([Gi]).

2.5. Generalized Jacquet functors.

Proposition 2.4. *For any $M \in \widehat{\mathcal{O}}_c(W, \mathfrak{h})$, a vector $v \in M$ is \mathfrak{h} -finite if and only if it is \mathfrak{h} -nilpotent.*

Proof. The “if” part actually holds for any $H_c(W, \mathfrak{h})$ -module M . Namely, if v is \mathfrak{h} -nilpotent then consider the finite dimensional space $S\mathfrak{h} \cdot v$. We prove that v is \mathfrak{h} -finite by induction in the dimension d of this space. We can use $d = 0$ as the base, so we only need to do the induction step. The space $S\mathfrak{h} \cdot v$ must contain a nonzero vector u such that $yu = 0$ for all $y \in \mathfrak{h}$. Let $U \subset M$ be the subspace of vectors with this property. Formula (1) for \mathfrak{h} implies that \mathfrak{h} acts in U by an element of the group algebra of W , hence locally finitely. So it is sufficient to prove that the image of v in $M / \langle U \rangle$ is \mathfrak{h} -finite (where $\langle U \rangle$ is the submodule generated by U). But this is true by the induction assumption, as $u = 0$ in $M / \langle U \rangle$.

To prove the “only if” part, assume that $(\mathfrak{h} - \mu)^N v = 0$. Then for any $u \in S^r \mathfrak{h} \cdot v$, we have $(\mathfrak{h} - \mu + r)^N v = 0$. But by Theorem 2.3, the real parts of generalized eigenvalues of \mathfrak{h} in M are bounded below. Hence $S^r \mathfrak{h} \cdot v = 0$ for large enough r , as desired. \square

According to Proposition 2.4, the functor E can be alternatively defined by setting $E(M)$ to be the subspace of M which is locally nilpotent under the action of \mathfrak{h} .

This gives rise to the following generalization of E : for any $\lambda \in \mathfrak{h}^*$ we define the functor $E_\lambda : \widehat{\mathcal{O}}_c(W, \mathfrak{h}) \rightarrow \mathcal{O}_c(W, \mathfrak{h})_\lambda$ by setting $E_\lambda(M)$ to be the space of generalized eigenvectors of $\mathbb{C}[\mathfrak{h}^*]^W$ in M with eigenvalue λ . This way, we have $E_0 = E$.

We can also define the generalized Jacquet functor $\mathcal{J}_\lambda : \mathcal{O}_c(W, \mathfrak{h}) \rightarrow \mathcal{O}_c(W, \mathfrak{h})_\lambda$ by the formula $\mathcal{J}_\lambda(M) = E_\lambda(\widehat{M})$. Then we have $\mathcal{J}_0 = \mathcal{J}$, and the restriction of \mathcal{J}_λ to $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ is the identity functor.

2.6. The duality functors. Let $\bar{c} \in \mathbb{C}[S]^W$ be defined by $\bar{c}(s) = c(s^{-1})$. Then we have a natural isomorphism $\gamma : H_{\bar{c}}(W, \mathfrak{h}^*)^{op} \rightarrow H_c(W, \mathfrak{h})$, acting trivially on \mathfrak{h} and \mathfrak{h}^* , and sending $w \in W$ to w^{-1} ([GGOR], 4.2). Thus, if M is an $H_c(W, \mathfrak{h})$ -module, then the full dual space M^* is naturally an $H_{\bar{c}}(W, \mathfrak{h}^*)$ -module, via $\pi_{M^*}(a) = \pi_M(\gamma(a))^*$.

It is clear that the duality functor $*$ defines an equivalence between the category $\mathcal{O}_c(W, \mathfrak{h})_0$ and $\widehat{\mathcal{O}}_{\bar{c}}(W, \mathfrak{h}^*)^{op}$, and that we can define the functor of restricted dual $\dagger : \mathcal{O}_c(W, \mathfrak{h}) \rightarrow \mathcal{O}_{\bar{c}}(W, \mathfrak{h}^*)^{op}$, given by the formula $M^\dagger = E(M^*)$. This functor assigns to M its restricted dual space under the grading by generalized eigenvalues of \mathfrak{h} . It is clear that this functor is an equivalence of categories, and $\dagger^2 = \text{id}$.

3. PARABOLIC INDUCTION AND RESTRICTION FUNCTORS

3.1. Parabolic subgroups. For a point a of \mathfrak{h} or \mathfrak{h}^* , let W_a denote the stabilizer of a in W . Define a parabolic subgroup of W to be the stabilizer W_b of a point $b \in \mathfrak{h}$. The set of conjugacy classes of parabolic subgroups in W will be denoted by $\text{Par}(W)$.

Suppose $W' \subset W$ is a parabolic subgroup, and $b \in \mathfrak{h}$ is such that $W_b = W'$. In this case, we have a natural W' -invariant decomposition

$$\mathfrak{h} = \mathfrak{h}^{W'} \oplus (\mathfrak{h}^{*W'})^\perp,$$

and $b \in \mathfrak{h}^{W'}$. Thus we have a nonempty open set $\mathfrak{h}_{\text{reg}}^{W'}$ of all $a \in \mathfrak{h}^{W'}$ for which $W_a = W'$; this set is nonempty because it contains b . We also have a W' -invariant decomposition $\mathfrak{h}^* = \mathfrak{h}^{*W'} \oplus (\mathfrak{h}^{W'})^\perp$, and we can define the open set $\mathfrak{h}_{\text{reg}}^{*W'}$ of all $\lambda \in \mathfrak{h}^{*W'}$ for which $W_\lambda = W'$. It is clear that this set is nonempty. This implies, in particular, that one can make an alternative definition of a parabolic subgroup of W as the stabilizer of a point in \mathfrak{h}^* .

3.2. The centralizer construction. For a finite group H , let $e_H = \frac{1}{|H|} \sum_{h \in H} h$ be the idempotent of the trivial representation in $\mathbb{C}[H]$.

If $G \supset H$ are finite groups, and A is an algebra containing $\mathbb{C}[H]$, then define the algebra $Z(G, H, A)$ to be the centralizer $\text{End}_A(P)$ of A in the right A -module $P = \text{Fun}_H(G, A)$ of H -invariant A -valued functions on G , i.e. such functions $f : G \rightarrow A$ that $f(hg) = hf(g)$. Clearly, P is a free A -module of rank $|G/H|$, so the algebra $Z(G, H, A)$ is isomorphic to $\text{Mat}_{|G/H|}(A)$, but this isomorphism is not canonical.

The following lemma is trivial.

Lemma 3.1. (i) *The functor $N \mapsto I(N) := P \otimes_A N = \text{Fun}_H(G, N)$ defines an equivalence of categories $A\text{-mod} \rightarrow Z(G, H, A)\text{-mod}$.*

(ii) $e_G Z(G, H, A) e_G = e_H A e_H$.

(iii) $Z(G, H, A) e_G Z(G, H, A) = Z(G, H, A)$ if and only if $A e_H A = A$.

3.3. Completion of rational Cherednik algebras at arbitrary points of \mathfrak{h}/W . The following result is, in essence, a consequence of the geometric approach to rational Cherednik algebras, described in Subsection 2.2. It should be regarded as a direct generalization to the case of Cherednik algebras of Theorem 8.6 of [L] for affine Hecke algebras.

Theorem 3.2. *Let $b \in \mathfrak{h}$, and c' be the restriction of c to the set S_b of reflections in W_b . Then one has a natural isomorphism*

$$\theta : \widehat{H}_c(W, \mathfrak{h})_b \rightarrow Z(W, W_b, \widehat{H}_{c'}(W_b, \mathfrak{h})_0),$$

defined by the following formulas. Suppose that $f \in P = \text{Fun}_{W_b}(W, \widehat{H}_c(W_b, \mathfrak{h})_0)$. Then

$$(\theta(u)f)(w) = f(wu), u \in W;$$

for any $\alpha \in \mathfrak{h}^$,*

$$(\theta(x_\alpha)f)(w) = (x_{w\alpha}^{(b)} + (w\alpha, b))f(w),$$

where $x_\alpha \in \mathfrak{h}^ \subset H_c(W, \mathfrak{h})$, $x_\alpha^{(b)} \in \mathfrak{h}^* \subset H_{c'}(W_b, \mathfrak{h})$ are the elements corresponding to α ; and for any $a \in \mathfrak{h}^*$,*

$$(3) \quad (\theta(y_a)f)(w) = y_{wa}^{(b)}f(w) + \sum_{s \in S: s \notin W_b} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(wa)}{x_{\alpha_s}^{(b)} + \alpha_s(b)} (f(sw) - f(w)).$$

where $y_a \in \mathfrak{h} \subset H_c(W, \mathfrak{h})$, $y_a^{(b)} \in \mathfrak{h} \subset H_{c'}(W_b, \mathfrak{h})$.

Proof. The proof is by a direct computation. We note that in the last formula, the fraction $\frac{\alpha_s(wa)}{x_{\alpha_s}^{(b)} + \alpha_s(b)}$ is viewed as a power series (i.e., an element of $\mathbb{C}[[\mathfrak{h}]]$), and that only the entire sum, and not each summand separately, is in the centralizer algebra. \square

Remark. Let us explain how to see the existence of θ without writing explicit formulas, and how to guess the formula (3) for θ . It is explained in [E2] (see e.g. [E2], Section 2.9) that the sheaf of algebras obtained by sheafification of $H_c(W, \mathfrak{h})$ over \mathfrak{h}/W is generated (on every affine open set in \mathfrak{h}/W) by regular functions on \mathfrak{h} , elements of W , and Dunkl-Opdam operators. Therefore, this statement holds for formal neighborhoods, i.e., it is true on the formal neighborhood of the image in \mathfrak{h}/W of any point $b \in \mathfrak{h}$. However, looking at the formula for Dunkl-Opdam operators near b , we see that the summands corresponding to $s \in S, s \notin W_b$ are actually regular at b , so they can be safely deleted without changing the generated algebra (as all regular functions on the formal neighborhood of b are included into the system of generators). But after these terms are deleted, what remains is nothing but the Dunkl operators for (W_b, \mathfrak{h}) , which, together with functions on the formal neighborhood of b and the group W_b , generate the completion of $H_c(W_b, \mathfrak{h})$. This gives a construction of θ without using explicit formulas.

Also, this argument explains why θ should be defined by the formula (3) of Theorem 3.2. Indeed, what this formula does is just restores the terms with $s \notin W_b$ that have been previously deleted.

The map θ defines an equivalence of categories

$$\theta_* : \widehat{H}_c(W, \mathfrak{h})_b - \text{mod} \rightarrow Z(W, W_b, \widehat{H}_{c'}(W_b, \mathfrak{h})_0) - \text{mod}.$$

Corollary 3.3. *We have a natural equivalence of categories $\psi_\lambda : \mathcal{O}_c(W, \mathfrak{h})_\lambda \rightarrow \mathcal{O}_{c'}(W_\lambda, \mathfrak{h}/\mathfrak{h}^{W_\lambda})_0$.*

Proof. The category $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ is the category of modules over $H_c(W, \mathfrak{h})$ which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and extend by continuity to the completion of the algebra $H_c(W, \mathfrak{h})$ at λ . So it follows from Theorem 3.2 that we have an equivalence $\mathcal{O}_c(W, \mathfrak{h})_\lambda \rightarrow \mathcal{O}_{c'}(W_\lambda, \mathfrak{h})_0$. Composing this equivalence with the equivalence $\zeta : \mathcal{O}_{c'}(W_\lambda, \mathfrak{h})_0 \rightarrow \mathcal{O}_{c'}(W_\lambda, \mathfrak{h}/\mathfrak{h}^{W_\lambda})_0$, we obtain the desired equivalence ψ_λ . \square

Remark 3.4. Note that in this proof, we take the completion of $H_c(W, \mathfrak{h})$ at a point of $\lambda \in \mathfrak{h}^*$ rather than $b \in \mathfrak{h}$.

3.4. The completion functor. Let $\widehat{\mathcal{O}}_c(W, \mathfrak{h})^b$ be the category of modules over $\widehat{H}_c(W, \mathfrak{h})_b$ which are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]_b}$.

Proposition 3.5. *The duality functor $*$ defines an anti-equivalence of categories $\mathcal{O}_c(W, \mathfrak{h})_\lambda \rightarrow \widehat{\mathcal{O}}_{\bar{c}}(W, \mathfrak{h}^*)^\lambda$.*

Proof. This follows from the fact (already mentioned above) that $\mathcal{O}_c(W, \mathfrak{h})_\lambda$ is the category of modules over $H_c(W, \mathfrak{h})$ which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and extend by continuity to the completion of the algebra $H_c(W, \mathfrak{h})$ at λ . \square

Let us denote the functor inverse to $*$ also by $*$; it is the functor of continuous dual (in the formal series topology).

We have an exact functor of completion at b , $\mathcal{O}_c(W, \mathfrak{h})_0 \rightarrow \widehat{\mathcal{O}}_c(W, \mathfrak{h})^b$, $M \mapsto \widehat{M}_b$. We also have a functor $E^b : \widehat{\mathcal{O}}_c(W, \mathfrak{h})^b \rightarrow \mathcal{O}_c(W, \mathfrak{h})_0$ in the opposite direction, sending a module N to the space $E^b(N)$ of \mathfrak{h} -nilpotent vectors in N .

Proposition 3.6. *The functor E^b is right adjoint to the completion functor $\widehat{}_b$.*

Proof. Straightforward. \square

Remark 3.7. Recall that by Theorem 2.3, if $b = 0$ then these functors are not only adjoint but also inverse to each other.

Proposition 3.8. (i) *For $M \in \mathcal{O}_{\bar{c}}(W, \mathfrak{h}^*)_b$, one has $E^b(M^*) = (\widehat{M})^*$ in $\mathcal{O}_c(W, \mathfrak{h})_0$.*

(ii) *For $M \in \mathcal{O}_c(W, \mathfrak{h})_0$, $(\widehat{M}_b)^* = E_b(M^*)$ in $\mathcal{O}_{\bar{c}}(W, \mathfrak{h}^*)_b$.*

(iii) *The functors E_b , E^b are exact.*

Proof. (i),(ii) are straightforward from the definitions. (iii) follows from (i),(ii), since the completion functors are exact. \square

3.5. Parabolic induction and restriction functors for rational Cherednik algebras. Theorem 3.2 allows us to define analogs of parabolic restriction functors for rational Cherednik algebras.

Namely, let $b \in \mathfrak{h}$, and $W_b = W'$. Define a functor $\text{Res}_b : \mathcal{O}_c(W, \mathfrak{h})_0 \rightarrow \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})_0$ by the formula

$$\text{Res}_b(M) = (\zeta \circ E \circ I^{-1} \circ \theta_*)(\widehat{M}_b).$$

We can also define the parabolic induction functors in the opposite direction. Namely, let $N \in \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})_0$. Then we can define the object $\text{Ind}_b(N) \in \mathcal{O}_c(W, \mathfrak{h})_0$ by the formula

$$\text{Ind}_b(N) = (E^b \circ \theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)}_0).$$

Proposition 3.9. (i) *The functors Ind_b , Res_b are exact.*

(ii) *One has $\text{Ind}_b(\text{Res}_b(M)) = E^b(\widehat{M}_b)$.*

Proof. Part (i) follows from the fact that the functor E^b and the completion functor $\widehat{}_b$ are exact (see Proposition 3.8). Part (ii) is straightforward from the definition. \square

Theorem 3.10. *The functor Ind_b is right adjoint to Res_b .*

Proof. We have

$$\begin{aligned} \mathrm{Hom}(\mathrm{Res}_b(M), N) &= \mathrm{Hom}((\zeta \circ E \circ I^{-1} \circ \theta_*)(\widehat{M}_b), N) = \\ &= \mathrm{Hom}((E \circ I^{-1} \circ \theta_*)(\widehat{M}_b), \zeta^{-1}(N)) = \\ &= \mathrm{Hom}((I^{-1} \circ \theta_*)(\widehat{M}_b), \widehat{\zeta^{-1}(N)}_0) = \mathrm{Hom}(\widehat{M}_b, (\theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)}_0)) = \\ &= \mathrm{Hom}(M, (E^b \circ \theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)}_0)) = \mathrm{Hom}(M, \mathrm{Ind}_b(N)). \end{aligned}$$

At the end we used Proposition 3.6. \square

Corollary 3.11. *The functor Res_b maps projective objects to projective ones, and the functor Ind_b maps injective objects to injective ones.*

We can also define functors $\mathrm{res}_\lambda : \mathcal{O}_c(W, \mathfrak{h})_0 \rightarrow \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})_0$ and $\mathrm{ind}_\lambda : \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})_0 \rightarrow \mathcal{O}_c(W, \mathfrak{h})_0$, attached to $\lambda \in \mathfrak{h}_{\mathrm{reg}}^{*W'}$, by

$$\mathrm{res}_\lambda := \dagger \circ \mathrm{Res}_\lambda \circ \dagger, \mathrm{ind}_\lambda := \dagger \circ \mathrm{Ind}_\lambda \circ \dagger,$$

where \dagger is as in Subsection 2.6.

Corollary 3.12. *The functors $\mathrm{res}_\lambda, \mathrm{ind}_\lambda$ are exact. The functor ind_λ is left adjoint to res_λ . The functor ind_λ maps projective objects to projective ones, and the functor res_λ injective objects to injective ones.*

We also have the following proposition, whose proof is straightforward.

Proposition 3.13. *We have*

$$\mathrm{ind}_\lambda(N) = (\mathcal{J} \circ \psi_\lambda^{-1})(N),$$

and

$$\mathrm{res}_\lambda(M) = (\psi_\lambda \circ E_\lambda)(\widehat{M}),$$

where ψ_λ is defined in Corollary 3.3.

3.6. Some evaluations of the parabolic induction and restriction functors. For generic c , the category $\mathcal{O}_c(W, \mathfrak{h})$ is semisimple, and naturally equivalent to the category $\mathrm{Rep}W$ of finite dimensional representations of W , via the functor $\tau \mapsto M_c(W, \mathfrak{h}, \tau)$. (If W is a Coxeter group, the exact set of such c (which are called regular) is known from [GGOR] and [Gy]).

Proposition 3.14. *(i) Suppose that c is generic. Upon the above identification, the functors $\mathrm{Ind}_b, \mathrm{ind}_\lambda$ and $\mathrm{Res}_b, \mathrm{res}_\lambda$ go to the usual induction and restriction functors between categories $\mathrm{Rep}W$ and $\mathrm{Rep}W'$. In other words, we have*

$$\mathrm{Res}_b(M_c(W, \mathfrak{h}, \tau)) = \bigoplus_{\xi \in \widehat{W}'} n_{\tau\xi} M_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'}, \xi),$$

and

$$\mathrm{Ind}_b(M_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'}, \xi)) = \bigoplus_{\tau \in \widehat{W}} n_{\tau\xi} M_c(W, \mathfrak{h}, \tau),$$

where $n_{\tau\xi}$ is the multiplicity of occurrence of ξ in $\tau|_{W'}$, and similarly for $\mathrm{res}_\lambda, \mathrm{ind}_\lambda$.

(ii) The equations of (i) hold at the level of Grothendieck groups for all c .

Proof. Part (i) is easy for $c = 0$, and is obtained for generic c by a deformation argument. Part (ii) is also obtained by deformation argument, taking into account that the functors Res_b and Ind_b are exact and flat with respect to c . \square

Example 3.15. Suppose that $W' = 1$. Then $\text{Res}_b(M)$ is the fiber of M at b , while $\text{Ind}_b(\mathbb{C}) = P_{KZ}$, the object defined in [GGOR], which is projective and injective (see Remark 3.19). This shows that Proposition 3.14 (i) does not hold for special c , as P_{KZ} is not, in general, a direct sum of standard modules.

3.7. Dependence of the functor Res_b on b . Let $W' \subset W$ be a parabolic subgroup. In the construction of the functor Res_b , the point b can be made a variable which belongs to the open set $\mathfrak{h}_{\text{reg}}^{W'}$.

Namely, let $\widehat{\mathfrak{h}}_{\text{reg}}^{W'}$ be the formal neighborhood of the locally closed set $\mathfrak{h}_{\text{reg}}^{W'}$ in \mathfrak{h} , and let $\pi : \widehat{\mathfrak{h}}_{\text{reg}}^{W'} \rightarrow \mathfrak{h}/W$ be the natural map (note that this map is an étale covering of the image with the Galois group $N_W(W')/W'$, where $N_W(W')$ is the normalizer of W' in W). Let $\widehat{H}_c(W, \mathfrak{h})_{\mathfrak{h}_{\text{reg}}^{W'}}$ be the pullback of the sheaf $H_{c,W,\mathfrak{h}}$ under π . We can regard it as a sheaf of algebras over $\mathfrak{h}_{\text{reg}}^{W'}$. Similarly to Theorem 3.2 we have an isomorphism

$$\theta : \widehat{H}_c(W, \mathfrak{h})_{\mathfrak{h}_{\text{reg}}^{W'}} \rightarrow Z(W, W', \widehat{H}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})_0) \hat{\otimes} D(\mathfrak{h}_{\text{reg}}^{W'}),$$

where $D(\mathfrak{h}_{\text{reg}}^{W'})$ is the sheaf of differential operators on $\mathfrak{h}_{\text{reg}}^{W'}$, and $\hat{\otimes}$ is an appropriate completion of the tensor product.

Thus, repeating the construction of Res_b , we can define the functor

$$\text{Res} : \mathcal{O}_c(W, \mathfrak{h})_0 \rightarrow \mathcal{O}_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})_0 \boxtimes \text{Loc}(\mathfrak{h}_{\text{reg}}^{W'}),$$

where $\text{Loc}(\mathfrak{h}_{\text{reg}}^{W'})$ stands for the category of local systems (i.e. O-coherent D-modules) on $\mathfrak{h}_{\text{reg}}^{W'}$. This functor has the property that Res_b is the fiber of Res at b . Namely, the functor Res is defined by the formula

$$\text{Res}(M) = (E \circ I^{-1} \circ \theta_*)(\widehat{M}_{\mathfrak{h}_{\text{reg}}^{W'}}),$$

where $\widehat{M}_{\mathfrak{h}_{\text{reg}}^{W'}}$ is the restriction of the sheaf M on \mathfrak{h} to the formal neighborhood of $\mathfrak{h}_{\text{reg}}^{W'}$.

Remark 3.16. If W' is the trivial group, the functor Res is just the KZ functor from [GGOR]. Thus, Res is a relative version of the KZ functor.

Thus, we see that the functor Res_b does not depend on b , up to an isomorphism. A similar statement is true for the functors $\text{Ind}_b, \text{res}_\lambda, \text{ind}_\lambda$.

Conjecture 3.17. For any $b \in \mathfrak{h}, \lambda \in \mathfrak{h}^*$ such that $W_b = W_\lambda$, we have isomorphisms of functors $\text{Res}_b \cong \text{res}_\lambda, \text{Ind}_b \cong \text{ind}_\lambda$.

Remark 3.18. Conjecture 3.17 would imply that Ind_b is left adjoint to Res_b , and that Res_b maps injective objects to injective ones, while Ind_b maps projective objects to projective ones.

Remark 3.19. If b and λ are generic (i.e., $W_b = W_\lambda = 1$) then the conjecture holds. Indeed, in this case the conjecture reduces to showing that we have an isomorphism of functors $\text{Fiber}_b(M) \cong \text{Fiber}_\lambda(M^\dagger)^*$ ($M \in \mathcal{O}_c(W, \mathfrak{h})$). Since both functors are exact functors to the category of vector spaces, it suffices to check that $\dim \text{Fiber}_b(M) = \dim \text{Fiber}_\lambda(M^\dagger)$. But this is true because both dimensions are given by the leading coefficient of the Hilbert polynomial of M (characterizing the growth of M).

It is important to mention, however, that although Res_b is isomorphic to $\text{Res}_{b'}$ if $W_b = W_{b'}$, this isomorphism is not canonical. So let us examine the dependence of Res_b on b a little more carefully.

Theorem 3.14 implies that if c is generic, then

$$\text{Res}(M_c(W, \mathfrak{h}, \tau)) = \oplus_\xi M_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'}, \xi) \otimes \mathcal{L}_{\tau\xi},$$

where $\mathcal{L}_{\tau\xi}$ is a local system on $\mathfrak{h}_{\text{reg}}^{W'}$ of rank $n_{\tau\xi}$. Let us characterize the local system $\mathcal{L}_{\tau\xi}$ explicitly.

Proposition 3.20. *The local system $\mathcal{L}_{\tau\xi}$ is given by the “partial” KZ connection on the trivial bundle, with the connection form*

$$\sum_{s \in S: s \notin W'} \frac{2c_s}{1 - \lambda_s} \frac{d\alpha_s}{\alpha_s} (s - 1).$$

with values in $\text{Hom}_{W'}(\xi, \tau|_{W'})$.

Proof. This follows immediately from formula (3). □

3.8. Supports of modules. The following two basic propositions are proved in [Gi], Section 6. We will give different proofs of them, based on the restriction functors.

Proposition 3.21. *Consider the stratification of \mathfrak{h} with respect to stabilizers of points in W . Then the support $\text{Supp}M$ of any object M of $\mathcal{O}_c(W, \mathfrak{h})$ in \mathfrak{h} is a union of strata of this stratification.*

Proof. This follows immediately from the existence of the flat connection along the set of points b with a fixed stabilizer W' on the bundle $\text{Res}_b(M)$. □

Proposition 3.22. *For any irreducible object M in $\mathcal{O}_c(W, \mathfrak{h})$, $\text{Supp}M/W$ is an irreducible algebraic variety.*

Proof. Let X be a component of $\text{Supp}M/W$. Let M' be the subspace of elements of M whose restriction to a neighborhood of a generic point of X is zero. It is obvious that M' is an $H_c(W, \mathfrak{h})$ -submodule in M . By definition, it is a proper submodule. Therefore, by the irreducibility of M , we have $M' = 0$. Now let $f \in \mathbb{C}[\mathfrak{h}]^W$ be a function that vanishes on X . Then there

exists a positive integer N such that f^N maps M to M' , hence acts by zero on M . This implies that $\text{Supp}M/W = X$, as desired. \square

Propositions 3.21 and 3.22 allow us to attach to every irreducible module $M \in \mathcal{O}_c(W, \mathfrak{h})$, a conjugacy class of parabolic subgroups, $C_M \in \text{Par}(W)$, namely, the conjugacy class of the stabilizer of a generic point of the support of M . Also, for a parabolic subgroup $W' \subset W$, denote by $\mathcal{S}(W')$ the set of points $b \in \mathfrak{h}$ whose stabilizer contains a subgroup conjugate to W' .

The following proposition is immediate.

Proposition 3.23. (i) *Let $M \in \mathcal{O}_c(W, \mathfrak{h})_0$ be irreducible. If b is such that $W_b \in C_M$, then $\text{Res}_b(M)$ is a nonzero finite dimensional module over $H_{c'}(W_b, \mathfrak{h}/\mathfrak{h}^{W_b})$.*

(ii) *Conversely, let $b \in \mathfrak{h}$, and L be a finite dimensional module $H_c(W_b, \mathfrak{h}/\mathfrak{h}^{W_b})$. Then the support of $\text{Ind}_b(L)$ in \mathfrak{h} is $\mathcal{S}(W_b)$.*

Let $FD(W, \mathfrak{h})$ be the set of c for which $H_c(W, \mathfrak{h})$ admits a finite dimensional representation.

Corollary 3.24. (i) *Let W' be a parabolic subgroup of W . Then $\mathcal{S}(W')$ is the support of some irreducible representation from $\mathcal{O}_c(W, \mathfrak{h})_0$ if and only if $c' \in FD(W', \mathfrak{h}/\mathfrak{h}^{W'})$.*

(ii) *Suppose that W is a Coxeter group. Then the category $\mathcal{O}_c(W, \mathfrak{h})_0$ is semisimple if and only if $c \notin \cup_{W' \in \text{Par}(W)} FD(W', \mathfrak{h}/\mathfrak{h}^{W'})$.*

Proof. (i) is immediate from Proposition 3.23, and (ii) follows from (i), since by the combination of results from [DJO],[Gy], and [GGOR], the category $\mathcal{O}_c(W, \mathfrak{h})_0$ is not semisimple if and only if there exists a nonzero representation in $\mathcal{O}_c(W, \mathfrak{h})_0$ whose support is not equal to \mathfrak{h} . \square

Example 3.25. Let $W = S_n$, $\mathfrak{h} = \mathbb{C}^{n-1}$. In this case, the set $\text{Par}(W)$ is the set of partitions of n . Assume that $c = r/m$, $(r, m) = 1$, $2 \leq m \leq n$. By a result of [BEG2], finite dimensional representations of $H_c(W, \mathfrak{h})$ exist if and only if $m = n$. Thus the only possible classes C_M for irreducible modules M have stabilizers $S_m \times \dots \times S_m$, i.e., correspond to partitions into parts, where each part is equal to m or 1. So there are $[n/m] + 1$ possible supports for modules, where $[a]$ denotes the integer part of a .

3.9. Cuspidal numbers. Let W be a real reflection group, \mathfrak{h} its reflection representation. Let us say that a function c is *singular* if the category $\mathcal{O}_c(W, \mathfrak{h})_0$ is not semisimple. It follows from [GGOR, Gy, DJO] that if c is constant and $c > 0$ then c is singular if and only if the polynomial representation $M_c(W, \mathfrak{h}, \mathbb{C})$ is reducible. The paper [DJO] determines the set of singular values. In particular, it is shown in [DJO] that constant $c > 0$ is singular if and only if $c \in \mathbb{Q}$, and the denominator of c divides one of the degrees d_i of W .

In this subsection we assume that c is a constant function. Let $\text{Div}(W, \mathfrak{h})$ be the set of all divisors of the degrees d_i of W .

Let us say that d is a *cuspidal number* for W if $d \in \text{Div}(W, \mathfrak{h})$, but $d \notin \text{Div}(W', \mathfrak{h})$ for any proper parabolic subgroup $W' \subset W$. Thus, constant c with denominator being a cuspidal number is a special kind of singular values.

Proposition 3.26. *The following two conditions on c are equivalent:*

(a) *The category $\mathcal{O}_c(W, \mathfrak{h})_0$ is not semisimple, but any representation $M \in \mathcal{O}_c(W, \mathfrak{h})_0$ is either finite dimensional or has full support in \mathfrak{h} .*

(b) *the denominator of c , when written as an irreducible fraction, is a cuspidal number of W .*

Proof. As we have mentioned, $\mathcal{O}_c(W, \mathfrak{h})_0$ is not semisimple iff the denominator of c divides a degree of d_i of W . Thus, by Corollary 3.24, condition (a) holds if and only if the denominator of c divides a degree of W , but does not divide a degree of a proper parabolic subgroup, which proves the proposition. \square

A basic example of a cuspidal number for any irreducible W is the Coxeter number h of W , since it is greater than any of the degrees for parabolic subgroups. Let us call any other cuspidal number non-Coxeter, and denote the set of such numbers $NC(W)$.

The non-Coxeter cuspidal numbers are found by inspecting tables. Let us enumerate them. Classical Weyl groups (of type A,B=C,D) do not have non-Coxeter cuspidal numbers. Here are the non-Coxeter cuspidal numbers for other irreducible Coxeter groups:

$$NC(E6) = \{9\}, NC(E7) = \{14\}, NC(E8) = \{15, 20, 24\}, NC(F4) = \{8\}, \\ NC(I(m)) = \{2 < d < m : m/d \in \mathbb{Z}\}, NC(H3) = \{6\}, NC(H4) = \{12, 15, 20\}.$$

Corollary 3.27. *Suppose that $c > 0$ and the denominator of c , when written as an irreducible fraction, is a cuspidal number of W . Then the representation $L_c(W, \mathfrak{h}, \mathbb{C})$ is finite dimensional.*

Proof. Since the denominator of c divides a degree of W , c is a singular value, and since $c > 0$, by [DJO], the polynomial representation $M_c(W, \mathfrak{h}, \mathbb{C})$ is reducible, so $L_c(W, \mathfrak{h}, \mathbb{C})$ cannot have full support. So by Proposition 3.26, it is finite dimensional, as desired. \square

Remark 3.28. If W is a Weyl group, this proposition follows from the main result of [VV], which states that for $c > 0$, $L_c(W, \mathfrak{h}, \mathbb{C})$ is finite dimensional if and only if the denominator of c is an elliptic number, because every cuspidal number is an elliptic number².

Remark 3.29. We note that in the case when W is a Weyl group and $d = h$, i.e. $c = j/h$, $j \in \mathbb{N}$, $(j, h) = 1$, the fact that the representations $L_c(W, \mathfrak{h}, \mathbb{C})$ are finite dimensional follows from the work of Cherednik (see [Ch1]); these

²An element $w \in W$ is *elliptic* if it has no nonzero invariants in the reflection representation \mathfrak{h} , and is *regular* if it has a regular eigenvector in \mathfrak{h} . An *elliptic number* is, by definition, the order of an elliptic regular element (see [VV]).

are the so-called perfect representations, of dimension j^r , where r is the rank of W . More precisely, Cherednik works with the true double affine Hecke algebras $\mathcal{H}_{q,t}$ (not with their rational degenerations), but it is known ([Ch1]) that finite dimensional representations for the two kinds of algebras have the same structure if $q = e^{\hbar}$, $t = e^{\hbar c}$, where \hbar is a formal parameter.

Now suppose c is as in Corollary 3.27, and consider the KZ functor $\mathcal{O}_c(W, \mathfrak{h})_0 \rightarrow \text{Rep}\mathcal{H}_q(W)$, where $q = e^{2\pi ic}$, and $\mathcal{H}_q(W)$ is the corresponding finite Hecke algebra. Then it follows from the results of [GGOR] and Proposition 3.26 that this functor kills finite dimensional irreducible modules, and sets up a bijection between other irreducible modules (with full support) and irreducible representations of $\mathcal{H}_q(W)$. Thus we get

Corollary 3.30. *The number of irreducible finite dimensional representations of $H_c(W, \mathfrak{h})$ equals $N(W) - N_q(W)$, where $N(W)$ is the number of irreducible representations of W , and $N_q(W)$ is the number of irreducible representations of $\mathcal{H}_q(W)$.*

Remark 3.31. It turns out (see [GP]) that if the denominator of c is a cuspidal number then $N(W) - N_q(W)$ is always 1 or 2, and it is 2 only in the cases when W is of type E_8 or H_4 and $d = 15$. In both of these cases, the additional finite dimensional irreducible representation is the one whose highest weight is the reflection representation of W .

Remark 3.32. The results of this subsection can also be found in the latest version of the paper [Rou], Section 5.2.4, which appeared while this paper was being written.

4. THE GORDON-STAFFORD THEOREM

4.1. Aspherical parameter values. Let M be a nonzero $H_c(W, \mathfrak{h})$ -module. Let us say that M is *aspherical* if $e_W M = 0$. Let c be called aspherical if $H_c(W, \mathfrak{h})$ admits an aspherical representation which belongs to the category $\mathcal{O}_c(W, \mathfrak{h})_0$. Let $\Sigma(W, \mathfrak{h})$ be the set of aspherical values. If $W' \subset W$ is a parabolic subgroup, then denote by $\Sigma'(W', \mathfrak{h})$ the preimage of $\Sigma(W', \mathfrak{h})$ in $\mathbb{C}[S]^W$ under the restriction map $c \mapsto c'$.

Let also $FDA(W, \mathfrak{h})$ be the set of c for which $H_c(W, \mathfrak{h})$ admits a finite dimensional aspherical representation.

Theorem 4.1. (i) $c \in \Sigma(W, \mathfrak{h})$ if and only if $H_c(W, \mathfrak{h})e_W H_c(W, \mathfrak{h}) \neq H_c(W, \mathfrak{h})$.
(ii) We have

$$\Sigma(W, \mathfrak{h}) = FDA(W, \mathfrak{h}) \cup \bigcup_{W' \in \text{Par}(W)} \Sigma'(W', \mathfrak{h}/\mathfrak{h}^{W'}).$$

Proof. (i) This is essentially proved in [BEG1]. Only the “if” direction requires proof. Let $B = H_c(W, \mathfrak{h})/H_c(W, \mathfrak{h})e_W H_c(W, \mathfrak{h})$; we have $B \neq 0$. Let us regard B as a $(\mathbb{C}[\mathfrak{h}]^W, \mathbb{C}[\mathfrak{h}^*]^W)$ -bimodule; then it is finitely generated. Thus if I is the maximal ideal in $\mathbb{C}[\mathfrak{h}^*]^W$ corresponding to the point 0,

then $B/BI \neq 0$. So B/BI is a module from category $\mathcal{O}_c(W, \mathfrak{h})_0$ which is aspherical. Hence c is aspherical.

(ii) By Lemma 3.1 and Theorem 3.2 if $c \notin \Sigma(W, \mathfrak{h})$ then

$$H_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'})e_{W'}H_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'}) = H_{c'}(W', \mathfrak{h}/\mathfrak{h}^{W'}),$$

which by (i) implies that c' is not aspherical.

Thus, $\Sigma(W, \mathfrak{h})$ contains the union $FDA(W, \mathfrak{h}) \cup \bigcup_{W' \in \text{Par}(W)} \Sigma'(W', \mathfrak{h}/\mathfrak{h}^{W'})$. It remains to show that it is also contained in this union. To this end, let $c \in \Sigma(W, \mathfrak{h})$. Then there exists a module $M \neq 0$ from category $\mathcal{O}_c(W, \mathfrak{h})_0$ such that $e_W M = 0$. If M is finite dimensional, then $c \in FDA(W, \mathfrak{h})$, and we are done. Otherwise, M must have a nonzero support in \mathfrak{h} . Let $b \in \mathfrak{h}$ be a nonzero point of this support, and $M_b = \text{Res}_b(M)$. This is a module from category $\mathcal{O}_{c'}(W_b, \mathfrak{h}/\mathfrak{h}^{W_b})$, which is killed by e_{W_b} . Thus, $c' \in \Sigma(W_b, \mathfrak{h}/\mathfrak{h}^{W_b})$, and $c \in \Sigma'(W_b, \mathfrak{h}/\mathfrak{h}^{W_b})$, as desired. \square

Corollary 4.2. *If $W = S_n$ and \mathfrak{h} its reflection representation, then $\Sigma(W, \mathfrak{h})$ is the set Q_n of rational numbers in $(-1, 0)$ with denominator $\leq n$.*

This is a slight strengthening of the result of Gordon and Stafford [GS] who proved that $\Sigma(S_n, \mathfrak{h}) \setminus Q_n$ is a (finite) set contained in $\frac{1}{2} + \mathbb{Z}$. It was proved earlier in [DJO], Theorem 4.9, that $\Sigma(S_n, \mathfrak{h}) \supset Q_n$.

Proof. It follows from the results of [BEG2] that

$$FDA(S_n, \mathfrak{h}) = \{r/n \mid -n < r < 0, \text{GCD}(n, r) = 1\}.$$

Thus the result follows from Theorem 4.1 immediately by induction in n . \square

Recall ([BEG1]) that we have translation (or shift) functors

$$F : H_c(S_n, \mathfrak{h})\text{-mod} \rightarrow H_{c+1}(S_n, \mathfrak{h})\text{-mod}, F_* : H_{c+1}(S_n, \mathfrak{h})\text{-mod} \rightarrow H_c(S_n, \mathfrak{h})\text{-mod}$$

defined by the formulas

$$F(V) = H_{c+1}e_- \otimes_{e_- H_{c+1}e_- = e_+ H_c e_+} e_+ V, \quad F_*(V) = H_c e_+ \otimes_{e_+ H_c e_+ = e_- H_{c+1}e_-} e_- V,$$

where we use a shorthand notation $H_c := H_c(S_n, \mathfrak{h})$, and e_+, e_- are the symmetrizer and antisymmetrizer for S_n .

Corollary 4.3. *If $c \notin Q_n$ then the translation functor F is an equivalence of categories.*

This corollary was proved in [GS] for $c \notin \frac{1}{2} + \mathbb{Z}$.

Proof. We have $F_* F(V) = H_c e_+ V$, and $FF_*(U) = H_{c+1}e_- U$. There is an automorphism of H_c sending c to $-c$ and e_+ to e_- ; also Q_n is stable under the map $c \rightarrow -1 - c$. This implies that $F_* F(V) = V$, $FF_*(U) = U$, so F is an equivalence. \square

4.2. Aspherical values of c for real reflection groups. For a general W , the determination of the set $\Sigma(W, \mathfrak{h})$ is an interesting open problem. For instance, let W be a real reflection group, \mathfrak{h} its reflection representation, and c a constant function. Let us say that c is *strongly singular* if the module $L_c(W, \mathfrak{h}, \text{sign})$ is aspherical. It follows from [DJO], Theorem 4.9, that c is strongly singular if and only if $c = -j/d_i$, where $1 < j < d_i - 1$, and d_i are the degrees of the generators in $\mathbb{C}[\mathfrak{h}]^W$. Also, it is clear that any strongly singular c is aspherical. Thus, for any i, j as above, $-j/d_i \in \Sigma(W, \mathfrak{h})$. Finally, any aspherical value $c \in (-1, 0)$ is strongly singular, since for other $c \in (-1, 0)$, the category $\mathcal{O}_c(W, \mathfrak{h})_0$ is semisimple, [GGOR], and hence all simple objects have W -invariant vectors (as they coincide with the corresponding Verma modules).

However (contrary to what was conjectured in the previous version of this paper), an aspherical value of c need not belong to $(-1, 0)$ and may be positive. This was pointed out to us by M. Balagovic and A. Puranik, who found an aspherical representation for $c = 1/2$ and the Coxeter group of type H_3 . There are also the following examples for type B_n .

Example 4.4. Let $W = S_n \ltimes \mathbb{Z}_2^n$ be the Weyl group of type B_n , and \mathfrak{h} its reflection representation. It is shown in [EM] that if m, l are positive integers with $ml = n$, and $2c(m - l \pm 1) = 1$, then there exists a representation U of $H_c(W, \mathfrak{h})$ which is irreducible as a W -module. Namely, the action of \mathbb{Z}_2^n on U is by ± 1 (depending on the above choice of sign), and as a representation S_n , U is isomorphic to the irreducible representation π_λ , where λ is the rectangular Young diagram with m columns and n rows.

In particular, if $m - l \pm 1 > 0$ but $l > 1$, we get an aspherical representation with $c > 0$.

4.3. Aspherical representations for S_n . Let $W = S_n$, \mathfrak{h} be its reflection representation, and let $c = -r/m$, $2 \leq m \leq n$, $1 \leq r \leq m - 1$ (so $c \in (-1, 0)$).

Proposition 4.5. *An irreducible representation $L = L_c(\tau)$ of $H_c(W, \mathfrak{h})$ is aspherical if and only if its support is not equal to \mathfrak{h} .*

Proof. Suppose that the support of L is \mathfrak{h} . Then $L|_{\mathfrak{h}_{\text{reg}}} \neq 0$, so $(L|_{\mathfrak{h}_{\text{reg}}})^W \neq 0$, and hence $L^W \neq 0$, so L is not aspherical.

Conversely, suppose the support of L is $X \neq \mathfrak{h}$, i.e. X/W is an irreducible subvariety of \mathfrak{h}/W . Let $b \in X$ be a generic point. In this case, as we have seen, $\text{Res}_b(L)$ is a finite dimensional representation of $H_c(W_b)$. Since $-1 < c < 0$, and W_b is a product of symmetric groups, we see that $\text{Res}_b(L)$ is aspherical.

Let $X' \subset X$ be the open set of points with stabilizer conjugate to W_b . Because $\text{Res}_b(L)$ is aspherical, we have $(L|_{X'})^W = 0$. But since L is irreducible, the map $L \rightarrow L|_{X'}$ is injective, so $L^W = 0$, and L is aspherical. \square

Corollary 4.6. *For $-1 < c < 0$, the category $\mathcal{O}_{\text{spherical}}$ for the spherical subalgebra $e_W H_c(W, \mathfrak{h}) e_W$ is equivalent to the category of finite dimensional representations of the Hecke algebra $\mathcal{H}_q(W)$, where $q = e^{2\pi ic}$.*

Proof. According to Proposition 4.5 and the paper [GGOR], both categories are equivalent to $\mathcal{O}/\mathcal{O}_{\text{tor}}$, where \mathcal{O}_{tor} is the Serre subcategory of objects which are torsion as modules over $\mathcal{C}[\mathfrak{h}]$. \square

Corollary 4.7. *For $c = -r/m$ as above, $L_c(\lambda)$ is aspherical if and only if the corresponding partition λ is not m -regular, i.e., if it contains some part at least m times.*

Proof. Let $q = e^{2\pi ic}$, a primitive m -th root of unity. Recall from [DJ] that for every partition λ we have the Specht module S_λ over the Hecke algebra $\mathcal{H}_q := \mathcal{H}_q(S_n)$ and its quotient D_λ , which is either simple (if λ is m -regular) or zero (if not), and this gives an enumeration, without repetitions, of irreducible representations of \mathcal{H}_q . Moreover, it is known ([DJ], theorem 7.6) that all the composition factors of S_λ are D_μ with $\mu \geq \lambda$ (in the dominance ordering), and the multiplicity of D_λ in S_λ (when D_λ is nonzero) is 1.

Let us say that a simple object L of $\mathcal{O}_c(W, \mathfrak{h})$ is thin if $KZ(L) = 0$, otherwise let us say that it is thick. By Proposition 4.5, $L_c(\lambda)$ is aspherical if and only if it is thin.

Our job is to show that $L_c(\lambda)$ is thick iff λ is m -regular, and in this case $KZ(L_c(\lambda)) = D_\lambda$. This follows from the paper [Rou] (Section 5), but we give a proof here for reader's convenience.

Let $N(\lambda) := \frac{n(n-1)}{2} - c(\lambda)$, where $c(\lambda)$ is the content of λ . Note that if $\nu > \lambda$ then $N(\nu) < N(\lambda)$. We prove that the statement holds for $N(\lambda) \leq k$, by induction in k .

If $k = 0$ then $\lambda = (n)$ and the statement is clear. Now suppose the statement is known for $k - 1$ and let us prove it for k .

By [GGOR], Theorem 5.14, the KZ functor is exact and maps a simple object either to zero or to a simple object, so for any μ , $KZ(L_c(\mu)) = 0$ if $L_c(\mu)$ is thin, and $KZ(L_c(\mu)) = D_{\nu(\mu)}$ for some $\nu = \nu(\mu)$ if $L_c(\mu)$ is thick. Also, by [GGOR], Corollary 6.10, $KZ(M_c(\mu)) = S_\mu$. This means that $\nu(\mu) \geq \mu$ for all μ .

Let λ be such that $N(\lambda) = k$. If $L_c(\lambda)$ is thin then by the above argument, $KZ(M_c(\lambda))$ has composition factors D_μ with $\mu > \lambda$. Since $KZ(M_c(\lambda)) = S_\lambda$, this implies that S_λ has composition factors D_μ with $\mu > \lambda$. By Theorem 7.6 of [DJ], this implies that λ is not m -regular. On the other hand, if $L_c(\lambda)$ is thick, then $\nu(\lambda)$ is m -regular, and by the induction assumption, if $\nu(\lambda) > \lambda$ then $D_{\nu(\lambda)}$ also equals $KZ(L_c(\nu(\lambda)))$, so two irreducible modules have the same nonzero image under the KZ functor, which contradicts Theorem 5.14 of [GGOR]. Thus, $\nu(\lambda) = \lambda$, and λ is m -regular. This completes the induction step. \square

Remark 4.8. Note that it is well known (and easy to see) that the generating function for the number of m -regular partitions is

$$f_m(q) = \frac{\phi(q^m)}{\phi(q)},$$

where ϕ is the Euler function,

$$\phi(q) = \prod_{n \geq 1} (1 - q^n).$$

Remark 4.9. A. Okounkov and the first author conjectured that the number of aspherical representations in $\mathcal{O}_c(S_n, \mathfrak{h})_0$ for each n is given by the rank of the residue of the connection describing the equivariant small quantum cohomology of the Hilbert scheme of \mathbb{C}^2 at $q = -e^{2\pi ic}$ ([OP]). According to [OP], this residue is proportional to the operator

$$\sum_{s \geq 1} \alpha_{-ms} \alpha_{ms}$$

on the degree n part of the Fock representation of the Heisenberg Lie algebra, with commutation relations $[\alpha_i, \alpha_j] = \delta_{i,-j}$. Thus, the conjecture follows from Corollary 4.7. Indeed, by Corollary 4.7 and the previous remark, the conjecture is equivalent to saying that the kernel of this operator has character $f_m(q)$, which is obvious, since this kernel is the space of polynomials of α_{-i} , $i \geq 1$, such that i is not divisible by m .

One can also observe that the eigenvalues of this residue are proportional to the codimensions of supports of the modules in $\mathcal{O}_c(S_n, \mathfrak{h})_0$.

4.4. The simplicity of the spherical subalgebra for $-1 < c < 0$ in type A. In [BEG1], it is shown that the algebra $H_c(W, \mathfrak{h})$ is simple if and only if c is not a singular value, and in this case $H_c(W, \mathfrak{h})$ is Morita equivalent to its spherical subalgebra $e_W H_c(W, \mathfrak{h}) e_W$. This implies that if c is not singular, the spherical subalgebra is simple, while if c is singular and $H_c(W, \mathfrak{h})$ is Morita equivalent to $e_W H_c(W, \mathfrak{h}) e_W$, it is not. However, it turns out that when H_c and $e_W H_c e_W$ are not Morita equivalent, it can happen that $e_W H_c e_W$ is simple. Namely, we have the following result.

Theorem 4.10. *The spherical subalgebra $A_c(n) := e_{S_n} H_c(S_n, \mathfrak{h}) e_{S_n}$ is simple for $c \in (-1, 0)$.*

Proof. Let $I \subset A_c(n)$ be a proper two-sided ideal, and consider the $A_c(n)$ -bimodule $M := A_c(n)/I \neq 0$. Let us regard M as a module over $\mathbb{C}[x_1, \dots, x_n]^{S_n} \otimes \mathbb{C}[y_1, \dots, y_n]^{S_n}$ by acting with the first factor on the left side and with the second one on the right side. Obviously, M is finitely generated. Let $Z \subset (\mathbb{C}^n/S_n)^2$ be the support of M . Then Z is a nonempty closed subvariety of $(\mathbb{C}^n/S_n)^2$. Let $p_1, p_2 : Z \rightarrow \mathbb{C}^n/S_n$ be the two projections. Let $b \in \mathbb{C}^n$ be a point such that $p_2^{-1}(S_n b)$ is nonempty. Then the fiber M_b^3 of M is a nonzero left $A_c(n)$ -module, finitely generated over $\mathbb{C}[x_1, \dots, x_n]^{S_n}$, on which symmetric polynomials of y_i act locally finitely, with generalized eigenvalue b . So it belongs to the category $e_{S_n} \mathcal{O}_c(S_n, \mathfrak{h})_\lambda$. By Corollary 3.3 and Proposition 4.5, this implies that the support of M_b as a $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ -module is the entire \mathbb{C}^n/S_n . Thus, $p_2^{-1}(b) = \mathbb{C}^n/S_n$ if it

³We abuse notation by denoting the orbit of b under S_n also by b .

is nonempty. Similarly one proves that $p_1^{-1}(b) = \mathbb{C}^n/S_n$ if it is nonempty. Thus, we find that $Z = (\mathbb{C}^n/S_n)^2$, which implies that $I = 0$. \square

4.5. A strengthening of the result of [BFG]. In this subsection we apply Corollary 4.7 to enhance the main result of [BFG].

4.5.1. *A modification of a result of [GG].* We start with a slightly modified version of Theorem 6.6.1 of [GG]. Let K be a field of characteristic zero, $\mathfrak{g} = \mathfrak{gl}_n(K)$, and $D(\mathfrak{g})$ the algebra of differential operators on \mathfrak{g} . Let c be an indeterminate, and V_c be the representation of \mathfrak{g} on the space of “functions” of the form $(x_1 \dots x_n)^c f(x_1, \dots, x_n)$, where f is a Laurent polynomial with coefficients in $K[c]$ of total degree zero (namely, \mathfrak{g} acts through its projection to $\mathfrak{sl}_n(K)$). Let $\text{Ann}V_c$ be the annihilator of V_c in $U(\mathfrak{g})[c]$, and $J_c = D(\mathfrak{g})[c]\text{ad}(\text{Ann}V_c)$, where $\text{ad} : U(\mathfrak{g}) \rightarrow D(\mathfrak{g})$ is the adjoint action. It is shown in [EG] that one has a filtration preserving homomorphism

$$\Phi_c : (D(\mathfrak{g})/D(\mathfrak{g})J_c)^{\mathfrak{g}} \rightarrow A_c(n),$$

where $A_c(n)$ is the spherical rational Cherednik algebra with parameter c being an indeterminate (here $D(\mathfrak{g})$ is filtered by order of differential operators, and $A_c(n)$ inherits the filtration from the full Cherednik algebra $H_c(n)$, which is filtered by $\deg(\mathfrak{h}^*) = \deg(S_n) = 0$, $\deg(\mathfrak{h}) = 1$).

Theorem 4.11. *The associated graded map $\text{gr}\Phi_c$, and hence Φ_c itself, are isomorphisms.*

The difference between Theorem 4.11 and Theorem 6.6.1 of [GG] is that in [GG], c is any fixed element of K , while here c is an indeterminate. However, this distinction is inessential, and the proof of Theorem 4.11 is parallel to the proof of Theorem 6.6.1 of [GG].

4.5.2. *The case of positive characteristic.* Now fix a positive integer n and let k be an algebraically closed field of sufficiently large (compared to n) prime characteristic p .

Let X denote the Hilbert scheme of n points on the plane \mathbb{A}_k^2 . As above, let $A_c = A_c(n)$ denote the spherical rational Cherednik algebra with parameter c .

For $c \in \mathbb{F}_p$ an Azumaya algebra \mathbb{A}_c of rank p^{2n} on $X^{(1)}$ was defined in [BFG]; here the superscript (1) denotes the Frobenius twist. Furthermore, one has the following strengthened version of the second statement of Theorem 7.2.1 of [BFG].

Theorem 4.12. *For $p \gg n$, the algebra of global sections $\Gamma(\mathbb{A}_c)$ is canonically isomorphic to A_c .*

Theorem 7.2.1 of [BFG] claims that this statement holds for (reduction mod p of) any rational c and $p > d = d(c)$, where $d(c)$ is a constant depending on c . The proof of Theorem 4.12 is similar to the proof of Theorem 7.2.1 of [BFG], using Theorem 4.11, which had not been known when [BFG] appeared.

Theorem 4.12 implies that we have the functor $R\Gamma : D^b(\text{Coh}(X^{(1)}, \mathbb{A}_c)) \rightarrow D^b(A_c - \text{mod}^{fg})$ where $\text{Coh}(X^{(1)}, \mathbb{A}_c)$ is the category of coherent sheaves of \mathbb{A}_c -modules, and $A_c - \text{mod}^{fg}$ is the category of finitely generated modules over A_c . We say that \mathbb{A}_c is *derived affine* if this functor is an equivalence.

Corollary 4.13. *For $p \gg n$ the Azumaya algebra \mathbb{A}_c is derived affine if and only if the inequality $c \neq -\frac{r}{m}$ holds in \mathbb{F}_p for all integers r, m such that $0 < r < m \leq n$.*

Proof. The results of [BFG] show that \mathbb{A}_c is derived affine if and only if A_c is Morita equivalent to the full rational Cherednik algebra H_c , i.e. if and only if $H_c e H_c = H_c$, where $e = e_{S_n}$. Corollary 4.7 implies that over a characteristic zero field the last equality holds exactly when $c \neq -\frac{r}{m}$ for r, m as above. It follows that the same is true over a field of positive characteristic p for almost all p . \square

5. APPENDIX: REDUCIBILITY OF THE POLYNOMIAL REPRESENTATION OF
THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

Pavel Etingof

5.1. Introduction. In this appendix (which has been previously posted as arXiv:0706.4308) we determine the values of parameters c for which the polynomial representation of the degenerate double affine Hecke algebra (DAHA), i.e. the trigonometric Cherednik algebra, is reducible. Namely, we show that c is a reducibility point for the polynomial representation of the trigonometric Cherednik algebra for a root system R if and only if it is a reducibility point for the *rational* Cherednik algebra for the Weyl group of some root subsystem $R' \subset R$ of the same rank given by (one step of) the well known Borel-de Siebenthal algorithm, [BdS] (i.e., by deleting a vertex from the extended Dynkin diagram of R).⁴

This generalizes to the trigonometric case the result of [DJO], where the reducibility points are found for the rational Cherednik algebra. Together with the result of [DJO], our result gives an explicit list of reducibility points in the trigonometric case.

We emphasize that our result is contained in the recent previous work of I. Cherednik [Ch2], where reducibility points are determined for nondegenerate DAHA. Namely, the techniques of [Ch2], based on intertwiners, work equally well in the degenerate case. In fact, outside of roots of unity, the questions of reducibility of the polynomial representation for the degenerate and nondegenerate DAHA are equivalent (see e.g. [VV], 2.2.4), and thus our result is equivalent to that of [Ch2]. However, our proof is quite different from that in [Ch2]; it is based on the geometric approach to Cherednik algebras developed in [E2], and thus clarifies the results of [Ch2] from a geometric point of view. In particular, we explain that our result and its proof can be generalized to the much more general setting of Cherednik algebras for any smooth variety with a group action.

We note that in the non-simply laced case, it is not true that the reducibility points for R are the same in the trigonometric and rational settings. In the trigonometric setting, one gets additional reducibility points, which arise for type B_n , $n \geq 3$, F_4 , and G_2 , but not for C_n . This phenomenon was discovered by Cherednik (in the B_n case, see [Ch3], Section 5); in [Ch2], he gives a complete list of additional reducibility points. At first sight, this list looks somewhat mysterious; here we demystify it, by interpreting it in terms of the Borel - de Siebenthal classification of equal rank embeddings of root systems.

⁴It is known from the Borel-de Siebenthal theory that any maximal rank root subsystem is obtained by repeating this process several times; however, the root subsystems R' appearing in this appendix are the ones obtained by just one step of the process; clearly, this contains all the maximal proper root subsystems, which correspond to the case where the label of the removed vertex is a prime number.

The result of this appendix is a manifestation of the general principle that the representation theory of the trigonometric Cherednik algebra (degenerate DAHA) for a root system R reduces to the representation theory of the *rational* Cherednik algebra for Weyl groups of root subsystems $R' \subset R$ obtained by the Borel-de Siebenthal algorithm. This principle is the “double” analog of a similar principle in the representation theory of affine Hecke algebras, which goes back to the work of Lusztig [L], in which it is shown that irreducible representations of the affine Hecke algebra of a root system R may be described in terms of irreducible representations of the degenerate affine Hecke algebras for Weyl groups of root subsystems $R' \subset R$ obtained by the Borel - de Siebenthal algorithm. We illustrate this principle at the end of the note by applying it to finite dimensional representations of trigonometric Cherednik algebras.

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5.2. Preliminaries.

5.2.1. *Preliminaries on root systems.* Let W be an irreducible Weyl group, \mathfrak{h} its (complex) reflection representation, and $L \subset \mathfrak{h}$ a \mathbb{Z} -lattice invariant under W .

For each reflection $s \in W$, let L_s be the intersection of L with the -1 -eigenspace of s in \mathfrak{h} , and let α_s^\vee be a generator of L_s . Let α_s be the element in \mathfrak{h}^* such that $s\alpha_s = -\alpha_s$, and $(\alpha_s, \alpha_s^\vee) = 2$. Then we have

$$s(x) = x - (x, \alpha_s^\vee)\alpha_s, \quad x \in \mathfrak{h}.$$

Let $R \subset \mathfrak{h}^*$ be the collection of vectors $\pm\alpha_s$, and $R^\vee \subset \mathfrak{h}$ the collection of vectors $\pm\alpha_s^\vee$. It is well known that R, R^\vee are mutually dual reduced root systems. Moreover, we have $Q^\vee \subset L \subset P^\vee$, where P^\vee is the coweight lattice, and Q^\vee the coroot lattice.

Consider the simple complex Lie group G with root system R , whose center is P^\vee/L . The maximal torus of G can be identified with $H = \mathfrak{h}/2\pi iL$ via the exponential map.

For $g \in H$, let $C_{\mathfrak{g}}(g)$ be the centralizer of g in $\mathfrak{g} := \text{Lie}(G)$. Then $C_{\mathfrak{g}}(g)$ is a reductive subalgebra of \mathfrak{g} containing \mathfrak{h} , and its Weyl group is the stabilizer W_g of g in W .

Let $\Sigma \subset H$ be the set of elements whose centralizer $C_{\mathfrak{g}}(g)$ is semisimple (of the same rank as \mathfrak{g}). Σ can also be defined as the set of point strata for the stratification of H with respect to stabilizers. It is well known that the set Σ is finite, and the Dynkin diagram of $C_{\mathfrak{g}}(g)$ is obtained from the extended Dynkin diagram of \mathfrak{g} by deleting one vertex (the Borel-de Siebenthal algorithm). Moreover, any Dynkin diagram obtained in this way corresponds to $C_{\mathfrak{g}}(g)$ for some g .

5.2.2. *The degenerate DAHA.* Let W, L, H be as in subsection 5.2.1. A *reflection hypertorus* in H is a connected component T of the fixed set H^s for a reflection $s \in W$. Let c be a conjugation invariant function on the set of reflection hypertori. Denote by \mathcal{S} the set of reflection hypertori. For $T \in \mathcal{S}$, denote by s_T the corresponding reflection, and by χ_T the affine linear map $H \rightarrow \mathbb{C}^*$ such that $\chi_T^{-1}(1) = T$. Let H_{reg} denote the complement of reflection hypertori in H .

Definition 5.1. (Cherednik, [Ch1]) The degenerate DAHA $H_c(W, H)$ attached to W, H is the algebra generated inside $\mathbb{C}[W] \rtimes D(H_{reg})$ by polynomial functions on H , the group W , and trigonometric Dunkl operators

$$\partial_a + \sum_T c(T) \frac{d\chi_T(a)}{1 - \chi_T} (s_T - 1),$$

Using the geometric approach of [E2], which attaches a Cherednik algebra to any smooth affine algebraic variety with a finite group action, the degenerate DAHA can also be defined as the Cherednik algebra $H_{1,c}(W, H)$ attached to the variety H with the action of the finite group W .

Note that this setting includes the case of non-reduced root systems. Namely, in the case of a non-reduced root system the function c may take nonzero values on reflection hypertori which don't go through $1 \in H$.

5.3. The results.

5.3.1. *The main results.* The degenerate DAHA has a polynomial representation $M = \mathbb{C}[H]$ on the space of regular functions on H . We would like to determine for which c this representation is reducible.

Let $g \in \Sigma$. Denote by c_g the restriction of the function c to reflections in W_g ; that is, for $s \in W_g$, $c_g(s)$ is the value of c on the (unique) hypertorus $T_{g,s}$ passing through g and fixed by s .

Remark 5.2. If $c(T) = 0$ unless T contains $1 \in H$ (“the reduced case”), then c can be regarded as a function of reflections in W , and c_g is the usual restriction of c to reflections in W_g .

Denote by $\text{Red}(W, \mathfrak{h})$ the set of c at which the polynomial representation $M_c(W, \mathfrak{h}, \mathbb{C})$ of the rational Cherednik algebra $H_c(W, \mathfrak{h})$ is reducible. These sets are determined explicitly in [DJO]. Denote by $\text{Red}_g(W, L)$ the set of c such that $c_g \in \text{Red}(W_g, \mathfrak{h})$.

Our main result is the following.

Theorem 5.3. *The polynomial representation M of $H_c(W, H)$ is reducible if and only if $c \in \cup_{g \in \Sigma} \text{Red}_g(W, L)$.*

The proof of this theorem is given in the next subsection.

Corollary 5.4. *If c is a constant function (in particular, if R is simply laced), then the polynomial representation M of $H_c(W, H)$ is reducible if and only if so is the polynomial representation of the rational Cherednik*

algebra $H_c(W, \mathfrak{h})$, i.e. iff $c = j/d_i$, where d_i is a degree of W , and j is a positive integer not divisible by d_i .

Proof. The result follows from Theorem 5.3, the result of [DJO], and the well known fact⁵ that for any subgroup $W' \subset W$ generated by reflections, every degree of W' divides some degree of W . \square

However, if c is not a constant function, the answer in the trigonometric case may differ from the rational case, as explained below.

5.3.2. *Proof of Theorem 5.3.* Assume first that the polynomial representation M is reducible. Then there exists a nonzero proper submodule $I \subset M$, which is an ideal in $\mathbb{C}[H]$. This ideal defines a subvariety $Z \subset H$, which is W -invariant; it is the support of the module M/I . It is easy to show using the results of [E2] (parallel to Proposition 3.21 of the present paper) that Z is a union of strata of the stratification of H with respect to stabilizers. In particular, since Z is closed, it contains a stratum which consists of one point g . Thus $g \in \Sigma$. Consider the formal completion \widehat{M}_g of M at g . As follows from [E2] (parallel to Section 3 of the present paper), this module can be viewed as a module over the formal completion $\widehat{H}_{c_g}(W_g, \mathfrak{h})_0$ of the rational Cherednik algebra of the group W_g at 0, and it has a nonzero proper submodule \widehat{I}_g . Thus, \widehat{M}_g is reducible, which implies (by taking nilpotent vectors under \mathfrak{h}^*) that the polynomial representation \bar{M} over $H_{c_g}(W_g, \mathfrak{h})$ is reducible, hence $c_g \in \text{Red}(W_g, \mathfrak{h})$, and $c \in \text{Red}_g(W, L)$.

Conversely, assume that $c \in \text{Red}_g(W, L)$, and thus $c_g \in \text{Red}(W_g, \mathfrak{h})$. Then the polynomial representation \bar{M} of $H_{c_g}(W_g, \mathfrak{h})$ is reducible. This implies that the completion $\widehat{M}_g = \widehat{\mathbb{C}[H]}_g$ is a reducible module over $\widehat{H}_{c_g}(W_g, \mathfrak{h})_0$, i.e. it contains a nonzero proper submodule (=ideal) J . Let $I \subset \mathbb{C}[H]$ be the intersection of $\mathbb{C}[H]$ with J . Clearly, $I \subset M$ is a proper submodule (it does not contain 1). So it remains to show that it is nonzero. To do so, denote by Δ a regular function on H which has simple zeros on all the reflection hyperplanes. Then clearly $\Delta^n \in J$ for large enough n , so $\Delta^n \in I$. Thus $I \neq 0$ and the theorem is proved.

5.3.3. *Reducibility points in the non-simply laced case.* In this subsection we will consider the reduced non-simply laced case, i.e. the case of root systems of type B_n, C_n, F_4 , and G_2 . In this case, c is determined by two numbers k_1 and k_2 , the values of c on reflections for long and short roots, respectively.

⁵This fact is proved as follows. Let $P_W(t)$ be the Poincaré polynomial of W ; so

$$P_W(t) = \prod_i \frac{1 - t^{d_i(W)}}{1 - t},$$

where $d_i(W)$ are the degrees of W . Then by Chevalley's theorem, $P_W(t)/P_{W'}(t)$ is a polynomial (the Hilbert polynomial of the generators of the free module $\mathbb{C}[\mathfrak{h}]^{W'}$ over $\mathbb{C}[\mathfrak{h}]^W$). So, since the denominator vanishes at a root of unity of degree $d_i(W')$, so does the numerator, which implies the statement.

The set $\text{Red}(W, \mathfrak{h})$ is determined for these cases in [DJO], as the union of the following lines (where $l \geq 1$, $u = k_1 + k_2$, and $i = 1, 2$).

$B_n = C_n$:

$$2jk_1 + 2k_2 = l, \quad l \neq 0 \pmod{2}, \quad j = 0, \dots, n-1,$$

and

$$jk_1 = l, \quad (l, j) = 1, \quad j = 2, \dots, n.$$

F_4 :

$$\begin{aligned} 2k_i = l, \quad 2k_i + 2u = l, \quad l \neq 0 \pmod{2}; \quad 3k_i = l, \quad l \neq 0 \pmod{3}; \\ 2u = l, \quad 4u = l, \quad l \neq 0 \pmod{2}; \\ 6u = l, \quad l = 1, 5, 7, 11 \pmod{12}. \end{aligned}$$

G_2 :

$$2k_i = l, \quad l \neq 0 \pmod{2}; \quad 3u = l, \quad l \neq 0 \pmod{3}.$$

By using Theorem 5.3, we determine that the polynomial representation in the trigonometric case is reducible on these lines and also on the following additional lines:

B_n , $n \geq 3$:

$$(2p-1)k_1 = 2q, \quad n/2 < q \leq n-1, \quad p \geq 1, \quad (2p-1, q) = 1.$$

F_4 :

$$6k_1 + 2k_2 = l, \quad 4k_1 = l, \quad l \neq 0 \pmod{2}.$$

G_2 :

$$3k_1 = l, \quad l \neq 0 \pmod{3}.$$

In the C_n case, we get no additional lines.

Note that exactly the same list of additional reducibility points appears in [Ch2].

Remark 5.5. As explained above, the additional lines appear from particular equal rank embeddings of root systems. Namely, the additional lines for B_n appear from the inclusion $D_n \subset B_n$. The two series of additional lines for F_4 appear from the embeddings $B_4 \subset F_4$ and $A_3 \times A_1 \subset F_4$, respectively. Finally, the additional lines for G_2 appear from the embedding $A_2 \subset G_2$.

5.3.4. *Generalizations.* Theorem 5.3 can be generalized, with essentially the same proof, to the setting of any smooth variety with a group action, as defined in [E2].

Namely, let X be a smooth algebraic variety, and G a finite group acting faithfully on X . Let c be a conjugation invariant function on the set of pairs (g, Y) , where $g \in G$, and Y is a connected component of X^g which has codimension 1 in X . Let $H_{1,c,0,X,G}$ be the corresponding sheaf of Cherednik algebras defined in [E2]. We have the polynomial representation \mathcal{O}_X of this sheaf.

Let $\Sigma \in X$ be the set of points with maximal stabilizer, i.e. points whose stabilizer is bigger than that of nearby points. Then Σ is a finite set. For $x \in X$, let G_x be the stabilizer of x in G ; it is a finite subgroup of $GL(T_x X)$.

Let c_x be the function of reflections in G_x defined by $c_x(g) = c(g, Y)$, where Y is the reflection hypersurface passing through x and fixed by g pointwise. Let $\text{Red}_x(G, X)$ be the set of c such that $c_x \in \text{Red}(G_x, T_x X)$ (where, as before, $\text{Red}(G_x, T_x X)$ denotes the set of values of parameters c for which the polynomial representation of the rational Cherednik algebra $H_c(G_x, T_x X)$ is reducible).

Then we have the following theorem, whose statement and proof are direct generalizations of those of Theorem 5.3 (which is obtained when G is a Weyl group and X a torus).

Theorem 5.6. *The polynomial representation \mathcal{O}_X of $H_{1,c,0,X,G}$ is reducible if and only if $c \in \cup_{x \in \Sigma} \text{Red}_x(G, X)$.*

Note that this result generalizes in a straightforward way to the case when X is a complex analytic manifold, and G a discrete group of holomorphic transformations of X .

5.4. Finite dimensional representations of the degenerate double affine Hecke algebra. ⁶

Another application of the approach of this appendix is a description of the category of finite dimensional representations of the degenerate DAHA in terms of categories of finite dimensional representations of rational Cherednik algebras. Namely, let $FD(A)$ denote the category of finite dimensional representations of an algebra (or sheaf of algebras) A . Then in the setting of the previous subsection we have the following theorem (see also Proposition 2.22 of [E2]).

Let Σ' be a set of representatives of Σ/G in Σ .

Theorem 5.7. *One has*

$$FD(H_{1,c,0,X,G}) = \oplus_{x \in \Sigma'} FD(H_{c_x}(G_x, T_x X)).$$

Proof. Suppose V is a finite dimensional representation of $H_{1,c,0,X,G}$. Then the support of V is a union of finitely many points, and these points must be strata of the stratification of X with respect to stabilizers, so they belong to Σ . This implies that $V = \oplus_{\xi \in \Sigma/G} V_\xi$, where V_ξ is supported on the orbit ξ . Taking completion of the Cherednik algebra at ξ , we can regard the fiber $(V_\xi)_x$ for $x \in \xi$ as a module over the rational Cherednik algebra $H_{c_x}(G_x, T_x X)$ (as follows from [E2] and the main results of the present paper). In this way, V gives rise to an object of $\oplus_{x \in \Sigma'} FD(H_{c_x}(G_x, T_x X))$.

This procedure can be reversed; this implies the theorem. \square

Corollary 5.8. *One has*

$$FD(H_c(W, H)) = \oplus_{g \in \Sigma/W} FD(H_{c_g}(W_g, \mathfrak{h})).$$

⁶The contents of this subsection arose from a discussion of the author with M. Varagnolo and E. Vasserot.

Remark 5.9. Recall that a representation of $H_c(W, H)$ is said to be spherical if it is a quotient of the polynomial representation. It is clear that the categorical equivalence of Corollary 5.8 preserves sphericity of representations (in both directions). This implies that the results of the paper [VV], which classifies spherical finite-dimensional representations of the rational Cherednik algebras, in fact yield, through Corollary 5.8, the classification of spherical finite dimensional representations of degenerate DAHA, and hence of nondgenerate DAHA outside of roots of unity. We note that the general classification of finite dimensional representations of Cherednik algebras outside of type A remains an open problem.

REFERENCES

- [BdS] A. Borel and J. de Siebenthal, Les sous-groupes fermés de rang maximum des groupes de Lie clos. *Comment Math. Helv.*, 23 (1949), 200-221.
- [BEG1] Yu. Berest, P. Etingof, V. Ginzburg: Cherednik algebras and differential operators on quasi-invariants. *Duke Math J.* 118 (2003), 279–337, math.QA/0111005.
- [BEG2] Yu. Berest, P. Etingof, V. Ginzburg, Finite dimensional representations of rational Cherednik algebras, *Int. Math. Res. Not.* 2003, no. 19, 1053-1088.
- [BFG] Bezrukavnikov, R., Finkelberg, M.; Ginzburg, V., Cherednik algebras and Hilbert schemes in characteristic p . With an appendix by Pavel Etingof. *Represent. Theory* 10 (2006), 254–298.
- [Ch1] I. Cherednik, Double affine Hecke algebras, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, Cambridge, 2005.
- [Ch2] I. Cherednik, Non-semisimple Macdonald polynomials, arXiv:0709.1742.
- [Ch3] Cherednik, I., Irreducibility of perfect representations of double affine Hecke algebras. *Studies in Lie theory*, 79–95, *Progr. Math.*, 243, Birkhäuser Boston, Boston, MA, 2006.
- [DJO] C.F. Dunkl, M.F.E. de Jeu and E.M. Opdam, Singular polynomials for finite reflection groups, *Trans. Amer. Math. Soc.* 346 (1994), 237-256.
- [DJ] R. Dipper and G. D. James, Representations of Hecke algebras of the general linear groups, *Proc. London Math. Soc.* 52 (1986), 20-52.
- [E1] P. Etingof, Calogero-Moser systems and representation theory, Zurich lectures in advanced mathematics, European Mathematical Society, Zurich, 2007, arXiv:math/0606233.
- [E2] P. Etingof, Cherednik and Hecke algebras of varieties with a finite group action, math.QA/0406499.
- [EG] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.* 147 (2002), no. 2, 243–348.
- [Gi] V. Ginzburg, On primitive ideals. *Selecta Math. (N.S.)* 9 (2003), no. 3, 379–407.
- [GG] W. L. Gan, V. Ginzburg, Almost-Commuting Variety, D-Modules, and Cherednik Algebras, *International Math Research Papers*, 2006, p.1-54, math/0409262.
- [GGOR] V. Ginzburg, N. Guay, E. Opdam, R. Rouquier, On the category \mathcal{O} for rational Cherednik algebras. *Invent. Math.* 154 (2003), no. 3, 617–651.
- [GP] M. Geck, G. Pfeiffer, Characters of finite Coxeter groups and Iwahori-Hecke algebras, London Mathematical Society Monographs. New Series, 21. The Clarendon Press, Oxford University Press, New York, 2000.
- [GS] I. Gordon, J. T. Stafford, *Rational Cherednik algebras and Hilbert schemes*, *Adv. Math.* **198** (2005), no. 1, 222–274.
- [Gy] A. Gyoja, Modular representation theory over a ring of higher dimension with applications to Hecke algebras. *J. Algebra* 174 (1995), no. 2, 553–572.

- [L] G. Lusztig, Affine Hecke algebras and their graded version. J. Amer. Math. Soc. 2 (1989), no. 3, 599–635.
- [EM] P. Etingof; S. Montarani, Finite dimensional representations of symplectic reflection algebras associated to wreath products, Represent. Theory 9 (2005), 457-467.
- [MS] D. Milicic, W. Soergel, The composition series of modules induced from Whittaker modules. Comment. Math. Helv. 72 (1997), no. 4, 503–520.
- [OP] A. Okounkov, R. Pandharipande, Quantum cohomology of the Hilbert scheme of points in the plane, math/0411210.
- [Rou] R. Rouquier, q-Schur algebras and complex reflection groups, I, math/0509252.
- [VV] M. Varagnolo, E. Vasserot, Finite dimensional representations of DAHA and affine Springers fibers : the spherical case, arXiv:0705.2691

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