THE DECOMPOSITION GROUP OF A LINE IN THE PLANE

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Abstract. We show that the decomposition group of a line L in the plane, i.e. the subgroup of plane birational transformations that send L to itself birationally, is generated by its elements of degree 1 and one element of degree 2, and that it does not decompose as a non-trivial amalgamated product.

1. Introduction

We denote by $\mathrm{Bir}(\mathbb{P}^2)$ the group of birational transformations of the projective plane $\mathbb{P}^2 = \mathrm{Proj}(k[x, y, z])$, where k is an algebraically closed field. Let $C \subset \mathbb{P}^2$ be a curve, and let

$$\mathrm{Dec}(C) = \{\varphi \in \mathrm{Bir}(\mathbb{P}^2),\ \varphi(C) \subset C \text{ and } \varphi|_C : C \dashrightarrow C \text{ is birational}\}.$$

This group has been studied for curves of genus ≥ 1 in [BPV2009], where it is linked to the classification of finite subgroups of $\mathrm{Bir}(\mathbb{P}^2)$. It has a natural subgroup $\mathrm{Ine}(C)$, the *inertia group* of C, consisting of elements that fix C, and Blanc, Pan and Vust give the following result: for any line $L \subset \mathbb{P}^2$, the action of $\mathrm{Dec}(L)$ on L induces a split exact sequence

$$0 \longrightarrow \mathrm{Ine}(L) \longrightarrow \mathrm{Dec}(L) \longrightarrow \mathrm{PGL}_2 = \mathrm{Aut}(L) \longrightarrow 0$$

and $\mathrm{Ine}(L)$ is neither finite nor abelian and also it doesn't leave any pencil of rational curves invariant [BPV2009, Proposition 4.1]. Further they ask the question whether $\mathrm{Dec}(L)$ is generated by its elements of degree 1 and 2 [BPV2009, Question 4.1.2].

We give an affirmative answer to their question in the form of the following result, similar to the Noether-Castelnuovo theorem [Cas1901] which states that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\sigma\colon [x : y : z] \mapstochar\dashrightarrow [yz : xz : xy]$ and $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$.

Theorem 1. *For any line $L \subset \mathbb{P}^2$, the group $\mathrm{Dec}(L)$ is generated by $\mathrm{Dec}(L) \cap \mathrm{PGL}_3$ and any of its quadratic elements having three proper base points in $\mathbb{P}^2$.*

The similarities between $\mathrm{Dec}(L)$ and $\mathrm{Bir}(\mathbb{P}^2)$ go further than this. Cornulier shows in [Cor2013] that $\mathrm{Bir}(\mathbb{P}^2)$ cannot be written as an amalgamated product in any nontrivial way, and we modify his proof to obtain an analogous result for $\mathrm{Dec}(L)$.

Theorem 2. *The decomposition group $\mathrm{Dec}(L)$ of a line $L \subset \mathbb{P}^2$ does not decompose as a non-trivial amalgam.*

The article is organised as follows: in Section 2 we show that for any element of $\mathrm{Dec}(L)$ we can find a decomposition in $\mathrm{Bir}(\mathbb{P}^2)$ into quadratic maps such that the successive images of L are curves (Proposition 2.6), i.e. the line is not contracted to a point at any

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time. We then show in Section 3 that we can modify this decomposition, still in $\mathrm{Bir}(\mathbb{P}^2)$, into de Jonquières maps where all of the successive images of L have degree 1, i.e. they are lines. Finally we prove Theorem 1. Our main sources of inspiration for techniques and ideas in Section 3 have been [AC2002, §8.4, §8.5] and [Bla2012]. In Section 4 we prove Theorem 2 using ideas that are strongly inspired by [Cor2013].

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2. Avoiding to contract L

Given a birational map $\rho\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, the Noether-Castelnuovo theorem states that there is a decomposition $\rho = \rho_m\rho_{m-1}\dots\rho_1$ of ρ where each ρ_i is a quadratic map with three proper base points. This decomposition is far from unique, and the aim of this section is to show that if $\rho \in \mathrm{Dec}(L)$, we can choose the ρ_i so that none of the successive birational maps $(\rho_i\dots\rho_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2)_{i=1}^m$ contracts L to a point. This is Proposition 2.6.

Given a birational map $\varphi : X \dashrightarrow Y$ between smooth projective surfaces, and a curve $C \subset X$ which is contracted by φ, we denote by $\pi_1\colon Z_1 \to Y$ the blowup of the point $\varphi(C) \in Y$. If C is contracted also by the birational map $\pi_1^{-1}\varphi\colon X \dashrightarrow Z_1$, we denote by $\pi_2\colon Z_2 \to Z_1$ the blowup of $(\pi_1^{-1}\varphi)(C) \in Z_1$ and consider the birational map $(\pi_1\pi_2)^{-1}\varphi\colon X \dashrightarrow Z_2$. If this map too contracts C, we denote by $\pi_3\colon Z_3 \to Z_2$ the blowup of the point onto which C is contracted. Repeating this procedure a finite number of times $D \in \mathbb{N}$, we finally arrive at a variety $Z := Z_D$ and a birational morphism $\pi := \pi_1\pi_2\cdots\pi_D\colon Z \to Y$ such that $(\pi^{-1}\varphi)$ does not contract C. Then $(\pi^{-1}\varphi)|_C\colon C \dashrightarrow (\pi^{-1}\varphi)(C)$ is a birational map.

Definition 2.1. In the above situation, we denote by $D(C,\varphi) \in \mathbb{N}$ the minimal number of blowups which are needed in order to not contract the curve C and we say that C is contracted $D(C,\varphi)$ times by φ. In particular, a curve C is sent to a curve by φ if and only if $D(C,\varphi) = 0$.

Remark 2.2. The integer $D(C,\varphi)$ can equivalently be defined as the order of vanishing of $K_Z - \pi^*(K_Y)$ along $(\pi^{-1}\varphi)(C)$.

We recall the following well known fact, which will be used a number of times in the sequel.

Lemma 2.3. *Let $\varphi_1, \varphi_2 \in \mathrm{Bir}(\mathbb{P}^2)$ be birational maps of degree 2 with proper base points p_1, p_2, p_3 and q_1, q_2, q_3 respectively. If φ_1 and φ_2 have (exactly) two common base points, say $p_1 = q_1$ and $p_2 = q_2$, then the composition $\tau = \varphi_2\varphi_1^{-1}$ is quadratic. Furthermore the three base points of τ are proper points of $\mathbb{P}^2$ if and only if q_3 is not on any of the lines joining two of the p_i.*

Proof. The lemma is proved by Figure 1, where squares and circles in $\mathbb{P}^2_2$ denote the base points of φ_1 and φ_2 respectively. The crosses in $\mathbb{P}^2_1$ denote the base points of φ_1^{-1} (corresponding to the lines in $\mathbb{P}^2_2$), and the conics in $\mathbb{P}^2_1$ and $\mathbb{P}^2_2$ denote the pullback of a general line $\ell \in \mathbb{P}^2_3$.

If q_3 is not on any of the three lines, the base points of τ are $E_1, E_2, \varphi_1(q_3)$. If q_3 is on one of the three lines, then the base points of τ are E_1, E_2 and a point infinitely close to the E_i which corresponds to the line that q_3 is on. $\square$

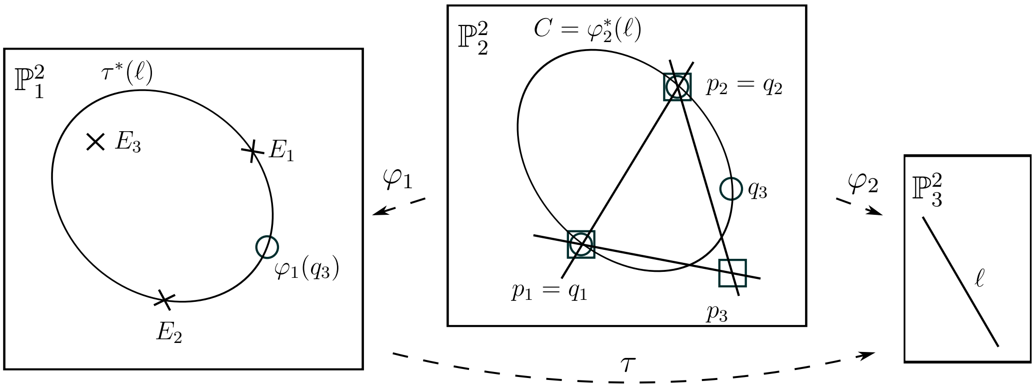

FIGURE 1. The composition of φ_1 and φ_2 in Lemma 2.3

The following lemma describes how the number of times that a line is contracted changes when composing with a quadratic transformation of $\mathbb{P}^2$ with three proper base points.

Lemma 2.4. *Let $\rho\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map and let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a quadratic birational map with base points $q_1, q_2, q_3 \in \mathbb{P}^2$. For $1 \le i < j \le 3$ we denote by $\ell_{ij} \subset \mathbb{P}^2$ the line which joins the base points q_i and q_j. If $D(L, \rho) = k \ge 1$, we have*

$$D(L, \varphi\rho) = \begin{cases} k+1 & \text{if } \rho(L) \in (\ell_{12} \cup \ell_{13} \cup \ell_{23}) \setminus \mathrm{Bp}(\varphi), \\ k & \text{if } \rho(L) \notin \ell_{12} \cup \ell_{13} \cup \ell_{23}, \\ k & \text{if } \rho(L) = q_i \text{ for some } i, \text{ and } (\rho\varphi)(L) \in \mathrm{Bp}(\varphi^{-1}), \\ k-1 & \text{if } \rho(L) = q_i \text{ for some } i, \text{ and } (\rho\varphi)(L) \notin \mathrm{Bp}(\varphi^{-1}). \end{cases}$$

Proof. We consider the minimal resolutions of φ; in Figures 2-5, the filled black dots denote the successive images of L, i.e. $\rho(L)$, $(\pi^{-1}\rho)(L)$ and $(\eta\pi^{-1}\rho)(L)$ respectively.

We argue by Figure 2 and 3 in the case where $\rho(L)$ does not coincide with any of the base points of φ. If $\rho(L) \in \ell_{ij}$ for some i, j, then $D(L, \varphi\rho) = D(L, \rho) + 1$, since ℓ_{ij} is contracted by φ. Otherwise, the number of times L is contracted does not change. Suppose that $\rho(L) = q_i$ for some i. If $D(L, \rho) = 1$, we have $(\pi^{-1}\rho)(L) = E_i$, and then

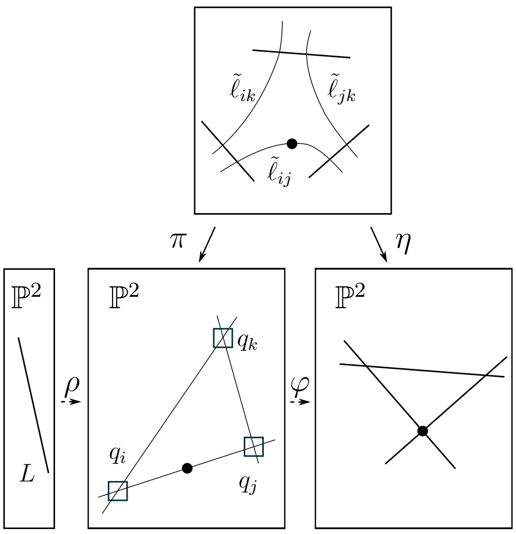

FIGURE 2. $D(L, \varphi\rho) = k+1$; $\rho(L) \in (\ell_{12} \cup \ell_{13} \cup \ell_{23}) \setminus \mathrm{Bp}(\varphi)$.

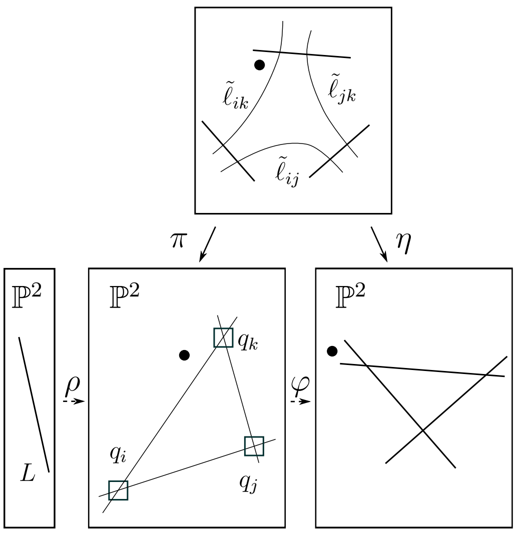

FIGURE 3. $D(L, \varphi\rho) = k$; $\rho(L) \notin \ell_{12} \cup \ell_{13} \cup \ell_{23}$.

clearly $D(L, \varphi\rho) = 0$ since E_i is not contracted by η. If $D(L, \rho) \geq 2$ we argue by the Figures 4 and 5.

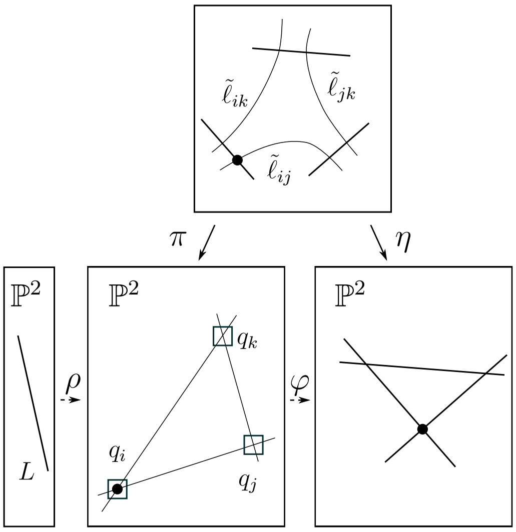

FIGURE 4. $D(L, \varphi\rho) = k$; $\rho(L) = q_i$ and $(\rho\varphi)(L) \in \mathrm{Bp}(\varphi^{-1})$.

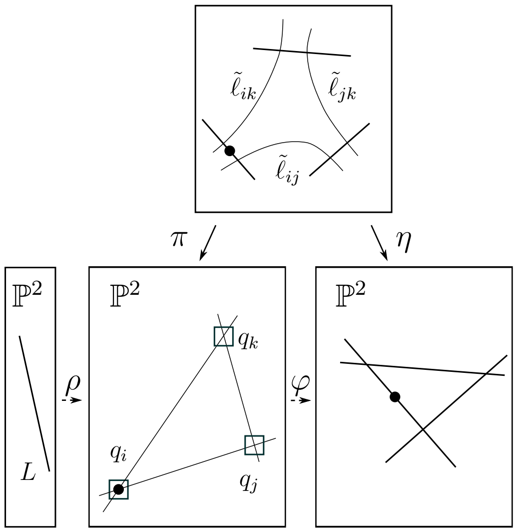

FIGURE 5. $D(L, \varphi\rho) = k - 1$; $\rho(L) = q_i$ and $(\rho\varphi)(L) \notin \mathrm{Bp}(\varphi^{-1})$.

□

Remark 2.5. If $D(L, \rho) \geq 2$, then the point $(\pi^{-1}\rho)(L)$ in the first neighbourhood of $\rho(L)$ defines a tangent direction at $\rho(L) \in \mathbb{P}^2$. If we take φ as in Lemma 2.4 with $q_i \in \mathrm{Bp}(\varphi)$ for some i, then this tangent direction coincides with the direction of one of ℓ_{ij}, ℓ_{ik} if and only if $(\rho\varphi)(L) \in \mathrm{Bp}(\varphi^{-1})$.

Proposition 2.6. *For any given element $\rho \in \mathrm{Dec}(L)$, there is a decomposition of ρ into quadratic maps $\rho = \rho_m \dots \rho_1$ with three proper base points such that none of the successive compositions $(\rho_i \dots \rho_1)_{i=1}^m$ contract L to a point.*

Proof. Let $\rho = \rho_m \dots \rho_1$ be a decomposition of ρ into quadratic maps with only proper base points. We can assume that $d := \max\{D(L, \rho_j \dots \rho_1) \mid 1 \leq j \leq m\} > 0$, otherwise we are done. Let $n := \max\{j \mid D(L, \rho_j \dots \rho_1) = d\}$. We denote the base points of ρ_n^{-1} and ρ_{n+1} by p_1, p_2, p_3 and q_1, q_2, q_3 respectively.

We first look at the case where $D(L, \rho_{n-1} \dots \rho_1) = D(L, \rho_{n+1} \dots \rho_1) = d - 1$. Then composition with ρ_n and ρ_{n+1} fall under Cases 1 and 4 of Lemma 2.4, so both ρ_n^{-1} and ρ_{n+1} have a base point at $(\rho_n \dots \rho_1)(L) \in \mathbb{P}^2$. We may assume that this point is $p_1 = q_1$, as in Figure 6. Interchanging the roles of q_2 and q_3 if necessary, we may assume that p_1, p_2, q_2 are not collinear. Let $r \in \mathbb{P}^2$ be a general point, and let c_1 and c_2 denote quadratic maps with base points $[p_1, p_2, r]$ and $[p_1, q_2, r]$ respectively; then the maps τ_1, τ_2, τ_3 (defined by the commutative diagram in Figure 6) are quadratic with three proper base points in $\mathbb{P}^2$. Note that $D(L, \tau_i \dots \tau_1 \rho_{n-1} \dots \rho_1) = d - 1$ for $i = 1, 2, 3$. Thus we obtained a new decomposition of ρ into quadratic maps with three proper base points

$$\rho = \rho_m \dots \rho_{n+2} \tau_3 \tau_2 \tau_1 \rho_{n-1} \dots \rho_1,$$

where the number of instances where L is contracted d times has decreased by 1.

Now assume instead that $D(L, \rho_{n-1} \dots \rho_1) = d$ and $D(L, \rho_{n+1} \dots \rho_1) = d - 1$. Then composition with ρ_{n+1} falls under Case 4 of Lemma 2.4, so $(\rho_n \dots \rho_1)(L)$ is a base point of ρ_{n+1}, which we may assume to be q_1. Furthermore composition with ρ_n falls under

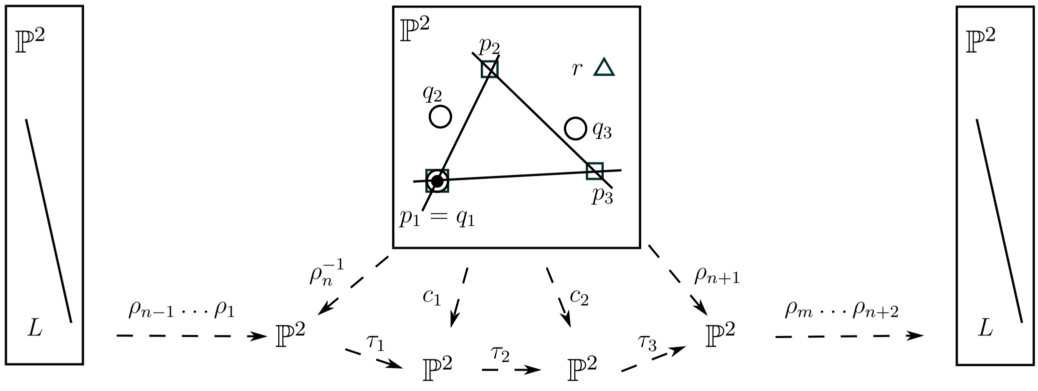

FIGURE 6. The decomposition of $\rho_{n+1}\rho_n$ into quadratic maps τ_1, τ_2, τ_3

Cases 2 or 3 of Lemma 2.4, so $(\rho_n\ldots\rho_1)(L)$ either does not lie on a line joining two base points of ρ_n^{-1}, or $D(L, \rho_n\ldots\rho_1) \geq 2$ and $(\rho_n\ldots\rho_1)(L)$ is a base point of ρ_n^{-1} (which we may assume to be p_1, and equal to q_1), at the same time as $(\rho_{n-1}\ldots\rho_1)(L)$ is a base point of ρ_n.

We consider the first case. If $D(L, \rho_n\ldots\rho_1) \geq 2$ so that L defines a tangent direction at $(\rho_n\ldots\rho_1)(L)$, then this tangent direction has to be different from at least two of the three directions at q_1 that are defined by the lines through q_1 and the p_i, $i = 1, 2, 3$. By renumbering the p_i, we may assume that p_2, p_3 define these two directions (no renumbering is needed if $D(L, \rho_n\ldots, \rho_1) = 1$). Then with a quadratic map $c_1 := [q_1, p_2, p_3]$ with base points q_1, p_2, p_3, we are in Case 4 of Lemma 2.4 and obtain $D(L, c_1\rho_n\ldots\rho_1) = D(L, \rho_n\ldots\rho_1) - 1$. Let $r, s \in \mathbb{P}^2$ be two general points and define c_2, c_3, c_4 with three proper base points respectively as $[q_1, r, p_3]$, $[q_1, r, s]$, $[q_1, q_2, s]$. Note that the corresponding maps $\tau_1, \ldots, \tau_5$, defined in an analogous way as in Figure 6, are quadratic with three proper base points. Note also that $D(L, c_i\rho_n\ldots\rho_1) = D(L, \rho_n\ldots\rho_1) - 1$ for $i = 2, 3, 4$. Only for $i = 4$ this is not immediately clear, so suppose that this is not the case, i.e. $D(L, c_4\rho_n\ldots\rho_1) = D(L, \rho_n\ldots\rho_1)$. It follows that $D(L, \rho_n\ldots\rho_1) \geq 2$ and that the tangent direction corresponding to $(\rho_n\ldots\rho_1)(L)$ is given by the line through q_1 and q_2, but this is not possible by the assumption that $D(L, \rho_{n+1}\ldots\rho_1) = d - 1$.

In the second case we have $p_1 = q_1$ and the tangent direction at $p_1 = q_1$ corresponding to $(\rho_n\ldots\rho_1)(L)$ is the direction either of the line through p_1 and p_2 or the line through p_1 and p_3 (see Figure 4). By interchanging the roles of p_2 and p_3 if necessary, we may assume that it corresponds to the direction of the line through p_1 and p_3. Interchanging the roles of q_2 and q_3 if necessary, we may assume that p_1, q_2, p_3 are not collinear. Let $r, s \in \mathbb{P}^2$ be general points and define quadratic maps c_1, c_2, c_3 with three proper base points respectively by $[p_1, p_2, s]$, $[p_1, r, s]$, $[p_1, r, q_2]$. Then the corresponding maps $\tau_1, \tau_2, \tau_3, \tau_4$ are quadratic with three proper base points and $D(L, c_i\rho_n\ldots\rho_1) = D(L, \rho_n\ldots\rho_1) - 1$ for $i = 1, 2, 3$. The latter holds for c_1 since the direction given by p_1 and p_2 is different from the tangent direction corresponding to $(\rho_n\ldots\rho_1)(L)$, and for c_3 it follows from the assumption that the image of L is contracted $d - 1$ times by $(\rho_{n+1}\ldots\rho_1)$ and that p_1, q_2, p_3 are not collinear.

Both in the first and second case, we again arrive at a new decomposition into quadratic maps with three proper base points

$$\rho = \rho_m\ldots\rho_{n+2}\tau_j\ldots\tau_1\rho_{n-1}\ldots\rho_1 \qquad (j \in \{4, 5\}),$$

where the number of instances where L is contracted d times has decreased by 1, and we conclude by induction. $\square$

3. Avoiding to send L to a curve of degree higher than 1.

By Proposition 2.6, any element $\rho \in \mathrm{Dec}(L)$ can be decomposed as

$$\rho = \rho_m \dots \rho_1$$

where each ρ_j is quadratic with three proper base points, and all of the successive images $((\rho_i \dots \rho_1)(L))_{i=1}^m$ of L are curves. The aim of this section is to show that the ρ_j even can be chosen so that all of these curves have degree 1. That is, we find a decomposition of ρ into quadratic maps such that all the successive images of L are lines. This means in particular that $\mathrm{Dec}(L)$ is generated by its elements of degree 1 and 2.

Definition 3.1. A birational transformation of $\mathbb{P}^2$ is called de Jonquières if it preserves the pencil of lines passing through $[1:0:0] \in \mathbb{P}^2$. These transformations form a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ which we denote by $\mathcal{J}$.

Remark 3.2. In [AC2002], a de Jonquières map is defined by the slightly less restrictive property that it sends a pencil of lines to a pencil of lines. Given a map with this property, we can always obtain an element in $\mathcal{J}$ by composing from left and right with elements of PGL_3.

For a curve $C \subset \mathbb{P}^2$ and a point p in $\mathbb{P}^2$ or infinitely near, we denote by $m_C(p)$ the multiplicity of C in p. If it is clear from context which curve we are referring to, we will use the notation $m(p)$.

Lemma 3.3. *Let $\varphi \in \mathcal{J}$ be of degree $e \geq 2$, and $C \subset \mathbb{P}^2$ a curve of degree d. Suppose that*

$$\deg(\varphi(C)) \leq d.$$

Then there exist two base points q_1, q_2 of φ different from $[1:0:0]$ such that

$$m_C([1:0:0]) + m_C(q_1) + m_C(q_2) \geq d.$$

This inequality can be made strict in case $\deg(\varphi(C)) < d$, with a completely analogous proof.

Proof. Since $\varphi \in \mathcal{J}$ is of degree e, it has exactly $2e-1$ base points $r_0 := [1:0:0], r_1, \dots, r_{2e-2}$ of multiplicity $e-1, 1, \dots, 1$ respectively. Then

$$\begin{aligned} d \geq \deg(\varphi(C)) =& ed - (e-1)m_C(r_0) - \sum_{i=1}^{e-1}(m_C(r_{2i-1}) + m_C(r_{2i})) \\ =& d + \sum_{i=1}^{e-1}(d - m_C(r_0) - m_C(r_{2i-1}) - m_C(r_{2i})) \end{aligned}$$

Hence there exist i_0 such that $d \leq m_C(r_0) + m_C(r_{2i_0-1}) + m_C(r_{2i_0})$. □

Remark 3.4. Note also that we can choose the points q_1, q_2 such that q_1 either is a proper point in $\mathbb{P}^2$ or in the first neighbourhood of $[1:0:0]$, and that q_2 either is proper point of $\mathbb{P}^2$ or is in the first neighbourhood of $[1:0:0]$ or q_1.

Remark 3.5. A quadratic map sends a pencil of lines through one of its base points to a pencil of lines, and we conclude from Proposition 2.6 and Remark 3.2 that there exists maps $\alpha_1, \dots, \alpha_{m+1} \in \mathrm{PGL}_3$ and $\rho_i \in \mathcal{J} \setminus \mathrm{PGL}_3$ such that

$$\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1} \dots \alpha_2\rho_1\alpha_1$$

and such that all of the successive images of L with respect to this decomposition are curves.

The following proposition is an analogue of the classical Castelnuovo's Theorem stating that any map in $\mathrm{Bir}(\mathbb{P}^2)$ is a product of de Jonquières maps.

Proposition 3.6. *Let $\rho \in \mathrm{Dec}(L)$. Then there exists $\rho_i \in \mathcal{J} \setminus \mathrm{PGL}_3$ and $\alpha_i \in \mathrm{PGL}_3$ such that $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\dots\alpha_2\rho_1\alpha_1$ and all of the successive images of L are lines.*

Proof. Start with a decomposition $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\dots\alpha_2\rho_1\alpha_1$ as in Remark 3.5.

Denote $C_i := (\rho_i\alpha_i\cdots\rho_1\alpha_1)(L) \subset \mathbb{P}^2$, $d_i := \deg(C_i)$ and let

$$D := \max\{d_i \mid i = 1,\dots,m\}, \quad n := \max\{i \mid D = d_i\}, \quad k := \sum_{i=1}^{n}(\deg\rho_i - 1).$$

We use induction on the lexicographically ordered pair (D,k).

We may assume that $D > 1$, otherwise our goal is already achieved. We may also assume that $\alpha_{n+1} \notin \mathcal{J}$, otherwise the pair (D,k) decreases as we replace the three maps $\rho_{n+1}, \alpha_{n+1}, \rho_n$ by their composition $\rho_{n+1}\alpha_{n+1}\rho_n \in \mathcal{J}$. Indeed, either D decreases, or D stays the same while k decreases at least by $\deg\rho_n - 1$. Using Lemma 3.3, we find simple base points p_1, p_2 of ρ_n^{-1} and simple base points $\tilde{q}_1, \tilde{q}_2$ of ρ_{n+1}, all different from $p_0 := [1:0:0]$, such that

$$m_{C_n}(p_0) + m_{C_n}(p_1) + m_{C_n}(p_2) \geq D$$

and

$$m_{\alpha_{n+1}(C_n)}(p_0) + m_{\alpha_{n+1}(C_n)}(\tilde{q}_1) + m_{\alpha_{n+1}(C_n)}(\tilde{q}_2) > D.$$

We choose $p_1, p_2, \tilde{q}_1, \tilde{q}_2$ as in Remark 3.4. By slight abuse of notation, we denote by $q_0 = \alpha_{n+1}^{-1}(p_0)$, $q_1 = \alpha_{n+1}^{-1}(\tilde{q}_1)$ and $q_2 = \alpha_{n+1}^{-1}(\tilde{q}_2)$ respectively the (proper or infinitely near) points in $\mathbb{P}^2$ that correspond to $p_0, \tilde{q}_1$, and $\tilde{q}_2$ under the isomorphism α_{n+1}^{-1}. Note that p_0 and q_0 are two distinct points of $\mathbb{P}^2$ since $\alpha_{n+1} \notin \mathcal{J}$. We number the points so that $m(p_1) \geq m(p_2)$, $m(\tilde{q}_1) \geq m(\tilde{q}_2)$ and so that if p_i (resp. $\tilde{q}_i$) is infinitely near p_j (resp. $\tilde{q}_j$), then $j < i$.

We study two cases separately depending on the multiplicities of the base points.

Case (a): $m(q_0) \geq m(q_1)$ and $m(p_0) \geq m(p_1)$. Then we find two quadratic maps $\tau', \tau \in \mathcal{J}$ and $\beta \in \mathrm{PGL}_3$ so that $\rho_{n+1}\alpha_{n+1}\rho_n = (\rho_{n+1}\tau^{-1})\beta(\tau\rho_n)$ and so that the pair (D,k) is reduced as we replace the sequence $(\rho_{n+1}, \alpha_{n+1}, \rho_n)$ by $(\rho_{n+1}\tau^{-1}, \beta, \tau\rho_n)$. The procedure goes as follows.

If possible we choose a point $r \in \{p_1, q_1\} \setminus \{p_0, q_0\}$. Should this set be empty, i.e. $p_0 = q_1$ and $p_1 = q_0$, we choose $r = q_2$ instead. The ordering of the points implies that the point r is either a proper point in $\mathbb{P}^2$ or in the first neighbourhood of p_0 or q_0. Furthermore, the assumption implies that $m(p_0) + m(q_0) + m(r) > D$, so r is not on the line passing through p_0 and q_0. In particular, there exists a quadratic map $\tau \in \mathcal{J}$ with base points p_0, q_0, r; then

$$\deg(\tau(C_n)) = 2D - m(p_0) - m(q_0) - m(r) < D.$$

Choose $\beta \in \mathrm{PGL}_3$ so that the quadratic map $\tau' := \beta\tau(\alpha_{n+1})^{-1}$ in the below commutative diagram is de Jonquières – this is possible since τ has q_0 as a base point. This decreases the pair (D,k).

$$\begin{tikzcd}
& \mathbb{P}^2 \arrow[r, "\alpha_{n+1}"] \arrow[dl, dashed, "\rho_n^{-1}"'] \arrow[dd, dashed, "\tau"] & \mathbb{P}^2 \arrow[dr, dashed, "\rho_{n+1}"] \arrow[dd, dashed, "\tau'"] & \\
\mathbb{P}^2 \arrow[dr, dashed] & & & \mathbb{P}^2 \\
& \mathbb{P}^2 \arrow[r, "\beta"'] & \mathbb{P}^2 \arrow[ur, dashed] &
\end{tikzcd}$$

Case (b): $m(p_0) < m(p_1)$. Let τ be a quadratic de Jonquières map with base points p_0, p_1, p_2. This is possible since our assumption implies that p_1 is a proper base point and because p_0, p_1, p_2 are base points of ρ_n^{-1} of multiplicity $\deg \rho_n - 1, 1, 1$ respectively and hence not collinear. Choose $\beta_1 \in \mathrm{PGL}_3$ which exchanges p_0 and p_1, let $\gamma = \alpha_{n+1}\beta_1^{-1}$ and choose $\beta_2 \in \mathrm{PGL}_3$ so that $\tau' := \beta_2 \tau \beta_1^{-1} \in \mathcal{J}$. The latter is possible since $\beta_1^{-1}(p_0) = p_1$ is a base point of τ, and we have the following diagram.

$$\begin{tikzcd}
& \mathbb{P}^2 \arrow[rr, "\alpha_{n+1}"] \arrow[dl, dashed, "\rho_n^{-1}"'] \arrow[dd, dashed, "\tau"] \arrow[dr, "\beta_1"] & & \mathbb{P}^2 \arrow[dr, dashed, "\rho_{n+1}"] & \\
\mathbb{P}^2 \arrow[dr, dashed] & & \boxed{\mathbb{P}^2} \arrow[ur, "\gamma"'] \arrow[d, dashed, "\tau'"] & & \mathbb{P}^2 \\
& \mathbb{P}^2 \arrow[r, "\beta_2"] & \mathbb{P}^2 & &
\end{tikzcd}$$

Since $\deg(\tau\rho_n) = \deg \rho_n - 1$, the pair (D, k) stays unchanged as we replace the sequence (α_{n+1}, ρ_n) in the decomposition of ρ by the sequence $(\gamma, (\tau')^{-1}, \beta_2, \tau\rho_n)$. In the new decomposition of ρ the maps $(\tau')^{-1}$ and γ play the roles that ρ_n and α_{n+1} respectively played in the previous decomposition. In the squared $\mathbb{P}^2$, we have

$$m(p_0) = m(\beta_1(p_1)) > m(\beta_1(p_0)) = m(p_1).$$

Define $q_0' := \gamma^{-1}(p_0)$, $q_1' := \gamma^{-1}(\tilde{q}_1)$, $q_2' := \gamma^{-1}(\tilde{q}_2)$, and note that $q_0' = \beta_1(q_0)$, $q_1' = \beta_1(q_1)$ and $q_2' = \beta_1(q_2)$. In the new decomposition these points play the roles that q_0, q_1, q_2 played in the previous decomposition.

If $m(q_0') \geq m(q_1')$, we continue as in case (a) with the points $p_0, p_1, \beta_1(p_2)$ and q_0', q_1', q_2'.

If $m(q_0') < m(q_1')$, we replace the sequence (ρ_{n+1}, γ) by a new sequence such that, similar to case (a), the roles of q_0' and q_1' are exchanged, and we will do this without touching $p_0, p_1, \beta(b_2)$. The replacement will not change (D, k) and we can apply case (a) to the new sequence.

As $m(q_0') < m(q_1')$, the point q_1' is a proper point of $\mathbb{P}^2$. Analogously to the previous case, there exists $\sigma \in \mathcal{J}$ with base points $\gamma(q_0') = p_0, \gamma(q_1') = \tilde{q}_1, \gamma(q_2') = \tilde{q}_2$, and there exists $\delta_1 \in \mathrm{PGL}_3$ which exchanges p_0 and $\tilde{q}_1$. Since $\delta_1^{-1}(p_0) = \tilde{q}_1$ is a base point of σ, there furthermore exists $\delta_2 \in \mathrm{PGL}_3$ such that $\sigma' := \delta_2 \sigma \delta_1^{-1} \in \mathcal{J}$. Let $\gamma_2 := \delta_1 \gamma$.

$$
\begin{tikzcd}
 & \mathbb{P}^2 \arrow[rrr, "\alpha_{n+1}"] \arrow[dl, dashed, "\rho_n^{-1}"'] \arrow[dd, dashed, "\tau"] \arrow[dr, "\beta_1"] & & & \mathbb{P}^2 \arrow[dl, "\delta_1"'] \arrow[dd, dashed, "\sigma"] \arrow[dr, dashed, "\rho_{n+1}"] & \\
\mathbb{P}^2 \arrow[dr, dashed] & & \boxed{\mathbb{P}^2} \arrow[urr, "\gamma"] \arrow[r, "\gamma_2"'] \arrow[d, dashed, "\tau'"] & \mathbb{P}^2 \arrow[d, dashed, "\sigma'"'] & & \mathbb{P}^2 \arrow[dl, dashed] \\
 & \mathbb{P}^2 \arrow[r, "\beta_2"] & \mathbb{P}^2 & \mathbb{P}^2 & \mathbb{P}^2 \arrow[l, "\delta_2"'] &
\end{tikzcd}
$$

Replacing the sequence (ρ_{n+1},γ) with $(\rho_{n+1}\sigma^{-1},\delta_2^{-1},\sigma',\delta_1\gamma)$ does not change the pair (D,k). The latest position with the highest degree is still the squared $\mathbb{P}^2$ but in the new sequence we have

$$m(\gamma_2^{-1}(p_0)) = m(\beta_1(q_1)) > m(\beta_1(q_0)) = m(\gamma_2^{-1}(\delta_1(\tilde{q}_1)))$$

Since $p_0, p_1, \beta_1(p_2)$ were undisturbed, the inequality $m(p_0) > m(p_1)$ still holds, and we proceed as in case (a).

In this proof, we have used several different quadratic maps $\tau, \tau', \sigma, \sigma'$. Note that none of these can contract C (or an image of C), since quadratic maps only can contract curves of degree 1. $\square$

Remark 3.7. Suppose that $\rho \in \mathcal{J}$ preserves a line L. Then the Noether-equalities imply that L passes either through $[1:0:0]$ and no other base points of ρ, or that it passes through exactly $\deg\rho - 1$ simple base points of ρ and not through $[1:0:0]$.

Lemma 3.8. *Let $\rho \in \mathcal{J}$ be of degree ≥ 2 and let L be a line passing through exactly $\deg\rho - 1$ simple base points of ρ and not through $[1:0:0]$. Then there exist $\rho_1,\dots,\rho_i \in \mathcal{J}$ of degree 2 such that $\rho = \rho_m \cdots \rho_1$ and the successive images of L are lines.*

Proof. Note that the curve $\rho(L)$ is a line not passing through $\rho(L)$. Call $p_0 := [1:0:0], p_1, \dots, p_{2d-2}$ the base points of ρ. Without loss of generality, we can assume that $p_1,\dots,p_{d-1}$ are the simple base points of ρ that are contained in L and that p_1 is a proper base point in $\mathbb{P}^2$. We do induction on the degree of ρ.

If there is no simple proper base point p_i, $i \geq d$, of ρ in $\mathbb{P}^2$ that is not on L, choose a general point $r \in \mathbb{P}^2$. There exists a quadratic transformation $\tau \in \mathcal{J}$ with base points p_0, p_1, r. The transformation $\rho\tau^{-1} \in \mathcal{J}$ is of degree $\deg\rho$ and sends the line $\tau(L)$ (which does not contain $[1:0:0]$) onto the line $\rho(L)$. The point $\rho(r) \in \mathbb{P}^2$ is a base point of $(\rho\tau^{-1})^{-1}$ not on the line $\rho(L)$.

So, we can assume that there exists a proper base point of ρ in $\mathbb{P}^2$ that is not on L, lets call it p_d. The points p_0, p_1, p_d are not collinear (because of their multiplicities), hence there exists $\tau \in \mathcal{J}$ of degree 2 with base points p_0, p_1, p_d. The map $\rho\tau^{-1} \in \mathcal{J}$ is of degree $\deg\rho - 1$ and $\tau(L)$ is a line passing through exactly $\deg\rho - 2$ simple base points of $\rho\tau^{-1}$ and not through $[1:0:0]$. $\square$

Lemma 3.9. *Let $\rho \in \mathcal{J}$ be of degree ≥ 2 and let L be a line passing through $[1:0:0]$ and no other base points of ρ. Then there exist $\rho_1,\dots,\rho_m \in \mathcal{J}$ of degree 2 such that $\rho = \rho_m \cdots \rho_1$ and the successive images of L are lines.*

Proof. Note that the curve $\rho(L)$ is a line passing through $[1:0:0]$. We use induction on the degree of ρ.

Assume that ρ has no simple proper base points, i.e. all simple base points are infinitely near $p_0 := [1:0:0]$. There exists a base point p_1 of ρ in the first neighbourhood of p_0. Choose a general point $q \in \mathbb{P}^2$. There exists $\tau \in \mathcal{J}$ quadratic with base points

p_0, p_1, q. The map $\rho\tau^{-1} \in \mathcal{J}$ is of degree $\deg\rho$ and $\tau(L)$ is a line passing through the base point p_0 of $\rho\tau^{-1}$ of multiplicity $\deg\rho - 1$ and through no other base points of $\rho\tau^{-1}$. Moreover, the point $\rho(q)$ is a (simple proper) base point of $\tau\rho^{-1}$. Therefore, $\tau\rho^{-1}$ has a simple proper base point in $\mathbb{P}^2$ and sends the line $\rho(L)$ onto the line $\tau(L)$, both of which pass through p_0 and no other base points.

So, we can assume that ρ has at least one simple proper base point p_1. Let p_2 be a base point of ρ that is a proper point of $\mathbb{P}^2$ or in the first neighbourhood of p_0 or p_1. Because of their multiplicities, the points p_0, p_1, p_2 are not collinear. Hence there exists $\tau \in \mathcal{J}$ quadratic with base points p_0, p_1, p_2. The map $\rho\tau^{-1}$ is a map of degree $\deg\rho - 1$ and $\tau(L)$ is a line passing through p_0 and no other base points. □

Lemma 3.10. *Let $\rho \in \mathcal{J}$ be a map of degree 2 that sends a line L onto a line. Then there exist quadratic maps $\rho_1, \ldots, \rho_n \in \mathcal{J}$ with only proper base points such that*

$$\rho = \rho_n \cdots \rho_1,$$

and the successive images of L are lines.

Proof. Suppose first that exactly two of the three base points of ρ are proper. We number the base points so that $p_1, p_2 \in \mathbb{P}^2$ and so that p_3 is in the first neighbourhood of p_1, and denote by $\ell_1 \subset \mathbb{P}^2$ the line through p_1 which has the tangent direction defined by p_3. Choose a general point $r \in \mathbb{P}^2$, and define a quadratic map ρ_1 with three base points $p_1, p_2, r \in \mathbb{P}^2$. A minimal resolution of ρ is given by π and η as in Figure 7; it is obtained by blowing up, in order, p_1, p_2, p_3, and then contracting in order $\tilde{\ell}_2 := \eta_*^{-1}(\ell_2)$, $\tilde{\ell}_1 := \eta_*^{-1}(\ell_1)$ and the exceptional divisor corresponding to p_1. By looking at the pull back of a general line in $\mathbb{P}^2$ with respect to $\rho_2 := \rho_1\rho^{-1}$, we see that this map has three proper base points E_{p_1}, $\rho(r)$, $\pi_*(\tilde{\ell}_1)$. This gives us a decomposition of the desired form: $\rho = \rho_2^{-1}\rho_1$. Note that since ρ sends the line L onto a line, L has to pass through exactly one of the base points of ρ, and this base point has to be proper. Thus L is sent to a line by ρ_1. Using the diagram in Figure 7, we can see that this line is further sent by ρ_2^{-1} to a line through E_{p_1} if L passes through p_1 and a line through $\pi_*(\tilde{\ell}_1)$ if L passes through p_2.

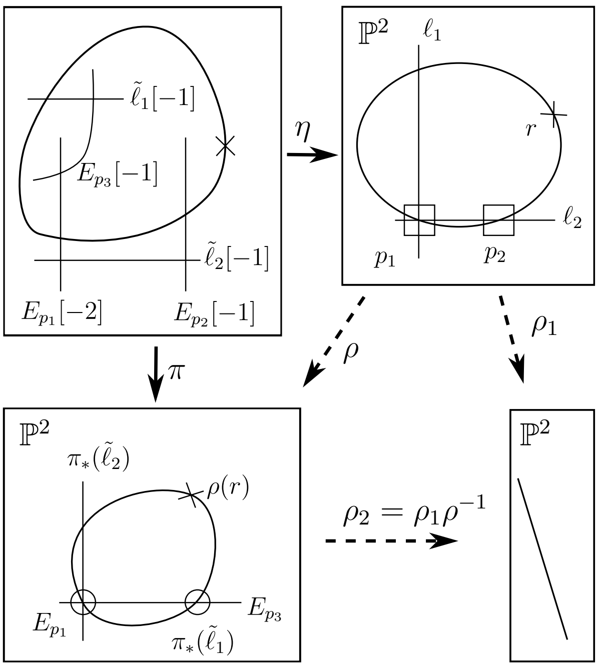

FIGURE 7. Numbers in square brackets denote self-intersection.

If $[1:0:0]$ is the only proper base point of ρ, we reduce to the first case as follows. Denote by q the base point in the first neighbourhood of $[1:0:0]$ and choose a general point $r \in \mathbb{P}^2$. Let ρ_1 be a quadratic map with base points $[1:0:0], q, r$, and let $\rho_2 := \rho_1\rho^{-1}$. If we denote the base points of ρ^{-1} by q_1, q_2, q_3 so that q_1 is the proper base point and q_2 the base point in the first neighbourhood of q_1, then the base points of ρ_2 are $q_1, q_2, \rho(r)$, i.e. it has exactly two proper base points.

It is also clear that ρ_1 sends L to a line, which is further sent by ρ_2^{-1} to a line through q_1. Thus we can apply the first part of this proof to each of ρ_2^{-1} and ρ_1 in $\rho = \rho_2^{-1}\rho_1$, and thus get a decomposition of the desired form.

$\square$

Theorem 1. *For any line L, the group $\mathrm{Dec}(L)$ is generated by $\mathrm{Dec}(L) \cap \mathrm{PGL}_3$ and any of its quadratic elements having three proper base points in $\mathbb{P}^2$.*

Proof. By conjugating with an appropriate automorphism of $\mathbb{P}^2$, we can assume that L is given by $x = y$. Note that the standard quadratic involution $\sigma\colon [x : y : z] \dashrightarrow [yz : xz : xy]$ is contained in $\mathrm{Dec}(L)$. It follows from Proposition 3.6, Remark 3.7, and Lemmata 3.8, 3.9 and 3.10 that every element $\rho \in \mathrm{Dec}(L)$ has a composition $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\cdots\alpha_2\rho_1\alpha_1$, where $\alpha_i \in \mathrm{PGL}_3$ and $\rho_i \in \mathcal{J}$ are quadratic with only proper base points in $\mathbb{P}^2$ such that the successive images of L are lines. By composing the ρ_i from the left and the right with linear maps, we obtain a decomposition

$$\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\cdots\alpha_2\rho_1\alpha_1$$

where $\alpha_i \in \mathrm{PGL}_3 \cap \mathrm{Dec}(L)$ and $\rho_i \in \mathrm{Dec}(L)$ are of degree 2 with only proper base points in $\mathbb{P}^2$. It therefore suffices to show that for any quadratic element $\rho \in \mathrm{Dec}(L)$ having three proper base points in $\mathbb{P}^2$ there exist $\alpha, \beta \in \mathrm{Dec}(L) \cap \mathrm{PGL}_3$ such that $\sigma = \beta\rho\alpha$.

By Remark 3.7, for any quadratic element of $\mathrm{Dec}(L)$ the line L passes through exactly one of its base points in $\mathbb{P}^2$.

Let $q_1 = [0 : 0 : 1]$, $q_2 = [0 : 1 : 0]$, $q_3 = [1 : 0 : 0]$. They are the base points of σ, and σ sends the pencil of lines through q_i onto itself. Furthermore, $q_1 \in L$ but $q_2, q_3 \notin L$. Let $s := [1 : 1 : 1] \in L$. Remark that $\sigma(s) = s$ and that no three of q_1, q_2, q_3, s are collinear.

Let $\rho \in \mathrm{Dec}(L)$ be another quadratic map having three proper base points in $\mathbb{P}^2$. Let p_1, p_2, p_3 (resp. p'_1, p'_2, p'_3) be its base points (resp. the ones of ρ^{-1}). Say L passes through p_1 and ρ sends the pencil of lines through p_i onto the pencil of lines through p'_i, $i = 1, 2, 3$. Pick a point $r \in L \setminus \{p_1\}$, not collinear with p_2, p_3. Then no three of p_1, p_2, p_3, r (resp. $p'_1, p'_2, p'_3, \rho(r)$) are collinear. In particular, there exist $\alpha, \beta \in \mathrm{PGL}_3$ such that

$$\alpha\colon \begin{cases} q_i \mapsto p_i \\ s \mapsto r \end{cases}, \quad \beta\colon \begin{cases} p'_i \mapsto q_i \\ \rho(r) \mapsto s \end{cases}$$

Note that $\alpha, \beta \in \mathrm{Dec}(L) \cap \mathrm{PGL}_3$. Furthermore, the quadratic maps $\sigma, \rho' := \beta\rho\alpha \in \mathrm{Dec}(L)$ and their inverse all have the same base points (namely q_1, q_2, q_3) and both σ, ρ' send the pencil through q_i onto itself. Since moreover $\rho'(s) = \sigma(s) = s$, we have $\sigma = \rho'$. $\square$

4. Dec(L) is not an amalgam

Just like $\mathrm{Bir}(\mathbb{P}^2)$, its subgroup $\mathrm{Dec}(L)$ is generated by its linear elements and one quadratic element (Theorem 1). In [Cor2013, Corollary A.2], it is shown that $\mathrm{Bir}(\mathbb{P}^2)$ is not an amalgamated product. In this section we adjust the proof to our situation and prove that the same statement holds for $\mathrm{Dec}(L)$.

The notion of being an amalgamated product is closely related to actions on trees, or, in this case, $\mathbb{R}$-trees.

Definition and Lemma 4.1. A *real tree*, or $\mathbb{R}$-tree, can be defined in the following three equivalent ways [Cis2001]:

(1) A geodesic space which is 0-hyperbolic in the sense of Gromov.
(2) A uniquely geodesic metric space for which $[a, c] \subset [a, b] \cup [b, c]$ for all a, b, c.

(3) A geodesic metric space with no subspace homeomorphic to the circle.

We say that a real tree is a *complete real tree* if it is complete as a metric space.

Every ordinary tree can be seen as a real tree by endowing it with the usual metric but not every real tree is isometric to an simplicial tree (endowed with the usual metric) [Cis2001, §II.2, Proposition 2.5, Example].

Definition 4.2. A group G has the *property* $(\mathrm{F}\mathbb{R})_\infty$ if for every isometric action of G on a complete real tree, every element has a fixed point.

We summarize the discussion in [Cor2013, before Remark A.3] in the following result.

Lemma 4.3. *If a group G has property $(\mathrm{F}\mathbb{R})_\infty$, it does not decompose as non-trivial amalgam.*

We will devote the rest of this section to proving Proposition 4.4 and thereby showing that $\mathrm{Dec}(L)$ is not an amalgam.

Proposition 4.4. *The decomposition group $\mathrm{Dec}(L)$ has property $(\mathrm{F}\mathbb{R})_\infty$.*

By convention, from now on, $\mathcal{T}$ will denote a complete real tree and all actions on $\mathcal{T}$ are assumed to be isometric.

Definition 4.5. Let $\mathcal{T}$ be a complete real tree.

(1) A *ray* in $\mathcal{T}$ is a geodesic embedding $(x_t)_{t\geq 0}$ of the half-line.
(2) An *end* in $\mathcal{T}$ is an equivalence class of rays, where we say that two rays x and y are equivalent if there exists $t, t' \in \mathbb{R}$ such that $\{x_s;\ s \geq t\} = \{y'_s;\ s' \geq t'\}$.
(3) Let G be a group of isometries of $\mathcal{T}$ and ω an end in $\mathcal{T}$ represented by a ray $(x_t)_{t\geq 0}$. The group G *stably fixes the end* ω if for every $g \in G$ there exists $t_0 := t_0(g)$ such that g fixes x_t for all $t \geq t_0$.

Remark 4.6. [Cor2013, Lemma A.9] For a group G, property $(\mathrm{F}\mathbb{R})_\infty$ is equivalent to each of the following statements:

(1) For every isometric action of G on a complete real tree, every finitely generated subgroup has a fixed point.
(2) Every isometric action of G on a complete real tree has a fixed point or stably fixes an end.

Definition 4.7. For a line $L \subset \mathbb{P}^2$, define $\mathcal{A}_L := \mathrm{PGL}_3 \cap \mathrm{Dec}(L)$. If L is given by the equation $f = 0$, we also use the notation $\mathcal{A}_{\{f=0\}}$.

Lemma 4.8. *For any line $L \subset \mathbb{P}^2$ the group $\mathcal{A}_L$ has property $(\mathrm{F}\mathbb{R})_\infty$.*

Proof. Since for two lines L and L' the groups $\mathrm{Dec}(L)$ and $\mathrm{Dec}(L')$ are conjugate, it is enough to prove the lemma for one line, say the line given by $x = 0$. Note that $A = (a_{ij})_{1\leq i,j\leq 3} \in \mathrm{PGL}_3$ is in $\mathcal{A}_{\{x=0\}}$ if and only if $a_{12} = a_{13} = 0$.

Let $\mathcal{A}_{\{x=0\}}$ act on $\mathcal{T}$ and let $F \subset \mathcal{A}_{\{x=0\}}$ be a finite subset. The elements of F can be written as a product of elementary matrices contained in $\mathcal{A}_{\{x=0\}}$; let R be the (finitely generated) subring of k generated by all entries of the elementary matrices contained in $\mathcal{A}_{\{x=0\}}$ that are needed to obtain the elements in F. Then F is contained in $\mathrm{EL}_3(R)$, the subgroup of $\mathrm{SL}_3(R)$ generated by elementary matrices. By the Shalom-Vaserstein theorem (see [EJZ010, Theorem 1.1]), $\mathrm{EL}_3(R)$ has Kazhdan's property (T) and in particular (as $\mathrm{EL}_3(R)$ is countable) has a fixed point in $\mathcal{T}$ [Wat1982, Theorem 2], so F has a fixed point in $\mathcal{T}$. It follows that the subgroup of $\mathcal{A}_{\{x=0\}}$ generated by F has a fixed point [Ser1977, §I.6.5, Corollary 3]. In particular, by Remark 4.6 (1), $\mathcal{A}_{\{x=0\}}$ has property $(\mathrm{F}\mathbb{R})_\infty$. □

From now on, we fix L to be the line given by $x=y$. It is enough to prove Proposition 4.4 for this line since $\mathrm{Dec}(L)$ and $\mathrm{Dec}(L')$ are conjugate groups (by linear elements) for all lines L and L'. As before, we denote the standard quadratic involution by $\sigma\in\mathrm{Bir}(\mathbb{P}^2)$; with our choice of L, it is contained in $\mathrm{Dec}(L)$.

Let $\mathcal{D}_L\subset\mathrm{PGL}_3$ be the subgroup of diagonal matrices that send L onto L, i.e.

$$\mathcal{D}_L:=\{\mathrm{diag}(s,s,t)\, s,t\in\mathbb{C}^*\}\subset\mathrm{PGL}_3.$$

Lemma 4.9. *We have $\langle\mathcal{D}_L,\mu_1,\mu_2,P\rangle=\mathcal{A}_L$, with the three involutions*

$$\mu_1:=\begin{bmatrix}-1&0&1\\0&-1&1\\0&0&1\end{bmatrix}\in\mathcal{A}_L,\ \mu_2:=\begin{bmatrix}-1&0&0\\0&-1&0\\1&0&1\end{bmatrix}\in\mathcal{A}_L,\quad \textit{and } P:=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\in\mathcal{A}_L.$$

Proof. Given any $\lambda\in\mathbb{C}^*$, the matrices

$$A_\lambda:=\begin{bmatrix}1&0&0\\0&1&0\\\lambda&0&1\end{bmatrix},\ B_\lambda:=\begin{bmatrix}1&0&0\\0&1&0\\0&\lambda&1\end{bmatrix},\ \text{and } C_\lambda:=\begin{bmatrix}1&0&\lambda\\0&1&\lambda\\0&0&1\end{bmatrix}$$

belong to $\langle\mathcal{D}_L,\mu_1,\mu_2,P\rangle$. Indeed, we have $A_\lambda=\mathrm{diag}(-\lambda^{-1},-\lambda^{-1},1)\cdot\mu_2\cdot\mathrm{diag}(\lambda,\lambda,1)$, $B_\lambda=PA_\lambda P$ and $C_\lambda=\mathrm{diag}(1,1,\lambda^{-1})\cdot\mu_1\cdot\mathrm{diag}(-1,-1,\lambda)$.

Left multiplication by these corresponds to three types of row operations on matrices in PGL_3 and right multiplication corresponds in the same way to three types of column operations. We denote them respectively by r_1,r_2,r_3,c_1,c_2,c_3, and we write d for multiplication by an element in $\mathcal{D}_L$.

Let $A=(a_{ij})_{1\le i,j\le 3}\in\mathrm{PGL}_3$ be a matrix which is in $\mathcal{A}_L$, i.e. such that $a_{13}=a_{23}$ and $a_{11}+a_{12}=a_{21}+a_{22}$. We proceed as follows, using only the above mentioned operations.

$$\begin{aligned}A\ =\ &\begin{bmatrix}*&*&*\\ *&*&*\\ *&*&*\end{bmatrix}\xrightarrow{d}\begin{bmatrix}*&*&*\\ *&*&*\\ *&*&1\end{bmatrix}\xrightarrow{r_3}\begin{bmatrix}*&*&0\\ y&z&0\\ *&*&1\end{bmatrix}\xrightarrow{c_1\text{ and }c_2}\begin{bmatrix}*&*&0\\ y&z&0\\ -y&-z&1\end{bmatrix}\\
\xrightarrow{r_3}&\begin{bmatrix}*&*&1\\ 0&0&1\\ -y&-z&1\end{bmatrix}\xrightarrow{d}\begin{bmatrix}1&-1&1\\ 0&0&1\\ *&*&1\end{bmatrix}\xrightarrow{r_1}\begin{bmatrix}1&-1&1\\ 0&0&1\\ 0&*&*\end{bmatrix}\xrightarrow{r_2}\begin{bmatrix}1&-1&1\\ 0&0&1\\ 0&*&0\end{bmatrix}\\
\xrightarrow{d}&\begin{bmatrix}1&-1&1\\ 0&0&1\\ 0&1&0\end{bmatrix}\xrightarrow{r_3}\begin{bmatrix}1&0&1\\ 0&1&1\\ 0&1&0\end{bmatrix}\xrightarrow{c_3}\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&1&-1\end{bmatrix}\xrightarrow{r_2}\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&0&-1\end{bmatrix}\xrightarrow{d}\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&0&1\end{bmatrix}\end{aligned}$$

In the first step (d) we assume that $a_{33}\neq 0$ – this can always be achieved by performing a row operation of type r_1 on A if necessary. In the second step (r_3), we use that $a_{13}=a_{23}$. The entries on place $(2,1)$ and $(2,2)$ after the second step are denoted by y and z respectively. In the fifth step (d), we use that the entry on place $(1,1)$ is nonzero; this follows from the assumption $a_{11}+a_{12}=a_{21}+a_{22}$ and that A is invertible. □

Lemma 4.10. *Suppose that* $\mathrm{Dec}(L)$ *acts on* $\mathcal{T}$ *so that* $\mathcal{A}_L$ *has no fixed points. Then* $\mathrm{Dec}(L)$ *stably fixes an end.*

Proof. Since $\mathcal{A}_L$ has property $(\mathrm{F}\mathbb{R})_\infty$ and has no fixed points, it stably fixes an end (Remark 4.6 (2)). Observe that this fixed end is unique: if $\mathcal{A}_L$ stably fixes two different ends ω_1,ω_2, then $\mathcal{A}_L$ pointwise fixes the line joining the two ends and has therefore fixed points (this uses that the only isometries on $\mathbb{R}$ are translations and reflections [Cis2001, §I.2, Lemma 2.1]).

Let ω, represented by the ray $(x_t)_{t\geq 0}$, be the unique end which is stably fixed by $\mathcal{A}_L$ and define $C := \langle \mathcal{D}_L, P\rangle$. Being a subgroup of $\mathcal{A}_L$, C obviously also stably fixes ω. Note that the end $\sigma\omega$ is stably fixed by $\sigma\mathcal{A}_L\sigma^{-1}$. In particular, since $\sigma C\sigma^{-1} = C$, the end $\sigma\omega$ is also stably fixed by C. If $\sigma\omega = \omega$, then ω is stably fixed by σ and by Theorem 1, ω is stably fixed by $\mathrm{Dec}(L)$. Otherwise, let l be the line joining ω and $\sigma\omega \neq \omega$. Since C stably fixes ω and $\sigma\omega$, it stably fixes both ends of l. In particular, the line l is pointwise fixed by C. Since $\mu_1, \mu_2 \in \mathcal{A}_L$, μ_1, μ_2 stably fix the end ω and therefore, x_t is fixed by μ_1, μ_2 for $t \geq t_0$ for some t_0, and hence, by Lemma 4.9, x_t is fixed by all of $\mathcal{A}_L$ for $t \geq t_0$, contradicting the assumption. □

Proof of Proposition 4.4. Recall that $\mu_1, \mu_2 \in \mathcal{A}_L$ and note that $\sigma\mu_1$ has order 3 and that $\sigma\mu_2$ has order 6. It follows that

$$\sigma = (\mu_1\sigma)\mu_1(\mu_1\sigma)^{-1}$$

By Theorem 1, $\mathrm{Dec}(L)$ is generated by σ and $\mathcal{A}_L$. It follows that $\mathcal{A}_1 := \mathcal{A}_L$ and $\mathcal{A}_2 := \sigma\mathcal{A}_L\sigma$ generate $\mathrm{Dec}(L)$.

Consider an action of $\mathrm{Dec}(L)$ on $\mathcal{T}$. It induces an action of $\mathcal{A}_L$, which has property $(\mathrm{F}\mathbb{R})_\infty$ by Lemma 4.8 (i.e. $\mathcal{A}_L$ has a fixed point or stably fixes an end by Remark 4.6 (2)). If $\mathcal{A}_L$ has no fixed point, Lemma 4.10 implies that $\mathrm{Dec}(L)$ stably fixes an end, and then we are done.

Assume that $\mathcal{A}_L$ has a fixed point. We conclude the proof by showing that in this case, even $\mathrm{Dec}(L)$ has a fixed point.

For $i = 1, 2$, let $\mathcal{T}_i$ be the set of fixed points of $\mathcal{A}_i$. The two trees are exchanged by σ. If $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \emptyset$, $\mathrm{Dec}(L)$ has a fixed point since $\langle \mathcal{A}_1, \mathcal{A}_2\rangle = \mathrm{Dec}(L)$. Let us consider the case where $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint.

Let $\mathcal{S} := [x_1, x_2]$, $x_i \in \mathcal{T}_i$, be the minimal segment joining the two trees and $s > 0$ its length. Let $C := \langle \mathcal{D}_L, P\rangle$. Then $\mathcal{S}$ is pointwise fixed by $C \subset \mathcal{A}_1 \cap \mathcal{A}_2$ and reversed by σ. For $i = 1, 2$, the image of $\mathcal{S}$ by μ_i is a segment $\mu_i(\mathcal{S}) = [x_1, \mu_i x_2]$. By Lemma 4.9, $\langle C, \mu_1, \mu_2\rangle = \mathcal{A}_1$, so it follows that for $i = 1$ or $i = 2$, we have $\mu_i(\mathcal{S}) \cap \mathcal{S} = \{x_1\}$. Otherwise, because $\mathcal{T}$ is a tree and $\mathcal{A}_1$ acts by isometries, both μ_1, μ_2 fix $\mathcal{S}$ pointwise and so $\mathcal{A}_1$ fixes $\mathcal{S}$ pointwise and in particular it fixes x_2 – this would contradict $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$. Choose an element $I \in \{1, 2\}$ such that $\mu_I(\mathcal{S}) \cap \mathcal{S} = \{x_1\}$.

Finally we arrive at a contradiction by computing $d(x_1, (\sigma\mu_I)^k x_1)$ in two different ways. On the one hand we see that this distance is sk, on the other hand we have $(\sigma\mu_I)^6 = 1$. More generally, we show that

$$d(\ (\sigma\mu_I)^k x_1,\ (\sigma\mu_I)^l x_1) = |k - l|s$$

for all k, l. Since we are on a real tree, it suffices to show this for k, l with $|k - l| \leq 2$ (cf. [Cor2013, Lemma A.4]). By translation, we only have to check it for $l = 0, k = 1, 2$. For $k = 1$, we have $d(\sigma\mu_I x_1, x_1) = d(\sigma x_1, x_1) = d(x_2, x_1) = s$. For $k = 2$, the segment $\mu_I(\mathcal{S}) = [x_1, \mu_I x_2]$ intersects $\mathcal{S}$ only at x_1. In particular, $d(\mu_I x_2, x_2) = 2s$ and hence

$$d(\sigma\mu_I\sigma\mu_I x_1, x_1) = d(\sigma\mu_I\sigma x_1, x_1) = d(\mu_I\sigma x_1, \sigma x_1) = d(\mu_I x_2, x_2) = 2s.$$

It follows that $\mathcal{T}_1$ and $\mathcal{T}_2$ cannot be disjoint, and we are done. □

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