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Weak-field gravity of revolving circular cosmic strings

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A weak-field solution of Einstein's equations is constructed. It is generated by a circular cosmic string revolving in its plane about the center of the circle. (The revolution is introduced to prevent the string from collapsing.) This solution exhibits a conical singularity, and the corresponding deficit angle is the same as for a straight string of the same linear energy density, irrespective of the angular velocity of the string.

I. INTRODUCTION

One of the most notable features of a straight cosmic string [1] is the presence in spacetime of an angular deficit, the magnitude of which is related to the linear energy density μ of the string by $\delta\psi = 8\pi G\mu$. Indeed, the deficit-angle structure of spacetime is central to many proposals for the possible observation of the gravitational effects of cosmic strings [2].

The deficit-angle model is generally accepted as a good approximation for describing the exterior gravitational field of cosmic strings. Frolov, Israel, and Unruh (FIU) used precisely this approximation when they considered a thin circular string at a moment of time symmetry [3].

Recently, we investigated further [4] the problem of the deficit angle produced by circular strings. We constructed a weak-field stationary solution of Einstein's equations generated by a thin circular string and established that external radial stresses had to be introduced to support the ring against collapse (thus allowing a stationary solution to exist). The form of the radial stresses was completely determined by stress-energy conservation. The main result of our study was that a circular string produces conical singularities with the same angular deficit as a straight string of identical linear energy density, fully supporting FIU's assumption. Thus we demonstrated, in the weak-field limit, the validity of the FIU hypothesis directly from the field equations. Furthermore, the external radial stresses were seen not to contribute to this angular deficit.

In the present work, we ask the question whether it is possible to extend the previous results to a self-supporting circular string, as opposed to an externally supported circular string: Can the stabilizing role previously played by external radial stresses be played by an internal mechanism? Centrifugal force is the simplest candidate; it should be possible to prevent gravitational collapse by spinning the ring at an appropriate angular velocity.

We construct a weak-field solution of Einstein's equations corresponding to an infinitely thin circular string revolving at a given angular velocity ω. We begin by examining the most general scenario where the string is partially supported by centrifugal force and partially by external radial stresses. The angular velocity is chosen arbitrarily, and the radial stresses are then determined by stress-energy conservation. The angular deficit produced by this solution is found to be equal to the deficit produced by a straight string (of the same linear energy density), irrespective of the value of the angular velocity ω (within the limitation of the weak-field approximation). Later, we also calculate and discuss the critical angular velocity ω_{crit} at which the ring is totally self-supporting, namely, the particular velocity at which no radial stresses are necessary to support the ring. The latter is then supported entirely by centrifugal force.

In this paper, we use units in which $\hbar = c = 1$, take the metric to have signature $(-, +, +, +)$, and adopt the geometrical conventions of Synge [5].

II. STRESS-ENERGY TENSOR

The stress-energy tensor $T^{\mu}{}_{\nu}$ generating the gravitational field of a revolving string partially supported against collapse by external radial stresses contains three contributions: $^{M}T^{\mu}{}_{\nu}$, the contribution from the circular motion of the ring (excluding radial stresses required to maintain circular motion); $^{A}T^{\mu}{}_{\nu}$, the contribution from the azimuthal flux through the string (which corresponds to the $T^{z}{}_{z}$ stresses for Vilenkin's straight string); and $^{E}T^{\mu}{}_{\nu}$, the contribution from the external radial stresses. Given that the diameter of the core of a string arising from a spontaneously broken gauge theory [6] is microscopically small, it is well justified to make the approximation that the stress-energy tensors $^{M}T^{\mu}{}_{\nu}$ and $^{A}T^{\mu}{}_{\nu}$ of the circular string be confined to the infinitely thin ring $r = a$, $z = 0$, where a denotes the radius of the ring. On

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the other hand, the radial stresses $^E T^\mu{}_\nu$, being external, are not confined to the core of the string. (See [4] for a more detailed discussion.)

For (pressure-free) dust of rest-energy density ρ_0, moving with four-velocity u^μ, the stress-energy tensor is $^M T^\mu{}_\nu = \rho_0 u^\mu u_\nu$, where $u^\mu u_\mu = -1$. In cylindrical coordinates $x^\mu \equiv (t, \phi, r, z)$, the four-velocity u^μ of an infinitely thin ring of radius a, centered at the origin, lying in the x-y plane, and revolving with angular velocity ω, is readily found as

$$u^\mu = \Gamma(\delta^\mu_t + \omega \delta^\mu_\phi) , \qquad (2.1)$$

$$-\Gamma^{-2} \equiv g_{tt}(a, 0) + 2\omega g_{t\phi}(a, 0) + \omega^2 g_{\phi\phi}(a, 0) , \qquad (2.2)$$

where $g_{\mu\nu}(a, 0)$ denotes the metric components evaluated at $r = a$, $z = 0$. (The metric enters this equation because of the normalization condition $u^\mu u_\nu = -1$.)

Furthermore, the ring has its rest-energy density ρ_0 purely localized:

$$\rho_0 = \mu_0 \delta(r - a) \delta(z) , \qquad (2.3)$$

where μ_0 is the linear rest-energy density. Consequently, the stress-energy tensor $^M T^\mu{}_\nu$ produced by the circular motion has the expression

$$^M T^\mu{}_\nu = \mu_0 \Gamma^2 (\delta^\mu_t + \omega \delta^\mu_\phi)[g_{\nu t}(a, 0) + \omega g_{\nu\phi}(a, 0)]$$
$$\times \delta(r - a) \delta(z) , \qquad (2.4)$$

where g is the metric and no summation is implied over repeated indices t and ϕ. (The summation convention will be suspended all throughout when t, ϕ, or r appear as subscripts or superscripts.)

The stress-energy tensor giving the contribution from the azimuthal flux through the string is elementary in the case of a nonrevolving ring [4], the simple argument being that $^A T^\phi{}_\phi$ plays the role that $T^z{}_z$ plays for a straight string. However, a more detailed reasoning is necessary when the string is revolving since a revolving frame is not inertial. The appropriate stress-energy tensor $^A T^\mu{}_\nu$ can be established by employing the general method applicable to anisotropic fluids [5]. (The "fluid" must be anisotropic since for physical reasons we expect a pressure along the ϕ axis but none along the r and z axes.)

The stress-energy tensor [5] of a fluid of rest-energy density ρ_0, moving with a four-velocity u^μ, is

$$T^\mu{}_\nu = \rho_0 u^\mu u_\nu - S^\mu{}_\nu , \qquad (2.5)$$

in which the tensor $S^\mu{}_\nu$, the stress tensor, satisfies $S^\mu{}_\nu u^\nu = 0$. The eigenvectors $\lambda^\mu_{(i)}$, assumed spacelike, and the corresponding eigenvalues $-P_{(i)}$, $1 \leq i \leq 3$, are called, respectively, the "principal axes" and the "principal stresses" of $S^\mu{}_\nu$. In terms of these quantities, the stress tensor can be expressed as

$$S^\mu{}_\nu = -\sum_{i=1}^3 P_{(i)} \lambda^\mu_{(i)} \lambda_{(i)\nu} , \qquad (2.6)$$

where $\lambda^\mu_{(i)}$ satisfies

$$u_\mu \lambda^\mu_{(i)} = 0 , \qquad (2.7)$$

$$\lambda^\mu_{(i)} \lambda_{(j)\mu} = \delta_{ij} . \qquad (2.8)$$

The stress-energy tensor $^M T^\mu{}_\nu$ of (2.4) already takes into account the motion of the ring; therefore, we only consider the stress part $S^\mu{}_\nu$ of (2.5). The physical requirement that there be no pressure along the r and z axes implies that the principal pressures $P_{(2)}$ and $P_{(3)}$ vanish. Thus the only nonzero principal pressure is the azimuthal pressure $P_{(1)} \equiv P_{(\phi)}$. For an infinitely thin ring, the azimuthal pressure is confined to the core of the string, which implies

$$P_{(1)} = k_0 \delta(r - a) \delta(z) , \qquad (2.9)$$

where k_0 is a constant with dimensions of force. (We interpret $-k_0$ as the tension in the ring.)

Moreover, the only relevant principal axis $\lambda_{(1)}$ must point spatially along the ϕ direction. The constraint (2.8), together with (2.7) applied to the velocity (2.1), determines $\lambda_{(1)}$ as

$$\lambda^\mu_{(1)} = K \delta^\mu_t + L \delta^\mu_\phi , \qquad (2.10)$$

where K and L are solutions of the system

$$K g_{tt}(a, 0) + (L + \omega K) g_{t\phi}(a, 0) + \omega L g_{\phi\phi}(a, 0) = 0 , \qquad (2.11)$$

$$K^2 g_{tt}(a, 0) + 2KL g_{t\phi}(a, 0) + L^2 g_{\phi\phi}(a, 0) = 1 . \qquad (2.12)$$

Thus we reach the conclusion that the stress-energy tensor $^A T^\mu{}_\nu$ generated by the azimuthal flux along the string reads

$$^A T^\mu{}_\nu = k_0 (K \delta^\mu_t + L \delta^\mu_\phi)[K g_{\nu t}(a, 0) + L g_{\nu\phi}(a, 0)]$$
$$\times \delta(r - a) \delta(z) \qquad (2.13)$$

(with no summation on t or ϕ). This tensor depends on two scalar parameters: ω, the angular velocity of the ring, and $-k_0$, the tension in the ring. When ω vanishes, (2.13) becomes identical to the expression obtained in the nonrevolving case [4], so that it is consistent to identify the scalar k_0 with the parameter k in [4]. Consequently, for string matter, it is reasonable to generalize the equation of state $k = -\mu$ of the nonrevolving case as $k_0 = -\mu_0$ in the revolving case.

In order to support the ring partially by external radial stresses, we assume the existence of a radial component of $^E T^\mu{}_\nu$ confined to the x-y plane (as in the case [4] of a nonrevolving ring):

$$^E T^\mu{}_\nu \equiv \Delta(r, z) \delta^\mu_r \delta^r_\nu \equiv f(r) \delta(z) \delta^\mu_r \delta^r_\nu , \qquad (2.14)$$

where f is a function to be determined later by stress-energy conservation, and there is no summation over r.

The complete stress-energy tensor $T^\mu{}_\nu$ generating the gravitational field of the revolving string is the sum of all the above contributions (2.4), (2.13), and (2.14):

$$T^\mu{}_\nu = {}^M T^\mu{}_\nu + {}^A T^\mu{}_\nu + {}^E T^\mu{}_\nu . \qquad (2.15)$$

We now turn to the form of the spacetime metric g.

III. METRIC AND FIELD EQUATIONS

The most general stationary metric produced by an axially symmetric revolving source may be written [7]

$$ds^2 = -e^{2v}dt^2 + e^{2\zeta-2v}r^2(d\phi - A\,dt)^2$$
$$+ e^{2\eta-2v}(dr^2+dz^2) \ , \tag{3.1}$$

where v, ζ, η, and A are functions of r and z only.

We proceed as in the nonrevolving case [4] and make the weak-field approximation. Thus we calculate the Einstein tensor $G^\mu{}_\nu$ [7] for the metric (3.1) and retain only first-order terms in v, η, ζ, and A.

Given that A is not dimensionless, it is not immediately apparent that terms of order A^2 may be neglected. However, for the stress-energy tensor (2.4), (2.13)–(2.15), the linearized Einstein field equations show that this is indeed the case at first order in the dimensionless quantities $G\mu_0$ and Gk_0. (A complete discussion is presented in the Appendix.) Therefore, in all our future considerations, the expression "weak-field approximation" will refer to the expansion of the field equations at first order in v, η, ζ, A, $G\mu_0$, and Gk_0.

With this approximation, Eqs. (2.2), (2.4), and (2.11)–(2.13) simplify greatly, and after some manipulations the field equations (see the Appendix) reduce to

$$\nabla^2 v = 4\pi G\{(C_1+C_2)\delta(r-a)\delta(z)+\Delta(r,z)\} \ , \tag{3.2}$$

$$\tilde{\nabla}^2\eta = 8\pi GC_2\delta(r-a)\delta(z) \ , \tag{3.3}$$

$$\nabla^2\zeta + \frac{1}{r}\zeta_r = 8\pi G\Delta(r,z) \ , \tag{3.4}$$

$$\frac{\partial}{\partial r}(r\zeta) = \eta - \eta_0 \ , \tag{3.5}$$

$$\nabla^2 A + \frac{2}{r}A_r = -16\pi GC_3 a^{-1}\delta(r-a)\delta(z) \ , \tag{3.6}$$

in which Δ is the radial-stress function defined in (2.14), $\tilde{\nabla} \equiv \partial_r^2 + \partial_z^2$, $\nabla \equiv \tilde{\nabla}+(1/r)\partial_r$ is the Laplacian, η_0 is an arbitrary constant of integration, and the constants C_i, $1 \le i \le 3$, are related to the parameters μ_0, k_0, ω, and a of the problem by

$$C_1 \equiv (\mu_0+\omega^2 a^2 k_0)\Gamma_0^2 \ , \tag{3.7}$$

$$C_2 \equiv (\omega^2 a^2\mu_0+k_0)\Gamma_0^2 \ , \tag{3.8}$$

$$C_3 \equiv (\mu_0+k_0)\Gamma_0^2\omega a \ , \tag{3.9}$$

$$\Gamma_0^{-2} \equiv 1-\omega^2 a^2 \ . \tag{3.10}$$

(The quantities GC_i, $1 \le i \le 3$, are dimensionless.)

The compatibility condition for (3.3)–(3.5), or

equivalently the stress-energy conservation law, determines the radial-stress function Δ of (2.14) as

$$\Delta(r,z) = f(r)\delta(z) = (C_2/r)\Theta(r-1)\delta(z) \ , \tag{3.11}$$

in which Θ denotes the Heaviside step function. This expression, which is similar to the one obtained in the nonrevolving case [4], may now be substituted into (3.2), so that we have a complete set of equations of which the metric functions v, ζ, η, and A are solutions.

IV. ANGULAR DEFICIT

Our main purpose in solving the field equations (3.2)–(3.6) is to investigate the metric for conical singularities and to calculate the corresponding angular deficit. Because of the fact that canonical singularities involve only the metric (3.1) at constant time t and constant azimuth ϕ, it is sufficient to restrict attention to obtaining explicitly the functions v and η given by (3.2) and (3.3). We solved these equations, for different values of the constants C_i, in our previous work on the nonrevolving ring [4], and it is therefore not necessary to repeat the analysis here. We only recall that, in toroidal [8] coordinates (t,ϕ,σ,ψ), which are related to the cylindrical coordinates (t,ϕ,r,z) by

$$z/a \equiv N^{-2}\sin\psi, \quad r/a \equiv N^{-2}\sinh\sigma \ ,$$
$$N^2 \equiv N^2(\sigma,\psi) \equiv \cosh\sigma - \cos\psi \ , \tag{4.1}$$

the solutions near the string (namely, for $\sigma \to \infty$) read

$$v(\sigma,\psi) \to -2G(C_1+C_2)\sigma \ , \tag{4.2}$$

$$\eta(\sigma,\psi) \to -4GC_2\sigma \ . \tag{4.3}$$

(As in the nonrevolving case [4], the radial stresses (2.14) and (3.11) do not contribute to these asymptotic forms for v and η, and thus have no influence on the conical singularities.)

It follows from (4.2) and (4.3) that the combination $\eta-v$, which determines the metric (3.1) at constant t and ϕ, becomes, after substituting the definitions (3.7), (3.8), (3.10) and noting the nontrivial cancellation of the ω-dependent terms,

$$\eta-v \to 2G(\mu_0-k_0)\sigma \ . \tag{4.4}$$

The fact that $\eta-v$ is proportional to the toroidal coordinate σ indicates the presence of a conical singularity [3,4]. The corresponding angular deficit $\delta\psi$, which is related to the ratio of the perimeter of a circle centered at the core of the string to the radius of this circle [3,4] is given by

$$\delta\psi = 2\pi - \lim_{\sigma_0 \to \infty}\left[\int_{-\pi}^{\pi}d\psi(N^{-2}e^{\eta-v})\Big|_{\sigma=\sigma_0} \Big/ \int_{\sigma_0}^{\infty}d\sigma\,N^{-2}e^{\eta-v}\right] \tag{4.5}$$

and is easily calculated as

$$\delta\psi = 4\pi G(\mu_0 - k_0) . \tag{4.6}$$

As announced earlier, this expression is independent of the angular velocity ω at which the ring revolves. Moreover, the angular deficit is also identical with Vilenkin's results [1] for a straight string. (For string matter, $k_0 = -\mu_0$, as explained in [4].) We have thus demonstrated, in the weak-field approximation, that a revolving circular string produces the same angular deficit as a straight string of the same linear density.

V. SELF-SUPPORTING RING

Finally, we address the problem of whether a revolving string can be totally self-supporting. Up to this point, we considered a ring partially supported by the centrifugal force produced by the revolution and partially supported by the external radial stresses Δ of (2.14) and (3.11).

For the discussion that follows, the equation of state relating k_0 and μ_0 will conveniently be written as

$$k_0 = (\alpha - 1)\mu_0 , \tag{5.1}$$

where α is a parameter. (This particular parametrization excludes the possibility of the physically uninteresting case $\mu_0 = 0$, $k_0 \neq 0$.) String matter is characterized by $\alpha = 0$, whereas nonstring matter has negative pressure for $0 < \alpha < 1$ and positive pressure if $\alpha > 1$.

By definition, the ring is self-supporting when no external radial stresses are necessary to prevent collapse. This happens, by virtue of (3.11), if and only if ω takes the critical value ω_{crit} that forces C_2 to vanish. Equations (3.8) and (3.10), upon inserting (5.1), imply that ω_{crit} is the solution of

$$0 = \mu_0 \left[-1 + \frac{\alpha}{1 - (\omega_{crit}a)^2} \right] . \tag{5.2}$$

The above constraint always has the trivial solution $\mu_0 = 0$, which in turn implies $k_0 = 0$, so that spacetime is flat everywhere. For string matter, $\alpha = 0$, and this trivial solution is the only solution that (5.2) admits. Consequently, we have established that a ring of string matter cannot be made self-supporting exclusively by centrifugal force, but that a certain amount of external stress is necessary to prevent collapse. In other words, trying to support a ring of string matter purely by inducing a revolution requires the string to be massless, which is nonphysical.

The solution of (5.2) in the physically interesting case $\mu_0 \neq 0$ is

$$(\omega_{crit}a)^2 = 1 - \alpha . \tag{5.3}$$

We observe that ω_{crit} is independent of G and, therefore, of the gravitational field. This is a simple manifestation of the well-known [9] "motion of the source" problem in the linearized Einstein equations: The linear approximation is sufficient to calculate the first metric correction produced by the source, but neglects the back reaction of gravity onto the source, so that the source moves as if gravity were absent. Taking this back reaction into account requires using at least second-order terms as done, for instance, in the Einstein-Infeld-Hoffman procedure [10] for the motion of point masses. It is important, however, to insist on the fact that, although the first-order framework is not appropriate to study gravitational influences on the *motion of the source*, it is perfectly valid to study gravitational corrections to the *metric* and, thus, our result on the angular deficit holds.

Further insight on the physical significance of (5.3) may be gained by studying, in classical mechanics, the equilibrium condition for a revolving ring of radius a, linear mass density μ, and tension T. Consider a small arc of angular width θ along the circle. The two extremities of this arc are subjected to a tension T which is tangential, but the resultant force T_R at the midpoint along the arc is purely radial and is given by $T_R = 2T \sin(\theta/2)$. On the other hand, the centrifugal force F_C acting on the arc reads $F_C = \mu\theta\omega^2 a^2$, and consequently, equilibrium is attained when ω reaches the critical value ω_{crit} satisfying

$$(\omega_{crit}a)^2 = \frac{T}{\mu} \lim_{\theta \to 0} [(2/\theta)\sin(\theta/2)] = \frac{T}{\mu} . \tag{5.4}$$

This result is identical with (5.3) for $T = -k_0$ and $\mu = \mu_0$ since $\alpha = 1 + k_0/\mu_0$ by (5.1).

It follows from (5.3) that matter must have a negative pressure to lead to a self-supporting ring [since α must be less than one for (5.3) to admit a solution for ω_{crit}]. This is physically reasonable since a positive pressure would tend to create an expansion of the ring, whereas a negative pressure would tend to create a collapse. Only a collapse, and therefore a negative pressure, could be counteracted by centrifugal force. (The gravitational force does not contribute to the collapse at first order, as just mentioned.)

Moreover, as the pressure becomes more negative and approaches $-\mu_0$, the critical velocity increases. The extreme case of self-supporting string matter, namely, $\alpha = 0$, formally implies that $\omega_{crit}a = \pm 1$, which means that the string revolves at the speed of light. This is impossible for a massive body. Thus we reach the same conclusion as before that string matter cannot be made self-supporting by revolution only. (This may no longer hold if gravity is explicitly taken into account in the motion of the source.) The present argument, albeit physically enlightening, is only formal since the weak-field approximation breaks down for a massive body if $\omega a = \pm 1$.

We emphasize that the stress-energy tensor used in our treatment contains terms proportional to $\mu_0(1 - a^2\omega^2)^{-1}$ and $k_0(1 - a^2\omega^2)^{-1}$. Therefore our results are valid when the dimensionless quantities $G\mu_0(1 - a^2\omega^2)^{-1}$ and $Gk_0(1 - a^2\omega^2)^{-1}$ are small. In particular, the value of ωa, for a ring partially supported by revolution and partially by external stresses, is not determined by any equation [in contrast with the totally self-supporting case (5.2), (5.3)], but is a free parameter. Therefore our main result, namely, that the angular deficit produced by a revolving ring is the same as for a straight string of the same linear density, is valid, within the confines of the

weak-field limit, for a large range of the values of the parameter $a\omega$.

VI. CONCLUSION

In this work we extended our previous results [4] on the deficit-angle structure of spacetime of a nonrevolving circular string to the case of a revolving circular string. The circular string was prevented from collapse partially by the centrifugal force produced by the revolution and partially by external radial stresses that were determined by stress-energy conservation. We established, in a weak-field treatment, that a conical singularity exists along the string and that the magnitude of the corresponding deficit angle is the same as that produced by a straight string of the same linear energy density.

We also investigated whether a revolving circular ring could be totally self-supporting, that is, whether there existed a critical angular velocity ω_{crit} at which the ring is in equilibrium without the presence of external radial stresses. We took string matter to have the equation of state, $k_0 = -\mu_0$, where $-k_0$ and μ_0 denote the tension and the linear rest energy of the string, respectively, and found that it was impossible to have a self-supporting string. However, for nonstring matter characterized by the equation of state, $k_0 \neq -\mu_0$, the critical angular velocity was established to be

$$(\omega_{\text{crit}} a)^2 = -k_0/\mu_0 , \qquad (6.1)$$

where a is the radius of the ring. [The weak-field treatment was shown to be valid as long as the dimensionless quantities $G\mu_0(1-a^2\omega^2)^{-1}$ and $Gk_0(1-a^2\omega^2)^{-1}$ are small compared to unity.]

APPENDIX: FRAME COMPONENTS OF THE FIELD EQUATIONS

The most general stationary metric produced by an axially symmetric source, namely, (3.1), may be written in terms of an orthonormal basis $\{\theta^{(\hat{\mu})}\}$ as

$$ds^2 = \eta_{\mu\nu}\theta^{(\hat{\mu})} \otimes \theta^{(\hat{\nu})} , \qquad (A1)$$

where $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ and

$$\theta^{(\hat{0})} \equiv e^\nu dt, \quad \theta^{(\hat{1})} \equiv re^{\zeta-\nu}(d\phi - A\, dt) ,$$
$$\theta^{(\hat{2})} \equiv e^{\eta-\nu}dr, \quad \theta^{(\hat{3})} \equiv e^{\eta-\nu}dz . \qquad (A2)$$

We define the transformation matrix $e^{\hat{\mu}}{}_\nu$ (and its inverse $e^\mu{}_{\hat{\nu}}$) by

$$\theta^{(\hat{\mu})} \equiv e^{\hat{\mu}}{}_\nu dx^\nu . \qquad (A3)$$

The components of the Einstein and stress-energy tensors in the frame $\theta^{(\hat{\mu})}$ are then simply

$$G_{\hat{\mu}\hat{\nu}} = G_{\alpha\beta}e^\alpha{}_{\hat{\mu}}e^\beta{}_{\hat{\nu}}, \quad T_{\hat{\mu}\hat{\nu}} = T_{\alpha\beta}e^\alpha{}_{\hat{\mu}}e^\beta{}_{\hat{\nu}} , \qquad (A4)$$

and Einstein's equations read $G_{\hat{\mu}\hat{\nu}} = -8\pi GT_{\hat{\mu}\hat{\nu}}$.

We now turn to the weak-field approximation. The metric (A1) is flat if $\nu = \eta = \zeta = 0$ and A is constant. Furthermore, the quantities ν, η, ζ are dimensionless, whereas A has dimensions of 1/length. Therefore it is not immediately clear that a meaningful approximation to Einstein's equations can be obtained by neglecting terms of order A^2.

To clarify this point, we begin by calculating the Einstein tensor in the orthonormal frame (A2) up to terms of first order in ν, η, ζ and all orders in A. The results are

$$G_{\hat{0}\hat{0}} = -2(\nabla^2\nu) + \left[\nabla^2 + \frac{1}{r}\partial_r\right]\zeta + \tilde{\nabla}^2\eta$$
$$+ \frac{r^2}{4}(1+2\epsilon)(A_r^2 + A_z^2) , \qquad (A5)$$

$$G_{\hat{1}\hat{1}} = -\tilde{\nabla}^2\eta + \frac{3r^2}{4}(1+2\epsilon)(A_r^2 + A_z^2) , \qquad (A6)$$

$$G_{\hat{2}\hat{2}} = -\zeta_{zz} - \frac{\eta_r}{r} + \frac{r^2}{4}(1+2\epsilon)(A_z^2 - A_r^2) , \qquad (A7)$$

$$G_{\hat{3}\hat{3}} = -\zeta_{rr} + \frac{1}{r}(\eta_r - 2\zeta_r) + \frac{r^2}{4}(1+2\epsilon)(A_r^2 - A_z^2) , \qquad (A8)$$

$$G_{\hat{2}\hat{3}} = \zeta_{rz} + \frac{1}{r}(\zeta_z - \eta_z) - \frac{r^2}{2}(1+2\epsilon)A_rA_z , \qquad (A9)$$

$$G_{\hat{0}\hat{1}} = -\frac{r}{2}\mathcal{L}A , \qquad (A10)$$

with

$$\mathcal{L} \equiv e^{-2(\eta+\zeta-2\nu)}\left\{\partial_r[e^{(3\zeta-4\nu)}\partial_r] + \partial_z[e^{(3\zeta-4\nu)}\partial_z]\right.$$
$$\left. + \frac{3}{r}[e^{(3\zeta-4\nu)}\partial_r]\right\} , \qquad (A11)$$

where $\epsilon \equiv \zeta - \nu - \eta$, $\tilde{\nabla}^2 \equiv \partial_r^2 + \partial_z^2$, and $\nabla^2 \equiv \tilde{\nabla}^2 + (1/r)\partial_r$ is the Laplacian operator. [In Eq. (A11), the exponentials are only to be taken to first order in their arguments.]

The stress-energy tensor (2.4) and (2.13) are proportional to μ_0 and k_0, respectively, and thus their contributions to the Einstein equations are proportional to the dimensionless quantities $G\mu_0$ and Gk_0. Therefore the metric appearing in the expressions for the stress-energy tensors (2.4) and (2.13) may be replaced by the flat-space metric, namely, $\nu = \eta = \zeta = 0$, $A = \text{const}$, if attention is restricted to a first-order treatment in $G\mu_0$ and Gk_0. Hence, without making any approximation on the order of A, we calculate that the $T_{\hat{0}\hat{1}}$ component of the total stress-energy tensor (2.15) is

$$GT_{\hat{0}\hat{1}} = G(\mu_0 + k_0)\frac{a(A-\omega)}{1-a^2(A-\omega)^2}\delta(r-a)\delta(z) . \qquad (A12)$$

Consequently, the Einstein equation $G_{\hat{0}\hat{1}} = -8\pi GT_{\hat{0}\hat{1}}$ becomes

$$\mathcal{L} A = 16\pi G(\mu_0 + k_0) \frac{A - \omega}{1 - a^2(A - \omega)^2} \delta(r - a)\delta(z) .$$

(A13)

We now introduce the dimensionless function B by

$$A \equiv -16\pi G(\mu_0 + k_0)\omega B .$$

(A14)

Note that A tends to zero with ω, so that the above parametrization guarantees compatibility between the revolving and nonrevolving cases [4], where A vanishes identically. Thus the following equation for B is equivalent to (A13):

$$\mathcal{L} B = \frac{1 + 16\pi G(\mu_0 + k_0)\omega B}{1 - a^2\omega^2[1 + 16\pi G(\mu_0 + k_0)B]^2} \delta(r - a)\delta(z) .$$

(A15)

Expanding B in powers of the dimensionless quantity $G(\mu_0 + k_0)$, we clearly see from (A15) that the leading term ^0B is of zeroth order in $G(\mu_0 + k_0)$ and satisfies

$$\mathcal{L} \, {}^0B = \frac{1}{1 - a^2\omega^2} \delta(r - a)\delta(z) .$$

(A16)

As a result, by virtue of (A14), the leading term in A is proportional to $G(\mu_0 + k_0)$. Therefore we may neglect second powers of A and products of A with v, η, or ζ in our approximation scheme, since we only retain first powers of $G\mu_0$ and Gk_0.

[1] Ya. B. Zel'dovich, Mon. Not. R. Astron. Soc. **192**, 663 (1980); A. Vilenkin, Phys. Rep. **121**, 263 (1985); R. H. Brandenberger, Phys. Scr. **T36**, 141 (1991); A. Vilenkin, Phys. Rev. Lett. **46**, 1169 (1981); **46**, 1496(E) (1981); Phys. Rev. D **23**, 852 (1981).

[2] A. Vilenkin, Phys. Rev. D **24**, 2082 (1981); Phys. Lett. **107B**, 47 (1981); Astrophys. J. **282**, L51 (1984); J. R. Gott, ibid. **288**, 422 (1985); N. Kaiser and A. Stebbins, Nature **310**, 391 (1984); C. J. Hogan and M. J. Rees, ibid. **311**, 109 (1984); T. Vachaspati and A. Vilenkin, Phys. Rev. D **31**, 3052 (1985); R. Brandenberger, A. Albrecht, and N. Turok, Nucl. Phys. **B277**, 605 (1986); F. Accetta and L. Krauss, ibid. **B319**, 747 (1989); N. Sanchez and M. Signore, Phys. Lett. B **219**, 413 (1989).

[3] V. P. Frolov, W. Israel, and W. G. Unruh, Phys. Rev. D **39**, 1084 (1989).

[4] S. J. Hughes, D. J. McManus, and M. A. Vandyck, Phys. Rev. D **47**, 468 (1993).

[5] J. L. Synge, Relativity: The General Theory (North-Holland, Amsterdam, 1960), Chap. 8, p. 310 and Chap. 4, p. 186.

[6] J. Preskill, in Architecture of the Fundamental Interactions at Short Distances, Proceedings of the Les Houches Summer School, Les Houches, France, 1985, edited by P. Ramond and R. Stora, Les Houches Summer School Proceedings Vol. 44 (North-Holland, Amsterdam, 1987).

[7] S. Chandrasekhar and J. L. Friedman, Astrophys. J. **175**, 379 (1972).

[8] H. Bateman, Partial Differential Equations of Mathematical Physics (Cambridge University Press, Cambridge, England, 1952), p. 461.

[9] H. C. Ohanian, Gravitation and Spacetime (Norton, New York, 1976), Chap. 3, p. 103.

[10] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938); A. Einstein and L. Infeld, Can. J. Math. **1**, 209 (1949).