A Dynamic Contagion Process

for Modelling Contagion Risk in Finance and Insurance

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Submitted for the degree of Doctor of Philosophy in the subject of Mathematical Finance,

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Abstract

We introduce a new point process, the dynamic contagion process, by generalising the Hawkes process and Cox process with shot noise intensity. Our process includes both self-excited and externally excited jumps, which could be used to model the dynamics of contagion impact from endogenous and exogenous factors of the underlying system. We systematically analyse the theoretical distributional properties of this new process, based on the piecewise-deterministic Markov process theory developed in Davis (1984), and the extension of the martingale methodology used in Dassios and Embrechts (1989). The analytic expressions of the Laplace transform of the intensity process and probability generating function of the point process are derived. A simulation algorithm is provided for further industrial implementation and statistical analysis. Some extensions of this process and comparison with other similar processes are also investigated. The major object of this study is to produce a general mathematical framework for modelling the dependence structure of arriving events with dynamic contagion, which has the potential to be applicable to a variety of problems in economics, finance and insurance. We apply our research to the default probability of credit risk and ruin probability of risk theory.

Notations

$\mathbb{P}$	original natural probability measure
P	probability under probability measure $\mathbb{P}$
$\mathbb{E}$	expectation under probability measure $\mathbb{P}$
$\widetilde{\mathbb{P}}$	equivalent probability measure $\widetilde{\mathbb{P}} \sim \mathbb{P}$
$\widetilde{P}$	probability under probability measure $\widetilde{\mathbb{P}}$
$\widetilde{\mathbb{E}}$	probability under probability measure $\widetilde{\mathbb{P}}$
$\mathcal{A}$	operator of infinitesimal generator
$\mathcal{L}$	operator of Laplace transform
$\mathbb{I}$	indicator function
a	constant rate of premium payment per time unit
c	constant reversion level
ρ	constant rate of a standard Poisson process
σ	constant volatility
μ_i	the i^{th} moment of a distribution
$Y^{(1)}$	random externally excited jump size
$T^{(1)}$	random time of the externally excited jump
$H(y)$	cumulative distribution function of random externally excited jump size $Y^{(1)}$
$h(y)$	density function of random externally excited jump size $Y^{(1)}$
$Y^{(2)}$	random self-excited jump size
$T^{(2)}$	random time of the self-excited jump
$G(y)$	cumulative distribution function of random self-excited jump size $Y^{(2)}$
$g(y)$	density function of random self-excited jump size $Y^{(2)}$
Z	random claim size
$Z(z)$	cumulative distribution function of random claim size Z
$z(z)$	density function of random claim size Z
τ^*	random time
t	variable time
T	fixed time / maturity time
τ	$T - t$/ time to maturity
Δt	sufficient small time interval
N_t	point process
M_t	point process
λ_t	intensity process
X_t	surplus process
W_t	standard Brownian motion / Wiener process

Contents

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Introduction

The behavior of default contagion through business links is more obvious during a financial crisis, especially after the collapse of Lehman Brothers in September 2008. The Greek sovereign debt crisis starting from 2010 has the contagion impact spreading to EU members, such as Portugal, Spain, Italy and even to United Kingdom. More recently, triggered by the United States debt ceiling crisis in 2011, an even much bigger contagion risk seems to emerge.

From the mathematical perspective, a point process with its intensity dependent on the point process itself could provide a more effective model to capture this contagion phenomenon of these clustering 'bad' events. However, only a few examples exist in the literature. These include the pioneering work of Jarrow and Yu (2001) and the more recent one of Errais, Giesecke and Goldberg (2009). Jarrow and Yu (2001) pointed out that, a model with the default intensity only linearly depending on a set of macroeconomic variables is not sufficient to explain the phenomena of clustering defaults around an economic recession; therefore, they introduced the concept of credit contagion, whereby upon default of a given name, the contagion jump shocks will impact immediately to the counterpart's default intensity. Furthermore, Errais, Giesecke and Goldberg (2009) found that, by using the self-excited Hawkes process, originally introduced by Hawkes (1971) (see also Hawkes and Oakes (1974), Oakes (1975)), the clustering of defaults observed from real financial data could be modelled more consistently. On the other hand, there are plenty of papers, including Duffie and Gârleanu (2001), and Longstaff and Rajan (2008), suggesting that, the default intensity could be impacted exogenously by multiple common factors, such as idiosyncratic, sector specific or market-wide events.

In this thesis, we combine both ideas above and introduce a new point process, named a "*dynamic contagion process*", by generalising the Hawkes process (with exponential decay) and the Cox process with shot noise intensity (with exponential decay), to include both the self-excited and externally excited jumps. We use it to model the dynamics of contagion impact from both endogenous (self-excited) and exogenous (externally excited) factors of the underlying system. This approach also extends the idea of default contagion by Jarrow and Yu (2001), to have a richer set of parameters, capable to capture some key aspects of the behavior of arriving events, such as the frequency, magnitude of the impact, and the decay with time.

To define and characterise the dynamic contagion process mathematically, we give a cluster process representation, implement the piecewise deterministic Markov process theory developed by Davis (1984) (and see also Davis (1993)), and then extend the martingale methodology used by Dassios and Embrechts (1989) (see also Dassios and Jang (2003), Dassios and Jang (2005)), to obtain the distributional properties for this new process. This process is analysed by deriving the first and second moments and, more importantly, the Laplace transform of the intensity process and the probability generating function of the point process. A possible way of change of measure has been found via the infinitesimal generator. Furthermore, an explicit example of jumps with exponential distributions and the simulation algorithm are provided for further industrial implementation and statistical analysis. An application to credit risk for a single company is given, some possible approaches for the multiple-name case are also discussed.

Meanwhile, applications of the dynamic contagion process of course are not limited to the areas of credit risk in finance. As substantially discussed in the literature of insurance world, the classical Cramér-Lundberg risk model with the arrival of claims modelled by a Poisson process is often not realistic in practice, and hence a variety of extensions have been studied. Many researchers, such as Björk and Grandell (1988), Embrechts, Grandell and Schmidli (1993) had already suggested using the Cox process to model the arrival of claims, see also the book by Grandel (1991). Schmidli (1996) investigated the case for a Cox process with a piecewise constant intensity. More recently, Albrecher and Asmussen (2006) discussed a Cox process with shot noise intensity. On the other hand, a few researchers have proposed risk models using self-excited processes, due to the observation of the clustering arrival of claims in reality, a similar pattern in the credit risk from the financial market, particularly during the current economic crisis. Stabile and Torrisi (2010) looked at the ruin problem in a model using the Hawkes process, a self-excited point process introduced by Hawkes (1971). To capture the clustering phenomenon as well as some common external factors involved for the arrival of claims within one single consistent framework, we extend further to use the dynamic contagion process and try to generalise results obtained for the classical model of infinite horizon. Some classical ruin problems, such as the net profit condition, (generalised) Lundberg's fundamental equation, ruin probability, Cramér-Lundberg approximation have been studied via the martingale approach and change of measure. Special attention is given to the case of exponential jumps and two numerical examples are also provided.

During our previous distributional analysis and applications in finance and insurance for the dynamic contagion process, we realise that many theoretical as well as applied results obtained inevitably involve some inverse functions of inconvenient unexplicit forms, whereas the Cox process with CIR intensity has explicit counterparts. Hence, by comparing some special cases of these two processes, we find some interesting analogies and inequalities between them. The tools of super-martingales and sub-martingales are deployed during this comparison analysis.

Moreover, we generalise the original dynamic contagion process to allow the intensity process perturbed by diffusion. Then the new point process becomes a hybrid of the Cox process with CIR intensity and Hawkes process with exponential decay. Some key distributional properties such as the Laplace transform of the intensity process and the probability generating function of the point process have been derived.

Interestingly, based on our analysis on the dynamic contagion process via a simple transformation, we discover a more general class of point processes. We name this new point process as a "*discretised dynamic contagion process*" and obtain some fundamental distributional properties, such as moments and probability generating functions. Finally, some special cases of this process are particularly discussed and then applied to model the delayed claims of ruin problem.

This thesis is organised as follows:

Chapter 2 acts as the core chapter of the whole thesis and introduces a new point process named *dynamic contagion process*, which has been mathematically defined as a branching process via the cluster process representation and stochastic intensity representation. Key distributional properties, such as moments, Laplace transforms and probability generating functions of the intensity process and the point process have been derived. Simulation algorithm has also been provided for future statistical analysis and implementation in practice.

Chapter 3 mainly applies the dynamic contagion process to model the credit risk for a single company and the default probability can be derived. The potential approaches for financial applications to multiple names in a portfolio level are also discussed and proposed as future research.

Chapter 4 provides applications of the dynamic contagion process to ruin problem for an insurance company. The classical problems, such as net profit condition, (generalised) Lundberg's fundamental equation, ruin probability, have been investigated. In addition, the approach of change of measure is discussed and some numerical examples are also represented.

Chapter 5 compares some special cases of the dynamic contagion process with the Cox process with CIR intensity, and discovers some interesting analogies as well as inequalities between them.

Chapter 6 generalises the original dynamic contagion process to allow the intensity process perturbed by diffusion. Some key distributional properties are discussed.

Chapter 7 extends the original dynamic contagion process to a new class of point processes named *discretised dynamic contagion process*. A fundamental transformation between the two processes has been be found, and some key distributional properties of this process and connections to the dynamic contagion process have also been obtained.

Chapter 8 applies a special case of discretised dynamic contagion process and some generalisation to model the delayed claims for ruin problem, and derives exact formulas for the asymptotics of ruin probability.

Chapter 9 concludes this thesis.

2

A Dynamic Contagion Process

We introduce a new point process, named a "*dynamic contagion process*", by generalising the Hawkes process (with exponential decay) and the Cox process with shot noise intensity (with exponential decay), to include both the self-excited and externally excited jumps. It could be used to model the dynamic contagion impact from both endogenous (self-excited) and exogenous (externally excited) factors of the underlying system. To define and characterise the dynamic contagion process mathematically, we give a cluster process representation, implement the piecewise deterministic Markov process theory developed by Davis (1984), and then extend the martingale methodology used by Dassios and Jang (2003), to obtain the distributional properties for this new process. This process is analysed by deriving the first and second moments, and then more importantly the Laplace transform of the intensity process and the probability generating function of the point process, respectively. A possible way of change of measure has been found via the infinitesimal generator. Furthermore, an explicit example of jumps with exponential distributions, and the simulation algorithm is provided for further industrial implementation and statistical analysis.

This chapter is organised as follows. Section 2.1 gives the mathematical definition of the process. Section 2.2 as the main section, analyses and derives some key distributional properties. The joint Laplace transform - probability generating function of the intensity process and the point process is derived in Section 2.2.1. The Laplace transform of the intensity process and the probability generating function of the point process are obtained in Section 2.2.2 and Section 2.2.3, respectively; the Hawkes process with exponential decay is included as an important special case and a brief summary of its distributional properties is also given. In Section 2.2.4, we obtain the first and second moments of the intensity process and the point process. We also provide an explicit example of jumps with exponential distributions in Section 2.3, and the algorithm for simulating the process in Section 2.4.1.

2.1 Definition

The dynamic contagion process includes both the self-excited jumps, which are distributed according to the branching structure of a Hawkes process with exponential fertility rate, and the externally excited jumps, which are distributed according to a particular shot noise Cox process.

Daley and Vere-Jones (2003) (see also Hawkes and Oakes (1974)) give a cluster process representation for a general Hawkes process, now we extend it to represent the mathematical definition for our process in *Definition 2.1.1* as a cluster point process, additionally characterised by the stochastic intensity representation and infinitesimal generator.

Definition 2.1.1. *The* ***dynamic contagion process*** *is a cluster point process $\mathbb{D}$ on $\mathbb{R}_+$: The number of points in the time interval $(0,t]$ is defined by $N_t = N_{\mathbb{D}(0,t]}$. The cluster centers of $\mathbb{D}$ are the particular points called immigrants, the other points are called offspring. They have the following structure:*

(a) The immigrants are distributed according to a Cox process A with points $\{D_m\}_{m=1,2,...} \in (0,\infty)$ and shot noise stochastic intensity process

$$a + (\lambda_0 - a)\, e^{-\delta t} + \sum_{i\geq 1} Y_i^{(1)} e^{-\delta(t-T_i^{(1)})} \mathbb{I}\left\{T_i^{(1)} \leq t\right\},$$

where

- *$a \geq 0$ is the constant reversion level;*
- *$\lambda_0 > 0$ is a constant as the initial value of the stochastic intensity process (defined later by (2.1));*
- *$\delta > 0$ is the constant rate of exponential decay;*
- *$\left\{Y_i^{(1)}\right\}_{i=1,2,...}$ is a sequence of independent identical distributed positive (externally excited) jumps with distribution function $H(y), y > 0$, at the corresponding random times $\left\{T_i^{(1)}\right\}_{i=1,2,...}$ following a homogeneous Poisson process with constant intensity $\rho > 0$;*
- *$\mathbb{I}$ is the indicator function.*

(b) Each immigrant D_m generates a cluster $C_m = C_{D_m}$, which is the random set formed by the points of generations $0, 1, 2, ...$ with the following branching structure:

the immigrant D_m is said to be of generation 0. Given generations $0, 1, ..., j$ in C_m, each point $T^{(2)} \in C_m$ of generation j generates a Cox process on $(T^{(2)}, \infty)$ of offspring of generation $j+1$ with the stochastic intensity $Y^{(2)} e^{-\delta(\cdot - T^{(2)})}$ where $Y^{(2)}$ is a positive (self-excited) jump at time $T^{(2)}$ with distribution function $G(y), y > 0$, independent of the points of generation $0, 1, ..., j$.

(c) Given the immigrants, the centered clusters

$$C_m - D_m = \left\{T^{(2)} - D_m : T^{(2)} \in C_m\right\}, \quad D_m \in A,$$

are independent identical distributed, and independent of A.

(d) $\mathbb{D}$ consists of the union of all clusters, i.e.

$$\mathbb{D} = \bigcup_{m=1,2,...} C_{D_m}.$$

Therefore, the dynamic contagion process can also be defined as a point process $N_t \equiv \left\{T_k^{(2)}\right\}_{k\geq 1}$ on $\mathbb{R}_+$, with the non-negative $\mathcal{F}_t-$stochastic intensity process λ_t following the piecewise deterministic dynamics with positive jumps, i.e.

$$\lambda_t = a + (\lambda_0 - a)\, e^{-\delta t} + \sum_{i\geq 1} Y_i^{(1)} e^{-\delta(t-T_i^{(1)})} \mathbb{I}\left\{T_i^{(1)} \leq t\right\} + \sum_{k\geq 1} Y_k^{(2)} e^{-\delta(t-T_k^{(2)})} \mathbb{I}\left\{T_k^{(2)} \leq t\right\}, \quad (2.1)$$

where

- $\{\mathcal{F}_t\}_{t\geq 0}$ *is a history of the process* N_t*, with respect to which* $\{\lambda_t\}_{t\geq 0}$ *is adapted,*
- $\left\{Y_k^{(2)}\right\}_{k=1,2,\ldots}$ *is a sequence of independent identical distributed positive (self-excited) jumps with distribution function* $G(y), y>0$*, at the corresponding random times* $\left\{T_k^{(2)}\right\}_{k=1,2,\ldots}$*,*
- *the sequences* $\left\{Y_i^{(1)}\right\}_{i=1,2,\ldots}$*,* $\left\{T_i^{(1)}\right\}_{i=1,2,\ldots}$ *and* $\left\{Y_k^{(2)}\right\}_{k=1,2,\ldots}$ *are assumed to be independent of each other.*

From the definition above and because of the exponential decay, we can see that λ_t is a Markov process. In particular, it decreases with rate $\delta(\lambda_t - a)$, and incurs additive upward (externally excited) jumps that have distribution function H with rate ρ, and additive upward (self-excited) jumps that have distribution function G with rate λ_t. Moreover, when jumps of the latter type occur, N_t increases by 1. Hence, (N_t, λ_t) is also a Markov process.

With the aid of piecewise deterministic Markov process theory and using the results in Davis (1984), the infinitesimal generator of the dynamic contagion process (λ_t, N_t, t) acting on a function $f(\lambda, n, t)$ within its domain $\Omega(\mathcal{A})$ is given by

$$\mathcal{A}f(\lambda,n,t)=\frac{\partial f}{\partial t}-\delta(\lambda-a)\frac{\partial f}{\partial \lambda}+(\rho+\lambda)\left[\int_0^\infty f(\lambda+y,n,t)\mathrm{d}\left(\frac{\rho}{\rho+\lambda}H(y)+\frac{\lambda}{\rho+\lambda}G(y)\right)-f(\lambda,n,t)\right],$$

or,

$$\begin{aligned}\mathcal{A}f(\lambda,n,t) &= \frac{\partial f}{\partial t}-\delta(\lambda-a)\frac{\partial f}{\partial \lambda}+\rho\left(\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y)-f(\lambda,n,t)\right)\\ &\quad+\lambda\left(\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y)-f(\lambda,n,t)\right), \qquad (2.2)\end{aligned}$$

where $\Omega(\mathcal{A})$ is the domain for the generator $\mathcal{A}$ such that $f(\lambda,n,t)$ is differentiable with respect to λ, t for all λ, n and t, and

$$\begin{aligned}\left|\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y)-f(\lambda,n,t)\right| &< \infty,\\ \left|\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y)-f(\lambda,n,t)\right| &< \infty.\end{aligned}$$

Remark 2.1.1. We could alternatively define the dynamic contagion process as a special case (without the diffusion terms) of the general affine point processes by Duffie, Filipović and Schachermayer (2003), with the infinitesimal generator specified by (2.2).

Remark 2.1.2. The dynamic contagion process is a point process N_t such that

$$\begin{aligned}P\left\{N_{t+\Delta t}-N_t=1\middle|N_t\right\} &= \lambda_t\Delta t+o(\Delta t),\\ P\left\{N_{t+\Delta t}-N_t>1\middle|N_t\right\} &= o(\Delta t),\end{aligned}$$

where Δt is a sufficient small time interval and λ_t is given by (2.1).

Remark 2.1.3. Note that, this point process is not a doubly stochastic Poisson process, or Cox process, as the point process N_t conditional on λ_t is not a Poisson process, and it does not satisfy its definition. In particular for all $0\leq t\leq T$,

$$\mathbb{E}\left[\theta^{(N_T-N_t)}\middle|\mathcal{F}_t\right]\neq\mathbb{E}\left[e^{-(1-\theta)(\Lambda_T-\Lambda_t)}\middle|\mathcal{F}_t\right], \qquad (2.3)$$

where $\Lambda_t =: \int_0^t \lambda_s \mathrm{d}s$ is the aggregated process, and the expectation $\mathbb{E}$ is based on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the information set $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. This is because, the infinitesimal generator acting on a function $f(\lambda, \Lambda, n)$ is given by

$$\begin{aligned}\mathcal{A}f(\lambda,\Lambda,n) &= \lambda\frac{\partial f}{\partial \Lambda} - \delta(\lambda - a)\frac{\partial f}{\partial \lambda} + \rho\Big(\int_0^\infty f(\lambda+y,\Lambda,n)\mathrm{d}H(y) - f(\lambda,\Lambda,n)\Big) \\ &\quad +\lambda\Big(\int_0^\infty f(\lambda+y,\Lambda,n+1)\mathrm{d}G(y) - f(\lambda,\Lambda,n)\Big),\end{aligned}$$

and the generators for the processes $e^{-(1-\theta)\Lambda_t}$ and θ^{N_t} are then given by

$$\begin{aligned}\mathcal{A}\left(e^{-(1-\theta)\Lambda}\right) &= -(1-\theta)e^{-(1-\theta)\Lambda}\lambda, \\ \mathcal{A}\left(\theta^n\right) &= -(1-\theta)\theta^n\lambda,\end{aligned}$$

where furthermore the generators for the processes $e^{-(1-\theta)\Lambda_t}\lambda_t$ and $\theta^{N_t}\lambda_t$ are given by

$$\mathcal{A}\left(e^{-(1-\theta)\Lambda}\lambda\right) = e^{-(1-\theta)\Lambda}\Big(-(1-\theta)\lambda^2 + \delta(a-\lambda) + \rho\mu_{1_H} + \lambda\mu_{1_G}\Big), \tag{2.4}$$

$$\mathcal{A}\left(\theta^n\lambda\right) = \theta^n\Big(-(1-\theta)\lambda^2 + \delta(a-\lambda) + \rho\mu_{1_H} + \lambda\mu_{1_G}\theta\Big). \tag{2.5}$$

If it was an equality in (2.3) for all t, then (2.4) and (2.5) would have been the same equation. However, (2.4) and (2.5) can not be the same as there is an extra term θ in (2.5). Therefore, the intensity λ_t given here in (2.1) is different from the stochastic intensity of the Cox process.

Remark 2.1.4. Note that, the intensity process λ_t is always above the level a, i.e. $\lambda_t \in E = [a, \infty)$ for any time t.

Remark 2.1.5. The parameters in the intensity process λ_t measure some key aspects of the events: the long term mean-reverting effect, the frequency of the underlying events, the magnitude of the impact from the events, and the time effect with exponential memory decay.

Remark 2.1.6. An economic interpretation from the perspective of the cluster process representation for the dynamic contagion process is the following: For a certain company, there are two classes of economic shocks: the primary shocks directly to this company and the common market-wide shocks. The arrivals of these primary shocks to this company are modelled by the generation 0 of the dynamic contagion process, i.e. the point process A (as described by **(a)**) with the intensity process modelled based on the external economic evolution including a stream of market-wide shocks: a shock at time $T_i^{(1)}$ has the magnitude of impact $Y_i^{(1)}$ with distribution H and decays exponentially with rate δ. In the aftermath of each primary shock to this company, it could further trigger a series of subsidiary internal turbulences in this company following the branching structure (as described by **(b)**): similarly a turbulence at time $T_k^{(2)}$ has the magnitude of impact $Y_k^{(2)}$ with distribution G and decays exponentially with rate δ.

To give an intuitive picture of this right-continuous process from the perspective of the stochastic intensity representation, we present *Figure 2.1* for illustrating how the externally excited jumps $\left\{Y_i^{(1)}\right\}_{i=1,2,\dots}$ (marked by single arrow $\downarrow$) and self-excited jumps $\left\{Y_k^{(2)}\right\}_{k=1,2,\dots}$ (marked by double arrow $\updownarrow$) in the intensity process λ_t interact with its dynamic contagion point process N_t.

Now, in this more general framework of the dynamic contagion process, the classic Cox process with shot noise intensity (with exponential decay), used by Dassios and Jang (2003) for

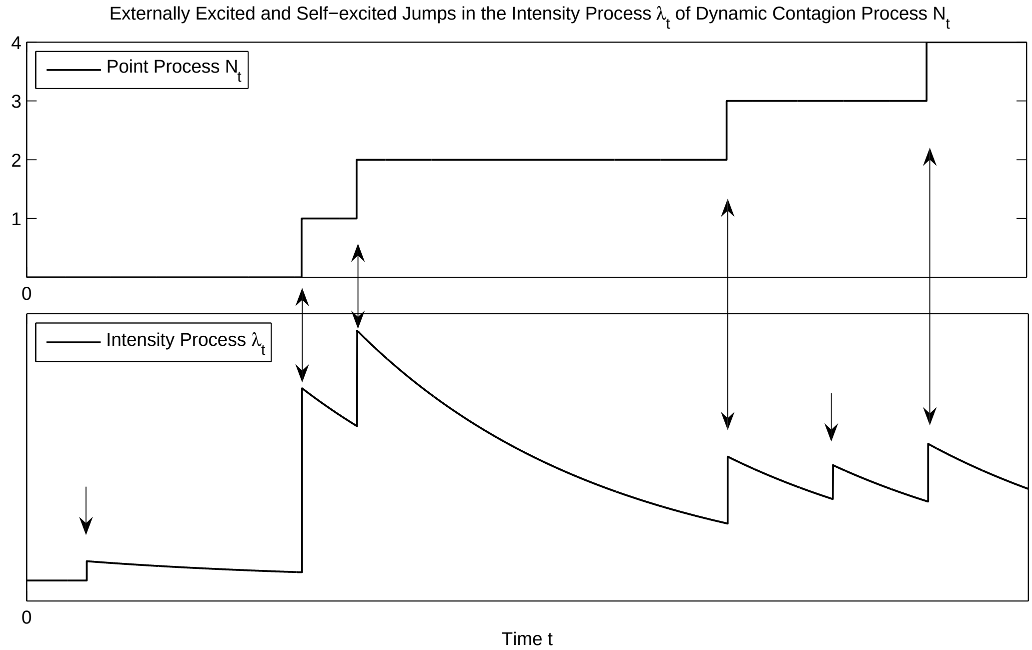

Fig. 2.1: Externally Excited and Self-excited Jumps in Intensity Process λ_t of Dynamic Contagion Process N_t

pricing catastrophe reinsurance and derivatives, can be recovered, by setting reversion level $a = 0$ and eliminating the self-excited jumps $\left\{Y_k^{(2)}\right\}_{k=1,2,\ldots}$; the Hawkes process (with exponential decay), used by Errais, Giesecke and Goldberg (2009) for modelling the portfolio credit risk, can be recovered, by setting the intensity $\rho = 0$ of the externally excited jumps $\left\{Y_i^{(1)}\right\}_{i=1,2,\ldots}$.

2.2 Distributional Properties

2.2.1 Joint Laplace Transform - Probability Generating Function of (λ_T, N_T)

We derive the joint Laplace transform - probability generating function of (λ_T, N_T) for a fixed time T in *Theorem 2.2.1* below, which leads to the key results of this paper, Laplace transform of λ_T and probability generating function of N_T in Section 2.2.2 and Section 2.2.3, respectively.

Theorem 2.2.1. *For the constants $0 \leq \theta \leq 1$, $v \geq 0$ and time $0 \leq t \leq T$, the conditional joint Laplace transform - probability generating function for the process λ_t (defined in Definition 2.1.1) and the point process N_t is given by*

$$\mathbb{E}\left[\theta^{(N_T - N_t)} e^{-v\lambda_T} \middle| \mathcal{F}_t\right] = e^{-\left(c(T)-c(t)\right)} e^{-B(t)\lambda_t}, \tag{2.6}$$

where $B(t)$ is determined by the non-linear ODE

$$-B'(t) + \delta B(t) + \theta \hat{g}\Big(B(t)\Big) - 1 = 0, \tag{2.7}$$

$$\hat{g}(u) =: \int_0^{\infty} e^{-uy} \mathrm{d}G(y),$$

with boundary condition $B(T) = v$; and $c(t)$ is determined by

$$c(t) = a\delta \int_0^t B(s)\mathrm{d}s + \rho \int_0^t \left[1 - \hat{h}\Big(B(s)\Big)\right] \mathrm{d}s, \tag{2.8}$$

$$\hat{h}(u) =: \int_0^\infty e^{-uy}\mathrm{d}H(y).$$

Proof. Consider a function $f(\lambda, n, t)$ with an exponential affine form

$$f(\lambda, n, t) = e^{c(t)}A^n(t)e^{-B(t)\lambda},$$

where $A(t), B(t), c(t)$ are all deterministic functions of time t. Substitute into $\mathcal{A}f = 0$ in (2.2), we then have

$$\frac{A'(t)}{A(t)}n + \Big(-B'(t) + \delta B(t) + A(t)\hat{g}(B(t)) - 1\Big)\lambda + \Big(c'(t) + \rho\hat{h}(B(t)) - \rho - a\delta B(t)\Big) = 0. \tag{2.9}$$

Since this equation holds for any n and λ, it is equivalent to solving three separated equations

$$\begin{cases} \frac{A'(t)}{A(t)} = 0 & (.1) \\ -B'(t) + \delta B(t) + A(t)\hat{g}(B(t)) - 1 = 0 & (.2) \\ c'(t) + \rho\hat{h}(B(t)) - \rho - a\delta B(t) = 0 & (.3) \end{cases}. \tag{2.10}$$

We have $A(t) = \theta$ immediately from (2.10.1); and substitute into (2.10.2) by adding the boundary condition $B(T) = v$, we have the ODE as (2.7); then, by (2.10.3) with boundary condition $c(0) = 0$, the integration as (2.8) follows. Since $e^{c(t)}\theta^{N_t}e^{-B(t)\lambda_t}$ is a $\mathcal{F}-$martingale by the property of the infinitesimal generator, we have

$$\mathbb{E}\left[e^{c(T)}\theta^{N_T}e^{-B(T)\lambda_T}\middle|\mathcal{F}_t\right] = e^{c(t)}\theta^{N_t}e^{-B(t)\lambda_t}. \tag{2.11}$$

Then, by the boundary condition $B(T) = v$, (2.6) follows. □

2.2.2 Laplace Transform of λ_T

Theorem 2.2.2. *The conditional Laplace transform λ_T given λ_0 at time $t = 0$, under the condition $\delta > \mu_{1_G}$, is given by*

$$\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right] = \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{a\delta u + \rho[1 - \hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right) \times \exp\Big(-\mathcal{G}_{v,1}^{-1}(T)\lambda_0\Big), \tag{2.12}$$

where

$$\mu_{1_G} =: \int_0^\infty y\mathrm{d}G(y),$$

$$\mathcal{G}_{v,1}(L) =: \int_L^v \frac{\mathrm{d}u}{\delta u + \hat{g}(u) - 1}{}^{1}. \tag{2.13}$$

Proof. By setting $t = 0$ and $\theta = 1$ in *Theorem 2.2.1*, we have

$$\mathbb{E}\left[e^{-v\lambda_T}\middle|\mathcal{F}_0\right] = e^{-c(T)}e^{-B(0)\lambda_0}, \tag{2.14}$$

where $B(0)$ is uniquely determined by the non-linear ODE

$$-B'(t) + \delta B(t) + \hat{g}\Big(B(t)\Big) - 1 = 0,$$

with boundary condition $B(T) = v$. It can be solved, under the condition $\delta > \mu_{1_G}$, by the following steps:

[1] It will be clear in the proof later that $\mathcal{G}_{v,1}(L)$ is a one by one function of L and hence its inverse function $\mathcal{G}_{v,1}^{-1}(T)$ exists.

1. Set $B(t) = L(T-t)$ and $\tau = T - t$, it is equivalent to the initial value problem

$$\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau} = 1 - \delta L(\tau) - \hat{g}(L(\tau)) =: f_1(L), \tag{2.15}$$

   with initial condition $L(0) = v$; we define the right-hand side as the function $f_1(L)$.

2. Under the condition $\delta > \mu_{1_G}$, we have

$$\frac{\partial f_1(L)}{\partial L} = \int_0^\infty y e^{-Ly}\mathrm{d}G(y) - \delta \leq \int_0^\infty y\mathrm{d}G(y) - \delta = \mu_{1_G} - \delta < 0, \quad L \geq 0,$$

   then, $f_1(L) < 0$ for $L > 0$, since $f_1(0) = 0$.

3. Rewrite (2.15) as

$$\frac{\mathrm{d}L}{\delta L + \hat{g}(L) - 1} = -\mathrm{d}\tau,$$

   by integrating both sides from time 0 to τ with initial condition $L(0) = v > 0$, we have

$$\int_L^v \frac{\mathrm{d}u}{\delta u + \hat{g}(u) - 1} = \tau,$$

   where $L \geq 0$, we define the function on left hand side as

$$\mathcal{G}_{v,1}(L) =: \int_L^v \frac{\mathrm{d}u}{\delta u + \hat{g}(u) - 1},$$

   then,

$$\mathcal{G}_{v,1}(L) = \tau,$$

   obviously $L \to v$ when $\tau \to 0$; by convergence test,

$$\lim_{u\to 0} \frac{\frac{1}{u}}{\frac{1}{\delta u + \hat{g}(u) - 1}} = \delta + \lim_{u \to 0} \frac{\hat{g}(u) - 1}{u} = \delta - \mu_{1_G} > 0,$$

   and we know that $\int_0^v \frac{1}{u}\mathrm{d}u = \infty$, then,

$$\int_0^v \frac{\mathrm{d}u}{\delta u + \hat{g}(u) - 1} = \infty,$$

   hence, $L \to 0$ when $\tau \to \infty$; the integrand is positive in the domain $u \in (0, v]$ and also for $L \leq v$, $\mathcal{G}_{v,1}(L)$ is a strictly decreasing function; therefore, $\mathcal{G}_{v,1}(L) : (0, v] \to [0, \infty)$ is a well defined (monotone) function, and its inverse function $\mathcal{G}_{v,1}^{-1}(\tau) : [0, \infty) \to (0, v]$ exists.

4. The unique solution is found by

$$L(\tau) = \mathcal{G}_{v,1}^{-1}(\tau),$$

   or,

$$B(t) = \mathcal{G}_{v,1}^{-1}(T - t).$$

5. $B(0)$ is obtained,

$$B(0) = L(T) = \mathcal{G}_{v,1}^{-1}(T).$$

Then, $c(T)$ is determined by

$$c(T) = a\delta \int_0^T \mathcal{G}_{v,1}^{-1}(\tau)\mathrm{d}\tau + \rho \int_0^T \left[1 - \hat{h}\left(\mathcal{G}_{v,1}^{-1}(\tau)\right)\right] \mathrm{d}\tau, \tag{2.16}$$

by the change of variable $\mathcal{G}_{v,1}^{-1}(\tau) = u$, we have $\tau = \mathcal{G}_{v,1}(u)$, and

$$\int_0^T \left[1 - \hat{h}\left(\mathcal{G}_{v,1}^{-1}(\tau)\right)\right] \mathrm{d}\tau = \int_{\mathcal{G}_{v,1}^{-1}(0)}^{\mathcal{G}_{v,1}^{-1}(T)} [1 - \hat{h}(u)] \frac{\partial \tau}{\partial u} \mathrm{d}u = \int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{1 - \hat{h}(u)}{\delta u + \hat{g}(u) - 1} \mathrm{d}u,$$

similarly,

$$\int_0^T \mathcal{G}_{v,1}^{-1}(\tau)\mathrm{d}\tau = \int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{u}{\delta u + \hat{g}(u) - 1} \mathrm{d}u.$$

Finally, substitute $B(0)$ and $c(T)$ into (2.14), and *Theorem 2.2.2* follows. □

Theorem 2.2.3. *If $\delta > \mu_{1_G}$, then the Laplace transform of the asymptotic distribution of λ_T is given by*

$$\lim_{T \to \infty} \mathbb{E}\left[e^{-v\lambda_T} \middle| \lambda_0\right] = \exp\left(- \int_0^v \frac{a\delta u + \rho[1 - \hat{h}(u)]}{\delta u + \hat{g}(u) - 1} \mathrm{d}u \right), \tag{2.17}$$

and this is also the Laplace transform of the stationary distribution of the process $\{\lambda_t\}_{t \geq 0}$.

Proof. Let $T \to \infty$ in *Theorem 2.2.2*, then $\mathcal{G}_{v,1}^{-1}(T) \to 0$ and the Laplace transform of the asymptotic distribution follows immediately as given by (2.17).

To further prove the stationarity, by *Proposition 9.2* of Ethier and Kurtz (1986) (and see also Costa (1990)), we need to prove that, for any function f within its domain $\Omega(\mathcal{A})$, we have

$$\int_E \mathcal{A}f(\lambda)\Pi(\lambda)\mathrm{d}\lambda = 0, \tag{2.18}$$

where $E = [a, \infty)$ is the domain for λ, $\mathcal{A}f(\lambda)$ is the infinitesimal generator of the dynamic contagion process acting on $f(\lambda)$, i.e.

$$\begin{aligned} \mathcal{A}f(\lambda) &= -\delta(\lambda - a)\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} + \rho\left(\int_0^\infty f(\lambda + y)\mathrm{d}H(y) - f(\lambda)\right) \\ &\quad + \lambda\left(\int_0^\infty f(\lambda + y)\mathrm{d}G(y) - f(\lambda)\right), \end{aligned}$$

and $\Pi(\lambda)$ is the density function of λ with the Laplace transform given by (2.17).

We will now try to solve equation (2.18). For the first term of (2.18), we have

$$\begin{aligned} \int_E \left[-\delta(\lambda - a)\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\right]\Pi(\lambda)\mathrm{d}\lambda &= -\delta \int_a^\infty (\lambda - a)f'(\lambda)\Pi(\lambda)\mathrm{d}\lambda \\ &= -\delta \int_{\lambda=a}^\infty f'(\lambda) \int_{u=a}^\lambda \Big[(u - a)\Pi(u)\Big]'\mathrm{d}u\mathrm{d}\lambda \\ &= -\delta \int_{u=a}^\infty \int_{\lambda=u}^\infty f'(\lambda)\Big[(u - a)\Pi(u)\Big]'\mathrm{d}\lambda\mathrm{d}u \\ &= \delta \int_a^\infty f(u)\Big[(u - a)\Pi(u)\Big]'\mathrm{d}u \\ &= \delta \int_a^\infty f(\lambda)\Big[(\lambda - a)\Pi(\lambda)\Big]'\mathrm{d}\lambda, \end{aligned}$$

since for a density function Π, obviously,

$$\lim_{u\to a}(u-a)\Pi(u)=0.$$

For the second term of (2.18), by change variable $\lambda+y=s$ $(y\le s)$ in the double integral, we have

$$\begin{aligned}\int_E\left[\rho\int_0^{\infty}f(\lambda+y)\mathrm{d}H(y)\right]\Pi(\lambda)\mathrm{d}\lambda &= \rho\int_{\lambda=a}^{\infty}\Pi(\lambda)\int_{y=0}^{\infty}f(\lambda+y)\mathrm{d}H(y)\mathrm{d}\lambda\\ &= \rho\int_{s=a}^{\infty}f(s)\int_{y=0}^{s}\Pi(s-y)\mathrm{d}H(y)\mathrm{d}s\\ &= \rho\int_{\lambda=a}^{\infty}f(\lambda)\int_{y=0}^{\lambda}\Pi(\lambda-y)\mathrm{d}H(y)\mathrm{d}\lambda.\end{aligned}$$

For the third term of (2.18), by change variable $\lambda+y=s$ $(y\le s)$ in the double integral, we have

$$\begin{aligned}\int_E\left[\lambda\left(\int_0^{\infty}f(\lambda+y)\mathrm{d}G(y)\right)\right]\Pi(\lambda)\mathrm{d}\lambda &= \int_{\lambda=a}^{\infty}\lambda\Pi(\lambda)\int_{y=0}^{\infty}f(\lambda+y)\mathrm{d}G(y)\mathrm{d}\lambda\\ &= \int_{s=a}^{\infty}f(s)\int_{y=0}^{s}(s-y)\Pi(s-y)\mathrm{d}G(y)\mathrm{d}s\\ &= \int_{\lambda=a}^{\infty}f(\lambda)\int_{y=0}^{\lambda}(\lambda-y)\Pi(\lambda-y)\mathrm{d}G(y)\mathrm{d}\lambda.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_E\mathcal{A}f(\lambda)\Pi(\lambda)\mathrm{d}\lambda\\ =&\int_E\left[-\delta\,(\lambda-a)\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}+\rho\left(\int_0^{\infty}f(\lambda+y)\mathrm{d}H(y)-f(\lambda)\right)+\lambda\left(\int_0^{\infty}f(\lambda+y)\mathrm{d}G(y)-f(\lambda)\right)\right]\Pi(\lambda)\mathrm{d}\lambda\\ =&\int_a^{\infty}f(\lambda)\Big[\delta\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big((\lambda-a)\Pi(\lambda)\Big)+\rho\left(\int_0^{\lambda}\Pi(\lambda-y)\mathrm{d}H(y)-\Pi(\lambda)\right)\\ &+\left(\int_0^{\lambda}(\lambda-y)\Pi(\lambda-y)\mathrm{d}G(y)-\lambda\Pi(\lambda)\right)\Big]\mathrm{d}\lambda.\end{aligned}$$

Set

$$\int_E\mathcal{A}f(\lambda)\Pi(\lambda)\mathrm{d}\lambda=0,$$

for any function $f(\lambda)\in\Omega(\mathcal{A})$, then,

$$\delta\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big((\lambda-a)\Pi(\lambda)\Big)+\rho\left(\int_0^{\lambda}\Pi(\lambda-y)\mathrm{d}H(y)-\Pi(\lambda)\right)+\left(\int_0^{\lambda}(\lambda-y)\Pi(\lambda-y)\mathrm{d}G(y)-\lambda\Pi(\lambda)\right)=0,$$

by the Laplace transform

$$\hat{\Pi}(v)=:\mathcal{L}\{\Pi(\lambda)\}=\int_E\Pi(\lambda)e^{-v\lambda}\mathrm{d}\lambda,$$

we have

$$\begin{aligned}\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big((\lambda-a)\Pi(\lambda)\Big)\right\} &= v\mathcal{L}\{(\lambda-a)\Pi(\lambda)\}=v\left(-\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}-a\hat{\Pi}(v)\right),\\ \mathcal{L}\left\{\int_0^{\lambda}\Pi(\lambda-y)\mathrm{d}H(y)\right\} &= \mathcal{L}\left\{\int_0^{\lambda}\Pi(\lambda-y)h(y)\mathrm{d}y\right\}=\hat{\Pi}(v)\hat{h}(v),\\ \mathcal{L}\left\{\int_0^{\lambda}(\lambda-y)\Pi(\lambda-y)\mathrm{d}G(y)\right\} &= \mathcal{L}\left\{\int_0^{\lambda}(\lambda-y)\Pi(\lambda-y)g(y)\mathrm{d}y\right\}\\ &= \mathcal{L}\{\lambda\Pi(\lambda)\}\,\hat{g}(v)\\ &= -\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}\hat{g}(v),\end{aligned}$$

then,

$$\delta v\left(-\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}-a\hat{\Pi}(v)\right)+\rho[\hat{h}(v)-1]\hat{\Pi}(v)+\Big(1-\hat{g}(v)\Big)\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}=0,$$

or,

$$\Big(1-\delta v-\hat{g}(v)\Big)\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}+\Big(-a\delta v+\rho[\hat{h}(v)-1]\Big)\hat{\Pi}(v)=0,$$

which is an ODE with the solution given by

$$\hat{\Pi}(v)=\hat{\Pi}(0)\exp\left(-\int_0^v\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right).$$

Note that, given the initial condition

$$\hat{\Pi}(0)=\int_E\Pi(\lambda)\mathrm{d}\lambda=1,$$

we have the unique solution

$$\hat{\Pi}(v)=\exp\left(-\int_0^v\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right),$$

which is exactly given by (2.17).

Since Π is the unique solution to (2.18), we have the stationarity for the intensity process $\{\lambda_t\}_{t\geq 0}$. □

Remark 2.2.1. The integral of (2.17) exists, since we have

$$\lim_{u\to 0}\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}=a\delta\lim_{u\to 0}\frac{1}{\delta-\frac{1-\hat{g}(u)}{u}}+\rho\lim_{u\to 0}\frac{1-\hat{h}(u)}{u}\frac{1}{\delta-\frac{1-\hat{g}(u)}{u}}=\frac{a\delta+\mu_{1_H}\rho}{\delta-\mu_{1_G}}>0$$

Remark 2.2.2. We can also prove the Laplace transform of the distribution of the stationary intensity process λ_t of (2.17) as follows.

Proof. Assume $f(\lambda,n,t)=e^{-v\lambda}$ and we have

$$\mathcal{A}\left(e^{-v\lambda}\right)=e^{-v\lambda}\left[-a\delta v+\rho[\hat{h}(v)-1]+\Big(\delta v+\hat{g}(v)-1\Big)\lambda\right],$$

then,

$$\begin{aligned}&\mathbb{E}\left[e^{-v\lambda_t}\big|\mathcal{F}_0\right]\\=&\int_0^t\mathbb{E}\left[\mathcal{A}\left(e^{-v\lambda_s}\right)\big|\mathcal{F}_0\right]\mathrm{d}s+e^{-v\lambda_0}\\=&\int_0^t\left[\Big(-a\delta v+\rho[\hat{h}(v)-1]\Big)\mathbb{E}\left[e^{-v\lambda_s}\big|\mathcal{F}_0\right]+(\delta v+\hat{g}(v)-1)\,\mathbb{E}\left[\lambda_s e^{-v\lambda_s}\big|\mathcal{F}_0\right]\right]\mathrm{d}s+e^{-v\lambda_0}.\end{aligned}$$

Differentiate two sides with respect to t, as

$$\frac{\partial}{\partial t}\int_0^t\mathbb{E}\left[\lambda_s e^{-v\lambda_s}\big|\mathcal{F}_0\right]=-\frac{\partial}{\partial v}\mathbb{E}\left[e^{-v\lambda_s}\big|\mathcal{F}_0\right],$$

we have

$$\frac{\partial\mathbb{E}\left[e^{-v\lambda_t}\big|\mathcal{F}_0\right]}{\partial t}=\Big(-a\delta v+\rho[\hat{h}(v)-1]\Big)\mathbb{E}\left[e^{-v\lambda_t}\big|\mathcal{F}_0\right]-(\delta v+\hat{g}(v)-1)\frac{\partial}{\partial v}\mathbb{E}\left[e^{-v\lambda_s}\big|\mathcal{F}_0\right].$$

Denote $\hat{\Pi}(v,t)=\mathbb{E}\left[e^{-v\lambda_t}\middle|\mathcal{F}_0\right]$, we have the first-order PDE

$$\frac{\partial\hat{\Pi}(v,t)}{\partial t}=\left(-a\delta v+\rho[\hat{h}(v)-1]\right)u(v,t)-(\delta v+\hat{g}(v)-1)\,\frac{\partial\hat{\Pi}(v,t)}{\partial v},$$

with the boundary conditions $\hat{\Pi}(0,t)=1$ and $\hat{\Pi}(v,0)=e^{-v\lambda_0}$. Because of the stationarity, $\hat{\Pi}(v,t)$ should be independent of time t, then,

$$\left(-a\delta v+\rho[\hat{h}(v)-1]\right)\hat{\Pi}(v)-(\delta v+\hat{g}(v)-1)\,\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}=0,$$

with the boundary condition $\hat{\Pi}(0)=1$, and we have

$$\hat{\Pi}(v)=\exp\left(-\int_0^v\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right),$$

which recovers (2.17). □

Remark 2.2.3. For instance, if λ_0 follows the distribution given by (2.17), then, based on *Theorem 2.2.2* for any time $T\geq 0$, we have

$$\begin{aligned}\mathbb{E}\left[e^{-v\lambda_T}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right]\right]\\ &= \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^{v}\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right)\mathbb{E}\left[e^{-\mathcal{G}_{v,1}^{-1}(T)\lambda_0}\right]\\ &= \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^{v}\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right)\times\exp\left(-\int_0^{\mathcal{G}_{v,1}^{-1}(T)}\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right)\\ &= \exp\left(-\int_0^{v}\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right).\end{aligned}$$

Alternative approaches for proving the stationarity for the special case of the Hawkes process and other related processes can be found in Hawkes and Oakes (1974), Brémaud and Massoulié (1996) and Massoulié (1998).

The self-excited Hawkes process was introduced theoretically by Hawkes (1971), and applied to risk theory by Chavez-Demoulin, Davison and Mc Neil (2005), and then only very recently applied to credit risk for modelling the default contagion by Errais, Giesecke and Goldberg (2009). It can be considered as an important special case under this more general framework of dynamic contagion process, all of the counterpart results can be obtained, by eliminating the impact from the externally excited jumps, i.e. setting its intensity $\rho=0$ in the corresponding results. Here we give the Laplace transform of the stationary distribution of the intensity process λ_t for the Hawkes process with exponential decay in *Corollary 2.2.1.* The probability generating function of the Hawkes point process N_t will be given by *Corollary 2.2.3* of Section 2.2.3.

Corollary 2.2.1. *If $\delta>\mu_{1_G}$, then the Laplace transform of the intensity λ_T of the Hawkes process with exponential decay conditional on λ_0 at time $t=0$ is given by*

$$\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right]=\exp\left(-a\delta\int_{\mathcal{G}_{v,1}^{-1}(T)}^{v}\frac{u}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right)\times\exp\left(-\mathcal{G}_{v,1}^{-1}(T)\lambda_0\right),$$

and the Laplace transform of the asymptotic distribution of λ_T is given by

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-v\lambda_T}\big|\lambda_0\right] = \exp\left(-a\delta\int_0^v \frac{u}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right), \tag{2.19}$$

which is also the Laplace transform of the stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$.

Proof. By setting the intensity of the externally excited jumps $\rho = 0$ in *Theorem 2.2.2* and *Theorem 2.2.3*, the results follow immediately. □

The limit of the log-Laplace transform for Hawkes processes with a general fertility rate can be found in Bordenave and Torrisi (2007) and Stabile and Torrisi (2010).

2.2.3 Probability Generating Function of N_T

Theorem 2.2.4. *The conditional probability generating function of N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under the condition $\delta > \mu_{1_G}$, is given by*

$$\mathbb{E}\Big[\theta^{N_T}\big|\lambda_0\Big] = \exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{a\delta u+\rho[1-\hat{h}(u)]}{1-\delta u-\theta\hat{g}(u)}\mathrm{d}u\right)\times\exp\Big(-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0\Big),$$

where

$$\mathcal{G}_{0,\theta}(L) =: \int_0^L \frac{\mathrm{d}u}{1-\delta u-\theta\hat{g}(u)}, \quad 0\leq\theta<1. \tag{2.20}$$

Proof. By setting $t = 0$, $v = 0$ and assuming $N_0 = 0$ in *Theorem 2.2.1*, we have

$$\mathbb{E}\Big[\theta^{N_T}\big|\mathcal{F}_0\Big] = e^{-c(T)}e^{-B(0)\lambda_0},$$

where $B(0)$ is uniquely determined by the non-linear ODE

$$-B'(t)+\delta B(t)+\theta\hat{g}\Big(B(t)\Big)-1 = 0,$$

with boundary condition $B(T) = 0$. It can be solved, under the condition $\delta > \mu_{1_G}$, by the following steps:

1. Set $B(t) = L(T-t)$ and $\tau = T-t$,

$$\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau} = 1-\delta L(\tau)-\theta\hat{g}(L(\tau)) =: f_2(L), \quad 0\leq\theta<1, \tag{2.21}$$

   with initial condition $L(0) = 0$; we define the right-hand side as the function $f_2(L)$.

2. There is only one positive singular point, denoted by $v^* > 0$, obtained by solving the equation $f_2(L) = 0$. This is because, for the case $0 < \theta < 1$, the equation $f_2(L) = 0$ is equivalent to

$$\hat{g}(u) = \frac{1}{\theta}(1-\delta u), \quad 0<\theta<1,$$

   note that $\hat{g}(\cdot)$ is a convex function, then it is clear that there is only one positive solution to this equation; for the case $\theta = 0$, there is only one singular point $v^* = \frac{1}{\delta} > 0$; and for both cases,

$$0 < \frac{1-\theta}{\delta} < v^* \leq \frac{1}{\delta}; \tag{2.22}$$

   then, we have $f_2(L) > 0$ for $0 \leq L < v^*$ and $f_2(L) < 0$ for $L > v^*$.

3. Rewrite (2.21) as
$$\frac{\mathrm{d}L}{1-\delta L-\theta\hat{g}(L)}=\mathrm{d}\tau,$$
and integrate,
$$\int_0^L \frac{\mathrm{d}u}{1-\delta u-\theta\hat{g}(u)}=\tau,$$
where $0 \le L < v^*$, we define the function on left-hand side as
$$\mathcal{G}_{0,\theta}(L) =: \int_0^L \frac{\mathrm{d}u}{1-\delta u-\theta\hat{g}(u)}, \tag{2.23}$$
then,
$$\mathcal{G}_{0,\theta}(L)=\tau,$$
as $L \to 0$ when $\tau \to 0$, and $L \to v^*$ when $\tau \to \infty$; the integrand is positive in the domain $u \in [0, v^*)$ and $L \ge 0$, $\mathcal{G}_{0,\theta}(L)$ is a strictly increasing function; therefore, $\mathcal{G}_{0,\theta}(L) : [0, v^*) \to [0, \infty)$ is a well defined function, and its inverse function $\mathcal{G}_{0,\theta}^{-1}(\tau) : [0,\infty) \to [0, v^*)$ exists.

4. The unique solution is found by
$$L(\tau)=\mathcal{G}_{0,\theta}^{-1}(\tau),$$
or,
$$B(t)=\mathcal{G}_{0,\theta}^{-1}(T-t).$$

5. $B(0)$ is obtained,
$$B(0)=L(T)=\mathcal{G}_{0,\theta}^{-1}(T).$$

Then, $c(T)$ is determined by
$$c(T)=a\delta\int_0^T \mathcal{G}_{0,\theta}^{-1}(\tau)\mathrm{d}\tau+\rho\int_0^T \left[1-\hat{h}\left(\mathcal{G}_{0,\theta}^{-1}(\tau)\right)\right]\mathrm{d}\tau, \tag{2.24}$$
where, by the change of variable,
$$\begin{aligned}\int_0^T \mathcal{G}_{0,\theta}^{-1}(\tau)\mathrm{d}\tau &= \int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{u}{1-\delta u-\theta\hat{g}(u)}\mathrm{d}u,\\ \int_0^T \left[1-\hat{h}\left(\mathcal{G}_{0,\theta}^{-1}(\tau)\right)\right]\mathrm{d}\tau &= \int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{1-\hat{h}(u)}{1-\delta u-\theta\hat{g}(u)}\mathrm{d}u.\end{aligned}$$
Finally, substitute $B(0)$ and $c(T)$ into (2.14), and the result follows. □

Remark 2.2.4. $\mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right]$ is a strictly decreasing function of time T and
$$\lim_{T\to 0}\mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right]=1,\quad \lim_{T\to\infty}\mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right]=0,\quad 0\le\theta<1.$$

Proof. The integrand is positive within its domain $u \in [0, v^*)$, where v^* is the only positive singular point, such that $1-\delta v^*-\theta\hat{g}(v^*)=0$. $\mathcal{G}_{0,\theta}^{-1}(T)$ is a strictly increasing function of time T. $\mathcal{G}_{0,\theta}^{-1}(T) \to 0$ when $T \to 0$; and $\mathcal{G}_{0,\theta}^{-1}(T) \to v^*$ when $T \to \infty$, and
$$\int_0^{v^*} \frac{\mathrm{d}u}{1-\delta u-\theta\hat{g}(u)}=\infty,$$
also,
$$\lim_{u\to v^*}\frac{a\delta u+\rho[1-\hat{h}(u)]}{1-\delta u-\theta\hat{g}(u)}\Big/\frac{1}{1-\delta u-\theta\hat{g}(u)}=a\delta v^*+\rho[1-\hat{h}(v^*)]>0.$$

Therefore,

$$\lim_{T\to\infty}\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{a\delta u+\rho[1-\hat{h}(u)]}{1-\delta u-\theta\hat{g}(u)}\mathrm{d}u=\infty,$$

and

$$\lim_{T\to\infty}\mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right]=\exp(-\infty)\times\exp\left(-v^*\lambda_0\right)=0.$$

□

Corollary 2.2.2. *If $\delta>\mu_{1_G}$ and the intensity process λ_t is stationary, then the probability generating function of N_T given $N_0=0$ is given by*

$$\mathbb{E}\Big[\theta^{N_T}\Big]=\exp\left(-(1-\theta)\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{\hat{g}(u)\left(a\delta u+\rho[1-\hat{h}(u)]\right)}{\left(1-\delta u-\theta\hat{g}(u)\right)\left(\delta u+\hat{g}(u)-1\right)}\mathrm{d}u\right).$$

Proof. Since the process λ_t is stationary, the Laplace transform of λ_0 is give by (2.17), by *Theorem 2.2.4*, we have

$$\begin{aligned}
\mathbb{E}\Big[\theta^{N_T}\Big] &= \mathbb{E}\left[\mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right]\right]\\
&= \exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{a\delta u+\rho[1-\hat{h}(u)]}{1-\delta u-\theta\hat{g}(u)}\mathrm{d}u\right)\times\mathbb{E}\left[\exp\left(-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0\right)\right]\\
&= \exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{a\delta u+\rho[1-\hat{h}(u)]}{1-\delta u-\theta\hat{g}(u)}\mathrm{d}u\right)\times\mathbb{E}\left[e^{-v\lambda_0}\right]\Big|_{v=\mathcal{G}_{0,\theta}^{-1}(T)}\\
&= \exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{a\delta u+\rho[1-\hat{h}(u)]}{1-\delta u-\theta\hat{g}(u)}\mathrm{d}u\right)\times\exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right)\\
&= \exp\left(-(1-\theta)\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{\hat{g}(u)\left(a\delta u+\rho[1-\hat{h}(u)]\right)}{\left(1-\delta u-\theta\hat{g}(u)\right)\left(\delta u+\hat{g}(u)-1\right)}\mathrm{d}u\right).
\end{aligned}$$

□

Corollary 2.2.3. *If $\delta>\mu_{1_G}$, then the probability generating function of N_T of the Hawkes process with exponential decay conditional on λ_0 and $N_0=0$ is given by*

$$\mathbb{E}\Big[\theta^{N_T}\big|\lambda_0\Big]=\exp\left(-a\delta\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{u}{1-\delta u-\theta\hat{g}(u)}\mathrm{d}u\right)\times e^{-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0};$$

and if the intensity process λ_t is stationary, then,

$$\mathbb{E}\Big[\theta^{N_T}\Big]=\exp\left(-(1-\theta)a\delta\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)}\frac{u\hat{g}(u)}{\left(1-\delta u-\theta\hat{g}(u)\right)\left(\delta u+\hat{g}(u)-1\right)}\mathrm{d}u\right).$$

Proof. By setting the intensity of the externally excited jumps $\rho=0$ in *Theorem 2.2.4* and *Corollary 2.2.2*, the results follow immediately. □

The probability $P\{N_T=0|\lambda_0\}$ can be derived by simply letting $\theta=0$ in the probability generating function of N_T in *Theorem 2.2.4*.

Corollary 2.2.4. *The conditional probability of no jump given λ_0 and $N_0=0$, under the condition $\delta>\mu_{1_G}$, is given by*

$$P\left\{N_T=0\middle|\lambda_0\right\}=\exp\left(-\int_0^{u_T}\frac{a\delta u+\rho[1-\hat{h}(u)]}{1-\delta u}\mathrm{d}u\right)\times e^{-u_T\lambda_0}, \tag{2.25}$$

where

$$u_T =: \frac{1}{\delta}\left(1 - e^{-\delta T}\right).$$

Proof. Since

$$P\left\{N_T = 0 \middle| \lambda_0\right\} = \mathbb{E}\left[\theta^{N_T}|\lambda_0\right]\Bigg|_{\theta=0},$$

and

$$\mathcal{G}_{0,0}(L) =: \mathcal{G}_{0,\theta}(L)\Bigg|_{\theta=0} = \int_0^L \frac{1}{1-\delta u}\mathrm{d}u = -\frac{1}{\delta}\ln(1-\delta L),$$

then, the inverse function

$$u_T = \mathcal{G}_{0,0}^{-1}(T) = \frac{1}{\delta}\left(1 - e^{-\delta T}\right),$$

by letting $\theta = 0$ in *Theorem 2.2.4*, (2.25) follows. □

Remark 2.2.5. Note that, since there is no jump in the point process N_t from time $t = 0$ to $t = T$, the conditional probability $P\left\{N_T = 0 \middle| \lambda_0\right\}$ is not dependent on the distribution of the self-excited jumps, and the result is similar to the non-self-excited case by Dassios and Jang (2003).

Remark 2.2.6. $P\left\{N_T = 0 \middle| \lambda_0\right\}$ is a strictly decreasing function of time T and

$$\lim_{T\to 0} P\left\{N_T = 0 \middle| \lambda_0\right\} = 1, \quad \lim_{T\to\infty} P\left\{N_T = 0 \middle| \lambda_0\right\} = 0.$$

Proof. Rewrite (2.25) as

$$P\left\{N_T = 0 \middle| \lambda_0\right\} = \exp\left(-\int_0^{u_T} \frac{a\delta + \rho\frac{1-\hat{h}(u)}{u}}{\frac{1}{u} - \delta}\mathrm{d}u\right) \times e^{-u_T\lambda_0}.$$

Since $0 < u < u_T < \frac{1}{\delta}$ and

$$\frac{1-\hat{h}(u)}{u} > 0, \quad \frac{1}{u} - \delta > 0,$$

the integrand is positive, and also u_T is a strictly increasing function of time T, therefore, $P\{N_T = 0\}$ is a strictly decreasing function of T. When $T \to 0$, $u_T \to 0$, then $P\{N_T = 0\} \to \exp(-0) = 1$; when $T \to \infty$, $u_T \to \frac{1}{\delta}$, since

$$\int_0^{\frac{1}{\delta}} \frac{1}{1-\delta u}\mathrm{d}u = \infty,$$

and

$$\lim_{u\to\frac{1}{\delta}} \frac{a\delta u + \rho[1-\hat{h}(u)]}{1-\delta u}\Big/\frac{1}{1-\delta u} = a + \rho\left[1 - \hat{h}\left(\frac{1}{\delta}\right)\right] > 0,$$

then,

$$\lim_{T\to 0}\int_0^{u_T} \frac{a\delta u + \rho[1-\hat{h}(u)]}{1-\delta u}\mathrm{d}u = \infty,$$

and

$$\lim_{T\to 0} P\{N_T = 0\} = \exp(-\infty) \times e^{-\frac{1}{\delta}\lambda_0} = 0.$$

□

Theoretically, the probability $P\left\{N_T = n \middle| \lambda_0\right\}$ for any natural number $n \in \mathbb{N}$ can be derived by

$$P\left\{N_T = n \middle| \lambda_0\right\} = \frac{\partial^n}{\partial\theta^n}\mathbb{E}\left[\theta^{N_T}|\lambda_0\right]\Bigg|_{\theta=0},$$

here, we derive the result of $P\left\{N_T = 1 \middle| \lambda_0\right\}$ in *Corollary 2.2.5*, for instance.

Corollary 2.2.5. *The conditional probability of exactly one jump given* λ_0 *and* $N_0 = 0$*, under the condition* $\delta > \mu_{1_G}$*, is given by*

$$\begin{aligned} P\left\{N_T = 1\middle|\lambda_0\right\} &= P\left\{N_T = 0\middle|\lambda_0\right\} \times \left\{ \left[a\left(1 - e^{-\delta T}\right) + \rho[1 - \hat{h}(u_T)] + \lambda_0 e^{-\delta T}\right] \right. \\ &\quad \left. \times \int_0^{u_T} \frac{\hat{g}(u)}{(1-\delta u)^2}\mathrm{d}u - \int_0^{u_T} \frac{\hat{g}(u)}{(1-\delta u)^2}\left(a\delta u + \rho[1 - \hat{h}(u)]\right)\mathrm{d}u \right\}, \end{aligned}$$

where

$$u_T = \frac{1}{\delta}\left(1 - e^{-\delta T}\right).$$

Proof. To simplify the notation, we define

$$\varphi(u, \theta) =: \frac{a\delta u + \rho[1 - \hat{h}(u)]}{1 - \delta u - \theta \hat{g}(u)}.$$

Then,

$$\begin{aligned} P\left\{N_T = 1\middle|\lambda_0\right\} &= \frac{\partial}{\partial\theta} \exp\left[-\int_0^{\mathcal{G}^{-1}_{0,\theta}(T)} \varphi(u,\theta)\mathrm{d}u - \mathcal{G}^{-1}_{0,\theta}(T)\lambda_0\right]\Bigg|_{\theta=0} \\ &= P\left\{N_T = 0\middle|\lambda_0\right\} \times (-1)\left[\int_0^{\mathcal{G}^{-1}_{0,\theta}(T)} \frac{\partial\varphi(u,\theta)}{\partial\theta}\mathrm{d}u + \left(\varphi\left(\mathcal{G}^{-1}_{0,\theta}(T), \theta\right) + \lambda_0\right)\frac{\partial}{\partial\theta}\mathcal{G}^{-1}_{0,\theta}(T)\right]\Bigg|_{\theta=0} \\ &= P\left\{N_T = 0\middle|\lambda_0\right\} \times (-1)\left[\int_0^{u_T} \frac{\partial\varphi(u,\theta)}{\partial\theta}\bigg|_{\theta=0}\mathrm{d}u + \left(\varphi(u_T, 0) + \lambda_0\right)\frac{\partial}{\partial\theta}\mathcal{G}^{-1}_{0,\theta}(T)\bigg|_{\theta=0}\right], \end{aligned}$$

where

$$\begin{aligned} \frac{\partial\varphi(u,\theta)}{\partial\theta}\bigg|_{\theta=0} &= \frac{\hat{g}(u)\left(a\delta u + \rho[1 - \hat{h}(u)]\right)}{\left(1 - \delta u - \theta\hat{g}(u)\right)^2}\Bigg|_{\theta=0} = \frac{\hat{g}(u)\left(a\delta u + \rho[1 - \hat{h}(u)]\right)}{(1-\delta u)^2}, \\ \varphi(u_T, 0) &= e^{\delta T}\left(a\left(1 - e^{-\delta T}\right) + \rho(1 - \hat{h}(u_T))\right), \end{aligned}$$

and $\frac{\partial}{\partial\theta}\mathcal{G}^{-1}_{0,\theta}(T)\Big|_{\theta=0}$ can be derived as below. Since $L(T;\theta) = \mathcal{G}^{-1}_{0,\theta}(T)$, we have the non-linear ODE of $L(\tau;\theta)$,

$$L(\tau;\theta)' = 1 - \delta L(\tau;\theta) - \theta\hat{g}(L(\tau;\theta)), \quad 0 \le \theta < 1,$$

with the initial condition $L(0;\theta) = 0$, differentiate both sides with respect to θ,

$$L^{(1)}(\tau;\theta)' = -\delta L^{(1)}(\tau;\theta) - \left[\hat{g}(L(\tau;\theta)) + \theta\hat{g}^{(1)}(L(\tau;\theta))\right], \quad 0 \le \theta < 1,$$

where

$$\begin{aligned} L^{(1)}(\tau;\theta) &= \frac{\partial}{\partial\theta}L(\tau;\theta), \\ \hat{g}^{(1)}(L(\tau;\theta)) &= \frac{\partial}{\partial\theta}\hat{g}(L(\tau;\theta)), \end{aligned}$$

by setting $\theta = 0$, we have the ODE for $L^{(1)}(\tau;0)$,

$$L^{(1)}(\tau;0)' = -\delta L^{(1)}(\tau;0) - \hat{g}(L(\tau;0)),$$

with the initial condition $L^{(1)}(0;0) = 0$, given $L(\tau;0) = \frac{1}{\delta}\left(1 - e^{-\delta\tau}\right)$, then, $L^{(1)}(\tau;0)$ can be uniquely solved,

$$\frac{\partial}{\partial\theta}\mathcal{G}^{-1}_{0,\theta}(T)\bigg|_{\theta=0} = L^{(1)}(\tau;0) = -e^{-\delta T}\int_0^T \hat{g}\left(\frac{1 - e^{-\delta s}}{\delta}\right)e^{\delta s}\mathrm{d}s < 0;$$

equivalently, by the change of variable $u = \frac{1-e^{-\delta s}}{\delta}$,

$$\int_0^T \hat{g}\left(\frac{1-e^{-\delta s}}{\delta}\right) e^{\delta s} \mathrm{d}s = \int_0^{u_T} \frac{\hat{g}(u)}{(1-\delta u)^2} \mathrm{d}u.$$

□

Remark 2.2.7. $P\left\{N_T = 1 \middle| \lambda_0\right\}$ is positive, since

$$\begin{aligned} & \int_0^{u_T} \frac{\hat{g}(u)}{(1-\delta u)^2} \left(a\delta u + \rho[1-\hat{h}(u)]\right) \mathrm{d}u \\ \leq & \int_0^{u_T} \frac{\hat{g}(u)}{(1-\delta u)^2} \mathrm{d}u \int_0^{u_T} \left(a\delta u + \rho[1-\hat{h}(u)]\right) \mathrm{d}u \\ < & \left[a\left(1-e^{-\delta T}\right) + \rho[1-\hat{h}(u_T)] + \lambda_0 e^{-\delta T}\right] \int_0^{u_T} \frac{\hat{g}(u)}{(1-\delta u)^2} \mathrm{d}u. \end{aligned}$$

$P\left\{N_T = 1 \middle| \lambda_0\right\}$ is not a strictly monotonous function of time T, since

$$\lim_{T\to 0} P\left\{N_T = 1 \middle| \lambda_0\right\} = 0, \quad \lim_{T\to\infty} P\left\{N_T = 1 \middle| \lambda_0\right\} = 0.$$

Similarly to the point process N_t, the probability generating function of the size of a cluster generated by a point of any generation can also be derived as follows.

Theorem 2.2.5. *For the size of a cluster generated by a point of any generation, $\check{N}_t$, under the condition $\delta > \mu_{1_G}$, we have*

$$\mathbb{E}\left[\theta^{\check{N}_T} \middle| \check{\lambda}_0\right] = e^{-\mathcal{G}_{0,\theta}^{-1}(T)\check{\lambda}_0}, \tag{2.26}$$

$$\mathbb{E}\left[\theta^{\check{N}_\infty} \middle| \check{\lambda}_0\right] = e^{-v^*\check{\lambda}_0}, \tag{2.27}$$

where $\mathcal{G}_{0,\theta}(\cdot)$ and v^* are given by (2.20) and (2.22), respectively, and $\check{\lambda}_0$ is the value of one of the associated externally excited or self-excited jumps. In particular, for a cluster generated by a point of generation 0, we have

$$\mathbb{E}\left[\theta^{\check{N}_\infty}\right] = \hat{h}(v^*);$$

for a cluster generated by a point of subsequent generations, we have

$$\mathbb{E}\left[\theta^{\check{N}_\infty}\right] = \frac{1-\delta v^*}{\theta}. \tag{2.28}$$

Proof. For the size of a cluster generated by a point of any generation, the infinitesimal generator of the process $(\check{\lambda}_t, \check{N}_t, t)$ acting on a function $f(\check{\lambda}, \check{n}, t)$ within its domain $\Omega(\mathcal{A})$ is given by

$$\mathcal{A}f(\check{\lambda}, \check{n}, t) = \frac{\partial f}{\partial t} - \delta\check{\lambda}\frac{\partial f}{\partial \check{\lambda}} + \check{\lambda}\left(\int_0^\infty f(\check{\lambda}+y, \check{n}+1, t)\mathrm{d}G(y) - f(\check{\lambda}, \check{n}, t)\right),$$

as it is just a special case of *Theorem 2.2.1* and *Theorem 2.2.4*. By setting $a = 0$ and $\rho = 0$, we can derive (2.26) immediately. By the proof of *Theorem 2.2.4*, we know that

$$\lim_{T\to\infty} \mathcal{G}_{0,\theta}^{-1}(T) = v^*,$$

then,

$$\mathbb{E}\left[\theta^{\check{N}_\infty} \middle| \check{\lambda}_0\right] = \lim_{T\to\infty} \mathbb{E}\left[\theta^{\check{N}_T} \middle| \check{\lambda}_0\right] = \lim_{T\to\infty} e^{-\mathcal{G}_{0,\theta}^{-1}(T)\check{\lambda}_0} = e^{-v^*\check{\lambda}_0}.$$

In particular, for a cluster generated by a point of generation 0, we have

$$\mathbb{E}\Big[\theta^{\check{N}_\infty}\Big] = \mathbb{E}\left[\mathbb{E}\left[\theta^{\check{N}_\infty}\big|\check{\lambda}_0\right]\right] = \mathbb{E}\Big[e^{-v^*\check{\lambda}_0}\Big] = \mathbb{E}\Big[e^{-v^*Y_1^{(1)}}\Big] = \hat{h}(v^*);$$

for a cluster generated by a point of subsequent generations, we have

$$\mathbb{E}\Big[\theta^{\check{N}_\infty}\Big] = \mathbb{E}\Big[e^{-v^*Y_1^{(2)}}\Big] = \hat{g}(v^*) = \frac{1-\delta v^*}{\theta}.$$

□

Remark 2.2.8. The size of a cluster generated by a point of any generation actually is a pure Hawkes process with the reversion level $a = 0$, a special case of dynamic contagion process. As time $t \to \infty$, the distribution of $\check{\lambda}_t$ converges to the distribution of a degenerate random variable at 0.

Remark 2.2.9. Based on the decaying distributional property of the intensity process $\check{\lambda}_t$ in *Remark 2.2.8*, we have an alternative approach to prove (2.27): By *Theorem 2.2.1* with the general boundary condition $B(T) = v$ $(0 < v < v^*)$ and using the similar method as given in the proof of *Theorem 2.2.4*, we have

$$\mathbb{E}\left[\theta^{\check{N}_T}e^{-v\check{\lambda}_T}\big|\check{\lambda}_0\right] = e^{-B(0)\check{\lambda}_0},$$

where

$$B(t) = \mathcal{G}^{-1}_{v,\theta}(T-t),$$

$$\mathcal{G}_{v,\theta}(L) =: \int_v^L \frac{\mathrm{d}u}{1-\delta u-\theta\hat{g}(u)}, \quad 0 \le \theta < 1, 0 < v < v^*,$$

and

$$\lim_{T\to\infty} B(0) = \lim_{T\to\infty} \mathcal{G}^{-1}_{v,\theta}(T) = v^*.$$

Then, as the intensity process decays, we have

$$\mathbb{E}\Big[\theta^{\check{N}_\infty}\big|\check{\lambda}_0\Big] = \lim_{T\to\infty}\mathbb{E}\left[\theta^{\check{N}_T}e^{-v\check{\lambda}_T}\big|\check{\lambda}_0\right] = \lim_{T\to\infty} e^{-B(0)\check{\lambda}_0} = e^{-v^*\check{\lambda}_0}.$$

Remark 2.2.10. (2.28) can also be derived from the perspective of the cluster process definition given by *Definition 2.1.1*, and we observe that each subcluster has the same distribution

$$\mathcal{E}(\theta) =: \mathbb{E}\Big[\theta^{\check{N}_\infty}\Big]$$

as its ancestor (for a cluster generated by a point of subsequent generation $1, 2, ...$), and hence $\mathcal{E}(\theta)$ satisfies the functional equation

$$\mathcal{E}(\theta) = \hat{g}\left(\frac{1-\theta\mathcal{E}(\theta)}{\delta}\right)$$

which also leads to (2.28).

We also provide an explicit example for *Theorem 2.2.5* in *Theorem 2.3.3* by assuming the jumps with the exponential distributions.

2.2.4 Moments of λ_t and N_t

Any moment of λ_t and N_t can be obtained by differentiating the Laplace transform of λ_t and the probability generating function of N_t with respect to v and θ, and then setting v and θ equal to zero, respectively. Alternatively, we can obtain the first and second moments of λ_t and N_t directly by solving ODEs, and also this method is slightly easier to generalise to derive higher moments beyond the condition $\delta > \mu_{1_G}$, therefore we will proceed with this method here.

Theorem 2.2.6. *The conditional expectation of the process λ_t given λ_0 at time $t = 0$, is given by*

$$\mathbb{E}\left[\lambda_t \middle| \lambda_0\right] = \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}} + \left(\lambda_0 - \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\right) e^{-(\delta-\mu_{1_G})t}, \quad \delta \neq \mu_{1_G}, \tag{2.29}$$

$$\mathbb{E}\left[\lambda_t \middle| \lambda_0\right] = \lambda_0 + (\mu_{1_H}\rho + a\delta)\, t, \quad \delta = \mu_{1_G}, \tag{2.30}$$

where

$$\mu_{1_H} =: \int_0^\infty y \mathrm{d}H(y).$$

Proof. By the martingale property of the infinitesimal generator as given in (2.2), we have a $\mathcal{F}-$martingale

$$f(\lambda_t, N_t, t) - f(\lambda_0, N_0, 0) - \int_0^t \mathcal{A}(\lambda_s, N_s, s)\mathrm{d}s$$

for $f \in \Omega(\mathcal{A})$. Now, by particularly setting $f(\lambda, n, t) = \lambda$, we have

$$\mathcal{A}\lambda = -(\delta - \mu_{1_G})\lambda + \mu_{1_H}\rho + a\delta,$$

then, $\lambda_t - \lambda_0 - \int_0^t \mathcal{A}\lambda_s \mathrm{d}s$ is a $\mathcal{F}-$martingale, and we have

$$\mathbb{E}\left[\lambda_t - \int_0^t \mathcal{A}\lambda_s \mathrm{d}s \middle| \lambda_0\right] = \lambda_0.$$

Hence,

$$\mathbb{E}\left[\lambda_t \middle| \lambda_0\right] = \lambda_0 + \mathbb{E}\left[\int_0^t \mathcal{A}\lambda_s \mathrm{d}s \middle| \lambda_0\right] = \lambda_0 - (\delta - \mu_{1_G}) \int_0^t \mathbb{E}\left[\lambda_s \middle| \lambda_0\right] \mathrm{d}s + (\mu_{1_H}\rho + a\delta)\, t,$$

by differentiating with respect to t, we obtain the non-linear inhomogeneous ODE,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\left(\delta - \mu_{1_G}\right) u(t) + \mu_{1_H}\rho + a\delta,$$

where $u(t) = \mathbb{E}\left[\lambda_t \middle| \lambda_0\right]$, with the initial condition $u(0) = \lambda_0$. This ODE has the solution given by (2.29) and (2.30). □

Lemma 2.2.1. *The second moment of the process λ_t given λ_0 at time $t = 0$, is given by*

$$\begin{aligned}
&\mathbb{E}\left[\lambda_t^2 \middle| \lambda_0\right] \\
= \;& \lambda_0^2 e^{-2(\delta-\mu_{1_G})t} + \frac{2(\mu_{1_H}\rho + a\delta) + \mu_{2_G}}{\delta - \mu_{1_G}} \left(\lambda_0 - \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\right)\left(e^{-(\delta-\mu_{1_G})t} - e^{-2(\delta-\mu_{1_G})t}\right) \\
&+ \left(\frac{(2(\mu_{1_H}\rho + a\delta) + \mu_{2_G})(\mu_{1_H}\rho + a\delta)}{2(\delta - \mu_{1_G})^2} + \frac{\mu_{2_H}\rho}{2(\delta - \mu_{1_G})}\right)\left(1 - e^{-2(\delta-\mu_{1_G})t}\right), \delta \neq \mu_{1_G}, \quad (2.31) \\
&\mathbb{E}\left[\lambda_t^2 \middle| \lambda_0\right] \\
= \;& \lambda_0^2 + \left(2(\mu_{1_H}\rho + a\delta) + \mu_{2_G}\right)\left(\lambda_0 t + \frac{1}{2}\left(\mu_{1_H}\rho + a\delta\right) t^2\right) + \mu_{2_H}\rho t, \quad \delta = \mu_{1_G}, \quad (2.32)
\end{aligned}$$

where

$$\mu_{2_H} =: \int_0^\infty y^2 \mathrm{d}H(y); \quad \mu_{2_G} =: \int_0^\infty y^2 \mathrm{d}G(y).$$

Proof. By setting $f(\lambda, n, t) = \lambda^2$ in (2.2), we have

$$\mathcal{A}\lambda^2 = -2(\delta - \mu_{1_G})\lambda^2 + \Big(2(\mu_{1_H}\rho + a\delta) + \mu_{2_G}\Big)\lambda + \mu_{2_H}\rho.$$

Since $\lambda_t^2 - \lambda_0^2 - \int_0^t \mathcal{A}\lambda_s^2 \mathrm{d}s$ is a $\mathcal{F}-$martingale by the martingale property of the generator, we have

$$\mathbb{E}\left[\lambda_t^2 - \int_0^t \mathcal{A}\lambda_s^2 \mathrm{d}s \middle| \lambda_0\right] = \lambda_0^2.$$

Hence,

$$\mathbb{E}\left[\lambda_t^2 \middle| \lambda_0\right] = \lambda_0^2 - 2(\delta - \mu_{1_G}) \int_0^t \mathbb{E}\left[\lambda_s^2 \middle| \lambda_0\right] \mathrm{d}s + \Big(2(\mu_{1_H}\rho + a\delta) + \mu_{2_G}\Big) \int_0^t \mathbb{E}\left[\lambda_s \middle| \lambda_0\right] \mathrm{d}s + \mu_{2_H}\rho t,$$

by differentiating with respect to t, we have the ODE,

$$\begin{aligned} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + 2(\delta - \mu_{1_G})u(t) \quad &= \quad \Big(2(\mu_{1_H}\rho + a\delta) + \mu_{2_G}\Big)\left(\lambda_0 - \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\right) e^{-(\delta - \mu_{1_G})t} \\ &\quad + \frac{\big(2(\mu_{1_H}\rho + a\delta) + \mu_{2_G}\big)\left(\mu_{1_H}\rho + a\delta\right)}{\delta - \mu_{1_G}} + \mu_{2_H}\rho, \end{aligned}$$

where $u(t) = \mathbb{E}\left[\lambda_t^2 \middle| \lambda_0\right]$, with the initial condition $u(0) = \lambda_0^2$. This ODE has the solution given by (2.31) and (2.32). □

Theorem 2.2.7. *The conditional variance of the process λ_t given λ_0 at time $t = 0$, is given by*

$$\begin{aligned} \mathrm{Var}\left[\lambda_t \middle| \lambda_0\right] \quad &= \quad \frac{1}{2(\delta - \mu_{1_G})}\left(\frac{\mu_{2_G}(\mu_{1_H}\rho + a\delta)}{\delta - \mu_{1_G}} - \mu_{2_H}\rho - 2\mu_{2_G}\lambda_0\right) e^{-2(\delta - \mu_{1_G})t} \\ &\quad + \frac{\mu_{2_G}}{\delta - \mu_{1_G}}\left(\lambda_0 - \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\right) e^{-(\delta - \mu_{1_G})t} \\ &\quad + \frac{1}{2(\delta - \mu_{1_G})}\left(\mu_{2_H}\rho + \frac{\mu_{2_G}(\mu_{1_H}\rho + a\delta)}{\delta - \mu_{1_G}}\right), \quad \delta \neq \mu_{1_G}, \qquad (2.33) \\ \mathrm{Var}\left[\lambda_t \middle| \lambda_0\right] \quad &= \quad \frac{1}{2}\mu_{2_G}\left(\mu_{1_H}\rho + a\delta\right) t^2 + \left(\mu_{2_G}\lambda_0 + \mu_{2_H}\rho\right) t, \quad \delta = \mu_{1_G}. \qquad (2.34) \end{aligned}$$

Proof. By $\mathrm{Var}\left[\lambda_t \middle| \lambda_0\right] = \mathbb{E}\left[\lambda_t^2 \middle| \lambda_0\right] - \Big(\mathbb{E}\left[\lambda_t \middle| \lambda_0\right]\Big)^2$ based on *Theorem 2.2.6* and *Lemma 2.2.1*, the result follows. □

Corollary 2.2.6. *Assume $\delta > \mu_{1_G}$, then the first and second moments and the variance of the stationary distribution of the process λ_t are given by*

$$\begin{aligned} \mathbb{E}\left[\lambda_t\right] \quad &= \quad \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}, \qquad (2.35) \\ \mathbb{E}\left[\lambda_t^2\right] \quad &= \quad \frac{(2(\mu_{1_H}\rho + a\delta) + \mu_{2_G})(\mu_{1_H}\rho + a\delta)}{2(\delta - \mu_{1_G})^2} + \frac{\mu_{2_H}\rho}{2(\delta - \mu_{1_G})}, \qquad (2.36) \\ \mathrm{Var}\left[\lambda_t\right] \quad &= \quad \frac{1}{2(\delta - \mu_{1_G})}\left(\mu_{2_H}\rho + \frac{\mu_{2_G}(\mu_{1_H}\rho + a\delta)}{\delta - \mu_{1_G}}\right). \end{aligned}$$

Proof. By setting time $t \to \infty$ in (2.29), (2.30), (2.31), (2.32), and (2.33), (2.34), respectively, then the results follow. □

We will now derive the moments for the point process N_t assuming that $\delta > \mu_{1_G}$.

Theorem 2.2.8. *For the stationary distribution of the process λ_t, given the condition $\delta > \mu_{1_G}$ and $N_0 = 0$, the expectation of the point process N_t is given by*

$$\mathbb{E}[N_t] = \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}} t. \tag{2.37}$$

Proof. By setting $f(\lambda, n, t) = n$ in (2.2), we have $\mathcal{A}n = \lambda$. Since $N_t - N_0 - \int_0^t \lambda_s \mathrm{d}s$ is a martingale by the martingale property of the intensity process λ_t of the point process N_t given by the definition (2.1), we have

$$\mathbb{E}\left[N_t - N_0 \middle| \mathcal{F}_0\right] = \mathbb{E}\left[\int_0^t \lambda_s \mathrm{d}s \middle| \mathcal{F}_0\right],$$

and also we know $\mathbb{E}[\lambda_t]$ from *Corollary 2.2.6*, then, by assuming $N_0 = 0$, we have

$$\mathbb{E}[N_t] = \mathbb{E}[N_t - N_0] = \int_0^t \mathbb{E}[\lambda_s] \mathrm{d}s = \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}} t.$$

□

Lemma 2.2.2. *For the stationary distribution of the process λ_t, given the condition $\delta > \mu_{1_G}$ and $N_0 = 0$, we have*

$$\mathbb{E}[\lambda_t N_t] = \bar{k}\left(1 - e^{-(\delta - \mu_{1_G})t}\right) + \left(\frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\right)^2 t, \tag{2.38}$$

where

$$\bar{k} =: \frac{2\mu_{1_G}(\mu_{1_H}\rho + a\delta) + \mu_{2_H}\rho}{2(\delta - \mu_{1_G})^2} + \frac{\mu_{2_G}(\mu_{1_H}\rho + a\delta)}{2(\delta - \mu_{1_G})^3}. \tag{2.39}$$

Proof. By setting $f(\lambda, n, t) = \lambda n$ in (2.2), we have

$$\mathcal{A}(\lambda n) = -(\delta - \mu_{1_G})\lambda n + (\mu_{1_H}\rho + a\delta) n + \lambda^2 + \mu_{1_G}\lambda.$$

Since $\lambda_t N_t - \lambda_0 N_0 - \int_0^t \mathcal{A}(\lambda_s N_s) \mathrm{d}s$ is a $\mathcal{F}-$martingale by the martingale property of the generator, given $N_0 = 0$, we have the ODE,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -(\delta - \mu_{1_G}) u(t) + (\mu_{1_H}\rho + a\delta)\mathbb{E}[N_t] + \mathbb{E}\left[\lambda_t^2\right] + \mu_{1_G}\mathbb{E}[\lambda_t],$$

where $u(t) = \mathbb{E}[\lambda_t N_t]$, with the initial condition $u(0) = 0$. Note that, $\mathbb{E}[N_t]$, $\mathbb{E}\left[\lambda_t^2\right]$ and $\mathbb{E}[\lambda_t]$ are already given by (2.37), (2.36) and (2.35), respectively, therefore, this ODE has the solution given by (2.38). □

Theorem 2.2.9. *For the stationary distribution of the process λ_t, given the condition $\delta > \mu_{1_G}$ and $N_0 = 0$, the second moment and the variance of the point process N_t are given by*

$$\begin{aligned}
\mathbb{E}\left[N_t^2\right] &= \frac{2}{\delta - \mu_{1_G}}\left(e^{-(\delta - \mu_{1_G})t} - 1\right) + 2\bar{k}t + \left(\frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\right)^2 t^2, \\
\mathrm{Var}[N_t] &= \frac{2}{\delta - \mu_{1_G}}\left(e^{-(\delta - \mu_{1_G})t} - 1\right) + 2\bar{k}t,
\end{aligned}$$

where constant $\bar{k}$ is given by (2.39).

Proof. By setting $f(\lambda,n,t) = n^2$ in (2.2), we have $\mathcal{A}\left(n^2\right) = (2n+1)\lambda$. Since $N_t^2 - N_0^2 - \int_0^t (2N_s+1)\lambda_s \mathrm{d}s$ is a $\mathcal{F}-$martingale by the martingale property of the generator, given $N_0 = 0$, we have

$$\mathbb{E}\left[N_t^2\right] = 2\int_0^t \mathbb{E}\left[\lambda_s N_s\right] \mathrm{d}s + \int_0^t \mathbb{E}\left[\lambda_s\right] \mathrm{d}s,$$

where $\mathbb{E}[\lambda_t N_t]$ and $\mathbb{E}[\lambda_t]$ are given by(2.38) and (2.35), respectively, then $\mathbb{E}\left[N_t^2\right]$ follows. Since $\mathrm{Var}[N_t] = \mathbb{E}\left[N_t^2\right] - \mathbb{E}[N_t]^2$ given $\mathbb{E}[N_t]$ in (2.37), $\mathrm{Var}[N_t]$ follows. □

The moments for the special case Hawkes process and other similar processes can also be found in Oakes (1975) and Azizpour and Giesecke (2008), and more generally in Brémaud, Massoulié and Ridolfi (2002).

2.3 Example: Jumps with Exponential Distributions

To give an explicit example for the key distributional properties derived above, in this section we assume both externally excited and self-excited jumps follow exponential distributions, i.e. the density functions

$$h(y) = \alpha e^{-\alpha y}; \quad g(y) = \beta e^{-\beta y}, \quad \text{where } y, \alpha, \beta > 0, \tag{2.40}$$

the Laplace transforms have the explicit forms

$$\hat{h}(u) = \frac{\alpha}{\alpha+u}; \quad \hat{g}(u) = \frac{\beta}{\beta+u}. \tag{2.41}$$

Then the corresponding Laplace transform of λ_T, conditional probability generating function of N_T, conditional probability $P\left\{N_T = 0 \middle| \lambda_0\right\}$ and $P\left\{N_T = 1 \middle| \lambda_0\right\}$ are obtained respectively as below. Note that, there are parameters $(a,\rho,\delta;\alpha,\beta;\lambda_0)$ for the general dynamic contagion process and $(a,\delta;\beta;\lambda_0)$ for the Hawkes process.

2.3.1 Laplace Transform of λ_T

Lemma 2.3.1. *If both the self-excited and externally excited jumps follow exponential distributions, i.e. the density functions are specified by (2.40), then the conditional Laplace transform of λ_T given λ_0 at time $t = 0$, under the condition $\delta\beta > 1$, is given by*

$$\mathbb{E}\left[e^{-v\lambda_T} \middle| \lambda_0\right] = e^{-\left(\mathcal{C}_1(v) - \mathcal{C}_1\left(\mathcal{G}_{v,1}^{-1}(T)\right)\right)} e^{-\mathcal{G}_{v,1}^{-1}(T)\lambda_0},$$

where

$$\mathcal{C}_1(u) =: \begin{cases} au + \frac{\rho(\alpha-\beta)}{\delta(\alpha-\beta)+1}\ln(\alpha+u) + \frac{1}{\delta}\left(a + \frac{\rho}{\delta(\alpha-\beta)+1}\right)\ln\left(u + \frac{\delta\beta-1}{\delta}\right), & \alpha \neq \beta - \frac{1}{\delta} \\ au + \frac{\rho\beta}{\delta\beta-1}\ln(\alpha+u) - \frac{\alpha\rho}{\delta(\delta\beta-1)}\frac{1}{\alpha+u} + \frac{1}{\delta}\left(a - \frac{\rho}{\delta\beta-1}\right)\ln\left(u + \frac{\delta\beta-1}{\delta}\right), & \alpha = \beta - \frac{1}{\delta} \end{cases}, \tag{2.42}$$

and

$$\mathcal{G}_{v,1}(L) = \frac{1}{\delta(\delta\beta-1)}\left[\delta\beta\ln\left(\frac{v}{L}\right) - \ln\left(\frac{\delta v + (\delta\beta-1)}{\delta L + (\delta\beta-1)}\right)\right].$$

Proof. By *Theorem 2.2.2* and $\mu_{1_G} = \frac{1}{\beta}$, the condition is $\delta > \frac{1}{\beta}$; and substitute (2.41), into *Theorem 2.2.2*, we have

$$\mathcal{G}_{v,1}(L) = \int_L^v \frac{u+\beta}{\delta u\left(u + \frac{\delta\beta-1}{\delta}\right)}\mathrm{d}u = \frac{1}{\delta(\delta\beta-1)}\left[\delta\beta\ln u - \ln\left(u + \frac{\delta\beta-1}{\delta}\right)\right]\Bigg|_{u=L}^{u=v},$$

and

$$\mathcal{C}_1(v) - \mathcal{C}_1\left(\mathcal{G}_{v,1}^{-1}(T)\right) = \int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{\left(a + \frac{\rho}{\delta}\frac{1}{u+\alpha}\right)(\beta+u)}{u + \frac{\delta\beta-1}{\delta}} du.$$

Note that, when calculating the integral, we need consider the special case when $\alpha = \beta - \frac{1}{\delta}$. Then, the result follows. □

Theorem 2.3.1. *If both the externally excited and self-excited jumps follow exponential distributions, i.e. the density functions are specified by (2.40), then, under the condition $\delta\beta > 1$, the stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$ is given by*

$$\lambda_t \stackrel{\mathcal{D}}{=} \begin{cases} a + \tilde{\Gamma}_1 + \tilde{\Gamma}_2, & \alpha \geq \beta \\ a + \tilde{\Gamma}_3 + \tilde{B}, & \alpha < \beta \text{ and } \alpha \neq \beta - \frac{1}{\delta} \\ a + \tilde{\Gamma}_4 + \tilde{P}, & \alpha = \beta - \frac{1}{\delta} \end{cases},$$

where independent random variables

$$\begin{aligned}
\tilde{\Gamma}_1 &\sim \text{Gamma}\left(\frac{1}{\delta}\left(a + \frac{\rho}{\delta(\alpha-\beta)+1}\right), \frac{\delta\beta-1}{\delta}\right); \\
\tilde{\Gamma}_2 &\sim \text{Gamma}\left(\frac{\rho(\alpha-\beta)}{\delta(\alpha-\beta)+1}, \alpha\right); \\
\tilde{\Gamma}_3 &\sim \text{Gamma}\left(\frac{a+\rho}{\delta}, \frac{\delta\beta-1}{\delta}\right); \\
\tilde{\Gamma}_4 &\sim \text{Gamma}\left(\frac{a+\rho}{\delta}, \alpha\right); \\
\tilde{B} &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_1} X_i^{(1)}, \quad N_1 \sim \text{NegBin}\left(\frac{\rho}{\delta}\frac{\beta-\alpha}{\gamma_1-\gamma_2}, \frac{\gamma_2}{\gamma_1}\right), X_i^{(1)} \sim \text{Exp}(\gamma_1), \\
&\quad \gamma_1 = \max\left\{\alpha, \frac{\delta\beta-1}{\delta}\right\}, \gamma_2 = \min\left\{\alpha, \frac{\delta\beta-1}{\delta}\right\}; \\
\tilde{P} &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_2} X_i^{(2)}, \quad N_2 \sim \text{Poisson}\left(\frac{\rho}{\delta^2\alpha}\right), X_i^{(2)} \sim \text{Exp}(\alpha).
\end{aligned}$$

$\tilde{B}$ follows a compound negative binomial distribution with underlying exponential jumps; $\tilde{P}$ follows a compound Poisson distribution with underlying exponential jumps.

Proof. By *Lemma 2.3.1*, *Theorem 2.2.3*, and as $\mathcal{G}_{v,1}^{-1}(T) \to 0$ when $T \to \infty$, we use the explicit function $C_1(u)$ in (2.42) to derive the Laplace transform of the stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$ by $\hat{\Pi}(v) = e^{-(C_1(v)-C_1(0))}$, then,

$$\hat{\Pi}(v) = \begin{cases} e^{-va}\left(\frac{\alpha}{\alpha+v}\right)^{\frac{\rho(\alpha-\beta)}{\delta(\alpha-\beta)+1}}\left(\frac{\frac{\delta\beta-1}{\delta}}{v+\frac{\delta\beta-1}{\delta}}\right)^{\frac{1}{\delta}\left(a+\frac{\rho}{\delta(\alpha-\beta)+1}\right)}, & \alpha \geq \beta \\ e^{-va}\left(\frac{\frac{\delta\beta-1}{\delta}}{v+\frac{\delta\beta-1}{\delta}}\right)^{\frac{a+\rho}{\delta}}\left(\frac{\frac{\gamma_2}{\gamma_1}}{1-\left(1-\frac{\gamma_2}{\gamma_1}\right)\frac{\gamma_1}{\gamma_1+v}}\right)^{\frac{\rho}{\delta}\frac{\beta-\alpha}{\gamma_1-\gamma_2}}, & \alpha < \beta \text{ and } \alpha \neq \beta - \frac{1}{\delta} \\ e^{-va}\left(\frac{\alpha}{\alpha+v}\right)^{\frac{\rho+a}{\delta}}\exp\left[\frac{\rho}{\delta^2\alpha}\left(\frac{\alpha}{\alpha+v}-1\right)\right], & \alpha = \beta - \frac{1}{\delta} \end{cases}. \tag{2.43}$$

If $\alpha \geq \beta$, it is obvious that, (2.43) is the Laplace transform of two independent Gamma distributions $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ shifted by a constant a. If $\alpha < \beta$ and $\alpha \neq \beta - \frac{1}{\delta}$, then always $\gamma_1 > \gamma_2$, and the second term is the Laplace transform of Gamma distribution with two parameters $\frac{a+\rho}{\delta}$ and $\frac{\delta\beta-1}{\delta}$; the third term is the Laplace transform of a compound negative binomial distribution with two parameters $\frac{\rho}{\delta}\frac{\beta-\alpha}{\gamma_1-\gamma_2}$ and $\frac{\gamma_2}{\gamma_1}$, and the underlying jumps follows an exponential distribution with parameter γ_1, since we know that the Laplace transform of negative binomial distribution N_1 with two parameters (r, p) is

$$\mathbb{E}\left[e^{-vN_1}\right] = \left(\frac{p}{1-(1-p)e^{-v}}\right)^r.$$

Then

$$\begin{aligned}
\mathbb{E}\left[e^{-v\tilde{B}}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-v\sum_{i=1}^{N_1} X_i^{(1)}}\right]\middle| N_1\right] \\
&= \mathbb{E}\left[\left(\frac{\gamma_1}{\gamma_1+v}\right)^{N_1}\right] \\
&= \mathbb{E}\left[e^{-\ln\left(\frac{\gamma_1+v}{\gamma_1}\right)N_1}\right] \\
&= \left(\frac{p}{1-(1-p)e^{-\ln\left(\frac{\gamma_1+v}{\gamma_1}\right)}}\right)^r \\
&= \left(\frac{p}{1-(1-p)\frac{\gamma_1}{\gamma_1+v}}\right)^r,
\end{aligned}$$

where

$$p = \frac{\gamma_1}{\gamma_2} \in (0,1); \quad r = \frac{\rho}{\delta}\frac{\beta-\alpha}{\gamma_1-\gamma_2} \in \mathbb{R}^+.$$

Also, it is also easy to identify the corresponding Laplace transforms for the case when $\alpha = \beta - \frac{1}{\delta}$. □

We discuss some important special cases below.

Remark 2.3.1. If both jumps follows the same exponential distribution, i.e. $\alpha = \beta$, then $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ combine as one single Gamma random variable $\tilde{\Gamma}_3$, since

$$\hat{\Pi}(v) = e^{-va}\left(\frac{\frac{\delta\beta-1}{\delta}}{v+\frac{\delta\beta-1}{\delta}}\right)^{\frac{a+\rho}{\delta}}.$$

Remark 2.3.2. For the non-self-excited case, i.e. when $\beta = \infty$, we have the Laplace transform of the stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$ given by

$$\hat{\Pi}(v) = e^{-va}\left(\frac{\alpha}{\alpha+v}\right)^{\frac{\rho}{\delta}},$$

then, λ_t follows a shifted Gamma distribution,

$$\lambda_t \stackrel{\mathcal{D}}{=} a + \tilde{\Gamma}_5,$$

where

$$\tilde{\Gamma}_5 \sim \text{Gamma}\left(\frac{\rho}{\delta}, \alpha\right),$$

which recovers the result by Dassios and Jang (2003) by setting $a = 0$.

Remark 2.3.3. For the Hawkes process, i.e. the non-externally-excited case when $\alpha = \infty$, or $\rho = 0$, we have the Laplace transform of the stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$ given by

$$\hat{\Pi}(v) = e^{-va}\left(\frac{\frac{\delta\beta-1}{\delta}}{v+\frac{\delta\beta-1}{\delta}}\right)^{\frac{a}{\delta}}, \tag{2.44}$$

then, λ_t follows a shifted Gamma distribution,

$$\lambda_t \stackrel{\mathcal{D}}{=} a + \tilde{\Gamma}_6,$$

where

$$\tilde{\Gamma}_6 \sim \text{Gamma}\left(\frac{a}{\delta}, \frac{\delta\beta - 1}{\delta}\right).$$

The result for the particular case $\alpha = \beta - \frac{1}{\delta}$ is actually the limit version of the result for the case when $\alpha < \beta$ and $\alpha \neq \beta - \frac{1}{\delta}$. In the following sections, we only focus on the main case when $\alpha \neq \beta - \frac{1}{\delta}$, with the Laplace transform of the stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$ specified by (2.43).

2.3.2 Probability Generating Function of N_T

Theorem 2.3.2. *If both the externally excited and self-excited jumps follow exponential distributions, i.e. the density functions are specified as (2.40), then the conditional probability generating function of N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under the condition $\delta\beta > 1$, is given by*

$$\mathbb{E}\Big[\theta^{N_T}\Big|\lambda_0\Big] = e^{-\left(\mathcal{C}_2\left(\mathcal{G}_{0,\theta}^{-1}(T)\right) - \mathcal{C}_2(0)\right)} e^{-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0}, \quad \alpha \neq -v_-^*,$$

where

$$\begin{aligned}
\mathcal{C}_2(u) \quad =: \quad & -au + \frac{\alpha(\beta-\alpha)\rho}{\delta\left(\alpha+v_-^*\right)(\alpha+v^*)}\ln(u+\alpha) \\
& + \frac{1}{\delta(v^*-v_-^*)}\Bigg\{\left[a\Big(v_-^* + (1-\theta)\beta\Big) + \rho v_-^*\frac{\beta+v_-^*}{\alpha+v_-^*}\right]\ln\left(u-v_-^*\right) \\
& - \left[a\Big(v^* + (1-\theta)\beta\Big) + \rho v^*\frac{\beta+v^*}{\alpha+v^*}\right]\ln\left(v^*-u\right)\Bigg\},
\end{aligned}$$

and

$$\mathcal{G}_{0,\theta}(L) = K(L) - K(0), \quad 0 \leq L < v^*,$$

where

$$K(u) =: -\frac{1}{\delta\left(v^*-v_-^*\right)}\Big[(v^*+\beta)\ln\left(v^*-u\right) - \left(v_-^*+\beta\right)\ln\left(u-v_-^*\right)\Big], \quad 0 \leq u < v^*,$$

$$\begin{aligned}
v^* \quad &= \quad \frac{\sqrt{\Delta} - (\delta\beta-1)}{2\delta} > 0, \qquad (2.45)\\
-\beta \leq v_-^* \quad &= \quad -\frac{\sqrt{\Delta} + (\delta\beta-1)}{2\delta} < 0, \\
\Delta \quad &= \quad (\delta\beta+1)^2 - 4\theta\delta\beta > 0, \quad 0 \leq \theta < 1.
\end{aligned}$$

Proof. Since $0 < u < v^*$, by substituting the explicit results of (2.41) into *Theorem 2.2.4*, we have

$$\mathcal{G}_{0,\theta}(L) = \int_0^L \frac{\beta+u}{-\delta u^2 - (\delta\beta-1)u + (1-\theta)\beta}\mathrm{d}u = K(L) - K(0),$$

and

$$\begin{aligned}
\mathcal{C}_2(u) = & -a\left\{u - K(u) - \frac{\theta\beta}{\delta}\frac{1}{v^*-v_-^*}\ln\frac{v^*-u}{u-v_-^*}\right\} \\
& +\rho\left\{K(u) + \frac{\alpha}{\delta}\frac{1}{v^*-v_-^*}\left[\ln\frac{v^*-u}{u-v_-^*} + (\beta-\alpha)\left(\frac{1}{\alpha+v^*}\ln\frac{v^*-u}{u+\alpha} - \frac{1}{\alpha+v_-^*}\ln\frac{u-v_-^*}{u+\alpha}\right)\right]\right\},
\end{aligned}$$

and also,

$$\begin{aligned} v^* &= \sqrt{\Delta}-\frac{\delta\beta-1}{2\delta}=\frac{\sqrt{(\delta\beta-1)^2+4(1-\theta)\delta\beta}-(\delta\beta-1)}{2\delta}>\frac{(\delta\beta-1)-(\delta\beta-1)}{2\delta}=0; \\ -v^*_- &= \sqrt{\Delta}+\frac{\delta\beta-1}{2\delta}=\frac{\sqrt{(\delta\beta+1)^2-4\theta\delta\beta}+(\delta\beta-1)}{2\delta}\leq\frac{(\delta\beta+1)+(\delta\beta-1)}{2\delta}=\beta, \end{aligned}$$

where $v^*_-=-\beta$ only when $\theta=0$. □

Remark 2.3.4. We need to assume $\alpha\neq -v^*_-$ in *Theorem 2.3.2*, since

$$-v^*_-=\frac{\sqrt{(\delta\beta+1)^2-4\theta\delta\beta}+(\delta\beta-1)}{2\delta},$$

and, for each $\theta\in[0,1)$ we have the unique v^*_-, where

$$-v^*_-\in\left(\beta-\frac{1}{\delta},\beta\right].$$

Therefore, if $\alpha\in\left(\beta-\frac{1}{\delta},\beta\right]$, there exists the unique $\theta\in[0,1)$, such that $\alpha+v^*_-=0$.

$\alpha=-v^*_-$ is a very particular case, and we will not consider it here and assume $\alpha\neq -v^*_-$ in the sequel.

Now we derive the probability $P\left\{N_T=0\middle|\lambda_0\right\}$ in *Corollary 2.3.1*, and $P\left\{N_T=1\middle|\lambda_0\right\}$ for case $\alpha\neq\beta$ in *Corollary 2.3.2*, a discussion for the special case $\alpha=\beta$ is given in *Remark 2.3.5*.

Corollary 2.3.1. *If both the externally excited and self-excited jumps follow exponential distributions, i.e. the density functions are specified by (2.40), then the conditional probability of no jump given λ_0 and $N_0=0$, under the condition $\delta\beta>1$, is given by*

$$P\left\{N_T=0\middle|\lambda_0\right\}=e^{-\left(a+\frac{\rho}{1+\delta\alpha}\right)T}e^{\frac{a-\lambda_0}{\delta}\left(1-e^{-\delta T}\right)}\left(\frac{1-e^{-\delta T}+\delta\alpha}{\delta\alpha}\right)^{\frac{\alpha\rho}{1+\delta\alpha}}.$$

Proof. By *Theorem 2.3.2* and setting $\theta=0$, then, $\Delta=(\delta\beta+1)^2$, $v^*=\frac{1}{\delta}$, $v^*_-=-\beta$,

$$\begin{aligned} \mathcal{G}^{-1}_{0,0}(T) &= \frac{1}{\delta}\left(1-e^{-\delta T}\right), \\ K(u) &= -\frac{1}{\delta}\ln(1-\delta u),\quad 0\leq u<\frac{1}{\delta}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}_2(u) &= -au+\frac{\alpha(\beta-\alpha)\rho}{\delta\left(\alpha+v^*_-\right)(\alpha+v^*)}\ln(u+\alpha)-\frac{1}{\delta(v^*-v^*_-)}\left(a+\frac{\rho v^*}{v^*+\alpha}\right)(v^*+\beta)\ln\left(v^*-u\right) \\ &= -au-\frac{\alpha\rho}{\delta\alpha+1}\ln\left(u+\alpha\right)-\frac{1}{\delta}\left(a+\frac{\rho}{\delta\alpha+1}\right)\ln\left(\frac{1}{\delta}-u\right), \end{aligned}$$

and the result follows. □

Corollary 2.3.2. *If both the externally excited and self-excited jumps follow exponential distributions, i.e. the density functions are specified by (2.40) ($\alpha\neq\beta$), then the conditional probability of exactly one jump given λ_0 and $N_0=0$, under the condition $\delta\beta>1$, is given by*

$$\begin{aligned} P\left\{N_T=1\middle|\lambda_0\right\} &= P\left\{N_T=0\middle|\lambda_0\right\}\times\Bigg[\left(H_T+a\delta\beta-\rho\right)Q_T-a\beta\left(e^{\delta T}-1\right) \\ &\quad +\rho\frac{\alpha\beta}{1+\delta\beta}\left(\bar{a}\ln\left(\frac{\alpha+u_T}{\alpha}\right)-\bar{b}\ln\left(\frac{\beta+u_T}{\beta}\right)+\bar{c}T+\bar{d}\left(e^{\delta T}-1\right)\right)\Bigg], \end{aligned}$$

where

$$\begin{aligned}
H_T &= \left(a + \frac{\rho}{\delta\alpha + 1 - e^{-\delta T}}\right)\left(1 - e^{-\delta T}\right) + \lambda_0 e^{-\delta T}, \\
Q_T &= \frac{\beta}{1+\delta\beta}\left[\frac{1}{1+\delta\beta}\ln\left(\frac{\beta+u_T}{\beta}\right) + \delta T + \left(e^{\delta T} - 1\right)\right], \\
u_T &= \frac{1}{\delta}\left(1 - e^{-\delta T}\right), \\
\bar{a} &= \frac{1}{1+\delta\beta}\frac{1}{\beta-\alpha} + \frac{\delta}{1+\delta\alpha}\left(\frac{1}{1+\delta\beta} + \frac{1}{1+\delta\alpha}\right), \\
\bar{b} &= \frac{1}{1+\delta\beta}\frac{1}{\beta-\alpha}, \\
\bar{c} &= \frac{\delta^2}{1+\delta\alpha}\left(\frac{1}{1+\delta\beta} + \frac{1}{1+\delta\alpha}\right), \\
\bar{d} &= \frac{\delta}{1+\delta\alpha}.
\end{aligned}$$

Proof. By *Corollary 2.2.5*, and

$$\frac{1}{(\beta+u)(1-\delta u)^2} = \frac{1}{1+\delta\beta}\left[\frac{1}{1+\delta\beta}\left(\frac{1}{\beta+u} + \frac{\delta}{1-\delta u}\right) + \frac{\delta}{(1-\delta u)^2}\right],$$

we have Q_T by

$$\begin{aligned}
\int_0^{u_T} \frac{\hat{g}(u)}{(1-\delta u)^2}\mathrm{d}u &= \beta\int_0^{u_T}\frac{1}{(\beta+u)(1-\delta u)^2}\mathrm{d}u \\
&= \frac{\beta}{1+\delta\beta}\left\{\frac{1}{1+\delta\beta}\left[\ln\left(\frac{\beta+u_T}{\beta}\right) + \delta T\right] + e^{\delta T} - 1\right\},
\end{aligned}$$

and

$$\int_0^{u_T}\frac{\hat{g}(u)u}{(1-\delta u)^2}\mathrm{d}u = \frac{\beta}{\delta}\left(e^{\delta T} - 1\right) - \beta Q_T,$$

also, when $\alpha \neq \beta$,

$$\begin{aligned}
\int_0^{u_T}\frac{\hat{g}(u)\hat{h}(u)}{(1-\delta u)^2}\mathrm{d}u &= \alpha\beta\int_0^{u_T}\frac{1}{(\alpha+u)(\beta+u)(1-\delta u)^2}\mathrm{d}u \\
&= \frac{\alpha\beta}{1+\delta\beta}\left(\bar{a}\ln\left(\frac{\alpha+u_T}{\alpha}\right) - \bar{b}\ln\left(\frac{\beta+u_T}{\beta}\right) + \bar{c}T + \bar{d}\left(e^{\delta T} - 1\right)\right),
\end{aligned}$$

then, the result follows. □

Remark 2.3.5. In particular, if $\alpha = \beta$, then,

$$\begin{aligned}
P\left\{N_T = 1\middle|\lambda_0\right\} &= P\left\{N_T = 0\middle|\lambda_0\right\} \times \Bigg\{\left(H_T + a\delta\beta - \rho\right)Z_T - a\beta\left(e^{\delta T} - 1\right) \\
&\quad +\rho\left(\frac{\beta}{1+\delta\beta}\right)^2\left[\frac{u_T}{\beta(\beta+u_T)} + \delta\left(e^{\delta T} - 1\right) + \frac{2\delta}{\delta\beta+1}\left(\ln\left(\frac{\beta+u_T}{\beta}\right) + \delta T\right)\right]\Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
H_T &= \left(a + \frac{\rho}{\delta\beta + 1 - e^{-\delta T}}\right)\left(1 - e^{-\delta T}\right) + \lambda_0 e^{-\delta T}, \\
Q_T &= \frac{\beta}{1+\delta\beta}\left[\frac{1}{1+\delta\beta}\ln\left(\frac{\beta+u_T}{\beta}\right) + \delta T + \left(e^{\delta T} - 1\right)\right].
\end{aligned}$$

Note that, when $\alpha=\beta$,

$$\begin{aligned}\int_0^{u_T}\frac{\hat{g}(u)\hat{h}(u)}{(1-\delta u)^2}\mathrm{d}u &= \beta^2\int_0^{u_T}\left(\frac{1}{(\beta+u)(1-\delta u)}\right)^2\mathrm{d}u\\ &= \left(\frac{\beta}{1+\delta\beta}\right)^2\left[\frac{u_T}{\beta(\beta+u_T)}+\delta\left(e^{\delta T}-1\right)+\frac{2\delta}{\delta\beta+1}\left(\ln\left(\frac{\beta+u_T}{\beta}\right)+\delta T\right)\right].\end{aligned}$$

Remark 2.3.6. For the Hawkes process, we have the conditional probability of no jump and exactly one jump, by setting $\rho=0$ in *Corollary 2.3.1* and *Corollary 2.3.2*, respectively,

$$\begin{aligned}&P\left\{N_T=0\middle|\lambda_0\right\}=e^{-aT}e^{\frac{a-\lambda_0}{\delta}\left(1-e^{-\delta T}\right)},\\ &P\left\{N_T=1\middle|\lambda_0\right\}=P\left\{N_T=0\middle|\lambda_0\right\}\\ &\times\beta\left[\frac{a(1-e^{-\delta T}+\delta\beta)+\lambda_0e^{-\delta T}}{1+\delta\beta}\left(\frac{1}{1+\delta\beta}\ln\left(\frac{\beta+u_T}{\beta}\right)+\delta T+\left(e^{\delta T}-1\right)\right)-a(e^{\delta T}-1)\right].\end{aligned}$$

We will state and prove the results for the size of clusters based on *Theorem 2.2.5* for this exponential distribution case as below.

Theorem 2.3.3. *If both the externally excited and self-excited jumps follow exponential distributions, i.e. the density functions are specified as (2.40), then for the size of a cluster generated by a point of any generation, $\check{N}_t$, under the condition $\delta\beta>1$, we have*

$$\mathbb{E}\left[\theta^{\check{N}_\infty}\middle|\check{\lambda}_0\right]=\exp\left(-\frac{\sqrt{(\delta\beta-1)^2+4\delta\beta(1-\theta)}-(\delta\beta-1)}{2\delta}\check{\lambda}_0\right); \tag{2.46}$$

and $\check{N}_\infty$ conditional on $\check{\lambda}_0$ actually follows a mixed Poisson distribution,

$$P\left\{\check{N}_\infty=k\middle|\check{\lambda}_0\right\}=\int_0^\infty\frac{v^ke^{-v}}{k!}m(v)\mathrm{d}v,\quad k=0,1,2,... \tag{2.47}$$

where $m(v)$ is the density function of the mixing distribution,

$$m(v)=e^{\frac{\delta\beta-1}{2\delta}\check{\lambda}_0}e^{-\left(\frac{\delta\beta-1}{2\delta}\right)^2\frac{\delta}{\beta}v}\frac{\sqrt{\frac{\beta}{2\delta}}\check{\lambda}_0}{\sqrt{2\pi}v^{\frac{3}{2}}}e^{-\frac{\frac{\beta}{2\delta}\check{\lambda}_0^2}{2v}}, \tag{2.48}$$

which is an inverse Gaussian distribution with parameters $\left(\frac{\beta}{\delta\beta-1}\check{\lambda}_0,\frac{\beta}{2\delta}\check{\lambda}_0^2\right)$.

Proof. By substituting the explicit exponential distribution functions of (2.41) and the constant v^* of (2.45) into Theorem *2.2.5*, we obtain (2.46) immediately.

To prove that $\check{N}_\infty$ follows a mixed Poisson distribution, we rewrite (2.46) by

$$\mathbb{E}\left[\theta^{\check{N}_\infty}\middle|\check{\lambda}_0\right]=e^{\frac{\delta\beta-1}{2\delta}\check{\lambda}_0}e^{-\sqrt{2\xi}\check{\lambda}_0},$$

where

$$\xi=\frac{1}{2}\left(\frac{\delta\beta-1}{2\delta}\right)^2+\frac{\beta}{2\delta}(1-\theta),$$

and identify that

$$e^{-\sqrt{2\xi}\check{\lambda}_0}=\mathbb{E}\left[e^{-\xi\widetilde{\mathrm{IG}}}\right]=\int_0^\infty e^{-\xi u}\frac{\left(\check{\lambda}_0^2\right)^{\frac{1}{2}}}{\sqrt{2\pi}u^{\frac{3}{2}}}e^{-\frac{\check{\lambda}_0^2}{2u}}\mathrm{d}u,$$

where $\widetilde{\mathrm{IG}}$ follows the (infinite mean) inverse Gaussian distribution with parameters $\left(\infty, \check{\lambda}_0^2\right)$, then, we have

$$\begin{aligned}
\mathbb{E}\left[\theta^{\check{N}_\infty} \middle| \check{\lambda}_0\right] &= e^{\frac{\delta\beta-1}{2\delta}\check{\lambda}_0} \int_0^\infty e^{-\xi u} \frac{\left(\check{\lambda}_0^2\right)^{\frac{1}{2}}}{\sqrt{2\pi} u^{\frac{3}{2}}} e^{-\frac{\check{\lambda}_0^2}{2u}} \mathrm{d}u \\
&= \int_0^\infty e^{-\left[\frac{1}{2}\left(\frac{\delta\beta-1}{2\delta}\right)^2 + \frac{\beta}{2\delta}(1-\theta)\right]u} e^{\frac{\delta\beta-1}{2\delta}\check{\lambda}_0} \frac{\left(\check{\lambda}_0^2\right)^{\frac{1}{2}}}{\sqrt{2\pi} u^{\frac{3}{2}}} e^{-\frac{\check{\lambda}_0^2}{2u}} \mathrm{d}u, \\
&= \int_0^\infty e^{-\frac{\beta}{2\delta}(1-\theta)u} e^{\frac{\delta\beta-1}{2\delta}\check{\lambda}_0} e^{-\frac{1}{2}\left(\frac{\delta\beta-1}{2\delta}\right)^2 u} \frac{\left(\check{\lambda}_0^2\right)^{\frac{1}{2}}}{\sqrt{2\pi} u^{\frac{3}{2}}} e^{-\frac{\check{\lambda}_0^2}{2u}} \mathrm{d}u
\end{aligned}$$

and let $v = \frac{\beta}{2\delta}u$,

$$\begin{aligned}
\mathbb{E}\left[\theta^{\check{N}_\infty} \middle| \check{\lambda}_0\right] &= \int_0^\infty e^{-(1-\theta)v} e^{\frac{\delta\beta-1}{2\delta}\check{\lambda}_0} e^{-\left(\frac{\delta\beta-1}{2\delta}\right)^2 \frac{\delta}{\beta} v} \frac{\left(\frac{\beta}{2\delta}\check{\lambda}_0^2\right)^{\frac{1}{2}}}{\sqrt{2\pi} v^{\frac{3}{2}}} e^{-\frac{\frac{\beta}{2\delta}\check{\lambda}_0^2}{2v}} \mathrm{d}v \\
&= \int_0^\infty e^{-(1-\theta)v} m(v) \mathrm{d}v = \hat{m}(\theta - 1),
\end{aligned}$$

where

$$\hat{m}(u) = \int_0^\infty e^{-uv} m(v) \mathrm{d}v.$$

Hence, by the definition of the mixed Poisson distribution, we have (2.47) and (2.48); set $u = 1-\theta$, we have

$$\begin{aligned}
\hat{m}(u) &= \exp\left(-\frac{\sqrt{(\delta\beta-1)^2 + 4\delta\beta u} - (\delta\beta - 1)}{2\delta} \check{\lambda}_0 \right) \\
&= \exp\left[\frac{\frac{\beta}{2\delta}\check{\lambda}_0^2}{\frac{\beta}{\delta\beta-1}\check{\lambda}_0} \left(1 - \sqrt{1 + 2\frac{\left(\frac{\beta}{2\delta-1}\check{\lambda}_0\right)^2}{\frac{\beta}{2\delta}\check{\lambda}_0^2} u} \right) \right],
\end{aligned}$$

which is exactly the Laplace transform of an inverse Gaussian distribution with parameters $\left(\frac{\beta}{\delta\beta-1}\check{\lambda}_0, \frac{\beta}{2\delta}\check{\lambda}_0^2\right)$. □

Corollary 2.3.3. *In particular, for a cluster generated by a point of generation* 0*, we have*

$$\mathbb{E}\left[\theta^{\check{N}_\infty}\right] = \frac{2\delta\alpha}{\delta(2\alpha - \beta) + 1 + \sqrt{(\delta\beta - 1)^2 + 4\delta\beta(1-\theta)}}; \tag{2.49}$$

for a cluster generated by a point of subsequent generations, we have

$$\mathbb{E}\left[\theta^{\check{N}_\infty}\right] = \frac{2\delta\beta}{1 + \sqrt{1 - \frac{4\delta\beta}{(\delta\beta+1)^2}\theta}}, \tag{2.50}$$

and

$$P\left\{\check{N}_\infty = k\right\} = \frac{(\delta\beta)^{k+1}}{(\delta\beta+1)^{2k}} \frac{(2k)!}{k!(k+1)!}, \quad k = 0, 1, ... \tag{2.51}$$

Proof. By substituting the explicit exponential distribution functions of (2.41) and the constant v^* of (2.45) into Theorem *2.2.5*, we obtain (2.49). In particular, by setting $\alpha = \beta$ in (2.46) we have (2.50); by stationarity condition $\delta > \frac{1}{\beta}$ we have

$$0 < \frac{4\delta\beta}{(\delta\beta+1)^2} < 1,$$

and additionally $0 \le \theta < 1$, we have

$$0 \le \frac{4\delta\beta}{(\delta\beta+1)^2}\theta < 1.$$

Then, by referring the formula provided by Gradshteyn and Ryzhik (2007), (2.50) can be expanded explicitly as given by (2.51). □

Remark 2.3.7. We can also expand (2.46) explicitly for some other special cases. For instance, if $2\delta\alpha + (1-\delta\beta) = 0$, we have

$$\mathbb{E}\Big[\theta^{\check{N}_\infty}\Big] = \frac{\alpha}{\sqrt{\alpha^2 + \frac{\beta}{\delta}}}\left(1 - \frac{\theta}{\frac{\delta}{\beta}\alpha^2 + 1}\right)^{-\frac{1}{2}},$$

and

$$P\left\{\check{N}_\infty = k\right\} = \frac{\delta\beta - 1}{2\sqrt{\delta\beta}}\,\frac{(2k)!}{(k!2^k)^2}\left[\frac{(\delta\beta+1)^2}{4\delta\beta}\right]^{-\left(k+\frac{1}{2}\right)}, \quad k = 0, 1, \ldots.$$

Remark 2.3.8. For the general case, we can expand (2.49) with respect to θ by Taylor expansion function in `Matlab`. An example with the parameter setting $(\delta;\alpha,\beta) = (2.0; 2.0, 1.5)$ for $P\{\check{N}_\infty = k\}$ is given by *Table 2.1*.

Tab. 2.1: Probability $P\{\check{N}_\infty = k\}$ for $k = 0, 1, 2, \ldots$; $(\delta;\alpha,\beta) = (2.0; 2.0, 1.5)$

k	$P\{\check{N}_\infty = k\}$ (%)	k	$P\{\check{N}_\infty = k\}$ (%)
0	80.0000	13	0.0124
1	12.0000	14	0.0083
2	4.0500	15	0.0056
3	1.7888	16	0.0039
4	0.9043	17	0.0026
5	0.4956	18	0.0018
6	0.2866	19	0.0013
7	0.1722	20	0.0009
8	0.1064	21	0.0006
9	0.0672	22	0.0004
10	0.0432	23	0.0003
11	0.0282	24	0.0002
12	0.0186	25	0.0001

The corresponding moments of λ_t and N_t based on exponential jump distributions are omitted as they can be easily obtained using the results in Section 2.2.4.

2.4 Simulation

We also provide the simulation algorithm for the dynamic contagion process (N_t, λ_t)

- to make easier for further practical implementation and industrial applications;
- for statistical analysis, such as parameter calibration;
- to validate the theoretical results we have derived;

- for investigating more complex situations where there could be no analytic result, such as the cases when the jump sizes follow various heavy-tailed distributions (e.g. log-normal, Pareto, Weibull distributions).

2.4.1 Simulation Algorithm

Algorithm 2.4.1. The simulation algorithm for one sample path $\left\{(N_t, \lambda_t)\right\}_{t\geq 0}$ of the dynamic contagion process conditional on λ_0 and $N_0 = 0$, with m jump times $\{T_1^*, T_2^*, ..., T_m^*\}$ in the process λ_t:

1. Set the initial conditions $T_0^* = 0$, $\lambda_{T_0^{*\pm}} = \lambda_0 > a$, $N_0 = 0$ and $i \in \{0, 1, 2, ..., m-1\}$.
2. Simulate the $(i+1)^{\text{th}}$ externally excited jump waiting time E_{i+1}^* by
$$E_{i+1}^* = -\frac{1}{\rho}\ln U, \quad U \sim \text{U}[0,1].$$
3. Simulate the $(i+1)^{\text{th}}$ self-excited jump waiting time S_{i+1}^* by
$$S_{i+1}^* = \begin{cases} S_{i+1}^{*(1)} \wedge S_{i+1}^{*(2)}, & d_{i+1} > 0 \\ S_{i+1}^{*(2)}, & d_{i+1} < 0 \end{cases},$$
where
$$d_{i+1} = 1 + \frac{\delta \ln U_1}{\lambda_{T_i^{*+}} - a}, \quad U_1 \sim \text{U}[0,1],$$
and
$$S_{i+1}^{*(1)} = -\frac{1}{\delta}\ln d_{i+1}; \quad S_{i+1}^{*(2)} = -\frac{1}{a}\ln U_2, \quad U_2 \sim \text{U}[0,1].$$
4. Record the $(i+1)^{\text{th}}$ jump time T_{i+1}^* in the process λ_t by
$$T_{i+1}^* = T_i^* + S_{i+1}^* \wedge E_{i+1}^*.$$
5. Record the change at the jump time T_{i+1}^* in the process λ_t by
$$\lambda_{T_{i+1}^{*+}} = \begin{cases} \lambda_{T_{i+1}^{*-}} + Y_{i+1}^{(2)}, & Y_{i+1}^{(2)} \sim G(y) \quad \left(S_{i+1}^* \wedge E_{i+1}^* = S_{i+1}^*\right) \\ \lambda_{T_{i+1}^{*-}} + Y_{i+1}^{(1)}, & Y_{i+1}^{(1)} \sim H(y) \quad \left(S_{i+1}^* \wedge E_{i+1}^* = E_{i+1}^*\right) \end{cases}, \tag{2.52}$$
where
$$\lambda_{T_{i+1}^{*-}} = \left(\lambda_{T_i^{*+}} - a\right) e^{-\delta\left(T_{i+1}^* - T_i^*\right)} + a.$$
6. Record the change at the jump time T_{i+1}^* in the point process N_t by
$$N_{T_{i+1}^{*+}} = \begin{cases} N_{T_{i+1}^{*-}} + 1 & \left(S_{i+1}^* \wedge E_{i+1}^* = S_{i+1}^*\right) \\ N_{T_{i+1}^{*-}} & \left(S_{i+1}^* \wedge E_{i+1}^* = E_{i+1}^*\right) \end{cases}. \tag{2.53}$$

Proof. Given T_i^*, the i^{th} jump time in λ_t.

- Case E_{i+1}^*: The $(i+1)^{\text{th}}$ jump in λ_t is caused by an externally-excited jump, then it follows a Poisson distribution with constant intensity ρ, and E_{i+1}^* is the $(i+1)^{\text{th}}$ waiting time, where $E_{i+1}^* \sim \text{Exp}(\rho)$, or,
$$E_{i+1}^* \stackrel{\mathcal{D}}{=} -\frac{1}{\rho}\ln U.$$
Hence, the $(i+1)^{\text{th}}$ jump time is $T_i^* + E_{i+1}^*$.

- Case S^*_{i+1}: The $(i+1)^{\text{th}}$ jump in λ_t is caused by a self-excited jump, then S^*_{i+1} is the $(i+1)^{\text{th}}$ waiting time and can be derived as follows:

  Between time T^*_i and $T^*_i + S^*_{i+1}$, the intensity process $\left\{\lambda_t\right\}_{T^*_i \le t < T^*_i + S^*_{i+1}}$ follows the ODE

$$\frac{\mathrm{d}\lambda_t}{\mathrm{d}t} = -\delta\left(\lambda_t - a\right),$$

  with the initial condition $\lambda_t\big|_{t=T^*_i} = \lambda_{T^*_i}$. It has the unique solution

$$\lambda_t = \left(\lambda_{T^{*+}_i} - a\right)e^{-\delta(t-T^*_i)} + a, \quad T^*_i \le t < T^*_i + S^*_{i+1},$$

  and then, the cumulative distribution function of the waiting time S^*_{i+1} in the point process N_t is given by

$$\begin{aligned} F_{S^*_{i+1}}(s) &= P\left\{S^*_{i+1} \le s\right\} = 1 - P\left\{S^*_{i+1} > s\right\} \\ &= 1 - P\left\{N_{T^*_i+s} - N_{T^*_i} = 0\right\} \\ &= 1 - \exp\left(-\int_{T^*_i}^{T^*_i+s} \lambda_u \mathrm{d}u\right) \\ &= 1 - \exp\left(-\int_0^s \lambda_{T^{*+}_i+v}\mathrm{d}v\right) \\ &= 1 - \exp\left(-\left(\lambda_{T^{*+}_i} - a\right)\frac{1-e^{-\delta s}}{\delta} - as\right). \end{aligned}$$

  By the inverse transformation method, we have

$$S^*_{i+1} \stackrel{\mathcal{D}}{=} F^{-1}_{S^*_{i+1}}(U).$$

  However, we can avoid inverting the function $F_{S^*_{i+1}}(\cdot)$ by separating S^*_{i+1} into two simpler and independent random variables $S^{*(1)}_{i+1}$ and $S^{*(2)}_{i+1}$ as

$$S^*_{i+1} \stackrel{\mathcal{D}}{=} S^{*(1)}_{i+1} \wedge S^{*(2)}_{i+1},$$

  where

$$\begin{aligned} P\left\{S^{*(1)}_{i+1} > s\right\} &= \exp\left(-\left(\lambda_{T^{*+}_i} - a\right)\frac{1-e^{-\delta s}}{\delta}\right), \\ P\left\{S^{*(2)}_{i+1} > s\right\} &= e^{-as}, \end{aligned}$$

  since

$$\begin{aligned} P\{S^*_{i+1} > s\} &= \exp\left(-\left(\lambda_{T^{*+}_i} - a\right)\frac{1-e^{-\delta s}}{\delta}\right) \times e^{-as} \\ &= P\left\{S^{*(1)}_{i+1} > s\right\} \times P\left\{S^{*(2)}_{i+1} > s\right\} \\ &= P\left\{S^{*(1)}_{i+1} \wedge S^{*(2)}_{i+1} > s\right\}. \end{aligned}$$

  - For $S^{*(1)}_{i+1}$, since

$$F_{S^{*(1)}_{i+1}}(s) = P\left\{S^{*(1)}_{i+1} \le s\right\} = 1 - \exp\left(-\left(\lambda_{T^{*+}_i} - a\right)\frac{1-e^{-\delta s}}{\delta}\right),$$

we have

$$\exp\left(-\left(\lambda_{T_i^{*+}}-a\right)\frac{1-e^{-\delta S_{i+1}^{*(1)}}}{\delta}\right)\stackrel{\mathcal{D}}{=}U_1,$$

by inverting the function, then we have

$$S_{i+1}^{*(1)}\stackrel{\mathcal{D}}{=}-\frac{1}{\delta}\ln\left(1+\frac{\delta\ln U_1}{\lambda_{T_i^{*+}}-a}\right). \tag{2.54}$$

Note that, $S_{i+1}^{*(1)}$ is a defective random variable as

$$\lim_{s\to\infty}F_{S_{i+1}^{*(1)}}(s)=P\left\{S_{i+1}^{*(1)}\leq\infty\right\}=1-\exp\left(-\frac{\lambda_{T_i^{*+}}-a}{\delta}\right)<1,$$

and the condition for simulating $S_{i+1}^{*(1)}$ is

$$d_{i+1}=:1+\frac{\delta\ln U_1}{\lambda_{T_i^{*+}}-a}>0.$$

– For $S_{i+1}^{*(2)}$, since $S_{i+1}^{*(2)}\sim\mathrm{Exp}(a)$, we have

$$S_{i+1}^{*(2)}\stackrel{\mathcal{D}}{=}-\frac{1}{a}\ln U_2. \tag{2.55}$$

Hence, for S_{i+1}^{*}, we have

$$S_{i+1}^{*}\stackrel{\mathcal{D}}{=}\begin{cases}S_{i+1}^{*(1)}\wedge S_{i+1}^{*(2)}, & d_{i+1}>0\\ S_{i+1}^{*(2)}, & d_{i+1}<0\end{cases},$$

where $S_{i+1}^{*(1)}$ and $S_{i+1}^{*(2)}$ are given by (2.54) and (2.55), respectively.

Based on the two cases discussed above, T_{i+1}^{*}, the $(i+1)^{\text{th}}$ jump time in λ_t, is given by

$$T_{i+1}^{*}=T_i^{*}+S_{i+1}^{*}\wedge E_{i+1}^{*},$$

and the changes in λ_t and N_t at time T_{i+1}^{*} then can be easily derived as given by (2.52) and (2.53), respectively.

□

Remark 2.4.1. Note that, this simulation procedure given by *Algorithm 2.4.1* applies to the general distribution assumption for externally and self-excited jump-sizes $H(y)$ and $G(y)$, respectively.

2.4.2 Example: Jumps with Exponential Distributions

By the simulation algorithm *Algorithm 2.4.1* and assuming both the externally excited and self-excited jump sizes follow exponential distributions with density functions specified by (2.40), we provide the some simulated examples below with parameter setting

$$(a,\rho,\delta;\alpha,\beta;\lambda_0)=(0.5;1.0,1.5;0.5,1.0;1.5).$$

One Simulated Sample Path

For instance, one simulated sample path (N_t,λ_t) with $T=50$ is provided in *Figure 2.2.* For comparison, the theoretical expectations $\mathbb{E}[\lambda_t]$, $\mathbb{E}[\lambda_t|\lambda_0]$ and $\mathbb{E}[N_t]$ (derived by *Corollary 2.2.6*, *Theorem 2.2.6* and *Theorem 2.2.8*, respectively) are also plotted.

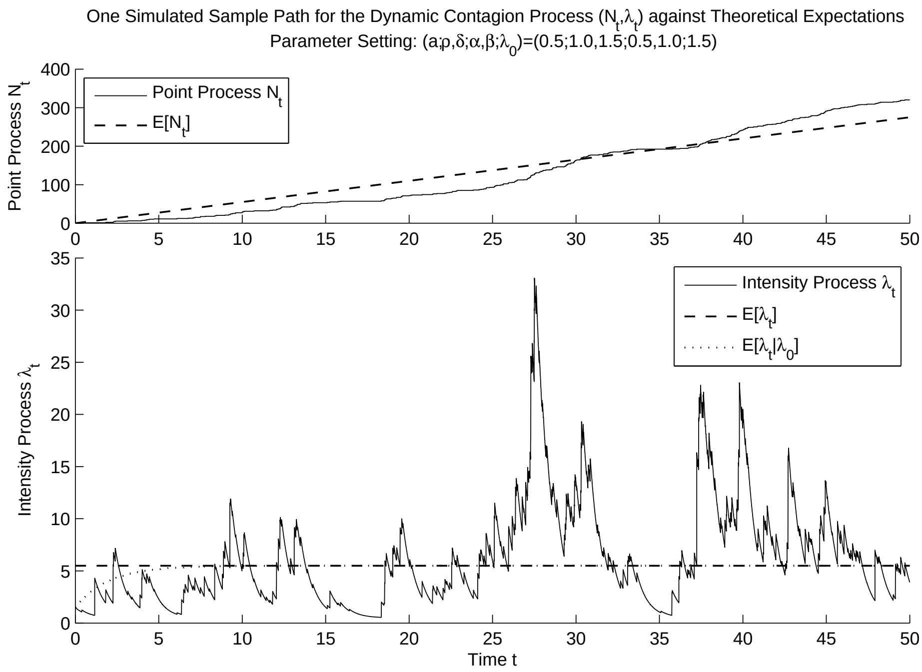

Fig. 2.2: One Simulated Sample Path of the Dynamic Contagion Process (N_t, λ_t)

Simulation Mean and Variance v.s. Theoretical Results for Intensity λ_t

To compare the simulated results with their theoretical counterparts, for instance, we calculate the simulation mean and the theoretical expectation $\mathbb{E}[\lambda_t|\lambda_0]$ (given by *Theorem 2.2.6*) in *Figure 2.3*, and the simulation variance and the theoretical variance $\text{Var}[\lambda_t|\lambda_0]$ (given by *Theorem 2.2.7*) in *Figure 2.4*, respectively. Every point (marked by a star $*$) is based on 10000 simulate sample paths of dynamic contagion process (N_t, λ_t).

2.5 Change of Measure

In this section, we provide one way to change measure for the dynamic contagion process (λ_t, N_t) via Esscher transform and scaling for the jump-size distributions.

By *Theorem 2.2.1*, *Theorem 2.2.2* and *Theorem 2.2.4*, we have a $\mathcal{F}_t^{\mathbb{P}}-$martingale

$$e^{c(t)}\theta^{N_t}e^{-B(t)\lambda_t}, \tag{2.56}$$

where parameters $c(t)$ and $B(t)$ follow the equations

$$\begin{cases} -B'(t) + \delta B(t) + \theta\hat{g}(B(t)) - 1 = 0 & (.1) \\ c'(t) + \rho\hat{h}(B(t)) - \rho - a\delta B(t) = 0 & (.2) \end{cases}, \tag{2.57}$$

and can be uniquely determined for the two cases (*I*, *II*) under the stationarity condition $\delta > \mu_{1_G}$ for $0 \leq t \leq T$:

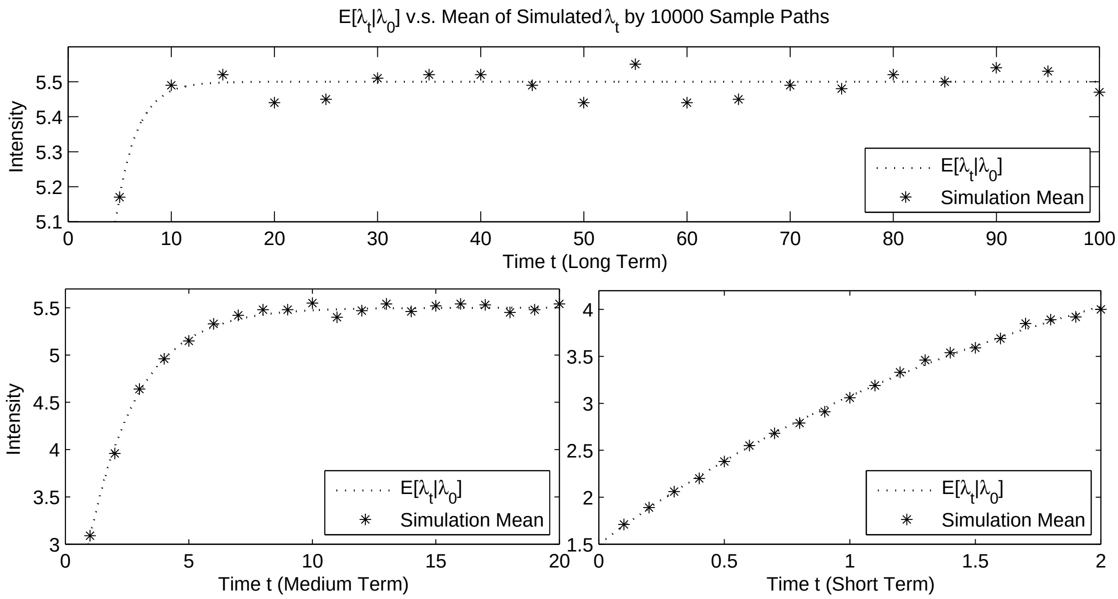

Fig. 2.3: Simulation Mean v.s. Theoretical Expectation of Intensity λ_t

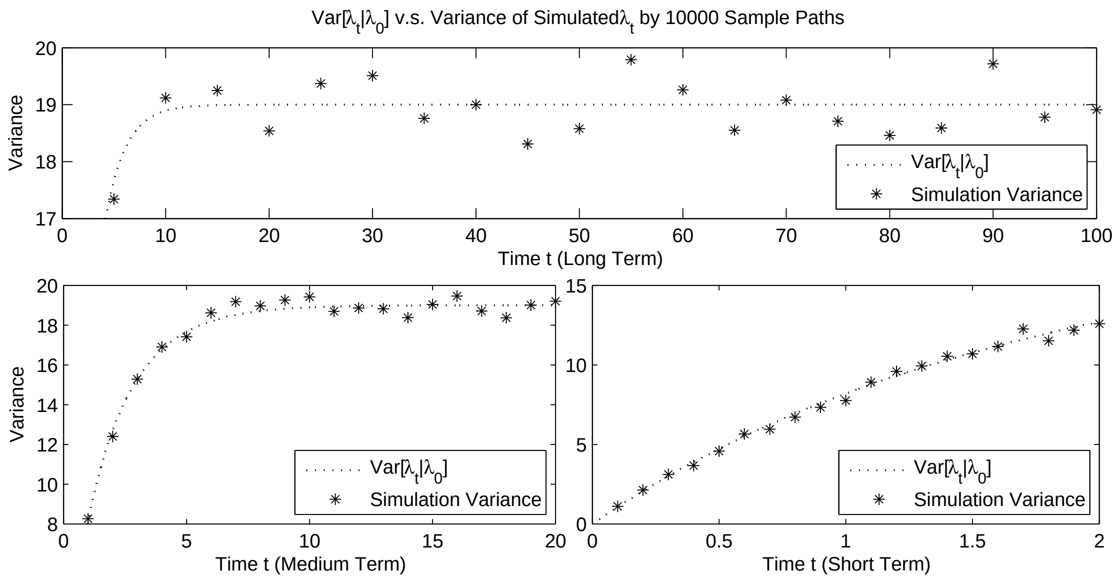

Fig. 2.4: Simulation Variance v.s. Theoretical Variance of Intensity λ_t

I. $\theta = 1, B(T) = v > 0$:

$$B(t) \in (0, v], \quad c(t) \in \left[0, \int_0^v \frac{a\delta u + \rho[1 - \hat{h}(u)]}{\delta u + \hat{g}(u) - 1} \mathrm{d}u\right), \quad t \in [0, T = \infty), \tag{2.58}$$

II. $0 \leq \theta < 1, B(T) = v = 0$:

$$B(t) \in [0, v^*), \quad c(t) \in \left[0, \int_0^{v^*} \frac{a\delta u + \rho[1 - \hat{h}(u)]}{1 - \delta u - \theta\hat{g}(u)} \mathrm{d}u\right), \quad t \in [0, T = \infty), \tag{2.59}$$

where v^* is the unique positive solution to $1 - \delta u - \theta\hat{g}(u) = 0$.

Theorem 2.5.1. *Define an equivalent probability measure* $\widetilde{\mathbb{P}}$*, via the Radon-Nikodym derivative*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} =: e^{c(t)-c(0)}\theta^{N_t - N_0}e^{-(B(t)\lambda_t - B(0)\lambda_0)}, \quad 0 \leq \theta \leq 1,$$

then, we have the parameter transformation for the dynamic contagion process (N_t, λ_t) *from* $\mathbb{P} \to \widetilde{\mathbb{P}}$ *by*

- $a \to \theta\hat{g}(B(t))a$,
- $\delta \to \delta$,
- $\rho \to \hat{h}(B(t))\rho$,
- $h(u) \to \dfrac{\widetilde{h}\left(\frac{1}{\theta\hat{g}(B(t))}u\right)}{\theta\hat{g}(B(t))}$,
- $g(u) \to \dfrac{\widetilde{g}\left(\frac{1}{\theta\hat{g}(B(t))}u\right)}{\theta\hat{g}(B(t))}$.

Proof. We use the martingale given by (2.56) to define an equivalent martingale probability measure $\widetilde{\mathbb{P}}$, via the Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} =: L_t =: \frac{e^{c(t)}\theta^{N_t}e^{-B(t)\lambda_t}}{\mathbb{E}\left[e^{c(t)}\theta^{N_t}e^{-B(t)\lambda_t}\right]} = e^{c(t)-c(0)}\theta^{N_t - N_0}e^{-(B(t)\lambda_t - B(0)\lambda_0)},$$

which is a $\mathcal{F}_t^{\mathbb{P}}-$martingale with mean value 1. Assume

$$f(\lambda, n, t) = e^{c(t)}\theta^n e^{-B(t)\lambda}\widetilde{f}(\lambda, n, t),$$

and set $\mathcal{A}f(\lambda, n, t) = 0$ for all λ, n and t in (2.2), we have

$$\begin{aligned}
&\Big(c'(t) - B'(t)\lambda\Big)\widetilde{f} + \frac{\partial\widetilde{f}}{\partial t} + \delta\,(a - \lambda)\left(-B(t)\widetilde{f} + \frac{\partial\widetilde{f}}{\partial\lambda}\right) \\
&+\rho\left(\int_0^\infty \widetilde{f}(\lambda + y, n, t)e^{-B(t)y}\mathrm{d}H(y) - \widetilde{f}(\lambda, n, t)\right) \\
&+\lambda\left(\theta\int_0^\infty \widetilde{f}(\lambda + y, n + 1, t)e^{-B(t)y}\mathrm{d}G(y) - \widetilde{f}(\lambda, n, t)\right) = 0.
\end{aligned}$$

Given the parameter relationship by (2.57) (without explicitly solving the equations) and *Esscher Transform*

$$\mathrm{d}\widetilde{H}(y) =: \frac{e^{-B(t)y}}{\hat{h}(B(t))}\mathrm{d}H(y); \quad \mathrm{d}\widetilde{G}(y) =: \frac{e^{-B(t)y}}{\hat{g}(B(t))}\mathrm{d}G(y),$$

we have

$$\frac{\partial \widetilde{f}}{\partial t} + \delta\,(a-\lambda)\,\frac{\partial \widetilde{f}}{\partial \lambda} \quad + \quad \hat{h}(B(t))\rho\left(\int_0^\infty \widetilde{f}(\lambda+y,n,t)\mathrm{d}\widetilde{H}(y) - \widetilde{f}(\lambda,n,t)\right)$$
$$+ \quad \theta\hat{g}(B(t))\lambda\left(\int_0^\infty \widetilde{f}(\lambda+y,n+1,t)\mathrm{d}\widetilde{G}(y) - \widetilde{f}(\lambda,n,t)\right) = 0.$$

Let $\widetilde{\lambda} = \theta\hat{g}(B(t))\lambda$, we have

$$\frac{\partial \widetilde{f}}{\partial t} + \delta\Big(\theta\hat{g}(B(t))a - \widetilde{\lambda}\Big)\frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} \quad + \quad \hat{h}(B(t))\rho\left(\int_0^\infty \widetilde{f}\left(\widetilde{\lambda} + \theta\hat{g}(B(t))y,n,t\right)\mathrm{d}\widetilde{H}(y) - \widetilde{f}(\widetilde{\lambda},n,t)\right)$$
$$+ \quad \widetilde{\lambda}\left(\int_0^\infty \widetilde{f}\left(\widetilde{\lambda} + \theta\hat{g}(B(t))y,n+1,t\right)\mathrm{d}\widetilde{G}(y) - \widetilde{f}(\widetilde{\lambda},n,t)\right) = 0.$$

Change variable by $u = \theta\hat{g}(B(t))y$, we have

$$\frac{\partial \widetilde{f}}{\partial t} + \delta\Big(\theta\hat{g}(B(t))a - \widetilde{\lambda}\Big)\frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} \quad + \quad \hat{h}(B(t))\rho\left(\int_0^\infty \widetilde{f}\left(\widetilde{\lambda} + u,n,t\right)\mathrm{d}\widetilde{H}(y) - \widetilde{f}(\widetilde{\lambda},n,t)\right)$$
$$+ \quad \widetilde{\lambda}\left(\int_0^\infty \widetilde{f}\left(\widetilde{\lambda} + u,n+1,t\right)\mathrm{d}\widetilde{G}(y) - \widetilde{f}(\widetilde{\lambda},n,t)\right) = 0.$$

Since

$$\mathrm{d}\widetilde{H}(y) = \widetilde{h}(y)\mathrm{d}y; \quad \mathrm{d}\widetilde{G}(y) = \widetilde{g}(y)\mathrm{d}y,$$

we finally have

$$\frac{\partial \widetilde{f}}{\partial t} + \delta\Big(\theta\hat{g}(B(t))a - \widetilde{\lambda}\Big)\frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} \quad + \quad \hat{h}(B(t))\rho\left(\int_0^\infty \widetilde{f}\left(\widetilde{\lambda} + u,n,t\right)\frac{\widetilde{h}\left(\frac{1}{\theta\hat{g}(B(t))}u\right)}{\theta\hat{g}(B(t))}\mathrm{d}u - \widetilde{f}(\widetilde{\lambda},n,t)\right)$$
$$+ \quad \widetilde{\lambda}\left(\int_0^\infty \widetilde{f}\left(\widetilde{\lambda} + u,n+1,t\right)\frac{\widetilde{g}\left(\frac{1}{\theta\hat{g}(B(t))}u\right)}{\theta\hat{g}(B(t))}\mathrm{d}u - \widetilde{f}(\widetilde{\lambda},n,t)\right) = 0.$$

Therefore, by comparing with (2.2), we have the parameter transform from the original measure $\mathbb{P} \rightarrow$ to the new measure $\widetilde{\mathbb{P}}$ as given by *Theorem 2.5.1.*

□

Remark 2.5.1. Base on *Theorem 2.5.1*, for any event $\mathcal{S} \in \mathcal{F}_t$, we have

$$\widetilde{P}\{\mathcal{S}\} = \mathbb{E}\Big[L_t\mathbb{I}\{\mathcal{S}\}\Big],$$

and inversely,

$$P\Big\{\mathcal{S}\Big|N_0 = n, \lambda_0 = \lambda\Big\} = \widetilde{\mathbb{E}}\left[\frac{1}{L_t}\mathbb{I}\{A\}\middle|N_0 = n, \lambda_0 = \lambda\right]$$
$$= e^{c(0)}\theta^n e^{-B(0)\lambda}\widetilde{\mathbb{E}}\left[e^{-c(t)}\theta^{-N_t}e^{B(t)\lambda_t}\mathbb{I}\{A\}\middle|N_0 = n, \lambda_0 = \lambda\right].$$

Stationarity Condition

Theorem 2.5.2. *If the stationarity condition holds under the original measure* $\mathbb{P}$*, i.e.*

$$\delta > \mu_{1_G},$$

then, it still holds under the new measure $\widetilde{\mathbb{P}}$*, i.e.*

$$\widetilde{\delta} > \mu_{1_{\widetilde{G}}}.$$

Proof. Under the new measure $\widetilde{\mathbb{P}}$, by the parameter transformation given by *Theorem 2.5.1* and change variable $y = \frac{1}{\theta\hat{g}(B(t))}u$ we have

$$\begin{aligned}
\mu_{1_{\widetilde{G}}} &= \int_0^\infty u \frac{\widetilde{g}\left(\frac{1}{\theta\hat{g}(B(t))}u\right)}{\theta\hat{g}(B(t))} \mathrm{d}u \\
&= \int_0^\infty u \frac{1}{\theta\hat{g}(B(t))} \frac{e^{-B(t)\frac{1}{\theta\hat{g}(B(t))}u}}{\hat{g}(B(t))} g\left(\frac{1}{\theta\hat{g}(B(t))}u\right) \mathrm{d}u \\
&= \theta \int_0^\infty y e^{-B(t)y} \mathrm{d}G(y).
\end{aligned}$$

Since $0 \leq \theta \leq 1$ and $B(T) = v \geq 0$ as given by (2.58) and (2.59) and the stationarity condition holds under the measure $\mathbb{P}$, we have

$$\widetilde{\delta} = \delta > \mu_{1_G} = \int_0^\infty y \mathrm{d}G(y) > \theta \int_0^\infty y e^{-B(t)y} \mathrm{d}G(y) = \mu_{1_{\widetilde{G}}}.$$

□

Example: Jumps with Exponential Distributions

Assume the jump-sizes follows essential distributions, $H \sim \mathrm{Exp}(\alpha)$ and $G \sim \mathrm{Exp}(\beta)$ as given by (2.40), by *Theorem 2.5.1* we have the parameter transform from the measure $\mathbb{P}$ to the new measure $\widetilde{\mathbb{P}}$ as follows:

- $a \to \frac{\theta\beta}{\beta+B(t)}a$,
- $\delta \to \delta$,
- $\rho \to \frac{\alpha}{\alpha+B(t)}\rho$,
- $H \sim \mathrm{Exp}(\alpha) \to \mathrm{Exp}\left(\frac{(\alpha+B(t))(\beta+B(t))}{\theta\beta}\right)$,
- $G \sim \mathrm{Exp}(\beta) \to \mathrm{Exp}\left(\frac{(\beta+B(t))^2}{\theta\beta}\right)$.

3

Applications to Finance: Credit Risk

The dynamic contagion process introduced by Chapter 2 provides us a new and proper tool for modelling the contagion and clustering of the arrivals of events. In this chapter, we mainly apply the dynamic contagion process to model the default credit risk of a single name (company) in Section 3.1. We also have a brief investigation and discussion on possible applications to portfolio credit risk in Section 3.2.

3.1 Single-Name Default Probability

Our motivation of applying the dynamic contagion process to model the credit risk in this section is a combination of Duffie and Singleton (1999) and Lando (1998). Duffie and Singleton (1999) introduced the affine processes to model the default intensity. Lando (1998), the extension of Jarrow, Lando and Turnbull (1997), used the state of credit ratings as an indicator of the likelihood of default, and modelled the underlying credit rating migration driven by a probability transition matrix with Cox processes in a finite-state Markov process framework. However, we go beyond this and model the bad events that can possibly lead to credit default, and the number and the intensity of these events are modelled by the dynamic contagion process.

Based on this idea, we proceed with the following modification of the intensity models. We assume that the final default or bankruptcy is caused by a number of bad events relating to the underlying company. The bad events are not only restricted to the credit rating downgrades announced by rating agencies, but also could be other bad news relevant to this company, such as bad corporate financial reports. The frequency of these bad events is dependent both on the common bad news in the market exogenously and the company's bad events endogenously. Each company has a certain level of capability or resistance to overcome some its bad events to avoid bankruptcy, for example, if we use the credit rating system as the indicator to quantify this level, usually the higher rated companies have higher capability level. We provide an application in credit risk for this idea by using the dynamic contagion process, based on the explicit results obtained in Section 2.3 for the case of exponential jumps.

The point process N_t is to model the number of bad events released from the underlying

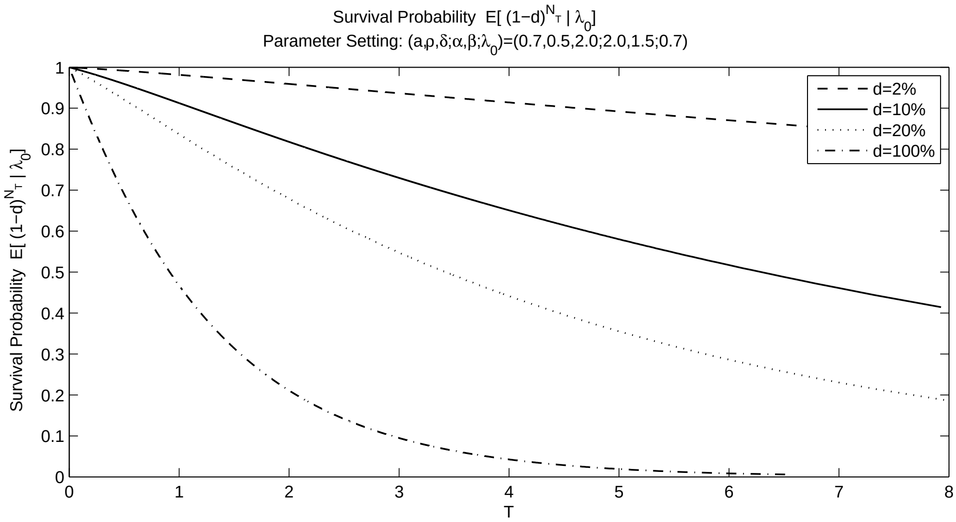

Fig. 3.1: Survival Probability $P_s(T)$; $(a, \rho, \delta; \alpha, \beta; \lambda_0) = (0.7, 0.5, 2.0; 2.0, 1.5; 0.7)$

company. It is driven by a series of bad events $\left\{Y_j^{(2)}\right\}_{j=1,2,\ldots}$ from itself and the common bad events $\left\{Y_i^{(1)}\right\}_{i=1,2,\ldots}$ widely in the whole market via its intensity process λ_t. The impact of each event decays exponentially with constant rate δ. We assume each jump, or bad event, can result to default with a constant probability d, $0 < d \leq 1$, which measures and quantifies the resistance level. Therefore, the survival probability conditional on the (initial) current intensity λ_0 at time T is given by

$$P_s(T) = \mathbb{E}\left[(1-d)^{N_T}\middle|\lambda_0\right] \tag{3.1}$$

which can be calculated simply by letting $\theta = 1 - d$ in the conditional probability generating function derived in *Theorem 2.3.2*.

By setting the parameters $(a, \rho, \delta; \alpha, \beta; \lambda_0) = (0.7, 0.5, 2.0; 2.0, 1.5; 0.7)$ in (3.1), the term structure of the survival probabilities $p_s(T)$ based on $d = 2\%$, 10%, 20% and 100% are shown in *Figure 3.1*, with the corresponding numerical results in *Table 3.1*.

Tab. 3.1: Survival Probability $P_s(T)$; $(a, \rho, \delta; \alpha, \beta; \lambda_0) = (0.7, 0.5, 2.0; 2.0, 1.5; 0.7)$

Time T	1	2	3	4	5	6
$d = 2\%$	98.15%	95.92%	93.65%	91.40%	89.21%	87.06%
$d = 10\%$	91.26%	81.78%	72.99%	65.07%	58.01%	51.70%
$d = 20\%$	83.66%	67.91%	54.78%	44.13%	35.54%	28.63%
$d = 100\%$	46.73%	21.10%	9.48%	4.26%	1.92%	0.86%

Alternatively, by using the same parameter setting, we can regenerate the survival probabilities $P_s(T)$ in *Table 3.2* based on 10000 simulated sample paths (truncated at time T), which are very close to the analytical results in *Table 3.1*. One of the underlying simulated sample paths is provided in *Figure 3.2*. For comparison, the theoretical expectations $\mathbb{E}[\lambda_t]$, $\mathbb{E}[\lambda_t|\lambda_0]$ and

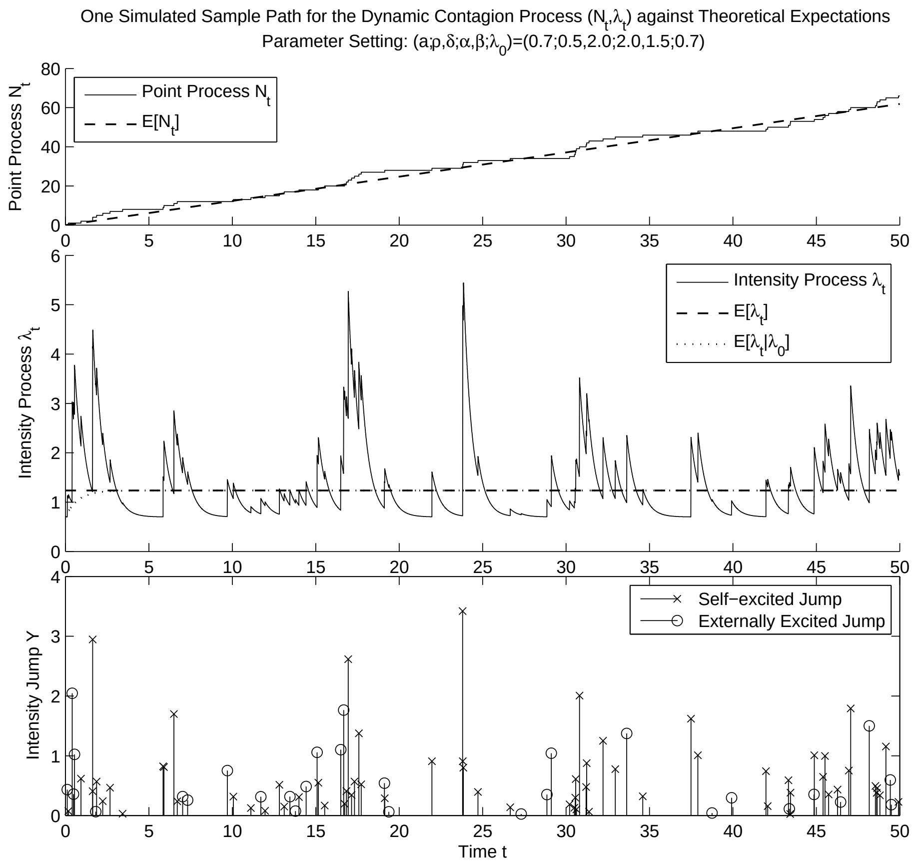

Fig. 3.2: One Simulated Sample Path of the Dynamic Contagion Process (N_t, λ_t)

$\mathbb{E}[N_t]$ (derived by *Corollary 2.2.6*, *Theorem 2.2.6* and *Theorem 2.2.8*, respectively) are also plotted.

Tab. 3.2: Survival Probability $P_s(T)$ by 10000 Simulated Sample Paths

Time T	1	2	3	4	5	6
$d = 2\%$	98.13%	95.89%	93.60%	91.46%	89.18%	87.04%
$d = 10\%$	91.18%	81.71%	72.97%	65.24%	58.00%	51.67%
$d = 20\%$	83.65%	67.85%	54.83%	43.85%	35.26%	28.81%
$d = 100\%$	46.66%	21.68%	9.98%	4.39%	1.77%	0.84%

As in Lando (1998), we could consider different values of d correspond to different credit ratings, by assuming these bad events are all related to the company's credit ratings.

We also provide a comparison for the survival probabilities based on three main processes discussed in this paper: dynamic contagion process, Hawkes process (by setting $\rho = 0$) and non-self-excited process (by setting $\beta = \infty$), with the same parameter setting and fixed $d = 10\%$. The results are shown in *Figure 3.3*, with numerical output in *Table 3.3*.

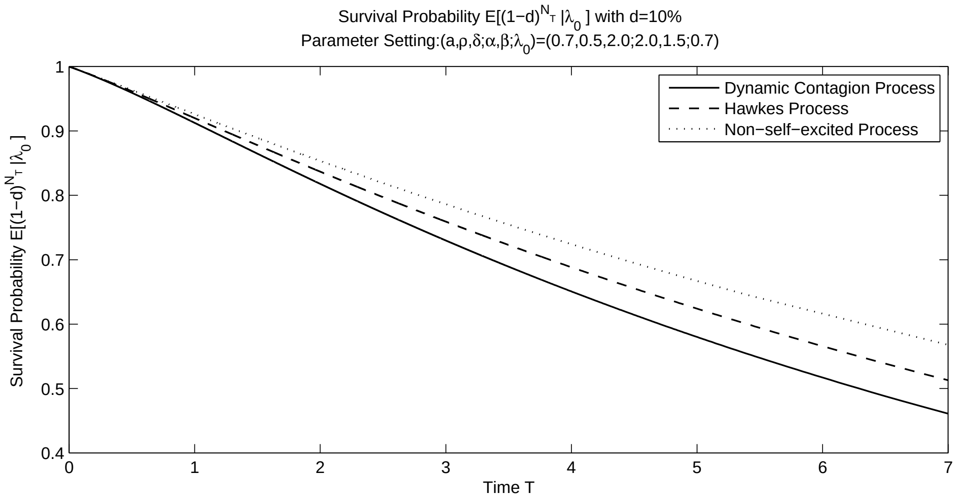

Fig. 3.3: Survival Probability Comparison for the Dynamic Contagion, Hawkes and Non-self-excited Processes

Tab. 3.3: Survival Probability Comparison for the Dynamic Contagion, Hawkes and Non-self-excited Process

Time T	1	2	3	4	5	6
Dynamic Contagion Process	91.26%	81.78%	72.99%	65.07%	58.01%	51.70%
Hawkes Process	91.99%	83.68%	75.92%	68.84%	62.40%	56.57%
Non-self-excited Process	92.59%	85.34%	78.62%	72.41%	66.70%	61.72%

We can see that, the dynamic contagion process, as the most general case of the three processes, generates the lowest survival probability, and the differences between the other two processes explain the impact from the endogenous and exogenous factors respectively. This process is capable to capture more aspects of the risk, which is particularly useful for modelling the risks during the economic downturn involving more clusters of bad economic events.

3.2 Multiple-Name Default Probability

For applications to portfolio credit risk involving multiple underlying names, such as pricing CDO, it is crucial to find the quantity $P\{N_T = k\}$, i.e. the probability of k names of default in the portfolio up to time t. Theoretically, we can obtain it explicitly by simply differentiating k times to the probability generating function of N_t derived in *Theorem 2.2.4*, however, in practice, the result could become extremely messy for a large number of k. Therefore, in this section, we propose three alternative methods below as future research to estimate this probability.

Monte Carlo Simulation under the Original Measure The probabilities $P\{N_T = k\}$ can be estimated based on the crude Monte Carlo simulation (CMC) by using *Algorithm 2.4.1*, and an example is provided as below.

With parameter setting $(a, \rho, \delta; \alpha, \beta; \lambda_0) = (0.7, 0.5, 2.0; 2.0, 1.5; 0.7)$, we estimate $P\{N_T = k\}$ for time $T = 1$, $T = 5$, $T = 10$ and $T = 20$, respectively, based on $1,000,000$ sample paths for each

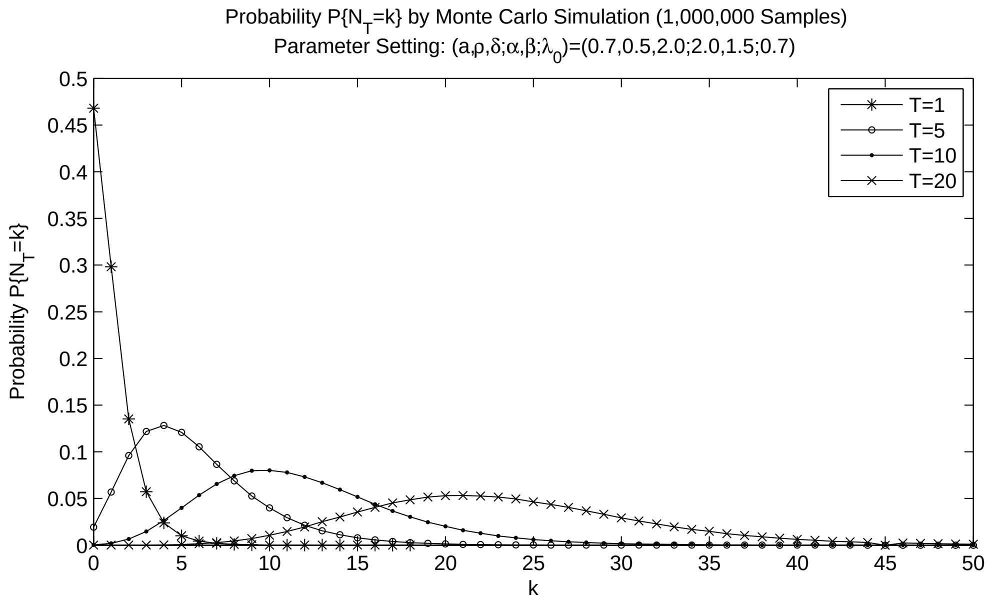

Fig. 3.4: Probability $P\{N_T = k\}$ by Monte Carlo Simulation $(1,000,000$ Samples)

point as represented in *Figure 3.4* and *Table 3.4*.

Remark 3.2.1. This CMC approach becomes difficult to estimate precisely the rare event probabilities $P\{N_T = k\}$ when k is very large (> 50), and it should be better if we could make a suitable change of measure and simulate with importance sampling.

Faà di Bruno's Formula Rather than differentiating k times to the probability generating function of N_t directly, we could differentiate the Riccati ordinary differential equations of the probability generating function of N_t, with aid of *Faà di Bruno's Formula*. This approach is also adopted by Errais, Giesecke and Goldberg (2010) for pricing CDO via a Hawkes-process model.

We slightly modify *Theorem 2.2.4* to be the Riccati ordinary differential equations of the probability generating function of N_t.

Theorem 3.2.1. *The probability generating function of N_T conditional on $\mathcal{F}_t$, under the condition $\delta > \mu_{1_G}$, is given by*

$$\mathbb{E}\Big[\theta^{(N_T - N_t)}\Big|\mathcal{F}_t\Big] = e^{-\Big(c(T)-c(t)\Big)} e^{-B(t)\lambda_t}, \quad 0 \leq \theta < 1,$$

where $B(t)$ is uniquely determined by the non-linear ODE

$$B'(t) = \delta B(t) + \theta \hat{g}(B(t)) - 1,$$

with the boundary condition $B(T) = 0$; and $c(T)$ is uniquely determined by

$$c'(t) = a\delta B(t) + \rho\Big[1 - \hat{h}(B(t))\Big],$$

with the boundary condition $c(0) = 0$.

Tab. 3.4: Probability $P\{N_T = k\}$ by Monte Carlo Simulation $(1,000,000$ Samples)

k	$P\{N_{T=1} = k\}$	$P\{N_{T=5} = k\}$	$P\{N_{T=10} = k\}$	$P\{N_{T=20} = k\}$
0	46.68%	1.92%	0.03%	0.00%
1	29.75%	5.75%	0.21%	0.00%
2	13.69%	9.66%	0.67%	0.00%
3	5.72%	12.10%	1.47%	0.01%
4	2.37%	12.76%	2.62%	0.03%
5	1.02%	12.11%	3.98%	0.07%
6	0.44%	10.52%	5.30%	0.14%
7	0.19%	8.71%	6.51%	0.26%
8	0.08%	6.86%	7.45%	0.46%
9	0.04%	5.27%	7.98%	0.72%
10	0.02%	3.98%	8.03%	1.05%
11	0.01%	2.89%	7.81%	1.49%
12	0.00%	2.13%	7.34%	1.94%
13	0.00%	1.56%	6.66%	2.49%
14	0.00%	1.11%	5.91%	3.05%
15	0.00%	0.79%	5.14%	3.58%
16	0.00%	0.56%	4.41%	4.04%
17	0.00%	0.40%	3.69%	4.51%
18	0.00%	0.27%	3.04%	4.85%
19	0.00%	0.19%	2.48%	5.08%
20	0.00%	0.14%	2.01%	5.29%

Define $B(t) = B(t;\theta)$, $c(t) = c(t;\theta)$, and

$$\mathcal{E}_t(k,T;\theta) =: \frac{\partial^k}{\partial\theta^k}\mathbb{E}\Big[\theta^{(N_T-N_t)}\big|\mathcal{F}_t\Big] = \frac{\partial^k}{\partial\theta^k}\exp\Big(-\Big(c(T;\theta)-c(t;\theta)\Big)-B(t;\theta)\lambda_t\Big),$$

and we need calculate

$$P\{N_T - N_t = k\big|\lambda_t\} = \frac{1}{k!}\mathcal{E}_t(k,T;0),$$

where

$$\begin{cases} B(t;\theta)' = \delta B(t;\theta) + \theta\hat{g}(B(t;\theta)) - 1 \qquad (0\le\theta<1, \delta>\mu_{1_G}) & (.1)\\ c(t;\theta)' = a\delta B(t;\theta) + \rho\Big[1-\hat{h}(B(t;\theta))\Big] & (.2)\end{cases}, \tag{3.2}$$

with boundary conditions $B(T;\theta) = 0$ and $c(0;\theta) = 0$. Note that, we have the expansion for the probability generating function

$$\mathbb{E}\Big[\theta^{(N_T-N_t)}\big|\mathcal{F}_t\Big] = \exp\Big(-\Big(c(T;\theta)-c(t;\theta)\Big)-B(t;\theta)\lambda_t\Big) = \sum_{k=0}^{\infty}\theta^k P\{N_T-N_t=k\big|\lambda_t\},$$

and $\{\mathcal{E}_t(k,T;0)\}_{k=0,1,\dots}$ can be obtained as follows.

- $\mathcal{E}_t(0,T;0)$: We have

$$\begin{aligned} \frac{\partial^0}{\partial\theta^0}\bigg|_{\theta=0}: \quad B(t;0)' &= \delta B(t;0) - 1,\\ c(t;0)' &= a\delta B(t;0) + \rho\Big[1-\hat{h}(B(t;0))\Big], \end{aligned}$$

  with boundary conditions $B(T;0) = 0$ and $c(0;0) = 0$. The solution is given by

$$\begin{aligned} B(t;0) &= \frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right),\\ c(t;0) &= \int_0^t \left[a\left(1-e^{-\delta(T-s)}\right)+\rho\left[1-\hat{h}\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right)\right)\right]\right]\mathrm{d}s, \end{aligned}$$

hence,

$$\begin{aligned}
&\frac{\partial^0}{\partial\theta^0}\mathbb{E}\Big[\theta^{(N_T-N_t)}\big|\mathcal{F}_t\Big]\Big|_{\theta=0}\\
=\ & \exp\Big(-\Big(c(T;0)-c(t;0)\Big)-B(t;0)\lambda_t\Big)\\
=\ & \exp\Big(-a(T-t)+\frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right)a-\frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right)\lambda_t\Big).
\end{aligned}$$

- $\mathcal{E}_t(1,T;0)$: We have

$$\begin{aligned}
\frac{\partial^1}{\partial\theta^1}:\quad B^{(1)}(t;\theta)' &= \delta B^{(1)}(t;\theta)+\Big[\hat{g}(B(t;\theta))+\theta\hat{g}^{(1)}(B(t;\theta))\Big],\\
c^{(1)}(t;\theta)' &= a\delta B^{(1)}(t;\theta)-\rho\hat{h}^{(1)}(B(t;\theta)),
\end{aligned}$$

with boundary condition $B^{(1)}(T;\theta)=0$ and $c^{(1)}(0;\theta)=0$,

$$\begin{aligned}
\frac{\partial^1}{\partial\theta^1}\Big|_{\theta=0}:\quad B^{(1)}(t;0)' &= \delta B^{(1)}(t;0)+\hat{g}(B(t;0)),\\
c^{(1)}(t;0)' &= a\delta B^{(1)}(t;0)-\rho\hat{h}^{(1)}(B(t;0)),
\end{aligned}$$

with boundary conditions $B^{(1)}(T;0)=0$ and $c^{(1)}(0;0)=0$. The solution is given by

$$\begin{aligned}
B^{(1)}(t;0) &= \int_0^t \hat{g}(B(s;0))e^{\delta(t-s)}\mathrm{d}s,\\
c^{(1)}(t;0) &= \int_0^t \Big[a\delta B^{(1)}(s;0)-\rho\hat{h}^{(1)}(B(s;0))\Big]\mathrm{d}s,
\end{aligned}$$

hence,

$$\begin{aligned}
&\frac{\partial^1}{\partial\theta^1}\mathbb{E}\Big[\theta^{(N_T-N_t)}\big|\mathcal{F}_t\Big]\Big|_{\theta=0}\\
=\ & \Big(-\Big(c^{(1)}(T;0)-c^{(1)}(t;0)\Big)-B^{(1)}(t;0)\lambda_t\Big)\exp\Big(-\Big(c(T;0)-c(t;0)\Big)-B(t;0)\lambda_t\Big).
\end{aligned}$$

- $\mathcal{E}_t(k,T;0)$: for larger k, we can use *Faà di Bruno's Formula* for differentiating multiple times w.r.t. θ.

Proposition 3.2.1. *Faà di Bruno's Formula:*

$$\frac{\partial^k}{\partial\theta^k}g\big(f(\theta)\big)=\sum\frac{k!}{b_1!b_2!\cdots b_k!}g^{(b_1+\cdots+b_k)}\big(f(\theta)\big)\left(\frac{f'(\theta)}{1!}\right)^{b_1}\left(\frac{f''(\theta)}{2!}\right)^{b_2}\cdots\left(\frac{f^{(k)}(\theta)}{k!}\right)^{b_k},$$

where the sum is over all different solutions in non-negative integers $b_1,\cdots,b_k$ of $b_1+2b_2+\cdots+kb_k=k$. Alternatively, this formula above can be expressed by

$$\frac{\partial^k}{\partial\theta^k}g\big(f(\theta)\big)=\sum_{d=1}^k S_{kd}f^{(d)}\big(g(\theta)\big),$$

where the coefficients

$$\begin{aligned}
S_{kd} &= \sum_{(b_1,\cdots,b_k)\in\mathcal{B}_{kd}}\frac{k!}{b_1!b_2!\cdots b_k!}\left(\frac{f'(\theta)}{1!}\right)^{b_1}\left(\frac{f''(\theta)}{2!}\right)^{b_2}\cdots\left(\frac{f^{(k)}(\theta)}{k!}\right)^{b_k},\\
\mathcal{B}_{kd} &= \Big\{(b_1,\cdots,b_k):\sum_{i=1}^k b_i=d,\sum_{i=1}^k ib_i=k,b_i\in\mathbb{N}_0\Big\}.
\end{aligned}$$

It is key to efficiently generate all vectors $(b_1, b_2, \cdots, b_k)$ in the set

$$\mathcal{B}_k = \bigcup_{d=1}^{k} \mathcal{B}_{kd},$$

and we adopt the algorithm of Klimko (1973) to generate all indices in *Faà di Bruno's Formula*. Solutions as vectors $(b_1, b_2, \cdots, b_k)$ stored in the matrices M_k from case $k = 1$ to case $k = 8$ are provided for instance. We can observe from *Figure 3.5*, *Table 3.5* and the matrices M_k, the cardinality of $\mathcal{B}_k$, i.e. the total number of vectors $(b_1, b_2, \cdots, b_k)$, increases exponentially with respect to k, and also the recursive structure in the ODEs, so we do not adopt this method for calculating $P\{N_T = k\}$ as it has inconvenience for processing in practice particularly when k is very large ($k > 50$).

$$M_k =: \begin{bmatrix} b_1 & \cdots & \cdots \\ \vdots & \vdots & \ldots \\ b_k & \cdots & \cdots \end{bmatrix}_{k \times \text{Cardinality of } \mathcal{B}_k}$$

$$M_{k=2} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

$$M_{k=3} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$M_{k=4} = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 5}$$

$$M_{k=5} = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 7}$$

$$M_{k=6} = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 11}$$

$$M_{k=7} = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{7\times 15}$$

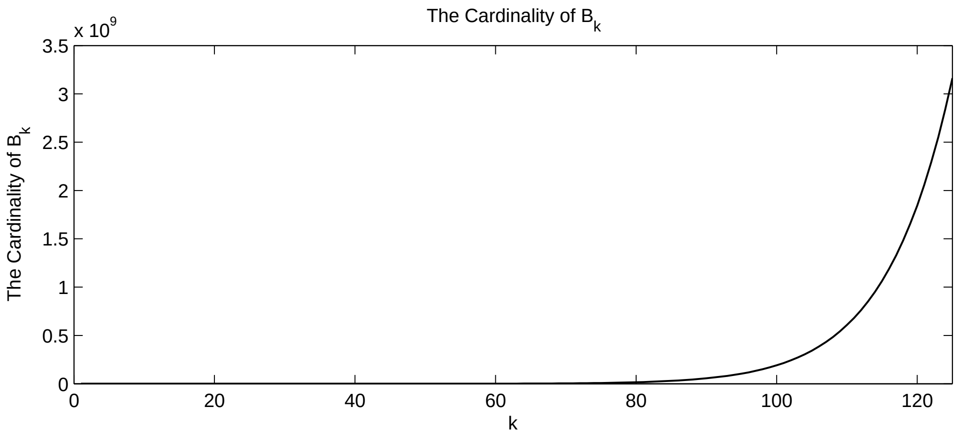

Fig. 3.5: The Cardinality of $\mathcal{B}_k$

$$M_{k=8} = \begin{bmatrix} 0&1&0&2&0&1&0&0&1&0&2&1&1&1&1&3&2&2&2&2&2&8 \\ 0&0&1&0&0&1&0&0&0&2&1&0&1&0&2&1&0&4&3&2&1&0 \\ 0&0&0&0&1&0&0&0&1&0&0&0&2&2&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&2&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}_{8\times 22}$$

Tab. 3.5: The Cardinality of $\mathcal{B}_k$

k	2	3	4	5	10	20	30	40	50	60	80	100	125
Cardinality	2	3	5	7	42	627	5,604	37,338	204,226	966,467	15,796,476	190,569,292	3,163,127,352

Asymptotic Approximation The major difficulty we meet in two aprroaches above is estimating the probability $P\{N_t = k\}$ precisely for a larger k, although in variety of situations of applications in practice, we would not meet the case when k is very large above 50, or, the probability is extremely small and could be neglected. For future research, we suggest to investigate the asymptotic property of $P\{N_T = k\}$ or $P\{N_t < k\}$ when $k \to \infty$ for a fixed time T by using *Tauberian Theorem*, *Monotone Density Theorem* or *Large Deviation Theory*.

4

Applications to Insurance: Ruin by Dynamic Contagion Claims

To capture the clustering phenomenon as well as some common external factors involved for the arrival of claims within one single consistent framework, we extend further to use the dynamic contagion process and try to generalise results obtained for the classical model of infinite horizon. In this chapter, we consider a risk process with the arrival of claims modelled by a dynamic contagion process. We derive results for the infinite horizon model that are generalisations of the Cramér-Lundberg approximation, Lundberg's fundamental equation, some asymptotics as well as bounds for the probability of ruin. Special attention is given to the case of exponential jumps and a numerical example is provided.

We organise this chapter as follows. Section 4.1 formulates the problem. It also provides a numerical example and some asymptotics that are based on simulations. In Section 4.2, we use the martingale method and generalise Lundberg's fundamental equation. We derive bounds for the ruin probability in Section 4.3. In Section 4.4, we derive all results via a change of measure. This makes simulations more efficient as ruin is certain under the new measure. Section 4.5 concentrates on exponentially distributed claims. Our results are illustrated by a numerical example.

4.1 Ruin Problem

We consider a company with its surplus process X_t in continuous time on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0, \tag{4.1}$$

where

- $X_0 = x \geq 0$ is the initial reserve at time $t = 0$;
- $c > 0$ is the constant rate of premium payment per time unit;
- N_t is a point process ($N_0 = 0$) counting the number of cumulative arrived claims in the time interval $(0, t]$, driven by a dynamic contagion process with its stochastic intensity process λ_t and the initial intensity $\lambda_0 = \lambda > 0$;

- $\left\{Z_i\right\}_{i=1,2,\dots}$ is a sequence of independent identical distributional positive random variables (claim sizes) with distribution function $Z(z), z>0$, and also independent of N_t; the mean, Laplace transform of density function and tail are denoted respectively by

$$\mu_{1_Z} =: \int_0^\infty z \mathrm{d}Z(z), \quad \hat{z}(u) =: \int_0^\infty e^{-uz} \mathrm{d}Z(z), \quad \overline{Z}(x) =: \int_x^\infty \mathrm{d}Z(s).$$

The surplus process X_t is a right-continuous function of time t.

Definition 4.1.1 (Ruin Time)**.** *The ruin (stopping) time τ^* is defined by*

$$\tau^* =: \begin{cases} \inf\left\{t>0 \middle| X_t \le 0\right\} \\ \inf\{\varnothing\} = \infty & \text{if } X_t > 0 \text{ for all } t; \end{cases}$$

in particular, $\tau^* = \infty$ means ruin does not occur.

We are interested in the ruin probability in finite time,

$$P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\};$$

in particular, the ultimate ruin probability in infinite time,

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} =: \lim_{t\to\infty} P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\};$$

and also the ultimate ruin probability in infinite time when the intensity process λ_t has stationary distribution,

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 \sim \Pi\right\},$$

where Π is the stationary distribution of λ_t given by *Theorem 2.2.3*.

4.1.1 Net Profit Condition

Theorem 4.1.1. *If $\delta > \mu_{1_G}$ and the arrival of claims is driven by a dynamic contagion process, then, the net profit condition is given by*

$$c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}} \mu_{1_Z}. \tag{4.2}$$

Proof. Obviously, we have the expectation of surplus process X_t defined by (4.1),

$$\mathbb{E}[X_t] = x + ct - \mu_{1_Z}\mathbb{E}[N_t],$$

since the sequence $\{Z_i\}_{i=1,2,\dots}$ and the process N_t are independent,

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{N_t} Z_i\right] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_t} Z_i\right] \middle| N_t\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[Z_1 + \cdots + Z_{N_t} \middle| N_t = n\right] P\{N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[Z_1 + \cdots + Z_n \middle| N_t = n\right] P\{N_t = n\} \\
&= \sum_{n=0}^{\infty} \mathbb{E}[Z_1 + \cdots + Z_n] P\{N_t = n\} \\
&= \sum_{n=0}^{\infty} n\mu_{1_Z} P\{N_t = n\} \\
&= \mu_{1_Z}\mathbb{E}[N_t].
\end{aligned}$$

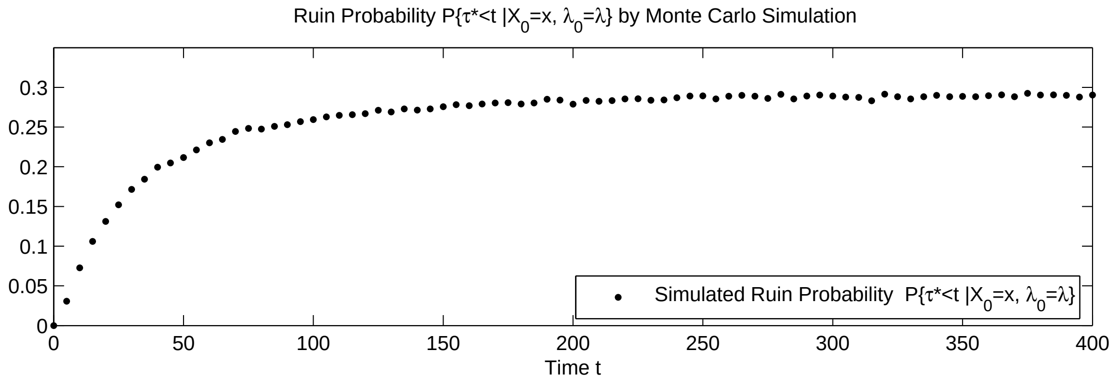

Fig. 4.1: Ruin Probability $P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\}$ by 50,000 Simulated Dynamic Contagion Processes

If $\delta > \mu_{1_G}$ and the net profit condition holds, by *Corollary 2.2.6* and *Theorem 2.2.8*, we have

$$\lim_{t\to\infty} \frac{\mathbb{E}[X_t]}{t} = c - \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z} > 0.$$

□

4.1.2 Simulation Examples

Before giving mathematical proofs, we can have a first glance at this ruin problem via Monte Carlo simulation. Assume the stationarity condition for λ_t and net profit condition for X_t both hold, and the two types of jump sizes and claim sizes all follow exponential distributions, i.e. $H \sim \text{Exp}(\alpha)$, $G \sim \text{Exp}(\beta)$ and $Z \sim \text{Exp}(\gamma)$. We implement the simulation algorithm for a dynamic contagion process provided by *Algorithm 2.4.1*, with parameters set by

$$(a, \lambda_0, \rho, \delta; \alpha, \beta, \gamma; X_0, c) = (0.7, 0.7, 0.5, 2.0; 2.0, 1.5, 1.0; 10, 1.5).$$

In *Figure 4.1*, we plot the ruin probability $P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\}$ against the time from $t = 0$ to $t = 400$. We can observe that the probability increases and converges around 30% when time t increases. Note that, each point is calculated based on 50,000 replications of dynamic contagion processes. For instance, one example of simulated surplus process X_t with the underlying point process of claim arrival N_t and intensity process λ_t from time $t = 0$ to $t = 100$ is represented by *Figure 4.2*, and the pattern of clustering arrival of claims generated by a dynamic contagion process is also shown in the histogram. For comparison, the theoretical expectations of λ_t and N_t (given by *Corollary 2.2.6*) are plotted together with their simulated paths. More numerical examples are provided later by Section 4.5.3.

Remark 4.1.1. It is impossible to simulate infinitely long ($t = \infty$) paths for estimating the ultimate ruin probability $P\left\{\tau^* < t = \infty \middle| X_0 = x, \lambda_0 = \lambda\right\}$. However, thanks to the convergence of simulation observed in *Figure 4.1*, we can truncate the simulated paths at a large time (say, $t = 400$) as an approximation to $t = \infty$.

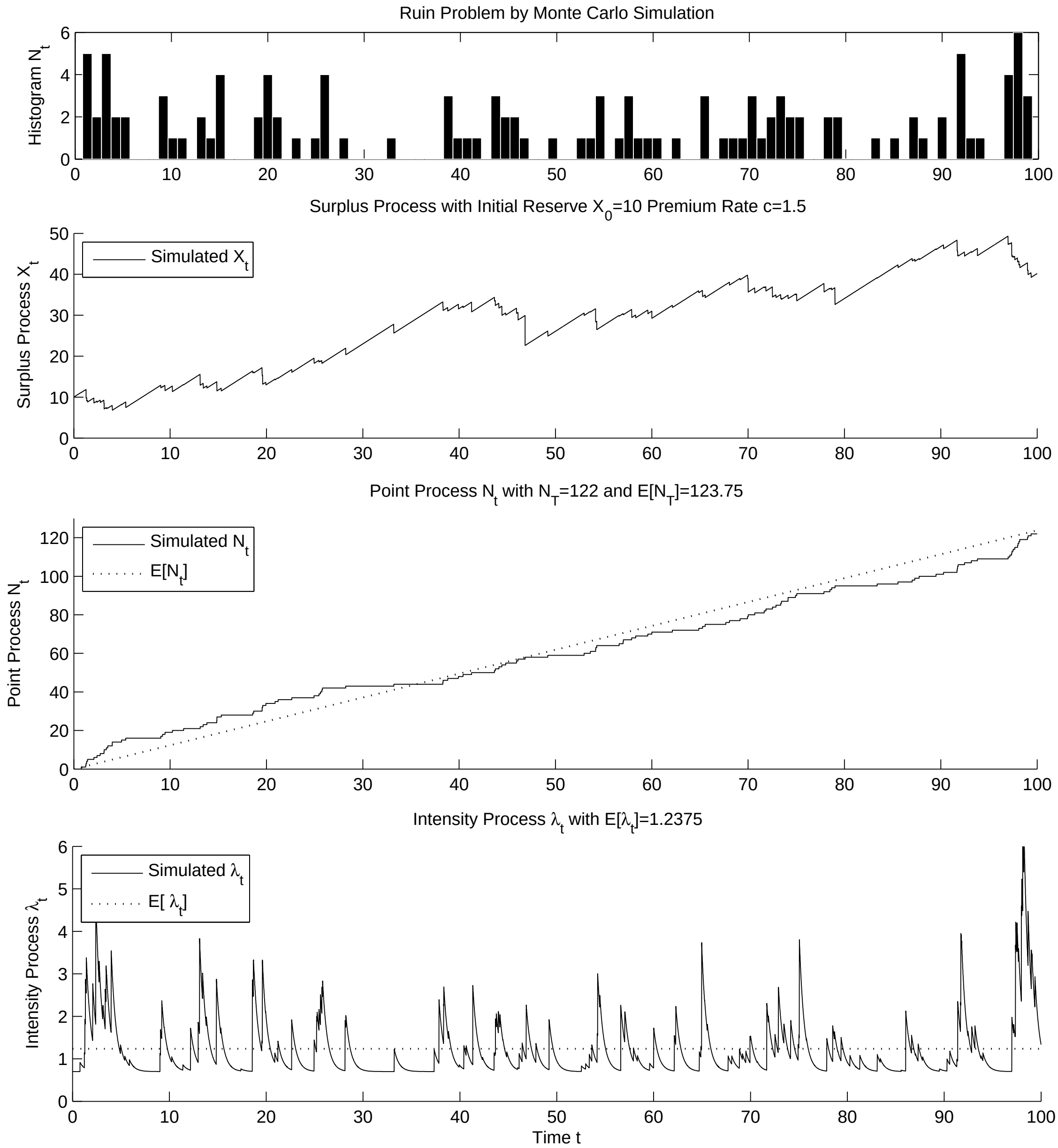

Fig. 4.2: Example: Ruin Problem by One Simulated Dynamic Contagion Process

4.2 Exponential Martingales and Generalised Lundberg's Fundamental Equation

In this section, we find some useful exponential martingales which link to the generalised Lundberg's fundamental equation. More importantly, they are crucial for deriving some key results of the ruin problem in the later sections.

Theorem 4.2.1. *Assume $\delta > \mu_{1_G}$ and the net profit condition (4.2), we have a martingale*

$$e^{-v_r X_t} e^{\eta_r \lambda_t} e^{-rt}, \quad r \geq 0,$$

where constants r, v_r and η_r satisfy a generalised Lundberg's fundamental equation

$$\begin{cases} -\delta\eta_r + \hat{z}(-v_r)\hat{g}(-\eta_r) - 1 = 0 \\ \rho\left(\hat{h}(-\eta_r) - 1\right) - r + a\delta\eta_r - cv_r = 0 \end{cases} \qquad \left(c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}, \quad \delta > \mu_{1_G}\right). \tag{4.3}$$

If $0 \leq r < r^*$, then (4.3) has a unique positive solution $(v_r^+ > 0, \eta_r^+ > 0)$, where

$$r^* =: \rho\left(\hat{h}(-\eta^*) - 1\right) + a\delta\eta^*, \tag{4.4}$$

and η^* is the unique positive solution to

$$1 + \delta\eta_r = \hat{g}(-\eta_r). \tag{4.5}$$

Proof. The (Model-1 type) infinitesimal generator of the process (X_t, λ_t, t) acting on a function $f(x, \lambda, t) \in \Omega(\mathcal{A})$ is given by

$$\begin{aligned} \mathcal{A}f(x,\lambda,t) &= \frac{\partial f}{\partial t} - \delta(\lambda - a)\frac{\partial f}{\partial \lambda} + c\frac{\partial f}{\partial x} + \lambda\left(\int_{y=0}^{\infty}\int_{z=0}^{\infty} f(x-z, \lambda+y, t)\mathrm{d}Z(z)\mathrm{d}G(y) - f(x,\lambda,t)\right) \\ &\quad + \rho\left(\int_0^{\infty} f(x, \lambda+y, t)\mathrm{d}H(y) - f(x,\lambda,t)\right). \end{aligned} \tag{4.6}$$

For the classification of Model-1 type and Model-2 type generators for ruin problem, see Dassios and Embrechts (1989).

Assume the form

$$f(x,\lambda,t) = e^{-v_r x} e^{\eta_r \lambda} e^{-rt},$$

and plug into the generator (4.6). To be a martingale, set $\mathcal{A}f(x,\lambda,t) = 0$, then,

$$-r - \delta(\lambda - a)\eta_r - cv_r + \lambda\left(\int_{y=0}^{\infty}\int_{z=0}^{\infty} e^{v_r z} e^{\eta_r y}\mathrm{d}Z(z)\mathrm{d}G(y) - 1\right) + \rho\left(\int_0^{\infty} e^{\eta_r y}\mathrm{d}H(y) - 1\right) = 0,$$

and rewrite as

$$\left(-\delta\eta_r + \hat{z}(-v_r)\hat{g}(-\eta_r) - 1\right)\lambda + \left(\rho\left(\hat{h}(-\eta_r) - 1\right) - r + a\delta\eta_r - cv_r\right) = 0,$$

holding for any λ. Hence, we have (4.3). The proofs of the uniqueness and the associated conditions for the solution to (4.3) are given by *Lemma 4.2.1* and *Lemma 4.2.2* as below. □

Lemma 4.2.1. *Under $\delta > \mu_{1_G}$ and the net profit condition (4.2), there are unique positive solution η_r^+ and unique negative solution η_r^- to η_r of the generalised Lundberg's fundamental equation (4.3); In particular, for $r = 0$, there is a unique positive solution η_0^+ and the solution zero.*

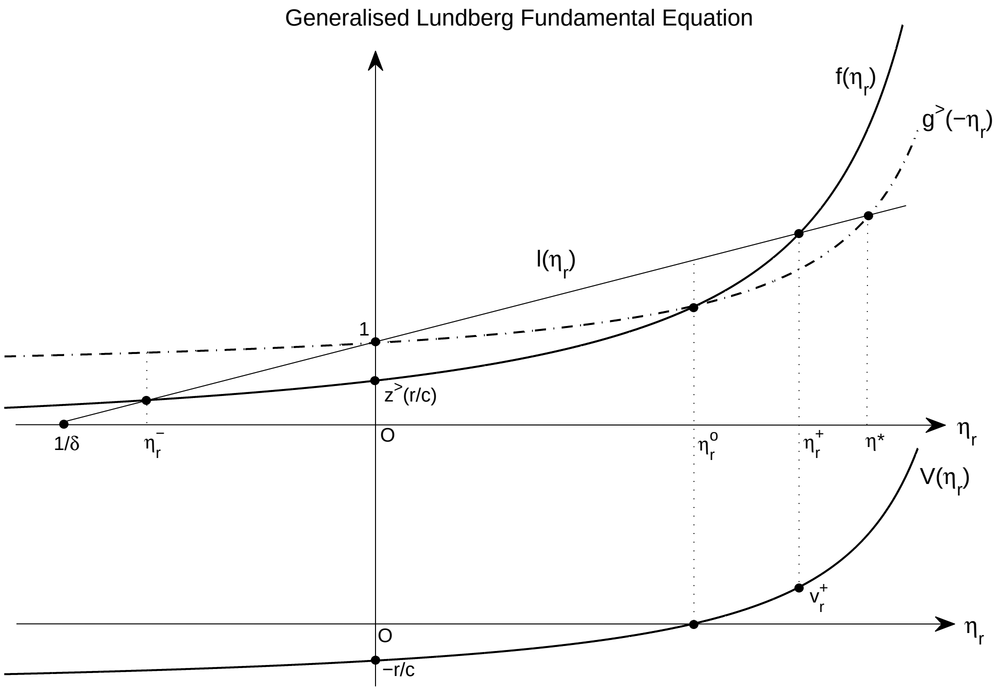

Fig. 4.3: Generalised Lundberg Fundamental Equation

Proof. Rewrite the generalised Lundberg's fundamental equation (4.3) w.r.t. η_r,

$$\begin{cases} \hat{z}\left(\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}\right)\hat{g}(-\eta_r) = 1+\delta\eta_r \\ -v_r = \frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c} \end{cases} \qquad \left(c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}, \quad \delta > \mu_{1_G}\right).$$

Consider the first equation above, i.e.

$$f(\eta_r) = l(\eta_r), \quad r > 0,$$

where

$$\begin{aligned} f(\eta_r) &=: \hat{z}\left(\frac{r - a\delta\eta_r + \rho\left(1-\hat{h}(-\eta_r)\right)}{c}\right)\hat{g}(-\eta_r), \\ l(\eta_r) &=: 1+\delta\eta_r. \end{aligned}$$

Obviously, $f(\eta_r)$ is a strictly increasing and strictly convex function of η_r, since

$$\frac{\partial\hat{h}(-u)}{\partial u} > 0, \quad \frac{\partial\hat{g}(-u)}{\partial u} > 0, \quad \frac{\partial\hat{z}(u)}{\partial u} < 0,$$

$$\frac{\partial^2\hat{h}(-u)}{\partial u^2} > 0, \quad \frac{\partial^2\hat{g}(-u)}{\partial u^2} > 0, \quad \frac{\partial^2\hat{z}(u)}{\partial u^2} > 0,$$

and

$$\begin{aligned}
\frac{\partial f(\eta_r)}{\partial \eta_r} &= \frac{-a\delta - \rho\frac{\partial \hat{h}(-\eta_r)}{\partial \eta_r}}{c} \frac{\partial \hat{z}(u)}{\partial u}\bigg|_{u=\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}} \hat{g}(-\eta_r) \\
&+\hat{z}\left(\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}\right)\frac{\partial \hat{g}(-\eta_r)}{\partial \eta_r} > 0, \\
\frac{\partial^2 f(\eta_r)}{\partial \eta_r^2} &= -\frac{\rho}{c}\frac{\partial^2 \hat{h}(-\eta_r)}{\partial \eta_r^2}\frac{\partial \hat{z}(u)}{\partial u}\bigg|_{u=\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}} \hat{g}(-\eta_r) \\
&+\frac{-a\delta - \rho\frac{\partial \hat{h}(-\eta_r)}{\partial \eta_r}}{c}\Bigg[\frac{-a\delta - \rho\frac{\partial \hat{h}(-\eta_r)}{\partial \eta_r}}{c}\frac{\partial^2 \hat{z}(u)}{\partial u^2}\bigg|_{u=\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}} \\
&+2\frac{\partial \hat{z}(u)}{\partial u}\bigg|_{u=\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}}\frac{\partial \hat{g}(-\eta_r)}{\partial \eta_r}\Bigg] \\
&+\hat{z}\left(\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}\right)\frac{\partial^2 \hat{g}(-\eta_r)}{\partial \eta_r^2} > 0.
\end{aligned}$$

Furthermore, we have $f(\eta_r) > 0$, $f(-\infty) = 0$, $f(+\infty) = +\infty$, and also $l(\eta_r)$ is a strictly linearly increasing function of η_r.

We discuss the solutions for the two cases $r > 0$ and $r = 0$ separately as below.

- For $r > 0$, we have

$$0 < f(0) = \hat{z}\left(\frac{r}{c}\right) < 1 = l(0),$$

and the slope of the tangent at $\eta_r = 0$,

$$\frac{\partial l(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} > \frac{\partial f(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} > 0.$$

By the stationarity condition $\delta > \mu_{1_G}$ and the net profit condition (4.2), we have

$$\begin{aligned}
\frac{\partial f(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} &= \frac{-a\delta - \mu_{1_H}\rho}{c}\frac{\partial \hat{z}(u)}{\partial u}\bigg|_{u=\frac{r}{c}} + \hat{z}\left(\frac{r}{c}\right)\mu_{1_G} \\
&< \frac{-a\delta - \mu_{1_H}\rho}{c}\frac{\partial \hat{z}(u)}{\partial u}\bigg|_{u=0} + \hat{z}\,(0)\,\mu_{1_G} \\
&= \frac{a\delta + \mu_{1_H}\rho}{c}\mu_{1_Z} + \mu_{1_G} \\
&< \delta = \frac{\partial l(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0}.
\end{aligned}$$

It is clear that there is a unique positive solution η_r^+ and a unique negative solution η_r^- by plotting $f(\eta_r)$ and $l(\eta_r)$, see *Figure 4.3*.

- For $r = 0$, we have

$$0 < f(0) = \hat{z}\,(0) = 1 = l(0),$$

and the slope of the tangent at $\eta_r = 0$,

$$\frac{\partial l(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} > \frac{\partial f(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} > 0.$$

By the stationarity condition and the net profit condition, we have

$$\frac{\partial f(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} < \frac{a\delta + \mu_{1_H}\rho}{c}\mu_{1_Z} + \mu_{1_G} < \delta = \frac{\partial l(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0}.$$

It is clear that there are unique positive solution η_0^+ and solution 0 by plotting $f(\eta_r)$ and $l(\eta_r)$.

□

In order to find the positive solution to v_r, we will only consider the unique positive solution η_r^+ for $r \geq 0$ in the sequel.

Lemma 4.2.2. *If* $0 \leq r < r^*$,

$$r^* =: \rho\left(\hat{h}(-\eta^*) - 1\right) + a\delta\eta^*, \tag{4.7}$$

where the constant η^* *is the unique positive solution to*

$$1 + \delta\eta_r = \hat{g}(-\eta_r), \quad \delta > \mu_{1_G},$$

then, there exists a unique positive solution v_r^+ *to* v_r *of the generalised Lundberg's fundamental equation (4.3),*

$$v_r^+ = -\frac{r - a\delta\eta_r^+ + \rho\Big(1 - \hat{h}(-\eta_r^+)\Big)}{c}. \tag{4.8}$$

Proof. By substituting η_r^+ (from *Lemma 4.2.1*) into the second equation of the generalised Lundberg's fundamental equation (4.3), we have the solution to v_r, i.e. (4.8). Define

$$V(\eta_r) =: -\frac{r - a\delta\eta_r + \rho\Big(1 - \hat{h}(-\eta_r)\Big)}{c}.$$

Obviously, $V(\eta_r)$ is a strictly increasing and strictly convex function of η_r, as $\frac{\partial V(\eta_r)}{\partial \eta_r} > 0$ and $\frac{\partial^2 V(\eta_r)}{\partial \eta_r^2} > 0$; also, $V(-\infty) = -\infty$, $V(+\infty) = +\infty$; $v(0) = -\frac{r}{c} < 0$; hence, there is unique (positive) root $\eta_r^o > 0$ such that $V(\eta_r^o) = 0$, also see *Figure 4.3*.

In order to find the unique positive solution v_r^+, such that $v_r^+ = V(\eta_r^+) > 0$, we have the condition $\eta_r^+ > \eta_r^o$, which also is equivalent to the condition

$$l(\eta_r^o) > f(\eta_r^o), \quad \eta_r^o > 0,$$

or,

$$1 - \delta\eta_r^o > \hat{g}(-\eta_r^o), \quad \eta_r^o > 0,$$

note that, $f(\eta_r^o) = \hat{g}(-\eta_r^o)$. Under the stationarity condition $\delta > \mu_{1_G}$, the equation $1 + \delta\eta_r = \hat{g}(-\eta_r)$ has the unique positive solution η^* (independent from $r > 0$) and the solution 0. Therefore, we have the condition

$$0 < \eta_r^o < \eta^*,$$

such that

$$1 + \delta\eta_r^o > \hat{g}(-\eta_r^o), \quad \eta_r^o > 0.$$

We discuss the two cases $r > 0$ and $r = 0$ separately as below.

- If $r = 0$, we have $\eta_0^o = \eta_r^o\big|_{r=0} = 0$, and it is clear that $\eta_0^+ > \eta_0^o > 0$ holds, therefore, $v_0^+ > 0$ exists without any condition.

- If $r > 0$, then the condition $\eta^* > \eta_r^o > 0$ is also equivalent to the condition $V(\eta^*) > 0$ since $V(\cdot)$ is a strictly increasing function, i.e.

$$V(\eta^*) = -\frac{r - a\delta\eta^* + \rho\Big(1 - \hat{h}(-\eta^*)\Big)}{c} > 0.$$

  Hence, we can obtain the upper bound r^* for $r > 0$ explicitly, i.e. $0 < r < r^*$, where r^* is given by (4.7), note that, $r^* > 0$ as $\eta^* > 0$, also see *Figure 4.3*.

□

Remark 4.2.1. Given the existence and uniqueness of solution (η_r^+, v_r^+) to the generalised Lundberg's fundamental equation (4.3), we have $\eta^* > \eta_r^+$, since

$$1 + \delta\eta_r^+ = \hat{z}\left(-v_r^+\right)\hat{g}\left(-\eta_r^+\right) > \hat{g}(-\eta_r^+),$$

we know that, if $\delta > \mu_{1_G}$ the equation $1 + \delta\eta_r = \hat{g}(-\eta_r)$ has solution 0 and $\eta^* > 0$, then, η_r^+ should be between them, i.e. $\eta^* > \eta_r^+ > 0$, also see *Figure 4.3*. Therefore, we have the full ranking

$$0 < \eta_r^o < \eta_r^+ < \eta^*.$$

Remark 4.2.2. In particular, for $r = 0$, we have a martingale $e^{-v_0^+ X_t} e^{\eta_0^+ \lambda_t}$, where (v_0^+, η_0^+) is the unique positive solution to the equations

$$\begin{cases} \delta\eta_0^+ = \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right) - 1 \\ cv_0^+ = a\delta\eta_0^+ + \rho\left(\hat{h}(-\eta_0^+) - 1\right) \end{cases} \qquad \left(c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}, \quad \delta > \mu_{1_G}\right).$$

The martingales and generalised Lundberg's fundamental equation derived in this section are the building blocks of the martingale method and change of measure, two key approaches adopted in the following sections.

4.3 Ruin Probability via Original Measure

Theorem 4.3.1. *The ruin probability conditional on λ_0 and X_0 is given by*

$$P\left\{\tau^* < \infty \big| X_0 = x, \lambda_0 = \lambda\right\} = \frac{e^{-v_0^+ x} e^{\eta_0^+ \lambda}}{\mathbb{E}\left[e^{-v_0^+ X_{\tau^*}} e^{\eta_0^+ \lambda_{\tau^*}} \big| \tau^* < \infty; X_0 = x, \lambda_0 = \lambda\right]}. \tag{4.9}$$

Proof. By the optional stopping theorem, a bounded martingale stopped at a stopping time is still a martingale. Now we consider the martingale found by *Theorem 4.2.1* stopped at the ruin time, i.e.

$$e^{-v_r^+ X_{(\tau^* \wedge t)}} e^{\eta_r^+ \lambda_{(\tau^* \wedge t)}} e^{-r(\tau^* \wedge t)}, \quad 0 \le r < r^*.$$

By the martingale property, we have

$$\begin{aligned} & \mathbb{E}\left[e^{-v_r^+ X_{(\tau^* \wedge t)}} e^{\eta_r^+ \lambda_{(\tau^* \wedge t)}} e^{-r(\tau^* \wedge t)}\right] \\ = \ & \mathbb{E}\left[e^{-v_r^+ X_{(\tau^* \wedge t)}} e^{\eta_r^+ \lambda_{(\tau^* \wedge t)}} e^{-r(\tau^* \wedge t)} \Big| X_0 = x, \lambda_0 = \lambda\right] \\ = \ & e^{-v_r^+ x} e^{\eta_r^+ \lambda}, \end{aligned}$$

and

$$\mathbb{E}\left[e^{-v_r^+X_{\tau^*}}e^{\eta_r^+\lambda_{\tau^*}}e^{-r\tau^*}\middle|\tau^*\leq t\right]P\{\tau^*\leq t\}+\mathbb{E}\left[e^{-v_r^+X_t}e^{\eta_r^+\lambda_t}e^{-rt}\middle|\tau^*>t\right]P\{\tau^*>t\}=e^{-v_r^+x}e^{\eta_r^+\lambda},$$

or,

$$\mathbb{E}\left[e^{-v_r^+X_{\tau^*}}e^{\eta_r^+\lambda_{\tau^*}}e^{-r\tau^*}\middle|\tau^*\leq t\right]P\{\tau^*\leq t\}+e^{-rt}\mathbb{E}\left[e^{-v_r^+X_t}e^{\eta_r^+\lambda_t}\middle|\tau^*>t\right]P\{\tau^*>t\}=e^{-v_r^+x}e^{\eta_r^+\lambda},\tag{4.10}$$

where

$$\mathbb{E}\left[e^{-v_r^+X_t}e^{\eta_r^+\lambda_t}\middle|\tau^*>t\right]P\{\tau^*>t\}=\mathbb{E}\left[e^{-v_r^+X_t}e^{\eta_r^+\lambda_t}\mathbb{I}\left(\tau^*>t\right)\right]\leq\mathbb{E}\left[e^{\eta_r^+\lambda_t}\right].$$

Note that, by *Theorem 2.2.3*, we have

$$\lim_{t\to\infty}\mathbb{E}\left[e^{\eta_r^+\lambda_t}\right]=\exp\left(\int_{-\eta_r^+}^0\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right)<\infty,$$

since by *Remark 4.2.1*, for $0<r<r^*$, we have $-\eta^*<-\eta_r^+<0$ where $-\eta^*$ is the negative singular point of the integrand function above, i.e. the unique negative solution to $\delta u+\hat{g}(u)-1=0$. Hence, for the second term in (4.10),

$$\lim_{t\to\infty}e^{-rt}\mathbb{E}\left[e^{-v_r^+X_t}e^{\eta_r^+\lambda_t}\middle|\tau^*>t\right]P\{\tau^*>t\}=0.$$

Let $t\to\infty$ in (4.10), then, $\{\tau^*\leq t\}\to\{\tau^*<\infty\}$, and

$$\mathbb{E}\left[e^{-v_r^+X_{\tau^*}}e^{\eta_r^+\lambda_{\tau^*}}e^{-r\tau^*}\middle|\tau^*<\infty\right]P\{\tau^*<\infty\}=e^{-v_r^+x}e^{\eta_r^+\lambda}.$$

Let $r\to 0$, we have

$$\mathbb{E}\left[e^{-v_0^+X_{\tau^*}}e^{\eta_0^+\lambda_{\tau^*}}\middle|\tau^*<\infty\right]P\{\tau^*<\infty\}=e^{-v_0^+x}e^{\eta_0^+\lambda},$$

then (4.9) follows. □

Corollary 4.3.1. *If* $Z\sim\mathrm{Exp}(\gamma)$*, then,*

$$P\left\{\tau^*<\infty\middle|X_0=x,\lambda_0=\lambda\right\}=\frac{\gamma-v_0^+}{\gamma}\frac{e^{\eta_0^+\lambda}e^{-v_0^+x}}{\mathbb{E}\left[e^{\eta_0^+\lambda_{\tau^*}}\middle|\tau^*<\infty;X_0=x,\lambda_0=\lambda\right]}.$$

Proof. If $Z\sim\mathrm{Exp}(\gamma)$, due to the memoryless property of the exponential distribution, the overshoot $-X_{\tau^*}>0$ then follows the same exponential distribution, i.e. $-X_{\tau^*}\sim\mathrm{Exp}(\gamma)$. Hence, for (4.9) we have

$$\begin{aligned}\mathbb{E}\left[e^{-v_0^+X_{\tau^*}}e^{\eta_0^+\lambda_{\tau^*}}\middle|\tau^*<\infty\right]&=\mathbb{E}\left[e^{-v_0^+X_{\tau^*}}\right]\mathbb{E}\left[e^{\eta_0^+\lambda_{\tau^*}}\middle|\tau^*<\infty\right]\\&=\frac{\gamma}{\gamma-v_0^+}\mathbb{E}\left[e^{\eta_0^+\lambda_{\tau^*}}\middle|\tau^*<\infty\right].\end{aligned}$$

□

Remark 4.3.1. Note that, the overshoot $-X_{\tau^*}>0$, $\lambda_{\tau^*}>0$, then, $e^{-v_0^+X_{\tau^*}}>1$, $e^{\eta_0^+\lambda_{\tau^*}}>1$, we have an inequality for the ruin probability,

$$P\left\{\tau^*<\infty\middle|X_0=x,\lambda_0=\lambda\right\}<\frac{e^{\eta_0^+\lambda}e^{-v_0^+x}}{\mathbb{E}\left[e^{\eta_0^+\lambda_{\tau^*}}\middle|\tau^*<\infty;X_0=x,\lambda_0=\lambda\right]}<e^{\eta_0^+\lambda}e^{-v_0^+x}.$$

$e^{\eta_0^+\lambda}e^{-v_0^+x}$ is a rough up bound of ruin probability, as it could be greater than one when λ_0 is relatively large. In order to obtain a more precise upper bound, it is better to find the distribution property of $\mathbb{E}\left[e^{\eta_0^+\lambda_{\tau^*}}\middle|\tau^*<\infty\right]$ but it would be not easy and we leave it as future research.

If $Z \sim \mathrm{Exp}(\gamma)$, then,

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} < \frac{\gamma}{\gamma - v_0^+} e^{\eta_0^+\lambda} e^{-v_0^+ x}.$$

For instance, the comparison between the bounds and the ruin probability $P\left\{\tau^* < \infty \middle| X_0 = 10, \lambda_0 = \lambda\right\}$ simulated by 50,000 sample paths with parameter setting

$$(a;\rho,\delta;\alpha,\beta,\gamma;X_0,c) = (0.7;0.5,2.0;2.0,1.5,1.0;10,1.5), \quad (\eta_0^+, v_0^+) = (0.0842, 0.0932),$$

is given by *Table 4.1* and *Figure 4.4*.

Tab. 4.1: Example: The Comparison between the Bounds and the Simulated Ruin Probability

$\lambda_0=\lambda$	$P\left\{\tau^*<\infty \middle| X_0=10,\lambda_0=\lambda\right\}$	Up Bound $e^{\eta_0^+\lambda_0}e^{-v_0^+X_0}$	Up Bound $\frac{\gamma-v_0^+}{\gamma}e^{\eta_0^+\lambda_0}e^{-v_0^+X_0}$
1	28.83%	42.84%	38.84%
2	31.34%	46.60%	42.26%
3	34.39%	50.69%	45.97%
4	37.34%	55.15%	50.01%
5	40.01%	59.99%	54.40%
6	43.46%	65.26%	59.18%
7	46.67%	70.99%	64.38%
8	50.45%	77.23%	70.03%
9	53.34%	84.01%	76.18%
10	56.83%	91.39%	82.88%
11	60.56%	99.42%	90.16%
12	63.66%	**108.16%**	98.08%

4.4 Ruin Probability via Change of Measure

In this section, we investigate the ruin probability and asymptotics by change of measure via the martingale derived by *Theorem 4.2.1.* We will find that under this new measure the ruin becomes certain, and this makes the simulation more efficient than under the original measure where the ruin is not certain and even rare. Similar ideas of improving simulation of rare events by a change of measure can also be found in Asmussen (1985) and more recently Asmussen and Glynn (2007).

4.4.1 Ruin Probability by Change of Measure

Theorem 4.4.1. *The ruin probability conditional on X_0 and λ_0 can be expressed under new measure $\tilde{\mathbb{P}}$ by*

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} = e^{-v_0^+x}e^{m_0^+\tilde{\lambda}}\tilde{\mathbb{E}}\left[\Psi\left(X_{\tau^*_-}\right)\frac{e^{-m_0^+\tilde{\lambda}_{\tau^*_-}}}{\hat{g}(-\eta_0^+)}\middle| X_0 = x, \tilde{\lambda}_0 = \tilde{\lambda}\right], \quad (4.11)$$

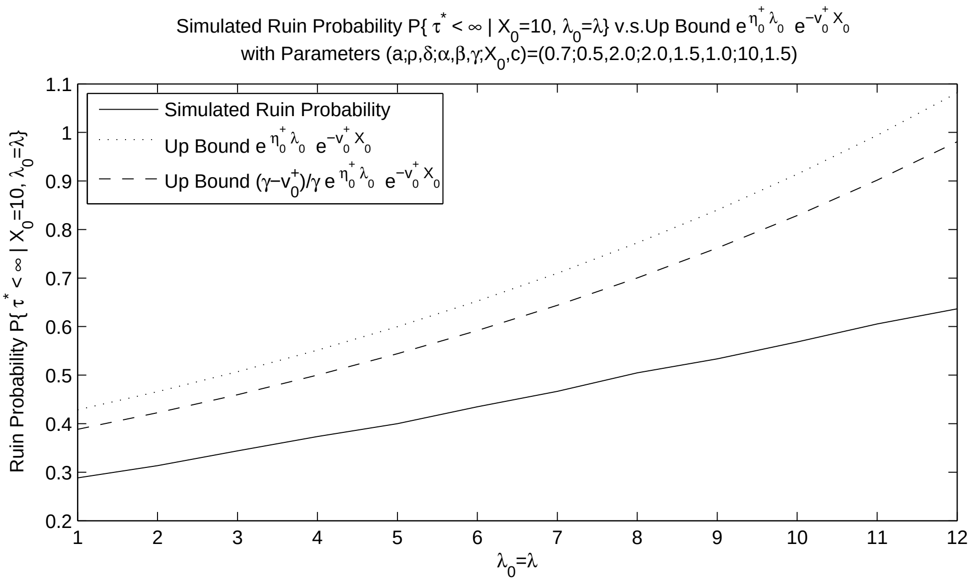

Fig. 4.4: Simulated Ruin Probability $P\left\{\tau^* < \infty \middle| X_0 = 10, \lambda_0 = \lambda\right\}$ v.s. Up Bounds

where $\tilde{\lambda} =: (1 + \delta\eta_0^+)\lambda$, $m_0^+ =: \frac{\eta_0^+}{\delta\eta_0^+ + 1}$,

$$\Psi(x) =: \frac{\overline{Z}(x)e^{v_0^+ x}}{\int_x^\infty e^{v_0^+ z}\mathrm{d}Z(z)}, \tag{4.12}$$

assuming the net profit condition holds under the original measure $\mathbb{P}$*, and the stationarity condition holds under both measures* $\mathbb{P}$ *and* $\tilde{\mathbb{P}}$*. The parameter setting for the process* (X_t, λ_t) *under* $\mathbb{P}$ *transforms to the new parameter setting for the process* $\left(X_t, \tilde{\lambda}_t\right)$ *under* $\tilde{\mathbb{P}}$ *as follows:*

- $a \nearrow \tilde{a} =: \left(1 + \delta\eta_0^+\right) a$,
- $c \to \tilde{c} =: c$,
- $\delta \to \tilde{\delta} =: \delta$,
- $\rho \nearrow \tilde{\rho} =: \hat{h}(-\eta_0^+)\rho$,
- $Z(z) \to \tilde{Z}(z)$,
- $g(u) \to \tilde{\tilde{g}}(u) =: \frac{\tilde{g}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$,
- $h(u) \to \tilde{\tilde{h}}(u) =: \frac{\tilde{h}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$,

where

$$\mathrm{d}\tilde{Z}(z) =: \frac{e^{v_0^+ z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)}, \quad \mathrm{d}\tilde{G}(u) =: \frac{e^{\eta_0^+ u}\mathrm{d}G(u)}{\hat{g}\left(-\eta_0^+\right)}, \quad \mathrm{d}\tilde{H}(u) =: \frac{e^{\eta_0^+ u}\mathrm{d}H(u)}{\hat{h}\left(-\eta_0^+\right)}, \tag{4.13}$$

and $\mathrm{d}\tilde{H}(u) =: \tilde{h}(u)\mathrm{d}u$, $\mathrm{d}\tilde{G}(u) =: \tilde{g}(u)\mathrm{d}u$; $\mathrm{d}\tilde{\tilde{H}}(u) =: \tilde{\tilde{h}}(u)\mathrm{d}u$, $\mathrm{d}\tilde{\tilde{G}}(u) =: \tilde{\tilde{g}}(u)\mathrm{d}u$.

Proof. We consider the (Model-2 type) generator

$$\begin{aligned}\mathcal{A}f(x,\lambda) &= -\delta(\lambda-a)\frac{\partial f}{\partial \lambda}+c\frac{\partial f}{\partial x}+\lambda\left(\int_{y=0}^{\infty}\int_{z=0}^{x} f(x-z,\lambda+y)\mathrm{d}Z(z)\mathrm{d}G(y)+\overline{Z}(x)-f(x,\lambda)\right)\\ &\quad +\rho\left(\int_0^{\infty} f(x,\lambda+y)\mathrm{d}H(y)-f(x,\lambda)\right), \quad x>0. \qquad (4.14)\end{aligned}$$

The solution of the integro-differential equation $\mathcal{A}f(x,\lambda)=0$ is the ruin probability

$$f(x,\lambda)=P\left\{\tau^*<\infty \middle| X_0=x,\lambda_0=\lambda\right\}.$$

Change Measure from $\mathbb{P}$ *to* $\tilde{\mathbb{P}}$ Substituting the function

$$f(x,\lambda)=e^{-v_0^+x}e^{\eta_0^+\lambda}\tilde{f}(x,\lambda)$$

into the generator (4.14), we have

$$\begin{aligned}&-\ \delta(\lambda-a)\left(\eta_0^+\tilde{f}+\frac{\partial \tilde{f}}{\partial \lambda}\right)+c\left(-v_0^+\tilde{f}+\frac{\partial \tilde{f}}{\partial x}\right)\\ &+\ \lambda\left(\int_0^{\infty}\int_0^{x}\tilde{f}(x-z,\lambda+y)e^{v_0^+z}e^{\eta_0^+y}\mathrm{d}Z(z)\mathrm{d}G(y)+\overline{Z}(x)e^{v_0^+x}e^{-\eta_0^+\lambda}-\tilde{f}\right)\\ &+\ \rho\left(\int_0^{\infty}\tilde{f}(x,\lambda+y)e^{\eta_0^+y}\mathrm{d}H(y)-\tilde{f}\right)=0. \qquad (4.15)\end{aligned}$$

Remind that, by *Theorem 4.2.1* for $r=0$, we have a $\mathcal{F}_t^{\mathbb{P}}$-martingale $e^{-v_0^+X_t}e^{\eta_0^+\lambda_t}$ where (v_0^+,η_0^+) is the unique positive solution to the equations

$$\begin{cases}\delta\eta_0^+ = \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)-1\\ cv_0^+ = a\delta\eta_0^+ +\rho\left(\hat{h}(-\eta_0^+)-1\right)\end{cases}\qquad \left(c>\frac{\mu_{1_H}\rho+a\delta}{\delta-\mu_{1_G}}\mu_{1_Z},\quad \delta>\mu_{1_G}\right).$$

Substitute $cv_0^+ = a\delta\eta_0^+ +\rho\left(\hat{h}(-\eta_0^+)-1\right)$ and $\delta\eta_0^+ = \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)-1$ into (4.15), we have

$$\begin{aligned}&-\ \delta(\lambda-a)\frac{\partial \tilde{f}}{\partial \lambda}+c\frac{\partial \tilde{f}}{\partial x}\\ &+\ \lambda\left(\int_0^{\infty}\int_0^{x}\tilde{f}(x-z,\lambda+y)e^{v_0^+z}e^{\eta_0^+y}\mathrm{d}Z(z)\mathrm{d}G(y)+\overline{Z}(x)e^{v_0^+x}e^{-\eta_0^+\lambda}-\hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)\tilde{f}\right)\\ &+\ \rho\left(\int_0^{\infty}\tilde{f}(x,\lambda+y)e^{\eta_0^+y}\mathrm{d}H(y)-\hat{h}(-\eta_0^+)\tilde{f}\right)=0.\end{aligned}$$

Change measure (*Esscher transform*) by (4.13), and rewrite as

$$\begin{aligned}&-\ \delta(\lambda-a)\frac{\partial \tilde{f}}{\partial \lambda}+c\frac{\partial \tilde{f}}{\partial x}\\ &+\ \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)\lambda\left(\int_0^{\infty}\int_0^{x}\tilde{f}(x-z,\lambda+y)\mathrm{d}\tilde{Z}(z)\mathrm{d}\tilde{G}(y)+\overline{Z}(x)\frac{e^{v_0^+x}e^{-\eta_0^+\lambda}}{\hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)}-\tilde{f}\right)\\ &+\ \hat{h}(-\eta_0^+)\rho\left(\int_0^{\infty}\tilde{f}(x,\lambda+y)\mathrm{d}\tilde{H}(y)-\tilde{f}\right)=0.\end{aligned}$$

Since $\hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)=1+\delta\eta_0^+$, we have

$$\begin{aligned}&-\ \delta(\lambda-a)\frac{\partial \tilde{f}}{\partial \lambda}+c\frac{\partial \tilde{f}}{\partial x}\\ &+\ (1+\delta\eta_0^+)\lambda\left(\int_0^{\infty}\int_0^{x}\tilde{f}(x-z,\lambda+y)\mathrm{d}\tilde{Z}(z)\mathrm{d}\tilde{G}(y)+\overline{Z}(x)\frac{e^{v_0^+x}e^{-\eta_0^+\lambda}}{\hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)}-\tilde{f}\right)\\ &+\ \hat{h}(-\eta_0^+)\rho\left(\int_0^{\infty}\tilde{f}(x,\lambda+y)\mathrm{d}\tilde{H}(y)-\tilde{f}\right)=0.\end{aligned}$$

Note that,

$$\overline{\tilde{Z}}(x) =: \int_x^{\infty} \mathrm{d}\tilde{Z}(z) = \int_x^{\infty} \frac{e^{v_0^+ z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)} = \frac{\int_x^{\infty} e^{v_0^+ z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)},$$

we have

$$\overline{Z}(x)\frac{e^{v_0^+ x}e^{-\eta_0^+\lambda}}{\hat{z}(-v_0^+)\hat{g}(-\eta_0^+)} = \frac{\overline{Z}(x)e^{v_0^+ x}}{\int_x^{\infty} e^{v_0^+ z}\mathrm{d}Z(z)}\frac{\int_x^{\infty} e^{v_0^+ z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)}\frac{e^{-\eta_0^+\lambda}}{\hat{g}(-\eta_0^+)} = \Psi(x)\frac{e^{-\eta_0^+\lambda}}{\hat{g}(-\eta_0^+)}\overline{\tilde{Z}}(x),$$

where $\Psi(x)$ is defined by (4.12). Hence, we have

$$\begin{aligned}
- &\quad \delta(\lambda-a)\frac{\partial \tilde{f}}{\partial \lambda} + c\frac{\partial \tilde{f}}{\partial x} \\
+ &\quad (1+\delta\eta_0^+)\lambda\left(\int_0^{\infty}\int_0^x \tilde{f}(x-z,\lambda+y)\mathrm{d}\tilde{Z}(z)\mathrm{d}\tilde{G}(y) + \Psi(x)\frac{e^{-\eta_0^+\lambda}}{\hat{g}(-\eta_0^+)}\overline{\tilde{Z}}(x) - \tilde{f}\right) \\
+ &\quad \hat{h}(-\eta_0^+)\rho\left(\int_0^{\infty}\tilde{f}(x,\lambda+y)\mathrm{d}\tilde{H}(y) - \tilde{f}\right) = 0. \qquad (4.16)
\end{aligned}$$

This integro-differential equation has the solution

$$\tilde{f}(x,\lambda) = \tilde{\mathbb{E}}\left[\Psi(X_{\tau_-^*})\frac{e^{-\eta_0^+\lambda_{\tau_-^*}}}{\hat{g}(-\eta_0^+)}\mathbb{I}\left(\tau^*<\infty\right)\middle|\lambda_0=\lambda, X_0=x\right].$$

It is similar to the expectation of a Gerber-Shiu penalty function (see Gerber and Shiu (1998)). Therefore, by comparing (4.16) with (4.14), we have the parameters for the process (X_t,λ_t) under $\mathbb{P}$ transformed to the parameters for the process (X_t,λ_t) under $\tilde{\mathbb{P}}$ as follows:

- $a \to \tilde{a} = a$,
- $c \to \tilde{c} = c$,
- $\delta \to \tilde{\delta} = \delta$,
- $\rho \to \tilde{\rho} = \hat{h}(-\eta_0^+)\rho$,
- $Z(z) \to \tilde{Z}(z)$,
- $G(y) \to \tilde{G}(y)$,
- $H(y) \to \tilde{H}(y)$,

and the ruin probability is given by

$$P\left\{\tau^*<\infty\middle|X_0=x,\lambda_0=\lambda\right\} = e^{-v_0^+ x}e^{\eta_0^+\lambda}\tilde{\mathbb{E}}\left[\Psi(X_{\tau_-^*})\frac{e^{-\eta_0^+\lambda_{\tau_-^*}}}{\hat{g}(-\eta_0^+)}\mathbb{I}\left(\tau^*<\infty\right)\middle|X_0=x,\lambda_0=\lambda\right].$$

Expression by $\tilde{\lambda}$ Alternatively, we can express the results above w.r.t. $\tilde{\lambda}$ where $\tilde{\lambda} = (1+\delta\eta_0^+)\lambda$. Consider $\frac{1}{1+\delta\eta_0^+}\tilde{\lambda}$ as a process, redefine the function $\tilde{f}(\tilde{\lambda},x) = \tilde{f}\left(\frac{1}{1+\delta\eta_0^+}\tilde{\lambda},x\right)$, and rewrite (4.16) as

$$\begin{aligned}
- &\quad \delta\left(\tilde{\lambda}-(1+\delta\eta_0^+)a\right)\frac{\partial \tilde{f}}{\partial \tilde{\lambda}} + c\frac{\partial \tilde{f}}{\partial x} \\
+ &\quad \tilde{\lambda}\left(\int_0^{\infty}\int_0^x \tilde{f}\left(x-z,\tilde{\lambda}+(1+\delta\eta_0^+)y\right)\mathrm{d}\tilde{Z}(z)\mathrm{d}\tilde{G}(y) + \Psi(x)\frac{e^{-\frac{\eta_0^+}{\delta\eta_0^++1}\tilde{\lambda}}}{\hat{g}(-\eta_0^+)}\overline{\tilde{Z}}(x) - \tilde{f}\right) \\
+ &\quad \hat{h}(-\eta_0^+)\rho\left(\int_0^{\infty}\tilde{f}\left(x,\tilde{\lambda}+(1+\delta\eta_0^+)y\right)\mathrm{d}\tilde{H}(y) - \tilde{f}\right) = 0.
\end{aligned}$$

Given $\mathrm{d}\tilde{H}(y) = \tilde{h}(y)\mathrm{d}y$ and $\mathrm{d}\tilde{G}(y) = \tilde{g}(y)\mathrm{d}y$, change variable by $u = (1+\delta\eta_0^+)y$, we have the equation of $\tilde{f}\left(\tilde{\lambda}, x\right)$,

$$
\begin{aligned}
& - \delta\left(\tilde{\lambda} - (1+\delta\eta_0^+)a\right)\frac{\partial \tilde{f}}{\partial \tilde{\lambda}} + c\frac{\partial \tilde{f}}{\partial x} \\
& + \tilde{\lambda}\left(\int_0^\infty \int_0^x \tilde{f}\left(x-z, \tilde{\lambda}+u\right)\mathrm{d}\tilde{Z}(z)\frac{\tilde{g}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}\mathrm{d}u + \Psi(x)\frac{e^{-\frac{\eta_0^+}{\delta\eta_0^+ +1}\tilde{\lambda}}}{\hat{g}(-\eta_0^+)}\overline{\tilde{Z}}(x) - \tilde{f}\right) \\
& + \hat{h}(-\eta_0^+)\rho\left(\int_0^\infty \tilde{f}\left(x, \tilde{\lambda}+u\right)\frac{\tilde{h}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}\mathrm{d}u - \tilde{f}\right) = 0. \qquad (4.17)
\end{aligned}
$$

This integro-differential equation has the solution

$$
\tilde{f}(x,\tilde{\lambda}) = \tilde{\mathbb{E}}\left[\Psi\left(X_{\tau_-^*}\right)\frac{e^{-\frac{\eta_0^+}{\delta\eta_0^+ +1}\tilde{\lambda}_{\tau_-^*}}}{\hat{g}(-\eta_0^+)}\mathbb{I}\left(\tau^* < \infty\right)\middle|\lambda_0 = \lambda, X_0 = x\right].
$$

Therefore, by comparing (4.17) with (4.14), we have the parameters for the process (X_t, λ_t) under $\mathbb{P}$ transformed to the parameters for the process $\left(X_t, \tilde{\lambda}_t\right)$ under $\tilde{\mathbb{P}}$ as follows:

- $a \nearrow \tilde{a} = \left(1+\delta\eta_0^+\right)a$,
- $c \rightarrow \tilde{c} = c$,
- $\delta \rightarrow \tilde{\delta} = \delta$,
- $\rho \nearrow \tilde{\rho} = \hat{h}(-\eta_0^+)\rho$,
- $Z(z) \rightarrow \tilde{Z}(z)$,
- $g(u) \rightarrow \tilde{\tilde{g}}(u) = \frac{\tilde{g}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$,
- $h(u) \rightarrow \tilde{\tilde{h}}(u) = \frac{\tilde{h}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$,

and the ruin probability is given by

$$
\begin{aligned}
& P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} \\
= \; & e^{-v_0^+ x}e^{\eta_0^+\lambda}\tilde{\mathbb{E}}\left[\Psi\left(X_{\tau_-^*}\right)\frac{e^{-m_0^+\tilde{\lambda}_{\tau_-^*}}}{\hat{g}(-\eta_0^+)}\mathbb{I}\left(\tau^* < \infty\right)\middle| X_0 = x, \lambda_0 = \lambda\right] \\
= \; & e^{-v_0^+ x}e^{m_0^+\tilde{\lambda}}\tilde{\mathbb{E}}\left[\Psi\left(X_{\tau_-^*}\right)\frac{e^{-m_0^+\tilde{\lambda}_{\tau_-^*}}}{\hat{g}(-\eta_0^+)}\mathbb{I}\left(\tau^* < \infty\right)\middle| X_0 = x, \tilde{\lambda}_0 = \tilde{\lambda}\right], \quad m_0^+ = \frac{\eta_0^+}{\delta\eta_0^+ + 1}.
\end{aligned}
$$

By *Theorem 4.4.3* (derived later in this section), if the net profit condition holds under $\mathbb{P}$ and the stationarity condition holds under $\mathbb{P}$ and $\tilde{\mathbb{P}}$, then the net profit condition cannot hold under $\tilde{\mathbb{P}}$, i.e. $\mathbb{I}\left(\tau^* < \infty\right) = 1$, hence, we have the ruin probability (4.11).

□

Remark 4.4.1. If $Z \sim \text{Exp}(\gamma)$, then, the expression of the ruin probability (4.11) can be greatly simplified, as $\Psi(x)$ is a constant, i.e.

$$\Psi(x) = \frac{e^{-\gamma x} e^{v_0^+ x}}{\int_x^\infty e^{v_0^+ z} \gamma e^{-\gamma z} \mathrm{d}z} = \frac{\gamma - v_0^+}{\gamma}.$$

4.4.2 Generalised Cramér-Lundberg Approximation for Exponentially Distributed Claims

Based on *Theorem 4.4.1*, if $Z \sim \text{Exp}(\gamma)$ and the initial intensity follows the stationary distribution under $\widetilde{\mathbb{P}}$, i.e. $\widetilde{\lambda} \sim \Pi$, then, the ruin probability is given by

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} = \frac{\gamma - v_0^+}{\gamma \hat{g}(-\eta_0^+)} \widetilde{\mathbb{E}}\left[e^{m_0^+ \widetilde{\lambda}}\right] \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau^*_-}} \middle| X_0 = x\right] e^{-v_0^+ x}. \tag{4.18}$$

Assumption 4.4.1. Assume $\lim_{x\to\infty} \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau^*_-}} \middle| X_0 = x, \lambda_0 = \lambda\right]$ exists and independent of λ.

Remark 4.4.2. Assumption 4.4.1 intuitively should hold as τ^* is long time in the future when $x \to \infty$, however, we leave it as an open problem to find the conditions under which it is true. Moreover, since $\widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau^*_-}} \middle| X_0 = x\right]$ given by (4.18) is bounded, then, there exists a sequence of $x_1 < x_2 < ... < x_n < ...$ with $x_n \to \infty, n \to \infty$ such that $\lim_{n\to\infty} \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau^*_-}} \middle| X_0 = x_n\right]$ exists.

Remark 4.4.3. Under Assumption 4.4.1 and by (4.18), there exists a constant C such that $P\left\{\tau^* < \infty \middle| X_0 = x\right\} \sim Ce^{-v_0^+ x}, x \to \infty$, and we obtain C in *Theorem 4.4.2.*

Theorem 4.4.2. *Under Assumption 4.4.1, if the claim sizes follows exponential distribution and the initial intensity follows the stationary distribution under* $\widetilde{\mathbb{P}}$, *i.e.* $\widetilde{Z} \sim \text{Exp}(\widetilde{\gamma})$ *and* $\widetilde{\lambda} \sim \Pi$, *then, the generalised Cramér-Lundberg approximation is given by*

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} \sim Ce^{-v_0^+ x}, \quad x \to \infty,$$

where

$$C =: \frac{\gamma - v_0^+}{\gamma \hat{g}(-\eta_0^+)} \widetilde{\mathbb{E}}\left[e^{m_0^+ \widetilde{\lambda}}\right] \frac{\frac{1}{\widetilde{\gamma}} \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}} \widetilde{\lambda}\right] - \widetilde{c} \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau^*_-}} \middle| X_0 = 0\right]}{\frac{1}{\widetilde{\gamma}} \widetilde{\mathbb{E}}[\widetilde{\lambda}] - \widetilde{c}}. \tag{4.19}$$

Proof. Use the new set of parameters under $\widetilde{\mathbb{P}}$ given by *Theorem 4.4.1*, and rewrite (4.17) as

$$\begin{aligned}
&- \delta\left(\widetilde{\lambda} - \widetilde{a}\right) \frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} + \widetilde{c} \frac{\partial \widetilde{f}}{\partial x} \\
&+ \widetilde{\lambda} \left(\int_0^\infty \int_0^x \widetilde{f}\left(x - z, \widetilde{\lambda} + u\right) \mathrm{d}\widetilde{Z}(z) \mathrm{d}\widetilde{\widetilde{G}}(u) + \Psi(x) \frac{e^{-m_0^+ \widetilde{\lambda}}}{\hat{g}(-\eta_0^+)} \overline{\widetilde{Z}}(x) - \widetilde{f} \right) \\
&+ \widetilde{\rho} \left(\int_0^\infty \widetilde{f}\left(x, \widetilde{\lambda} + u\right) \mathrm{d}\widetilde{\widetilde{H}}(u) - \widetilde{f} \right) = 0.
\end{aligned}$$

If $\widetilde{Z} \sim \text{Exp}(\widetilde{\gamma})$, $\widetilde{\gamma} = \gamma - v_0^+$ under $\widetilde{\mathbb{P}}$ (equivalent to $Z \sim \text{Exp}(\gamma)$ under $\mathbb{P}$), then, by *Remark 4.4.1*, we have

$$\begin{aligned}
&- \delta\left(\widetilde{\lambda} - \widetilde{a}\right) \frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} + \widetilde{c} \frac{\partial \widetilde{f}}{\partial x} \\
&+ \widetilde{\lambda} \left(\int_0^\infty \int_0^x \widetilde{f}\left(x - z, \widetilde{\lambda} + u\right) \widetilde{\gamma} e^{-\widetilde{\gamma} z} \mathrm{d}z \mathrm{d}\widetilde{\widetilde{G}}(u) + \frac{\gamma - v_0^+}{\gamma} \frac{e^{-m_0^+ \widetilde{\lambda}}}{\hat{g}(-\eta_0^+)} e^{-\widetilde{\gamma} x} - \widetilde{f} \right) \\
&+ \widetilde{\rho} \left(\int_0^\infty \widetilde{f}\left(x, \widetilde{\lambda} + u\right) \mathrm{d}\widetilde{\widetilde{H}}(u) - \widetilde{f} \right) = 0.
\end{aligned}$$

Take Laplace transform w.r.t. x, i.e.

$$\hat{\widetilde{f}}(w,\widetilde{\lambda}) =: \mathcal{L}\left\{\widetilde{f}(x,\widetilde{\lambda})\right\} = \int_0^\infty \widetilde{f}(u,\widetilde{\lambda})e^{-wu}\mathrm{d}u,$$

we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial \widetilde{f}(x,\widetilde{\lambda})}{\partial x}\right\} &= w\hat{\widetilde{f}}(w,\widetilde{\lambda}) - \widetilde{f}(0,\widetilde{\lambda}),\\
\mathcal{L}\left\{\int_0^x \widetilde{f}(x-z,\widetilde{\lambda}+u)\widetilde{\gamma}e^{-\widetilde{\gamma}z}\mathrm{d}z\right\} &= \frac{\widetilde{\gamma}}{\widetilde{\gamma}+w}\hat{\widetilde{f}}(w,\widetilde{\lambda}+u),\\
\mathcal{L}\left\{e^{-\widetilde{\gamma}x}\right\} &= \frac{1}{\widetilde{\gamma}+w},
\end{aligned}$$

then,

$$\begin{aligned}
&- \delta\left(\widetilde{\lambda}-\widetilde{a}\right)\frac{\partial \hat{\widetilde{f}}(w,\widetilde{\lambda})}{\partial \widetilde{\lambda}} + \widetilde{c}\left(w\hat{\widetilde{f}}(w,\widetilde{\lambda}) - \widetilde{f}(0,\widetilde{\lambda})\right)\\
&+ \widetilde{\lambda}\left(\frac{\widetilde{\gamma}}{\widetilde{\gamma}+w}\int_0^\infty \hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u) + \frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}+w} - \hat{\widetilde{f}}(w,\widetilde{\lambda})\right)\\
&+ \widetilde{\rho}\left(\int_0^\infty \hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{H}}(u) - \hat{\widetilde{f}}(w,\widetilde{\lambda})\right) = 0,
\end{aligned}$$

or,

$$\widetilde{\mathcal{A}}\hat{\widetilde{f}}(w,\widetilde{\lambda}) + \widetilde{c}\left(w\hat{\widetilde{f}}(w,\widetilde{\lambda}) - \widetilde{f}(0,\widetilde{\lambda})\right) + \widetilde{\lambda}\left(-\frac{w}{\widetilde{\gamma}+w}\int_0^\infty \hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u) + \frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}+w}\right) = 0.$$

If $\widetilde{\lambda}\sim\Pi$, then,

$$\widetilde{\mathbb{E}}\left[\widetilde{\mathcal{A}}\hat{\widetilde{f}}(w,\widetilde{\lambda}) + \widetilde{c}\left(w\hat{\widetilde{f}}(w,\widetilde{\lambda}) - \widetilde{f}(0,\widetilde{\lambda})\right) + \widetilde{\lambda}\left(-\frac{w}{\widetilde{\gamma}+w}\int_0^\infty \hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u) + \frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}+w}\right)\right] = 0,$$

and

$$\lim_{w\to 0}\widetilde{\mathbb{E}}\left[\widetilde{\mathcal{A}}\hat{\widetilde{f}}(w,\widetilde{\lambda}) + \widetilde{c}\left(w\hat{\widetilde{f}}(w,\widetilde{\lambda}) - \widetilde{f}(0,\widetilde{\lambda})\right) + \widetilde{\lambda}\left(-\frac{w}{\widetilde{\gamma}+w}\int_0^\infty \hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u) + \frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}+w}\right)\right] = 0.$$

Since under Assumption 4.4.1,

$$\widetilde{C} =: \lim_{x\to\infty}\widetilde{f}(x,\widetilde{\lambda}) = \lim_{w\to 0} w\hat{\widetilde{f}}(w,\widetilde{\lambda}),$$

$$\lim_{w\to 0}\frac{w}{\widetilde{\gamma}+w}\int_0^\infty \hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u) = \int_0^\infty \lim_{w\to 0}\frac{w}{\widetilde{\gamma}+w}\hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u) = \int_0^\infty \frac{1}{\widetilde{\gamma}}\widetilde{C}\mathrm{d}\widetilde{\widetilde{G}}(u) = \frac{\widetilde{C}}{\widetilde{\gamma}},$$

and by the property (2.18), we also have $\mathbb{E}\left[\widetilde{\mathcal{A}}\hat{\widetilde{f}}(0,\widetilde{\lambda})\right] = 0$, then,

$$\widetilde{\mathbb{E}}\left[\widetilde{c}\left(\widetilde{C} - \widetilde{f}(0,\widetilde{\lambda})\right) + \widetilde{\lambda}\left(-\frac{\widetilde{C}}{\widetilde{\gamma}} + \frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}}\right)\right] = 0,$$

and

$$\widetilde{C} = \frac{\gamma-v_0^+}{\gamma\hat{g}(-\eta_0^+)}\frac{\frac{1}{\widetilde{\gamma}}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\widetilde{\lambda}\right] - \widetilde{c}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau_-^*}}\middle|\widetilde{\lambda}_0 = \widetilde{\lambda}\sim\Pi, X_0 = 0\right]}{\frac{1}{\widetilde{\gamma}}\widetilde{\mathbb{E}}[\widetilde{\lambda}] - \widetilde{c}}, \tag{4.20}$$

note that, by definition,

$$\widetilde{\mathbb{E}}\left[\widetilde{f}(0,\widetilde{\lambda})\right] = \frac{\gamma - v_0^+}{\gamma \hat{g}(-\eta_0^+)} \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau^*_-}} \middle| \widetilde{\lambda}_0 = \widetilde{\lambda} \sim \Pi, X_0 = 0\right].$$

Hence, we have the generalised Cramér-Lundberg constant (4.19) for $\widetilde{\lambda} \sim \Pi$, as

$$C =: \lim_{x\to\infty} \frac{P\left\{\tau^* < \infty \middle| X_0 = x\right\}}{e^{-v_0 x}} = \lim_{x\to\infty} \widetilde{\mathbb{E}}\left[e^{m_0^+ \widetilde{\lambda}} \widetilde{f}(x,\widetilde{\lambda})\right] = \widetilde{\mathbb{E}}\left[e^{m_0^+ \widetilde{\lambda}}\right]\widetilde{C}.$$

□

Remark 4.4.4. For the Cramér-Lundberg constant (4.19), by *Theorem 2.2.3* and *Corollary 2.2.6*, we can explicitly calculate the terms

$$\begin{aligned}
\widetilde{\mathbb{E}}[\widetilde{\lambda}] &= \frac{\mu_{1_{\widetilde{\widetilde{H}}}} \widetilde{\rho} + \widetilde{a}\widetilde{\delta}}{\widetilde{\delta} - \mu_{1_{\widetilde{\widetilde{G}}}}}, \\
\widetilde{\mathbb{E}}\left[e^{m_0^+ \widetilde{\lambda}}\right] &= \exp\left(\int_{-m_0^+}^{0} \frac{\widetilde{a}\widetilde{\delta}u + \widetilde{\rho}[1 - \hat{\widetilde{\widetilde{h}}}(u)]}{\widetilde{\delta}u + \hat{\widetilde{\widetilde{g}}}(u) - 1} \mathrm{d}u\right), \\
\widetilde{\mathbb{E}}\left[\widetilde{\lambda} e^{-m_0^+ \widetilde{\lambda}}\right] &= -\frac{\mathrm{d}}{\mathrm{d}m} \widetilde{\mathbb{E}}\left[e^{-m\widetilde{\lambda}}\right]\Big|_{m=m_0^+} = \frac{\widetilde{a}\widetilde{\delta}m_0^+ + \widetilde{\rho}[1 - \hat{\widetilde{\widetilde{h}}}(m_0^+)]}{\widetilde{\delta}m_0^+ + \hat{\widetilde{\widetilde{g}}}(m_0^+) - 1} \exp\left(-\int_0^{m_0^+} \frac{\widetilde{a}\widetilde{\delta}u + \widetilde{\rho}[1 - \hat{\widetilde{\widetilde{h}}}(u)]}{\widetilde{\delta}u + \hat{\widetilde{\widetilde{g}}}(u) - 1} \mathrm{d}u\right).
\end{aligned}$$

Also, by *Theorem 4.4.3* for the net profit condition under the measure $\widetilde{\mathbb{P}}$, we have

$$\frac{1}{\widetilde{\widetilde{\gamma}}} \widetilde{\mathbb{E}}[\widetilde{\lambda}] - \widetilde{c} > 0.$$

4.4.3 Net Profit Condition under $\mathbb{P}$ and $\tilde{\mathbb{P}}$

Theorem 4.4.3. *If the net profit condition and the stationarity condition both hold under $\mathbb{P}$, i.e.*

$$c > \frac{\mu_{1_H} \rho + a\delta}{\delta - \mu_{1_G}} \mu_{1_Z}, \quad \delta > \mu_{1_G},$$

and the stationarity condition also holds under the new measure $\tilde{\mathbb{P}}$, i.e. $\tilde{\delta} > \mu_{1_{\tilde{\tilde{G}}}}$, then, under $\tilde{\mathbb{P}}$, we have

$$\frac{\mu_{1_{\tilde{\tilde{H}}}} \tilde{\rho} + \tilde{a}\tilde{\delta}}{\tilde{\delta} - \mu_{1_{\tilde{\tilde{G}}}}} \mu_{1_{\tilde{Z}}} > \tilde{c}, \tag{4.21}$$

and the ruin becomes certain (almost surely), i.e.

$$\tilde{\mathbb{P}}\{\tau^* < \infty\} =: \lim_{t\to\infty} \tilde{\mathbb{P}}\{\tau^* \le t\} = 1.$$

Proof. By the transformation between two measures from *Theorem 4.4.1*, we have

$$\mu_{1_{\tilde{Z}}} =: \tilde{\mathbb{E}}[Z_i] = \int_0^\infty z \mathrm{d}\tilde{Z}(z) = \int_0^\infty z \frac{e^{v_0^+ z} \mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)} = \frac{1}{\hat{z}\left(-v_0^+\right)} \int_0^\infty z e^{v_0^+ z} \mathrm{d}Z(z) = \frac{\hat{z}'\left(-v_0^+\right)}{\hat{z}\left(-v_0^+\right)}.$$

Change variable $y = \frac{1}{1+\delta\eta_0^+} u$, then,

$$\begin{aligned}
\mu_{1_{\tilde{\tilde{H}}}} &= \tilde{\mathbb{E}}\left[Y^{(1)}\right] = \int_0^\infty u \frac{\tilde{h}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+} \mathrm{d}u = \frac{\int_0^\infty u e^{\frac{\eta_0^+}{1+\delta\eta_0^+} u} h\left(\frac{1}{1+\delta\eta_0^+} u\right) \mathrm{d}u}{\left(1+\delta\eta_0^+\right) \hat{h}(-\eta_0^+)} = \frac{1+\delta\eta_0^+}{\hat{h}(-\eta_0^+)} \int_0^\infty y e^{\eta_0^+ y} \mathrm{d}H(y); \\
\mu_{1_{\tilde{\tilde{G}}}} &= \tilde{\mathbb{E}}\left[Y^{(2)}\right] = \frac{1+\delta\eta_0^+}{\hat{g}(-\eta_0^+)} \int_0^\infty y e^{\eta_0^+ y} \mathrm{d}G(y) = \hat{z}(-v_0^+)\hat{g}'(-\eta_0^+). \qquad \left(\because \hat{z}(-v_0^+)\hat{g}(-\eta_0^+) = 1+\delta\eta_0^+\right)
\end{aligned}$$

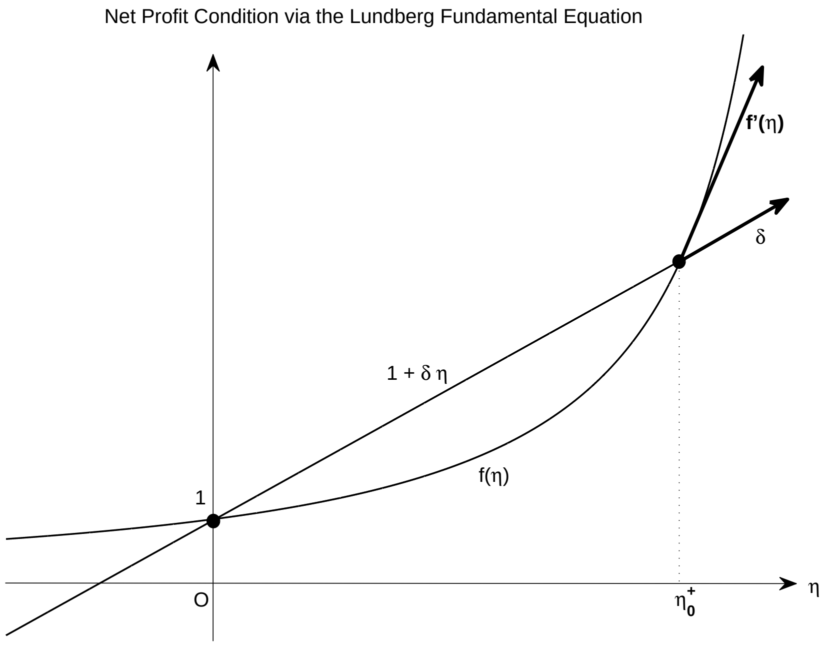

Fig. 4.5: Net Profit Condition via the Generalised Lundberg Fundamental Equation

The mean of self-excited jump sizes under $\tilde{\mathbb{P}}$ is greater than the one under $\mathbb{P}$, since

$$\mu_{1_{\tilde{\tilde{G}}}} > \hat{g}'(-\eta_0^+) = \int_0^\infty y e^{\eta_0^+ y} \mathrm{d}G(y) > \int_0^\infty y \mathrm{d}G(y) = \mu_{1_G}.$$

Hence,

$$\begin{aligned} & \frac{\mu_{1_{\tilde{\tilde{H}}}} \tilde{\rho} + \tilde{a}\tilde{\tilde{\delta}}}{\tilde{\delta} - \mu_{1_{\tilde{\tilde{G}}}}} \mu_{1_{\tilde{Z}}} \\ = \;& \frac{\rho \int_0^\infty y e^{\eta_0^+ y} \mathrm{d}H(y) + a\delta}{\delta - \hat{z}(-v_0^+)\hat{g}'(-\eta_0^+)} \frac{1 + \delta\eta_0^+}{\hat{z}(-v_0^+)} \int_0^\infty z e^{v_0^+ z} \mathrm{d}Z(z) \qquad \left(\because \hat{z}(-v_0^+)\hat{g}(-\eta_0^+) = 1 + \delta\eta_0^+ \right) \\ = \;& \hat{z}'(-v_0^+)\hat{g}(-\eta_0^+) \frac{\hat{h}'(-\eta_0^+)\rho + a\delta}{\delta - \hat{z}(-v_0^+)\hat{g}'(-\eta_0^+)}. \end{aligned} \tag{4.22}$$

From the generalised Lundberg's fundamental equation, we have

$$1 + \delta\eta_0^+ = \hat{z}\left(\frac{-a\delta\eta_0^+ + \rho\left(1 - \hat{h}(-\eta_0^+)\right)}{c} \right) \hat{g}(-\eta_0^+).$$

If the net profit condition and stationarity condition both hold under $\mathbb{P}$, the right-hand-side function is a strictly increasing and convex function of η_0^+ as obviously a convex function of a function convex function is still a convex function; it was also proved formally in the proof of *Lemma 4.2.1.* Hence, as shown in *Figure 4.5*, at the point η_0^+ the slope of the left-hand-side function is greater than the slope of the right-hand-side function, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\Big(1+\delta\eta\Big)\Bigg|_{\eta=\eta_0^+} < \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\hat{z}\left(\frac{-a\delta\eta+\rho\left(1-\hat{h}(-\eta)\right)}{c}\right)\hat{g}(-\eta)\right)\Bigg|_{\eta=\eta_0^+},$$

or,

$$\begin{aligned}\delta \quad < \quad & -\left(\frac{a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}{c}\right)\frac{\mathrm{d}\hat{z}(u)}{\mathrm{d}u}\Bigg|_{u=\frac{-a\delta\eta_0^+ +\rho\left(1-\hat{h}(-\eta_0^+)\right)}{c}}\hat{g}(-\eta_0^+)+\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+}\\
&= -\left(\frac{a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}{c}\right)\frac{\mathrm{d}\hat{z}(u)}{\mathrm{d}u}\Bigg|_{u=-v_0^+}\hat{g}(-\eta_0^+)+\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+}\\
&= \left(\frac{a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}{c}\right)\frac{\mathrm{d}\hat{z}(-v_0^+)}{\mathrm{d}v_0^+}\hat{g}(-\eta_0^+)+\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+},\end{aligned}$$

and

$$c\left(\delta-\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+}\right) < \left(a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}\right)\frac{\mathrm{d}\hat{z}(-v_0^+)}{\mathrm{d}v_0^+}\hat{g}(-\eta_0^+).$$

Since the stationarity condition also holds under $\tilde{\mathbb{P}}$, i.e.

$$\delta > \hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+},$$

then,

$$c < \frac{a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}{\delta-\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}\hat{g}(-\eta_0^+)\frac{\mathrm{d}\hat{z}(-v_0^+)}{\mathrm{d}v_0^+},$$

and by (4.22), we have (4.21).

□

Remark 4.4.5. If the net profit condition and the stationarity condition hold under $\mathbb{P}$, but the stationarity condition does not hold under $\tilde{\mathbb{P}}$, i.e. $\tilde{\delta} < \mu_{1_{\tilde{G}}}$, then, the intensity $\tilde{\lambda}_t$ under $\tilde{\mathbb{P}}$ will increase arbitrarily. It does not mean the measures are not equivalent, as we are only considering them till a fixed time T anyway in the optional stopping theorem; also, ruin does occur with probability one and pretty fast (which will manifest itself in the simulation).

In particular, for the special case of shot noise intensity, interestingly, we find a conjugate relationship between the expected loss rates under the two measures.

Corollary 4.4.1. *For the shot noise case with $H \sim \mathrm{Exp}(\alpha)$ and $Z \sim \mathrm{Exp}(\gamma)$, if the net profit condition holds under the original measure $\mathbb{P}$, i.e.*

$$c > \frac{\rho}{\delta\alpha\gamma},$$

then, under the new measure $\tilde{\mathbb{P}}$, we have

$$\tilde{c} < \frac{\tilde{\rho}}{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}},$$

and

$$\frac{\rho}{\delta\alpha\gamma}\frac{\tilde{\rho}}{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}} = c^2. \tag{4.23}$$

Proof. In particular, for the shot noise case with jump-size distributions $H \sim \text{Exp}(\alpha)$ and $Z \sim \text{Exp}(\gamma)$ (by setting $a = 0$ and $\hat{g}(\cdot) = 1$ in *Theorem 4.4.3*), we have the parameters transformed by

- $c \to \tilde{c} = c$,
- $\delta \to \tilde{\delta} = \delta$,
- $\rho \nearrow \tilde{\rho} = \frac{\alpha}{\alpha - \eta_0^+}\rho$,
- $\gamma \searrow \tilde{\gamma} = \gamma - v_0^+$,
- $\alpha \searrow \tilde{\alpha} = \frac{\alpha - \eta_0^+}{1 + \delta\eta_0^+}$,

where the constants are restricted by the generalised Lundberg's fundamental equation

$$\begin{cases} \delta\eta_0^+ = \frac{\gamma}{\gamma - v_0^+} - 1 \\ cv_0^+ = \rho\left(\frac{\alpha}{\alpha - \eta_0^+} - 1\right) \end{cases} \qquad \left(c > \frac{\rho}{\delta\alpha\gamma}\right).$$

The net profit condition holds under $\mathbb{P}$, i.e. $c > \frac{\rho}{\delta\alpha\gamma}$, but under $\tilde{\mathbb{P}}$ we have $\frac{\tilde{\rho}}{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}} > \tilde{c}$, since

$$\begin{aligned}
\frac{\tilde{\rho}}{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}} &= \frac{\frac{\alpha}{\alpha - \eta_0^+}\rho}{\frac{\alpha - \eta_0^+}{1 + \delta\eta_0^+}\left(\gamma - v_0^+\right)\delta} \\
&= \frac{\alpha\rho}{\delta}\frac{1 + \delta\eta_0^+}{(\alpha - \eta_0^+)^2\frac{\gamma}{\delta\eta_0^+ + 1}} \qquad \left(\because \gamma - v_0^+ = \frac{\gamma}{\delta\eta_0^+ + 1}\right) \\
&= \frac{\alpha\rho}{\delta\gamma}\left(\frac{1 + \delta\eta_0^+}{\alpha - \eta_0^+}\right)^2 \\
&= \frac{\alpha\rho}{\delta\gamma}\left(\frac{c\delta\gamma}{\rho}\right)^2 \\
&= \frac{\delta\alpha\gamma}{\rho}c^2 \qquad \left(\because c = \frac{1 + \delta\eta_0^+}{\alpha - \eta_0^+}\frac{\rho}{\delta\gamma}\right) \\
&> \frac{\delta\alpha\gamma}{\rho}\frac{\rho}{\delta\alpha\gamma}c = \tilde{c}.
\end{aligned}$$

Hence, we also find (4.23).

□

4.5 Example: Jumps with Exponential Distributions

To represent the previous results in explicit forms, in this section, we further assume the externally excited and self-excited jumps in the intensity process λ_t and the claim sizes all follow exponential distributions, i.e. $H \sim \text{Exp}(\alpha)$, $G \sim \text{Exp}(\beta)$ and $Z \sim \text{Exp}(\gamma)$, with the density functions

$$h(y) = \alpha e^{-\alpha y}, \quad g(y) = \beta e^{-\beta y}, \quad z(z) = \gamma e^{-\gamma z}, \quad y, z; \alpha, \beta, \gamma > 0,$$

and the Laplace transforms

$$\hat{h}(u) = \frac{\alpha}{\alpha + u}, \quad \hat{g}(u) = \frac{\beta}{\beta + u}, \quad \hat{z}(u) = \frac{\gamma}{\gamma + u}.$$

4.5.1 Generalised Lundberg's Fundamental Equation

We discuss the general case $0 \le r < r^*$ and the special case $r = 0$ for the generalised Lundberg's fundamental equation (from *Theorem 4.2.1*) respectively.

Case $0 \le r < r^*$ By *Theorem 4.2.1*, we have the generalised Lundberg's fundamental equation for $0 \le r < r^*$,

$$\begin{cases} \frac{\gamma}{\gamma - v_r}\frac{\beta}{\beta - \eta_r} = 1 + \delta\eta_r \\ -v_r = \frac{r - a\delta\eta_r + \rho\left(1 - \frac{\alpha}{\alpha - \eta_r}\right)}{c} \end{cases} \qquad \left(v_r < \gamma, \eta_r < (\alpha \wedge \beta); c > \frac{\beta(\rho + a\alpha\delta)}{\alpha\gamma(\delta\beta - 1)}, \delta\beta > 1\right),$$

or, rewrite it w.r.t. η_r as

$$\begin{aligned} 1 + \delta\eta_r &= \frac{c\gamma\beta(\alpha - \eta_r)}{\left(a\delta\eta_r^2 - (\gamma c + \rho + a\delta\alpha + r)\eta_r + \gamma c\alpha + \alpha r\right)(\beta - \eta_r)}, \quad \eta_r < (\alpha \wedge \beta), \\ v_r &= \frac{\eta_r}{c}\left(\frac{\rho}{\alpha - \eta_r} + a\delta\right) - \frac{r}{c}, \quad v_r < \gamma, \end{aligned}$$

with parameters restricted by

$$c > \frac{\beta(\rho + a\alpha\delta)}{\alpha\gamma(\delta\beta - 1)}, \quad \delta\beta > 1.$$

Solve (4.5) of *Lemma 4.2.2* and substitute the unique negative solution $\eta^* = \frac{\delta\beta - 1}{\delta}$ into (4.4), we obtain the constant r^*,

$$r^* = (\delta\beta - 1)\left(a + \frac{\rho}{\delta(\alpha - \beta) + 1}\right).$$

Case $r = 0$ Set $r \to 0$, we have the generalised Lundberg's fundamental equation for $r = 0$,

$$\begin{cases} \frac{\gamma}{\gamma - v_0}\frac{\beta}{\beta - \eta_0} = 1 + \delta\eta_0 \\ -v_0 = \frac{-a\delta\eta_0 + \rho\left(1 - \frac{\alpha}{\alpha - \eta_0}\right)}{c} \end{cases} \qquad \left(v_0 < \gamma, \eta_0 < (\alpha \wedge \beta); c > \frac{\beta(\rho + a\alpha\delta)}{\alpha\gamma(\delta\beta - 1)}, \delta\beta > 1\right),$$

or, rewrite w.r.t. η_0 as

$$\begin{aligned} 1 + \delta\eta_0 &= \frac{c\gamma\beta(\alpha - \eta_0)}{\left(a\delta\eta_0^2 - (\gamma c + \rho + a\delta\alpha)\eta_0 + \gamma c\alpha\right)(\beta - \eta_0)}, \quad \eta_0 < (\alpha \wedge \beta), \\ v_0 &= \frac{\eta_0}{c}\left(\frac{\rho}{\alpha - \eta_0} + a\delta\right), \quad v_0 < \gamma, \end{aligned}$$

with parameters restricted by

$$c > \frac{\beta(\rho + a\alpha\delta)}{\alpha\gamma(\delta\beta - 1)}, \quad \delta\beta > 1.$$

The results of case $r = 0$ here will be used later in Section 4.5.3 for numerical calculations.

4.5.2 Ruin Probability and Generalised Cramér-Lundberg Approximation via Measure $\widetilde{\mathbb{P}}$

The *Corollary 4.5.1* below is an example of *Theorem 4.4.1* and *Theorem 4.4.2* by additionally assuming the exponential distributions.

Corollary 4.5.1. *If $H \sim \mathrm{Exp}(\alpha)$, $G \sim \mathrm{Exp}(\beta)$, $Z \sim \mathrm{Exp}(\gamma)$, $\alpha \ge \beta$, the net profit condition holds under $\mathbb{P}$, and stationarity condition holds under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, and the initial intensity follows the stationary distribution under $\widetilde{\mathbb{P}}$, i.e. $\widetilde{\lambda} \stackrel{\mathcal{D}}{=} \widetilde{a} + \widetilde{\Gamma}_1 + \widetilde{\Gamma}_2$ where*

$$\widetilde{\Gamma}_1 \sim \mathrm{Gamma}\left(\frac{1}{\widetilde{\delta}}\left(\widetilde{a} + \frac{\widetilde{\rho}}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}\right), \frac{\widetilde{\delta}\widetilde{\beta} - 1}{\widetilde{\delta}}\right), \quad \widetilde{\Gamma}_2 \sim \mathrm{Gamma}\left(\frac{\widetilde{\rho}(\widetilde{\alpha} - \widetilde{\beta})}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}, \widetilde{\alpha}\right),$$

then, we have the ruin probability

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} = \frac{\gamma - v_0^+}{\gamma}\frac{\beta - \eta_0^+}{\beta}\widetilde{\mathbb{E}}\left[e^{m_0^+\widetilde{\lambda}}\right]\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau_-^*}}\middle| X_0 = x\right]e^{-v_0^+ x}, \tag{4.24}$$

and the generalised Cramér-Lundberg approximation

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} \sim Ce^{-v_0^+ x}, \quad x \to \infty,$$

where

$$C =: \frac{\gamma - v_0^+}{\gamma}\frac{\beta - \eta_0^+}{\beta}\widetilde{\mathbb{E}}\left[e^{m_0^+\widetilde{\lambda}}\right]\frac{\frac{1}{\widetilde{\gamma}}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\widetilde{\lambda}\right] - \widetilde{c}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau_-^*}}\middle| X_0 = 0\right]}{\frac{1}{\widetilde{\gamma}}\widetilde{\mathbb{E}}[\widetilde{\lambda}] - \widetilde{c}}. \tag{4.25}$$

The transformation from $\mathbb{P}$ *to* $\widetilde{\mathbb{P}}$ *is given by*

- $a \nearrow \widetilde{a} =: \left(1 + \delta\eta_0^+\right)a$,
- $c \to \widetilde{c} =: c$,
- $\delta \to \widetilde{\delta} =: \delta$,
- $\rho \nearrow \widetilde{\rho} =: \frac{\alpha}{\alpha - \eta_0^+}\rho$,
- $\gamma \searrow \widetilde{\gamma} =: \gamma - v_0^+$,
- $\beta \searrow \widetilde{\beta} =: \frac{\beta - \eta_0^+}{1 + \delta\eta_0^+}$,
- $\alpha \searrow \widetilde{\alpha} =: \frac{\alpha - \eta_0^+}{1 + \delta\eta_0^+}$.

Proof. If $H \sim \text{Exp}(\alpha)$, $G \sim \text{Exp}(\beta)$, $Z \sim \text{Exp}(\gamma)$, by *Theorem 2.3.1* for the case when $\alpha \geq \beta$, we have the Laplace transform

$$\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\right] = e^{-m_0^+\widetilde{a}}\left(\frac{\widetilde{\alpha}}{\widetilde{\alpha} + m_0^+}\right)^{\frac{\widetilde{\rho}(\widetilde{\alpha} - \widetilde{\beta})}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}}\left(\frac{\frac{\widetilde{\delta}\widetilde{\beta} - 1}{\widetilde{\delta}}}{m_0^+ + \frac{\widetilde{\delta}\widetilde{\beta} - 1}{\widetilde{\delta}}}\right)^{\frac{1}{\widetilde{\delta}}\left(\widetilde{a} + \frac{\widetilde{\rho}}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}\right)}.$$

Use *Theorem 4.4.1* and *Theorem 4.4.2*, the ruin probability and generalised Cramér-Lundberg approximation can be derived immediately. ☐

We only discuss the case when $\alpha \geq \beta$ for instance. It is similar to derive the corresponding results for other cases when $\alpha < \beta$ and we omit them here.

Remark 4.5.1. We can calculate explicitly for the terms in (4.24) and (4.25) of *Corollary 4.5.1*,

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[e^{m_0^+\widetilde{\lambda}}\right] &= e^{m_0^+\widetilde{a}}\left(\frac{\widetilde{\alpha}}{\widetilde{\alpha} - m_0^+}\right)^{\frac{\widetilde{\rho}(\widetilde{\alpha} - \widetilde{\beta})}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}}\left(\frac{\frac{\widetilde{\delta}\widetilde{\beta} - 1}{\widetilde{\delta}}}{\frac{\widetilde{\delta}\widetilde{\beta} - 1}{\widetilde{\delta}} - m_0^+}\right)^{\frac{1}{\widetilde{\delta}}\left(\widetilde{a} + \frac{\widetilde{\rho}}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}\right)}, \\
\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\widetilde{\lambda}\right] &= e^{-m_0^+\widetilde{a}}\left(\frac{\widetilde{\alpha}}{\widetilde{\alpha} + m_0^+}\right)^{\frac{\widetilde{\rho}(\widetilde{\alpha} - \widetilde{\beta})}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}}\left(\frac{\frac{\widetilde{\delta}\widetilde{\beta} - 1}{\widetilde{\delta}}}{m_0^+ + \frac{\widetilde{\delta}\widetilde{\beta} - 1}{\widetilde{\delta}}}\right)^{\frac{1}{\widetilde{\delta}}\left(\widetilde{a} + \frac{\widetilde{\rho}}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}\right)}\frac{\widetilde{a}\widetilde{\delta} + \frac{\widetilde{\rho}}{\widetilde{\alpha} + m_0^+}}{\widetilde{\delta} - \frac{1}{\widetilde{\beta} + m_0^+}}, \\
\widetilde{\mathbb{E}}[\widetilde{\lambda}] &= \frac{\frac{\widetilde{\rho}}{\widetilde{\alpha}} + \widetilde{a}\widetilde{\delta}}{\widetilde{\delta} - \frac{1}{\widetilde{\beta}}},
\end{aligned}$$

except the term $\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau^*_-}}\Big|\lambda_0 = \widetilde{\lambda} \sim \Pi, X_0 = x\right]$. However, this term can be easily estimated by simulation under $\widetilde{\mathbb{P}}$ where ruin becomes certain.

4.5.3 Numerical Example

Now we provide a numerical example of *Corollary 4.5.1* for the case of exponential distribution when $\alpha \geq \beta$, with parameters under the original measure $\mathbb{P}$ set by

$$(a, \rho, \delta; \alpha, \beta, \gamma; c) = (0.7, 0.5, 3; 2, 1.5, 1; 1.5).$$

It is easy to check that the stationarity and net profit condition hold. Then, we can obtain $(\eta_0^+, v_0^+) = (0.1441, 0.2276)$ (the unique solution of the generalised Lundberg's fundamental equation given in Case $r = 0$ of Section 4.5.1), and $m_0^+ = 0.1006$ (defined in *Theorem 4.4.1*). By *Corollary 4.5.1*, the parameters under the new measure $\widetilde{\mathbb{P}}$ are given by

$$(\widetilde{a}, \widetilde{\rho}, \widetilde{\delta}; \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}; \widetilde{c}) = (1.0026, 0.5388, 3; 1.2957, 0.9467, 0.7724; 1.5000).$$

It is also easy to check that under $\widetilde{\mathbb{P}}$ the stationarity condition holds but the net profit condition does not hold, hence ruin is certain i.e. $\widetilde{P}\{\tau^* < \infty\} = 1$. By *Remark 4.5.1*, we can explicitly calculate $\widetilde{\mathbb{E}}\left[e^{m_0^+\widetilde{\lambda}}\right] = 1.2019$, $\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\widetilde{\lambda}\right] = 1.3974$, and estimate $\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau^*_-}}|\widetilde{\lambda}_0 = \widetilde{\lambda} \sim \Pi, X_0 = 0\right] \approx 0.8330$ from simulation of $10,000$ replications under $\widetilde{\mathbb{P}}$. Therefore, we have $\widetilde{C} \approx 0.5006$ (defined by (4.20)), and by (4.25) the estimated Cramér constant $C \approx 1.2019 \times 0.5006 = 0.6017$ with estimated standard error 1.44×10^{-5}, then,

$$P\left\{\tau^* < \infty \big| X_0 = x\right\} \sim 0.6017e^{-0.2276x}, \quad x \to \infty.$$

By (4.24), the estimated ruin probability $P\{\tau^* < \infty \big| X_0 = x\}$ and the estimated standard error are also given by Table 4.2 based on simulation of $10,000$ replications under $\widetilde{\mathbb{P}}$.

Tab. 4.2: Estimation of Ruin Probability $P\{\tau^* < \infty | X_0 = x\}$ by Our Method

x	$P\{\tau^* < \infty \| X_0 = x\}$	Standard Error ($\times 10^{-4}$)
4	0.2576	4.01
6	0.1609	2.58
8	0.1013	1.68
10	0.064	1.07
12	0.0405	0.71
14	0.0256	0.45
16	0.0162	0.28
18	0.0103	0.18
20	0.0065	0.11
22	0.0041	0.07
24	0.0026	0.05
26	0.0017	0.03
28	0.0011	0.02
30	0.0007	0.01

For comparison, the estimated ruin probability $P\{\tau^* < \infty \big| X_0 = x\}$ and the estimated standard error based on the simulation of $10,000$ replications under the original measure $\mathbb{P}$ are given by Table 4.3, and the ratio of estimated standard errors under the two methods is given by Table 4.4. We can see that the estimated ruin probabilities based on simulations under the two methods are very close. However, by using our method, the estimated standard error has been massively reduced, particularly for a larger x, as the ratio of the estimated standard errors is increasing rapidly as x becomes larger.

Moreover, the computer time needed for each replication is shorter because ruin is certain. Under $\widetilde{\mathbb{P}}$, the average time to ruin and hence the average replication length is approximately 3, all replications had ended before time 100 and 97.5% before time 20, while under $\mathbb{P}$ we had to run replications for longer than that as we had to extend the time horizon to 100 for the probability of ruin only to stabilise.

Tab. 4.3: Estimation of Ruin Probability $P\{\tau^* < \infty | X_0 = x\}$ by Direct Simulation under the Original Measure $\mathbb{P}$

x	$P\{\tau^* < \infty \vert X_0 = x\}$	Standard Error ($\times 10^{-4}$)
4	0.2572	43.85
6	0.1612	36.79
8	0.1016	30.53
10	0.0642	24.51
12	0.0405	19.74
14	0.0258	15.73
16	0.0164	12.89
18	0.0103	10.38
20	0.0065	8.10
22	0.0041	6.54
24	0.0026	4.89
26	0.0017	3.87
28	0.0011	3.16
30	0.0007	2.45

Tab. 4.4: Ratio of the Estimated Standard Errors of Ruin Probability $P\{\tau^* < \infty | X_0 = x\}$ under the Two Methods

x	4	6	8	10	12	14	16	18	20	22	24	26	28	30
Ratio of Errors	10.94	14.25	18.12	22.80	27.88	35.20	45.62	58.02	72.36	89.81	106.90	133.83	169.64	206.95

5

Comparison of Dynamic Contagion Process and Cox Processes with CIR Intensity

As represented in Chapter 2, many results of the dynamic contagion process (DCP) have no explicit formulas, mainly because of involving the non-explicit inverse functions $\mathcal{G}_{\cdot}^{-1}$ from (2.13) and (2.20) in *Theorem 2.2.2* and *Theorem 2.2.4*. In this chapter, we carry out a parallel analysis for the Cox processes with the intensity following some special Cox-Ingersoll-Ross (CIR) processes and discover that they behave very similarly to the DCP case and at the same time have explicit formulas. These formulas thus can be used to find the explicit upper or lower boundaries for the corresponding DCP results, via these links between DCP and CIR constructed by martingale approach and properties of sub-martingale and super-martingale.

We provide a comparison for the two processes: DCP and CIR, both for two sub-cases: the decaying and stationary intensity processes. In particular, the decaying processes have potential to be applicable to the investigation of boundaries of DCP ruin probability. Interestingly, we find that, for some special cases, the two types of processes are sharing the same distributional properties, such as the probability generating functions of point processes, Laplace transform of the intensity processes, and the first moments of intensity processes and point processes.

5.1 Introduction

We consider a Cox process with the point process N_t and the intensity process λ_t following a Cox-Ingersoll-Ross (CIR) process, i.e.

$$\mathrm{d}\lambda_t = \kappa(\mu - \lambda_t)\mathrm{d}t + \sigma\sqrt{\lambda_t}\mathrm{d}W_t,$$

where W_t is a Brownian motion, and constants κ, μ, σ are the speed of adjustment, mean, volatility, respectively, with stationarity condition $2\kappa\mu > \sigma^2$. The generator of process (λ_t, N_t, t) acting on a function $f(\lambda, n, t)$ within its domain is given by

$$\mathrm{CIR}: \mathcal{A}f(\lambda, n, t) = \frac{\partial f}{\partial t} - \kappa(\lambda - \mu)\frac{\partial f}{\partial \lambda} + \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial \lambda^2} + \lambda\Big(f(\lambda, n+1, t) - f(\lambda, n, t)\Big).$$

For comparison, the generator for a general dynamic contagion process (DCP) is given by (2.2), i.e.

$$\begin{aligned}\text{DCP}: \mathcal{A}f(\lambda, n, t) &= \frac{\partial f}{\partial t} + \delta(a-\lambda)\frac{\partial f}{\partial \lambda} + \rho\left(\int_0^\infty f(\lambda+y, n, t)\mathrm{d}H(y) - f(\lambda, n, t)\right) \\ &+\lambda\left(\int_0^\infty f(\lambda+y, n+1, t)\mathrm{d}G(y) - f(\lambda, n, t)\right).\end{aligned}$$

In the following two sections, we will compare the distributional properties for two special cases of CIR and DCP under the stationarity condition with parameter setting as below.

1. *Decaying Intensity Case* (in Section 5.2):

   - DCP: $a = 0$, $\rho = 0$ and $G \sim \text{Exp}(\beta)$,
   - CIR:

$$\begin{aligned}\mu &= 0, \\ \kappa &= \delta - \frac{1}{\beta} \quad (>0), \qquad (5.1)\\ \sigma^2 &= 2\frac{\delta}{\beta}, \qquad (5.2)\end{aligned}$$

   with stationarity condition $\delta\beta > 1$.

2. *Stationary Intensity Case* (in Section 5.3):

   - DCP: $a = 0$, $\rho = 0$ and $H \stackrel{\mathcal{D}}{=} G \sim \text{Exp}(\beta)$,
   - CIR:

$$\begin{aligned}\mu &= \frac{\rho}{\delta\beta - 1} \quad (>0), \\ \kappa &= \delta - \frac{1}{\beta} \quad (>0), \\ \sigma^2 &= 2\frac{\delta}{\beta},\end{aligned}$$

   with stationarity condition $\delta\beta > 1$.

We will find the two special cases of DCP and CIR above behave similarly at some circumstances, and this provides us an alternative aspect to investigate the distributional properties of DCP.

5.2 Decaying Intensity Case

5.2.1 Asymptotic Distribution of N_t

Theorem 5.2.1. *For the decaying intensity case, N_t of CIR and DCP have the same asymptotic distribution, i.e.*

$$\text{DCP}: \mathbb{E}\left[\theta^{N_\infty}\middle|\lambda_0\right] = \text{CIR}: \mathbb{E}\left[\theta^{N_\infty}\middle|\lambda_0\right].$$

Proof. The generator of DCP of the decaying intensity case is given by

$$\text{DCP}: \mathcal{A}f(\lambda, n) = -\delta\lambda\frac{\partial f}{\partial \lambda} + \lambda\left(\int_0^\infty f(\lambda+y, n+1)\mathrm{d}G(y) - f(\lambda, n)\right).$$

Assume the form $f(\lambda, n) = \theta^n e^{-B\lambda}$, and let $\mathcal{A}f(\lambda, n) = 0$, we have

$$\delta B + \theta\hat{g}(B) - 1 = 0,$$

which has unique positive solution v^*. In particular, if $G \sim \text{Exp}(\beta)$, we have

$$\frac{\delta}{\beta}B^2 + \left(\delta - \frac{1}{\beta}\right)B + (\theta - 1) = 0, \tag{5.3}$$

and v^* is given by (2.45). Then, we have martingale $\theta^{N_t}e^{-v^*\lambda_t}$ and

$$\mathbb{E}\left[\theta^{N_\infty}e^{-v^*\lambda_\infty}\big|\lambda_0\right] = e^{-v^*\lambda_0},$$

since $\lambda_\infty = 0$, we have

$$\text{DCP}: \mathbb{E}\left[\theta^{N_\infty}\big|\lambda_0\right] = e^{-v^*\lambda_0}.$$

On the other hand, the generator of CIR of the decaying intensity case is given by

$$\text{CIR}: \mathcal{A}f(\lambda, n) = -\kappa\lambda\frac{\partial f}{\partial\lambda} + \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial\lambda^2} + \lambda\Big(f(\lambda, n+1) - f(\lambda, n)\Big).$$

Assume the form $f(\lambda, n) = \theta^n e^{-B\lambda}$, and let $\mathcal{A}f(\lambda, n) = 0$, we have a similar form as

$$\frac{1}{2}\sigma^2 B^2 + \kappa B + (\theta - 1) = 0. \tag{5.4}$$

By comparing (5.4) with (5.3) and letting $\kappa = \delta - \frac{1}{\beta}$, $\sigma^2 = 2\frac{\delta}{\beta}$, then, (5.4) and (5.3) become identical. □

5.2.2 Conditional Distribution of N_T

Proposition 5.2.1. *$f(\lambda, t)$ is a sub-martingale if $\mathcal{A}f(\lambda, t) \geq 0$ for all λ and t; $f(\lambda, t)$ is a super-martingale if $\mathcal{A}f(\lambda, t) \leq 0$ for all λ and t.*

Theorem 5.2.2. *For the decaying intensity case, we have*

$$\text{DCP}: \mathbb{E}\left[\theta^{N_T}\big|\lambda_0\right] \quad > \quad \text{CIR}: \mathbb{E}\left[\theta^{N_T}\big|\lambda_0\right], \quad 0 < T < \infty.$$

Proof. The generator of the CIR case is given by

$$\text{CIR}: \mathcal{A}f(\lambda, n, t) = \frac{\partial f}{\partial t} - \kappa\lambda\frac{\partial f}{\partial\lambda} + \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial\lambda^2} + \lambda\Big(f(\lambda, n+1, t) - f(\lambda, n, t)\Big).$$

Assume the form $f(\lambda, n, t) = e^{\bar{c}(t)}\theta^n e^{-\bar{B}(t)\lambda}$, and set $\mathcal{A}f(\lambda, n, t) = 0$, then,

$$\bar{c}'(t) - \bar{B}'(t)\lambda - \kappa\lambda(-\bar{B}(t)) + \frac{1}{2}\sigma^2\lambda\bar{B}^2(t) + \lambda(\theta - 1) = 0,$$

and

$$\begin{aligned} \bar{B}'(t) &= \frac{1}{2}\sigma^2\bar{B}^2(t) + \kappa\bar{B}(t) + (\theta - 1), \\ \bar{c}'(t) &= 0. \end{aligned}$$

Set $\kappa = \delta - \frac{1}{\beta}$, $\sigma^2 = 2\frac{\delta}{\beta}$, then,

$$\begin{aligned} \bar{B}'(t) &= \frac{\delta}{\beta}\bar{B}^2(t) + \left(\delta - \frac{1}{\beta}\right)\bar{B}(t) + (\theta - 1), \\ \bar{c}'(t) &= 0, \end{aligned}$$

with boundary condition $\bar{B}(T)=0$ and condition $\delta\beta>1$. By (2.59), $\bar{B}'(t)<0$ when $0\leq\bar{B}(t)<v^*$.

On the other hand, the generator of DCP case is given by

$$\text{DCP}: \mathcal{A}f(\lambda,n,t)=\frac{\partial f}{\partial t}-\delta\lambda\frac{\partial f}{\partial\lambda}+\lambda\left(\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y)-f(\lambda,n,t)\right),$$

where $G\sim\text{Exp}(\beta)$. We have

$$\begin{aligned}
&\text{DCP}: \mathcal{A}\left(e^{\bar{c}(t)}\theta^n e^{-\bar{B}(t)\lambda}\right)\\
= \;& e^{\bar{c}(t)}\theta^n e^{-\bar{B}(t)\lambda}\left[\bar{c}'(t)-\bar{B}'(t)\lambda+\delta\bar{B}(t)\lambda+\lambda\left(\theta\frac{\beta}{\beta+\bar{B}(t)}-1\right)\right]\\
= \;& e^{\bar{c}(t)}\theta^n e^{-\bar{B}(t)\lambda}\left[-\left(\frac{\delta}{\beta}\bar{B}^2(t)+\left(\delta-\frac{1}{\beta}\right)\bar{B}(t)+(\theta-1)\right)\lambda+\delta\bar{B}(t)\lambda+\lambda\left(\theta\frac{\beta}{\beta+\bar{B}(t)}-1\right)\right]\\
= \;& e^{\bar{c}(t)}\theta^n e^{-\bar{B}(t)\lambda}\lambda\bar{B}(t)\frac{-1}{\beta+\bar{B}(t)}\left[\frac{\delta}{\beta}\bar{B}^2(t)+\left(\delta-\frac{1}{\beta}\right)\bar{B}(t)+(\theta-1)\right]\\
= \;& -e^{\bar{c}(t)}\theta^n e^{-\bar{B}(t)\lambda}\lambda\bar{B}(t)\frac{\bar{B}'(t)}{\beta+\bar{B}(t)}>0,
\end{aligned}$$

as $\bar{B}(t)>0,\bar{B}'(t)<0$. Therefore, $e^{\bar{c}(t)}\theta^{N_t}e^{-\bar{B}(t)\lambda_t}$ is a sub-martingale in the DCP case, and we have a martingale

$$e^{\bar{c}(t)}\theta^{N_t}e^{-\bar{B}(t)\lambda_t}-e^{\bar{c}(0)}\theta^{N_0}e^{-\bar{B}(0)\lambda_0}-\int_0^t\mathcal{A}\left(e^{\bar{c}(s)}\theta^{N_s}e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s.$$

Hence,

$$\begin{aligned}
\text{DCP}: \mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right] &= \mathbb{E}\left[e^{\bar{c}(T)}\theta^{N_T}e^{-\bar{B}(T)\lambda_T}\middle|\lambda_0\right]\\
&= e^{\bar{c}(0)}\theta^{N_0}e^{-\bar{B}(0)\lambda_0}+\mathbb{E}\left[\int_0^T\mathcal{A}\left(e^{\bar{c}(s)}\theta^{N_s}e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s\middle|\lambda_0\right]\\
&> e^{\bar{c}(0)}\theta^{N_0}e^{-\bar{B}(0)\lambda_0}=\text{CIR}: \mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right],
\end{aligned}$$

since

$$\mathbb{E}\left[\int_0^T\mathcal{A}\left(e^{\bar{c}(s)}\theta^{N_s}e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s\middle|\lambda_0\right]>0.$$

□

5.2.3 Conditional Distribution of λ_T

Theorem 5.2.3. *For the decaying intensity case, we have*

$$\text{DCP}: \mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right] \quad < \quad \text{CIR}: \mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right], \quad 0<T<\infty.$$

Proof. The generator of the CIR case is given by

$$\text{CIR}: \mathcal{A}f(\lambda,t)=\frac{\partial f}{\partial t}-\kappa\lambda\frac{\partial f}{\partial\lambda}+\frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial\lambda^2}.$$

Assume the form $f(\lambda,t)=e^{\bar{c}(t)}e^{-\bar{B}(t)\lambda}$, and set $\mathcal{A}f(\lambda,t)=0$, then,

$$\bar{c}'(t)-\bar{B}'(t)\lambda-\kappa\lambda(-\bar{B}(t))+\frac{1}{2}\sigma^2\lambda\bar{B}^2(t)=0,$$

and

$$\begin{aligned}\bar{B}'(t) &= \frac{1}{2}\sigma^2\bar{B}^2(t)+\kappa\bar{B}(t),\\ \bar{c}'(t) &= 0.\end{aligned}$$

Set $\kappa=\delta-\frac{1}{\beta}$, $\sigma^2=2\frac{\delta}{\beta}$, then,

$$\begin{aligned}\bar{B}'(t) &= \frac{\delta}{\beta}\bar{B}^2(t)+\left(\delta-\frac{1}{\beta}\right)\bar{B}(t),\\ \bar{c}'(t) &= 0,\end{aligned}$$

with boundary condition $\bar{B}(T)=v>0$ and stationarity condition $\delta\beta>1$, $\bar{B}'(t)>0$.

On the other hand, the generator of the DCP case is given by

$$\mathrm{DCP}:\mathcal{A}f(\lambda,t)=\frac{\partial f}{\partial t}-\delta\lambda\frac{\partial f}{\partial \lambda}+\lambda\left(\int_0^\infty f(\lambda+y,t)\mathrm{d}G(y)-f(\lambda,t)\right),$$

where $G\sim\mathrm{Exp}(\beta)$. We have

$$\begin{aligned}&\mathrm{DCP}:\mathcal{A}\left(e^{\bar{c}(t)}e^{-\bar{B}(t)\lambda}\right)\\ =\;& e^{\bar{c}(t)}e^{-\bar{B}(t)\lambda}\left[\bar{c}'(t)-\bar{B}'(t)\lambda+\delta\bar{B}(t)\lambda+\lambda\left(\frac{\beta}{\beta+\bar{B}(t)}-1\right)\right]\\ =\;& e^{\bar{c}(t)}e^{-\bar{B}(t)\lambda}\left[-\left(\frac{\delta}{\beta}\bar{B}^2(t)+\left(\delta-\frac{1}{\beta}\right)\bar{B}(t)\right)\lambda+\delta\bar{B}(t)\lambda+\lambda\left(\frac{\beta}{\beta+\bar{B}(t)}-1\right)\right]\\ =\;& e^{\bar{c}(t)}e^{-\bar{B}(t)\lambda}\lambda\bar{B}(t)\frac{-1}{\beta+\bar{B}(t)}\left[\frac{\delta}{\beta}\bar{B}^2(t)+\left(\delta-\frac{1}{\beta}\right)\bar{B}(t)\right]\\ =\;& -e^{\bar{c}(t)}e^{-\bar{B}(t)\lambda}\lambda\bar{B}(t)\frac{\bar{B}'(t)}{\beta+\bar{B}(t)}<0,\end{aligned}$$

as $\bar{B}(t)>0, \bar{B}'(t)>0$. Therefore, $e^{\bar{c}(t)}e^{-\bar{B}(t)\lambda_t}$ is a super-martingale in the DCP case, and we have a martingale

$$e^{\bar{c}(t)}e^{-\bar{B}(t)\lambda_t}-e^{\bar{c}(0)}e^{-\bar{B}(0)\lambda_0}-\int_0^t\mathcal{A}\left(e^{\bar{c}(s)}e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s,$$

then,

$$\begin{aligned}\mathrm{DCP}:\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right] &= \mathbb{E}\left[e^{\bar{c}(T)}e^{-\bar{B}(T)\lambda_T}\middle|\lambda_0\right]\\ &= e^{\bar{c}(0)}e^{-\bar{B}(0)\lambda_0}+\mathbb{E}\left[\int_0^T\mathcal{A}\left(e^{\bar{c}(s)}e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s\middle|\lambda_0\right]\\ &< e^{\bar{c}(0)}e^{-\bar{B}(0)\lambda_0}=\mathrm{CIR}:\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right],\end{aligned}$$

since

$$\mathbb{E}\left[\int_0^T\mathcal{A}\left(e^{\bar{c}(s)}e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s\middle|\lambda_0\right]<0.$$

□

5.2.4 Moments of λ_t and N_t

Theorem 5.2.4. *For $0 < t < \infty$, the comparison of the moments conditional on λ_0 between CIR and DCP for the decaying intensity case is summarised by*

$$\begin{aligned}
\text{CIR} : \mathbb{E}[\lambda_t|\lambda_0] &= \text{DCP} : \mathbb{E}[\lambda_t|\lambda_0], \\
\text{CIR} : \mathbb{E}[N_t|\lambda_0] &= \text{DCP} : \mathbb{E}[N_t|\lambda_0], \\
\text{CIR} : \mathbb{E}[\lambda_t^2|\lambda_0] &> \text{DCP} : \mathbb{E}[\lambda_t^2|\lambda_0], \\
\text{CIR} : \mathbb{E}[\lambda_t N_t|\lambda_0] &< \text{DCP} : \mathbb{E}[\lambda_t N_t|\lambda_0], \\
\text{CIR} : \mathbb{E}[N_t^2|\lambda_0] &< \text{DCP} : \mathbb{E}[N_t^2|\lambda_0], \\
\text{CIR} : \text{Var}[N_t|\lambda_0] &< \text{DCP} : \text{Var}[N_t|\lambda_0].
\end{aligned}$$

Proof. The proofs are given separately as below.

Remind that, the generator for CIR is given by

$$\text{CIR} : \mathcal{A}f(\lambda, n, t) = \frac{\partial f}{\partial t} - \kappa\lambda\frac{\partial f}{\partial \lambda} + \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial \lambda^2} + \lambda\Big(f(\lambda, n+1, t) - f(\lambda, n, t)\Big). \tag{5.5}$$

$\mathbb{E}[\lambda_t|\lambda_0]$*:* For CIR, set $f(\lambda, n, t) = \lambda$ in (5.5), we have

$$\mathcal{A}\lambda = -\kappa\lambda.$$

Since $\lambda_t - \lambda_0 - \int_0^t \mathcal{A}\lambda_s \mathrm{d}s$ is a martingale, we have

$$\mathbb{E}[\lambda_t|\lambda_0] = \lambda_0 + \mathbb{E}\left[\int_0^t \mathcal{A}\lambda_s \mathrm{d}s \middle| \lambda_0\right] = \lambda_0 - \kappa\int_0^t \mathbb{E}[\lambda_s|\lambda_0]\mathrm{d}s,$$

and

$$u(t) = \lambda_0 - \kappa\int_0^t u(s)\mathrm{d}s,$$

where $u(t) = \mathbb{E}[\lambda_t|\lambda_0]$, then, the ODE

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\kappa u(t),$$

with the initial condition $u(0) = \lambda_0$, we have

$$\text{CIR} : \mathbb{E}[\lambda_t|\lambda_0] = \lambda_0 e^{-\kappa t}. \tag{5.6}$$

Comparing with the DCP case from (2.29), i.e.

$$\text{DCP} : \mathbb{E}[\lambda_t|\lambda_0] = \lambda_0 e^{-\left(\delta - \mu_{1_G}\right)t} = \lambda_0 e^{-\left(\delta - \frac{1}{\beta}\right)t}, \tag{5.7}$$

let $\kappa = \delta - \frac{1}{\beta}$, then CIR (5.6) = DCP (5.7), i.e.

$$\text{CIR} : \mathbb{E}[\lambda_t|\lambda_0] = \text{DCP} : \mathbb{E}[\lambda_t|\lambda_0], \quad t > 0.$$

$\mathbb{E}[N_t|\lambda_0]$*:* For CIR, set $f(\lambda, n, t) = n$ in (5.5), we have

$$\mathcal{A}n = \lambda,$$

since $N_t - N_0 - \int_0^t \lambda_s \mathrm{d}s$ is a martingale, and assume $N_0 = 0$ we have

$$\mathbb{E}[N_t|\lambda_0] = \int_0^t \mathbb{E}[\lambda_s|\lambda_0]\mathrm{d}s = \lambda_0 \int_0^t e^{-\kappa s}\mathrm{d}s = \lambda_0 \frac{1-e^{-\kappa t}}{\kappa}.$$

Set $\kappa = \delta - \frac{1}{\beta}$,

$$\mathrm{CIR}: \mathbb{E}[N_t|\lambda_0] = \lambda_0 \frac{1}{\delta - \frac{1}{\beta}} \left(1 - e^{-\left(\delta - \frac{1}{\beta}\right)t}\right),$$

which is the same for DCP and CIR, i.e.

$$\mathrm{CIR}: \mathbb{E}[N_t|\lambda_0] = \mathrm{DCP}: \mathbb{E}[N_t|\lambda_0], \quad t > 0.$$

$\mathbb{E}[\lambda_t^2|\lambda_0]$: For CIR, set $f(\lambda, n, t) = \lambda^2$ in (5.5), we have

$$\mathcal{A}\lambda = -2\kappa\lambda^2 + \sigma^2\lambda.$$

Since $\lambda_t^2 - \lambda_0^2 - \int_0^t \mathcal{A}\lambda_s^2 \mathrm{d}s$ is a martingale, we have

$$\mathbb{E}[\lambda_t^2|\lambda_0] = \lambda_0^2 + \mathbb{E}\left[\int_0^t \mathcal{A}\lambda_s^2 \mathrm{d}s \middle| \lambda_0\right] = \lambda_0^2 - 2\kappa \int_0^t \mathbb{E}[\lambda_s^2|\lambda_0]\mathrm{d}s + \sigma^2 \int_0^t \mathbb{E}[\lambda_s|\lambda_0]\mathrm{d}s,$$

then,

$$u(t) = \lambda_0^2 - 2\kappa \int_0^t u(s)\mathrm{d}s + \sigma^2 \int_0^t \mathbb{E}[\lambda_s|\lambda_0]\mathrm{d}s,$$

where $u(t) = \mathbb{E}[\lambda_t^2|\lambda_0]$, and the ODE

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -2\kappa u(t) + \sigma^2 \mathbb{E}[\lambda_t|\lambda_0],$$

i.e.

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -2\kappa u(t) + \sigma^2 \lambda_0 e^{-\kappa t},$$

with the initial condition $u(0) = \lambda_0^2$, we have

$$\mathbb{E}[\lambda_t^2|\lambda_0] = \lambda_0^2 e^{-2\kappa t} + \frac{\sigma^2}{\kappa}\lambda_0 \left(e^{-\kappa t} - e^{-2\kappa t}\right).$$

Set $\kappa = \delta - \frac{1}{\beta}$ and $\sigma^2 = 2\frac{\delta}{\beta}$, we have

$$\mathrm{CIR}: \mathbb{E}[\lambda_t^2|\lambda_0] = \lambda_0^2 e^{-2\left(\delta - \frac{1}{\beta}\right)t} + \frac{2\frac{\delta}{\beta}}{\delta - \frac{1}{\beta}}\lambda_0 \left(e^{-\left(\delta - \frac{1}{\beta}\right)t} - e^{-2\left(\delta - \frac{1}{\beta}\right)t}\right), \tag{5.8}$$

comparing with the DCP case from (2.31), i.e.

$$\mathrm{DCP}: \mathbb{E}[\lambda_t^2|\lambda_0] = \lambda_0^2 e^{-2\left(\delta - \frac{1}{\beta}\right)t} + \frac{\frac{2}{\beta^2}}{\delta - \frac{1}{\beta}}\lambda_0 \left(e^{-\left(\delta - \frac{1}{\beta}\right)t} - e^{-2\left(\delta - \frac{1}{\beta}\right)t}\right). \tag{5.9}$$

Compare (5.9) with (5.8), since the stationarity condition $\delta\beta > 1$, we have

$$\frac{2\frac{\delta}{\beta}}{\delta - \frac{1}{\beta}} > \frac{\frac{2}{\beta^2}}{\delta - \frac{1}{\beta}},$$

then CIR (5.8) > DCP (5.9), i.e.

$$\mathrm{CIR}: \mathbb{E}[\lambda_t^2|\lambda_0] > \mathrm{DCP}: \mathbb{E}[\lambda_t^2|\lambda_0], \quad t > 0.$$

$\mathbb{E}[\lambda_t N_t|\lambda_0]$: For CIR, set $f(\lambda, n, t) = \lambda n$ in (5.5), we have

$$\mathcal{A}(\lambda n) = -\kappa n\lambda + \lambda^2,$$

since $\lambda_t N_t - \lambda_0 N_0 - \int_0^t \mathcal{A}(\lambda_s N_s)\mathrm{d}s$ is a martingale, and assume $N_0 = 0$, we have

$$\mathbb{E}[\lambda_t N_t|\lambda_0] = \mathbb{E}\left[\int_0^t \mathcal{A}(\lambda_s N_s)\mathrm{d}s\middle|\lambda_0\right] = -\kappa\int_0^t \mathbb{E}[\lambda_s N_s|\lambda_0]\mathrm{d}s + \int_0^t \mathbb{E}[\lambda_s^2|\lambda_0]\mathrm{d}s,$$

then, the ODE

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\kappa u(t) + \mathbb{E}[\lambda_t^2|\lambda_0], \tag{5.10}$$

or,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + \kappa u(t) = \lambda_0 e^{-\kappa t}\left[\left(\lambda_0 - \frac{\sigma^2}{\kappa}\right)e^{-\kappa t} + \frac{\sigma^2}{\kappa}\right],$$

where $u(t) = \mathbb{E}[\lambda_t N_t|\lambda_0]$ and $u(0) = 0$, then,

$$\begin{aligned}\text{CIR}: \mathbb{E}[\lambda_t N_t|\lambda_0] &= \lambda_0\left[\left(\lambda_0 - \frac{\sigma^2}{\kappa}\right)\frac{e^{-\kappa t} - e^{-2\kappa t}}{\kappa} + \frac{\sigma^2}{\kappa}te^{-\kappa t}\right]\\ &= \lambda_0\left[\left(\lambda_0 - \frac{2\frac{\delta}{\beta}}{\delta - \frac{1}{\beta}}\right)\frac{e^{-\left(\delta-\frac{1}{\beta}\right)t} - e^{-2\left(\delta-\frac{1}{\beta}\right)t}}{\delta - \frac{1}{\beta}} + \frac{2\frac{\delta}{\beta}}{\delta-\frac{1}{\beta}}te^{-\left(\delta-\frac{1}{\beta}\right)t}\right].\end{aligned}$$

For DCP, set $f(\lambda, n, t) = \lambda n$ in the generator, we have

$$\mathcal{A}(\lambda n) = -(\delta - \mu_{1_G})n\lambda + \lambda^2 + \mu_{1_G}\lambda,$$

similarly, we have the ODE

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -(\delta - \mu_{1_G})u(t) + \mathbb{E}[\lambda_t^2|\lambda_0] + \mu_{1_G}\mathbb{E}[\lambda_t|\lambda_0], \tag{5.11}$$

where $u(t) = \mathbb{E}[\lambda_t N_t|\lambda_0]$ and $u(0) = 0$, then, we have

$$\begin{aligned}&\text{DCP}: \mathbb{E}[\lambda_t N_t|\lambda_0]\\ = \;&\lambda_0\left[\left(\lambda_0 - \frac{\mu_{2_G}}{\delta-\mu_{1_G}}\right)\frac{e^{-(\delta-\mu_{1_G})t} - e^{-2(\delta-\mu_{1_G})t}}{\delta-\mu_{1_G}} + \left(\frac{\mu_{2_G}}{\delta-\mu_{1_G}} + \mu_{1_G}\right)te^{-(\delta-\mu_{1_G})t}\right]\\ = \;&\lambda_0\left[\left(\lambda_0 - \frac{\frac{2}{\beta^2}}{\delta-\frac{1}{\beta}}\right)\frac{e^{-(\delta-\frac{1}{\beta})t} - e^{-2(\delta-\frac{1}{\beta})t}}{\delta-\frac{1}{\beta}} + \left(\frac{\frac{2}{\beta}}{\delta-\frac{1}{\beta}} + \frac{1}{\beta}\right)te^{-(\delta-\frac{1}{\beta})t}\right].\end{aligned}$$

To compare CIR with DCP, we firstly compare their corresponding $\mathcal{A}(\lambda n)$ via (5.10) and (5.11), i.e.

$$\begin{aligned}&\text{CIR}: \frac{\mathrm{d}\mathbb{E}[\lambda_t N_t|\lambda_0]}{\mathrm{d}t} - \text{DCP}: \frac{\mathrm{d}\mathbb{E}[\lambda_t N_t|\lambda_0]}{\mathrm{d}t}\\ = \;&\frac{2\frac{\delta}{\beta}}{\delta-\frac{1}{\beta}}\lambda_0\left(e^{-\left(\delta-\frac{1}{\beta}\right)t} - e^{-2\left(\delta-\frac{1}{\beta}\right)t}\right) - \left[\frac{\frac{2}{\beta^2}}{\delta-\frac{1}{\beta}}\lambda_0\left(e^{-\left(\delta-\frac{1}{\beta}\right)t} - e^{-2\left(\delta-\frac{1}{\beta}\right)t}\right) + \frac{1}{\beta}\lambda_0 e^{-\left(\delta-\frac{1}{\beta}\right)t}\right]\\ = \;&\frac{1}{\beta}\lambda_0 e^{-\left(\delta-\frac{1}{\beta}\right)t}\left(1 - 2e^{-\left(\delta-\frac{1}{\beta}\right)t}\right),\end{aligned} \tag{5.12}$$

where $t^* = \frac{\ln 2}{\delta - \frac{1}{\beta}}$ is the critical point. Note that, CIR : $\mathbb{E}[\lambda_t N_t|\lambda_0]$ − DCP : $\mathbb{E}[\lambda_t N_t|\lambda_0]$ is the integration of (5.12). From time $t = 0$ to $t = \infty$, (5.12) is first negative and then positive, meanwhile the integration of (5.12) starts from 0 and ends up at 0, hence the difference CIR : $\mathbb{E}[\lambda_t N_t|\lambda_0]$ − DCP : $\mathbb{E}[\lambda_t N_t|\lambda_0]$ is first decreasing and then increasing. A function that is first decreasing and then increasing (quasi-convex) and starts from 0 and ends to 0 has to be negative. Therefore, we have

$$\text{CIR}: \mathbb{E}[\lambda_t N_t|\lambda_0] < \text{DCP}: \mathbb{E}[\lambda_t N_t|\lambda_0], \quad t > 0.$$

$\mathbb{E}[N_t^2|\lambda_0]$: For CIR, set $f(\lambda,n,t)=n^2$ in (5.5), we have

$$\mathcal{A}(n^2)=\lambda(2n+1),$$

since $N_t^2-N_0^2-\int_0^t\mathcal{A}(N_s^2)\mathrm{d}s$ is a martingale, and assume $N_0=0$, we have

$$\text{CIR}:\mathbb{E}[N_t^2|\lambda_0]=\mathbb{E}\left[\int_0^t\mathcal{A}(N_s^2)\mathrm{d}s\middle|\lambda_0\right]=2\int_0^t\mathbb{E}[\lambda_sN_s|\lambda_0]\mathrm{d}s+\int_0^t\mathbb{E}[\lambda_s|\lambda_0]\mathrm{d}s.$$

For DCP, set $f(\lambda,n,t)=n^2$ in the generator, we have

$$\mathcal{A}(n^2)=\lambda(2n+1),$$

since $N_t^2-N_0^2-\int_0^t\mathcal{A}(N_s^2)\mathrm{d}s$ is a martingale, and assume $N_0=0$, we have

$$\text{DCP}:\mathbb{E}[N_t^2|\lambda_0]=\mathbb{E}\left[\int_0^t\mathcal{A}(N_s^2)\mathrm{d}s\middle|\lambda_0\right]=2\int_0^t\mathbb{E}[\lambda_sN_s|\lambda_0]\mathrm{d}s+\int_0^t\mathbb{E}[\lambda_s|\lambda_0]\mathrm{d}s.$$

Since $\text{CIR}:\mathbb{E}[\lambda_tN_t|\lambda_0]<\text{DCP}:\mathbb{E}[\lambda_tN_t|\lambda_0]$, $t>0$, we also have

$$\text{CIR}:\mathbb{E}[N_t^2|\lambda_0]<\text{DCP}:\mathbb{E}[N_t^2|\lambda_0],\quad t>0.$$

$\mathrm{Var}[N_t|\lambda_0]$: Since $\mathrm{Var}[N_t|\lambda_0]=\mathbb{E}[N_t^2|\lambda_0]-\mathbb{E}[N_t|\lambda_0]^2$, where $\text{CIR}:\mathbb{E}[N_t^2|\lambda_0]<\text{DCP}:\mathbb{E}[N_t^2|\lambda_0]$ and $\text{CIR}:\mathbb{E}[N_t|\lambda_0]=\text{DCP}:\mathbb{E}[N_t|\lambda_0]$, we have

$$\text{CIR}:\mathrm{Var}[N_t|\lambda_0]<\text{DCP}:\mathrm{Var}[N_t|\lambda_0],t>0.$$

□

5.2.5 The Probability of the First Jump Time of N_t

Theorem 5.2.5. *For CIR of the decaying intensity case, the probability of no jump conditional on λ_0 is given by*

$$\text{CIR}:P\left\{T_1^*>T\middle|\lambda_0\right\}=P\left\{N_T=0\middle|\lambda_0\right\}=\exp\left(-\frac{1}{\delta}\frac{1-e^{-\left(\delta+\frac{1}{\beta}\right)T}}{1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)T}}\lambda_0\right).\tag{5.13}$$

Proof. Define $\Lambda_t=:\int_0^t\lambda_s\mathrm{d}s$, the generator of CIR is given by

$$\text{CIR}:\mathcal{A}f(\lambda,\Lambda,t)=\frac{\partial f}{\partial t}+\lambda\frac{\partial f}{\partial\Lambda}-\left(\frac{\delta\beta-1}{\beta}\right)\lambda\frac{\partial f}{\partial\lambda}+\frac{\delta}{\beta}\lambda\frac{\partial^2f}{\partial\lambda^2}.$$

Assume the form $f(\lambda,\Lambda,t)=e^{-\Lambda}e^{-B(t)\lambda}$, and set $\mathcal{A}f(\lambda,\Lambda,t)=0$, we have

$$B'(t)=\frac{\delta}{\beta}B^2(t)+\left(\delta-\frac{1}{\beta}\right)B(t)-1,$$

or, factorise as

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t}=\frac{\delta}{\beta}\Big(B(t)+\beta\Big)\left(B(t)-\frac{1}{\delta}\right),$$

then, with boundary condition $B(T)=0$, we have

$$B(t)=\frac{1}{\delta}\frac{1-e^{-\left(\delta+\frac{1}{\beta}\right)(T-t)}}{1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)(T-t)}}.$$

Hence,

$$B(0)=\frac{1}{\delta}\frac{1-e^{-\left(\delta+\frac{1}{\beta}\right)T}}{1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)T}}.$$

Since

$$\mathbb{E}\left[e^{-\Lambda_T}e^{-B(T)\lambda_T}\middle|\lambda_0\right]=e^{-\Lambda_0}e^{-B(0)\lambda_0},$$

or,

$$\mathbb{E}\left[e^{-\Lambda_T}\middle|\lambda_0\right]=e^{-B(0)\lambda_0},$$

we obtain the probability of no jump explicitly for CIR,

$$\text{CIR}: P\left\{N_T=0\middle|\lambda_0\right\}=P\left\{T_1^*>T\middle|\lambda_0\right\}=\mathbb{E}\left[e^{-\Lambda_T}\middle|\lambda_0\right]=\exp\left(-\frac{1}{\delta}\frac{1-e^{-\left(\delta+\frac{1}{\beta}\right)T}}{1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)T}}\lambda_0\right).$$

Note that, it is an increasing function of T as

$$\frac{\mathrm{d}}{\mathrm{d}T}\left\{\frac{1-e^{-\left(\delta+\frac{1}{\beta}\right)T}}{1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)T}}\right\}=\frac{\delta+\frac{1}{\beta}}{1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)T}}\left(e^{-\left(\delta+\frac{1}{\beta}\right)T}+\frac{\frac{1}{\delta\beta}\left(1-e^{-\left(\delta+\frac{1}{\beta}\right)T}\right)}{1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)T}}\right)>0.$$

□

Theorem 5.2.6. *For the decaying intensity case, we have*

$$\text{DCP}: P\left\{T_1^*>T\middle|\lambda_0\right\} \quad > \quad \text{CIR}: P\left\{T_1^*>T\middle|\lambda_0\right\}, \quad 0<T<\infty.$$

Proof. Compare (5.13) with the decaying contagion case from *Corollary 2.2.4*, i.e.

$$\text{DCP}: P\left\{N_T=0\middle|\lambda_0\right\}=P\left\{T_1^*>T\middle|\lambda_0\right\}=\exp\left(-\frac{1}{\delta}\left(1-e^{-\delta T}\right)\lambda_0\right). \tag{5.14}$$

By the inequality

$$\begin{aligned} e^x &> 1+x, \quad x\neq 0,\\ e^x &= 1+x, \quad x=0, \end{aligned}$$

if $T\neq 0$, then,

$$\beta e^{\frac{1}{\beta}T}+\frac{1}{\delta}e^{-\delta T}>\beta\left(1+\frac{1}{\beta}T\right)+\frac{1}{\delta}\left(1-\delta T\right)=\beta+\frac{1}{\delta},$$

and

$$\begin{aligned} e^{\frac{1}{\beta}T}+\frac{1}{\delta\beta}\left(e^{-\delta T}-1\right) &> 1\\ e^{-\delta T}+\frac{1}{\delta\beta}\left(e^{-\left(2\delta+\frac{1}{\beta}\right)T}-e^{-\left(\delta+\frac{1}{\beta}\right)T}\right) &> e^{-\left(\delta+\frac{1}{\beta}\right)T}\\ \left(1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)T}\right)\left(1-e^{-\delta T}\right) &< 1-e^{-\left(\delta+\frac{1}{\beta}\right)T}\\ 1-e^{-\delta T} &< \frac{1-e^{-\left(\delta+\frac{1}{\beta}\right)T}}{1+\frac{1}{\delta\beta}e^{-\left(\delta+\frac{1}{\beta}\right)T}}\\ (5.14) &> (5.13); \end{aligned}$$

if $T=0$, then (5.14) = (5.13). □

Remark 5.2.1. Note that, the super-martingale approach does not work here as

$$\text{DCP}: P\left\{T_1^*<T\middle|\lambda_0\right\}\neq\mathbb{E}\left[e^{-\Lambda_T}\middle|\lambda_0\right].$$

5.2.6 The Probability of the Last Jump Time of N_t

Note that, for both CIR and DCP cases, we have

$$\begin{aligned} P\left\{T_L^* < T \middle| \lambda_0\right\} &= \mathbb{E}\left[\exp\left(-\int_T^\infty \lambda_s \mathrm{d}s\right) \middle| \lambda_0\right] \\ &= \mathbb{E}\left[\exp\left(-\int_T^\infty \lambda_T e^{-\delta(s-T)} \mathrm{d}s\right) \middle| \lambda_0\right] \\ &= \mathbb{E}\left[e^{-\frac{\lambda_T}{\delta}} \middle| \lambda_0\right], \end{aligned}$$

and we can use the super-martingale method for comparison.

Theorem 5.2.7. *For CIR of the decaying intensity case, the probability of the last jump time conditional on λ_0 is given by*

$$\mathrm{CIR}: P\left\{T_L^* < T \middle| \lambda_0\right\} = \exp\left(-\frac{1}{\delta}\frac{\delta\beta - 1}{\delta\beta e^{\left(\delta - \frac{1}{\beta}\right)T} - 1}\lambda_0\right).$$

Proof. Assume the form $f(\lambda, t) = e^{-\bar{B}(t)\lambda}$, and set $\mathcal{A}f(\lambda, \Lambda, t) = 0$, for CIR, we have

$$\bar{B}'(t) = \frac{\delta}{\beta}\bar{B}^2(t) + \left(\delta - \frac{1}{\beta}\right)\bar{B}(t),$$

then, with the boundary condition $\bar{B}(T) = \frac{1}{\delta}$, we have the solution

$$\bar{B}(t) =: \frac{1}{\delta}\frac{\delta\beta - 1}{\delta\beta e^{\left(\delta - \frac{1}{\beta}\right)(T-t)} - 1} > 0, \quad 0 < t < T.$$

$\bar{B}'(t) > 0$ as it is a strictly increasing function of time t. We have

$$\begin{aligned} \mathrm{CIR}: P\left\{T_L^* < T \middle| \lambda_0\right\} &= \mathbb{E}\left[e^{-\frac{1}{\delta}\lambda_T} \middle| \lambda_0\right] \\ &= \mathbb{E}\left[e^{-\bar{B}(T)\lambda_T} \middle| \lambda_0\right] \\ &= e^{-\bar{B}(0)\lambda_0} \\ &= \exp\left(-\frac{1}{\delta}\frac{\delta\beta - 1}{\delta\beta e^{\left(\delta - \frac{1}{\beta}\right)T} - 1}\lambda_0\right). \end{aligned}$$

□

Remark 5.2.2. For DCP case, by setting $v = \frac{1}{\delta}$, $a = 0$, $\rho = 0$ in *Theorem 2.2.2*, we have

$$\mathrm{DCP}: P\left\{T_L^* < T \middle| \lambda_0\right\} = \mathbb{E}\left[e^{-\frac{1}{\delta}\lambda_T} \middle| \lambda_0\right] = \exp\left(-\mathcal{G}^{-1}_{v=\frac{1}{\delta},1}(T)\lambda_0\right),$$

where

$$\mathcal{G}_{v=\frac{1}{\delta},1}(L) =: \int_L^{v=\frac{1}{\delta}} \frac{\mathrm{d}u}{\delta u + \hat{g}(u) - 1};$$

in particular, when $G \sim \mathrm{Exp}(\beta)$, by *Lemma 2.3.1*, we have

$$\mathcal{G}_{v,1}(L) = \frac{1}{\delta(\delta\beta - 1)}\left[\delta\beta \ln\left(\frac{v}{L}\right) - \ln\left(\frac{\delta v + (\delta\beta - 1)}{\delta L + (\delta\beta - 1)}\right)\right],$$

which can not be explicitly inverse. Hence, it is hard to compare with the CIR case directly. Below, we alternatively adopt the super-martingale method for this comparison.

Theorem 5.2.8. *For the decaying intensity case, we have*

$$\text{DCP}: P\left\{T_L^* < T \middle| \lambda_0\right\} \quad < \quad \text{CIR}: P\left\{T_L^* < T \middle| \lambda_0\right\}, \quad 0 < T < \infty.$$

Proof. Plug $f(\lambda, t) = e^{-\bar{B}(t)\lambda}$ into DCP's generator, then, for all λ and $t > 0$,

$$\begin{aligned}
\text{DCP}: \mathcal{A}\left(e^{-\bar{B}(t)\lambda}\right) &= \lambda e^{-\bar{B}(t)\lambda}\left[-\bar{B}'(t) + \delta\bar{B}(t) + \frac{\beta}{\beta + \bar{B}(t)} - 1\right] \\
&= \lambda e^{-\bar{B}(t)\lambda}\left[-\bar{B}'(t) + \frac{\delta}{\beta + \bar{B}(t)}\bar{B}(t)\left(\bar{B}(t) + \left(\beta - \frac{1}{\delta}\right)\right)\right] \\
&= \lambda e^{-\bar{B}(t)\lambda}\left[-\bar{B}'(t) + \frac{\beta\bar{B}'(t)}{\beta + \bar{B}(t)}\right] \\
&= -\lambda e^{-\bar{B}(t)\lambda}\frac{\bar{B}(t)}{\beta + \bar{B}(t)}\bar{B}'(t) < 0,
\end{aligned}$$

as $\bar{B}(t) > 0, \bar{B}'(t) > 0$. Therefore, $e^{-\bar{B}(t)\lambda_t}$ is a super-martingale in the DCP case, and we have a martingale

$$e^{-\bar{B}(t)\lambda_t} - e^{-\bar{B}(0)\lambda_0} - \int_0^t \mathcal{A}\left(e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s,$$

then,

$$\begin{aligned}
\text{DCP}: P\left\{T_L^* < T \middle| \lambda_0\right\} &= \mathbb{E}\left[e^{-\bar{B}(T)\lambda_T} \middle| \lambda_0\right] \\
&= e^{-\bar{B}(0)\lambda_0} + \mathbb{E}\left[\int_0^T \mathcal{A}\left(e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s \middle| \lambda_0\right] \\
&< e^{-\bar{B}(0)\lambda_0} = \text{CIR}: P\left\{T_L^* < T \middle| \lambda_0\right\},
\end{aligned}$$

since

$$\mathbb{E}\left[\int_0^T \mathcal{A}\left(e^{-\bar{B}(s)\lambda_s}\right)\mathrm{d}s \middle| \lambda_0\right] < 0.$$

□

5.3 Stationary Intensity Case

This section provides an example of two different Markov processes sharing an identical asymptotic and stationary distribution.

5.3.1 Asymptotic Distribution of λ_t

Theorem 5.3.1. *For the stationary intensity case, DCP and CIR have the same asymptotic distribution of λ_t, i.e.*

$$\text{DCP}: \lim_{T\to\infty}\mathbb{E}\left[e^{-v\lambda_T} \middle| \lambda_0\right] = \text{CIR}: \lim_{T\to\infty}\mathbb{E}\left[e^{-v\lambda_T} \middle| \lambda_0\right].$$

Proof. For DCP, by setting $a = 0$ and $\alpha = \beta$ in *Theorem 2.3.1*, we have the Gamma distribution

$$\text{DCP}: \lambda_\infty \sim \text{Gamma}\left(\frac{\rho}{\delta}, \frac{\delta\beta - 1}{\delta}\right),$$

with the Laplace transform give by

$$\text{DCP}: \lim_{T\to\infty}\mathbb{E}\left[e^{-v\lambda_T} \middle| \lambda_0\right] = \left(\frac{\frac{\delta\beta-1}{\delta}}{v + \frac{\delta\beta-1}{\delta}}\right)^{\frac{\rho}{\delta}}, \tag{5.15}$$

which is also the Laplace transform of the stationary distribution of $\{\lambda_t\}_{t\geq 0}$.

On the other hand, the infinitesimal generator of a general CIR for the process (λ_t, t) is given by

$$\text{CIR}: \mathcal{A}f(\lambda,t) = \frac{\partial f}{\partial t} - \kappa(\lambda-\mu)\frac{\partial f}{\partial \lambda} + \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial \lambda^2}.$$

Assume the form $f(\lambda,t) = e^{c(t)}e^{-B(t)\lambda}$, and let $\mathcal{A}f(\lambda,t)=0$, we have

$$\Big(-B'(t) + \kappa B(t) + \frac{1}{2}\sigma^2 B^2(t)\Big)\lambda + \Big(c'(t) - \kappa\mu B(t)\Big) = 0,$$

holding for any λ, then,

$$\begin{aligned} B'(t) &= B(t)\left(\kappa + \frac{1}{2}\sigma^2 B(t)\right), \\ c'(t) &= \kappa\mu B(t). \end{aligned}$$

With boundary condition $B(T) = v > 0$,we have

$$B(t) = \frac{\frac{2}{\sigma^2}\kappa v}{\left(v + \frac{2}{\sigma^2}\kappa\right)e^{\kappa(T-t)} - v},$$

and

$$B(0) = \frac{\frac{2}{\sigma^2}\kappa v}{\left(v + \frac{2}{\sigma^2}\kappa\right)e^{\kappa T} - v},$$

then,

$$\lim_{T\to\infty} B(0) = 0.$$

Also, we have

$$\begin{aligned} c(T) - c(0) &= \int_0^T \kappa\mu B(t)\mathrm{d}t \\ &= \kappa\mu\frac{2}{\sigma^2}\kappa\int_0^T \frac{1}{\left(1+\frac{1}{v}\frac{2}{\sigma^2}\kappa\right)e^{\kappa T}e^{-\kappa t} - 1}\mathrm{d}t \qquad \left(u = e^{\kappa t}\right) \\ &= \kappa\mu\frac{2}{\sigma^2}\kappa\int_1^{e^{\kappa T}} \frac{1}{\left(1+\frac{1}{v}\frac{2}{\sigma^2}\kappa\right)e^{\kappa T} - u}\mathrm{d}u \\ &= -\kappa\mu\frac{2}{\sigma^2}\ln\left[\frac{\left(1+\frac{1}{v}\frac{2}{\sigma^2}\kappa\right)e^{\kappa T} - e^{\kappa T}}{\left(1+\frac{1}{v}\frac{2}{\sigma^2}\kappa\right)e^{\kappa T} - 1}\right], \end{aligned}$$

and

$$\lim_{T\to\infty} e^{-[c(T)-c(0)]} = \exp\left(\kappa\mu\frac{2}{\sigma^2}\ln\left[\frac{\frac{2}{\sigma^2}\kappa}{v+\frac{2}{\sigma^2}\kappa}\right]\right) = \left(\frac{\frac{2}{\sigma^2}\kappa}{v+\frac{2}{\sigma^2}\kappa}\right)^{\frac{2}{\sigma^2}\kappa\mu}.$$

Note that,

$$\mathbb{E}\left[e^{c(T)}e^{-B(T)\lambda_T}\middle|\lambda_0\right] = e^{c(0)}e^{-B(0)\lambda_0},$$

or,

$$\mathbb{E}\left[e^{-B(T)\lambda_T}\middle|\lambda_0\right] = e^{-[c(T)-c(0)]}e^{-B(0)\lambda_0},$$

then,

$$\text{CIR}: \lim_{T\to\infty}\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right] = \left(\frac{\frac{2}{\sigma^2}\kappa}{v+\frac{2}{\sigma^2}\kappa}\right)^{\frac{2}{\sigma^2}\kappa\mu},$$

which is also the Laplace transform of the stationary distribution of $\{\lambda_t\}_{t\geq 0}$ (with proof given by *Theorem 5.3.2*), and

$$\text{CIR}: \lambda_\infty \sim \text{Gamma}\left(\frac{2}{\sigma^2}\kappa\mu, \frac{2}{\sigma^2}\kappa\right).$$

Compare with DCP given by (5.15), if we set

$$\mu = \frac{\rho}{\delta\beta - 1}, \quad \frac{2}{\sigma^2}\kappa = \frac{\delta\beta - 1}{\delta},$$

given the parameters (5.1) and (5.2) derived from Section 5.2 for κ and σ^2, we have

$$\mu = \frac{\rho}{\delta\beta - 1}, \quad \kappa = \delta - \frac{1}{\beta}, \quad \sigma^2 = 2\frac{\delta}{\beta}.$$

Hence, DCP and CIR share the same asymptomatic (Gamma) distribution of λ_t.

□

5.3.2 Stationary Distribution of λ_t

Theorem 5.3.2. *For the stationary intensity case, DCP and CIR have the same stationary distribution of λ_t, i.e.*

$$\text{DCP}: \mathbb{E}\left[e^{-v\lambda_t}\right] = \text{CIR}: \mathbb{E}\left[e^{-v\lambda_t}\right].$$

Proof. We adopt the same approach as the proof for *Theorem 2.2.3* to prove the stationarity. If λ follows a distribution with density $\Pi(\lambda)$, we have

$$\mathbb{E}\big[\mathcal{A}(\lambda)\big] = \int_0^\infty \mathcal{A}(\lambda)\Pi(\lambda)\mathrm{d}\lambda = \int_0^\infty \left[-\kappa(\lambda-\mu)\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} + \frac{1}{2}\sigma^2\lambda\frac{\mathrm{d}^2 f(\lambda)}{\mathrm{d}\lambda^2}\right]\Pi(\lambda)\mathrm{d}\lambda.$$

Since

$$\begin{aligned}
\int_0^\infty \left[-\kappa(\lambda-\mu)\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\right]\Pi(\lambda)\mathrm{d}\lambda &= -\kappa\int_0^\infty f'(\lambda)(\lambda-\mu)\Pi(\lambda)\mathrm{d}\lambda \qquad \because \Pi(0)=0 \\
&= -\kappa\int_{\lambda=0}^\infty f'(\lambda)\int_{u=0}^\lambda \left[(u-\mu)\Pi(u)\right]'\mathrm{d}u\mathrm{d}\lambda \\
&= -\kappa\int_{\lambda=0}^\infty \int_{u=0}^\lambda f'(\lambda)\left[(u-\mu)\Pi(u)\right]'\mathrm{d}u\mathrm{d}\lambda \\
&= -\kappa\int_{u=0}^\infty \int_{\lambda=u}^\infty f'(\lambda)\left[(u-\mu)\Pi(u)\right]'\mathrm{d}\lambda\mathrm{d}u \\
&= -\kappa\int_{u=0}^\infty \int_{\lambda=u}^\infty f'(\lambda)\mathrm{d}\lambda\left[(u-\mu)\Pi(u)\right]'\mathrm{d}u \qquad \because f(\infty)=0 \\
&= \kappa\int_0^\infty f(u)\Big[(u-\mu)\Pi(u)\Big]'\mathrm{d}u \\
&= \kappa\int_0^\infty f(\lambda)\Big[(\lambda-\mu)\Pi(\lambda)\Big]'\mathrm{d}\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \left[\frac{1}{2}\sigma^2\lambda\frac{\mathrm{d}^2 f(\lambda)}{\mathrm{d}\lambda^2}\right]\Pi(\lambda)\mathrm{d}\lambda &= \frac{1}{2}\sigma^2\int_0^\infty \lambda\frac{\mathrm{d}^2 f(\lambda)}{\mathrm{d}\lambda^2}\Pi(\lambda)\mathrm{d}\lambda \\
&= \frac{1}{2}\sigma^2\int_0^\infty f''(\lambda)\lambda\Pi(\lambda)\mathrm{d}\lambda \\
&= \frac{1}{2}\sigma^2\int_{\lambda=0}^\infty f''(\lambda)\int_{u=0}^\lambda \Big[u\Pi(u)\Big]'\mathrm{d}u\mathrm{d}\lambda \\
&= \frac{1}{2}\sigma^2\int_{\lambda=0}^\infty \int_{u=0}^\lambda f''(\lambda)\Big[u\Pi(u)\Big]'\mathrm{d}u\mathrm{d}\lambda \\
&= \frac{1}{2}\sigma^2\int_{u=0}^\infty \int_{\lambda=u}^\infty f''(\lambda)\Big[u\Pi(u)\Big]'\mathrm{d}\lambda\mathrm{d}u \\
&= \frac{1}{2}\sigma^2\int_{u=0}^\infty \int_{\lambda=u}^\infty f''(\lambda)\mathrm{d}\lambda\Big[u\Pi(u)\Big]'\mathrm{d}u \qquad \because f'(\infty)=0 \\
&= -\frac{1}{2}\sigma^2\int_{u=0}^\infty f'(u)\Big[u\Pi(u)\Big]'\mathrm{d}u \\
&= -\frac{1}{2}\sigma^2\int_{u=0}^\infty \Big[u\Pi(u)\Big]'\mathrm{d}f(u) \\
&= -\frac{1}{2}\sigma^2\left[f(u)\Big[u\Pi(u)\Big]'\Big|_{u=0}^{u=\infty} - \int_{u=0}^\infty f(u)\Big[u\Pi(u)\Big]''\mathrm{d}u\right] \\
&= \frac{1}{2}\sigma^2\int_{u=0}^\infty f(u)\Big[u\Pi(u)\Big]''\mathrm{d}u \\
&= \frac{1}{2}\sigma^2\int_0^\infty f(\lambda)\Big[\lambda\Pi(\lambda)\Big]''\mathrm{d}\lambda,
\end{aligned}$$

we have

$$\mathbb{E}\Big[\mathcal{A}(\lambda)\Big] = \int_0^\infty f(\lambda)\Big[\kappa\Big[(\lambda-\mu)\Pi(\lambda)\Big]' + \frac{1}{2}\sigma^2\Big[\lambda\Pi(\lambda)\Big]''\Big]\mathrm{d}\lambda.$$

Set $\mathbb{E}\Big[\mathcal{A}(\lambda)\Big]=0$ for any $f\in\Omega(\mathcal{A})$, we have

$$\kappa\Big[(\lambda-\mu)\Pi(\lambda)\Big]' + \frac{1}{2}\sigma^2\Big[\lambda\Pi(\lambda)\Big]'' = 0.$$

By Laplace transform

$$\hat{\Pi}(v) =: \mathcal{L}\{\Pi(\lambda)\} = \int_0^\infty \Pi(\lambda)e^{-v\lambda}\mathrm{d}\lambda,$$

$$\begin{aligned}
\mathcal{L}\Big\{\Big[(\lambda-\mu)\Pi(\lambda)\Big]'\Big\} &= v\mathcal{L}\Big\{(\lambda-\mu)\Pi(\lambda)\Big\} = v\left(-\hat{\Pi}'(v)-\mu\hat{\Pi}(v)\right), \\
\mathcal{L}\Big\{\Big[\lambda\Pi(\lambda)\Big]''\Big\} &= -v^2\hat{\Pi}'(v),
\end{aligned}$$

we have the ODE of $\hat{\Pi}(v)$,

$$\kappa v\left(-\hat{\Pi}'(v)-\mu\hat{\Pi}(v)\right) - \frac{1}{2}\sigma^2v^2\hat{\Pi}'(v) = 0,$$

rewrite as

$$\hat{\Pi}'(v) + \frac{\kappa\mu}{\kappa+\frac{1}{2}\sigma^2 v}\hat{\Pi}(v) = 0,$$

with boundary condition $\hat{\Pi}(0)=1$, we have

$$\text{CIR}: \hat{\Pi}(v) = \left(\frac{\frac{2}{\sigma^2}\kappa}{\frac{2}{\sigma^2}\kappa+v}\right)^{\frac{2}{\sigma^2}\kappa\mu}.$$

□

6

A Dynamic Contagion Process with Diffusion

In this chapter, we investigate a dynamic contagion process with diffusion (DCPD), an extension of the original dynamic contagion process (DCP) introduced by Chapter 2 with the intensity process perturbed by diffusion. However, DCPD is a point process that cannot be classified as a branching process as DCP defined in *Definition 2.1.1*, so, rather than combining the results of DCP and DCPD within a single chapter, here we seperately derive the Laplace transform of the intensity process and the probability generating function of the point process for DCPD.

6.1 Introduction

The infinitesimal generator of the dynamic contagion process with diffusion (λ_t, N_t, t) acting on a function $f(\lambda, n, t)$ within its domain $\Omega(\mathcal{A})$ is given by

$$\begin{aligned}\mathcal{A}f(\lambda,n,t) \quad = \quad & \frac{\partial f}{\partial t} - \delta\left(\lambda - a\right)\frac{\partial f}{\partial \lambda} + \rho\left(\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y) - f(\lambda,n,t)\right) \\ & +\lambda\left(\int_0^\infty f(\lambda+z,n+1,t)\mathrm{d}G(z) - f(\lambda,n,t)\right) + \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial\lambda^2}, \end{aligned} \tag{6.1}$$

where constant $\sigma > 0$ is the volatility of the intensity process perturbed by diffusion and $\Omega(\mathcal{A})$ is the domain for the generator $\mathcal{A}$ such that $f(\lambda, n, t)$ is differentiable with respect to λ, t for all λ, n and t, and

$$\begin{aligned}\left|\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y) - f(\lambda,n,t)\right| \quad & < \quad \infty, \\ \left|\int_0^\infty f(\lambda+z,n+1,t)\mathrm{d}G(z) - f(\lambda,n,t)\right| \quad & < \quad \infty.\end{aligned}$$

6.2 Distributional Properties

6.2.1 Joint Laplace Transform - Probability Generating Function of (λ_T, N_T)

Theorem 6.2.1. *For the constants $0 \le \theta \le 1$, $v \ge 0$ and time $0 \le t \le T$, we have the conditional joint Laplace transform - probability generating function for the process λ_t and the point process*

N_t,

$$\mathbb{E}\Big[\theta^{(N_T-N_t)}e^{-v\lambda_T}\Big|\mathcal{F}_t\Big] = e^{-\big(c(T)-c(t)\big)}e^{-B(t)\lambda_t}, \tag{6.2}$$

where $B(t)$ is determined by the non-linear ODE

$$-B'(t)+\delta B(t)+\theta\hat{g}\big(B(t)\big)-1+\frac{1}{2}\sigma^2B^2(t)=0, \tag{6.3}$$

$$\hat{g}(u) =: \int_0^\infty e^{-uz}\mathrm{d}G(z),$$

with boundary condition $B(T)=v$; and $c(t)$ is determined by

$$c(t) = a\delta\int_0^t B(s)\mathrm{d}s + \rho\int_0^t \Big[1-\hat{h}\big(B(s)\big)\Big]\,\mathrm{d}s, \tag{6.4}$$

$$\hat{h}(u) =: \int_0^\infty e^{-uy}\mathrm{d}H(y).$$

Proof. Consider a function $f(\lambda,n,t)$ with an exponential affine form

$$f(\lambda,n,t) = e^{c(t)}A^n(t)e^{-B(t)\lambda},$$

substitute into $\mathcal{A}f=0$ in (6.1); we then have

$$\frac{A'(t)}{A(t)}n + \Big(-B'(t)+\delta B(t)+A(t)\hat{g}(B(t))-1+\frac{1}{2}\sigma^2B^2(t)\Big)\lambda$$
$$+\Big(c'(t)+\rho\hat{h}(B(t))-\rho-a\delta B(t)\Big)=0. \tag{6.5}$$

Since this equation holds for any n and λ, it is equivalent to solving three separated equations

$$\begin{cases} \frac{A'(t)}{A(t)}=0 & (.1)\\ -B'(t)+\delta B(t)+A(t)\hat{g}(B(t))-1+\frac{1}{2}\sigma^2B^2(t)=0 & (.2)\\ c'(t)+\rho\hat{h}(B(t))-\rho-a\delta B(t)=0 & (.3)\end{cases}. \tag{6.6}$$

We have $A(t)=\theta$ immediately from (6.6.1); and substitute into (6.6.2) by adding the boundary condition $B(T)=v$, we have the ODE as (6.3); then, by (6.6.3) with boundary condition $c(0)=0$, the integration as (6.4) follows. Since $e^{c(t)}\theta^{N_t}e^{-B(t)\lambda_t}$ is a $\mathcal{F}-$martingale by the property of the infinitesimal generator, we have

$$\mathbb{E}\Big[e^{c(T)}\theta^{N_T}e^{-B(T)\lambda_T}\Big|\mathcal{F}_t\Big] = e^{c(t)}\theta^{N_t}e^{-B(t)\lambda_t}. \tag{6.7}$$

Then, by the boundary condition $B(T)=v$, (6.2) follows. □

6.2.2 Laplace Transform of λ_T

Theorem 6.2.2. *The conditional Laplace transform λ_T given λ_0 at time $t=0$, under the condition $\delta>\mu_{1_G}$, is given by*

$$\mathbb{E}\left[e^{-v\lambda_T}\big|\lambda_0\right] = \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^{v}\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}\mathrm{d}u\right)\times\exp\left(-\mathcal{G}_{v,1}^{-1}(T)\lambda_0\right), \tag{6.8}$$

where

$$\mu_{1_G} =: \int_0^\infty z\mathrm{d}G(z),$$

$$\mathcal{G}_{v,1}(L) =: \int_L^v \frac{\mathrm{d}u}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}\,[1].$$

[1] It will be clear in the proof later that $\mathcal{G}_{v,1}(L)$ is a one by one function of L and hence its inverse function $\mathcal{G}_{v,1}^{-1}(T)$ exsits.

Proof. By setting $t=0$ and $\theta=1$ in *Theorem 6.2.1*, we have

$$\mathbb{E}\left[e^{-v\lambda_T}\middle|\mathcal{F}_0\right]=e^{-c(T)}e^{-B(0)\lambda_0}, \tag{6.9}$$

where $B(0)$ is uniquely determined by the non-linear ODE

$$-B'(t)+\delta B(t)+\hat{g}\Big(B(t)\Big)-1+\frac{1}{2}\sigma^2B^2(t)=0,$$

with boundary condition $B(T)=v$. It can be solved, under the condition $\delta>\mu_{1_G}$, by the following steps:

1. Set $B(t)=L(T-t)$ and $\tau=T-t$, it is equivalent to the initial value problem
$$\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau}=1-\delta L(\tau)-\hat{g}(L(\tau))-\frac{1}{2}\sigma^2L^2(\tau)=:f_1(L), \tag{6.10}$$
with initial condition $L(0)=v$; we define the right-hand side as the function $f_1(L)$.

2. Under the condition $\delta>\mu_{1_G}$, we have
$$\frac{\partial f_1(L)}{\partial L}=\int_0^\infty ye^{-Lz}\mathrm{d}G(z)-\delta-\sigma^2L\le\int_0^\infty z\mathrm{d}G(z)-\delta=\mu_{1_G}-\delta<0,\quad \text{for } L\ge 0,$$
then, $f_1(L)<0$ for $L>0$, since $f_1(0)=0$.

3. Rewrite (6.10) as
$$\frac{\mathrm{d}L}{\delta L+\hat{g}(L)-1+\frac{1}{2}\sigma^2L^2}=-\mathrm{d}\tau,$$
by integrating both sides from time 0 to τ with initial condition $L(0)=v>0$, we have
$$\int_L^v\frac{\mathrm{d}u}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}=\tau,$$
where $L\ge 0$, we define the function on left hand side as
$$\mathcal{G}_{v,1}(L)=:\int_L^v\frac{\mathrm{d}u}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2},$$
then,
$$\mathcal{G}_{v,1}(L)=\tau,$$
obviously $L\to v$ when $\tau\to 0$; by convergence test,
$$\lim_{u\to 0}\frac{\frac{1}{u}}{\frac{1}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}}=\delta+\lim_{u\to 0}\frac{\hat{g}(u)-1}{u}=\delta-\mu_{1_G}>0,$$
and we know that $\int_0^v\frac{1}{u}\mathrm{d}u=\infty$, then,
$$\int_0^v\frac{\mathrm{d}u}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}=\infty,$$
hence, $L\to 0$ when $\tau\to\infty$; the integrand is positive in the domain $u\in(0,v]$ and also for $L\le v$, $\mathcal{G}_{v,1}(L)$ is a strictly decreasing function; therefore, $\mathcal{G}_{v,1}(L):(0,v]\to[0,\infty)$ is a well defined (monotone) function, and its inverse function $\mathcal{G}_{v,1}^{-1}(\tau):[0,\infty)\to(0,v]$ exists.

4. The unique solution is found by
$$L(\tau)=\mathcal{G}_{v,1}^{-1}(\tau),\quad \text{or,}\quad B(t)=\mathcal{G}_{v,1}^{-1}(T-t).$$

5. $B(0)$ is obtained,

$$B(0) = L(T) = \mathcal{G}_{v,1}^{-1}(T).$$

Then, $c(T)$ is determined by

$$c(T) = a\delta \int_0^T \mathcal{G}_{v,1}^{-1}(\tau)\mathrm{d}\tau + \rho \int_0^T \left[1 - \hat{h}\left(\mathcal{G}_{v,1}^{-1}(\tau)\right)\right]\mathrm{d}\tau, \tag{6.11}$$

by the change of variable $\mathcal{G}_{v,1}^{-1}(\tau) = u$, we have $\tau = \mathcal{G}_{v,1}(u)$, and

$$\int_0^T \left[1 - \hat{h}\left(\mathcal{G}_{v,1}^{-1}(\tau)\right)\right]\mathrm{d}\tau = \int_{\mathcal{G}_{v,1}^{-1}(0)}^{\mathcal{G}_{v,1}^{-1}(T)} [1 - \hat{h}(u)]\frac{\partial \tau}{\partial u}\mathrm{d}u = \int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{1 - \hat{h}(u)}{\delta u + \hat{g}(u) - 1 + \frac{1}{2}\sigma^2 u^2}\mathrm{d}u,$$

similarly,

$$\int_0^T \mathcal{G}_{v,1}^{-1}(\tau)\mathrm{d}\tau = \int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{u}{\delta u + \hat{g}(u) - 1 + \frac{1}{2}\sigma^2 u^2}\mathrm{d}u.$$

Finally, substitute $B(0)$ and $c(T)$ into (6.9), and *Theorem 6.2.2* follows. □

Theorem 6.2.3. *If $\delta > \mu_{1_G}$ and $\frac{2\delta}{\sigma^2}a > 1$, then the Laplace transform of the asymptotic distribution of λ_T is given by*

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right] = \exp\left(-\int_0^v \frac{a\delta u + \rho[1 - \hat{h}(u)]}{\delta u + \hat{g}(u) - 1 + \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right), \tag{6.12}$$

and this is also the Laplace transform of the stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$.

Proof. Let $T \to \infty$ in *Theorem 6.2.2*, then $\mathcal{G}_{v,1}^{-1}(T) \to 0$ and the Laplace transform of the asymptotic distribution follows immediately as given by (6.12).

To further prove the stationarity, by *Proposition 9.2* of Ethier and Kurtz (1986) (and see also Costa (1990)), we need to prove that, for any function f within its domain $\Omega(\mathcal{A})$, we have

$$\int_E \mathcal{A}f(\lambda)\Pi(\lambda)\mathrm{d}\lambda = 0, \tag{6.13}$$

where $E = [0, \infty)$ is the domain for λ, $\mathcal{A}f(\lambda)$ is the infinitesimal generator of this process acting on $f(\lambda)$, i.e.

$$\begin{aligned}\mathcal{A}f(\lambda) \quad = \quad & -\delta\lambda\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} + \delta a\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} + \rho\left(\int_0^\infty f(\lambda + y)\mathrm{d}H(y) - f(\lambda)\right) \\ & +\lambda\left(\int_0^\infty f(\lambda + z)\mathrm{d}G(z) - f(\lambda)\right) + \frac{1}{2}\sigma^2\lambda\frac{\mathrm{d}^2 f(\lambda)}{\mathrm{d}\lambda^2},\end{aligned} \tag{6.14}$$

and $\Pi(\lambda)$ is the density function of λ with the Laplace transform given by (6.12).

Note that, for the density function Π, since $\int_0^\infty \Pi(u)\mathrm{d}u = 1$, we have

$$\begin{aligned}\lim_{u\to\infty} u\Pi(u) \quad &= \quad 0, \\ \lim_{u\to\infty} [u\Pi(u)]' \quad &= \quad 0.\end{aligned}$$

By convergence test,

$$\lim_{u\to\infty} \frac{\frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1 + \frac{1}{2}\sigma^2 u^2}}{\frac{a\delta}{\delta + \frac{1}{2}\sigma^2 u}} = \lim_{u\to\infty} \frac{u + \frac{\rho}{a\delta}[1 - \hat{h}(u)]}{u + \frac{\hat{g}(u)-1}{\delta+\frac{1}{2}\sigma^2 u}} = \lim_{u\to\infty} \frac{\left(u + \frac{\rho}{a\delta}[1 - \hat{h}(u)]\right)'}{\left(u + \frac{\hat{g}(u)-1}{\delta+\frac{1}{2}\sigma^2 u}\right)'} = 1,$$

since

$$\lim_{v\to\infty}\int_0^v \frac{a\delta}{\delta+\frac{1}{2}\sigma^2 u}\mathrm{d}u=\infty,$$

we have

$$\lim_{v\to\infty}\int_0^v \frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}=\infty.$$

For constant $\epsilon>0$,

$$\begin{aligned}
& \epsilon^{-\frac{2\delta}{\sigma^2}a}\times\lim_{v\to\infty} v^{\frac{2\delta}{\sigma^2}a}\hat{\Pi}(v) \\
= & \lim_{v\to\infty}\exp\left(-\int_0^v \frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}\mathrm{d}u+\frac{2\delta}{\sigma^2}a\int_\epsilon^v\frac{1}{u}\mathrm{d}u\right) \\
= & \exp\left(-\int_0^\epsilon \frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}\mathrm{d}u\right) \\
& \times\lim_{v\to\infty}\exp\left(-\int_\epsilon^v \frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}\mathrm{d}u+\frac{2\delta}{\sigma^2}a\int_\epsilon^v\frac{1}{u}\mathrm{d}u\right) \\
= & \hat{\Pi}(\epsilon)\times\lim_{v\to\infty}\exp\left(-\int_\epsilon^v\left[\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}-\frac{\frac{2\delta}{\sigma^2}a}{u}\right]\mathrm{d}u\right) \\
= & \hat{\Pi}(\epsilon)\times\lim_{v\to\infty}\exp\left(-\int_\epsilon^v\frac{\rho[1-\hat{h}(u)]u-\frac{2\delta}{\sigma^2}a\left[\delta u-\hat{g}(u)-1\right]}{\left(\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2\right)u}\mathrm{d}u\right) \\
= & \hat{\Pi}(\epsilon)\times\mathcal{E}(\epsilon),
\end{aligned}$$

where

$$\mathcal{E}(\epsilon)=:\exp\left(-\int_\epsilon^\infty\frac{\rho[1-\hat{h}(u)]u-\frac{2\delta}{\sigma^2}a\left[\delta u-\hat{g}(u)-1\right]}{\left(\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2\right)u}\mathrm{d}u\right)<\infty,$$

hence,

$$\lim_{v\to\infty} v^{\frac{2\delta}{\sigma^2}a}\hat{\Pi}(v)=\epsilon^{\frac{2\delta}{\sigma^2}a}\hat{\Pi}(\epsilon)\mathcal{E}(\epsilon),$$

i.e.

$$\hat{\Pi}(v)\sim\epsilon^{\frac{2\delta}{\sigma^2}a}\hat{\Pi}(\epsilon)\mathcal{E}(\epsilon)\times v^{-\frac{2\delta}{\sigma^2}a},\quad v\to\infty,$$

by *Tauberian Theorem*, we have

$$\Pi(\lambda)\sim\frac{\epsilon^{\frac{2\delta}{\sigma^2}a}\hat{\Pi}(\epsilon)\mathcal{E}(\epsilon)}{\Gamma\left(\frac{2\delta}{\sigma^2}a\right)}\times\lambda^{\left(\frac{2\delta}{\sigma^2}a-1\right)},\quad\lambda\to 0.$$

Note that a constant is a slowly varying function. Then, if $\frac{2\delta}{\sigma^2}a>1$, we have

$$\begin{aligned}
\lim_{u\to 0}\Pi(u) &= 0, \\
\lim_{u\to 0}u\Pi(u) &= 0, \\
\lim_{u\to 0}[u\Pi(u)]' &= 0.
\end{aligned}$$

We will now try to solve equation (6.13).

For the first term of (6.13), we have

$$\begin{aligned}
\int_E \left[-\delta\lambda\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\right]\Pi(\lambda)\mathrm{d}\lambda &= -\delta\int_0^\infty f'(\lambda)[\lambda\Pi(\lambda)]\mathrm{d}\lambda \\
&= \delta\int_{\lambda=0}^\infty f'(\lambda)\int_{u=\lambda}^\infty [u\Pi(u)]'\mathrm{d}u\mathrm{d}\lambda \qquad \because [u\Pi(u)]\Big|_{u=\infty}=0 \\
&= \delta\int_{\lambda=0}^\infty\int_{u=\lambda}^\infty f'(\lambda)[u\Pi(u)]'\mathrm{d}u\mathrm{d}\lambda \\
&= \delta\int_{u=0}^\infty\int_{\lambda=u}^\infty f'(\lambda)[u\Pi(u)]'\mathrm{d}\lambda\mathrm{d}u \\
&= \delta\int_{u=0}^\infty\int_{\lambda=u}^\infty f'(\lambda)\mathrm{d}\lambda[u\Pi(u)]'\mathrm{d}u \\
&= \delta\int_0^\infty \Big[f(u)-f(0)\Big][u\Pi(u)]'\mathrm{d}u \\
&= \delta\int_0^\infty f(u)[u\Pi(u)]'\mathrm{d}u - \delta f(0)[u\Pi(u)]\Big|_{u=0}^{u=\infty} \qquad \because [u\Pi(u)]\Big|_{u=0}^{u=\infty}=0 \\
&= \delta\int_0^\infty f(u)[u\Pi(u)]'\mathrm{d}u.
\end{aligned}$$

For the second term of (6.13), we have

$$\begin{aligned}
\int_E \left[\delta a\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\right]\Pi(\lambda)\mathrm{d}\lambda &= \delta a\int_{\lambda=0}^\infty f'(\lambda)\Pi(\lambda)\mathrm{d}\lambda \\
&= -\delta a\int_{\lambda=0}^\infty f'(\lambda)\int_{u=\lambda}^\infty \Pi'(u)\mathrm{d}u\mathrm{d}\lambda \qquad \because \Pi(u)\Big|_{u=\infty}=0 \\
&= -\delta a\int_{\lambda=0}^\infty\int_{u=\lambda}^\infty f'(\lambda)\Pi'(u)\mathrm{d}u\mathrm{d}\lambda \\
&= -\delta a\int_{u=0}^\infty\int_{\lambda=0}^u f'(\lambda)\Pi'(u)\mathrm{d}u\mathrm{d}\lambda \\
&= -\delta a\int_{u=0}^\infty\int_{\lambda=0}^u f'(\lambda)\mathrm{d}\lambda\Pi'(u)\mathrm{d}u \\
&= -\delta a\int_{u=0}^\infty \Big[f(u)-f(0)\Big]\Pi'(u)\mathrm{d}u \\
&= -\delta a\int_{u=0}^\infty f(u)\Pi'(u)\mathrm{d}u + \delta a f(0)\int_{u=0}^\infty \Pi'(u)\mathrm{d}u \\
&= -\delta a\int_{u=0}^\infty f(u)\Pi'(u)\mathrm{d}u + \delta a f(0)\Pi(u)\Big|_{u=0}^{u=\infty} \qquad \because \Pi(u)\Big|_{u=\infty}=0 \\
&= -\delta a\int_{u=0}^\infty f(u)\Pi'(u)\mathrm{d}u + \delta a f(0)\Pi(0) \qquad \because \Pi(0)=0 \\
&= -\delta a\int_{u=0}^\infty f(u)\Pi'(u)\mathrm{d}u.
\end{aligned}$$

For the third term of (6.13), by change variable $\lambda+y=s$ $(y\le s)$ in the double integral,

$$\begin{aligned}
&\int_E \left[\rho\int_0^\infty f(\lambda+y)\mathrm{d}H(y)\right]\Pi(\lambda)\mathrm{d}\lambda \\
= \ &\rho\int_{\lambda=0}^\infty \Pi(\lambda)\int_{y=0}^\infty f(\lambda+y)\mathrm{d}H(y)\mathrm{d}\lambda = \rho\int_{s=0}^\infty f(s)\int_{y=0}^s \Pi(s-y)\mathrm{d}H(y)\mathrm{d}s,
\end{aligned}$$

or,

$$\int_E \left[\rho\int_0^\infty f(\lambda+y)\mathrm{d}H(y)\right]\Pi(\lambda)\mathrm{d}\lambda = \rho\int_{\lambda=0}^\infty f(\lambda)\int_{y=0}^\lambda \Pi(\lambda-y)\mathrm{d}H(y)\mathrm{d}\lambda.$$

For the fourth term of (6.13), by change variable $\lambda + z = s$ ($z \le s$) in the double integral,

$$\begin{aligned}
& \int_E \left[\lambda \left(\int_0^\infty f(\lambda+z)\mathrm{d}G(z)\right)\right]\Pi(\lambda)\mathrm{d}\lambda \\
= \; & \int_{\lambda=0}^\infty \lambda\Pi(\lambda)\int_{z=0}^\infty f(\lambda+z)\mathrm{d}G(z)\mathrm{d}\lambda = \int_{s=0}^\infty f(s)\int_{z=0}^s (s-z)\Pi(s-z)\mathrm{d}G(z)\mathrm{d}s,
\end{aligned}$$

or,

$$\int_E \left[\lambda \left(\int_0^\infty f(\lambda+z)\mathrm{d}G(z)\right)\right]\Pi(\lambda)\mathrm{d}\lambda = \int_{\lambda=0}^\infty f(\lambda)\int_{z=0}^\lambda (\lambda-z)\Pi(\lambda-z)\mathrm{d}G(z)\mathrm{d}\lambda.$$

For the fifth term of (6.13), we have

$$\begin{aligned}
& \int_E \left[\frac{1}{2}\sigma^2\lambda\frac{\mathrm{d}^2 f(\lambda)}{\mathrm{d}\lambda^2}\right]\Pi(\lambda)\mathrm{d}\lambda \\
= \; & \frac{1}{2}\sigma^2\int_0^\infty f''(\lambda)\lambda\Pi(\lambda)\mathrm{d}\lambda \\
= \; & -\frac{1}{2}\sigma^2\int_{\lambda=0}^\infty f''(\lambda)\int_{u=\lambda}^\infty [u\Pi(u)]'\mathrm{d}u\mathrm{d}\lambda \qquad \because [u\Pi(u)]\Big|_{u=\infty} = 0 \\
= \; & -\frac{1}{2}\sigma^2\int_{\lambda=0}^\infty \int_{u=\lambda}^\infty f''(\lambda)[u\Pi(u)]'\mathrm{d}u\mathrm{d}\lambda \\
= \; & -\frac{1}{2}\sigma^2\int_{u=0}^\infty \int_{\lambda=0}^u f''(\lambda)[u\Pi(u)]'\mathrm{d}\lambda\mathrm{d}u \\
= \; & -\frac{1}{2}\sigma^2\int_{u=0}^\infty \int_{\lambda=0}^u f''(\lambda)\mathrm{d}\lambda[u\Pi(u)]'\mathrm{d}u \\
= \; & -\frac{1}{2}\sigma^2\int_{u=0}^\infty \Big[f'(u) - f'(0)\Big][u\Pi(u)]'\mathrm{d}u \\
= \; & -\frac{1}{2}\sigma^2\int_{u=0}^\infty f'(u)[u\Pi(u)]'\mathrm{d}u + \frac{1}{2}\sigma^2 f'(0)\int_{u=0}^\infty [u\Pi(u)]'\mathrm{d}u \\
= \; & -\frac{1}{2}\sigma^2\int_{u=0}^\infty f'(u)[u\Pi(u)]'\mathrm{d}u + \frac{1}{2}\sigma^2 f'(0)[u\Pi(u)]\Big|_{u=0}^{u=\infty} \qquad \because [u\Pi(u)]\Big|_{u=0}^{u=\infty} = 0 \\
= \; & -\frac{1}{2}\sigma^2\int_{u=0}^\infty f'(u)[u\Pi(u)]'\mathrm{d}u \\
= \; & \frac{1}{2}\sigma^2\int_{u=0}^\infty f'(u)\int_{s=u}^\infty [s\Pi(s)]''\mathrm{d}s\mathrm{d}u \qquad \because [s\Pi(s)]'\Big|_{s=\infty} = 0 \\
= \; & \frac{1}{2}\sigma^2\int_{u=0}^\infty \int_{s=u}^\infty f'(u)[s\Pi(s)]''\mathrm{d}s\mathrm{d}u \\
= \; & \frac{1}{2}\sigma^2\int_{s=0}^\infty \int_{u=0}^s f'(u)[s\Pi(s)]''\mathrm{d}u\mathrm{d}s \\
= \; & \frac{1}{2}\sigma^2\int_{s=0}^\infty \int_{u=0}^s f'(u)\mathrm{d}u[s\Pi(s)]''\mathrm{d}s \\
= \; & \frac{1}{2}\sigma^2\int_{s=0}^\infty \Big[f(s) - f(0)\Big][s\Pi(s)]''\mathrm{d}s \\
= \; & \frac{1}{2}\sigma^2\int_{s=0}^\infty f(s)[s\Pi(s)]''\mathrm{d}s - \frac{1}{2}\sigma^2 f(0)\int_{s=0}^\infty [s\Pi(s)]''\mathrm{d}s \\
= \; & \frac{1}{2}\sigma^2\int_{s=0}^\infty f(s)[s\Pi(s)]''\mathrm{d}s - \frac{1}{2}\sigma^2 f(0)[s\Pi(s)]'\Big|_{s=0}^{s=\infty} \qquad \because [s\Pi(s)]'\Big|_{s=0}^{s=\infty} = 0 \\
= \; & \frac{1}{2}\sigma^2\int_{s=0}^\infty f(s)[s\Pi(s)]''\mathrm{d}s.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_E \mathcal{A}f(\lambda)\Pi(\lambda)\mathrm{d}\lambda \\
= \quad &\int_0^\infty f(\lambda)\Big[\delta\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big((\lambda-a)\Pi(\lambda)\Big)+\rho\left(\int_0^\lambda \Pi(\lambda-y)\mathrm{d}H(y)-\Pi(\lambda)\right) \\
&+\left(\int_0^\lambda(\lambda-z)\Pi(\lambda-z)\mathrm{d}G(z)-\lambda\Pi(\lambda)+\frac{1}{2}\sigma^2\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[\lambda\Pi(\lambda)\right]\right)\Big]\mathrm{d}\lambda.
\end{aligned}
$$

Set

$$
\int_E \mathcal{A}f(\lambda)\Pi(\lambda)\mathrm{d}\lambda = 0,
$$

for any function $f(\lambda)\in\Omega(\mathcal{A})$, then,

$$
\begin{aligned}
&\delta\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big((\lambda-a)\Pi(\lambda)\Big)+\rho\left(\int_0^\lambda \Pi(\lambda-y)\mathrm{d}H(y)-\Pi(\lambda)\right) \\
&+\left(\int_0^\lambda(\lambda-z)\Pi(\lambda-z)\mathrm{d}G(z)-\lambda\Pi(\lambda)+\frac{1}{2}\sigma^2\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[\lambda\Pi(\lambda)\right]\right)=0,
\end{aligned}
$$

by Laplace transform

$$
\hat{\Pi}(v) =: \mathcal{L}\left\{\Pi(\lambda)\right\} = \int_E \Pi(\lambda)e^{-v\lambda}\mathrm{d}\lambda,
$$

we have

$$
\begin{aligned}
\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big((\lambda-a)\Pi(\lambda)\Big)\right\} &= v\mathcal{L}\left\{(\lambda-a)\Pi(\lambda)\right\} = v\left(-\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}-a\hat{\Pi}(v)\right), \\
\mathcal{L}\left\{\int_0^\lambda \Pi(\lambda-y)\mathrm{d}H(y)\right\} &= \mathcal{L}\left\{\int_0^\lambda \Pi(\lambda-y)h(y)\mathrm{d}y\right\} = \hat{\Pi}(v)\hat{h}(v), \\
\mathcal{L}\left\{\int_0^\lambda(\lambda-z)\Pi(\lambda-z)\mathrm{d}G(z)\right\} &= \mathcal{L}\left\{\int_0^\lambda(\lambda-z)\Pi(\lambda-z)g(z)\mathrm{d}z\right\} \\
&= \mathcal{L}\left\{\lambda\Pi(\lambda)\right\}\hat{g}(v) = -\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}\hat{g}(v), \\
\mathcal{L}\left\{\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[\lambda\Pi(\lambda)\right]\right\} &= -v^2\frac{\mathrm{d}}{\mathrm{d}v}\hat{\Pi}(v),
\end{aligned}
$$

then,

$$
\delta v\left(-\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}-a\hat{\Pi}(v)\right)+\rho[\hat{h}(v)-1]\hat{\Pi}(v)+\left(1-\hat{g}(v)-\frac{1}{2}\sigma^2v^2\right)\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}=0,
$$

or,

$$
\left(1-\delta v-\hat{g}(v)-\frac{1}{2}\sigma^2v^2\right)\frac{\mathrm{d}\hat{\Pi}(v)}{\mathrm{d}v}+\Big(-a\delta v+\rho[\hat{h}(v)-1]\Big)\hat{\Pi}(v)=0,
$$

which is an ODE with the solution given by

$$
\hat{\Pi}(v)=\hat{\Pi}(0)\exp\left(-\int_0^v\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}\mathrm{d}u\right).
$$

Note that, given the initial condition

$$
\hat{\Pi}(0)=\int_E \Pi(\lambda)\mathrm{d}\lambda = 1,
$$

we have the unique solution

$$
\hat{\Pi}(v)=\exp\left(-\int_0^v\frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1+\frac{1}{2}\sigma^2u^2}\mathrm{d}u\right),
$$

which is exactly given by (6.12).

Since Π is the unique solution to (6.13), we have the stationarity for the intensity process $\{\lambda_t\}_{t\geq 0}$. □

Corollary 6.2.1. *For the case without externally excited and self-excited jumps, we have*

$$\left\{\lambda_t\right\}_{t\geq 0} \sim \text{Gamma}\left(\frac{2\delta}{\sigma^2}a, \frac{2\delta}{\sigma^2}\right).$$

Proof.

$$\hat{\Pi}(v) = \exp\left(-\int_0^v \frac{a\delta u}{\delta u + \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right) = \exp\left(-\frac{2\delta}{\sigma^2}a\int_0^v \frac{1}{\frac{2\delta}{\sigma^2}+u}\mathrm{d}u\right) = \left(\frac{\frac{2\delta}{\sigma^2}}{v+\frac{2\delta}{\sigma^2}}\right)^{\frac{2\delta}{\sigma^2}a}.$$

□

Corollary 6.2.2. *For the case without self-excited jumps and assume $H \sim \mathrm{Exp}(\alpha)$, we have*

$$\left\{\lambda_t\right\}_{t\geq 0} \sim \text{Gamma}\left(\frac{2\delta}{\sigma^2}a - \frac{2\rho}{2\delta-\alpha\sigma^2}, \frac{2\delta}{\sigma^2}\right) + \text{Gamma}\left(\frac{2\rho}{2\delta-\alpha\sigma^2}, \alpha\right).$$

Proof.

$$\begin{aligned}
\hat{\Pi}(v) &= \exp\left(-\int_0^v \frac{a\delta u + \rho\left[1-\frac{\alpha}{\alpha+u}\right]}{\delta u + \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right) \\
&= \exp\left(-\int_0^v \frac{a\delta u}{\delta u + \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right) \times \exp\left(-\int_0^v \frac{\rho\left[1-\frac{\alpha}{\alpha+u}\right]}{\delta u + \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right) \\
&= \left(\frac{\frac{2\delta}{\sigma^2}}{v+\frac{2\delta}{\sigma^2}}\right)^{\frac{2\delta}{\sigma^2}a} \times \left(\frac{\frac{2\delta}{\sigma^2}}{v+\frac{2\delta}{\sigma^2}}\right)^{-\frac{2\rho}{2\delta-\alpha\sigma^2}} \left(\frac{\alpha}{\alpha+v}\right)^{\frac{2\rho}{2\delta-\alpha\sigma^2}} \\
&= \left(\frac{\frac{2\delta}{\sigma^2}}{v+\frac{2\delta}{\sigma^2}}\right)^{\left(\frac{2\delta}{\sigma^2}a-\frac{2\rho}{2\delta-\alpha\sigma^2}\right)} \times \left(\frac{\alpha}{\alpha+v}\right)^{\frac{2\rho}{2\delta-\alpha\sigma^2}}.
\end{aligned}$$

□

Corollary 6.2.3. *For the case without externally excited jumps and assume $G \sim \mathrm{Exp}(\beta)$ and stationarity condition $\delta\beta > 1$, we have*

$$\hat{\Pi}(v) = \exp\left(-\frac{2\delta}{\sigma^2}a\int_0^v l(u)\mathrm{d}u\right),$$

where

$$l(u) =: \frac{\beta+u}{u^2+\left(\frac{2\delta}{\sigma^2}+\beta\right)u+\frac{2}{\sigma^2}(\delta\beta-1)}, \quad u\in[0,v],$$

and

$$\mathbb{E}[\lambda_t] = \frac{\delta\beta}{\delta\beta-1}a.$$

Proof.

$$\begin{aligned}
\hat{\Pi}(v) &= \exp\left(-\int_0^v \frac{a\delta u}{\delta u + \frac{\beta}{\beta+u} - 1 + \frac{1}{2}\sigma^2 u^2}\mathrm{d}u\right)\\
&= \exp\left(-\int_0^v \frac{a\delta}{\delta - \frac{1}{\beta+u} + \frac{1}{2}\sigma^2 u}\mathrm{d}u\right)\\
&= \exp\left(-\frac{2\delta}{\sigma^2}a\int_0^v \frac{\beta+u}{u^2 + \left(\frac{2\delta}{\sigma^2}+\beta\right)u + \frac{2}{\sigma^2}(\delta\beta-1)}\mathrm{d}u\right).
\end{aligned}$$

Note that,

$$l(0) = \frac{\sigma^2\beta}{2(\delta\beta-1)},$$

for $u \in [0, v]$, $l(u)$ is a positive strictly increasing function since

$$l'(u) = \frac{u^2 + 2\beta u + \left(\beta^2 + \frac{2}{\sigma^2}\right)}{\left[u^2 + \left(\frac{2\delta}{\sigma^2}+\beta\right)u + \frac{2}{\sigma^2}(\delta\beta-1)\right]^2} < 0, \quad u \in [0, v].$$

hence, $\hat{\Pi}(v)$ exists. For instance, we derive the first moment of stationary λ_t,

$$\mathbb{E}[\lambda_t] = -\hat{\Pi}'(v)\big|_{v=0} = -\hat{\Pi}(0)\left(-\frac{2\delta}{\sigma^2}al(0)\right) = \frac{2\delta}{\sigma^2}a \times \frac{\sigma^2\beta}{2(\delta\beta-1)} = \frac{\delta\beta}{\delta\beta-1}a,$$

which is independent of volatility parameter σ. □

Corollary 6.2.4. *If $\delta > \mu_{1_G}$ and $\lambda_0 \sim \Pi$, then $\lambda_T \sim \Pi$ for any time $T \geq 0$.*

Proof.

$$\begin{aligned}
\mathbb{E}\left[e^{-v\lambda_T}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right]\right]\\
&= \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^v \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1 + \frac{1}{2}\sigma^2u^2}\mathrm{d}u\right) \times \mathbb{E}\left[\exp\left(-\mathcal{G}_{v,1}^{-1}(T)\lambda_0\right)\right]\\
&= \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^v \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1 + \frac{1}{2}\sigma^2u^2}\mathrm{d}u\right)\\
&\quad \times \exp\left(-\int_0^{\mathcal{G}_{v,1}^{-1}(T)} \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1 + \frac{1}{2}\sigma^2u^2}\mathrm{d}u\right)\\
&= \exp\left(-\int_0^v \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1 + \frac{1}{2}\sigma^2u^2}\mathrm{d}u\right)\\
&= \hat{\Pi}(v).
\end{aligned}$$

□

6.2.3 Probability Generating Function of N_T

Theorem 6.2.4. *The conditional probability generating function of N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under the condition $\delta > \mu_{1_G}$, is given by*

$$\mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right] = \exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{a\delta u + \rho[1-\hat{h}(u)]}{1 - \delta u - \theta\hat{g}(u) - \frac{1}{2}\sigma^2u^2}\mathrm{d}u\right) \times \exp\left(-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0\right),$$

where

$$\mathcal{G}_{0,\theta}(L) =: \int_0^L \frac{\mathrm{d}u}{1 - \delta u - \theta\hat{g}(u) - \frac{1}{2}\sigma^2u^2}, \qquad 0 \leq \theta < 1. \tag{6.15}$$

Proof. By setting $t = 0$, $v = 0$ and assuming $N_0 = 0$ in *Theorem 6.2.1*, we have

$$\mathbb{E}\Big[\theta^{N_T}\Big|\mathcal{F}_0\Big] = e^{-c(T)}e^{-B(0)\lambda_0},$$

where $B(0)$ is uniquely determined by the non-linear ODE

$$-B'(t) + \delta B(t) + \theta\hat{g}\Big(B(t)\Big) - 1 + \frac{1}{2}\sigma^2 B^2(t) = 0,$$

with boundary condition $B(T) = 0$. It can be solved for $\sigma > 0$, under the condition $\delta > \mu_{1_G}$, by the following steps:

1. Set $B(t) = L(T-t)$ and $\tau = T - t$,

$$\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau} = 1 - \delta L(\tau) - \theta\hat{g}(L(\tau)) - \frac{1}{2}\sigma^2 L^2(\tau) =: f_2(L), \qquad 0 \le \theta < 1, \tag{6.16}$$

   with initial condition $L(0) = 0$; we define the right-hand side as the function $f_2(L)$.

2. There is only one positive singular point, denoted by $v^* > 0$, by solving the equation $f_2(L) = 0$. This is because,

   - for the case $0 < \theta < 1$, the equation $f_2(L) = 0$ is equivalent to

   $$\hat{g}(u) = \frac{1}{\theta}\left(1 - \delta u - \frac{1}{2}\sigma^2 u^2\right), \quad 0 < \theta < 1,$$

     note that $\hat{g}(\cdot)$ is a convex function, then it is clear that there is only one positive solution to this equation;

   - for the case $\theta = 0$, the equation $f_2(L) = 0$ is equivalent to

   $$1 - \delta u - \frac{1}{2}\sigma^2 u^2 = 0, \quad \theta = 0,$$

     which has only one positive solution

   $$v^* = \frac{-\delta + \sqrt{\delta^2 + 2\sigma^2}}{\sigma^2} > 0, \quad \sigma > 0;$$

   and for both cases,

$$0 < \frac{-\delta + \sqrt{\delta^2 + 2\sigma^2(1-\theta)}}{\sigma^2} < v^* \le \frac{-\delta + \sqrt{\delta^2 + 2\sigma^2}}{\sigma^2}, \quad \sigma > 0; \tag{6.17}$$

   then, we have $f_2(L) > 0$ for $0 \le L < v^*$ and $f_2(L) < 0$ for $L > v^*$.

3. Rewrite (6.16) as

$$\frac{\mathrm{d}L}{1 - \delta L - \theta\hat{g}(L) - \frac{1}{2}\sigma^2 L^2} = \mathrm{d}\tau,$$

   and integrate,

$$\int_0^L \frac{\mathrm{d}u}{1 - \delta u - \theta\hat{g}(u) - \frac{1}{2}\sigma^2 u^2} = \tau,$$

   where $0 \le L < v^*$, we define the function on left-hand side as

$$\mathcal{G}_{0,\theta}(L) =: \int_0^L \frac{\mathrm{d}u}{1 - \delta u - \theta\hat{g}(u) - \frac{1}{2}\sigma^2 u^2} \tag{6.18}$$

   then,

$$\mathcal{G}_{0,\theta}(L) = \tau,$$

   as $L \to 0$ when $\tau \to 0$, and $L \to v^*$ when $\tau \to \infty$; the integrand is positive in the domain $u \in [0, v^*)$ and $L \ge 0$, $\mathcal{G}_{0,\theta}(L)$ is a strictly increasing function; therefore, $\mathcal{G}_{0,\theta}(L) : [0, v^*) \to [0, \infty)$ is a well defined function, and its inverse function $\mathcal{G}^{-1}_{0,\theta}(\tau) : [0, \infty) \to [0, v^*)$ exists.

4. The unique solution is found by

$$L(\tau) = \mathcal{G}_{0,\theta}^{-1}(\tau), \quad \text{or,} \quad B(t) = \mathcal{G}_{0,\theta}^{-1}(T-t).$$

5. $B(0)$ is obtained,

$$B(0) = L(T) = \mathcal{G}_{0,\theta}^{-1}(T).$$

Then, $c(T)$ is determined by

$$c(T) = a\delta \int_0^T \mathcal{G}_{0,\theta}^{-1}(\tau)\mathrm{d}\tau + \rho \int_0^T \left[1 - \hat{h}\left(\mathcal{G}_{0,\theta}^{-1}(\tau)\right)\right] \mathrm{d}\tau, \tag{6.19}$$

where, by the change of variable,

$$\begin{aligned} \int_0^T \mathcal{G}_{0,\theta}^{-1}(\tau)\mathrm{d}\tau &= \int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{u}{1 - \delta u - \theta\hat{g}(u) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u, \\ \int_0^T \left[1 - \hat{h}\left(\mathcal{G}_{0,\theta}^{-1}(\tau)\right)\right] \mathrm{d}\tau &= \int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{1 - \hat{h}(u)}{1 - \delta u - \theta\hat{g}(u) - \frac{1}{2}\sigma^2 u^2}\mathrm{d}u. \end{aligned}$$

Finally, substitute $B(0)$ and $c(T)$ into (6.9), and the result follows. □

7

A Discretised Dynamic Contagion Process

In this chapter, we introduce a new point process named "*discretised dynamic contagion process*", a Markov chain model for contagion. We can prove (in *Theorem 7.2.3*) that it is a special class of branching processes that generalises a dynamic contagion process (of reversion level $a = 0$) introduced by Chapter 2. The transformation formulas between a discretised dynamic contagion process and the original dynamic contagion process ($a = 0$) are obtained. Some key distributional properties of this new point process have also been derived.

7.1 Introduction

We provide the definition of a discretised dynamic contagion process based on the transition probability within a sufficient small time interval.

Definition 7.1.1. *The discretised dynamic contagion process* (N_t, M_t) *is a point process on* $\mathbb{R}_+$ *such that*

$$
\begin{aligned}
P\Big\{M_{t+\Delta t} - M_t = k, N_{t+\Delta t} - N_t = 0 \Big| M_t, N_t\Big\} &= \rho p_k \Delta t + o(\Delta t), \quad k = 1, 2..., \\
P\Big\{M_{t+\Delta t} - M_t = k-1, N_{t+\Delta t} - N_t = 1 \Big| M_t, N_t\Big\} &= \delta M_t q_k \Delta t + o(\Delta t), \quad k = 0, 1..., \\
P\Big\{M_{t+\Delta t} - M_t = 0, N_{t+\Delta t} - N_t = 0 \Big| M_t, N_t\Big\} &= 1 - \Big(\rho(1-p_0) + \delta M_t\Big)\Delta t + o(\Delta t), \\
P\Big\{\text{Others} \Big| M_t, N_t\Big\} &= o(\Delta t),
\end{aligned}
$$

where Δt *is a sufficient small time interval,* $\delta, \rho > 0$ *are constants and batch-size distributions*

$$
p_k =: P\{K_P = k\}, \quad q_k =: P\{K_Q = k\}, \quad k = 0, 1...,
$$

with the probability generating functions are defined by

$$
\hat{p}(\theta) =: \sum_{k=0}^{\infty} \theta^k p_k, \quad \hat{q}(\theta) =: \sum_{k=0}^{\infty} \theta^k q_k.
$$

Remark 7.1.1. In the point process M_t, there are two types of jumps:

1. *Independent Jumps* K_P:

- jump independent of N_t,
- upward by $0, 1, 2, ..$ steps, with the corresponding probability $p_0, p_1, p_2, ...$;

2. *Joint Jumps K_Q*:

- jump simultaneously with N_t,
- upward by $-1, 0, 1, 2, ...$ steps, with the corresponding probability $q_0, q_1, q_2, ...$, in particular, -1 means decline by 1 step.

With the aid of the piecewise deterministic Markov process theory and using the results in Davis (1984), the infinitesimal generator of a discretised dynamic contagion process (M_t, N_t, t) acting on a function $f(m, n, t) \in \Omega(\mathcal{A})$ is given by

$$\begin{aligned}\mathcal{A}f(m,n,t) &= \frac{\partial f}{\partial t} + \rho\left(\sum_{k=0}^{\infty} f(m+k,n,t)p_k - f(m,n,t)\right) \\ &\quad +\delta m\left(\sum_{k=0}^{\infty} f(m+k-1,n+1,t)q_k - f(m,n,t)\right), \end{aligned} \tag{7.1}$$

where $\Omega(\mathcal{A})$ is the domain of the generator $\mathcal{A}$ such that $f(\lambda, n, t)$ is differentiable with respect to t, and for all m, n and t,

$$\left|\sum_{k=0}^{\infty} f(m+k,n,t)p_k - f(m,n,t)\right| < \infty,$$

$$\left|\sum_{k=0}^{\infty} f(m+k-1,n+1,t)q_k - f(m,n,t)\right| < \infty.$$

Similarly to the *Definition 2.1.1* for the dynamic contagion process, the discretised dynamic contagion process is a cluster point process on $\mathbb{R}_+$ and also can be defined as a branching process. A sample path of the discretised dynamic contagion process (N_t, M_t) is given by *Figure 7.1.*

Remark 7.1.2. The point process M_t is a non-negative process, as by *Definition 7.1.1*,

- if $M_t = 0$, there is no joint jump and M_t cannot be brought downward further by 1 step or more;
- if $M_t = 1, 2, ...$, there is possible downward movement with maximum 1 step.

Higher levels of M_t will bring more possible jumps in N_t, hence more joint jumps in M_t. So, for a general discretised dynamic contagion process, M_t could explode at time infinity, and a stationarity condition (given later by *Remark 7.2.1*) needed to keep the process M_t stationary:

- when M_t goes higher, there will be more possible jumps in N_t, and hence more simultaneous joint jumps with -1 step in M_t. So it finally will bring the level of M_t to be lower;
- when M_t goes lower, in particular, during the period when $M_t = 0$, there are no joint jumps; there are only possible independent jumps that bring M_t out of 0 level; once M_t is above 0, there will be possible joint jumps, and hence more possible joint jumps that bring the level of M_t higher.

Remark 7.1.3. We can consider the discrete piecewise non-negative process $\{\delta M_t\}_{t\geq 0}$ as the intensity process of the point process N_t (with the proof given later by (7.2)).

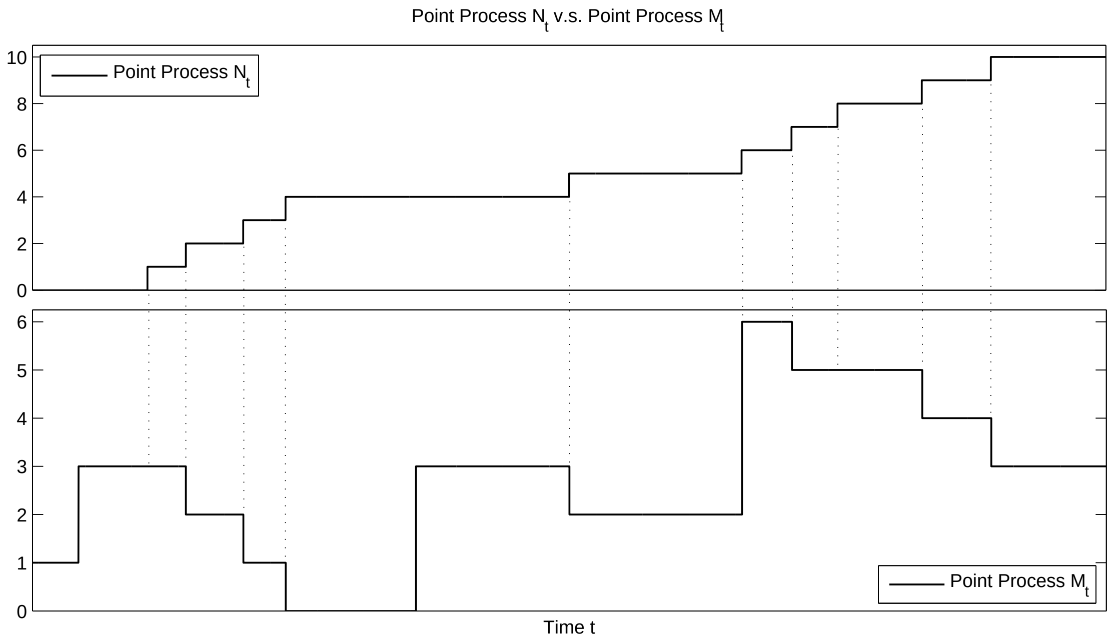

Fig. 7.1: Point Process N_t v.s. Point Process M_t

Remark 7.1.4. The discretised dynamic contagion process is a generalised birth-death process with multiple births and one single death of offsprings for each individual alive at time t.

Remark 7.1.5. The discretised dynamic contagion process is more general than the dynamic contagion process ($a = 0$) introduced by Chapter 2, as the intensity process $\{\delta M_t\}_{t\geq 0}$ allows to be zero, and also the dynamic contagion process ($a = 0$) is a special case of discretised dynamic contagion process when both K_P and K_Q follow mixed–Poisson distributions (with proof later given by *Theorem 7.2.3*).

7.2 Distributional Properties

7.2.1 Moments of M_t and N_t

We derive the first moments of M_t and N_t by ODE method and discuss the stationarity condition for the process M_t. Other moments can also be obtained similarly to Section 2.2.4 for dynamic contagion process, and we omit them here.

Theorem 7.2.1. *The expectation of M_t conditional on M_0 is given by*

$$\begin{aligned}\mathbb{E}[M_t|M_0] &= \frac{\mu_{1_P}\rho}{\delta(1-\mu_{1_Q})} + \left(M_0 - \frac{\mu_{1_P}\rho}{\delta(1-\mu_{1_Q})}\right)e^{-\delta(1-\mu_{1_Q})t}, \quad \mu_{1_Q}\neq 1,\\ \mathbb{E}[M_t|M_0] &= M_0 + \mu_{1_P}\rho t, \quad \mu_{1_Q} = 1.\end{aligned}$$

Proof. Set $f(m,n,t) = m$ and plug into the generator (7.1), we have

$$\mathcal{A}m = -\delta(1-\mu_{1_Q})m + \mu_{1_P}\rho.$$

Since $M_t - M_0 - \int_0^t \mathcal{A}M_s\mathrm{d}s$ is a $\mathcal{F}^{\mathbb{P}}-$martingale, then,

$$\mathbb{E}[M_t|M_0] = M_0 + \mathbb{E}\left[\int_0^t \mathcal{A}M_s\mathrm{d}s\bigg|M_0\right] = M_0 - \delta(1-\mu_{1_Q})\int_0^t \mathbb{E}[M_s|M_0]\mathrm{d}s + \mu_{1_P}\rho t,$$

and we can derive the expectation via the ODE

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\delta(1-\mu_{1_Q})u(t) + \mu_{1_P}\rho,$$

where $u(t) = \mathbb{E}[M_t|M_0]$ with the initial condition $u(0) = M_0$.

□

Remark 7.2.1. The stationarity condition of the process M_t is $\mu_{1_Q} < 1$ where

$$\mu_{1_Q} =: \sum_{k=0}^{\infty} k q_k.$$

Note that, by *Theorem 7.2.3*, it can be alternatively derived via the transformation from the stationarity condition $\delta > \mu_{1_G}$ for a dynamic contagion process, i.e.

$$\mu_{1_Q} = \mathbb{E}\left[K_Q\right] = \mathbb{E}\left[\frac{Y}{\delta}\middle| Y \sim G\right] = \frac{\mu_{1_G}}{\delta} < 1.$$

In particular, if $K_Q \sim \text{Geometric}\,(\acute{q})$, then, the stationarity condition is $\acute{q} > \frac{1}{2}$.

Corollary 7.2.1. *Assume $\mu_{1_Q} < 1$ and $N_0 = 0$, for the stationary distribution of M_t, we have*

$$\begin{aligned} \mathbb{E}[M_t] &= \frac{\mu_{1_P}\rho}{\delta(1-\mu_{1_Q})}, \\ \mathbb{E}[N_t] &= \frac{\mu_{1_P}\rho}{1-\mu_{1_Q}} t. \end{aligned}$$

Proof. Set $f(m, n, t) = n$ and plug into the generator (7.1), we have

$$\mathcal{A}n = \delta m.$$

Since $N_t - N_0 - \int_0^t \mathcal{A}N_s \mathrm{d}s$ is a $\mathcal{F}^{\mathbb{P}}-$martingale and $N_0 = 0$, then,

$$\mathbb{E}[N_t|M_0] = N_0 + \mathbb{E}\left[\int_0^t \mathcal{A}N_s \mathrm{d}s \middle| M_0\right] = \delta \int_0^t \mathbb{E}[M_s|M_0]\mathrm{d}s, \tag{7.2}$$

and

$$\mathbb{E}[N_t] = \delta \int_0^t \mathbb{E}[M_s]\mathrm{d}s = \mathbb{E}[N_t] = \frac{\mu_{1_P}\rho}{1-\mu_{1_Q}} t.$$

□

7.2.2 Joint Probability Generating Function of (M_T, N_T)

Theorem 7.2.2. *For constants $0 \le \theta, \varphi \le 1$ and time $0 \le t \le T$, we have the conditional joint probability generating function of (M_T, N_T),*

$$\mathbb{E}\left[\theta^{(N_T - N_t)} \varphi^{M_T} \middle| \mathcal{F}_t\right] = e^{-(c(T)-c(t))}[A(t)]^{M_t},$$

where $A(t)$ is determined by the non-linear ODE

$$A'(t) + \delta\theta\hat{q}(A(t)) - \delta A(t) = 0,$$

with boundary condition $A(T) = \varphi$; and $c(t)$ is determined by

$$c(t) = \rho \int_0^t \left[1 - \hat{p}(A(s))\right] \mathrm{d}s.$$

Proof. Assume the exponential affine form

$$f(m,n,t) = [A(t)]^m \theta^n e^{c(t)},$$

and set $\mathcal{A}f(m,n,t) = 0$, note that,

$$\sum_{k=0}^{\infty}[A(t)]^k p_k = \hat{p}\left(A(t)\right); \quad \sum_{k=0}^{\infty}[A(t)]^k q_k = \hat{q}\left(A(t)\right),$$

then, we have

$$\begin{cases} \frac{A'(t)}{A(t)} + \delta\left(\frac{\theta}{A(t)}\hat{q}(A(t)) - 1\right) = 0 & (.1) \\ c'(t) = \rho[1 - \hat{p}(A(t))] & (.2) \end{cases},$$

or,

$$\begin{cases} A'(t) + \delta\theta\hat{q}(A(t)) - \delta A(t) = 0 & (.1) \\ c'(t) = \rho[1 - \hat{p}(A(t))] & (.2) \end{cases}.$$

Since $[A(t)]^{M_t}\theta^{N_t}e^{c(t)}$ is a $\mathcal{F}^{\mathbb{P}}-$martingale, we have

$$\mathbb{E}\Big[[A(T)]^{M_T}\theta^{N_T}e^{c(T)}\Big|\mathcal{F}_t\Big] = [A(t)]^{M_t}\theta^{N_t}e^{c(t)},$$

with boundary conditions $A(T) = \varphi$ and $c(0) = 0$. □

7.2.3 Transformation to Dynamic Contagion Process

The discretised dynamic contagion process (N_t, M_t) (defined by *Definition 7.1.1*) is matching to a dynamic contagion process (N_t, λ_t) (defined by *Definition 2.1.1*) with reversion level $a = 0$. We discover the transformation formulas between the discretised dynamic contagion process and the dynamic contagion process ($a = 0$) by comparing their infinitesimal generators:

- The generator of a dynamic contagion process (λ_t, N_t) with $a = 0$ is given by (2.2), i.e.

$$\begin{aligned} \mathcal{A}f(\lambda,n,t) &= \frac{\partial f}{\partial t} - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho\left(\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y) - f(\lambda,n,t)\right) \\ &\quad +\lambda\bigg(\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y) - f(\lambda,n,t)\bigg). \end{aligned}$$

  Assume the form $f(\lambda,n,t) = e^{-B(t)\lambda}\theta^n e^{c(t)}$ and set $\mathcal{A}f(\lambda,n,t) = 0$, we have

$$\begin{cases} -B'(t) + \delta B(t) + \theta\hat{g}(B(t)) - 1 = 0 \\ c'(t) = \rho[1 - \hat{h}(B(t))] \end{cases} \tag{7.3}$$

  which has been used to derive $\mathbb{E}\left[\theta^{N_T}e^{-v\lambda_T}\middle|\lambda_0\right]$ (as given by *Theorem 2.2.1*).

- The generator of a discretised dynamic contagion process (M_t, N_t) is given by (7.1), i.e.

$$\begin{aligned} \mathcal{A}f(m,n,t) &= \frac{\partial f}{\partial t} + \rho\left(\sum_{k=0}^{\infty} f(m+k,n,t)p_k - f(m,n,t)\right) \\ &\quad +\delta m\left(\sum_{k=0}^{\infty} f(m+k-1,n+1,t)q_k - f(m,n,t)\right). \end{aligned}$$

  Similarly, assume $f(m,n,t) = [A(t)]^m\theta^n e^{c(t)}$ and set $\mathcal{A}f(m,n,t) = 0$, we have

$$\begin{cases} A'(t) + \delta\theta\hat{q}(A(t)) - \delta A(t) = 0 \\ c'(t) = \rho[1 - \hat{p}(A(t))] \end{cases} \tag{7.4}$$

  which has been used to derive $\mathbb{E}\left[\theta^{N_T}\varphi^{M_T}\middle|M_0\right]$ (as given by *Theorem 7.2.2*).

Compare (7.4) with (7.3), if we set

$$B(t) = \frac{1-A(t)}{\delta}, \tag{7.5}$$

$$\hat{p}\,(u) = \hat{h}\left(\frac{1-u}{\delta}\right), \quad \hat{q}\,(u) = \hat{g}\left(\frac{1-u}{\delta}\right), \tag{7.6}$$

or,

$$A(t) = 1 - \delta B(t), \tag{7.7}$$

$$\hat{p}\,(1-\delta u) = \hat{h}\,(u)\,, \quad \hat{q}\,(1-\delta u) = \hat{g}\,(u)\,, \tag{7.8}$$

then, (7.3) and (7.4) are equivalent. The transformation between the discretised dynamic contagion process and the dynamic contagion process ($a = 0$) is thus given by *Theorem 7.2.3*.

Theorem 7.2.3. *The discretised dynamic contagion process is a dynamic contagion process with $a = 0$, if*

$$K_P \sim \textit{Mixed–Poisson}\left(\frac{Y}{\delta}\middle| Y \sim H\right), \quad K_Q \sim \textit{Mixed–Poisson}\left(\frac{Y}{\delta}\middle| Y \sim G\right),$$

i.e.

$$p_k = \int_0^\infty \frac{e^{-\frac{y}{\delta}}}{k!}\left(\frac{y}{\delta}\right)^k \mathrm{d}H(y), \quad q_k = \int_0^\infty \frac{e^{-\frac{y}{\delta}}}{k!}\left(\frac{y}{\delta}\right)^k \mathrm{d}G(y).$$

Proof. By the transformation formula (7.6), we have

$$\begin{aligned}
\sum_{k=0}^{\infty}\theta^k p_k &= \hat{p}\,(\theta) = \hat{h}\left(\frac{1-\theta}{\delta}\right) = \int_0^\infty e^{-\frac{1-\theta}{\delta}y}\mathrm{d}H(y) = \mathbb{E}\left[e^{-\frac{Y}{\delta}(1-\theta)}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\theta^{K_P}\middle| K_P \sim \text{Poisson}\left(\frac{Y}{\delta}\right)\right]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty}\theta^k P\left\{K_P = k\middle| K_P \sim \text{Poisson}\left(\frac{Y}{\delta}\right)\right\}\right] \\
&= \sum_{k=0}^{\infty}\theta^k \mathbb{E}\left[P\left\{K_P = k\middle| K_P \sim \text{Poisson}\left(\frac{Y}{\delta}\right)\right\}\right] \\
&= \sum_{k=0}^{\infty}\theta^k P\left\{K_P = k\right\};
\end{aligned}$$

and similarly, for q_k. □

Corollary 7.2.2. *If $H \sim \mathrm{Exp}\,(\alpha)$ and $G \sim \mathrm{Exp}\,(\beta)$, then, the transformation is given by*

$$\begin{aligned}
\{p_k\}_{k=0,1,2...} &\sim \text{Geometric}\,(\acute{p})\,, \quad \acute{p} =: \frac{\delta\alpha}{\delta\alpha+1}; \\
\{q_k\}_{k=0,1,2...} &\sim \text{Geometric}\,(\acute{q})\,, \quad \acute{q} =: \frac{\delta\beta}{\delta\beta+1}.
\end{aligned}$$

Proof. If $H \sim \mathrm{Exp}\,(\alpha)$, then, by (7.6), we have

$$\hat{p}(u) = \hat{h}\left(\frac{1-u}{\delta}\right) = \frac{\alpha}{\alpha + \frac{1-u}{\delta}} = \frac{\frac{\delta\alpha}{\delta\alpha+1}}{1-\left(1-\frac{\delta\alpha}{\delta\alpha+1}\right)u} = \frac{\acute{p}}{1-(1-\acute{p})u};$$

and similarly, for $G \sim \mathrm{Exp}\,(\beta)$. □

7.2.4 Probability Generating Function of M_T

Theorem 7.2.4. *Under the condition $\mu_{1_Q} < 1$, the probability generating function of M_T conditional on M_0 is given by*

$$\mathbb{E}[\varphi^{M_T}|M_0] = \exp\left(-\int_{\varphi}^{\mathcal{Q}^{-1}_{\varphi,1}(T)} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right) \times \left[\mathcal{Q}^{-1}_{\varphi,1}(T)\right]^{M_0},$$

where

$$\mathcal{Q}_{\varphi,1}(L) =: \int_{\varphi}^{L} \frac{\mathrm{d}u}{\delta\hat{q}(u)-\delta u}. \tag{7.9}$$

Proof. In particular, set $t = 0$, $\theta = 1$ and assume $N_0 = 0$ in *Theorem 7.2.2*, we have

$$\mathbb{E}\Big[\varphi^{M_T}\Big|M_0\Big] = e^{-c(T)}[A(0)]^{M_0}, \tag{7.10}$$

where $A(0)$ is uniquely determined by the non-linear ODE

$$A'(t) + \delta\hat{q}(A(t)) - \delta A(t) = 0,$$

with boundary condition $A(T) = \varphi$. Under the condition $\mu_{1_Q} < 1$, it can be solved by the following steps:

1. Set $A(t) = L(T-t)$ and $\tau = T - t$, it is equivalent to the initial value problem

   $$\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau} = \delta\hat{q}(L(\tau)) - \delta L(\tau) =: f_1(L),$$

   with initial condition $L(0) = \varphi$; we define the right-hand side as the function $f_1(L)$.

2. Under the condition $\mu_{1_Q} < 1$, we have

   $$\frac{\partial f_1(L)}{\partial L} = \delta\left(\sum_{k=0}^{\infty} kL^{k-1}p_k - 1\right) \le \delta\left(\sum_{k=0}^{\infty} kp_k - 1\right) = \delta\left(\mu_{1_Q} - 1\right) < 0, \quad 0 < L \le 1,$$

   then, $f_1(L) > 0$ for $0 < L < 1$, since $f_1(1) = 0$.

3. Rewrite as

   $$\frac{\mathrm{d}L}{\delta\hat{q}(L)-\delta L} = \mathrm{d}\tau,$$

   by integrating both sides from time 0 to τ with initial condition $L(0) = \varphi > 0$, we have

   $$\int_{\varphi}^{L} \frac{\mathrm{d}u}{\delta\hat{q}(u)-\delta u} = \tau,$$

   where $0 < L \le 1$, we define the function on left hand side as

   $$\mathcal{Q}_{\varphi,1}(L) =: \int_{\varphi}^{L} \frac{\mathrm{d}u}{\delta\hat{q}(u)-\delta u},$$

   then,

   $$\mathcal{Q}_{\varphi,1}(L) = \tau,$$

   obviously $L \to \varphi$ when $\tau \to 0$; by convergence test,

   $$\lim_{u\to 1} \frac{\frac{1}{1-u}}{\frac{1}{\delta\hat{q}(u)-\delta u}} = \delta \lim_{u\to 1} \frac{\hat{q}(u)-u}{1-u} = \delta \lim_{u\to 1} \frac{\left(\hat{q}(u)-u\right)'}{(1-u)'} = \delta\left(1 - \lim_{u\to 1}\hat{q}'(u)\right) = 1 - \mu_{1_Q} > 0,$$

and we know that $\int_{\varphi}^{1} \frac{1}{1-u}\mathrm{d}u = \infty$, then,

$$\int_{\varphi}^{1} \frac{\mathrm{d}u}{\delta\hat{q}(u) - \delta u} = \infty,$$

hence, $L \to 1$ when $\tau \to \infty$; the integrand is positive in the domain $u \in [\varphi, 1)$ and also $\mathcal{Q}_{\varphi,1}(L)$ is a strictly increasing function; therefore, $\mathcal{Q}_{\varphi,1}(L) : [\varphi, 1) \to [0, \infty)$ is a well defined (monotone) function, and its inverse function $\mathcal{Q}_{\varphi,1}^{-1}(\tau) : [0, \infty) \to [\varphi, 1)$ exists.

4. The unique solution is found by

$$L(\tau) = \mathcal{Q}_{\varphi,1}^{-1}(\tau), \quad \text{or,} \quad A(t) = \mathcal{Q}_{\varphi,1}^{-1}(T - t).$$

5. $A(0)$ is obtained,

$$A(0) = L(T) = \mathcal{Q}_{\varphi,1}^{-1}(T).$$

Then, $c(T)$ is determined by

$$c(T) = \rho \int_0^T \left[1 - \hat{p}\left(\mathcal{Q}_{\varphi,1}^{-1}(\tau)\right)\right] \mathrm{d}\tau,$$

by the change of variable $\mathcal{Q}_{\varphi,1}^{-1}(\tau) = u$, we have $\tau = \mathcal{Q}_{\varphi,1}(u)$, and

$$\int_0^T \left[1 - \hat{h}\left(\mathcal{Q}_{\varphi,1}^{-1}(\tau)\right)\right] \mathrm{d}\tau = \int_{\mathcal{Q}_{\varphi,1}^{-1}(0)}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} [1 - \hat{p}(u)] \frac{\partial \tau}{\partial u} \mathrm{d}u = \int_{\varphi}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} \frac{1 - \hat{p}(u)}{\delta\hat{q}(u) - \delta u} \mathrm{d}u.$$

□

Alternatively, with the aid of the transformation, we can directly prove *Theorem 7.2.2* from *Theorem 2.2.2* for the dynamic contagion process.

Proof. Review the proof of *Theorem 2.2.2*, as given by (7.10), we have

$$\mathbb{E}\Big[[A(T)]^{M_T} \Big| M_0\Big] = e^{-c(T)}[A(0)]^{M_0},$$

where $A(T)$ and $A(0)$ can be alternatively solved via the transformation. Set $A(T) = \varphi$, by (7.7), we have

$$A(T) = 1 - \delta B(T) = 1 - \delta \mathcal{G}_{v,1}^{-1}(0) = 1 - \delta v = \varphi,$$

then, $v = \frac{1-\varphi}{\delta}$, and by (2.13),

$$\begin{aligned}
\mathcal{G}_{v,1}(L) &= \int_L^v \frac{1}{\delta u + \hat{g}(u) - 1} \mathrm{d}u \\
&= \int_L^v \frac{1}{\delta u + \hat{q}(1 - \delta u) - 1} \mathrm{d}u \qquad (s = 1 - \delta u) \\
&= \int_{1-\delta v}^{1-\delta L} \frac{1}{\delta\hat{q}(s) - \delta s} \mathrm{d}s \qquad \left(v = \frac{1-\varphi}{\delta}\right) \\
&= \int_{\varphi}^{1-\delta L} \frac{1}{\delta\hat{q}(s) - \delta s} \mathrm{d}s.
\end{aligned}$$

Since

$$\mathcal{G}_{v,1}(B) = \int_{\varphi}^{1-\delta B} \frac{1}{\delta\hat{q}(s) - \delta s} \mathrm{d}s,$$

and by transformation (7.7), $A = 1 - \delta B$, we have

$$\mathcal{G}_{v,1}\left(\frac{1-A}{\delta}\right) = \int_{\varphi}^{A} \frac{1}{\delta\hat{q}(s) - \delta s}\mathrm{d}s.$$

Define $\mathcal{Q}_{\varphi,1}(L)$ by (7.9), we have

$$\mathcal{Q}_{\varphi,1}(u) = \mathcal{G}_{v,1}\left(\frac{1-u}{\delta}\right),$$

or,

$$\mathcal{Q}_{\varphi,1}(1-\delta u) = \mathcal{G}_{v,1}(u).$$

then,

$$\mathcal{Q}_{\varphi,1}\left(1-\delta\mathcal{G}_{v,1}^{-1}(T)\right) = \mathcal{G}_{v,1}\left(\mathcal{G}_{v,1}^{-1}(T)\right) = T,$$

and

$$\mathcal{Q}_{\varphi,1}^{-1}(T) = 1 - \delta\mathcal{G}_{v,1}^{-1}(T).$$

By transformtation (7.7),

$$A(0) = 1 - \delta B(0) = 1 - \delta\mathcal{G}_{v,1}^{-1}(T) = \mathcal{Q}_{\varphi,1}^{-1}(T),$$

and

$$\begin{aligned}
c(T) &= \int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{\rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u \\
&= \int_{\frac{1-\mathcal{Q}_{\varphi,1}^{-1}(T)}{\delta}}^{\frac{1-\varphi}{\delta}} \frac{\rho[1-\hat{p}(1-\delta u)]}{\delta u + \hat{q}(1-\delta u) - 1}\mathrm{d}u \qquad (s = 1-\delta u) \\
&= \int_{\varphi}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} \frac{\rho[1-\hat{p}(s)]}{\delta\hat{q}(s) - \delta s}\mathrm{d}s.
\end{aligned}$$

□

Theorem 7.2.5. *If* $M_0 \sim \text{Poisson}\left(\frac{\lambda_0}{\delta}\right)$, *then,*

$$\mathbb{E}[\varphi^{M_T}] = \mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right].$$

Proof. If $M_0 \sim \text{Poisson}\left(\frac{\lambda_0}{\delta}\right)$, then, $\mathbb{E}\left[\psi^{M_0}\right] = e^{-\frac{1-\psi}{\delta}\lambda_0}$. By the transformation (7.5) and (7.7), we have

$$v = \frac{1-\varphi}{\delta}, \quad \mathcal{G}_{v,1}^{-1}(T) = \frac{1-\mathcal{Q}_{\varphi,1}^{-1}(T)}{\delta}, \quad A(0) = 1 - \delta B(0),$$

and

$$\mathcal{G}_{v,1}^{-1}(T) = B(0), \quad \mathcal{Q}_{\varphi,1}^{-1}(T) = A(0).$$

Then, by comparing *Theorem 7.2.2* and *Theorem 2.2.2*, we have

$$
\begin{aligned}
\mathbb{E}[\varphi^{M_T}] &= \mathbb{E}\left[\mathbb{E}[\varphi^{M_T}\big|M_0]\right] \\
&= \exp\left(-\int_{\varphi}^{\mathcal{Q}^{-1}_{\varphi,1}(T)} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right)\mathbb{E}\left[\left[\mathcal{Q}^{-1}_{\varphi,1}(T)\right]^{M_0}\right] \\
&= \exp\left(-\int_{\varphi}^{\mathcal{Q}^{-1}_{\varphi,1}(T)} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right)e^{-\frac{1-\mathcal{Q}^{-1}_{\varphi,1}(T)}{\delta}\lambda_0} \\
&= \exp\left(-\int_{\varphi}^{\mathcal{Q}^{-1}_{\varphi,1}(T)} \frac{\rho\left[1-\hat{h}\left(\frac{1-u}{\delta}\right)\right]}{\delta\hat{g}\left(\frac{1-u}{\delta}\right)-\delta u}\mathrm{d}u\right)e^{-\frac{1-\mathcal{Q}^{-1}_{\varphi,1}(T)}{\delta}\lambda_0} \qquad \left(s=\frac{1-u}{\delta}\right) \\
&= \exp\left(-\int_{\frac{1-\mathcal{Q}^{-1}_{\varphi,1}(T)}{\delta}}^{\frac{1-\varphi}{\delta}} \frac{\rho[1-\hat{h}(s)]}{\delta s+\hat{g}(s)-1}\mathrm{d}s\right)e^{-\frac{1-\mathcal{Q}^{-1}_{\varphi,1}(T)}{\delta}\lambda_0} \\
&= \exp\left(-\int_{\mathcal{G}^{-1}_{v,1}(T)}^{v} \frac{a\delta u+\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right)e^{-\mathcal{G}^{-1}_{v,1}(T)\lambda_0} \\
&= \mathbb{E}\left[e^{-v\lambda_T}\big|\lambda_0\right].
\end{aligned}
$$

□

Theorem 7.2.6. *If* $\mu_{1_Q} < 1$*, then, the probability generating function of the asymptotic distribution of* M_T *is given by*

$$
\lim_{T\to\infty}\mathbb{E}[\varphi^{M_T}\big|M_0] = \exp\left(-\int_{\varphi}^{1}\frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right), \tag{7.11}
$$

and this is also the probability generating function of the stationary distribution of the process $\{M_t\}_{t\geq 0}$*.*

Proof. Since

$$
\lim_{T\to\infty}\mathcal{Q}^{-1}_{\varphi,1}(T) = 1,
$$

and by *Theorem 7.2.4*, we have the probability generating function of the asymptotic distribution of M_T immediately.

To further prove the stationarity, by *Proposition 9.2* of Ethier and Kurtz (1986) (and see also Costa (1990)), we need to prove that, for any function $f \in \Omega(\mathcal{A})$, we have

$$
\sum_{m=0}^{\infty}\mathcal{A}f(m)\aleph_m = 0, \tag{7.12}
$$

where $\mathcal{A}f(m)$ is the infinitesimal generator of the discretised dynamic contagion process acting on $f(m)$, i.e.

$$
\mathcal{A}f(m) = \rho\left(\sum_{k=0}^{\infty}f(m+k)p_k - f(m)\right) + \delta m\left(\sum_{k=0}^{\infty}f(m+k-1)q_k - f(m)\right), \tag{7.13}
$$

and $\{\aleph_m\}_{k=0,1,2,...}$ are the probabilities of m with the probability generating function given by (7.11). Now, we try to solve equation (7.12).

For the first term of (7.12), we have

$$\begin{aligned}
\sum_{m=0}^{\infty}\left[\rho\left(\sum_{k=0}^{\infty} f(m+k)p_k\right)\right]\aleph_m &= \rho\sum_{m=0}^{\infty}\aleph_m\sum_{k=0}^{\infty} f(m+k)p_k \qquad (j=m+k)\\
&= \rho\sum_{j=0}^{\infty} f(j)\sum_{k=0}^{j}\aleph_{j-k}p_k\\
&= \rho\sum_{m=0}^{\infty} f(m)\sum_{k=0}^{m}\aleph_{m-k}p_k.
\end{aligned}$$

For the second term of (7.12), we have

$$\begin{aligned}
\sum_{m=0}^{\infty}\left[\delta m\left(\sum_{k=0}^{\infty} f(m+k-1)q_k\right)\right]\aleph_m &= \delta\sum_{m=0}^{\infty} m\aleph_m\sum_{k=0}^{\infty} f(m+k-1)q_k\\
&= \delta\sum_{m=-1}^{\infty}(m+1)\aleph_{m+1}\sum_{k=0}^{\infty} f(m+k)q_k\\
&= \delta\sum_{m=0}^{\infty}(m+1)\aleph_{m+1}\sum_{k=0}^{\infty} f(m+k)q_k \qquad (j=m+k)\\
&= \delta\sum_{j=0}^{\infty} f(j)\sum_{k=0}^{j}(j-k+1)\aleph_{j-k+1}q_k\\
&= \delta\sum_{m=0}^{\infty} f(m)\sum_{k=0}^{m}(m-k+1)\aleph_{m-k+1}q_k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\mathcal{A}f(m)\aleph_m\\
= &\sum_{m=0}^{\infty} f(m)\left[\rho\left(\sum_{k=0}^{m}\aleph_{m-k}p_k-\aleph_m\right)+\delta\left(\sum_{k=0}^{m}(m-k+1)\aleph_{m-k+1}q_k-m\aleph_m\right)\right]=0,
\end{aligned}$$

for any function $f(m)\in\Omega(\mathcal{A})$, then, we have recursive equation

$$\rho\left(\sum_{k=0}^{m}\aleph_{m-k}p_k-\aleph_m\right)+\delta\left(\sum_{k=0}^{m}(m-k+1)\aleph_{m-k+1}q_k-m\aleph_m\right)=0,$$

and

$$\sum_{m=0}^{\infty}\varphi^m\times\left[\rho\left(\sum_{k=0}^{m}\aleph_{m-k}p_k-\aleph_m\right)+\delta\left(\sum_{k=0}^{m}(m-k+1)\aleph_{m-k+1}q_k-m\aleph_m\right)\right]=0.$$

By the Laplace transform

$$\hat{\aleph}(\varphi)=:\mathcal{L}\{\aleph_m\}=\sum_{m=0}^{\infty}\aleph_m\varphi^m,$$

since

$$\begin{aligned}
\sum_{m=0}^{\infty}\varphi^m\sum_{k=0}^{m}\aleph_{m-k}p_k &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\varphi^k p_k\varphi^{m-k}\aleph_{m-k} \\
&= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\varphi^{m-k}\aleph_{m-k}\varphi^k p_k \qquad \left(\sum_{m=k}^{\infty}\varphi^{m-k}\aleph_{m-k}=\hat{\aleph}(\varphi)\right) \\
&= \hat{\aleph}(\varphi)\sum_{k=0}^{\infty}\varphi^k p_k \\
&= \hat{\aleph}(\varphi)\hat{p}(\varphi),
\end{aligned}$$

$$\begin{aligned}
\sum_{m=0}^{\infty}\varphi^m\sum_{k=0}^{m}(m-k+1)\aleph_{m-k+1}q_k &= \sum_{m=0}^{\infty}\varphi^m\sum_{j=1}^{m+1}j\aleph_j q_{m+1-j} \qquad (j=m-k+1) \\
&= \frac{1}{\varphi}\sum_{m=0}^{\infty}\varphi^j\sum_{j=1}^{m+1}j\aleph_j q_{m+1-j}\varphi^{m+1-j} \qquad (i=m+1) \\
&= \frac{1}{\varphi}\sum_{i=1}^{\infty}\sum_{j=1}^{i}\varphi^j j\aleph_j q_{i-j}\varphi^{i-j} \\
&= \frac{1}{\varphi}\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}q_{i-j}\varphi^{i-j}\varphi^j j\aleph_j \qquad \left(\sum_{i=j}^{\infty}q_{i-j}\varphi^{i-j}=\hat{q}(\varphi)\right) \\
&= \frac{1}{\varphi}\hat{q}(\varphi)\sum_{j=1}^{\infty}\varphi^j j\aleph_j \\
&= \hat{q}(\varphi)\sum_{j=1}^{\infty}j\varphi^{j-1}\aleph_j \\
&= \hat{q}(\varphi)\sum_{j=0}^{\infty}j\varphi^{j-1}\aleph_j \\
&= \hat{q}(\varphi)\hat{\aleph}'(\varphi),
\end{aligned}$$

and

$$\sum_{m=0}^{\infty}\varphi^m m\aleph_m = \varphi\sum_{m=0}^{\infty}m\aleph_m\varphi^{m-1}=\varphi\hat{\aleph}'(\varphi),$$

we have the ODE of $\hat{\aleph}(\varphi)$,

$$\rho\Big(\hat{\aleph}(\varphi)\hat{p}(\varphi)-\hat{\aleph}(\varphi)\Big)+\delta\Big(\hat{q}(\varphi)\hat{\aleph}'(\varphi)-\varphi\hat{\aleph}'(\varphi)\Big)=0,$$

or,

$$\delta\Big(\hat{q}(\varphi)-\varphi\Big)\frac{\mathrm{d}\hat{\aleph}(\varphi)}{\mathrm{d}\varphi}+\rho\left[\hat{p}(\varphi)-1\right]\hat{\aleph}(\varphi)=0,$$

then,

$$\hat{\aleph}(\varphi)=\hat{\aleph}(1)\exp\left(-\int_{\varphi}^{1}\frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right),$$

with the initial condition $\hat{\aleph}(1)=1$, hence, we have the unique solution

$$\hat{\aleph}(\varphi)=:\exp\left(-\int_{\varphi}^{1}\frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right)$$

which is exactly given by (7.11).

Since the distribution $\aleph$ is the unique solution to (7.12), we have the stationarity for the process $\{M_t\}_{t\geq 0}$. □

Corollary 7.2.3. *Under condition $\mu_{1_Q} < 1$, if $M_0 \sim \aleph$, then $M_T \sim \aleph$.*

Proof. If $M_0 \sim \aleph$, then, by *Theorem 7.2.6* and *Theorem 7.2.4*, we have

$$\begin{aligned}
\mathbb{E}\Big[\psi^{M_T}\Big] &= \mathbb{E}\Big[\Big[\psi^{M_T}\big|M_0\Big]\Big] \\
&= \exp\left(-\int_{\varphi}^{\mathcal{Q}^{-1}_{\varphi,1}(T)} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right) \times \mathbb{E}\left[\left[\mathcal{Q}^{-1}_{\varphi,1}(T)\right]^{M_0}\right] \\
&= \exp\left(-\int_{\varphi}^{\mathcal{Q}^{-1}_{\varphi,1}(T)} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right) \times \exp\left(-\int_{\mathcal{Q}^{-1}_{\varphi,1}(T)}^{1} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right) \\
&= \exp\left(-\int_{\varphi}^{1} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right) \\
&= \hat{\aleph}(\varphi).
\end{aligned}$$

□

Corollary 7.2.4. *The probability generating function of stationary distribution of $\{M_t\}_{t\geq 0}$ can be expressed by the Laplace transform of stationary distribution of $\{\lambda_t\}_{t\geq 0}$ as*

$$\hat{\aleph}(\varphi) = \hat{\Pi}(v),$$

or,

$$\hat{\aleph}(u) = \hat{\Pi}\left(\frac{1-u}{\delta}\right). \tag{7.14}$$

Proof. By *Theorem 7.2.6* and *Theorem 2.2.3*, we have

$$\begin{aligned}
\hat{\Pi}(v) &= \exp\left(-\int_0^v \frac{\rho[1-\hat{h}(u)]}{\delta u+\hat{g}(u)-1}\mathrm{d}u\right) \\
&= \exp\left(-\int_0^v \frac{\rho[1-\hat{p}(1-\delta u)]}{\delta u+\hat{q}(1-\delta u)-1}\mathrm{d}u\right) \qquad (s = 1-\delta u) \\
&= \exp\left(-\int_{1-\delta v}^{1} \frac{\rho[1-\hat{p}(s)]}{\delta\hat{q}(s)-\delta u}\mathrm{d}s\right) \\
&= \hat{\aleph}(1-\delta v) \\
&= \hat{\aleph}(\varphi).
\end{aligned}$$

□

7.2.5 Probability Generating Function of N_T

Theorem 7.2.7. *Assume $\mu_{1_Q} < 1$ and $N_0 = 0$, the probability generating function of N_T conditional on M_0 is given by*

$$\mathbb{E}\Big[\theta^{N_T}\big|M_0\Big] = \exp\left(-\int_{\mathcal{Q}^{-1}_{0,\theta}(T)}^{1} \frac{\rho[1-\hat{p}(u)]}{\delta u-\delta\theta\hat{q}(u)}\mathrm{d}u\right) \times \left[\mathcal{Q}^{-1}_{0,\theta}(T)\right]^{M_0},$$

where

$$\mathcal{Q}_{0,\theta}(L) =: \int_L^1 \frac{\mathrm{d}u}{\delta u-\delta\theta\hat{q}(u)}, \quad 0 \leq \theta < 1. \tag{7.15}$$

Proof. By setting $t = 0$, $\varphi = 1$ and assuming $N_0 = 0$ in *Theorem 7.2.2*, we have

$$\mathbb{E}\Big[\theta^{N_T}\Big|M_0\Big] = e^{-c(T)}[A(0)]^{M_0}, \tag{7.16}$$

where $A(0)$ is uniquely determined by the non-linear ODE

$$A'(t) + \delta\theta\hat{q}(A(t)) - \delta A(t) = 0,$$

with boundary condition $A(T) = 1$. It can be solved, under the condition $\mu_{1_Q} < 1$, by the following steps:

1. Set $A(t) = L(T-t)$ and $\tau = T - t$,

$$\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau} = \delta\theta\hat{q}(L(\tau)) - \delta L(\tau) =: f_2(L), \quad 0 \le \theta < 1, \tag{7.17}$$

   with initial condition $L(0) = 1$; we define the right-hand side as the function $f_2(L)$.

2. There is only one positive singular point in the interval $[0, 1]$, denoted by

$$0 \le \varphi^* \le 1, \tag{7.18}$$

   by solving the equation $f_2(L) = 0$. This is because, for the case $0 < \theta < 1$, the equation $f_2(L) = 0$ is equivalent to

$$\hat{q}(u) = \frac{1}{\theta}u, \quad 0 < \theta < 1,$$

   note that $\hat{q}(\cdot)$ is a convex function, then it is clear that there is only one positive solution within $[0, 1]$ to this equation; in particularly when $\theta \to 0$, $\varphi^* \to 0$; then, we have $f_2(L) < 0$ for $\varphi^* < L \le 1$.

3. Rewrite (7.17) as

$$\frac{\mathrm{d}L}{\delta L - \delta\theta\hat{q}(L)} = -\mathrm{d}\tau,$$

   and integrate,

$$\int_L^1 \frac{\mathrm{d}u}{\delta u - \delta\theta\hat{q}(u)} = \tau,$$

   where $\varphi^* < L \le 1$, we define the function on left-hand side as

$$\mathcal{Q}_{0,\theta}(L) =: \int_L^1 \frac{\mathrm{d}u}{\delta u - \delta\theta\hat{q}(u)},$$

   then,

$$\mathcal{Q}_{0,\theta}(L) = \tau,$$

   as $L \to 1$ when $\tau \to 0$, and $L \to \varphi^*$ when $\tau \to \infty$; the integrand is positive in the domain $u \in (\varphi^*, 1]$ and $L \ge 0$, $\mathcal{Q}_{0,\theta}(L)$ is a strictly decreasing function; therefore, $\mathcal{Q}_{0,\theta}(L) : (\varphi^*, 1] \to [0, \infty)$ is a well defined function, and its inverse function $\mathcal{Q}_{0,\theta}^{-1}(\tau) : [0, \infty) \to (\varphi^*, 1]$ exists.

4. The unique solution is found by

$$L(\tau) = \mathcal{Q}_{0,\theta}^{-1}(\tau),$$

   or,

$$A(t) = \mathcal{Q}_{0,\theta}^{-1}(T - t).$$

5. $A(0)$ is obtained,

$$A(0) = L(T) = \mathcal{Q}_{0,\theta}^{-1}(T).$$

Then, $c(T)$ is determined by

$$c(T) = \rho \int_0^T \left[1 - \hat{p}\left(\mathcal{Q}_{0,\theta}^{-1}(\tau)\right)\right] \mathrm{d}\tau,$$

where, by the change of variable,

$$\int_0^T \left[1 - \hat{p}\left(\mathcal{Q}_{0,\theta}^{-1}(\tau)\right)\right] \mathrm{d}\tau \quad = \quad \int_{\mathcal{Q}_{0,\theta}^{-1}(T)}^1 \frac{1 - \hat{p}(u)}{\delta u - \delta\theta\hat{q}(u)} \mathrm{d}u.$$

□

Alternatively, with the aid of the transformation, we can directly prove *Theorem 7.2.7* from *Theorem 2.2.4*.

Proof. Review the proof of *Theorem 2.2.4*, as given by (7.16), we have

$$\mathbb{E}\left[\theta^{N_T} \middle| M_0\right] = e^{-c(T)}[A(0)]^{M_0},$$

where $A(0)$ can be alternatively solved via the transformation. By definition (2.20), we have

$$\begin{aligned}
\mathcal{G}_{0,\theta}(L) &= \int_0^L \frac{1}{1 - \delta u - \theta\hat{g}(u)} \mathrm{d}u \\
&= \int_0^L \frac{1}{1 - \delta u - \theta\hat{q}(1 - \delta u)} \mathrm{d}u \qquad (s = 1 - \delta u) \\
&= \int_{1-\delta L}^1 \frac{1}{\delta s - \delta\theta\hat{q}(s)} \mathrm{d}s.
\end{aligned}$$

Define $\mathcal{Q}_{0,\theta}(L)$ by (7.15), then,

$$\mathcal{Q}_{0,\theta}(1 - \delta u) = \mathcal{G}_{0,\theta}(u).$$

Since

$$\mathcal{Q}_{0,\theta}\left(1 - \delta\mathcal{G}_{0,\theta}^{-1}(T)\right) = \mathcal{G}_{0,\theta}\left(\mathcal{G}_{0,\theta}^{-1}(T)\right) = T,$$

then,

$$\mathcal{Q}_{0,\theta}^{-1}(T) = 1 - \delta\mathcal{G}_{0,\theta}^{-1}(T).$$

Hence, by transformation (7.7),

$$A(0) = 1 - \delta B(0) = 1 - \delta\mathcal{G}_{0,\theta}^{-1}(T) = \mathcal{Q}_{0,\theta}^{-1}(T),$$

and

$$\begin{aligned}
c(T) &= \int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{\rho[1 - \hat{h}(u)]}{1 - \delta u - \theta\hat{g}(u)} \mathrm{d}u \\
&= \int_0^{\frac{1-\mathcal{Q}_{0,\theta}^{-1}(T)}{\delta}} \frac{\rho[1 - \hat{p}(1 - \delta u)]}{1 - \delta u - \theta\hat{q}(1 - \delta u)} \mathrm{d}u \qquad (s = 1 - \delta u) \\
&= \int_0^{\frac{1-\mathcal{Q}_{0,\theta}^{-1}(T)}{\delta}} \frac{\rho[1 - \hat{p}(1 - \delta u)]}{1 - \delta u - \theta\hat{q}(1 - \delta u)} \mathrm{d}u \\
&= \int_{\mathcal{Q}_{0,\theta}^{-1}(T)}^1 \frac{\rho[1 - \hat{p}(s)]}{\delta s - \delta\theta\hat{q}(s)} \mathrm{d}s.
\end{aligned}$$

□

7.3 Some Special Cases of Discretised Dynamic Contagion Process

In this section, we discuss the distributional properties for the following three special cases of the discretised dynamic contagion process:

$$\text{Case } p_1 = 1: \quad p_1 = 1, \{p_k\}_{k\neq 1} = 0; \quad q_0 = 1, \{q_k\}_{k\neq 0} = 0; \tag{7.19}$$
$$\text{Case } q_1 = q: \quad p_1 = 1, \{p_k\}_{k\neq 1} = 0; \quad q_0 = 1-q, q_1 = q, \{q_k\}_{k=2,3,\ldots} = 0; 0 \leq q < 1; \tag{7.20}$$
$$\text{Case } q_0 = 1: \quad q_0 = 1, \{q_k\}_{k\neq 0} = 0. \tag{7.21}$$

7.3.1 Case $p_1 = 1$

The case $p_1 = 1$ (7.19) of discretised dynamic contagion process can be defined by

$$\begin{aligned} P\left\{M_{t+\Delta t} - M_t = 1, N_{t+\Delta t} - N_t = 0 \middle| M_t, N_t\right\} &= \rho\Delta t + o(\Delta t), \\ P\left\{M_{t+\Delta t} - M_t = -1, N_{t+\Delta t} - N_t = 1 \middle| M_t, N_t\right\} &= \delta M_t \Delta t + o(\Delta t), \\ P\left\{M_{t+\Delta t} - M_t = 0, N_{t+\Delta t} - N_t = 0 \middle| M_t, N_t\right\} &= 1 - \Big(\rho + \delta M_t\Big)\Delta t + o(\Delta t), \\ P\left\{\text{Others} \middle| M_t, N_t\right\} &= o(\Delta t); \end{aligned}$$

with the generator

$$\mathcal{A}f(m,n,t) = \frac{\partial f}{\partial t} + \rho\Big(f(m+1,n,t) - f(m,n,t)\Big) + \delta m\Big(f(m-1,n+1,t) - f(m,n,t)\Big). \tag{7.22}$$

We will apply this case and extensions to ruin problem in Chapter 8.

Corollary 7.3.1. *Assume $N_0 = 0$, we have*

$$\mathbb{E}\left[\theta^{N_T}\varphi^{M_T}\middle|M_0\right] = \exp\left(-\rho\left((1-\theta)T - (\varphi-\theta)\frac{1-e^{-\delta T}}{\delta}\right)\right)\left[(\varphi-\theta)e^{-\delta T} + \theta\right]^{M_0}.$$

If $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, then,

$$\mathbb{E}\left[\theta^{N_T}\varphi^{M_T}\right] = \exp\left(-(1-\theta)(\rho T + \zeta) + (\varphi-\theta)\left[\frac{\rho}{\delta}\left(1-e^{-\delta T}\right) + \zeta e^{-\delta T}\right]\right);$$

in particular, for $\zeta = \frac{\rho}{\delta}$,

$$\mathbb{E}\left[\theta^{N_T}\varphi^{M_T}\right] = e^{-\rho T(1-\theta)}e^{-\frac{\rho}{\delta}(1-\varphi)}.$$

Proof. By *Theorem 7.2.2*, solve

$$A'(t) + \delta\theta - \delta A(t) = 0,$$

with boundary condition $A(T) = \varphi$, and we have the solution

$$A(t) = (\varphi-\theta)e^{-\delta(T-t)} + \theta,$$

then,

$$\begin{aligned} A(0) &= (\varphi-\theta)e^{-\delta T} + \theta, \\ C(T) &= \rho\left((1-\theta)T - (\varphi-\theta)\frac{1-e^{-\delta T}}{\delta}\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} \mathbb{E}\left[\theta^{N_T}\varphi^{M_T}\middle|M_0\right] &= e^{-c(T)}[A(0)]^{M_0} \\ &= \exp\left(-\rho\left((1-\theta)T - (\varphi-\theta)\frac{1-e^{-\delta T}}{\delta}\right)\right)\left[(\varphi-\theta)e^{-\delta T} + \theta\right]^{M_0}. \end{aligned}$$

If $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, then,

$$\begin{aligned}
\mathbb{E}\left[\theta^{N_T}\varphi^{M_T}\right] &= \exp\left(-\rho\left((1-\theta)T-(\varphi-\theta)\frac{1-e^{-\delta T}}{\delta}\right)\right)\mathbb{E}\left[\left[(\varphi-\theta)e^{-\delta T}+\theta\right]^{M_0}\right] \\
&= \exp\left(-\rho\left((1-\theta)T-(\varphi-\theta)\frac{1-e^{-\delta T}}{\delta}\right)\right)\exp\left(-\zeta\left(1-\theta-(\varphi-\theta)e^{-\delta T}\right)\right) \\
&= \exp\left(-(1-\theta)(\rho T+\zeta)+(\varphi-\theta)\left[\frac{\rho}{\delta}\left(1-e^{-\delta T}\right)+\zeta e^{-\delta T}\right]\right).
\end{aligned}$$

□

Corollary 7.3.2. *If* $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, *then,*

$$M_T \sim \text{Poisson}\left(\frac{\rho}{\delta}\left(1-e^{-\delta T}\right)+\zeta e^{-\delta T}\right);$$

in particular, the stationary distribution of M_t *is given*

$$\left\{M_t\right\}_{t\geq 0} \sim \text{Poisson}\left(\frac{\rho}{\delta}\right).$$

Proof. By *Theorem 7.2.4*, we have

$$\begin{aligned}
\mathcal{Q}_{\varphi,1}(L) &= -\frac{1}{\delta}\ln\left(\frac{1-L}{1-\varphi}\right), \\
\mathcal{Q}_{\varphi,1}^{-1}(T) &= 1-(1-\varphi)e^{-\delta T},
\end{aligned}$$

then,

$$\begin{aligned}
\mathbb{E}[\varphi^{M_T}|M_0] &= \exp\left(-\int_{\varphi}^{\mathcal{Q}_{\varphi,1}^{-1}(T)}\frac{\rho[1-u]}{\delta-\delta u}\mathrm{d}u\right)\left[\mathcal{Q}_{\varphi,1}^{-1}(T)\right]^{M_0} \\
&= \exp\left(-\frac{\rho}{\delta}\left(1-e^{-\delta T}\right)(1-\varphi)\right)\left[1-(1-\varphi)e^{-\delta T}\right]^{M_0}.
\end{aligned}$$

If $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, then,

$$\begin{aligned}
\mathbb{E}[\varphi^{M_T}] &= \mathbb{E}\left[\mathbb{E}[\varphi^{M_T}|M_0]\right] \\
&= \exp\left(-\frac{\rho}{\delta}\left(1-e^{-\delta T}\right)(1-\varphi)\right)\mathbb{E}\left[\left[1-(1-\varphi)e^{-\delta T}\right]^{M_0}\right] \\
&= \exp\left(-\left(\frac{\rho}{\delta}\left(1-e^{-\delta T}\right)+\zeta e^{-\delta T}\right)(1-\varphi)\right).
\end{aligned}$$

By *Theorem 7.2.6*, we have

$$\begin{aligned}
\hat{\aleph}(\varphi) &= \exp\left(-\int_{\varphi}^{1}\frac{\rho\left[1-\sum_{k=0}^{\infty}u^k p_k\right]}{\delta\sum_{k=0}^{\infty}u^k q_k-\delta u}\mathrm{d}u\right) \\
&= \exp\left(-\int_{\varphi}^{1}\frac{\rho\left[1-u\right]}{\delta-\delta u}\mathrm{d}u\right) \\
&= e^{-\frac{\rho}{\delta}(1-\varphi)},
\end{aligned}$$

which is the probability generating function of a Poisson distribution with constant intensity $\frac{\rho}{\delta}$. □

Corollary 7.3.3. *Assume* $N_0 = 0$, *we have*

$$\mathbb{E}\left[\theta^{N_T}|M_0\right] = \exp\left(-\frac{\rho}{\delta}\left[\delta T-1+e^{-\delta T}\right](1-\theta)\right)\left[\theta+(1-\theta)e^{-\delta T}\right]^{M_0}; \tag{7.23}$$

if $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, *then,*

$$N_T \sim \text{Poisson}\left(\rho\left(T - \frac{1-e^{-\delta T}}{\delta}\right) + \zeta\left(1-e^{-\delta T}\right)\right);$$

in particular, if $M_0 \sim \aleph \sim \text{Poisson}\left(\frac{\rho}{\delta}\right)$, *then* N_T *is a* ρ*-Poisson process, i.e.*

$$N_T \sim \text{Poisson}(\rho T).$$

Proof. By *Theorem 7.2.7*, we have

$$\begin{aligned}\mathcal{Q}_{0,\theta}(L) &= \frac{1}{\delta}\ln\left(\frac{1-\theta}{L-\theta}\right), \\ \mathcal{Q}^{-1}_{0,\theta}(T) &= \theta + (1-\theta)e^{-\delta T},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\big[\theta^{N_T}\big|M_0\big] &= \exp\left(-\int_{\mathcal{Q}^{-1}_{0,\theta}(T)}^{1}\frac{\rho[1-u]}{\delta u - \delta\theta}\mathrm{d}u\right)\left[\mathcal{Q}^{-1}_{0,\theta}(T)\right]^{M_0}, \\ &= \exp\left(-\frac{\rho}{\delta}\left[\delta T - 1 + e^{-\delta T}\right](1-\theta)\right)\left[\theta + (1-\theta)e^{-\delta T}\right]^{M_0};\end{aligned}$$

if $M_0 \sim \text{Poisson}(\zeta)$, then,

$$\begin{aligned}\mathbb{E}\big[\theta^{N_T}\big] &= \mathbb{E}\left[\mathbb{E}\big[\theta^{N_T}\big|M_0\big]\right] \\ &= \exp\left(-\frac{\rho}{\delta}\left[\delta T - 1 + e^{-\delta T}\right](1-\theta)\right)\mathbb{E}\left[\left[\theta + (1-\theta)e^{-\delta T}\right]^{M_0}\right] \\ &= \exp\left(-\frac{\rho}{\delta}\left[\delta T - 1 + e^{-\delta T}\right](1-\theta)\right)\exp\left(-\zeta\left(1 - \left[\theta + (1-\theta)e^{-\delta T}\right]\right)\right) \\ &= \exp\left(-\left[\rho\left(T - \frac{1-e^{-\delta T}}{\delta}\right) + \zeta\left(1-e^{-\delta T}\right)\right](1-\theta)\right);\end{aligned}$$

in particular, if $M_0 \sim \text{Poisson}\left(\frac{\rho}{\delta}\right)$, then, $\mathbb{E}[\theta^{N_T}] = e^{-\rho T(1-\theta)}$.

□

Corollary 7.3.4. *If* $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, *then* M_T *and* N_T *are independent.*

Proof. If $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, then, by *Corollary 7.3.1*, *Corollary 7.3.2* and *Corollary 7.3.3*, we have

$$\begin{aligned}\mathbb{E}\left[\theta^{N_T}\varphi^{M_T}\right] &= \exp\left(-(1-\theta)(\rho T + \zeta) + (\varphi - \theta)\left[\frac{\rho}{\delta}\left(1-e^{-\delta T}\right) + \zeta e^{-\delta T}\right]\right), \\ M_T &\sim \text{Poisson}\left(\frac{\rho}{\delta}\left(1-e^{-\delta T}\right) + \zeta e^{-\delta T}\right), \\ N_T &\sim \text{Poisson}\left(\rho\left(T - \frac{1-e^{-\delta T}}{\delta}\right) + \zeta\left(1-e^{-\delta T}\right)\right).\end{aligned}$$

Hence,

$$\mathbb{E}\left[\theta^{N_T}\varphi^{M_T}\right] = \mathbb{E}\left[\varphi^{M_T}\right]\mathbb{E}\left[\theta^{N_T}\right], \quad \forall\varphi,\theta \geq 0.$$

□

More importantly, by assuming $N_0 = 0$ and $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, we prove that for the case $p_1 = 1$ (7.19), N_t is a non-homogeneous Poisson process as follows.

Lemma 7.3.1. *For a constant* $0 \leq \theta \leq 1$, *we have a* $\mathcal{F}$*-martingale*

$$[A(t)]^{M_t}\theta^{N_t}e^{c(t)},$$

where $A(t)$ *and* $c(t)$ *are determined by the ODEs*

$$\begin{aligned} A'(t) + \delta\left(\theta - A(t)\right) &= 0, &(7.24)\\ c'(t) + \rho\Big(A(t) - 1\Big) &= 0. &(7.25)\end{aligned}$$

Proof. Assume the exponential affine form

$$f(m,n,t) = [A(t)]^m \theta^n e^{c(t)},$$

and set $\mathcal{A}f(m,n,t) = 0$ in the generator (7.22) such that $f(m,n,t)$ becomes a $\mathcal{F}$-martingale, then, for any m, we have

$$c'(t) + m\frac{A'(t)}{A(t)} + \rho\Big(A(t) - 1\Big) + \delta m\left(\frac{\theta}{A(t)} - 1\right) = 0,$$

hence, the equations of $A(t)$ and $c(t)$,

$$\begin{aligned} \frac{A'(t)}{A(t)} + \delta\left(\frac{\theta}{A(t)} - 1\right) &= 0,\\ c'(t) + \rho\Big(A(t) - 1\Big) &= 0.\end{aligned}$$

□

Theorem 7.3.1. *For any time* $t_2 > t_1 \geq 0$, *if* $M_{t_1} \sim \text{Poisson}(\upsilon)$, $\upsilon \geq 0$, *then,*

$$\begin{aligned} M_{t_2} &\sim \text{Poisson}\left(\upsilon e^{-\delta(t_2-t_1)} + \rho\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right),\\ N_{t_2} - N_{t_1} &\sim \text{Poisson}\left(\upsilon\left(1-e^{-\delta(t_2-t_1)}\right) + \rho\left((t_2-t_1) - \frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right)\right),\end{aligned}$$

and also they are independent.

Proof. Set the boundary condition $A(t_2) = \varphi$, $0 \leq \varphi \leq 1$, in *Lemma 7.3.1*, the equations (7.24) and (7.25) of $A(t)$ and $c(t)$ can be solved explicitly, and we have

$$\begin{aligned} A(t) &= (\varphi - \theta)e^{-\delta(t_2-t)} + \theta,\\ c(t_2) - c(t_1) &= \rho\left((1-\theta)(t_2-t_1) - (\varphi-\theta)\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right).\end{aligned}$$

Since $[A(t)]^{M_t}\theta^{N_t}e^{c(t)}$ is a $\mathcal{F}$-martingale, we have

$$\mathbb{E}\left[[A(t_2)]^{M_{t_2}}\theta^{N_{t_2}}e^{c(t_2)}\middle|\mathcal{F}_{t_1}\right] = [A(t_1)]^{M_{t_1}}\theta^{N_{t_1}}e^{c(t_1)},$$

and

$$\mathbb{E}\left[\varphi^{M_{t_2}}\theta^{N_{t_2}-N_{t_1}}\middle|M_{t_1}\right] = [A(t_1)]^{M_{t_1}}e^{-\Big(c(t_2)-c(t_1)\Big)}.$$

Then, the joint probability generating function of M_{t_2} and $N_{t_2} - N_{t_1}$ is given by

$$\begin{aligned}
& \mathbb{E}\left[\varphi^{M_{t_2}}\theta^{N_{t_2}-N_{t_1}}\right] \\
= & \mathbb{E}\left[\mathbb{E}\left[\varphi^{M_{t_2}}\theta^{N_{t_2}-N_{t_1}}\middle|M_{t_1}\right]\right] \\
= & \mathbb{E}\left[[A(t_1)]^{M_{t_1}}\right]e^{-\left(c(t_2)-c(t_1)\right)} \\
= & e^{-\upsilon(1-A(t_1))}e^{-\left(c(t_2)-c(t_1)\right)} \\
= & \exp\left(-\upsilon\left[(1-\theta)-(\varphi-\theta)e^{-\delta(t_2-t_1)}\right]-\rho\left[(1-\theta)(t_2-t_1)-(\varphi-\theta)\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right]\right).
\end{aligned}$$

Set $\theta = 1$ and $\varphi = 1$, respectively, we have the marginal distributions of M_{t_2} and $N_{t_2} - N_{t_1}$,

$$\begin{aligned}
\mathbb{E}\left[\varphi^{M_{t_2}}\right] &= \exp\left(-(1-\varphi)\left[\upsilon e^{-\delta(t_2-t_1)}+\rho\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right]\right), \\
\mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\right] &= \exp\left(-(1-\theta)\left[\upsilon\left(1-e^{-\delta(t_2-t_1)}\right)+\rho\left((t_2-t_1)-\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right)\right]\right) \quad (7.26)
\end{aligned}$$

Obviously, we also have

$$\mathbb{E}\left[\varphi^{M_{t_2}}\theta^{N_{t_2}-N_{t_1}}\right] = \mathbb{E}\left[\varphi^{M_{t_2}}\right]\mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\right].$$

Therefore, M_{t_2} and $N_{t_2} - N_{t_1}$ are Poisson distributed and also independent.

□

Corollary 7.3.5. *If $N_0 = 0$, $M_0 \sim \text{Poisson}(\zeta)$, then, M_t and N_t are independent and follow Poisson distributions given by*

$$\begin{aligned}
M_t &\sim \text{Poisson}\left(\zeta e^{-\delta t}+\rho\frac{1-e^{-\delta t}}{\delta}\right), \\
N_t &\sim \text{Poisson}\left(\zeta\left(1-e^{-\delta t}\right)+\rho\left(t-\frac{1-e^{-\delta t}}{\delta}\right)\right),
\end{aligned}$$

Proof. Given the initial conditions $N_0 = 0$ and $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, we set $t_1 = 0$, any time $t_2 = t > 0$ and $\upsilon = \zeta \geq 0$ in *Theorem 7.3.1*, and the results follow immediately.

□

Corollary 7.3.6. *If $N_0 = 0$, $M_0 \sim \text{Poisson}(\zeta)$, then, N_t is a non-homogeneous Poisson process of rate $\rho + (\zeta\delta - \rho)\, e^{-\delta t}$.*

Proof. For any time $t_2 > t_1 \geq 0$, by *Corollary 7.3.5*, we have

$$M_{t_1} \sim \text{Poisson}\left(\zeta e^{-\delta t_1}+\rho\frac{1-e^{-\delta t_1}}{\delta}\right).$$

By *Theorem 7.3.1*, set

$$\upsilon = \zeta e^{-\delta t_1}+\rho\frac{1-e^{-\delta t_1}}{\delta}$$

in (7.26), then,

$$\begin{aligned}
\mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\right] &= \exp\left(-(1-\theta)\left[-\zeta\left(e^{-\delta t_2}-e^{-\delta t_1}\right)+\rho\left((t_2-t_1)+\frac{e^{-\delta t_2}-e^{-\delta t_1}}{\delta}\right)\right]\right) \\
&= \exp\left(-(1-\theta)\int_{t_1}^{t_2}\left[\zeta\delta e^{-\delta s}+\rho\left(1-e^{-\delta s}\right)\right]\mathrm{d}s\right),
\end{aligned}$$

hence, the increments of N_t follow a Poisson distribution,

$$N_{t_2} - N_{t_1} \sim \text{Poisson}\left(\int_{t_1}^{t_2} \left[\zeta\delta e^{-\delta s} + \rho\left(1 - e^{-\delta s}\right)\right] \mathrm{d}s\right).$$

Based on *Theorem 7.3.1* and *Corollary 7.3.5*, we observe that M_{t_2} and $N_{t_2} - N_{t_1}$ are both Poisson distributed and crucially independent. Because of the Markov property, all the future increments after N_{t_2} only depend on M_{t_2}, they are independent of $N_{t_2} - N_{t_1}$ as well, i.e. for any random variable $X \in \sigma\left\{\mathcal{N}_s : N_s - N_{t_2}, s \geq t_2\right\}$, we have

$$\begin{aligned}
\mathbb{E}\left[X\theta^{N_{t_2}-N_{t_1}}\right] &= \mathbb{E}\left[\mathbb{E}\left[X\theta^{N_{t_2}-N_{t_1}}\middle|M_{t_2}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X\middle|M_{t_2}\right]\mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\middle|M_{t_2}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X\middle|M_{t_2}\right]\right]\mathbb{E}\left[\mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\middle|M_{t_2}\right]\right] \\
&= \mathbb{E}\left[X\right]\theta^{N_{t_2}-N_{t_1}}.
\end{aligned}$$

The increments of the point process N_t follow a Poisson distribution and also they are independent, therefore, N_t is a non-homogeneous Poisson process of rate $\zeta\delta e^{-\delta t} + \rho\left(1 - e^{-\delta t}\right)$.

□

Remark 7.3.1. In particular, if and only if $\zeta = \frac{\rho}{\delta}$, N_t is a Poisson process with rate of ρ independent from time t.

7.3.2 Case $q_1 = q$

Corollary 7.3.7. *For the case $q_1 = q$ (7.20), the stationary distribution of M_t is given*

$$\left\{M_t\right\}_{t\geq 0} \sim \text{Poisson}\left(\frac{\rho}{\delta(1-q)}\right).$$

Proof. By *Theorem 7.2.6*, we have

$$\begin{aligned}
\hat{\aleph}(\varphi) &= \exp\left(-\int_{\varphi}^{1} \frac{\rho\left[1 - \sum_{k=0}^{\infty} u^k p_k\right]}{\delta\sum_{k=0}^{\infty} u^k q_k - \delta u}\mathrm{d}u\right) \\
&= \exp\left(-\int_{\varphi}^{1} \frac{\rho\left[1 - u\right]}{\delta(1 - q + uq - u)}\mathrm{d}u\right) \\
&= e^{-\frac{\rho}{\delta(1-q)}(1-\varphi)},
\end{aligned}$$

which is the probability generating function of a Poisson distribution with constant intensity $\frac{\rho}{\delta(1-q)}$.

□

Corollary 7.3.8. *For the case $q_1 = q$ (7.20), we have*

$$\mathbb{E}\left[\theta^{N_T}\middle|M_0\right] = \exp\left(-\frac{\rho}{\delta}\frac{1-\theta}{1-\theta q}\left[\delta T - \frac{1 - e^{-(1-\theta q)\delta T}}{1-\theta q}\right]\right)\left[\frac{\theta(1-q) + (1-\theta)e^{-(1-\theta q)\delta T}}{1-\theta q}\right]^{M_0}; \tag{7.27}$$

if $M_0 \sim \aleph \sim$ Poisson $\left(\frac{\rho}{\delta(1-q)}\right)$, then we have

$$\mathbb{E}\left[\theta^{N_T}\right] = \exp\left(-\rho T\left(1 - \frac{1-q}{1-\theta q}\theta\right)\right)\exp\left(-\frac{\rho}{\delta}\frac{q}{1-q}\left(1 - \frac{1-q}{1-\theta q}\theta\right)^2\left(1 - e^{-(1-\theta q)\delta T}\right)\right). \tag{7.28}$$

Proof. Note that the stationarity condition holds as $\mu_{1_Q} = q < 1$. By *Theorem 7.2.7*, we have

$$\begin{aligned}
\mathcal{Q}_{0,\theta}(L) &= \frac{1}{\delta}\frac{1}{1-\theta q}\ln\left(\frac{1-\theta}{(1-\theta q)L-\theta(1-q)}\right), \quad 0 \le \theta < 1,\\
\mathcal{Q}_{0,\theta}^{-1}(T) &= \frac{\theta(1-q)+(1-\theta)e^{-(1-\theta q)\delta T}}{1-\theta q},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\Big[\theta^{N_T}\big|M_0\Big] &= \exp\left(-\int_{\mathcal{Q}_{0,\theta}^{-1}(T)}^{1}\frac{\rho[1-u]}{\delta u-\delta\theta(1-q+uq)}\mathrm{d}u\right)\Big[\mathcal{Q}_{0,\theta}^{-1}(T)\Big]^{M_0},\\
&= \exp\left(-\frac{\rho}{\delta}\frac{1-\theta}{1-\theta q}\left[\delta T-\frac{1-e^{-(1-\theta q)\delta T}}{1-\theta q}\right]\right)\left[\frac{\theta(1-q)+(1-\theta)e^{-(1-\theta q)\delta T}}{1-\theta q}\right]^{M_0},
\end{aligned}$$

then,

$$\begin{aligned}
\mathbb{E}\Big[\theta^{N_T}\Big] &= \mathbb{E}\left[\mathbb{E}\Big[\theta^{N_T}\big|M_0\Big]\right]\\
&= \exp\left(-\frac{\rho}{\delta}\frac{1-\theta}{1-\theta q}\left[\delta T-\frac{1-e^{-(1-\theta q)\delta T}}{1-\theta q}\right]\right)\mathbb{E}\left[\left[\frac{\theta(1-q)+(1-\theta)e^{-(1-\theta q)\delta T}}{1-\theta q}\right]^{M_0}\right]\\
&= \exp\left(-\rho T\left(1-\frac{1-q}{1-\theta q}\theta\right)\right)\exp\left(-\frac{\rho}{\delta}\frac{q}{1-q}\left(1-\frac{1-q}{1-\theta q}\theta\right)^2\left(1-e^{-(1-\theta q)\delta T}\right)\right).
\end{aligned}$$

□

Remark 7.3.2. The first term of $\mathbb{E}[\theta^{N_T}]$ of (7.28) is the probability generating function of a compound Poisson distribution N_1 with point $\mathring{N}_T \sim \text{Poisson}(\rho T)$ and underlying $X_1 \sim \text{Geometric}(1-q)$ where

$$P\{X_1 = j\} = q^{j-1}(1-q), \quad j = 1, 2, ..., \quad \mathbb{E}[\theta^{X_1}] = \frac{1-q}{1-\theta q}\theta;$$

the second term is the the probability generating function of a proper random variable $\tilde{O}$. Hence, $N_T = N_1 + \tilde{O}$, and N_T is stochastically larger than N_1, i.e.

$$N_T \succ N_1.$$

Note that, if $T \to \infty$, then $\mathbb{E}\Big[\theta^{N_T}\big|M_0\Big] \to 0$ and $\mathbb{E}\Big[\theta^{N_T}\Big] \to 0$. As we have explicit formulas of $\mathbb{E}\Big[\theta^{N_T}\big|M_0\Big]$ and $\mathbb{E}\Big[\theta^{N_T}\Big]$ for the case above, we can easily expand them by MatLab to obtain $P\{N_T = n\}$ for any $n = 0, 1, 2, ...$

7.3.3 Case $q_0 = 1$

The case $q_0 = 1$ (7.21) is an important case which matches to the Cox process with shot noise intensity (a special case of dynamic contagion process) by the transformation. It can be defined by

$$\begin{aligned}
P\left\{M_{t+\Delta t}-M_t = k, N_{t+\Delta t}-N_t = 0\big|M_t, N_t\right\} &= \rho p_k\Delta t + o(\Delta t), \quad k = 1, 2...,\\
P\left\{M_{t+\Delta t}-M_t = -1, N_{t+\Delta t}-N_t = 1\big|M_t, N_t\right\} &= \delta M_t\Delta t + o(\Delta t),\\
P\left\{M_{t+\Delta t}-M_t = 0, N_{t+\Delta t}-N_t = 0\big|M_t, N_t\right\} &= 1-\Big(\rho(1-p_0)+\delta M_t\Big)\Delta t + o(\Delta t),\\
P\left\{\text{Others}\big|M_t, N_t\right\} &= o(\Delta t),
\end{aligned}$$

with the generator

$$\mathcal{A}f(m,n,t) = \frac{\partial f}{\partial t} + \rho\left(\sum_{k=0}^{\infty} f(m+k,n,t)p_k - f(m,n,t)\right) + \delta m\Big(f(m-1,n+1,t) - f(m,n,t)\Big). \tag{7.29}$$

Corollary 7.3.9. *For the case* $q_0 = 1$ *(7.21), if* $\{p_k\}_{k=0,1,2,\dots} \sim \text{Geometric}(\acute{p})$*, then, the stationary distribution of* M_t *is given*

$$\Big\{M_t\Big\}_{t\geq 0} \sim \text{NegBin}\left(\frac{\rho}{\delta}, 1-\acute{p}\right).$$

Proof. Since

$$\hat{p}(u) = \frac{\acute{p}}{1-(1-\acute{p})u},$$

by *Theorem 7.2.6*, we have

$$\hat{\aleph}(\varphi) = \exp\left(-\int_{\varphi}^{1} \frac{\rho\left[1-\frac{\acute{p}}{1-(1-\acute{p})u}\right]}{\delta-\delta u}\text{d}u\right) = \left(\frac{\acute{p}}{1-(1-\acute{p})\varphi}\right)^{\frac{\rho}{\delta}},$$

which is the probability generating function of a negative binomial distribution with parameters $\frac{\rho}{\delta}$ and $1-\acute{p}$. □

Alternatively, we can use the direct distributional transformation between M_t and λ_t from *Corollary 7.2.4*.

Proof. By *Theorem 2.3.1*, for the shot noise case with externally excited jump sizes $Y^{(1)} \sim \text{Exp}(\alpha)$, we have

$$\{\lambda_t\}_{t\geq 0} \sim \text{Gamma}\left(\frac{\rho}{\delta}, \alpha\right),$$

and

$$\hat{\Pi}(v) = \left(\frac{\alpha}{\alpha+v}\right)^{\frac{\rho}{\delta}},$$

then, by the transformation (7.14), we have

$$\hat{\aleph}(\varphi) = \hat{\Pi}\left(\frac{1-\varphi}{\delta}\right) = \left(\frac{\alpha}{\alpha+\frac{1-\varphi}{\delta}}\right)^{\frac{\rho}{\delta}} = \left(\frac{1-\frac{1}{\alpha\delta+1}}{1-\frac{1}{\alpha\delta+1}\varphi}\right)^{\frac{\rho}{\delta}} = \left(\frac{\acute{p}}{1-(1-\acute{p})\varphi}\right)^{\frac{\rho}{\delta}}, \quad \acute{p} = \frac{\delta\alpha}{\delta\alpha+1}.$$

□

Corollary 7.3.10. *For the case* $q_0 = 1$ *(7.21), if* $\{p_k\}_{k=0,1,2,\dots} \sim \text{Geometric}(\acute{p})$*, then,*

$$\mathbb{E}\Big[\theta^{N_T}\Big|M_0\Big] = e^{-\rho T(1-\hat{p}(\theta))}\left(\frac{\acute{p}_T}{1-(1-\acute{p}_T\theta)}\right)^{-\frac{\rho}{\delta}\hat{p}(\theta)}\Big[(1-\theta)e^{-\delta T}+\theta\Big]^{M_0},$$

where

$$\acute{p}_T =: \frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}};$$

if $M_0 \sim \aleph \sim \text{NegBin}\left(\frac{\rho}{\delta}, 1-\acute{p}\right)$*, then,*

$$\mathbb{E}\Big[\theta^{N_T}\Big] = e^{-\rho T(1-\hat{p}(\theta))}\left(\frac{\acute{p}_T}{1-(1-\acute{p}_T\theta)}\right)^{\frac{\rho}{\delta}(1-\hat{p}(\theta))}. \tag{7.30}$$

Proof. By *Theorem 7.2.7*, we have

$$\begin{aligned} \mathcal{Q}_{0,\theta}(L) &= \frac{1}{\delta}\ln\left(\frac{1-\theta}{L-\theta}\right), \quad 0\le\theta<1, \\ \mathcal{Q}_{0,\theta}^{-1}(T) &= (1-\theta)e^{-\delta T}+\theta, \end{aligned}$$

since

$$\hat{p}(u)=\frac{\acute{p}}{1-(1-\acute{p})u},$$

we have

$$\begin{aligned} \mathbb{E}\Big[\theta^{N_T}\big|M_0\Big] &= \exp\left(-\int_{\mathcal{Q}_{0,\theta}^{-1}(T)}^{1}\frac{\rho\left[1-\frac{\acute{p}}{1-(1-\acute{p})u}\right]}{\delta u-\delta\theta}\mathrm{d}u\right)\Big[\mathcal{Q}_{0,\theta}^{-1}(T)\Big]^{M_0} \\ &= e^{-\rho T(1-\hat{p}(\theta))}\left(\frac{\acute{p}}{1-(1-\acute{p})\Big[(1-\theta)e^{-\delta T}+\theta\Big]}\right)^{-\frac{\rho}{\delta}\hat{p}(\theta)}\Big[(1-\theta)e^{-\delta T}+\theta\Big]^{M_0} \\ &= e^{-\rho T(1-\hat{p}(\theta))}\left(\frac{\frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}}}{1-\Big(1-\frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}}\theta\Big)}\right)^{-\frac{\rho}{\delta}\hat{p}(\theta)}\Big[(1-\theta)e^{-\delta T}+\theta\Big]^{M_0}. \end{aligned}$$

If $M_0\sim\aleph\sim\mathrm{NegBin}\left(\frac{\rho}{\delta},1-\acute{p}\right)$, then we have

$$\begin{aligned} \mathbb{E}\Big[\theta^{N_T}\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\theta^{N_T}\big|M_0\Big]\Big] \\ &= e^{-\rho T(1-\hat{p}(\theta))}\left(\frac{\frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}}}{1-\Big(1-\frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}}\theta\Big)}\right)^{-\frac{\rho}{\delta}\hat{p}(\theta)}\mathbb{E}\left[\Big[(1-\theta)e^{-\delta T}+\theta\Big]^{M_0}\right] \\ &= e^{-\rho T(1-\hat{p}(\theta))}\left(\frac{\frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}}}{1-\Big(1-\frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}}\theta\Big)}\right)^{-\frac{\rho}{\delta}\hat{p}(\theta)}\left(\frac{\acute{p}}{1-(1-\acute{p})\Big[(1-\theta)e^{-\delta T}+\theta\Big]}\right)^{\frac{\rho}{\delta}} \\ &= e^{-\rho T(1-\hat{p}(\theta))}\left(\frac{\frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}}}{1-\Big(1-\frac{\acute{p}}{1-(1-\acute{p})e^{-\delta T}}\theta\Big)}\right)^{\frac{\rho}{\delta}(1-\hat{p}(\theta))}. \end{aligned}$$

□

Remark 7.3.3. The first term of of $\mathbb{E}[\theta^{N_T}]$ of (7.30) is the probability generating function of a compound Poisson distribution N_2 with point $\mathring{N}_T\sim\mathrm{Poisson}(\rho T)$ and underlying $X_2\sim\mathrm{Geometric}(\acute{p})$ where

$$P\{X_2=j\}=(1-\acute{p})^j\acute{p},\quad j=0,1,2,...,\quad \mathbb{E}[\theta^{X_2}]=\frac{\acute{p}}{1-(1-\acute{p})\theta};$$

the second term of (7.30) is the probability generating function of a proper random variable $\tilde{O}$. Hence, we have $N_T=N_2+\tilde{O}$, and N_T is stochastically larger than N_2, i.e.

$$N_T\succ N_2.$$

Note that,

$$\lim_{T\to\infty}\acute{p}_T=\acute{p},$$

for a large T, we have

$$\mathbb{E}\Big[\theta^{N_T}\Big]\approx e^{-\rho T(1-\hat{p}(\theta))}\Big[\hat{p}(\theta)\Big]^{\frac{\rho}{\delta}(1-\hat{p}(\theta))}.$$

8

Applications to Insurance: Ruin by Delayed Claims

In this chapter, we apply a special case (when $p_1 = 1$ and some generalisation) of discretised dynamic contagion process introduced by Chapter 7 to ruin theory, by introducing a simple risk model with delayed claims, an extension of the classical Poisson model. The arrival of claims is assumed to be a Poisson process, claims follow a light-tailed distribution and each loss payment of the claims will be settled with a random period of delay. We obtain asymptotic expressions for the ruin probability by exploiting a connection to Poisson models that are not time-homogeneous. A finer asymptotic formula is obtained for the special case of exponentially delayed claims and an exact formula when the claims are also exponentially distributed.

8.1 Introduction

In a variety of real situations, claims could have already occurred but have not been settled or reported immediately. Many factors may lead to the delay of the actual loss payment of the claims. For instance, the acronyms, such as IBNR (Incurred But Not Reported) and IBNR (Reported But Not Settled) are typically used to classify the delayed claims by different reasonings.

In the literature, the issues of ruin problem involving delayed claim settlement have been studied. Waters and Papatriandafylou (1985) and Trufin, Albrecher and Denuit (2011) considered a discrete-time model for a risk process allowing claims being delayed. Boogaert and Haezendonck (1989) discussed a liability process with settling delay in the framework of economical environment. Yuen, Guo and Ng (2005) introduced a continuous-time model with one claim settled immediately and the other claim (named 'by-claim') settled with delay for the each time of claim occurrences. Delaying claims were also modelled by a Poisson shot noise process, see Klüppelberg and Mikosch (1995) and Brémaud (2000), or by a shot noise Cox process, see also Macci and Torrisi (2004) and Albrecher and Asmussen (2006).

This chapter introduces a simple delayed-claim model. We assume claims arrive as a Poisson process, claims follow a light-tailed distribution, i.e. the distribution of claims has moment generating function, and each of the claims will be settled in a randomly delayed period of time. The loss of each claim payment only occurs at the settlement time, rather than at the arrival time. In particular, we consider the special case of exponential delay where the ultimate ruin probability and asymptotics can be exactly obtained by a power series, and this is also a simplified version of the model by Yuen, Guo and Ng (2005) without the immediate settled claims.

This chapter is organised as follows. Section 8.2 introduces our model setting of the delayed-claim risk process and the underlying processes of claim arrival, delay and settlement. Section 8.3 derives an asymptotic formula for the ruin probability for the general case of delay, and in particular, exploit a well known connection to the non-homogeneous Poisson models. For the special case of exponential delay, the Laplace transform of non-ruin probability and a finer asymptotic expansion for the ruin probability are obtained in Section 8.4. Section 8.5 derives an exact formula of ruin probability by assuming the claims are exponentially delayed and sizes are exponentially distributed.

8.2 Risk Process

Consider a surplus process $\{X_t\}_{t\geq 0}$ in continuous time on a probability space $(\Omega, \mathcal{F}, P)$,

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0,$$

where

- $x = X_0 \geq 0$ is the initial reserve at time $t = 0$;
- $c > 0$ is the constant rate of premium payment per time unit;
- N_t is the number of cumulative settled claims within the time interval $[0, t]$ and assume $N_0 = 0$;
- $\{Z_i\}_{i=1,2,\dots}$ is a sequence of independent and identically distributed positive random variables (claims sizes), independent of N_t, following a light-tailed distribution with the cumulative distribution function $Z(z), z > 0$, i.e.

  $$\hat{z}(w) = \int_0^\infty e^{-wz} \mathrm{d}Z(z) < \infty, \quad \text{for some } w < 0;$$

  the mean and tail of Z are denoted respectively by

  $$\mu_{1_Z} = \int_0^\infty z \mathrm{d}Z(z), \quad \overline{Z}(x) = \int_x^\infty \mathrm{d}Z(s).$$

Assume the arrival of claims follows a Poisson process of rate ρ, and each of the claims will be settled with a random delay. Loss only occurs when claims are being settled. M_t is denoted as the number of cumulative unsettled claims within the time interval $[0, t]$ and assume the initial number $M_0 = 0$. $\left\{T_k\right\}_{k=1,2,\dots}$, $\left\{L_k\right\}_{k=1,2,\dots}$ and $\left\{T_k + L_k\right\}_{k=1,2,\dots}$ are denoted as the (random) times of claim arrival, delayed period and settlement, respectively, and hence,

$$\begin{aligned} M_t &= \sum_k \Big(\mathbb{I}\left\{T_k \leq t\right\} - \mathbb{I}\left\{T_k + L_k \leq t\right\} \Big), \\ N_t &= \sum_k \mathbb{I}\left\{T_k + L_k \leq t\right\}. \end{aligned}$$

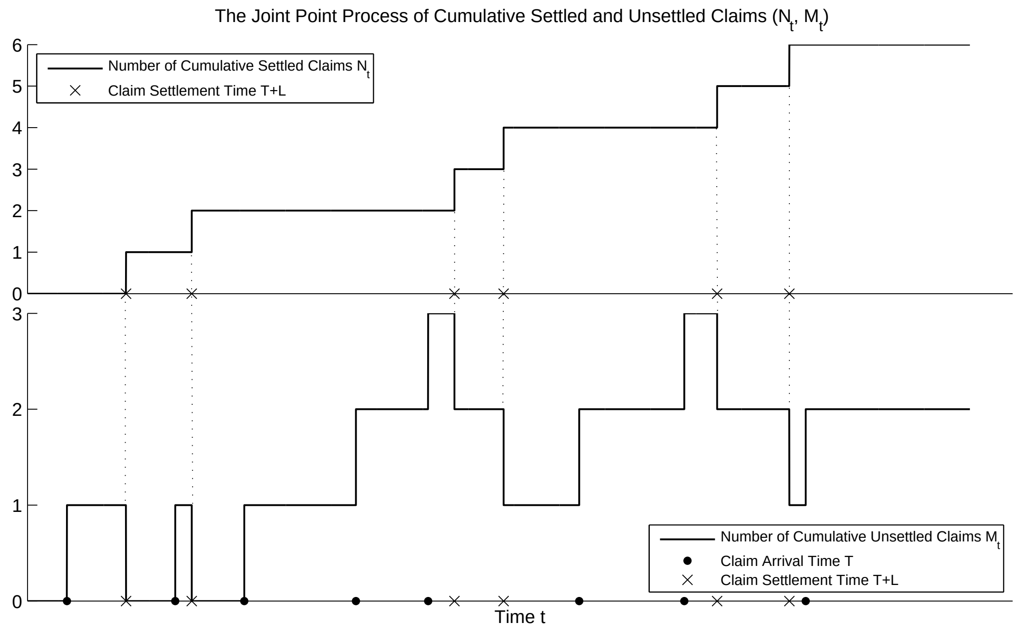

Fig. 8.1: A Sample Path of the Joint Point Processes of Cumulative Settled and Unsettled Claims (N_t, M_t)

$\left\{L_k\right\}_{k=1,2,\dots}$ are independent and identically distributed non-negative random variables with the cumulative distribution function L. A sample path of the joint point processes of the cumulative settled and unsettled claims (N_t, M_t) is given by *Figure 8.1*.

The ruin (stopping) time after time $t \geq 0$ is defined by

$$\tau_t^* =: \begin{cases} \inf\{s : s > t, X_s \leq 0\}, \\ \inf\{\varnothing\} = \infty, & \text{if } X_s > 0 \text{ for all } t; \end{cases}$$

in particular, $\tau_t^* = \infty$ means ruin does not occur. We are interested in the ultimate ruin probability at time t, i.e.

$$\psi(x, t) =: P\left\{\tau_t^* < \infty \middle| X_t = x\right\}, \tag{8.1}$$

or, the ultimate non-ruin probability at time t, i.e.

$$\phi(x, t) =: 1 - \psi(x, t). \tag{8.2}$$

Note that, $\psi(x, t)$ defined by (8.1) is the ultimate ruin probability at the general time $t \geq 0$, rather than the conventionally defined ruin probability of finite-horizon time t.

8.3 Ruin with Randomly Delayed Claims

8.3.1 Preliminaries

The net profit condition remains the same as the classical Poisson model, i.e. $c > \rho\mu_{1_Z}$, since, obviously,

$$\lim_{t\to\infty} \frac{\int_0^t \overline{L}(s)\mathrm{d}s}{t} = 0,$$

and sequence $\{Z_i\}_{i=1,2,...}$ and point process N_t are independent,

$$\lim_{t\to\infty}\frac{\mathbb{E}[X_t]}{t}=\lim_{t\to\infty}\frac{x+ct-\mu_{1Z}\mathbb{E}[N_t]}{t}=\lim_{t\to\infty}\frac{x+ct-\mu_{1Z}\rho\left(t-\int_0^t\overline{L}(s)\mathrm{d}s\right)}{t}=c-\rho\mu_{1Z}>0.$$

Lemma 8.3.1. *Assume $c>\rho\mu_{1Z}$ and $L\sim\mathrm{Exp}(\delta)$, we have a series of modified Lundberg fundamental equations*

$$cw-\rho\left[1-\hat{z}(w)\right]-\delta j=0,\quad j=0,1,...; \tag{8.3}$$

- *for $j=0$, (8.3) has solution zero and a unique negative solution (denoted by $W_0^+=0$ and $W_0^-<0$);*
- *for $j=1,2,...$, (8.3) has unique positive and negative solutions (denoted by $W_j^+>0$ and $W_j^-<0$).*

Proof. Rewrite (8.3) as

$$\hat{z}(w)=l_j(w), \tag{8.4}$$

where

$$l_j(w)=:-\frac{c}{\rho}w+\left(1+\frac{\delta}{\rho}j\right),\quad j=0,1,....$$

Note that,

$$\left.\frac{\mathrm{d}\hat{z}(w)}{\mathrm{d}w}\right|_{w=0}=-\mu_{1Z},\quad \left.\frac{\mathrm{d}l_j(w)}{\mathrm{d}w}\right|_{w=0}=-\frac{c}{\rho},$$

by the net profit condition $c>\rho\mu_{1Z}$, we have

$$\left.\frac{\mathrm{d}\hat{z}(w)}{\mathrm{d}w}\right|_{w=0}>\left.\frac{\mathrm{d}l_j(w)}{\partial w}\right|_{w=0}.$$

In particular, for $j=0$, we have $l_0(0)=\hat{z}(0)=1$. Then, further by the convexity of $\hat{z}(w)$ and the linearity of $l_j(w)$, the uniqueness of the positive and negative solutions to (8.3) follows immediately. It is more obvious by plotting (8.4) in *Figure 8.2*.

□

Denote the (modified) adjustment coefficients by

$$R_j=:-W_j^-,\quad j=0,1,...,$$

note that,

$$0<R_0<R_1<R_2<...<R_\infty,$$

where $R_\infty=:\inf\left\{R\middle|\hat{z}(-R)=\infty\right\}$.

If $Z\sim\mathrm{Exp}(\gamma)$, then, we have a series of the modified Lundberg fundamental equations

$$cw^2+(c\gamma-\rho-\delta j)w-\gamma\delta j=0,\quad j=0,1,...,$$

with explicit solutions

$$W_j^{\pm}=\frac{(\rho+\delta j-c\gamma)\pm\sqrt{(\rho+\delta j-c\gamma)^2+4c\gamma\delta j}}{2c},\quad j=0,1,...,$$

and

$$R_\infty=\lim_{j\to\infty}R_j=\gamma.$$

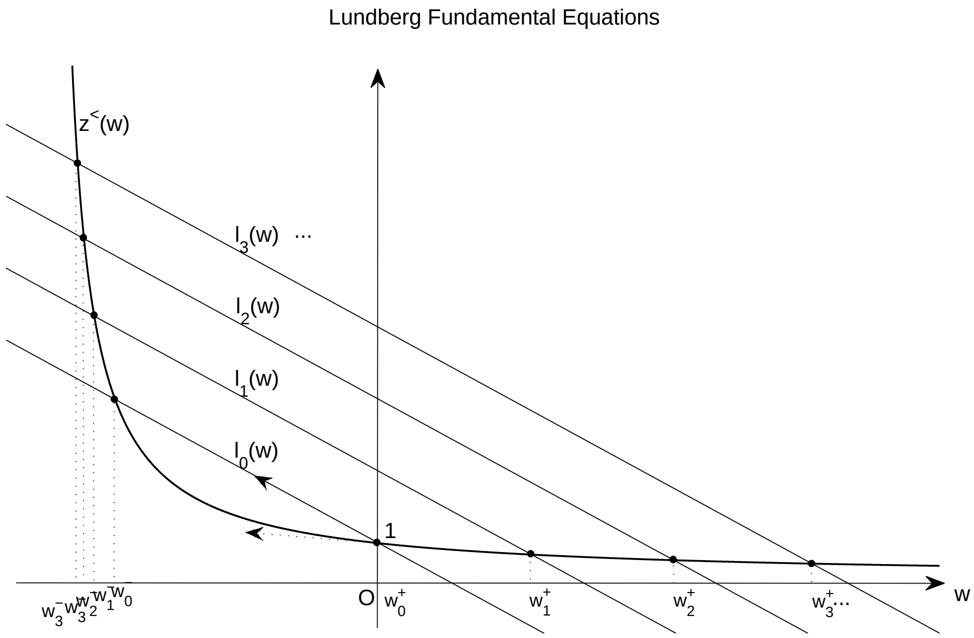

Fig. 8.2: Lundberg Fundamental Equations

8.3.2 Asymptotics of Ruin Probability

By Mirasol (1963), we know that, a delayed (or displaced) Poisson process is still a (non-homogeneous) Poisson process, which is also a special case of discretised dynamic contagion process introduced by Dassios and Zhao (2012), see also Newell (1966), Lawrance and Lewis (1975) and Dassios and Zhao (2011). According to the model setting in Section 8.2, the settlement process N_t hence is a non-homogeneous Poisson process with rate $\rho L(t)$, and we can obtain the asymptotics of the ruin probability as below.

Theorem 8.3.1. *Assume* $c > \rho\mu_{1_Z}$ *and the first, second moments of* L *exist, we have the asymptotics of ruin probability*

$$\psi(x,t) \sim e^{-cR_0\int_t^\infty \overline{L}(s)\mathrm{d}s} \frac{c-\rho\mu_{1_Z}}{\rho\int_0^\infty ze^{R_0 z}\mathrm{d}Z(z)-c} e^{-R_0 x} + o\left(e^{-R_0 x}\right), \quad x\to\infty,$$

where $\overline{L}(t) =: 1 - L(t)$.

Proof. By Markov property, the integro-differential equation of the ruin probability $\psi(x,t)$ defined by (8.1) is given by

$$\frac{\partial\psi(x,t)}{\partial t} + c\frac{\partial\psi(x,t)}{\partial x} + \rho L(t)\left(\int_0^x \psi(x-z,t)\mathrm{d}Z(z) + \overline{Z}(x) - \psi(x,t)\right) = 0.$$

By the Laplace transform

$$\hat{\psi}(w,t) =: \mathcal{L}_w\Big\{\psi(x,t)\Big\} = \int_0^\infty e^{-wx}\psi(x,t)\mathrm{d}x,$$

we have

$$\frac{\partial\hat{\psi}(w,t)}{\partial t} + c\left(w\hat{\psi}(w,t) - \psi(0,t)\right) - \rho L(t)\left(\left[1-\hat{z}(w)\right]\hat{\psi}(w,t) - \frac{1-\hat{z}(w)}{w}\right) = 0,$$

or,

$$\frac{\partial\hat{\psi}(w,t)}{\partial t} - c\psi(0,t) + \Big(cw - \rho L(t)\left[1-\hat{z}(w)\right]\Big)\hat{\psi}(w,t) + \rho L(t)\frac{1-\hat{z}(w)}{w} = 0. \tag{8.5}$$

Note that, the special case of $t \to \infty$ corresponds to the classical Poisson case as $L(t) \to 1$, i.e.

$$c\frac{\partial\psi(x,\infty)}{\partial x} + \rho\left(\int_0^x \psi(x-z,\infty)\mathrm{d}Z(z) + \overline{Z}(x) - \psi(x,\infty)\right) = 0,$$

and it is well known that the Laplace transform of the solution $\psi(x,\infty)$ is given by

$$\hat{\psi}(w,\infty) = \frac{\rho\left(\mu_{1_Z} - \frac{1-\hat{z}(w)}{w}\right)}{cw - \rho\left[1-\hat{z}(w)\right]}.$$

Define

$$\hat{\psi}(w,t) =: \frac{\rho\left(\mu_{1_Z} - \frac{1-\hat{z}(w)}{w}\right)}{cw - \rho\left[1-\hat{z}(w)\right]} e^{\rho\int_t^\infty [1-\hat{z}(w)]\overline{L}(s)\mathrm{d}s} + \hat{k}(w,t), \tag{8.6}$$

where $\hat{k}(w,t)$ is the Laplace transform of a function $k(x,t)$ and satisfies

$$\lim_{t\to\infty}\hat{k}(w,t) = 0. \tag{8.7}$$

Plug (8.6) into (8.5), then, we have the ODE of $\hat{k}(w,t)$,

$$\begin{aligned}&\frac{\partial\hat{k}(w,t)}{\partial t} + c\left(w\hat{k}(w,t) - \psi(0,t)\right) - \rho L(t)\left(\left[1-\hat{z}(w)\right]\hat{k}(w,t) - \frac{1-\hat{z}(w)}{w}\right)\\ &+\rho\left(\mu_{1_Z} - \frac{1-\hat{z}(w)}{w}\right)e^{\rho\int_t^\infty[1-\hat{z}(w)]\overline{L}(s)\mathrm{d}s} = 0,\end{aligned}$$

or,

$$\begin{aligned}&\frac{\partial\hat{k}(w,t)}{\partial t} + \Big(cw - \rho\left[1-\hat{z}(w)\right] + \rho\overline{L}(t)\left[1-\hat{z}(w)\right]\Big)\hat{k}(w,t)\\ =\ & c\left(\psi(0,t) - \frac{\rho\mu_{1_Z}}{c}\right) + \rho\left(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\right)\left(e^{\rho\int_t^\infty[1-\hat{z}(w)]\overline{L}(s)\mathrm{d}s} - 1\right) + \rho\overline{L}(t)\frac{1-\hat{z}(w)}{w}.\end{aligned}$$

By multiplying (multiplier factor) $e^{(cw-\rho[1-\hat{z}(w)])t}e^{-\rho\int_t^\infty \overline{L}(s)[1-\hat{z}(w)]\mathrm{d}s}$, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\hat{k}(w,t)e^{(cw-\rho[1-\hat{z}(w)])t}e^{-\rho\int_t^\infty \overline{L}(s)[1-\hat{z}(w)]\mathrm{d}s}\right)\\ =\ & \left[c\left(\psi(0,t) - \frac{\rho\mu_{1_Z}}{c}\right) + \rho\left(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\right)\left(e^{\rho\int_t^\infty[1-\hat{z}(w)]\overline{L}(s)\mathrm{d}s} - 1\right) + \rho\overline{L}(t)\frac{1-\hat{z}(w)}{w}\right]\\ &\times e^{(cw-\rho[1-\hat{z}(w)])t}e^{-\rho\int_t^\infty \overline{L}(s)[1-\hat{z}(w)]\mathrm{d}s},\end{aligned}$$

with the boundary condition (8.7), and then the solution

$$\begin{aligned}&\hat{k}(w,t) \qquad (8.8)\\ =\ & e^{-(cw-\rho[1-\hat{z}(w)])t}e^{\rho\int_t^\infty \overline{L}(s)[1-\hat{z}(w)]\mathrm{d}s}\int_t^\infty e^{(cw-\rho[1-\hat{z}(w)])s}e^{-\rho\int_s^\infty \overline{L}(u)[1-\hat{z}(w)]\mathrm{d}u}\\ &\times\left[-c\left(\psi(0,s) - \frac{\rho\mu_{1_Z}}{c}\right) - \rho\left(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\right)\left(e^{\rho\int_s^\infty[1-\hat{z}(w)]\overline{L}(u)\mathrm{d}u} - 1\right) - \rho\overline{L}(s)\frac{1-\hat{z}(w)}{w}\right]\mathrm{d}s.\end{aligned}$$

Obviously, from *Figure 8.2*, for $-R_0 < w < 0$, we have $l_0(w) > \hat{z}(w)$, i.e.

$$cw - \rho\left[1-\hat{z}(w)\right] < 0, \quad -R_0 < w < 0.$$

Now, we discuss the three terms of $\hat{k}(w,t)$ given by (8.8), respectively.

1. It is well known that (see Gerber (1979) and Grandel (1991)), in the classical model when the claim settlement follows a Poisson process with a constant rate λ, the ruin probability with the initial reserve $x = 0$ is simply $\frac{\mu_{1_Z}}{c}\lambda$, whereas $\psi(0,t)$ here in the first term of (8.8) is based on the realisation of the rate $\{\rho L(s)\}_{t \le s \le \infty}$. Also, the cumulative function $L(s)$ is an increasing function of s, then, the ruin probability $\psi(0,t)$ should be greater than the case $\lambda = \rho L(t)$ and smaller than the case $\lambda = \rho L(\infty) = \rho$ of the classical model, i.e.

$$\frac{\mu_{1_Z}}{c}\rho L(t) < \psi(0,t) < \frac{\mu_{1_Z}}{c}\rho,$$

or,

$$0 < \frac{\rho\mu_{1_Z}}{c} - \psi(0,t) < \frac{\rho\mu_{1_Z}}{c}\overline{L}(t).$$

If the first moment of L exists, then, we have

$$\int_t^\infty \left|\psi(0,s) - \frac{\rho\mu_{1_Z}}{c}\right| \mathrm{d}s < \frac{\rho\mu_{1_Z}}{c}\int_t^\infty \overline{L}(s)\mathrm{d}s < \frac{\rho\mu_{1_Z}}{c}\int_0^\infty \overline{L}(s)\mathrm{d}s < \infty.$$

2. For the second term of (8.8), if the second moment of L exists, then,

$$\begin{aligned} & \int_t^\infty e^{-\rho\int_s^\infty [1-\hat{z}(w)]\overline{L}(u)\mathrm{d}u}\left(e^{\rho\int_s^\infty [1-\hat{z}(w)]\overline{L}(u)\mathrm{d}u} - 1\right)\mathrm{d}s \\ = & \int_t^\infty \left(1 - e^{-\rho\int_s^\infty [1-\hat{z}(w)]\overline{L}(u)\mathrm{d}u}\right)\mathrm{d}s \\ < & \int_t^\infty \rho\int_s^\infty [1-\hat{z}(w)]\,\overline{L}(u)\mathrm{d}u\mathrm{d}s \\ < & \ \rho\,[1-\hat{z}(w)]\int_0^\infty\int_s^\infty \overline{L}(u)\mathrm{d}u\mathrm{d}s < \infty. \end{aligned}$$

3. For the third term of (8.8), if the first moment of L exists, then,

$$\int_t^\infty \rho\overline{L}(s)\frac{1-\hat{z}(w)}{w}\mathrm{d}s = \rho\frac{1-\hat{z}(w)}{w}\int_t^\infty \overline{L}(s)\mathrm{d}s < \rho\frac{1-\hat{z}(w)}{w}\int_0^\infty \overline{L}(s)\mathrm{d}s < \infty.$$

Therefore, for $-R_0 < w < 0$, we have

$$\hat{k}(w,t) < \infty,$$

and

$$\hat{k}(-R_0,t) = \lim_{w\downarrow -R_0}\hat{k}(w,t) = \int_0^\infty e^{R_0 x}k(x,t)\mathrm{d}x < \infty,$$

hence,

$$k(x,t) = o\left(e^{-R_0 x}\right).$$

By the *Final Value Theorem* and $\hat{\psi}(w,t)$ given by (8.6), we have

$$\begin{aligned} & \lim_{x\to\infty} e^{R_0 x}\psi(x,t) \\ = & \lim_{w\to 0} w\mathcal{L}_w\left\{e^{R_0 x}\psi(x,t)\right\} \\ = & \lim_{w\to 0} w\hat{\psi}(w-R_0,t) \\ = & \ e^{\rho\int_t^\infty [1-\hat{z}(-R_0)]\overline{L}(s)\mathrm{d}s}\lim_{w\to 0} w\frac{\rho\left(\mu_{1_Z} - \frac{1-\hat{z}(w-R_0)}{w-R_0}\right)}{c(w-R_0) - \rho\,[1-\hat{z}(w-R_0)]} + \lim_{w\to 0} w\hat{k}(w-R_0,t) \\ = & \ e^{\rho\int_t^\infty [1-\hat{z}(-R_0)]\overline{L}(s)\mathrm{d}s}\frac{c-\rho\mu_{1_Z}}{\rho\int_0^\infty z e^{R_0 z}\mathrm{d}Z(z) - c} + 0 \\ = & \ e^{-cR_0\int_t^\infty \overline{L}(s)\mathrm{d}s}\frac{c-\rho\mu_{1_Z}}{\rho\int_0^\infty z e^{R_0 z}\mathrm{d}Z(z) - c}. \end{aligned}$$

Note that, by definition, $-R_0$ is the solution to $cw-\rho[1-\hat{z}(w)]=0$, and we have

$$1-\hat{z}(-R_0)=-\frac{cR_0}{\rho}.$$

□

8.4 Ruin with Exponentially Delayed Claims

By specifying the distribution of the period of delay L, we could improve the result in *Theorem 8.3.1* with higher order of asymptotics. Here, for instance, we consider the special case when the claims are exponentially delayed, in order to derive $o\left(e^{-R_0x}\right)$ with more details.

8.4.1 Laplace Transform of Non-ruin Probability

We derive the the Laplace transform of non-ruin probability in two different expressions as given by *Theorem 8.4.1* and *Theorem 8.4.2*, respectively, and then, they will be used to derive the asymptotics of ruin probability.

Theorem 8.4.1. *Assume $c>\rho\mu_{1_Z}$ and $L\sim \mathrm{Exp}(\delta)$, we have the Laplace transform of non-ruin probability*

$$\hat{\phi}(w,t)=e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}\left(\frac{c-\rho\mu_{1_Z}}{cw-\rho\left[1-\hat{z}(w)\right]}+c\sum_{j=1}^{\infty}e^{-j\delta t}\frac{\sum_{\ell=0}^{j}r_\ell\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw-\rho\left[1-\hat{z}(w)\right]-\delta j}\right), \tag{8.9}$$

where $\vartheta=\frac{\rho}{\delta}$,

$$r_0=1-\frac{\rho}{c}\mu_{1_Z},\quad r_\ell=-\sum_{i=0}^{\ell-1}\frac{\left[\vartheta\hat{z}(W_\ell^+)\right]^{\ell-i}}{(\ell-i)!}r_i,\quad \ell=1,2,\ldots. \tag{8.10}$$

Proof. If $L\sim\mathrm{Exp}(\delta)$, then, $L(t)=1-e^{-\delta t}$, and N_t is a non-homogeneous Poisson process with rate $\rho-\vartheta\delta e^{-\delta t}$, and the non-ruin probability $\phi(x,t)$ defined by (8.2) satisfies the integro-differential equation

$$\frac{\partial\phi(x,t)}{\partial t}+c\frac{\partial\phi(x,t)}{\partial x}+\left(\rho-\vartheta\delta e^{-\delta t}\right)\left(\int_0^x\phi(x-z,t)\mathrm{d}Z(z)-\phi(x,t)\right)=0.$$

By the Laplace transform

$$\hat{\phi}(w,t)=:\mathcal{L}_w\left\{\phi(x,t)\right\}=\int_0^\infty e^{-wx}\phi(x,t)\mathrm{d}x, \tag{8.11}$$

we have

$$\frac{\partial\hat{\phi}(w,t)}{\partial t}+c\left(w\hat{\phi}(w,t)-\phi(0,t)\right)-\left(\rho-\vartheta\delta e^{-\delta t}\right)\left[1-\hat{z}(w)\right]\hat{\phi}(w,t)=0. \tag{8.12}$$

Define

$$\hat{h}(w,t)=:\hat{\phi}(w,t)\exp\left(\int_0^t\delta\vartheta e^{-\delta s}\left[1-\hat{z}(w)\right]\mathrm{d}s\right),$$

where $\hat{h}(w,t)$ is the Laplace transform of a function $h(x,t)$, then,

$$\hat{\phi}(w,t)=\hat{h}(w,t)e^{-\vartheta\left(1-e^{-\delta t}\right)[1-\hat{z}(w)]}. \tag{8.13}$$

Plug (8.13) into (8.12), we have

$$\frac{\partial\hat{h}(w,t)}{\partial t}+c\left(w\hat{h}(w,t)-\phi(0,t)e^{\vartheta\left(1-e^{-\delta t}\right)[1-\hat{z}(w)]}\right)-\rho\left[1-\hat{z}(w)\right]\hat{h}(w,t)=0. \tag{8.14}$$

Note that, by (8.13), we have

$$\begin{aligned}\hat{\phi}(w,t) &= \hat{h}(w,t)e^{-\vartheta\left(1-e^{-\delta t}\right)}e^{\vartheta\left(1-e^{-\delta t}\right)\hat{z}(w)}\\ &= \hat{h}(w,t)e^{-\vartheta\left(1-e^{-\delta t}\right)}\left(1+\sum_{k=1}^{\infty}\frac{\left(\vartheta\left(1-e^{-\delta t}\right)\right)^k}{k!}\hat{z}^k(w)\right)\\ &= e^{-\vartheta\left(1-e^{-\delta t}\right)}\left(\hat{h}(w,t)+\sum_{k=1}^{\infty}\frac{\left(\vartheta\left(1-e^{-\delta t}\right)\right)^k}{k!}\hat{h}(w,t)\hat{z}^k(w)\right),\end{aligned}$$

which is the Laplace transform of

$$\phi(x,t)=e^{-\vartheta\left(1-e^{-\delta t}\right)}\left(h(x,t)+\sum_{k=1}^{\infty}\frac{\left(\vartheta\left(1-e^{-\delta t}\right)\right)^k}{k!}\int_0^x h(x-z,t)\mathrm{d}Z^{(k)}(z)\right),$$

where $Z^{(k)}$ is the $k-$fold convolution of the distribution Z, i.e.

$$Z^{(k)}\overset{\mathcal{D}}{=}\sum_{i=1}^{k}Z_i,$$

then, we have

$$\phi(0,t)=h(0,t)e^{-\vartheta\left(1-e^{-\delta t}\right)}. \tag{8.15}$$

Plug (8.15) into (8.14), we have

$$\frac{\partial\hat{h}(w,t)}{\partial t}+\Big(cw-\rho\left[1-\hat{z}(w)\right]\Big)\hat{h}(w,t)-ce^{-\vartheta\hat{z}(w)}h(0,t)e^{\vartheta e^{-\delta t}\hat{z}(w)}=0.$$

This equation of $\hat{h}(w,t)$ has a power series solution

$$\hat{h}(w,t)=\sum_{j=0}^{\infty}e^{-j\delta t}\hat{h}_j(w),$$

the Laplace transform of

$$h(x,t)=\sum_{j=0}^{\infty}e^{-j\delta t}h_j(x).$$

Since

$$\begin{aligned}\frac{\partial\hat{h}(w,t)}{\partial t} &= -\delta\sum_{j=0}^{\infty}je^{-j\delta t}\hat{h}_j(w),\\ h(0,t)e^{\vartheta e^{-\delta t}\hat{z}(w)} &= \sum_{j=0}^{\infty}e^{-j\delta t}h_j(0)\times\sum_{k=0}^{\infty}\frac{e^{-k\delta t}\left[\vartheta\hat{z}(w)\right]^k}{k!}\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}e^{-(j+k)\delta t}h_j(0)\frac{\left[\vartheta\hat{z}(w)\right]^k}{k!}\qquad (j+k=i)\\ &= \sum_{i=0}^{\infty}e^{-i\delta t}\sum_{j=0}^{i}h_j(0)\frac{\left[\vartheta\hat{z}(w)\right]^{i-j}}{(i-j)!},\end{aligned}$$

we have

$$\sum_{j=0}^{\infty}e^{-j\delta t}\left[\Big(-\delta j+cw-\rho\left[1-\hat{z}(w)\right]\Big)\hat{h}_j(w)-ce^{-\vartheta\hat{z}(w)}\sum_{\ell=0}^{j}h_\ell(0)\frac{\left[\vartheta\hat{z}(w)\right]^{j-\ell}}{(j-\ell)!}\right]=0,$$

then, for any $j = 0, 1, ...,$

$$\Big(-\delta j + cw - \rho\,[1 - \hat{z}(w)] \Big)\hat{h}_j(w) - ce^{-\vartheta \hat{z}(w)} \sum_{\ell=0}^{j} h_\ell(0) \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!} = 0,$$

and hence,

$$\hat{h}_j(w) = \frac{ce^{-\vartheta \hat{z}(w)}}{cw - \rho\,[1-\hat{z}(w)] - \delta j} \sum_{\ell=0}^{j} h_\ell(0) \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!}, \quad j = 0, 1, ... \tag{8.16}$$

Note that, the denominator of (8.16) is the modified Lundberg fundamental equation given by *Lemma 8.3.1*.

By (8.13), we have

$$\hat{\phi}(w,t) = e^{-\vartheta\left(1-e^{-\delta t}\right)[1-\hat{z}(w)]} \left(\hat{h}_0(w) + \sum_{j=1}^{\infty} e^{-j\delta t} \hat{h}_j(w) \right). \tag{8.17}$$

Note that, if $t \to \infty$, it recovers the classical Poisson model. By (8.17), we have

$$\hat{\phi}(w,\infty) = e^{-\vartheta[1-\hat{z}(w)]}\hat{h}_0(w), \tag{8.18}$$

$$\hat{\phi}(w,0) = \sum_{j=0}^{\infty} \hat{h}_j(w). \tag{8.19}$$

The series of constants $\{h_\ell(0)\}_{\ell=0,1,...}$ in (8.16) can be obtained as follows.

For case $j = 0$, by (8.16), we have

$$\hat{h}_0(w) = \frac{ce^{-\vartheta \hat{z}(w)}}{cw - \rho\,[1-\hat{z}(w)]} h_0(0).$$

By (8.15) and (8.18), we have

$$\begin{aligned} \phi(0,\infty) &= h(0,\infty)e^{-\vartheta} = h_0(0)e^{-\vartheta}, \\ \hat{\phi}(w,\infty) &= \hat{h}_0(w)e^{-\vartheta[1-\hat{z}(w)]} = \frac{ce^{-\vartheta}h_0(0)}{cw - \rho\,[1-\hat{z}(w)]} = \frac{c\phi(0,\infty)}{cw - \rho\,[1-\hat{z}(w)]}. \end{aligned}$$

Since

$$\lim_{x\to\infty} \phi(x,\infty) = \lim_{w\to 0} w\hat{\phi}(w,\infty) = 1,$$

i.e.

$$\lim_{w\to 0} w \frac{c\phi(0,\infty)}{cw - \rho\,[1-\hat{z}(w)]} = \frac{c\phi(0,\infty)}{\lim_{w\to 0} \frac{1}{w}\left(cw - \rho\,[1-\hat{z}(w)]\right)} = \frac{c\phi(0,\infty)}{c - \rho\mu_{1Z}} = 1,$$

we have

$$\phi(0,t) = \frac{c - \rho\mu_{1Z}}{c},$$

$$h_0(0) = \frac{e^{\vartheta}\left(c - \rho\mu_{1Z}\right)}{c}, \tag{8.20}$$

and

$$\hat{\phi}(w,t) = \frac{c - \rho\mu_{1Z}}{cw - \rho\,[1-\hat{z}(w)]},$$

which is exactly the Laplace transform of ultimate non-ruin probability of the classical Poisson model. Hence, we have

$$\hat{h}_0(w) = e^{\vartheta[1-\hat{z}(w)]} \frac{c - \rho\mu_{1Z}}{cw - \rho\,[1-\hat{z}(w)]}. \tag{8.21}$$

For case $j = 1, 2, ...$, since $\hat{h}_j(w)$ of (8.16) exists at $w = W_j^+$, we have

$$\lim_{w \to W_j^+} \left(c e^{-\vartheta \hat{z}(w)} \sum_{\ell=0}^{j} h_\ell(0) \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!} \right) = 0, \quad j = 1, 2, ...,$$

or,

$$\sum_{\ell=0}^{j} \frac{\left[\vartheta \hat{z}(W_j^+)\right]^{j-\ell}}{(j-\ell)!} h_\ell(0) = 0, \quad j = 1, 2, ...$$

Given the initial value $h_0(0)$ by (8.20), obviously, the series of constants $\{h_\ell(0)\}_{\ell=1,2,...}$ can be solved uniquely and explicitly by recursion. Define the solution by

$$r_j =: e^{-\vartheta} h_j(0),$$

with the initial value $r_0 = 1 - \frac{\rho}{c}\mu_{1Z}$, and we have

$$\hat{h}_j(w) = \frac{c e^{\vartheta[1-\hat{z}(w)]}}{cw - \rho[1-\hat{z}(w)] - \delta j} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!}, \quad j = 1, 2, ..., \tag{8.22}$$

where

$$r_\ell = -\sum_{i=0}^{\ell-1} \frac{\left[\vartheta \hat{z}(W_\ell^+)\right]^{\ell-i}}{(\ell-i)!} r_i, \quad \ell = 1, 2, ...$$

Therefore, by (8.17), we have the Laplace transform of non-ruin probability

$$\hat{\phi}(w,t) = e^{-\vartheta\left(1-e^{-\delta t}\right)[1-\hat{z}(w)]} \left(\frac{e^{\vartheta[1-\hat{z}(w)]}(c - \rho\mu_{1Z})}{cw - \rho[1-\hat{z}(w)]} + \sum_{j=1}^{\infty} e^{-j\delta t} \frac{c e^{\vartheta[1-\hat{z}(w)]} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw - \rho[1-\hat{z}(w)] - \delta j} \right).$$

□

Remark 8.4.1. In particular, for $t = 0$, we have

$$\hat{\phi}(w,0) = e^{\vartheta[1-\hat{z}(w)]} \left(\frac{c - \rho\mu_{1Z}}{cw - \rho[1-\hat{z}(w)]} + c \sum_{j=1}^{\infty} \frac{\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw - \rho[1-\hat{z}(w)] - \delta j} \right);$$

and, for $t = \infty$,

$$\hat{\phi}(w,\infty) = \frac{c - \rho\mu_{1Z}}{cw - \rho[1-\hat{z}(w)]},$$

which recovers the result of the classic Poisson model.

Remark 8.4.2. (8.10) offers a numerically tractable formula for calculating the coefficients $\{r_j\}_{j=0,1,...}$. For instance, if $Z \sim \text{Exp}(\gamma)$ with parameter setting $(c, \delta, \rho, \gamma) = (1.5, 2.0, 0.5, 1.0)$, then, we have $r_0 = 0.6667$, $r_1 = -0.0657$, $r_2 = 0.0028$, $r_3 = -7.2560 \times 10^{-5}$,....

Alternatively, the Laplace transform of non-ruin probability can also be expressed by another power series as below.

Theorem 8.4.2. *Assume $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$, we have the Laplace transform of the non-ruin probability*

$$\hat{\phi}(w,t) = \sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_j(w),$$

where $\left\{\hat{\phi}_j(w)\right\}_{j=0,1,\dots}$ follow the recurrence

$$\hat{\phi}_j(w) = \rho\frac{\left[1-\hat{z}(W_j^+)\right]\hat{\phi}_{j-1}(W_j^+) - \left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w)}{cw - \rho\left[1-\hat{z}(w)\right] - \delta j}, \quad j = 1,2,..., \tag{8.23}$$

$$\hat{\phi}_0(w) = \frac{c\left(1-\frac{\rho}{c}\mu_{1_Z}\right)}{cw - \rho\left[1-\hat{z}(w)\right]}. \tag{8.24}$$

Proof. Rewrite (8.12) as

$$\frac{\partial\hat{\phi}(w,t)}{\partial t} + c\left(w\hat{\phi}(w,t) - \phi(0,t)\right) - \rho\left[1-\hat{z}(w)\right]\hat{\phi}(w,t) + \rho\left[1-\hat{z}(w)\right]e^{-\delta t}\hat{\phi}(w,t) = 0.$$

This equation has a power series solution

$$\hat{\phi}(w,t) = \sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_j(w),$$

the Laplace transform of the non-ruin probability

$$\phi(x,t) = \sum_{j=0}^{\infty} e^{-j\delta t}\phi_j(x).$$

Note that, by setting $\hat{\phi}_{-1}(w) = 0$, we have

$$\begin{aligned}
\frac{\partial\hat{\phi}(w,t)}{\partial t} &= -\delta\sum_{j=0}^{\infty} je^{-j\delta t}\hat{\phi}_j(w),\\
e^{-\delta t}\hat{\phi}(w,t) &= \sum_{j=0}^{\infty} e^{-(j+1)\delta t}\hat{\phi}_j(w) = \sum_{j=1}^{\infty} e^{-j\delta t}\hat{\phi}_{j-1}(w) = \sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_{j-1}(w),
\end{aligned}$$

then,

$$\begin{aligned}
&-\delta\sum_{j=0}^{\infty} je^{-j\delta t}\hat{\phi}_j(w) + c\left(w\sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_j(w) - \sum_{j=0}^{\infty} e^{-j\delta t}\phi_j(0)\right) - \rho\left[1-\hat{z}(w)\right]\sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_j(w)\\
&+\rho\left[1-\hat{z}(w)\right]\sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_{j-1}(w) = 0,
\end{aligned}$$

or,

$$\sum_{j=0}^{\infty} e^{-j\delta t}\Big[-\delta j\hat{\phi}_j(w) + c\left(w\hat{\phi}_j(w) - \phi_j(0)\right) - \rho\left[1-\hat{z}(w)\right]\hat{\phi}_j(w) + \rho\left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w)\Big] = 0,$$

and then, for any $j = 0,1,...$,

$$-\delta j\hat{\phi}_j(w) + c\left(w\hat{\phi}_j(w) - \phi_j(0)\right) - \rho\left[1-\hat{z}(w)\right]\hat{\phi}_j(w) + \rho\left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w) = 0.$$

Hence, we have

$$\hat{\phi}_j(w) = \frac{c\phi_j(0) - \rho\left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w)}{cw - \rho\left[1-\hat{z}(w)\right] - \delta j}, \quad j = 0,1,....$$

For the initial case $j=0$, note that $\hat{\phi}_{-1}(w)=0$, we have

$$\hat{\phi}_0(w)=\frac{c\phi_0(0)}{cw-\rho\left[1-\hat{z}(w)\right]}.$$

By the boundary condition

$$\lim_{w\to 0} w\hat{\phi}_0(w)=\lim_{x\to\infty}\phi_0(x)=1,$$

we have

$$\lim_{w\to 0} w\hat{\phi}_0(w)=\lim_{w\to 0}\frac{c\phi_0(0)}{c-\rho\frac{1-\hat{z}(w)}{w}}=\frac{c\phi_0(0)}{c-\rho\mu_{1_Z}}=1,$$

then,

$$\phi_0(0)=1-\frac{\rho}{c}\mu_{1_Z},$$

and $\hat{\phi}_0(w)$ as given by (8.24). Since $\hat{\phi}_j(w)$ exists at $w=W_j^+$ for any $j=1,2,...$, we have

$$\lim_{w\to W_j^+}\left(c\phi_j(0)-\rho\left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w)\right)=0,$$

and

$$\phi_j(0)=\frac{\rho}{c}\left[1-\hat{z}(W_j^+)\right]\hat{\phi}_{j-1}(W_j^+),\quad j=1,2,....$$

Hence, we have the recurrence relation between $\hat{\phi}_j(w)$ and $\hat{\phi}_{j-1}(w)$ as given by (8.23).

□

Remark 8.4.3. *Theorem 8.4.1* will be used to derive a general asymptotic formula (given by *Theorem 8.4.3*), whereas *Theorem 8.4.2* is more useful for obtaining an exact expression in the case of exponentially distributed claim sizes (given by *Theorem 8.5.1*).

8.4.2 Asymptotics of Ruin Probability

Theorem 8.4.3. *Assume $c>\rho\mu_{1_Z}$ and $L\sim \mathrm{Exp}(\delta)$, we have the asymptotics of the ruin probability*

$$\psi(x,t)\sim\sum_{j=0}^{\infty}\kappa_j(t)e^{-R_jx},\quad x\to\infty, \tag{8.25}$$

where

$$\kappa_0(t) =: e^{-\frac{cR_0}{\rho}\vartheta e^{-\delta t}}\frac{c-\rho\mu_{1_Z}}{\rho\int_0^\infty ze^{R_0z}\mathrm{d}Z(z)-c}, \tag{8.26}$$

$$\kappa_j(t) =: e^{-j\delta t}\frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(-R_j)]}}{\rho\int_0^\infty ze^{R_jz}\mathrm{d}Z(z)-c}\sum_{\ell=0}^{j}r_\ell\frac{[\vartheta\hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!},\quad j=1,2,.... \tag{8.27}$$

Proof. Denote

$$\phi(x,t)=:\sum_{j=0}^{\infty}\phi_j(x,t),$$

then,

$$\hat{\phi}(w,t)=\sum_{j=0}^{\infty}\hat{\phi}_j(w,t),$$

where every term $\hat{\phi}_j(w,t)$ is specified by (8.9), i.e.

$$\hat{\phi}_0(w,t) \quad =: \quad e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}\frac{c-\rho\mu_{1_Z}}{cw-\rho\,[1-\hat{z}(w)]}, \tag{8.28}$$

$$\hat{\phi}_j(w,t) \quad =: \quad ce^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}\frac{\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw-\rho\,[1-\hat{z}(w)]-\delta j}, \quad j=1,2,.... \tag{8.29}$$

Now, we discuss the asymptotics of the terms $\phi_0(x,t)$ and $\{\phi_j(x,t)\}_{j=1,2,...}$, respectively.

For $\phi_0(x,t)$, we have the asymptotics

$$1-\phi_0(x,t)\sim\kappa_0(t)e^{-R_0x}, \quad x\to\infty,$$

since by *Final Value Theorem*,

$$\begin{aligned}
\kappa_0(t) &= \lim_{x\to\infty} e^{R_0x}\Big(1-\phi_0(x,t)\Big)\\
&= \lim_{w\to 0} w\mathcal{L}_w\left\{e^{R_0x}\Big(1-\phi_0(x,t)\Big)\right\}\\
&= \lim_{w\to 0} w\left(\frac{1}{w-R_0}-\hat{\phi}_0(w-R_0,t)\right)\\
&= -\lim_{w\to 0} w\hat{\phi}_0(w-R_0,t)\\
&= -\lim_{w\to 0} w\frac{e^{\vartheta e^{-\delta t}[1-\hat{z}(w-R_0)]}\,(c-\rho\mu_{1_Z})}{c(w-R_0)-\rho\,[1-\hat{z}(w-R_0)]}\\
&= -\frac{e^{\vartheta e^{-\delta t}[1-\hat{z}(-R_0)]}\,(c-\rho\mu_{1_Z})}{\frac{\mathrm{d}}{\mathrm{d}w}\Big(c(w-R_0)-\rho\,[1-\hat{z}(w-R_0)]\Big)\Big|_{w=0}}\\
&= \frac{e^{-\frac{cR_0}{\rho}\vartheta e^{-\delta t}}\,(c-\rho\mu_{1_Z})}{\rho\int_0^\infty ze^{R_0z}\mathrm{d}Z(z)-c}.
\end{aligned}$$

For $\phi_j(x,t)$, $j=1,2,...$, we have the asymptotics

$$-\phi_j(x,t)\sim\kappa_j(t)e^{-R_jx}, \quad x\to\infty,$$

since, by *Final Value Theorem*,

$$\begin{aligned}
\kappa_j(t) &= \lim_{x\to\infty} e^{R_jx}(-\phi_j(x,t))\\
&= \lim_{w\to 0} w\mathcal{L}_w\left\{e^{R_jx}(-\phi_j(x,t))\right\}\\
&= -\lim_{w\to 0} w\hat{\phi}_j(w-R_j,t)\\
&= -\lim_{w\to 0}\left(w\frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(w-R_j)]}e^{-j\delta t}}{c(w-R_j)-\rho\,[1-\hat{z}(w-R_j)]-\delta j}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(w-R_j)]^{j-\ell}}{(j-\ell)!}\right)\\
&= -\frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(-R_j)]}e^{-j\delta t}}{\frac{\mathrm{d}}{\mathrm{d}w}\Big(c(w-R_j)-\rho\,[1-\hat{z}(w-R_j)]-\delta j\Big)\Big|_{w=0}}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!}\\
&= \frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(-R_j)]}e^{-j\delta t}}{\rho\int_0^\infty ze^{R_jz}\mathrm{d}Z(z)-c}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!}.
\end{aligned}$$

Therefore,

$$\psi(x,t)=1-\phi(x,t)=1-\phi_0(x,t)+\sum_{j=1}^{\infty}-\phi_j(x,t),$$

the result of asymptotics (8.25) follows immediately.

□

Remark 8.4.4. Set $L(t) = 1 - \frac{\vartheta}{\rho}\delta e^{-\delta t}$ and $t = 0$ in *Theorem 8.3.1*, then, $\int_0^\infty \overline{L}(s)\mathrm{d}s = \frac{\vartheta}{\rho}$ and it recovers $\kappa_0(t)e^{-R_0 x}$, the first-order asymptotics of the ruin probability obtained by *Theorem 8.4.3*. The higher orders of asymptotics depend on the distributional property of the general distribution function L.

Remark 8.4.5. We can rewrite $\hat{\phi}_0(w,t)$ of (8.28) by

$$\begin{aligned}\hat{\phi}_0(w,t) &= e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}\frac{1}{w}\frac{p_0}{1-(1-p_0)\frac{1-\hat{z}(w)}{\mu_{1_Z} w}}\\ &= e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}\frac{1}{w}\sum_{i=0}^{\infty} p_0\,(1-p_0)^i\left(\frac{1-\hat{z}(w)}{\mu_{1_Z} w}\right)^i, \quad p_0 = 1-\frac{\rho\mu_{1_Z}}{c}.\end{aligned}$$

The third term of $\hat{\phi}_0(w,t)$ above is the Laplace transform of a compound geometric distribution

$$\sum_{i=0}^{\infty} p_0\,(1-p_0)^i\, d_0^{(i)}(x),$$

where $d_0^{(i)}(x)$ is the $i-$fold convolution of a proper density function

$$d_0(x) =: \frac{\overline{Z}(x)}{\mu_{1_Z}},$$

since $0 < p_0 < 1$ and

$$\begin{aligned}\mathcal{L}_w\{d_0(x)\} &= \frac{1-\hat{z}(w)}{\mu_{1_Z} w},\\ \int_0^\infty d_0(x)\mathrm{d}x &= \mathcal{L}_w\{d_0(x)\}\Big|_{w=0} = \frac{1}{\mu_{1_Z}}\lim_{w\to 0}\frac{1-\hat{z}(w)}{w} = \frac{1}{\mu_{1_Z}}\mu_{1_Z} = 1.\end{aligned}$$

For, $j = 1, 2, \ldots$, we can also rewrite $\hat{\phi}_j(w,t)$ of (8.29) by

$$\begin{aligned}&\hat{\phi}_j(w,t)\\ =\;& \frac{w-W_j^+}{cw-\rho[1-\hat{z}(w)]-\delta j-\left(cW_j^+-\rho\left[1-\hat{z}(W_j^+)\right]-\delta j\right)}\frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}}{w-W_j^+}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}\\ =\;& \frac{w-W_j^+}{c(w-W_j^+)-\rho\left[\hat{z}(W_j^+)-\hat{z}(w)\right]}\frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}}{w-W_j^+}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}\\ =\;& \frac{p_j}{1-(1-p_j)\frac{W_j^+}{1-\hat{z}(W_j^+)}\frac{\hat{z}(W_j^+)-\hat{z}(w)}{w-W_j^+}}\frac{1}{p_j}\frac{e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}}{w-W_j^+}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!},\\ =\;& \sum_{i=0}^{\infty} p_j(1-p_j)^i\left(\frac{W_j^+}{1-\hat{z}(W_j^+)}\frac{\hat{z}(W_j^+)-\hat{z}(w)}{w-W_j^+}\right)^i\times\frac{1}{p_j}\frac{e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}}{w-W_j^+}\sum_{\ell=0}^{j} r_\ell\frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!},\end{aligned}$$

where $p_j = \frac{\delta j}{cW_j^+}$. The first term of $\hat{\phi}_j(w,t)$ above is the Laplace transform of a compound geometric distribution

$$\sum_{i=0}^{\infty} p_j\,(1-p_j)^i\, d_j^{(i)}(x),$$

where $d_j^{(i)}(x)$ is the $i-$fold convolution of a proper density function

$$d_j(x) =: \frac{W_j^+}{1-\hat{z}(W_j^+)} e^{W_j^+ x} \int_x^\infty e^{-W_j^+ z} \mathrm{d}Z(z),$$

since

$$0 < p_j = 1 - \frac{\rho}{c} \frac{1-\hat{z}(W_j^+)}{W_j^+} = \frac{\delta j}{cW_j^+} = \frac{\delta j}{\rho\left[1-\hat{z}(W_j^+)\right] + \delta j} < 1,$$

and

$$\begin{aligned} \mathcal{L}_w\left\{d_j(x)\right\} &= \frac{W_j^+}{1-\hat{z}(W_j^+)} \frac{\hat{z}(W_j^+)-\hat{z}(w)}{w-W_j^+}, \\ \int_0^\infty d_j(x)\mathrm{d}x &= \frac{W_j^+}{1-\hat{z}(W_j^+)} \frac{\hat{z}(W_j^+)-\hat{z}(w)}{w-W_j^+}\bigg|_{w=0} = 1. \end{aligned}$$

Note that, for a constant ν, we have

$$\mathcal{L}_w\left\{e^{\nu x} \int_x^\infty e^{-\nu z} \mathrm{d}Z(z)\right\} = \frac{\hat{z}(\nu)-\hat{z}(w)}{w-\nu},$$

which is a special case of the double Dickson-Hipp operator introduced by Dickson and Hipp (2001).

8.5 Ruin with Exponentially Delayed Claims and Exponentially Distributed Sizes

The asymptotic formula of (8.25) becomes exact if the claim sizes follow an exponential distribution.

Theorem 8.5.1. *Assume $c > \rho\mu_{1_Z}$, $L \sim \mathrm{Exp}(\delta)$ and Z follows an exponential distribution, we have the ruin probability*

$$\psi(x,t) = \sum_{j=0}^\infty \kappa_j(t) e^{-R_j x}. \tag{8.30}$$

Proof. By *Theorem 8.4.2*, if $Z \sim \mathrm{Exp}(\gamma)$, then, for $j = 0$, we have

$$\hat{\phi}_0(w) = \frac{c - \frac{\rho}{\gamma}}{cw - \rho\frac{w}{\gamma+w}} = \left(1 - \frac{\rho}{c\gamma}\right) \frac{\gamma+w}{(w+R_0)w}. \tag{8.31}$$

For $j = 1, 2, \ldots$, we have

$$\begin{aligned} \hat{\phi}_j(w) &= \rho \frac{\frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+) - \frac{w}{\gamma+w}\hat{\phi}_{j-1}(w)}{cw - \rho\frac{w}{\gamma+w} - \delta j} \\ &= \rho \frac{\frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+)(\gamma+w) - w\hat{\phi}_{j-1}(w)}{c(w+R_j)(w-W_j^+)} \\ &= \rho \frac{\frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+)(\gamma+W_j^+ + w - W_j^+) - w\hat{\phi}_{j-1}(w)}{c(w+R_j)(w-W_j^+)} \\ &= \rho \frac{W_j^+\hat{\phi}_{j-1}(W_j^+) - w\hat{\phi}_{j-1}(w) + \frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+)(w-W_j^+)}{c(w+R_j)(w-W_j^+)} \\ &= \rho \frac{\frac{W_j^+\hat{\phi}_{j-1}(W_j^+) - w\hat{\phi}_{j-1}(w)}{w-W_j^+} + \frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+)}{c(w+R_j)}. \end{aligned}$$

In particular, for $j=1$, we observe

$$\begin{aligned}\hat{\phi}_1(w) &= \rho\frac{\frac{W_1^+\hat{\phi}_0(W_1^+)-w\hat{\phi}_0(w)}{w-W_1^+}+\frac{W_1^+}{\gamma+W_1^+}\hat{\phi}_0(W_1^+)}{c(w+R_1)}\\ &= \rho\frac{\left(1-\frac{\rho}{c\gamma}\right)\frac{\gamma-R_0}{W_1^++R_0}\frac{1}{w+R_0}+\frac{W_1^+}{\gamma+W_1^+}\hat{\phi}_0(W_1^+)}{c(w+R_1)}\\ &= \rho\frac{\left(1-\frac{\rho}{c\gamma}\right)\frac{\gamma-R_0}{W_1^++R_0}+\frac{W_1^+}{\gamma+W_1^+}\hat{\phi}_0(W_1^+)(w+R_0)}{c(w+R_0)(w+R_1)},\end{aligned}$$

which is the Laplace transform of a linear combination of e^{-R_0x} and e^{-R_1x}.

In general, for $j=1,2,...$, assume

$$\hat{\phi}_j(w)=\frac{P_j(w)}{c\prod_{i=0}^{j}(w+R_i)},\quad j=1,2,...,$$

where $\{P_j(w)\}_{j=1,2,...}$ are functions of w, then,

$$\frac{P_j(w)}{c\prod_{i=0}^{j}(w+R_i)}=\rho\frac{\frac{W_j^+\frac{P_{j-1}(W_j^+)}{c\prod_{i=0}^{j-1}(W_j^++R_i)}-w\frac{P_{j-1}(w)}{c\prod_{i=0}^{j-1}(w+R_i)}}{w-W_j^+}+\frac{W_j^+}{\gamma+W_j^+}\frac{P_{j-1}(W_j^+)}{c\prod_{i=0}^{j-1}(W_j^++R_i)}}{c(w+R_j)},$$

or,

$$P_j(w)=\frac{\rho}{c}\prod_{i=0}^{j-1}(w+R_i)\left[\frac{W_j^+\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^++R_i)}-w\frac{P_{j-1}(w)}{\prod_{i=0}^{j-1}(w+R_i)}}{w-W_j^+}+\frac{W_j^+}{\gamma+W_j^+}\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^++R_i)}\right],$$

then, we have

$$\begin{aligned}&P_j(w)\\ =\;&\frac{\rho}{c}\left[\frac{W_j^+\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^++R_i)}\prod_{i=0}^{j-1}(w+R_i)-wP_{j-1}(w)}{w-W_j^+}+\frac{W_j^+}{\gamma+W_j^+}\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^++R_i)}\prod_{i=0}^{j-1}(w+R_i)\right],j=2,3,...,\\ &P_1(w)=\rho\left[\left(1-\frac{\rho}{c\gamma}\right)\frac{\gamma-R_0}{W_1^++R_0}+\frac{W_1^+}{\gamma+W_1^+}\hat{\phi}_0(W_1^+)(w+R_0)\right].\end{aligned}$$

Note that, for $j=2,3,...$, $w=W_j^+$ is one of the roots of the numerator of the first term, the denominator $w-W_j^+$ then is canceled. $P_1(w)$ is a polynomial function with degree of 1, and obviously, by the method of induction, $\{P_j(w)\}_{j=1,2,...}$ are polynomial functions of w with maximum degree of j. Hence, for any $j=1,2,...$, we can have a partial fraction decomposition

$$\frac{P_j(w)}{c\prod_{i=0}^{j}(w+R_i)}=\sum_{i=0}^{j}b_{ji}\frac{1}{w+R_i},$$

where $\{b_{ji}\}_{i=0,1,...,j}$ are all constants. Since

$$\mathcal{L}_w\left\{e^{-R_ix}\right\}=\frac{1}{w+R_i},\quad i=0,1,...,j,$$

we have

$$\phi_j(x) = \sum_{i=0}^{j} b_{ji} e^{-R_i x}, \quad j = 1, 2, ...$$

For $j = 0$, we have $R_0 = \gamma - \frac{\rho}{c}$, and rewrite (8.31) as

$$\hat{\phi}_0(w) = \left(1 - \frac{\rho}{c\gamma}\right)\left[\frac{\gamma}{R_0}\frac{1}{w} + \left(1 - \frac{\gamma}{R_0}\right)\frac{1}{w + R_0}\right] = \frac{1}{w} - \frac{\rho}{c\gamma}\frac{1}{w + R_0},$$

which is the Laplace transform of

$$\phi_0(x) = 1 - \frac{\rho}{c\gamma}e^{-R_0 x}.$$

Then, the ruin probability $\psi(x,t)$ is a linear combination of $\left\{e^{-R_j x}\right\}_{j=0,1,...}$, since

$$\psi(x,t) = 1-\phi(x,t) = 1-\phi_0(x)-\sum_{j=1}^{\infty} e^{-j\delta t}\phi_j(x) = \frac{\rho}{c\gamma}e^{-R_0 x}-\sum_{j=1}^{\infty} e^{-j\delta t}\sum_{i=0}^{j} b_{ji}e^{-R_i x} = \sum_{j=0}^{\infty} B_j(t)e^{-R_j x},$$

where $\{B_j(t)\}_{j=0,1,...}$ are all deterministic functions of time t. Then, (8.5.1) should hold, because the asymptotic representation given by *Theorem 8.4.3* is also a linear combination of $\left\{e^{-R_j x}\right\}_{j=0,1,...}$.

□

9

Conclusions and Future Research

This thesis produces a general mathematical framework for modelling the dependence structure of arriving events with contagion dynamics, mainly based on generalising the Hawkes process (with exponential decay) and the Cox process with shot noise intensity (with exponential decay). In Chapter 2, 5, 6 and 7, the dynamic contagion process as well as its extensions dynamic contagion process with diffusion and discretised dynamic contagion process newly introduced here have been systemically studied by representing different mathematical definitions, analysing various distributional properties and comparing with other processes. Theoretical results such as the moments, Laplace transforms, probability generating functions of the point processes and intensity processes, methods of change of measures as well as computational methodology such as Monte Carlo simulation algorithm and numerical examples are presented.

These new point processes newly introduced in the thesis could have significant potential to be applicable to a variety of problems in economics, finance and insurance. Here, we only look at some applications to credit risk in finance in Chapter 3 and ruin problem in insurance in Chapter 4 and 8: the probability of default for a single name and probability distribution of multiple-name defaults are investigated and calculated by using various methods; the ruin probabilities and estimations such as bounds and asymptotics are derived and expressed in different representations. However, other applications such as managing portfolio credit risk, pricing credit derivatives as well as modelling the dynamics of risk contagion in economics could be the object of further research.

Some problems are proposed as future research (particularly, from Section 2.5, 3.2, 5.3 and Chapter 6):

- develop methods of calibration and estimation for a dynamic contagion process;
- apply the one-dimensional dynamic contagion process to pricing derivatives (such as CDS and CDO) and portfolio risk management (such as capital reserve calculation), with the parameters calibrated on the real financial data;
- extend to two-dimension or higher-dimension of dynamic contagion processes, and apply to modelling the contagion risk of two underlying companies (such as counterparty risk);
- prove *Assumption 4.4.1*, i.e. $\lim_{x\to\infty}\widetilde{\mathbb{E}}\Big[e^{-m_0^+\widetilde{\lambda}_{\tau^*_-}}\big|X_0=x,\lambda_0=\lambda\Big]$ exists and independent of λ;

- investigate the distributional properties of the intensity process at ruin time, i.e. $\mathbb{E}\Big[e^{\eta_0^+ \lambda_{\tau^*}} | \tau^* < \infty\Big]$.
- change of measure for pricing financial derivatives and improving the simulation of the a dynamic contagion process;
- investigate the asymptotics of $\lim_{n\to\infty} P\{N_T = n\}$ for the point process N_t of the dynamic contagion process at a fixed time T;
- develop the simulation algorithm of the multi-dimensional dynamic contagion processes;
- extend to choose other distributions, rather than exponential distributions.

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