fuzzy geometry

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Being a thesis submitted to the University of Warwick for the degree of Doctor of Philosophy

in

june
1971

Appendix H
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CONTENTS

Introduction

Origin

The concept of fuzzy space is due independently to Poincaré [20,21,22,23] and Zeeman [31,32,33]. (Poincaré used the term "physical continuum", Zeeman the term "tolerance space". I have reluctantly introduced a third expression since my attempts to generate a vocabulary from either of these have all proved impossibly unwieldy.) Both were led to it by the nature of our perception of space, and both adapted to it tools current in topology. Unfortunately, neither examined the application of these tools in complete detail, and as a result the argument from analogy was somewhat over-extended by both. The resemblances to topology are strong; the differences are sometimes glaring and sometimes subtle. In the latter case the difficulties produced by a topologically-conditioned intuition can be severe obstacles to progress. (Certainly, having been reared mathematically as a topologist I have found it necessary to distrust any conclusion whose proof is not painfully precise.) For this reason many of the proofs in this paper are set out in somewhat more detail than would be natural in a more established field. For this reason also I have here not only set out the positive results I have so far obtained in the subject but, for the benefit of topologists, elaborated on the failures of analogy with topology where a more succinct exposition would have ignored them as dead ends (e.g., in Chap.I,§2).

Apologia

The reasons for studying fuzzy spaces are various. Firstly, they are all around us. With all five and more senses† and - even in principle - with instruments we discover a limit of discrimination. (To measure in centimetres to fifty decimal places would require an energy of 20,000,000 metric tons [4]. This represents as severe a breakdown of the concept of 'arbitrarily short length' as the predictions on black-body radiation were in the classical theory of radiation. A fortiori, distinctions depending on an infinite number of decimal places, such as that between the rational and irrational flows on the torus in dynamical system theory, become absurd when considered as physical statements.) This gives rise to a relation of 'indistinguishability' which is reflexive and symmetric but intransitive. Unlike the Euclidean plane or pseudo-Riemannian manifolds, then, fuzzy spaces occur as objects of direct experience. (Philosophical aside: this is not assertion that, necessarily, the objects of perception have an intrinsic existence. Deny that they have - even assume solipsistically that nothing outside yourself exists - and you can still perceive your illusory perceptions to constitute a fuzzy space.) Secondly, the concept of 'preserving indistinguishability' is a far better formalisation of the intuitive idea

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† E.g. hearing, touch, taste, smell, bodily position, sight, time. The last two illustrate particularly aptly the fuzzy basis of our perception of continuity: stand away from a row of dots and see a line; go watch a movie.

of continuity than is the topological one. This does not of course apply to the intuition of most readers of this paper, who have been as thoroughly conditioned to topological continuity as the mathematicians of the Eighteenth Century were to the parallel postulate. But I can explain the basic definitions and theorems of fuzzy geometry to an economist over a beer, in contrast to the term or more necessary to convince a reasonable proportion of mathematics freshmen about 'real' functions that are discontinuous at every rational but continuous everywhere else. (Intuitionists of course are unconvinced again; they claim that all real functions are continuous. <u>My</u> intuition says things can break.) Etymologically, one would suppose the more intuitive a concept, the less teaching necessary, and vice versa. The requirement of 'preserving indistinguishability' may be rephrased in a form more resembling topological continuity as a requirement that <u>the</u> neighbourhood of the image of a point must have <u>the</u> neighbourhood of the point mapped into it.

Thirdly, fuzzy spaces are a topic as rich in mathematical interest for its own sake as topology, and potentially as wide and various in application. Just as many mathematical structures other than the original Klein bottles and similar curiosities have a topology, in a natural manner that is useful in studying them (one might mention the successes of topological techniques in algebraic geometry), an intransitive reflexive symmetric relation is an object that may arise in any context, and the techniques discussed here may be of use. A possible example is considered in Chap.VI, §3.

Finally, for some centuries differential equations have been used in ever more descriptions, even where the assumption of infinite divisibility (necessary to define the differential) has been known to be untrue, as in fluid mechanics and electronics. These are now increasingly being solved in practice numerically by approximation by finite difference equations. Finite observations are thus being handled by finite computation to give finitistic, approximate, answers, which are all that can in any case be tested. But this admirable procedure is justified by the use of differential equations! To remove this singularity it is necessary to construct the appropriate general framework for finite difference equations, corresponding to the theory of differentiable manifolds for the differential case. There is much work to be done before it can be asserted that fuzzy spaces fully provide such a framework, but the results of Chapters III and IV, in providing a first coordinate-free expression of results in the calculus of finite differences, show some promise in this direction.

This paper, inevitably, is open-ended, and in its later parts every question answered or result proved suggests further work. A paper of 174 pages, or twice that, can no more hope to be exhaustive on fuzzy geometry than could a paper of similar length on general, algebraic and differential topology together.

Organisation

Chapter I discusses general structure and covers basic theorems that or more or less analogous to those central theorems of general topology that are not concerned with the different ways of not being locally nasty. Fuzzy spaces cannot be locally nasty, and hence no conditions on the spaces are necessary to make the product of identification maps an identification map, to make the exponential theorem hold, or (in II §7) to make possible the existence of a universal cover. Connectedness and path-connectedness are equivalent, and a metric with integer values is intrinsic. Chapter II contains the fundamentals of the fuzzy analogue to algebraic topology. It consists in part of material such as that on the homotopy groups, exploring the analogy with topology, and in part of results intended for use in difference-geometric contexts later. Thus the alternating simplicial cohomology functor recurs naturally in III §4 as the analogue of De Rham cohomology, and the cubical functor, useful in defining the Hurewicz map, seems to have applications also in the context of commuting vector fields, though this is not yet clear and is not developed in the text. The theory of covering spaces in §7 models closely the topological case for l.p.c., l.s.c spaces, with the interesting departure to be found in II.7.1. The claims made in [5] for the elegance and economy to be achieved by the use of groupoids are borne out even more dramatically in this category than in the topological case.

Chapter III covers the foundations of the finitistic analogue of differential geometry. The introduction of matroids as a local structure seems to be found equally unexpected by the combinatorialist, who has always considered them as essentially global objects, and by almost everybody else, who has never considered them. For the differential geometer, however, V §3 provides a large source of motivating examples.

Discrete potential theory was established in the 1920's [6] as a theory on functions on square or cubic lattices and has continued in this setting since, the major innovation being the application of Fourier integrals [8,9]. In 1969 this technique was extended to the body-centered cubic lattice

$\{(x,y,z) \in Z \times Z \times Z \mid (-1)^x = (-1)^y = (-1)^z\}$. [29]

Chapter IV generalises this theory in two ways; lattices are replaced by fuzzy spaces, giving a coordinate-free theory that covers all types of lattice as special case, and functions (0-forms) are extended by the consideration of forms of all orders. In this setting an analogue of Hodge's Theorem is proved, the deep methods necessary for the infinitistic version proving unnecessary here, and a number of other results established. This material can evidently be extended to a more general theory of elliptic difference operators, and it is my intention to do so.

Chapter V explores the relationship between fuzzy and continuous mathematics. This is a larger topic than either, since it subsumes both, and it is not possible here to do more than point out some of the possibilities and pitfalls. It has not been my major interest; the philosophy that has stimulated this work so far is that limits are physically meaningless and mathematically unnecessary, and the ideal is to displace Euclidean geometry in the physically small as Lobachevskian has displaced it in the large. This may be impossible, but the fuzzy-geometric approach to a finitistic physics seems promising, and at least as natural as taking space-time to be a lattice [24] or replacing the real number continuum with a finite field [1]†. I shall therefore continue for the present to pursue it.

Chapter VI is a coda, discussing some of the points for further work not already considered in the remainder of the text.

I have come to realise that some of the results in the first two chapters could be established more economically by more highly-powered categorical techniques. However, if I were now to start rewriting this paper I would be disposed to write less categorically rather than more so, since it is an unfamiliar vocabulary to some of those who have expressed interest in the later parts.

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† The following conjecture is due to David Fowler: the true base for counting in the cosmos is Z_p, for some large p, and there are (p-1) particles. When God wishes to end it, he will add another one.

Acknowledgements

I would like to thank my successive supervisors, Professor R. Brown, Dr. E. Rees, Professor W.H. Cockcroft and Professor E.C. Zeeman, for their assistance and encouragement, and also in particular Professor J. Eells for his very fruitful suggestions. An essential part of my own thinking being to explain it to anyone who will listen, and clarify my own ideas, in the process, I further gratefully thank and apologise to everyone I have bent the ear of, and thank them for their comments, doubts and suggestions. Finally my heartfelt gratitude goes to Christine and Gail, without whom the physical form of this paper would have been a hideous mess produced a great deal later.

Notations and Conventions

The symbol $\mathbb{N}$ is used throughout to denote the natural numbers, which are taken to include 0.

Considerable use is made of the 'Kronecker δ' δ_{ij} (sometimes δ_{ab}, δ_{xy} etc. in context) which is either, according to context, a function equal to the 1 of the context if $i = j$, the 0 otherwise, or the image of (i,j) by this function.

$A \setminus B$ is used to denote the set $\{a | a \in A , a \notin B\}$, whether or not $B \subseteq A$. $\emptyset$ denotes the empty set.

If $f : X \to Y$ is a function between sets (with additional structure or otherwise) and $A \subseteq X$, $B \subseteq Y$ have $f(A) \subseteq B$, then $B|f|A$ denotes the restriction and corestriction of f to A and B respectively: the map $A \to B$ induced by f. We abbreviate $Y|f|A$ to $f|A$, $B|f|X$ to $B|f$.

$f|A \equiv 0$ means that $f(a) = 0 \; \forall \, a \in A$.

I have been baffled too often by unclarity as to whether $f^{-1}(x)$ was an instruction to invert f or $(f(x))$, and therefore use $f^{\leftarrow}(Y)$ for the set $\{x | f(x) \in Y\}$, and $f^{\leftarrow}$ for the inverse map to f if it exists.

For any surjection f, a <u>section</u> of f is a map g such that $f(g(x)) = x$, $\forall \, x$.

Since different parts of this material may be of interest to readers of different backgrounds, it is very liberally cross-referenced, both to results and to definitions.

References given in proofs for known results are chosen for convenience of access; no attempt is made to identify first appearance. The common symbol ■ signifies the end or omission of a proof; in the case of a conjecture this is amended to ⍰ .

I. General Fuzzy

1. Fuzzy Spaces

1.0. Definitions

1.00. A physical continuum, tolerance space, or fuzzy space (X,τ) or (loosely) X is a set X with a symmetric reflexive relation $\tau \subset X \times X$, the tolerance or fuzzy on X. If $(x,y) \in \tau$ (which we shall also write as "$x\tau y$"), x is within fuzzy of y, or indistinguishable from y.

1.01. The set $N(x) = \{y|x\tau y\} \subset X$ is the fuzzy neighbourhood of x in X. If $A \subset X$, the fuzzy neighbourhood of A in X is $N(A) = \bigcup_{x \in A} N(x)$.

1.02. Evidently fuzzies on a set X are partially ordered by inclusion. If $\sigma \subseteq \tau$, τ is bigger than σ, σ is less than τ. $X \times X = \iota$ is the big fuzzy on a set X, and $\{(x,x)|x \in X\} = \delta$ the little fuzzy. A singleton will always be taken as a fuzzy space, with the unique fuzzy, as will the empty set.

1.03. X is the underlying set of the fuzzy space (X,τ), and a set-theoretic map to or from (X,τ) is a map to or from the set X.

1.04. A set-theoretic map $f : (X,\tau) \to (Y,\sigma)$ is fuzmic if $(f,f)\tau \subset \sigma$. A fuzmic map is a fuzmap or fuzmorphism. Evidently the set-theoretic identity on any fuzzy space, and composites of fuzmorphisms, are fuzmic, so that fuzzy spaces and fuzmorphisms form a category, which we shall denote by $\mathcal{F}uz$.

1.05. A fuzmap $f : (X,\tau) \to (Y,\sigma)$ is an embedding if it is monic and $(f,f)^{\leftarrow} \sigma = \tau$, a fuzziomorphism if it has a fuzmic two-sided inverse (i.e. if it is an isomorphism in the category).

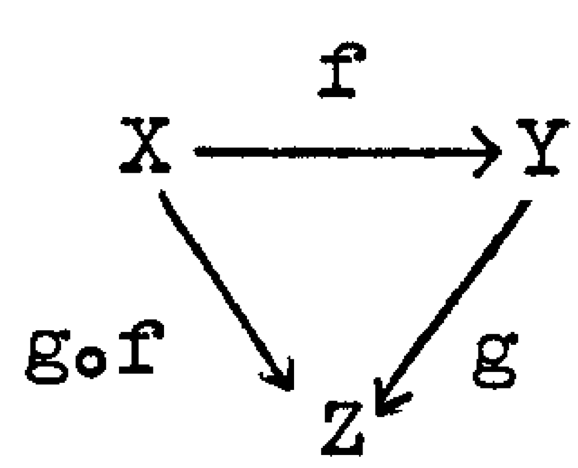
1.06. The cardinality $n(X,\tau)$ or $n(X)$ of a fuzzy space is the cardinality of its underlying set.

1.07. A fuzzy space (X,τ) is separable or Halphen (cf.[14]) if for all $x,y \in X$, $x \neq y \Rightarrow \exists\, z \in X$ such that $(z,x) \in \tau$, $(z,y) \notin \tau$.

1.1. Proposition

If $f : (X,\tau) \to Y$ is a set-theoretic map, the following definitions of the coinduced fuzzy $f_*\tau$ on Y are equivalent, and it is well defined by each.

1.11. $f_*\tau$ is the unique fuzzy on Y with the universal property that for all set-theoretic maps $g : Y \to Z$ and fuzzies σ on Z ,



$g \circ f : (X,\tau) \to (Z,\sigma)$ is fuzmic $\Leftrightarrow$ $g : (Y,f_*\tau) \to (Z,\sigma)$ is fuzmic.

1.12. $f_*\tau$ is the least fuzzy on Y s.t. $f : (X,\tau) \to (Y,f_*\tau)$ is fuzmic.

1.13. $f_*\tau = \delta \cup (f,f)\tau$.

Proof

Denote the universal property 1.11 by p .

i) Let 1_Y be the set-theoretic identity on Y , and let the fuzzy K on Y have the property p . Then

$$\begin{aligned} f : (X,\tau) \to (Y,\mu) \text{ fuzmic} &\Leftrightarrow 1_Y \circ f : (X,\tau) \to (Y,\mu) \text{ fuzmic} \\ &\Leftrightarrow 1_Y : (Y,K) \to (Y,\mu) \text{ fuzmic, by (p)} \\ &\Leftrightarrow K \subseteq \mu \end{aligned}$$

i.e. K is the least fuzzy on Y s.t. $f : (X,\tau) \to (Y,K)$ is fuzmic. If K' also has property p , the same argument applies, so that we have $K' \subseteq K$, $K \subseteq K'$, and hence $K' = K$. Thus if there is a fuzzy with property p it is unique, and $f_*\tau$ as defined by 1.11, if it exists, satisfies 1.12.

ii) Any relation including δ is reflexive, hence $\delta \cup (f,f)\tau$ is reflexive.

$$\begin{aligned} (y,y') \in \delta \cup (f,f)\tau &\Rightarrow y = y' \\ &\quad \text{or } (y = f(x), y' = f(x'), \text{where } (x,x') \in \tau) . \\ &\Rightarrow y' = y \\ &\quad \text{or } (y' = f(x'), y = f(x), \text{where } (x',x) \in \tau) , \\ &\quad \text{since } \tau \text{ is symmetric.} \\ &\Rightarrow (y',y) \in \delta \cup (f,f)\tau . \end{aligned}$$

Thus $\delta \cup (f,f)\tau$ is symmetric.

Hence $\delta \cup (f,f)\tau$ is a fuzzy.

For any fuzzy K on Y

$$\begin{aligned} f : (X,\tau) \to (Y,K) \text{ fuzmic} &\Leftrightarrow (f,f)\tau \subseteq K \qquad (1.04) \\ &\Leftrightarrow \delta \cup (f,f)\tau \subseteq K \\ &\qquad (\delta \subseteq K \text{ for all fuzzies } K) \end{aligned}$$

i.e. the least fuzzy K s.t. $f : (X,\tau) \to (Y,K)$ is fuzmic is $\delta \cup (f,f)\tau$.

$\therefore$ $f_*\tau$ as defined by 1.12 satisfies 1.13.

iii) $g \circ f : (X,\tau) \to (Z,\sigma)$ fuzmic $\Leftrightarrow (g \circ f, g \circ f)\tau \subseteq \sigma$

$\Leftrightarrow (g,g) \circ (f,f)\tau \subseteq \sigma$

$\Leftrightarrow (g,g)[(f,f)\tau] \subseteq \sigma$

$\Leftrightarrow (g,g)\delta \cup (g,g)[(f,f)\tau] \subseteq \sigma$, since $(g,g)\delta_Y \subseteq \delta_Z \subseteq \sigma$, σ a fuzzy

$\Leftrightarrow (g,g)[\delta \cup (f,f)\tau] \subseteq \sigma$

$\Leftrightarrow g : (Y, \delta \cup (f,f)\tau) \to (Z,\sigma)$ fuzmic. (1.04)

i.e. $\delta \cup (f,f)\tau$ has property p .

$\therefore$ $f_*\tau$ as defined in 1.13 satisfies 1.11, and establishes existence. Thus $f_*\tau$ is well defined by 1.11, and equivalently by 1.12 and 1.13.■

1.14. Corollary

$(1_X)_*\tau = \tau$.

If $(X,\tau) \underset{f}{\to} Y \underset{g}{\to} Z$ are set-theoretic maps,

$(g \circ f)_*\tau = g_*(f_*\tau)$.

Proof

Obvious.■

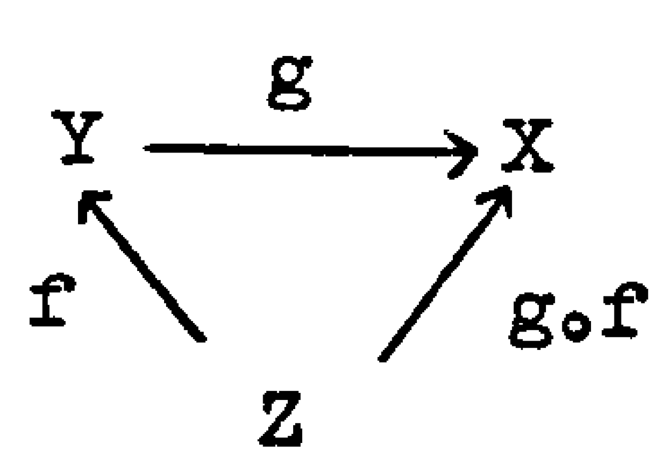
1.15. Definition

If $f : X \to Y$ is epic, and hence the quotient map of some equivalence relation ρ (or in particular the relation $(A \times A) \cup \delta$, for some $A \subseteq X$), the fuzzy space $(Y, f_* \tau)$ is the quotient space of (X, τ) by ρ (in particular, by the subset A) and denoted by $\frac{(X,\tau)}{\rho}$ (in particular, by $\frac{(X,\tau)}{A}$). Any epimorphism $f : (X,\tau) \to (Y,\sigma)$ such that $\sigma = f_* \tau$ is an identification map. Note that for an epimorphism $\delta \subseteq (f,f)\tau$, so that $f_* \tau = (f,f)\tau$ precisely. Similarly we have

1.2. Proposition

If $g : Y \to (X,\tau)$ is a set-theoretic map, the following definitions of the induced fuzzy $g^* \tau$ on Y are equivalent, and it is well defined by each.

1.21. $g^* \tau$ is the unique fuzzy on Y with the universal property that for all set-theoretic maps $f : Z \to Y$ and fuzzies σ on Z,



$g \circ f : (Z,\sigma) \to (X,\tau)$ is fuzmic $\Leftrightarrow f : (Z,\sigma) \to (Y, g^* \tau)$ is fuzmic.

1.22. $g^* \tau$ is the biggest fuzzy on Y s.t. $g : (Y, g^* \tau) \to (X,\tau)$ is fuzmic.

1.23. $g^* \tau = (g,g)^{\leftarrow} \tau$.

<u>Proof</u>

Denote the universal property 1.21 by q .

i) Let 1_Y be the set-theoretic identity on Y , and the fuzzy μ on Y have the property q . Then

$$\begin{aligned} g:(Y,K)\to(X,\tau) \text{ fuzmic } &\Leftrightarrow g\circ 1_Y:(Y,K)\to(X,\tau) \text{ fuzmic} \\ &\Leftrightarrow 1_Y:(Y,K)\to(Y,\mu) \text{ fuzmic, by (q)} \\ &\Leftrightarrow K\subseteq\mu . \end{aligned}$$

i.e. μ is the biggest fuzzy on Y s.t. $g:(Y,\mu)\to(X,\tau)$ is fuzmic. If μ' also has property q , the same argument applies, so that we have $\mu'\subseteq\mu$, $\mu\subseteq\mu'$, and hence $\mu=\mu'$. Thus if there is a fuzzy with property q it is unique, and $g^*\tau$ as defined by 1.21., if it exists, satisfies 1.22.

ii) $(g(y), g(y))\in\tau \quad \forall y\in Y$ (τ reflexive)

i.e. $(g,g)(y,y)\in\tau \quad \forall y\in Y$

i.e. $(y,y)\in(g,g)^{\leftarrow}\tau \quad \forall y\in Y$

$\therefore$ $(g,g)^{\leftarrow}\tau$ is reflexive.

$$\begin{aligned} (y,y')\in(g,g)^{\leftarrow}\tau &\Rightarrow (g(y), g(y'))\in\tau \\ &\Rightarrow (g(y'), g(y))\in\tau \quad (\tau \text{ symmetric}) \\ &\Rightarrow (y',y)\in(g,g)^{\leftarrow}\tau \end{aligned}$$

$\therefore$ $(g,g)^{\leftarrow}\tau$ is symmetric.

Thus $(g,g)^{\leftarrow}\tau$ is a fuzzy.

For any fuzzy K on Y

$$\begin{aligned} g:(Y,K)\to(X,\tau) \text{ fuzmic } &\Leftrightarrow (g,g)K\subseteq\tau \quad (1.04) \\ &\Leftrightarrow K\subseteq(g,g)^{\leftarrow}\tau \end{aligned}$$

i.e. the biggest fuzzy K s.t. $g:(Y,K)\to(X,\tau)$ is fuzmic is $(g,g)^{\leftarrow}\tau$.

$\therefore$ $g^*\tau$ as defined by 1.22 satisfies 1.23.

iii) $g \circ f : (Z,\sigma) \to (X,\tau)$ fuzmic $\Leftrightarrow (g\circ f, g\circ f)\sigma \subseteq \tau$ (1.04)

$\Leftrightarrow (g,g)\circ(f,f)\sigma \subseteq \tau$

$\Leftrightarrow (f,f)\sigma \subseteq (g,g)^{\leftarrow}\tau$

$\Leftrightarrow f : (Z,\sigma) \to (Y,(g,g)^{\leftarrow}\tau)$ fuzmic. (1.04)

i.e. $(g,g)^{\leftarrow}\tau$ has property q

$\therefore$ $g^{*}\tau$ as defined in 1.23. satisfies 1.21., and establishes existence. Thus $g^{*}\tau$ is well defined by 1.21., and equivalently by 1.22. and 1.23.■

1.24. Corollary

$(1_X)^{*}\tau = \tau$.

If $Z \underset{f}{\to} Y \underset{g}{\to} (X,\tau)$ are set-theoretic maps,

$(g\circ f)^{*}\tau = f^{*}(g^{*}\tau)$.

Proof

Obvious.■

1.25. Definition

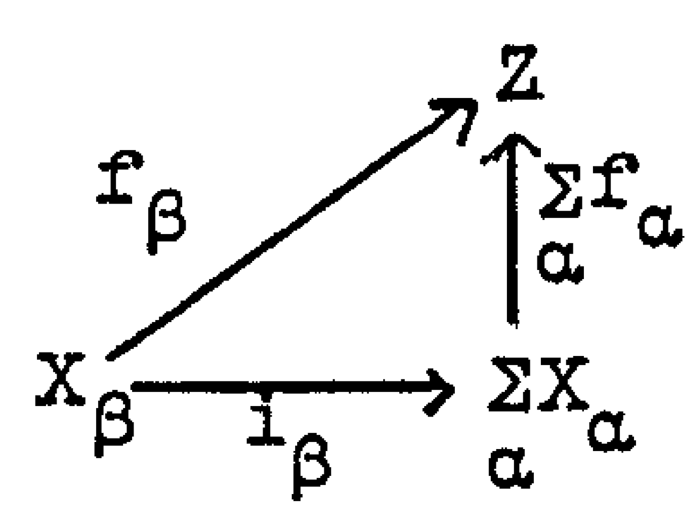
Note that if $g : Y \to X$ is injective, $g : (Y,g^{*}\tau) \to (X,\tau)$ is an embedding. If g is an inclusion, $g^{*}\tau$ is the subspace fuzzy on Y , and $(Y,g^{*}\tau)$ is a subspace of (X,τ) . We will often in this case denote $g^{*}\tau$ also by τ . The fuzzy neighbourhood $N(x)$ of $x \in (X,\tau)$ will always have the subspace fuzzy unless the contrary is explicitly stated.

1.3. Proposition

If $\{(X_\alpha,\tau_\alpha)\}$ is an arbitrary family of fuzzy spaces, the following definitions of their coproduct or disjoint union $\sum_\alpha (X_\alpha,\tau_\alpha)$ are equivalent, and it is well defined by each.

$\sum_\alpha (X_\alpha,\tau_\alpha)$ is the set-theoretic coproduct $\sum_\alpha X_\alpha$ of $\{X_\alpha\}$ with the coproduct fuzzy $\sum_\alpha \tau_\alpha$ defined as follows:

1.31. $\sum_\alpha \tau_\alpha$ is the unique fuzzy with the universal property that given any family $\{f_\alpha : X_\alpha \to Z\}$ of set-theoretic maps from the X_α , and any fuzzy σ on Z , then where $\sum_\alpha f_\alpha : \sum_\alpha X_\alpha \to Z$ is the unique set-theoretic map induced by the set coproduct structure, we have



$\sum_\alpha f_\alpha : (\sum_\alpha X_\alpha, \sum_\alpha \tau_\alpha) \to (Z,\sigma)$ is fuzmic $\Leftrightarrow$ each $f_\beta : (X_\beta,\tau_\beta) \to (Z,\sigma)$ is fuzmic.

1.32. $\sum_\alpha \tau_\alpha$ is the least fuzzy such that all the injections $i_\beta : (X_\beta,\tau_\beta) \to (\sum_\alpha X_\alpha, \sum_\alpha \tau_\alpha)$ are fuzmic.

1.33. $\sum_\alpha \tau_\alpha = \bigcup_\alpha (i_\alpha, i_\alpha)\tau_\alpha$.

Proof

Denote the universal property 1.31. by p .

i) Let the fuzzy K on $\sum_\alpha X_\alpha$ have the property p .

Then for any fuzzy μ on $\sum_\alpha X_\alpha$,

Each $i_\beta : (X_\beta, \tau_\beta) \to (\sum_\alpha X_\alpha, \mu)$ is fuzmic $\Leftrightarrow \sum_\alpha i_\alpha : (\sum_\alpha X_\alpha, K) \to (\sum_\alpha X_\alpha, \mu)$ is fuzmic, by p .

$\Leftrightarrow 1_{\sum_\alpha X_\alpha} : (\sum_\alpha X_\alpha, K) \to (\sum_\alpha X_\alpha, \mu)$ is fuzmic (same map)

$\Leftrightarrow K \subseteq \mu$.

i.e. K is the least fuzzy on $\sum_\alpha X_\alpha$ s.t. each $i_\beta : (X_\beta, \tau_\beta) \to (\sum_\alpha X_\alpha, K)$ is fuzmic.

If K' also has property p , the same argument applies, so that we have $K' \subseteq K$, $K \subseteq K'$, and hence $K' = K$. Thus if there is a fuzzy with property p it is unique, and $\sum_\alpha \tau_\alpha$ as defined by 1.31., if it exists, satisfies 1.32.

ii) $x \in \sum_\alpha X_\alpha \Rightarrow x = i_\beta(x')$ for some β and some $x' \in X_\beta$

$\Rightarrow (x,x) = (i_\beta(x'), i_\beta(x')) = (i_\beta, i_\beta)(x',x')$

$\Rightarrow (x,x) \in (i_\beta, i_\beta)\tau_\beta$ (τ_β reflexive)

$\Rightarrow (x,x) \in \bigcup_\alpha (i_\alpha, i_\alpha)\tau_\alpha$.

i.e. $\bigcup_\alpha (i_\alpha, i_\alpha)\tau_\alpha$ is reflexive.

$(x,y) \in \bigcup_\alpha (i_\alpha, i_\alpha)\tau_\alpha \Rightarrow (x,y) \in (i_\beta, i_\beta)\tau_\beta$ for some β

$\Rightarrow (x,y) = (i_\beta, i_\beta)(x',y')$ for some $(x',y') \in \tau_\beta$

$\Rightarrow (y,x) = (i_\beta, i_\beta)(y',x')$

$\Rightarrow (y,x) \in (i_\beta, i_\beta)\tau_\beta$ (τ_β symmetric)

$\Rightarrow (y,x) \in \bigcup_\alpha (i_\alpha, i_\alpha)\tau_\alpha$.

i.e. $\bigcup_\alpha (i_\alpha, i_\alpha)\tau_\alpha$ is symmetric.

Thus $\bigcup_{\alpha}(i_{\alpha},i_{\alpha})\tau_{\alpha}$ is a fuzzy.

For any fuzzy K

$i_{\beta} : (X_{\beta},\tau_{\beta}) \rightarrow (\sum_{\alpha} X_{\alpha},K)$ is fuzmic for each β

$\Leftrightarrow (i_{\beta},i_{\beta})\tau_{\beta} \subseteq K$ for each β

$\Leftrightarrow \bigcup_{\alpha}(i_{\alpha},i_{\alpha})\tau_{\alpha} \subseteq K$.

i.e. the least fuzzy K s.t. each

$i_{\beta} : (X_{\beta},\tau_{\beta}) \rightarrow (\sum_{\alpha} X_{\alpha},K)$ is fuzmic is $\bigcup_{\alpha}(i_{\alpha},i_{\alpha})\tau_{\alpha}$.

$\therefore$ $\sum_{\alpha}\tau_{\alpha}$ as defined by 1.32. satisfies 1.33.

iii) $\sum_{\beta} f_{\beta} : (\sum_{\alpha} X_{\alpha},\bigcup_{\alpha}(i_{\alpha},i_{\alpha})\tau_{\alpha}) \rightarrow (Z,\sigma)$ is fuzmic

$\Leftrightarrow (\sum_{\beta} f_{\beta},\sum_{\beta} f_{\beta})(\bigcup_{\alpha}(i_{\alpha},i_{\alpha})\tau_{\alpha}) \subseteq \sigma$ (1.04)

$\Leftrightarrow \bigcup_{\alpha}[(\sum_{\beta} f_{\beta},\sum_{\beta} f_{\beta})((i_{\alpha},i_{\alpha})\tau_{\alpha})] \subseteq \sigma$

$\Leftrightarrow \bigcup_{\alpha}(f_{\alpha},f_{\alpha})\tau_{\alpha} \subseteq \sigma$

$\Leftrightarrow (f_{\alpha},f_{\alpha})\tau_{\alpha} \subseteq \sigma$ for each α

$\Leftrightarrow$ each f_{α} is fuzmic.

i.e. $\bigcup_{\alpha}(i_{\alpha},i_{\alpha})\tau_{\alpha}$ has property p .

$\therefore$ $\sum_{\alpha}\tau_{\alpha}$ as defined by 1.33. satisfies 1.31., and establishes existence. Thus $\sum_{\alpha}\tau_{\alpha}$ is well defined by 1.31., and equivalently by 1.32. and 1.33.■

1.34. Corollary

Each injection $i_{\alpha}:(X_{\alpha},\tau_{\alpha}) \rightarrow \sum_{\alpha}(X_{\alpha},\tau_{\alpha})$ is an embedding.■

1.35. Notation

If the indexing set $\{\alpha\}$ is finite we may write

$$\underset{\alpha}{\Sigma}(X_\alpha,\tau_\alpha) = (X_{\alpha_1},\tau_\alpha) \underline{\cup} \dots \underline{\cup} (X_{\alpha_n},\tau_{\alpha_n})$$

$$= (X_{\alpha_1} \underline{\cup} \dots \underline{\cup} X_{\alpha_n} , \tau_{\alpha_1} \underline{\cup} \dots \underline{\cup} \tau_{\alpha_n}) \text{ , etc.}$$

1.36. Remark

Even if $i_1 : Y_1 \to (X,\tau)$, $i_2 : Y_2 \to (X,\tau)$ are the inclusions of disjoint subsets Y_1 , Y_2 of X , it does not follow that

$$i_1 \underline{\cup} i_2 : (Y_1 \underline{\cup} Y_2 , (i_1^*\tau) \underline{\cup} (i_2^*\tau)) \to (X,\tau)$$

is an embedding, unless $N(Y_1) \cap Y_2 = \emptyset$.

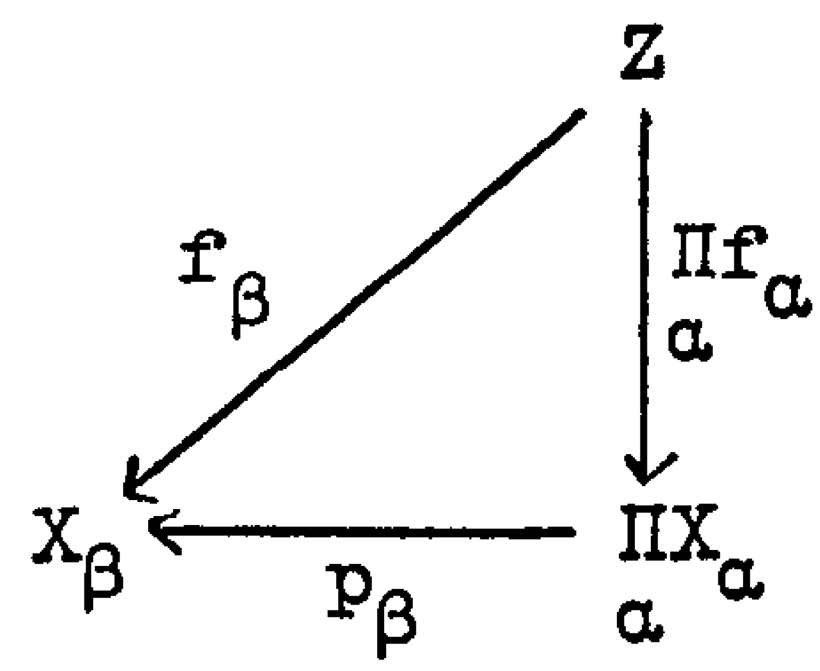
Similarly to Proposition 1.3., modulo the well-definition of products in $\mathcal{S}et$ (i.e. modulo the Axiom of Choice), we have

1.4. Proposition

If $\{(X_\alpha,\tau_\alpha)\}$ is an arbitrary family of fuzzy spaces, the following definitions of their product $\underset{\alpha}{\Pi}(X_\alpha,\tau_\alpha)$ are equivalent and it is well defined by each.

$\underset{\alpha}{\Pi}(X_\alpha,\tau_\alpha)$ is the set $\underset{\alpha}{\Pi}X_\alpha$ with the product fuzzy $\underset{\alpha}{\Pi}\tau_\alpha$ defined as follows:-

1.41. $\underset{\alpha}{\Pi}\tau_\alpha$ is the unique fuzzy with the universal property that given any family $\{f_\alpha : Z \to X_\alpha\}$ of set-theoretic maps from a set Z to the X_α , and any fuzzy σ on Z , if $\underset{\alpha}{\Pi}f_\alpha : Z \to \underset{\alpha}{\Pi}X_\alpha$ is the unique map induced by the f_α in the $\mathcal{S}et$ product structure, then



$\prod_\alpha f_\alpha : (Z,\sigma) \to (\prod_\alpha X_\alpha, \prod_\alpha \tau_\alpha)$ is fuzmic $\Leftrightarrow$ each $f_\beta : (Z,\sigma) \to (X_\beta,\tau_\beta)$ is fuzmic.

<u>1.42</u>. $\prod_\alpha \tau_\alpha$ is the biggest fuzzy s.t. all the projections $p_\beta : (\prod_\alpha X_\alpha, \prod_\alpha \tau_\alpha) \to (X_\beta,\tau_\beta)$ are fuzmic.

<u>1.43</u>. $\prod_\alpha \tau_\alpha = \bigcap_\alpha ((p_\alpha,p_\alpha)^{\leftarrow}\tau_\alpha)$.

<u>Proof</u>

Precisely dual to the proof of 1.3., this proof is left to the interested reader.■

<u>1.44. Corollary</u>

Each projection $p_\alpha : \prod_\alpha (X_\alpha,\tau_\alpha) \to (X_\alpha,\tau_\alpha)$ is an identification map.■

<u>1.45. Notation</u>

(i) With finite indexing sets we may write

$$\prod_\alpha (X_\alpha,\tau_\alpha) = (X_{\alpha_1},\tau_{\alpha_1}) \mathrm{X} \dots \mathrm{X} (X_{\alpha_n},\tau_{\alpha_n})$$
$$= (X_{\alpha_1} \times \dots \times X_{\alpha_n}, \tau_{\alpha_1}.\tau_{\alpha_2}. \dots .\tau_{\alpha_n}) \text{ etc.}$$

The f_α are as usual the <u>factors</u> of $\prod_\alpha f_\alpha$.

(ii) By a standard abuse of notation, given a family $\{f_\alpha : X_\alpha \to Y_\alpha\}$ of fuzmaps, if p_β is the projection $\prod_\alpha X_\alpha \to X_\beta$ we denote by $\prod_\alpha f_\alpha$ the map which is, strictly, $\prod_\alpha (f_\alpha \circ p_\alpha)$.

1.5. Proposition

The product of an arbitrary family $\{f_\alpha : (X_\alpha,\tau_\alpha) \to (Y_\alpha,\sigma_\alpha)\}$ of identification maps is an identification map.

Proof

$(y,y') \in \prod_\alpha \sigma_\alpha \Rightarrow (p_\alpha^Y(y),p_\alpha^Y(y')) \in \sigma_\alpha$, each α .

$[p_\alpha^Y : \prod_\alpha (Y_\alpha,\tau_\alpha) \to (Y_\alpha,\tau_\alpha)$ fuzmic$]$

$\Rightarrow (p_\alpha^Y(y),p_\alpha^Y(y') \in (f_\alpha)_*\tau_\alpha$, each α .

$[f_\alpha$ an identification map$]$

$\Rightarrow (p_\alpha^Y(y),p_\alpha^Y(y')) = (f_\alpha x_\alpha, f_\alpha x'_\alpha)$ for $(x_\alpha,x'_\alpha) \in \tau_\alpha$, each α .

$\Rightarrow (p_\alpha^Y(y),p_\alpha^Y(y')) = (f_\alpha p_\alpha(x), f\ p\ )x'))$, each α , (where x,x' are uniquely determined by $\{p_\alpha(x) = x_\alpha\}$ and $\{p_\alpha(x') = x'_\alpha\}$) , and $(x,x') \in \prod_\beta \tau_\beta$.

$\Rightarrow (p_\alpha^Y(y),p_\alpha^Y(y')) = (p_\alpha^Y \prod_\beta f_\beta(x), p_\alpha^Y \prod_\beta f_\beta(x'))$, each α .

$\Rightarrow (y,y') = (\prod_\beta f_\beta, \prod_\beta f_\beta)(x,x')$ (by the set-theoretic uniqueness property of the product), where $(x,x') \in \prod_\beta \tau_\beta$.

i.e. $\prod_\alpha \sigma_\alpha \subseteq (\prod_\beta f_\beta, \prod_\beta f_\beta)(\prod_\beta \tau_\beta)$.

But $\Pi_{\alpha}\sigma_{\alpha} \supseteq (\Pi_{\beta} f_{\beta}, \Pi_{\beta} f_{\beta})(\Pi_{\beta}\tau_{\beta})$ ($\Pi_{\beta} f_{\beta}$ fuzmic)

$\therefore \Pi_{\alpha}\sigma_{\alpha} = (\Pi_{\beta} f_{\beta}, \Pi_{\beta} f_{\beta})(\Pi_{\beta}\tau_{\beta})$.

i.e. $\Pi_{\beta} f_{\beta}$ is an identification map.■

2. Fuzzy Adjunction Spaces

The following material is included only for completeness, since it is far less significant than its topological analogue. The greatest use of adjunction spaces in topology occurs in the complexes of cell complexes. Fuzzy cell complexes may be constructed but, for reasons that will appear later, do not have the usefulness and hence importance possessed by topological cell complexes. I therefore state the salient facts without proof.

2.01. Definition

Given fuzzy spaces X, Y, Z and fuzmaps $f : X \to Y$, $g : X \to Z$, the adjunction space $Y_f \sqcup_g Z$ is defined by the pushout

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{g}} \\ Z & \xrightarrow{\;\bar{f}\;} & Y_f \sqcup_g Z \end{array} .$$

The material above concerning coproducts and identification spaces is sufficient to ensure that $\mathcal{F}uz$ has pushouts.

2.1. Properties of Adjunction Spaces, etc.

2.11. If $\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{g}} \\ Y & \xrightarrow[\bar{f}]{} & P \end{array}$ is a pushout in $\mathcal{F}$uz, and

Q is an arbitrary fuzzy space, then

$$\begin{array}{ccc} X \times Q & \xrightarrow{(f,1_Q)} & Z \times Q \\ {\scriptstyle (g,1_Q)}\downarrow & & \downarrow{\scriptstyle (\bar{g},1_Q)} \\ Y \times Q & \xrightarrow[(\bar{f},1_Q)]{} & P \times Q \end{array}$$ is also a pushout.

2.12. If g is the inclusion of a subspace, we write $Y_f \sqcup Z$ for $Y_f \sqcup_g Z$. Then:

2.13. If $f : X \to Z$ is an identification map, so is the map $\bar{f} : Y \to Y_f \sqcup Z$.

2.14. If Z', Y' are subspaces of Z, Y respectively s.t. $X \subseteq Z'$, $f(X) \subseteq Y'$, then if

$f' = (f$ corestricted to $Y')$

the space $Y'_{f'} \sqcup Z'$ is canonically a subspace of $Y_f \sqcup Z$.

Connectedness is not defined until §5, but I include

2.15. If X is non-empty and Z,Y are connected, $Y_f \sqcup_g Z$ is connected.

2.16. If $Y_f \sqcup_g Z$ and X are connected, Z is connected.

To go on to the definitions necessary to set up fuzzy cell complexes would be a fruitless multiplication of technicalities.

3. Function Spaces

3.01. Definitions

Given $f,g : (X,\tau) \to (Y,\sigma)$ fuzmorphisms, f and g are indistinguishable if $(f,g)\tau \subseteq \sigma$. Clearly indistinguishability is a fuzzy on the set Y^X of fuzmaps $X \to Y$. (Note that f is fuzmic if and only if it is indistinguishable from itself). This fuzzy is denoted by σ^τ, and the fuzzy space (Y^X,σ^τ), or $(Y,\sigma)^{(X,\tau)}$, is the function space of fuzmaps $X \to Y$.

The map $\varepsilon : Y^X \times X \to Y$ is the evaluation map.

$$(f,x) \mapsto f(x) .$$

3.1. Proposition

The evaluation map is fuzmic and an identification map.

Proof

(i) $((f,x),(g,x')) \in \sigma^\tau.\tau \Rightarrow (f,g) \in \sigma^\tau \;\&\; (x,x') \in \tau$ (1.43)

$\Rightarrow (f,g)(x,x') \in \sigma$ (3.01)

$\Rightarrow (f(x),g(x')) \in \sigma$

$\Rightarrow (\varepsilon(f,x),\varepsilon(g,x')) \in \sigma$

$\Rightarrow (\varepsilon,\varepsilon)((f,x),(g,x')) \in \sigma$.

$\therefore$ $(\varepsilon,\varepsilon)(\sigma^\tau.\tau) \subseteq \sigma$, i.e. ε is fuzmic.

(ii) For any $y \in Y$, let f_y be the map $X \to Y$

$x \mapsto y$

Then if $(y,y') \in \sigma$, we have

$$(f_y,f_{y'})\tau \subseteq (f_y,f_{y'})X = \{y,y'\} \subseteq \sigma ,$$

so that $(f_y,f_{y'}) \in \sigma^\tau$.

If we choose $x \in X$, we have $(x,x) \in \tau$ (reflexivity)

and hence $((f_y,x),(f_{y'},x)) \in \sigma^\tau.\tau$, (1.43)

thus by (i) $(\varepsilon,\varepsilon)((f_y,x),(f_{y'},x)) = (f_y(x),f_{y'}(x))$

$= (y,y')$.

Thus for all $(y,y') \in \sigma$, $(y,y') \in (\varepsilon,\varepsilon)(\sigma^\tau.\tau)$,

i.e. $\sigma \subseteq (\varepsilon,\varepsilon)(\sigma^\tau.\tau)$

$\therefore$ by (i), $\sigma = (\varepsilon,\varepsilon)(\sigma^\tau.\tau) = \varepsilon_*(\sigma^\tau.\tau)$ (ε surjective)

and hence ε is an identification map. (1.15.) ∎

3.2. Proposition (Exponential Law)

Given fuzzy spaces $(Y,\sigma),(X,\tau)$ and (T,ρ) there is a natural fuzziomorphism

$$\varphi : ((Y^X)^T,(\sigma^\tau)^\rho) \cong (Y^{X\times T},\sigma^{\tau.\rho}) .$$

Proof

i) Define $\varphi : (Y^X)^T \to Y^{X\times T}$

$f \rightsquigarrow [(x,t) \rightsquigarrow f(t)(x)]$

and $\Psi : Y^{X\times T} \to (Y^X)^T$

$g \rightsquigarrow [t \rightsquigarrow (x \rightsquigarrow g(x,t))]$.

Now $(f,f') \in (\sigma^\tau)^\rho \Leftrightarrow (f,f')\rho \subseteq \sigma^\tau$

$\Leftrightarrow (f,f')(t,t') \in \sigma^\tau$, $\forall (t,t') \in \rho$

$\Leftrightarrow (f,f')(t,t))\tau \subseteq \sigma$, $\forall (t,t') \in \sigma$

$\Leftrightarrow (f,f')(t,t')(x,x') \in \sigma$,

$\forall (t,t') \in \rho \,\&\, \forall (x,x') \in \tau$

$\Leftrightarrow (\varphi f,\varphi f')((x,t),(x',t')) \in \sigma$,

$\forall ((x,t),(x',t')) \in \tau.\rho$

$\Leftrightarrow (\varphi f,\varphi f')(\tau.\rho) \subseteq \sigma$

$\Leftrightarrow (\varphi f,\varphi f') \in \sigma^{\tau.\rho}$

Hence if $f' = f$, $(\varphi f,\varphi f)\tau.\rho \subseteq \sigma$, i.e.

$$\varphi f : (X \times T,\tau.\rho) \to (Y,\sigma) \quad \text{is fuzmic.}$$

Thus φ is well defined. Evidently Ψ is a two-sided inverse set-theoretically for φ , so that φ is a bijection. Since then in particular φ is surjective, the equivalence

$$(f,f') \in (\sigma^{\tau})^{\rho} \Leftrightarrow (\varphi f,\varphi f') \in \sigma^{\tau.\rho}$$

proved above shows that

$$(\varphi,\varphi)((\sigma^{\tau})^{\rho}) = \sigma^{\tau.\rho} .$$

Hence also $(\sigma^{\tau})^{\rho} = (\Psi,\Psi)\sigma^{\tau.\rho}$ so that Ψ is fuzmic. Thus we have a fuzmic two-sided inverse for φ , which is therefore a fuzziomorphism.

ii) For any ordered triple (Y,X,T) of fuzzy spaces let

$$\varphi(Y,X,T) : (Y^X)^T \to Y^{X\times T}$$
$$f \mapsto [(x,t) \mapsto f(t)(x)]$$

as above. Then for any triple

$$(p : Y \to Y' , q : X' \to X , r : T' \to T)$$

of fuzmaps the square

$$\begin{array}{ccccc}
f & (Y^X)^T & \xrightarrow[\cong]{\varphi(Y,X,T)} & Y^{X\times T} & g \\
\Big\downarrow & \Big\downarrow{\scriptstyle (p^q)^r} & & \Big\downarrow{\scriptstyle p^{q\times r}} & \Big\downarrow \\
(t' \mapsto p\circ f(r(t'))\circ q) & (Y'^{X'})^{T'} & \xrightarrow[\varphi(Y',X',T')]{\cong} & Y'^{X'\times T'} & p\circ g\circ(q\times r)
\end{array}$$

commutes, so that φ is a natural transformation between the functors (covariant in the first variable, contravariant in the second and third);

$$(Y,X,T) \to (Y^X)^T \text{ , correspondingly on maps}$$

and $$(Y,X,T) \to Y^{X\times T} \text{ , correspondingly on maps}$$

$$\mathcal{F}uz \times \mathcal{F}uz \times \mathcal{F}uz \to \mathcal{F}uz \ . \ \blacksquare$$

3.21. Corollary

Given fuzzy spaces $(Y,\sigma),(X_i,\tau_i)$, $i = 1,\dots,n$, there is a natural fuzziomorphism

$$\Big((\dots((Y^{X_1})^{X_2})\dots)^{X_n},(\dots((\sigma^{\tau_1})^{\tau_2})\dots)^{\tau_n}\Big)$$
$$\cong (Y^{X_1\times X_2\times\dots X_n},\sigma^{\tau_1\cdot\tau_2\cdot\ \dots\ \tau_n}) \ . \ \blacksquare$$

4. Pointed Fuzzy Spaces

4.00. Definition

A pointed fuzzy space (X,x_0,τ) is an inclusion map $\{x_0\} \to (X,\tau)$. x_0 is the base point of (X,x_0) .
A pointed fuzmap $(X,x_0,\tau) \to (Y,y_0,\sigma)$ is a commutative square

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{f} & (Y,\sigma) \\ \cup & & \cup \\ \{x_0\} & \longrightarrow & \{y_0\} \end{array}$$

of fuzmaps, which we will commonly identify with the map f. Clearly any fuzmap $g : (X,\tau) \to (Y,\sigma)$ such that $g(x_0) = y_0$ defines a pointed fuzmap $(X,x_0,\tau) \to (Y,y_0,\sigma)$, which we also denote by g. We shall when convenient denote the base point of any pointed fuzzy space by $*$.

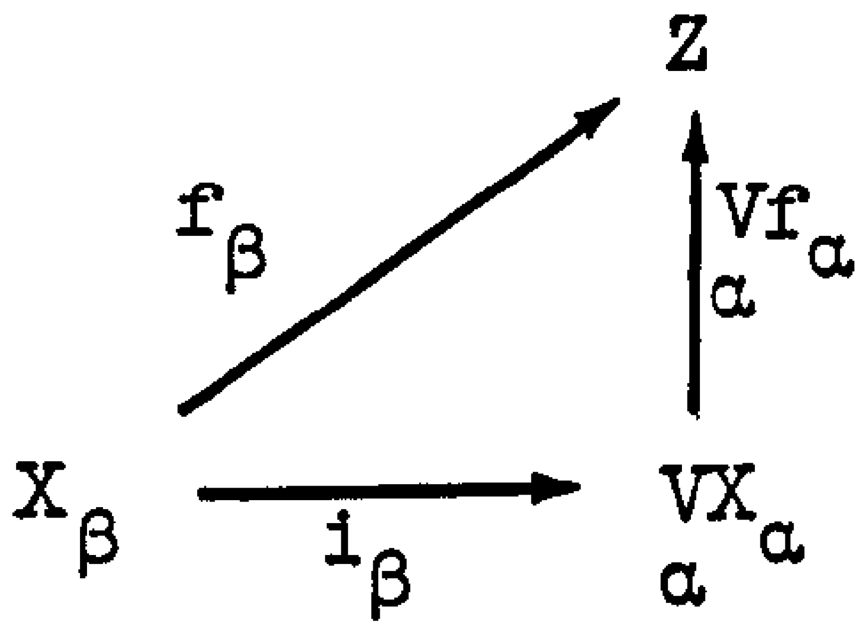
Evidently, pointed fuzzy spaces and pointed fuzmaps form a category, which we denote by $\mathcal{F}uz_*$. We have the obvious definitions of pointed embeddings and pointed fuzziomorphisms, and sets of definitions with equivalence proofs exactly as for the non-pointed case for induced and coinduced fuzzies in this context, together with the same corollaries, etc. We have proofs barely different from the non-pointed case ~~for~~ (3.4.-3.7) for:—

4.1. Proposition

If $\{(X_\alpha,x_\alpha,\tau_\alpha)\}$ is an arbitrary family of pointed fuzzy spaces, the following definitions of their coproduct or wedge product $\bigvee_\alpha(X_\alpha,x_\alpha,\tau_\alpha)$ are equivalent, and it is well defined by each.

$\bigvee(X_\alpha,x_\alpha,\tau_\alpha)$ is the set $\bigvee_\alpha X_\alpha = \dfrac{\sum_\alpha X_\alpha}{\{i_\alpha x_\alpha\}}$ with base point the set $\{i_\alpha,x_\alpha\}$, denoted by $*$, and the coproduct fuzzy $\bigvee_\alpha \tau_\alpha$ defined as follows:

4.11. $\underset{\alpha}{V}\tau_\alpha$ is the unique fuzzy with the universal property that given any family $\{f_\alpha : (X_\alpha, x_\alpha) \to (Z, z_0)\}$ of pointed set-theoretic maps from the X_α , and any fuzzy σ on Z , then where $\underset{\alpha}{V}f_\alpha : \underset{\alpha}{V}X_\alpha \to Z$ is the unique set-theoretic map induced by the f_α and the $\mathcal{S}et_*$ coproduct structure, we have



$\underset{\alpha}{V}f_\alpha : (\underset{\alpha}{V}X_\alpha, *, \underset{\alpha}{V}\tau_\alpha) \to (Z, z_0, \sigma)$ is fuzmic $\Leftrightarrow$ each $f_\beta : (X_\beta, x_\beta, \tau_\beta) \to (Z, z_0, \sigma)$ is fuzmic.

4.12. $\underset{\alpha}{V}\tau_\alpha$ is the least fuzzy such that all the injections $i_\beta : (X_\beta, x_\beta, \tau_\beta) \to (\underset{\alpha}{V}X_\alpha, *, \underset{\alpha}{V}\tau_\alpha)$ are fuzmic.

4.13. $\underset{\alpha}{V}\tau_\alpha = \underset{\alpha}{\cup}(i_\alpha, i_\alpha)\tau_\alpha$. ∎

4.14. Corollary

Each injection i_α is an embedding.∎

4.15. Corollary

$\underset{\alpha}{V}(X_\alpha, x_\alpha, \tau_\alpha) \cong \left(\dfrac{\underset{\alpha}{\Sigma}(X_\alpha, \tau_\alpha)}{\{i_\alpha x_\alpha\}}, \{i_\alpha x_\alpha\}\right)$ as pointed fuzzy spaces. ∎

4.2. Proposition

Exactly as for Proposition 1.4., with the base point * of $\prod_\alpha (x_\alpha, x_\alpha)$ defined as the image of the map

$\prod_\alpha p_\alpha$, where $p_\alpha : \{0\} \to X$.

$0 \rightsquigarrow x_\alpha$ ■

4.21. Corollary

Each projection $p_\beta : \prod_\alpha (X_\alpha, x_\alpha, \tau_\alpha) \to (X_\beta, x_\beta, \tau_\beta)$ is a pointed identification map.■

4.3. Proposition

The product of an arbitrary family of pointed identification maps is a pointed identification map.■

4.4. Pointed Adjunction Spaces

Adjunction spaces are defined for $\mathcal{J}uz_*$ as for $\mathcal{J}uz$, and have properties 2.02.-2.05. If $i : \{x_0\} \to (X,\tau)$ and $j : \{y_0\} \to (Y,\sigma)$ are pointed spaces,

$(X, x_0, \tau) \vee (Y, y_0, \sigma) \cong ((X,\tau)_i \sqcup_j (Y,\sigma), \{x_0\}_i \ {}_j\{y_0\})$.

4.50. Definition

The smash product $\bigwedge_\alpha (X_\alpha, *_\alpha, \tau_\alpha)$ of a family $\{(X_\alpha, *_\alpha, \tau_\alpha)\}$ is the quotient space

$$\frac{\prod_\alpha (X_\alpha, *_\alpha, \tau_\alpha)}{\underset{\alpha}{W} X_\alpha}$$

where $\underset{\alpha}{W} X_\alpha = \{x \in \prod_\alpha X_\alpha \mid p_\beta(x) = *_\beta \text{ for some } \beta \}$.

If the indexing set is finite, we may denote

$\underset{\alpha}{\boxtimes}(X_\alpha, x_\alpha, \tau_\alpha)$ by $(X_{\alpha_1}, x_{\alpha_1}, \tau_{\alpha_1}) \wedge \dots \wedge (X_{\alpha_n}, x_{\alpha_n}, \alpha_n)$

or $(X_{\alpha_1} \wedge \dots X_{\alpha_n}, *, \tau_{\alpha_1} . \tau_{\alpha_2} . \dots . \tau_{\alpha_n})$, and if p is the identification map $\prod_{i=1}^{n} (X_i, *_i, \tau_i) \to \underset{i=1}{\overset{n}{\boxtimes}} (X_i, *_i, \tau_i)$

we denote $p(x_1, \dots, x_n)$ by $[x_1, \dots, x_n]$.

As in $\mathcal{T}op$, the smash product is not a categorical product. The set-theoretic injection $\underset{\alpha}{V} X_\alpha \to \underset{\alpha}{\Pi} X_\alpha$ is not an embedding, even where $\alpha \in \{1,2\}$ only, in which case $\underset{\alpha}{W}(X_\alpha, x_\alpha) \cong \underset{\alpha}{V}(X_\alpha, x_\alpha)$ as sets: if $(x_0, x) \in \tau \setminus \delta$ and $(y_0, y) \in \sigma \setminus \delta$, then $(x,y) \notin \tau \vee \sigma$ in $(X,\tau) \vee (Y,\sigma)$, though $(i(x), i(y)) \in \tau . \sigma$ in $(X,\tau) \times (Y,\sigma)$.

4.51. Definition

The pointed function space $(Y, y_0, \sigma)^{(X, x_0, \tau)}$ is the set $(Y^X)_*$ of pointed fuzmaps $(X, x_0, \tau) \to (Y, y_0, \sigma)$, with base point the map $\underset{x \mapsto y_0, \forall x}{* : X \to Y}$, and the subspace fuzzy from the inclusion $(Y^X)_* \to (Y^X, \sigma^\tau)$.

The evaluation map ε restricted to $((Y^X, *) \times (X, x_0), \sigma^\tau . \tau)$ factors set-theoretically through the identification map

$$(Y^X, *) \times (X, x_0) \to (Y^X, *) \wedge (X, x_0)$$

so that by the universal property of the quotient fuzzy we have a fuzmic evaluation map

$$\varepsilon_*: (Y^X, *, \sigma^\tau) \wedge (X, x_0, \tau) \to (Y, y_0, \sigma) .$$

The map ε_* may not be an identification map; for instance if $(X,\tau) = (\{0,1\}, \iota)$, then $(\varepsilon_*, \varepsilon_*)(\sigma^\tau . \tau) = \{(y_0, y)/(y_0, y) \in \sigma\} \neq \sigma$ in general.

4.6. Proposition (Pointed Exponential Law)

Given pointed fuzzy spaces $(Y,y_0,\sigma),(X,x_0,\tau)$ and (T,t_0,ρ) there is a natural pointed fuzziomorphism.

$$\bar{\varphi}: ((Y^X)^T,*,(\sigma^\tau)^\rho) \cong (Y^{X\wedge T},*,\sigma^{\tau\cdot\rho}) .$$

Proof

i) Let φ,ψ be the (non-pointed) fuzziomorphisms defined in the proof of Proposition 3.2. Then for any $f \in ((Y^X)^T,*,(\sigma^\tau)^\rho)$, $\varphi f : X\times T \to Y$ is a fuzmap such that

$$\varphi f(x_0,\tau) = f(t)(x_0)$$

$= y_0$, since $f(t) \in (Y^X,*)$ and so is pointed,

$$\varphi f(x,t_0) = f(t_0)(x)$$

$= *(x)$, since $f \in ((Y^X)^T,*)$ and so is pointed

$= y_0$.

Hence $\varphi f(i(X\vee T)) = \{y_0\}$.

Therefore by the universal property of the identification map $p: X\times T \to X\wedge T$, there exists a unique fuzmap

$$\bar{\varphi}f : (X\wedge T,*,\tau.\rho) \to (Y,y_0,\sigma)$$

such that $(\bar{\varphi}f)\circ p = \varphi f$.

Evidently $(\varphi f,\varphi f') \in \sigma^{\tau\cdot\rho}$ in $Y^{X\times T} \Rightarrow (\bar{\varphi}f,\varphi f') \in \sigma^{\tau\cdot\rho}$ in $(Y^{X\wedge T},*)$.

Hence $\bar{\varphi}$ is a fuzmap

$$((Y^X)^T,*,(\sigma^\tau)^\rho) \to (Y^{X\wedge T},*,\sigma^{\tau\cdot\rho})$$

and is clearly pointed.

ii) Given $g \in Y^{X\wedge T}$, define $\overline{\psi}g = \psi(g\circ p)$

Then

$(g,g') \in \sigma^{\tau\cdot\rho}$ in $Y^{X\wedge T} \Rightarrow (g\circ p, g'\circ p) \in \sigma^{\tau\cdot\rho}$ in $Y^{X\times T}$. (p fuzmic)

$\Rightarrow (\overline{\psi}g, \overline{\psi}g') \in \sigma^{\tau\cdot\rho}$ in $Y^{X\wedge T}$. (ψ fuzmic)

Evidently g pointed $\Rightarrow \overline{\psi}g$ pointed, hence $\overline{\psi}$ is a fuzmap

$$(Y^{X\wedge T}, *, \sigma^{\tau\cdot\rho}) \to ((Y^X)^T, *, (\sigma^\tau)^\rho)$$

and is clearly pointed.

iii) It is immediate that $\overline{\psi}\circ\overline{\varphi} = 1$, $\overline{\varphi}\circ\overline{\psi} = 1$, so $\overline{\varphi}$ is the required fuzziomorphism.

iv) The inclusions $(Y^X_*)^T_* \to (Y^X)^T$ are natural, so that the naturality of $\overline{\varphi}$ follows from that of φ by composition. ■

4.61. Corollary

Given pointed fuzzy spaces $(Y, y_0, \sigma), (X_i, *_i, \tau_i)$, $i \in \{1, \ldots, n\}$, there is a natural pointed fuzziomorphism

$$\overline{\varphi}: ((\ldots((Y^{X_1})^{X_2})\ldots)^{X_n}, *, (\ldots((\sigma^{\tau_1})^{\tau_2})\ldots)^{\tau_n}$$
$$\cong (Y^{X_1\wedge\cdots\wedge X_n}, *, \sigma^{\tau_1\cdots\tau_n})$$

■

5. Paths and Connectedness

5.00. Definition

The standard fuzzy $\approx$ on the set $\mathcal{N}$ of non-negative integers is the relation $\{(m,n) : |m-n| \leqslant 1\}$. Any subset of $\mathcal{N}$ will have the subspace fuzzy induced from $\approx$ unless otherwise stated. This will also be denoted by $\approx$, without reference to the inclusion into $\mathcal{N}$.

The fuzzy space $(\{m, m+1, \ldots, n\}, \approx)$ will be denoted by $[m;n]$.

If the context requires $\mathcal{N}$ or $[0,m]$ pointed, the base point will be 0 .

5.01. Definition

A path of length m in a fuzzy space (X,τ) is a fuzmap $\omega: [0,m] \to (X,\tau)$. The path ω begins at $\omega(0)$ and ends at $\omega(m)$, or ω is a path from $\omega(0)$ to $\omega(m)$. The points $\omega(0)$ and $\omega(m)$ are connected by ω , and we may write $\omega : \omega(0) \to \omega(m)$.

The strict finiteness of this definition is in marked contrast to the nondenumerability involved in topological paths. For some purposes it is convenient to make the following essentially equivalent, definition:-

5.02. Defintion

An m-path in a fuzzy space (X,τ) is a fuzmap $\omega: (\mathcal{N},\approx) \to (X,\tau)$ in association with a number $m \in \mathcal{N}$ s.t. $n \geqslant m \Rightarrow \omega(n) = \omega(m)$. We then say ω begins at $\omega(0)$, ends at $\omega(m)$, etc. If $m' > m$, m' is a bound for ω , while m is the bound for ω . A path in (X,τ) is a fuzmap $\omega(\mathcal{N},\approx) \to (X,\tau)$ such that a bound exists, without a specific choice of bound.

A path in a pointed space will in general be required to be pointed, though-to avoid duplicating arguments - not in the construction in Chapter II of the homotopy groups via groupoids.

5.03. Definition

The set PX of paths in (X,τ) will carry the subspace fuzzy from the inclusion $PX \to (X^{\mathcal{N}},\tau^{\approx})$. With this fuzzy it is the path space on X . The set ΩX of paths in (X,x_0,τ) beginning and ending at x_0 , again with the subspace fuzzy from $(X^{\mathcal{N}},\tau^{\approx})$, is the loop space on (X,x_0,τ) . It has base point the path $* : \mathcal{N} \to \tau$, $i \rightsquigarrow x$.

5.04. Definition

(a) The constant m-path at $x \in X$ is the map $\omega_x^m : \mathcal{N} \to X$, $i \mapsto x$.

(b) If ω is an m-path in X, define the reverse m-path

$$\overleftarrow{\omega} : \mathcal{N} \to X$$
$$i \mapsto \omega(\max\{m-i,0\})$$

(c) If ω, ω' are an m-path and an m'-path in X, and $\omega'(0) = \omega(m)$, define their composite

$$\omega'\omega : \mathcal{N} \to X$$
$$i \mapsto \begin{cases} \omega(i) & i \leqslant m \\ \omega'(i-m) & i \geqslant m \end{cases}$$

Evidently $\omega'\omega$ is an $(m+m')$-path in X, and the set $\overline{P}X$ of paths with associated bounds is a category under this composition, with identities the 0-paths ω_x^0 for $x \in X$. Similarly the set $\overline{\Omega}X$ of loops with associated bounds is a monoid.

5.05. Definition

A fuzzy space X is connected if for all pairs $(x,x') \in X \times X$, there exists a path $x \to x'$. Given $x_0 \in X$, the component of x_0 in X is the set

$$\{x \in X \mid \exists \text{ a path } x_0 \to x\}$$

Evidently, ρ = 'connected by a path of some length' is an equivalence relation on X, and the component of x_0 is its ρ-equivalence class. A non-empty ρ-equivalence class is called a component of X, and is clearly a maximal connected subspace.

5.1. Proposition

A fuzzy space (X,τ) is connected if and only if there is no fuzmic epimorphism

$$(X,\tau) \to (\{0,2\},\mathcal{S}) .$$

Proof

$\Rightarrow$:

Suppose (X,τ) connected, and consider a set-theoretic epimorphism $\eta : X \to \{0,2\}$.

As η is epic, we may take $x_0 \in \eta^{\leftarrow}\{0\}$, $x_2 \in \eta^{\leftarrow}\{2\}$.

As (X,τ) is connected, there exists a fuzmap

$\omega : [0,m] \to (X,\tau)$, for some m, s.t. $\omega(0) = x_0, \omega(m) = x_2$.

Then if $K = \{n \in [0,m] | \eta_0\omega(m) = 0\}$, we have

$K \neq \emptyset$, and $m \notin K$.

Thus if $k = \max K$, $\{k,k+1\} \subseteq [0,m]$

$\therefore$, since ω is fuzmic and $k \approx k+1$, $(\omega(k),\omega(k+1)) \in \tau$

But $(\eta,\eta)(\omega(k),\omega(k+1)) = (\eta_0\omega(k),\eta_0\omega(k+1))$

$$= (0,2)$$

$$\notin \delta .$$

$\therefore$ η is not fuzmic.

$\therefore$ if (X,τ) is connected, there exists no fuzmic epimorphism $(X,\tau) \to (\{0,2\},\delta)$.

$\Leftarrow$:

Suppose (X,τ) not connected.

Then $\exists\, x_0,x_2 \in (X,\tau)$ with no path connecting them.

Let P_0 be the component in (X,τ) of x_0.

Then

$$[x \in P_0] \,\&\, [(x,x') \in \tau] \Rightarrow [x' \in P_0] ,$$

for $[0,1] \to (X,\tau)$ is a path from x to

$0 \mapsto x$

$1 \mapsto x'$

x', so that we have $x_0\rho x$, $x\rho x'$, hence $x' \in P_0$.

Conversely

$$[x \notin P_0] \,\&\, [(x,x') \in \tau] \Rightarrow [x' \in P_0] \,.$$

Now define

$$\eta : (X,\tau) \to (\{0,2\},\approx)$$

$$x \leadsto \begin{cases} 0 \text{ if } x \in P_0 \\ 2 \text{ if } x \notin P_0 \end{cases}$$

Then η is fuzmic, for

$$(x,x') \in \tau \Rightarrow [x \in P_0 \ni x'] \text{ or } [x \notin P_0 \not\ni x']$$

$$\Rightarrow \eta(x) = \eta(x')$$

$$\Rightarrow (\eta(x),\eta(x')) \in \approx \,,$$

and η is epic, for

$$\eta(x_0,x_2\} = \{0,2\} \,.$$

Hence if (X,τ) is not connected, there exists a fuzmic epimorphism $(X,\tau) \to (\{0,2\},\approx)$. Hence if there exists no fuzmic epimorphism $(X,\tau) \to (\{0,2\},\approx)$, (X,τ) is connected. ■

5.2. Proposition

If the set $\{X_\alpha\}$ of components of a fuzzy space X is indexed by $\alpha \in A$

$$X \cong \sum_A X_\alpha \text{ naturally.}$$ ■

5.3. (History)

The following definitions are due to Poincaré [22] .

5.30. Given a fuzzy space (C,K) (called by Poincaré a physical continuum), a cut is an arbitrary subset X of C .

5.31. A cut X divides C if for some component P of C , $P \backslash N(X)$ is not connected.

5.32. A physical continuum (C,K) is of one dimension if it can be divided by a cut X such that $(X,K) = (X,\delta)$.

5.33. Inductively, a continuum (C,K) is of (n+1) dimensions when a cut X can divide it only if (X,K) is of n dimensions.

These definitions do not pin down dimension with the success that Poincaré, misled by analogy with topology, claimed for them. For if $C = \mathcal{N} \times \mathcal{N}$ and $X = \{7\} \times \{3\mathcal{N}\}$, by ~~4.22~~ 5.22 C is one-dimensional. Moreover, by a similar device we may divide $\mathcal{N}^{\alpha}$, for $\alpha = 2, \ldots, \infty$ or uncountable, by a set of mutually distinguishable points, so that it would seem that all fuzzy spaces are one-dimensional. This may be remedied by substituting:-

5.32.* A physical continuum is of one dimension if it can be divided by a set X s.t. $\forall x,x' \in X$,

$$x \neq x' \Rightarrow N(N(x)) \cap N(x') = \emptyset .$$

5.33.* Inductively, a continuum (C,K) is of n+1 dimensions when a cut X can divide it only if (N(X),K) is of n dimensions.

This, however, still leaves problems, since if a space is not homogeneous it picks out only a minimal dimension (e.g. it describes $(\mathcal{N},0,\approx) \vee (\mathcal{N} \times \mathcal{N}, (0,0), \approx\cdot\approx)$ as one-dimensional) and it does not assign a dimension to small fuzzy spaces such as $[0,4]/\{0,4\}$.

The approach that now seems most natural in defining dimension for fuzzy spaces is to use local homology groups (see Chapter III) but for Poincaré to have done so would be remarkable. A man may stand on the shoulders of giants, he may - like Poincaré - be a giant, but not even Poincaré could stand on the shoulders of mathematicians standing on his.

6. Metric Properties

6.00. Definition

For any fuzzy space (X,τ) define the hop metric

$$d : X \times X \to \mathcal{N} \cup \{\omega\}$$

$$(x,x') \rightsquigarrow \begin{cases} \min \{m \mid \exists \text{ an } m\text{-path } x \to x'\} \\ \omega \text{ if there is no path } x \to x' . \end{cases}$$

It is immediate that

(i) $d(x,x') = 0 \Leftrightarrow x = x'$

(ii) $d(x,x') = d(x',x)$

(iii) $d(x,x'') \leqslant d(x,x') + d(x',x'')$

We call $d(x,x')$ the (hop) distance from x to x' .

6.01. Remark

Note that any fuzmorphism $f : X \to Y$ preserves m-paths for all m , hence for all $x, x' \in X$ we have

$$d(f(x),f(x')) \leqslant d(x,x') .$$

In particular, if f is a fuzziomorphism its inverse also has this property, so that

$$d(f(x),f(x')) = d(x,x') ,$$

always. Thus the metric structure of a fuzzy space is intrinsic and invariant, rather than the optional and somewhat arbitrary extra that it is when considering topological spaces.

This requirement that an isomorphism in the category be an isometry is the most basic structural difference between fuzzy geometry and topology, and the reason for the choice of the term 'geometry'. (The most basic logical difference is that for fuzzy spaces uncountability is largely irrelevant, and that even the denumerably infinite sets used could be expressed in terms of the potential infinity of counting, rather than an actual infinity of reference.)

It implies that the problem of classifying - for instance - surfaces up to isomorphism is of the complexity of that in Euclidean geometry, not the tidily solved question that it is in the topological, smooth and PL categories. Further implications will appear in 8.3.

6.02. Definition

(a) Given $A,B \subseteq (X,\tau)$, the distance $d(A,B)$ from A to B is $\min(d(A \times B))$.

(b) Given $x \in X$, $A \subseteq (X,\tau)$, the distance $d(x,A)$ of x from A is $d(\{x\},A)$.

(c) Given $x \in A$, $A \subseteq (X,\tau)$, x is an interior point of, or interior to, A if $d(x,X\setminus A) > 1$.

(d) Given $A,B \subseteq (X,\tau)$, the distinction $\delta(A,B)$ between A and B is $\max[\{d(a,B) \mid a \in A\} \cup \{d(b,A) \mid b \in B\}]$.

(e) The internal diameter $D(X,\tau)$ or $D(X)$ of a fuzzy space (X,τ) is $\max(d(X \times X))$.

(If (d) or (e) is undefined by the specification of a maximum, we set the quantity involved equal to 0 or ∞ according as the set without a maximum is empty or non-empty. We take $\omega > \infty$ in defining maxima).

(f) If 2^X is the set of subsets of X, the distance fuzzy on 2^X is $\{(A,B) \mid d(A,B) \leqslant 1\}$, the discrimination fuzzy is $\{(A,B) \mid \delta(A,B) \leqslant 1\}$.

7. Intermediate Value Theorem

7.0. Definitions

7.00. A total fuzzy order on a set X is a fuzzy τ and total order $>$ such that

$$a > b \, , \; b > c \, , \quad a \tau c \Rightarrow a \tau b \, , \; b \tau c \, .$$

Examples: The natural, rational, algebraic or real numbers with a fuzzy of the form $a \tau b \Leftrightarrow |a-b| < \delta$; any totally ordered set with the little fuzzy.

7.01. A semiordering on a set X is a pair of binary relations P, I on X such that for all $a,b,c,d \in X$:

S1) Exactly one of aPb , bPa , aIb obtains.

S2) aIa

S3) aPb , bIc , $cPd \Rightarrow aPd$

S4) aPb , bPc , $bId \Rightarrow$ not both aId & cId .

Evidently I is a fuzzy.

This definition has advantages in connection with preference and utility functions (real-valued order-preserving functions) and brings the abstract behavioural models of psychologists and economists a little closer to the real world of people who can't tell Stork from poor-grade butter (see [18]), but is stronger than necessary here. We therefore define:

7.02. A weak semiordering on a set X is a pair of binary relations P,I on X such that for all $a,b,c \in X$:

WS1) Exactly one of aPb , bPa , aIb obtains

WS2) aIa

WS3) aPb , $bPc \Rightarrow aPc$.

Evidently I is again a fuzzy, and $S(1-4) \Rightarrow WS(1-3)$.

Examples: 401 cups of coffee with $(1+\frac{i}{100})x$ gm. of sugar,

$i = 0,1,\ldots,400$, x = weight in gm. of one cube of sugar, P = preference, I = indifference ([18]); any heap of examination scripts not yet argued over by an examiners meeting.

7.1. Proposition

If (X,τ) is a connected fuzzy space, $(Y,\sigma,>)$ is a totally fuzzy ordered set, and (Z,I,P) is a weakly semiordered set, then if $f : X \to Y$, $g : X \to Z$ are fuzmic we have for any $x,x' \in X$

(i) $f(x) > y_0 > f(x') \Rightarrow \exists x_0 \in X$ s.t. $f(x_0)\,\sigma\, y_0$.

(ii) $g(x) P z_0 \, P g(x') \Rightarrow \exists \bar{x}_0 \in X$ s.t. $g(\bar{x}_0)\, I\, z_0$.

Proof

Suppose the contrary.

Then f,g corestrict to fuzmaps

$$f' : X \to Y \backslash N(y_0)$$
$$g' : X \to Z \backslash N(z_0) \ .$$

Define $\eta : Y\backslash N(y_0) \to (\{0,2\},\approx)$ and $\varepsilon : Z\backslash N(z_0) \to (\{0,2\},\approx)$

$$y \rightsquigarrow 0, y_0 > y \qquad\qquad z \rightsquigarrow 0,\ z_0 P z$$
$$2,\ y > y_0 \qquad\qquad 2,\ zPz_0$$

Now, for $y > y_0 > y'$,

$$y\,\sigma\, y' \Rightarrow y\,\sigma\, y_0 \ , \ y_0\,\sigma\, y' \qquad (7.00.)$$
$$\Rightarrow y,y' \in N(y_0) \qquad (1.01.)$$

Thus for $y,y' \in Y\backslash N(y_0)$,

$$y > y_0 > y' \Rightarrow (y,y') \notin \sigma \ .$$

Likewise zPz_0 , $z_0Pz' \Rightarrow zPz'$ (WS3)

$\Rightarrow zIz'$ does not obtain. (WS1)

Thus η, ε are fuzmic.

Hence $\eta\circ f'$, $\varepsilon\circ g$ are fuzmic.

But $\eta\circ f'(X) \supseteq \eta\circ f'\{x,x'\} = \{0,2\}$

$\varepsilon\circ g'(X) \supseteq \varepsilon\circ g'\{x,x'\} = \{0,2\}$,

so that $\eta\circ f'$, $\varepsilon\circ g'$ are fuzmic epimorphisms, contradicting the connectedness of X . ■

7.11. Corollary

Totally fuzzy ordered sets and weakly semiordered sets, if their internal diameter is at least four hops, are one-dimensional in the amended sense of Poincaré. (5.32*.) ■

7.12. Unverifiable Conjecture

Analogues of 7.1. and 7.11. will hold for any combination of the concepts of order and fuzzy that carries conviction. ⍰

8. Homotopy

8.0. Proposition

The following definitions are equivalent (in the sense of giving rise to isomorphic categories):

Given two fuzmaps

$$f,g : (X,\tau) \to (Y,\sigma)$$

a homotopy of length m (or bounded by m), $F: f \simeq g$ from f to g , is

(i) a fuzmap $F : X \times [0,m] \to Y$ s.t. $F(x,0) = f(x)$
$F(x,m) = g(x)$

(ii) a sequence F of fuzmaps

$$f = F_0, F_1, \ldots, F_m = g$$

such that $F_i \sigma^\tau F_{i+1} \quad \forall_i = 0, \ldots, m-1$.

(iii) an m-path from f to g in (Y^X, σ^τ) .

All homotopies must be of a specified finite length/be bounded by a specified $m \in \mathbb{N}$. If $\exists$ a homotopy $H : f \simeq g$, f and g are homotopic.

Proof

Immediate. ■

8.01. Definition

A homotopy $F : f \simeq g : X \to Y$ is relative to $A \subseteq X$ if $F(x,i) = f(x)\ \forall i$.

In $\mathcal{F}uz_*$ we have similarly:

8.1. Proposition

The following definitions are equivalent (in the same sense): Given two pointed fuzmaps

$$f,g : (X,x_0,\tau) \to (Y,y_0,\sigma)$$

a based homotopy of length m , or bounded by m , $F : f \simeq g$ is

(i) a fuzmap $F : X \times [0,m] \to Y$ s.t. $F(x,0) = f(x)$, $F(x,m) = g(x)$, $F(x_0,i) = y_0$

(ii) a sequence F of pointed fuzmaps $f = F_0, F_1, \ldots, F_m = g$ s.t. $\forall i$, $F_i \sigma^\tau F_{i+1}$.

(iii) an m-path F (not required to be pointed) from f to g in $(Y^X, *, \sigma^\tau)$.

Thus a based homotopy is a homotopy relative to the base point. ■

8.2. Definitions

8.20. If $f,g,h : X \to Y$ are fuzmaps, $H : f \simeq g$ $H' : g \simeq h$ homotopies, the composite $H'H$ of H and H' is their composite as paths in Y^X . It is evidently a homotopy $f \simeq h$.

If $f,g,h : (X,x_0) \to (Y,y_0)$ are pointed fuzmaps, $H : f \simeq g$ $H' : g \simeq h$ based homotopies, the <u>composite</u> $H'H$ of H and H' is their composite as paths in $(Y^X,*)$. It is evidently a based homotopy $f \simeq h$.

<u>8.21</u>. If $H : f \simeq g : X \to Y$ is a homotopy of length m, and $h : W \to X$, $d : Y \to Z$ are fuzmorphisms, define

$$Hh : N \to Y^W \quad \text{and} \quad dH : N \to Z^X$$
$$i \rightsquigarrow H(i)\circ h \qquad\qquad i \rightsquigarrow d\circ H(i) .$$

Then $Hh : f\circ h \simeq g\circ h$, $dH : d\circ f \simeq d\circ g$ are also homotopies of length m.

<u>8.22</u>. Thus, if we denoted by $[X,Y]$ the set of homotopy classes of maps $X \to Y$, since homotopy is evidently an equivalence relation (i.e. $[X,Y]$ is the set of components of the space Y^X), we have a well-defined composition.

$$[X,Y] \times [Y,Z] \to [X,Z]$$

so that we have the <u>fuzzy homotopy category</u> $\mathcal{F}$huz, with objects fuzzy spaces and morphisms homotopy classes of fuzmaps, and similarly

<u>8.23</u>. If $[X,Y]_*$ denotes the set of based homotopy classes of pointed maps $X \to Y$, we obtain the <u>pointed fuzzy homotopy category</u> $\mathcal{F}\text{huz}_*$.

<u>8.24. Definition</u>

A fuzmap $f : X \to Y$ is <u>null-homotopic</u> if it is homotopic to a constant map, a <u>homotopy equivalence</u> if $\exists$ a fuzmap $g : Y \to X$ s.t. $fg \simeq 1_Y$, $gf \simeq 1_X$. The spaces X and Y are then <u>homotopy equivalent</u> (isomorphic in $\mathcal{F}$huz.) A space is <u>contractible</u> if it is homotopy equivalent to a point. Similarly for $\mathcal{F}\text{huz}_*$.

8.3. Remark

By analogy with topology, one could make the following definition:

A fuzzy space X has the homotopy extension property w.r.t. a subspace $A \subseteq X$ if for all m the square of inclusions

$$\begin{array}{ccc} A \times \{0\} & \to & A \times [0,m] \\ \downarrow & & \downarrow \\ X \times \{0\} & \to & X \times [0,m] \end{array}$$

is a pushout.

However, the metric-decreasing character of fuzmorphisms, together with the equivalence (as in $\mathcal{T}op$) of the HEP to the existence of a retraction

$$X \times [0,m] \to X_f \sqcup (A \times [0,m]) ,$$

where $f : A \times \{0\} \to X ,$

$(a,0) \rightsquigarrow a$

implies that the HEP is impossible if $\delta(X,A) > 1$.

Thus even $([0,2],\{0\})$ does not have it. For example, there is a homotopy $(0 \rightsquigarrow 1) \simeq (0 \rightsquigarrow 0) : \{0\} \to [0,3]$ of length 1 , but the shortest homotopy

$$([0,2] \underset{\text{add } 1}{\to} [0,3]) \simeq ([0,2] \underset{\text{inclusion}}{\to} [0,3])$$

extending it is of length 3 .

If this difficulty is overcome by a wider definition, not requiring the extending homotopy to be of the same length, a more fundamental one appears. If

$$X = [0,2] \quad \text{and} \quad A = (\{0,2\},\approx) ,$$

while $f : X \to [0,4]$ and $g : A \to [0,4]$,

$x \rightsquigarrow x+1 \qquad a \rightsquigarrow 2a$

then f and g are fuzmic and there is a homotopy $g \simeq f|A$, but there is no fuzmic extension over X even of the map g , let alone of the homotopy.

The HEP thus exists only in thoroughly trivial cases, except where it arises from a deformation retraction, and is unproductive as a tool. Since in topology a space is represented as a cell complex chiefly in order to climb its skeleton with homotopy extensions, the lesser significance of fuzzy cell complexes mentioned in §2 becomes clear.

Homotopy in $\mathcal{F}uz$ has other limitations. For example, an annulus with a reasonable fuzzy is homotopy equivalent to its inner but not its outer edge, and the identity on $\frac{[0,4]}{\{0,4\}}$ is homotopic only to itself.

All this relates to the fact that homotopy is the 'stretchingest' area of topology, and differences of $\mathcal{F}uz$ from $\mathcal{T}op$ are thus at their most acute. Stretching is impossible (§6) for fuzzy spaces, where size is intrinsic (as it is in physics, though <u>not</u> in the mathematical language currently used to describe the subject). Where topology is 'india-rubber geometry', fuzzy theory may perhaps be described as 'chain-mail geometry'.

Non-trivial cofibrations do not, then, arise, but fibrations and coverings do not require stretching so essentially. Particularly for covering spaces, the theory is much more like the topological case. It can be developed more conveniently after some algebraic fuzzy.

II. Algebraic Fuzzy

1. Notation

1.00. Denote the category of groupoids by $\mathcal{G}pd$, that of pointed groupoids by $\mathcal{G}pd_*$, that of groups by $\mathcal{G}$, that of graded abelian groups by $\underline{\mathcal{G}}$, and that of chain complexes by $\mathcal{Q}$.

1.01. Denote the full subgroupoid of a groupoid G on a subset A of the objects of G by (G,A) , the set of objects of G by Ob(G) , and the object map of a groupoid morphism f by Ob(f) .

1.02. A fuzzy pair (X,A,τ) or (X,A) is an inclusion

$$i : (A,i^*\tau) \to (X,\tau)$$

and a map of pairs $f : (X,A) \to (Y,B)$ is a commutative square

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{f} & (Y,\sigma) \\ i_A \cup & & \cup\, i_B \\ (A,i_A^*\tau) & \xrightarrow{f|A} & (B,i_B^*\sigma) \end{array}$$

of fuzmaps. Denote the resulting category by $\mathcal{F}uzz$. We may embed $\mathcal{F}uz$ in $\mathcal{F}uzz$ by $X \rightsquigarrow (X,\emptyset)$, and shall therefore where convenient denote $(X,\emptyset)$ by X . The cardinality n(X,A) of (X,A) is defined to be n(X) . (cf. I.1.06.)

1.03. The definitions of subpair, quotient pair, induced fuzzy etc. are the obvious generalisations of those in $\mathcal{F}uz$. In particular, the function space $(Y,B)^{(X,A)}$ is the set of maps of pairs $(X,A) \to (Y,B)$ with the fuzzy induced by

$$[f : (X,A) \to (Y,B)] \rightsquigarrow [f : (X,\emptyset) \to (Y,\emptyset)] \rightsquigarrow [f : X \to Y] ,$$

and a pair homotopy $f \simeq g : (X,A) \to (Y,B)$ is a path

$f \to g$ in $(Y,B)^{(X,A)}$. A pair is contractible if there is a pair homotopy

$$Id(X,A) \simeq \text{(some constant map)} .$$

1.04. A pointed fuzzy pair (X,A,x_o,τ) or (X,A,x_o) is a square

$$\begin{array}{ccc} (A,i^*\tau) & \to & (X,\tau) \\ \cup & & \cup \\ \{x_o\} & \to & \{x_o\} \end{array}$$

of inclusions, and a map of pointed fuzzy spaces is the obvious commutative cube of fuzmorphisms. The category of pointed fuzzy pairs is denoted by $\mathcal{F}uzz_*$, in which we shall embed $\mathcal{F}uz_*$ and denote $(X,\emptyset,x_o)$ by (X,x_o) .

1.05. The definitions of product, coproduct, wedge product and smash product in $\mathcal{F}uz$ and $\mathcal{F}uz_*$ generalise immediately to $\mathcal{F}uzz$ and $\mathcal{F}uzz_*$, as do their properties such as the exponential laws.

1.06. For x a rational number, m,i natural numbers, define $[x] = \max\{n | n < x , n \in \mathcal{N}\}, \langle i,m\rangle = i-m[\frac{i}{m}]$.

2. Fundamental Groupoid

2.01. Definition

Two paths $\omega,\omega' \in PX$, for X a fuzzy space, are equivalent, written $\omega \equiv \omega'$ if there is a path $F : \omega \to \omega'$ in PX s.t. $F(i)(0) = F(0)(0)$, $\forall i$, and a number m such that

i) m is a bound for F and each $F(i)$,

ii) $F(i)(m) = F(m)(m)$, $\forall i$.

We say the equivalence is bounded by m , and we may write $\omega \underset{m}{\equiv} \omega'$ to specify the bound.

It is clear that '$\equiv$' is an equivalence relation.

2.1. Proposition

The composition in $\overline{P}X$ induces a composition in the set $\pi(X,\tau)$, or πX , of equivalence classes of paths. Under this composition πX is a groupoid having the points of X as objects, the <u>fundamental groupoid</u> of (X,τ) .

Proof

i) Evidently if a path ω bounded by m has $\omega(m) = x$, an equivalence

$$\omega_x^{m'}\omega \underset{m+m'}{\equiv} \omega_x^{0}\omega = \omega \qquad \text{(see I.5.04(a).)}$$

is established by a path of length one, for any m' .

If a path ω bounded by m has $\omega(0) = x$, define

$F : \mathcal{N} \to PX$ by

$$F(i)(j) = \begin{cases} \omega(0) & \text{if } i \leqslant mm' \text{ and } j \leqslant [\frac{i}{m}] , \\ \omega(j - [\frac{i}{m}] - 1) & \text{if } i \leqslant mm' \text{ and } [\frac{i}{m}] < j < \langle i,m\rangle , \\ \omega(j - [\frac{i}{m}]) & \text{if } i \leqslant mm' \text{ and } [\frac{i}{m}] < j \geqslant \langle i,m\rangle , \\ \omega(\max\{0,j-m'\}) & \text{if } i \geqslant mm' . \quad \text{(recall 1.06.)} \end{cases}$$

Then F is a path $\omega = \omega\omega_x^{0} \to \omega\omega_x^{m}$ such that $F(i)(0) = F(0)(0)\ \forall i$, mm' is a bound for F and each $F(i)$, and $F(i)(mm') = F(mm')(mm')\ \forall i$.

$$\therefore \quad \omega \equiv \omega\omega_x^{m'}$$

ii) Suppose $\omega_1(0) = \omega(m)$, ω bounded by m , and $\omega_1 \underset{m'}{\equiv} \omega_2$ an equivalence established by the path

$F : \omega_1 \to \omega_2$ in PX .

Then $\omega_2(0) = \omega_1(0) = \omega(m)$, so that $\omega_2\omega$ and $\omega_1\omega$ are both defined.

Define $F'(i)(j) = \omega(j)$ if $j \leqslant m$,

$F(i)(j-m)$ if $j \geqslant m$.

Then $F' : \mathcal{N} \to PX$ is a path $\omega_1\omega \to \omega_2\omega$ in PX such that

$F'(i)(0) = F'(0)(0)\ \forall i$,

$m + m'$ is a bound for m for each $F'(i)$,

$F'(i)(m+m') = F'(m+m')(m+m')\ \forall i$.

$\therefore\ \omega_1\omega \equiv \omega_2\omega$.

iii) Suppose $x = \omega_3(0) = \omega_1(m_1)$, and $\omega_1 \underset{m}{\equiv} \omega_2$ is an equivalence established by the path $F : \omega_1 \to \omega_2$ in PX , where ω_i is bounded by m_i , $i = 1,2,3$. Then $\omega_2(m_2) = \omega_2(m) = \omega_1(m) = \omega_1(m_1) = \omega_3(0)$, so that $\omega_3\omega_2$ and $\omega_3\omega_1$ are both defined.

By part (i) we have

$$\omega_3\omega_x^{m-m_i} \equiv \omega_3 \ , \ i = 1,2 \ .$$

Hence by part (ii) we have

$$§ \qquad \omega_3\omega_x^{m-m_i}\omega_i \equiv \omega_3\omega_i \ , \ i = 1,2 \ .$$

Define $F'(i)(j) = F(i)(j)$ if $j \leqslant m$

$\omega(j - m)$ if $j \geqslant m$

Then $F' : \mathcal{N} \to PX$ is a path $\omega_3\omega_x^{m-m_1}\omega_1 \to \omega_3\omega_x^{m-m_2}\omega_2$ in PX such that

$F'(i)(0) = F'(0)(0)\ \forall i$,

$m + m_3$ is a bound for F and each $F(i)$,

$F'(i)(m+m_3) = F'(m+m_3)(m+m_3)\ \forall i$.

$\therefore\ \omega_3\omega_x^{m-m_1}\omega_1 \equiv \omega_3\omega_x^{m-m_2}\omega_2$.

Hence by § above,

$$\omega_3\omega_1 \equiv \omega_2\omega_1 \ .$$

iv) By (ii) and (iii), composition of equivalence classes is well defined by the rule

$$\overline{\omega}\,\overline{\omega'} = \overline{\omega\omega'} \; .$$

By (i), therefore, the class $\overline{\omega_x^o}$ of the identity ω_x^o at x in $\overline{P}X$ is an identity at x in πX, and will be denoted by 1_x.

v) Consider an arbitrary path ω in (X,τ) from x to y, bounded by m.

Define
$$F(i)(j) = \begin{cases} \overleftarrow{\omega}\,\omega(j) & \text{if } |j-m| \geq i \\ \overleftarrow{\omega}\,\omega(m-i) & \text{if } |j-m| \leq i \leq m \\ x & \text{if } i \geq m \end{cases}$$
(see I.5.04 (v))

Then $F : \mathcal{N} \to PX$ is a path $\overleftarrow{\omega}\,\omega \to \omega_x^o$ such that

$F(i)(0) = F(0)(0)$, $\forall i$,

$2m$ is a bound for F and each $F(i)$,

$F(i)(2m) = F(2m)(2m)$, $\forall i$.

$$\therefore \overleftarrow{\omega}\,\omega = \omega_x^o$$
$$\therefore \overline{\overleftarrow{\omega}}\,\overline{\omega} = \overline{\overleftarrow{\omega}\,\omega} = \overline{\omega_x^o} = 1_x \; .$$

Similarly $\overline{\omega}\,\overline{\overleftarrow{\omega}} = 1_y$.

Hence $\overline{\overleftarrow{\omega}}$ is a two-sided inverse for $\overline{\omega}$, and may be denoted $(\overline{\omega})^{-1}$.

vi) By (iv) and (v), πX with the composition induced from $\overline{P}X$ is indeed a groupoid.■

2.11. Remark

It is necessary to include the requirement of a specific bound in the definition of a path, in orderly properly to define composition of paths; it is defined only in $\overline{P}X$, not PX. Complete rigour on this point, however, leads to very

cumbersome argument. As long as each path has a specified bound at every stage in a computation, composition is well defined at that stage. In this section we are interested in paths essentially only up to equivalence, and § in part (iii) of the proof above shows that up to equivalence, composition is well defined independently of the bound. Without loss of generality, therefore, we make the convention that whenever we make use of an equivalence $\omega \underset{m}{\equiv} \omega'$ established by a path F in PX, m becomes <u>the</u> bound for F and for each $F(i)$.
This lapse in rigour leads to less confusion than would the cumbersomeness of complete strictness.

<u>2.2. Proposition</u>

Any fuzmorphism $f : (X,\tau) \to (Y,\sigma)$ induces a groupoid homomorphism $\pi(f)$ or $f_* : \pi X \to \pi Y$, such that π becomes a functor $\mathcal{F}uz \to \mathcal{G}pd$, inducing a functor $\pi' : \mathcal{F}uz_* \to \mathcal{G}pd_*$.

<u>Proof</u>

i) For any $x \in X$, define $f_*(x) = f(x)$.
For any $p \in \pi X$, take $\omega \in p$ and define $f_*(p) = \overline{f_\circ\omega}$.
If $\omega' \in p$ also, $\omega' \underset{m}{\equiv} \omega$ for some m; there is a path $F : \omega' \to \omega$ in PX such that

$F(i)(0) = F(0)(0)$, $\forall i$,

m is a bound for F and each $F(i)$,

$F(i)(m) = F(m)(m)$, $\forall i$.

Then if $F'(i) = f_\circ F(i)$, F' is a path $f_\circ\omega' \to f_\circ\omega$ in PY such that

$F'(i)(0) = F'(0)(0)$, $\forall i$,

m is a bound for F' and each $F(i)$,

$F'(i)(m) = F'(m)(m)$.

Hence $f \circ \omega' \underset{m}{\equiv} f \circ \omega$

$\therefore \quad \overline{f \circ \omega'} = \overline{f \circ \omega}$

Thus $f_*(p)$ is independent of choice, so that f_* is well defined.

ii) For any $p, q \in \pi X$ s.t. pq is defined, if $\omega \in p$, $\omega' \in q$

$$\begin{aligned} f_*(pq) &= f_*(\bar{\omega}\,\bar{\omega}') \\ &= f_*(\overline{\omega\omega'}) \\ &= \overline{f \circ (\omega\omega')} \\ &= \overline{(f \circ \omega)(f \circ \omega')} \\ &= (\overline{f \circ \omega})(\overline{f \circ \omega'}) \\ &= f_*(p)\, f_*(q) \,. \end{aligned}$$

and
$$\begin{aligned} f_*(1_x) &= f_*(\overline{\omega_x^o}) \\ &= \overline{f \circ \omega_x^o} \\ &= \overline{\omega_{f(x)}^o} \\ &= 1_{f*(x)} \,. \end{aligned}$$

Hence f_* is a groupoid homomorphism.

Evidently $\pi(Id_X) = Id_{\pi X}$, $\pi(f \circ g) = \pi(f) \circ \pi(g)$.

$\therefore$ π is a functor $\mathcal{T}uz \to \mathcal{G}pd$.

iii) Since $\pi(f)$ on objects is just f , and preserves base-points if f does, $\pi : \mathcal{T}uz \to \mathcal{G}pd$ induces a functor $\pi' : \mathcal{T}uz_* \to \mathcal{G}pd_*$ by simple restriction on morphisms. We have

$$\pi'(X, x_o) = (\pi X, x_o) \,.$$

∎

3. Homotopy Groups

3.00. Definition

The first homotopy group, or fundamental group, $\pi_1(X,x_o,\tau)$ or $\pi_1(X,x_o)$ is the object group $(\pi X,\{x_o\})$ of πX at x_o. If (X,τ) is connected, so is πX, and hence all these object groups are (not, in general, naturally) isomorphic. In this case we may loosely refer to any one of the groups $\pi_1(X,x,\tau)$, $x \in X$ as "the fundamental group, $\pi_1 X$, of (X,τ)". If $\pi_1 X = 0$, X is simply connected.

3.01. Notation

Denote the double loop space $\Omega(\Omega X,*,\tau^{\approx})$ of (X,x_o,τ) (see I.5.03) by $\Omega^2 X$, and so on inductively for $\Omega^n X$. Set $\Omega^o X = X$.

3.02. Definition

For $0 < n \in \mathcal{N}$, the n^{th} homotopy group $\pi_n(X,x_o,\tau)$ or $\pi_n X$ of (X,τ) at x_o is the group $\pi_1(\Omega^{n-1}X)$. Note that as a set $\pi_n X$ is the set of components of $\Omega^n X$. If $\pi_n X = 0$ for $n = 1,\ldots,m$, X is m-connected.

3.1 Proposition

Each π_n, $0 < n \in \mathcal{N}$, is a functor $\mathcal{F}uz_* \to \mathcal{G}$.

Proof

The restriction R to the object group at the base point is a functor $\mathcal{G}pd_* \to \mathcal{G}$.
The passage L_m from pointed fuzzy spaces to their m-tuple loop spaces is a functor $\mathcal{F}uz_* \to \mathcal{F}uz_*$ by the naturality in the pointed exponential law (I.4.61.) and the naturality of restriction to loops.

Hence $\pi_n = R\pi' L_{m-1}$ is a functor. ∎

4. Track Groupoids

4.0. Definitions

4.00. An n-track on (X,τ) is a fuzmap

$$f : \left(\overset{n-1}{\overline{\ast}}\, \mathcal{N}\right) \times \mathcal{N} \to (X,\tau)$$

such that $f \mid \left(\overset{n-1}{\overline{\ast}}\, \mathcal{N}\right) \times \{0\}$ is constant, such that for some number m

$$f([x_1,\dots,x_{n-1}],x_n) = f(*,x_n) \text{ if } \max\{x_1,\dots,x_{n-1}\} \geqslant m$$

and $f([x_1,\dots,x_{n-1}],x_n) = f(*,m)$ if $x_n \geqslant m$.

f is then bounded by m .

By analogy with paths, we write

$$f : f([0,\dots,0],0) \to f([0,\dots,0],m)$$

and say f is from $f(*,0)$ to $f(*,m)$.

For further definition of tracks we shall denote $[x_1,\dots,x_{n-1}]$ by $[x]$ where this is possible without confusion.

4.01. Given an n-track f bounded by m , define the reverse track

$$\overleftarrow{f} : \left(\overset{n-1}{\overline{\ast}}\, \mathcal{N}\right) \times \mathcal{N} \to (X,\tau)$$

$$([x],x_n) \rightsquigarrow f([x],\max\{m-x_n,0\})$$

also bounded by m .

4.02. Define the constant n-track bounded by m at $p \in X$ as

$$f_p^m : \left(\overset{n-1}{\overline{\ast}}\, \mathcal{N}\right) \times \mathcal{N} \to (X,\tau)$$

$$([x],x_n) \rightsquigarrow p \ .$$

4.03. Denote the set of n-tracks on (X,τ) by $T_n(X,\tau)$, and give it the subspace fuzzy from the function space $X^{\left(\overset{n-1}{\overline{\ast}}\, \mathcal{N}\right) \times \mathcal{N}}$. Denote the set of n-tracks on (X,τ) in association with particular bounds by $\overline{T}_n(X,\tau)$.

4.04. The composite of the n-tracks $f : x \to y$, $g : y \to z$ (where $x,y,z \in X$) , bounded by m_f , m_g respectively, is the n-track

$$gf : (\overset{n-1}{\prod} \mathcal{N}) \times \mathcal{N} \to (X,\tau)$$

$$([x],x_n \rightsquigarrow \begin{cases} f([x],x_n) & \text{if } x_n \leq m_f \\ f([x],x_n - mf) & \text{if } x_n \geq m_f \end{cases}$$

from x to z , bounded by $(m_f + m_g)$.

4.05. Two n-tracks f,g in X are equivalent, $f \equiv g$, if there is a path F in T_nX connecting them such that $F(i)([x],0) = F(0)([x],0)$, $\forall i$, and a number m such that m is a bound for F and each $F(i)$, and $F(i)(*,m) = F(m)(*,m)$ $\forall i$. It is clear that '$\equiv$' is an equivalence relation. Denote the equivalence class of f by $\overline{f}$.

The next two propositions are proved exactly as for the fundamental groupoid (2.1, 2.2).

4.1. Proposition

The composition in $\overline{T}_nX$ induces a composition in the set $\pi^n(X,\tau)$, or π^nX , of equivalence classes of n-tracks, under which π^nX is a groupoid having the points of X as objects. It is called the n-track groupoid of (X,τ) .■

4.11. Remark

We allow a similar abuse of notation to that of Remark 2.11.

4.2. Proposition

Any fuzmorphism $f : (X,\tau) \to (Y,\sigma)$ induces a groupoid homomorphism $\pi^n(f)$ or $f_* : \pi^nX \to \pi^nY$ such that π^n becomes a functor $\mathcal{F}uz \to \mathcal{G}pd$, inducing a functor $\pi'^n : \mathcal{F}uz_* \to \mathcal{G}pd_*$.■

4.3. Proposition

For any $n > 1$ there exist natural groupoid homomorphisms

$$i_n : \pi X \to \pi^n X$$
$$p_n : \pi^n X \to \pi X$$

such that $p_n i_n = 1_{\pi X}$.

Proof

If $q \in \pi X$, take $\omega \in q$ and define

$$i_n(q) = \overline{\omega \circ p}$$

where p is the projection

$$(\overset{n-1}{\textstyle\prod_*} \mathcal{N}) \times \mathcal{N} \to \mathcal{N}$$
$$([x], x_n) \rightsquigarrow x_n$$

If $\omega' \in q$ also, then $\omega' \equiv \omega$; there is a path in PX $F : \omega' \to \omega$ such that $F(i)(0) = F(0)(0), \forall i$, and a number m such that m is a bound for F and each $F(i)$, and $F(i)(m) = F(m)(m), \forall i$.

If F' is the path

$$\mathcal{N} \to T_n X$$
$$i \rightsquigarrow F(i) \circ p$$

in $T_n X$, it has

$$\begin{aligned} F'(i)([x],0) &= F(i)(0) \\ &= F(0)(0) \\ &= F'(0)([x],0) \ \forall i , \end{aligned}$$

m is a bound for F' and each $F'(i)$,

and

$$\begin{aligned} F'(i)(*,m) &= F(i)(m) \\ &= F(m)(m) \\ &= F'(m)(*,m) \ \forall i . \end{aligned}$$

$$\therefore \omega \circ p \equiv \omega' \circ p$$

$$\text{i.e. } \overline{\omega \circ p} = \overline{\omega' \circ p}$$

Hence i_n is well defined.

Evidently if $\omega, \omega' \in \overline{P}X$ have $\omega\omega'$ defined, then the composite $(\omega \circ p)(\omega' \circ p)$ is defined and equal to $(\omega\omega') \circ p$, so that, taking equivalence classes, i_n is a homomorphism of groupoids.

Naturality follows from the commutativity of the square

$$\begin{array}{ccc} \pi X & \xrightarrow{\pi(f)} & \pi Y \\ i_n^X \downarrow & & \downarrow i_n^Y \\ \pi^n X & \xrightarrow{\pi^n(f)} & \pi^n Y \end{array}$$

which is immediate.

(i) If $t \in \pi^n X$, take $f \in t$ and define $p_n(t) = \overline{f \circ i}$, where i is the injection

$$\mathcal{N} \to (\overset{n-1}{\ast} \mathcal{N}) \times \mathcal{N} .$$
$$n \rightsquigarrow (*, n)$$

The proof that p_n is well-defined and a natural homomorphism is exactly as for i_n.

(ii) Since $p \circ i = 1_{\mathcal{N}}$, we have, if $\omega \in q \in \pi X$,

$$\begin{aligned} p_n i_n(q) &= p_n i_n(\overline{\omega}) \\ &= p_n(\overline{\omega \circ p}) \\ &= \overline{\omega \circ p \circ i} \\ &= q \end{aligned}$$

$$\therefore \; p_n i_n = 1_{\pi X} .$$ ∎

4.40. Definition

If $p_n(\overline{g}) = \overline{\omega}$, $\overline{g}$ is <u>along</u> $\overline{\omega}$, and $i_n(\overline{\omega})$ is the <u>trivial</u> n-track class along $\overline{\omega}$.

Without equivalences, g is <u>along</u> $g \circ i$, $\omega \circ p$ is the <u>trivial</u> n-track along ω.

5. Track Groups

Evidently p_n , i_n are the identity on objects, and hence restrict to natural group homomorphisms

$$p'_n : (\pi^n X, \{x_o\}) \to \pi_1(X, x_o)$$
$$i'_n : \pi_1(X, x_o) \to (\pi^n X, \{x_o\})$$

on object groups, still with the property that

$$p'_n \circ i'_n = 1_{\pi_1(X, x_o)} .$$

Thus the information carried by the object groups $(\pi^n X, \{x_o\})$ includes in each case the information carried by the fundamental group. This is unnecessary and obscuring, therefore we take

5.00. Definition

The n-track group $\overline{\pi}_n(X, x_o, \tau)$, or, loosely, $\overline{\pi}_n X$, of X at x_o is the normal subgroup $(p'_n)^{\leftarrow}(1)$ of the object group $(\pi^n X, \{x_o\})$.

It is clear that, together with restrictions of the groupoid maps, this defines a functor (since p'_n is natural).

5.01. Definition

A map $f : \overset{n}{\overline{\ast}}\, \mathcal{N} \to X$ or $(\overset{n}{\overline{\ast}}\, \mathcal{N}, *) \to (X, x_o)$ is bounded if there is a number m such that

$$\max\{x_1, \ldots, x_n\} \geqslant m \Rightarrow f([x]) = f(*) ,$$
$$\text{or } f([x]) = x_o ,$$

respectively. Denote the set of based homotopy classes of bounded pointed fuzmaps $(\overset{n}{\overline{\ast}}\, \mathcal{N}, *) \to (X, x_o)$ by $[\overset{n}{\overline{\ast}}\, \mathcal{N}, X]^b_*$. (This is unambiguous except under change of base point.)

5.1. Lemma

If g is an n-track along ω, and $\omega' \equiv \omega$, then $\exists\, g'$ along ω' such that $g' \equiv g$.

Proof

Take g as bounded by m_g.

We have a path $F : \omega' \to \omega$ in PX such that $F(i)(0) = F(0)(0)\ \forall i$, and a number m such that m is a bound for F and each $F(i)$, and $F(i)(m) = F(m)(m)\ \forall i$.

If $[x] = [x_1,\ldots,x_{n-1}] \in \overset{n-1}{\pi^*} \mathcal{N}$, let

$$M[x] = \max\{x_1,\ldots,x_{n-1}\}$$
$$m[x] = \min\{x_1,\ldots,x_{n-1}\}$$

Define g', g_i', g_{ij} as follows:

$g' : (\overset{n-1}{\pi^*} \mathcal{N}) \times \mathcal{N} \to X$

$$([x],x_n) \rightsquigarrow \begin{cases} F(M[x])(x_n) & \text{if } M[x] \leq m, \\ \omega(x_n) & \text{if } m \leq M[x] \leq m+m_g \quad m[x] \leq m, \\ g([x_1-m,\ldots,x_{n-1}-m],x_n) & \text{if } m \leq M[x] \leq m+m_g \quad m[x] \geq m, \\ F(2m+m_g-M[x])(x_n) & \text{if } m+m_g \leq M[x] \leq 2m+m_g, \\ \omega'(n) & \text{if } 2m+m_g \leq M[x]. \end{cases}$$

$g_i' : (\overset{n-1}{\pi^*} \mathcal{N}) \times \mathcal{N} \to X$

$$([x],x_n) \rightsquigarrow \begin{cases} F(\max\{M[x],i\})(x_n) & \text{if } M[x] \leq m, \\ \omega(x_n) & \text{if } m \leq M[x] \leq m+m_g \quad m[x] \leq m, \\ g([x_1-m,\ldots,x_{n-1}-m],x_n) & \text{if } m \leq M[x] \leq m+m_g \quad m[x] \geq m, \\ F(\max\{2m+m_g-M[x],i\})(x_n) & \text{if } m+m_g \leq M[x] \leq 2m+m_g, \\ F(i)(x_n) & \text{if } 2m+m_g \leq M[x]. \end{cases}$$

$g_{ij} : (\overset{n-1}{\pi^*} \mathcal{N}) \times \mathcal{N} \to X$

$$([x],x_n) \rightsquigarrow \begin{cases} g([x_1-m+j,\ldots,x_{n-1}-m+j],x_n) & \text{if } M[x] > m+m_g-j-i \\ & \quad m[x] \geq m-j, \\ \omega(x_n) & \text{if } M[x] > m+m_g-j-i \\ & \quad m[x] < m-j, \end{cases}$$

Continued/..

$$g([x_1-m-1+j,\dots,x_{n-1}-m-1+j],x_n) \quad \text{if } M[x]\leq m+m_g-j-i$$
$$m[x]>m-j\ ,$$
$$\omega(x_n) \quad \text{if } M[x]\leq m+m_g-j-i$$
$$m[x]\leq m-j\ .$$

Then g' is an n-track bounded by $m(m_g+1)$, along ω' taken as bounded by $(2m+m_g)$, and the sequence

$$g'=g'_o,g'_1,\dots,g'_m,g_{11},g_{21},\dots,g_{m_g1},g_{12},\dots,g_{m_gj},g_{1(j+1)},\dots,g_{m_gm} = g$$

defines a path $G : g' \to g$ in T_nX such that

$G(i)([x],0) = G(0)([x],0)\ \forall i$,

$m(m_g+1)$ is a bound for G and each $G(i)$,

$G(i)(*,m) = G(m)(*,m)\ \forall i$.

$$\therefore\ g' \equiv g\ . \quad \blacksquare$$

5.11. Corollary

Each element of $\bar{\pi}_n(X,x_o,\tau)$ has a representative along a constant path $\omega^m_{x_o}$ for some m . $\blacksquare$

5.12. Corollary

If $p : (\bigwedge^{n-1}\mathcal{N})\times\mathcal{N}\to\bigwedge^{n}\mathcal{N}$ is the identification map, the map

$$p^* : [\bigwedge^{n}\mathcal{N},X]^b_* \to \bar{\pi}_nX$$
$$\bar{g} \rightsquigarrow \overline{g_\circ p}$$

is well defined and a natural injection.

If $[\bigwedge^{n}\mathcal{N},X]^b_*$ is given the composition

$$\bar{f}\,\bar{g} = \overline{f.g}$$

where $f.g : (\bigwedge^{n}\mathcal{N},*)\to(X,\tau)$

$$[x] \rightsquigarrow \begin{cases} g([x]) & x_n\leq m \\ f([x_1,x_2,\dots,x_n-m]) & x_n\geq m \end{cases}$$ where m is the bound for g .

Then this composition is well defined, and under it $[\bigwedge^{n}\mathcal{N},X]^b_*$ becomes a group, p^* a natural group isomorphism.

Proof

The existence of a set-theoretic inverse to p^* follows from Corollary 5.11. All the other statements follow directly from the definitions involved.■

5.2. Proposition

$\bar{\pi}_n : \mathcal{F}uz_* \to \mathcal{S}$ is naturally equivalent to π_n .

Proof

The natural fuzziomorphism

$$\bar{\varphi}: ((\ldots((X^{\mathcal{N}})^{\mathcal{N}})\ldots)^{\mathcal{N}},*,(\ldots((\tau^{\approx})^{\approx})\ldots)^{\approx}) \cong (X^{\overset{n}{\bar{\wedge}}\mathcal{N}},*,\tau^{\approx\cdot\approx\cdot\ldots\cdot\approx})$$

which exists by the Pointed Exponential Law (I.4.6) restricts naturally to a fuzziomorphism

$$\psi:(\Omega^n X) \cong (\{f:(\overset{n}{\bar{\wedge}}\mathcal{N},*) \to (X,x_o)\,|\,f \text{ bounded}\},\ \tau^{\approx\cdot\approx\cdot\ldots\cdot\approx})$$

since evidently both $\bar{\varphi}$ and $(\bar{\varphi})^{\leftarrow}$ preserve boundedness.

Thus ψ induces a natural bijection

$$\psi_* : \pi_n X \to [\overset{n}{\bar{\wedge}}\mathcal{N},X]^b_*$$

on the sets of components.

Now consider ω_1, ω_2 , loops on $\Omega^{n-1}(X)$. We may without loss of generality assume a common bound m for ω_1, ω_2, $\psi(\omega_1)$, and $\psi(\omega_2)$.

Now $\psi(\omega)[x_1,\ldots,x_n]=(\ldots(((\omega(x_n))x_{n-1})x_{n-2})\ldots)x_1$

so $\psi(\omega_1\omega_2)[x_1,\ldots,x_n]=(\ldots(((\omega_1\omega_2(x_n))x_{n-1})x_{n-2})\ldots)x_1$

$$=\begin{cases}(\ldots(((\omega_2(x_n))x_{n-1})x_{n-2})\ldots)x_1, & x_n\leqslant m\\ (\ldots(((\omega_1(x_n-m)x_{n-1})x_{n-2})\ldots)x_1, & x_n\geqslant m\end{cases}$$

(I.5.04(c).)

$$=\begin{cases}\psi(\omega_2)[x_1,\ldots,x_n], & x_n\leqslant m\\ \psi(\omega_1)[x_1,\ldots,x_n-m], & x_n\geqslant m\end{cases}$$

$$= (\psi(\omega_1).\psi(\omega_2))[x_1,\ldots,x_n]$$

under the composition of 5.12.

$$\therefore\ \psi_*(\bar{\omega}_1\bar{\omega}_2) = \psi_*(\overline{\omega_1\omega_2})$$
$$= \overline{\psi(\omega_1\omega_2)}$$
$$= \overline{\psi(\omega_1)\cdot\psi(\omega_2)}$$
$$= \overline{\psi(\omega_1)}\ \overline{\psi(\omega_2)} \qquad (\text{def}^n)$$
$$= \psi_*(\omega_1)\psi_*(\omega_2)$$

Hence ψ_* is a natural bijective homomorphism

$$\pi_n X \to [\overset{n}{\overline{\underset{*}{\Lambda}}}\mathcal{N}, X]^b_*$$

and thus a natural isomorphism.

Therefore the composite

$$p^*\psi_* : \pi_n X \to \bar{\pi}_n X$$

is a natural isomorphism.

Hence π_n , $\overline{\pi_n}$ are naturally equivalent, and we shall denote both functors by π_n . ■

6. Commutativity

6.1. Proposition

For all $n > 1$ and fuzzy spaces (X,x_0,τ) , $\bar{\pi}_n(X)$ is abelian.

Proof

Consider $\omega_1, \omega_2 \in \Omega^n(X)$, $n > 1$.
Without loss of generality we may suppose $\omega_1, \omega_2, \omega_1(j), \omega_2(j)$, for all $j \in \mathcal{N}$, all to have a common bound m_1 .
Define

$$\omega_1 * \omega_2 : \mathcal{N} \to \Omega^{n-1}X$$
$$n \rightsquigarrow \omega_1(n)\ \omega_2(n)$$

with bound $2m_1$. (Note that $\omega_1(n)\ \omega_2(n)$ has bound $2m_1$ by the convention for the composition of paths.) Evidently $*$ is a binary operation on $\bar{\Omega}^n(X)$ with ω_*^o as identity.
If $\omega_1' \underset{m'}{\equiv} \omega_1$ by virtue of a path $F : \omega_1' \to \omega_1$ in $\Omega^n(X)$, then if $(F_*\omega_2)(i) = (n \rightsquigarrow F(i)(n)\omega_2(n))$, $F_*\omega_2$ is a path $\omega_1' {}_* \omega_2 \to \omega_1 {}_* \omega_n$ establishing an equivalence $\omega_1' {}_* \omega_2 \underset{m'}{\equiv} \omega_1 {}_* \omega_2$.

Similarly we have $\omega_2 {}_* \omega_1' \underset{m'}{\equiv} \omega_2 {}_* \omega_1$.

Thus $\bar{\omega}_1 {}_* \bar{\omega}_2 = \overline{\omega_1 {}_* \omega_2}$ is well defined, with identity the identity $1 = \overline{\omega_*^o}$ of the group structure of $\Pi_n(X)$.
Now consider loops $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega^n(X)$, again without loss of generality supposing $\omega_i, \omega_i(j)$, $i = 1, \ldots, 4$, $j \in \mathcal{N}$, all to have a common bound m .

Recalling Definition 1.06, and using the convention

$$(a - b) = 0 \quad \text{if} \quad a < b$$

define $F : \mathcal{N} \to \Omega^n(X)$ as follows:

$$
\begin{aligned}
F(i)(n)(n') &= \omega_1(n-2m)(n'-m-[\tfrac{i}{m}]) \text{ if } n \geq 2m\,, \\
&\quad n' \geq m+[\tfrac{i}{m}]+\langle i,m\rangle,\ 0 \leq i \leq m^2\,, \\
&= \omega_1(n-2m)(n'-m-[\tfrac{i}{m}]-1) \text{ if } n \geq 2m\,, \\
&\quad m < n' < m+[\tfrac{i}{m}]+\langle i,m\rangle,\ 0 \leq i \leq m^2\,, \\
&= \omega_2(n-2m+[\tfrac{i}{m}])(n') \text{ if } n' \leq m\,, \\
&\quad m \leq n \leq 3m-[\tfrac{i}{m}]-\langle i,m\rangle,\ 0 \leq i \leq m^2\,, \\
&= \omega_2(n-2m+[\tfrac{i}{m}]+1)(n') \text{ if } n' \leq m\,, \\
&\quad n > 3m-[\tfrac{i}{m}]-\langle i,m\rangle,\ 0 \leq i \leq m^2\,, \\
&= \omega_3(n)(n'-m-[\tfrac{i}{m}]) \text{ if } n \leq m\,, \\
&\quad n' \geq m+[\tfrac{i}{m}]+\langle i,m\rangle,\ 0 \leq i \leq m^2\,, \\
&= \omega_3(n)(n'-m-[\tfrac{i}{m}]-1) \text{ if } n \leq m\,, \\
&\quad m < n' < m+[\tfrac{i}{m}]+\langle i,m\rangle,\ 0 \leq i \leq m^2\,, \\
&= \omega_1(n-3m+[\tfrac{i}{m}])(n'-2m) \text{ if } n' \geq 2m\,, \\
&\quad m \leq n \leq 4m-[\tfrac{i}{m}]-\langle i,m\rangle,\ m^2 \leq i \leq 2m^2\,, \\
&= \omega_1(n-3m+[\tfrac{i}{m}]+1)(n'-2m) \text{ if } n' \geq 2m\,, \\
&\quad n > 4m-[\tfrac{i}{m}]-\langle i,m\rangle,\ m^2 \leq i \leq 2m^2\,, \\
&= \omega_2(n-m)(n') \text{ if } n \geq m\,, \\
&\quad n' \leq m\,,\ m^2 \leq i\,, \\
&= \omega_3(n)(n'-2m) \text{ if } n \leq m\,, \\
&\quad n' \geq 2m\,,\ m^2 \leq i\,, \\
&= \omega_4(n)(n') \text{ if } n \leq m\,, \\
&\quad n' \leq m\,,\ 0 \leq i\,, \\
&= \omega_1(n-m)(n'-2m) \text{ if } n \geq m\,, \\
&\quad n' \geq 2m\,,\ 2m^2 \leq i\,, \text{ and} \\
&= * \qquad\qquad \text{otherwise.}
\end{aligned}
$$

Then F is a path in $\Omega^n(X)$ from $(\omega_1 {}_* \omega_2)(\omega_3 {}_* \omega_4)$ to $(\omega_1\omega_3)_*(\omega_2\omega_4)$ such that

$$F(i)(0) = G(0)(0) \ \forall i$$

$2m^2$ is a bound for F and each $F(i)$

$$F(i)(2m^2) = F(2m^2)(2m^2) \ \forall i \ .$$

$$\therefore \ (\omega_1 {}_* \omega_2)(\omega_3 {}_* \omega_4) \equiv (\omega_1\omega_3)_*(\omega_2\omega_4)$$

$$\therefore \ (\overline{\omega}_1 {}_* \overline{\omega}_2)(\overline{\omega}_3 {}_* \overline{\omega}_4) = (\overline{\omega}_1\overline{\omega}_3)_*(\overline{\omega}_2\overline{\omega}_4)$$

$$\begin{aligned} \therefore \qquad \overline{\omega\omega'} &= (\overline{\omega}_* 1)(1_* \overline{\omega'}) \\ &= (\overline{\omega}1)_*(1\overline{\omega'}) \\ &= \overline{\omega}_* \overline{\omega'} \\ &= (1\overline{\omega})_*(\overline{\omega'}1) \\ &= (1_* \overline{\omega'})(\overline{\omega'}_* 1) \\ &= \overline{\omega'} \ \overline{\omega} \ , \text{ for any } \overline{\omega} \ , \overline{\omega'} \in \Pi_n(X) \ . \end{aligned}$$

i.e. $\pi_n X$ is abelian. ■

7. Covering Spaces

7.01. Definition

A fuzmap $f : \tilde{X} \to X$ is a local fuzziomorphism if for all $\tilde{x} \in \tilde{X}$, $N(f(\tilde{x}))|f|N(\tilde{x})$ is a fuzziomorphism.

7.02. Definition

A local fuzziomorphism $f : (\tilde{X},\tilde{\tau}) \to (X,\tau)$ is a covering if for all $x \in X$,

$$f^{\leftarrow}(N(x)) = \sum_{f(\tilde{x})=x} N(\tilde{x}) \ .$$

Evidently, if f is surjective $f_*(\tilde{\tau}) = \tau$, so that f is an identification map.

A pointed fuzmap is a covering in $\mathcal{F}uz_*$ if it defines a covering in $\mathcal{F}uz$.

7.1. Proposition

Every local fuzziomorphism $f : (\tilde{X},\tilde{\tau}) \to (X,\tau)$ is a covering.

Proof

(i) Suppose $x \in f(\tilde{X})$, $\tilde{x} \neq \tilde{x}' \in \tilde{X}$, $f(\tilde{x}) = f(\tilde{x}') = x$.

(For $f^{\leftarrow}(N(x)) = \sum\limits_{f(\tilde{x})=x} N(\tilde{x})$ trivially if $f^{\leftarrow}(x)$ is a singleton.)

If there exists $\tilde{x}'' \in N(\tilde{x}) \cap N(\tilde{x}')$, we have

$$\tilde{x} \neq \tilde{x}' \in N(\tilde{x}'') , \; f(\tilde{x}) = f(\tilde{x}')$$

so that $N(f(\tilde{x}''))|f|N(\tilde{x}'')$ is not a fuzziomorphism, contrary to assumption.

Thus $f^{\leftarrow}(N(x)) = \sum\limits_{f(\tilde{x})=x} N(\tilde{x})$ set-theoretically.

If $N(x) \cap N(x') = \emptyset$ but there exist $\bar{x} \in N(\tilde{x})$, $\bar{x}' \in N(\tilde{x}')$ such that $\bar{x} \, \tilde{\tau} \, \bar{x}'$, then

$$f(\bar{x}') \; \tau \; f(\tilde{x}') . \quad \text{(f fuzmic)}$$

But $\quad f(\tilde{x}') = f(\tilde{x})$.

$\therefore \quad f(\bar{x}') \; \tau \; f(\tilde{x})$.

But $\bar{x}'$, $\tilde{x} \in N(\bar{x})$ and $(\bar{x}',\tilde{x}) \notin \tilde{\tau}$ (since $\bar{x}' \notin N(\tilde{x})$)

$\therefore$ $N(f(\bar{x}))|f|N(\bar{x})$ is not a fuzziomorphism, contrary to assumption.

Thus $f^{\leftarrow}(N(x)) = \sum\limits_{f(\tilde{x})=x} N(\tilde{x})$ as fuzzy spaces, by I.5.2.

(ii) Suppose $x \notin f(\tilde{X})$.

Then $N(x) \cap f(\tilde{X}) = \emptyset$, for

$$x' \in N(x) \cap f(\tilde{X}) \Rightarrow x \in N(x') , \; x' = f(\tilde{x}) , \; \tilde{x} \in \tilde{X}$$
$$\Rightarrow x \in f(N(\tilde{x}))$$
$$\Rightarrow x \in f(\tilde{X}) .$$

$\therefore$ $f^{\leftarrow}(N(x)) = \emptyset = \sum\limits_{f(\tilde{x})=x} N(\tilde{x})$.

■

7.2. Proposition

If $f : \tilde{X} \to X$ is a covering, for all components $\tilde{X}'$ of $\tilde{X}$ we have $f(\tilde{X}')$ a component of X.

Proof

By connectedness $f(\tilde{X}') \subset X'$, some component of X.

Now

§ $\quad x \in X' \setminus f(\tilde{X}') \Rightarrow N(x) \cap f(\tilde{X}') = \emptyset$,

by an argument like that in (ii) of the proof of 7.1.,

$\therefore \quad x \in X' \setminus f(\tilde{X}') \Rightarrow N(N(x)) \cap f(\tilde{X}') = \emptyset$,

by § applied to each $x' \in N(x)$, and inductively

$x \in X' \setminus f(\tilde{X}') \Rightarrow X' \cap f(\tilde{X}') = \emptyset$

But $f(\tilde{X}) \subset X'$, so we have a contradiction since $\tilde{X}' \neq \emptyset$ by the definition of a component (I.5.05.)

■

7.3. Proposition

The fuzmap $f : \tilde{X} \to X$ is a covering if and only if $f(\tilde{X})|f|\tilde{X}'$ is a covering for every component $\tilde{X}'$ of $\tilde{X}$.

Proof

f is a covering if and only if

$$N(f(\tilde{x}))|f|N(\tilde{x})$$

is a fuzziomorphism for every $\tilde{x} \in \tilde{X}$, but since $N(\tilde{x}) \subset \tilde{X}'$, $N(f(\tilde{x})) \subset f(\tilde{X}')$, this holds if and only if

$$N(f(\tilde{x}'))|f|N(\tilde{x}')$$

is a fuzziomorphism for every $\tilde{x}' \in \tilde{X}'$, $N(\tilde{x}')$ and $N(f(\tilde{x}'))$ neighbourhoods of $\tilde{x}'$, $f(\tilde{x}')$ in the subspace fuzzy, for every component $\tilde{X}'$ of $\tilde{X}$.

■

As a consequence of 7.2 and 7.3 we may without loss of generality restrict attention entirely to the subcategory of connected fuzzy spaces for the rest of this section.

7.4. Proposition

If $\{f_\alpha : \tilde{X}_\alpha \to X_\alpha\}$ is an arbitrary family of coverings,

$$\prod_\alpha f_\alpha : \prod_\alpha \tilde{X}_\alpha \to \prod_\alpha X_\alpha$$

is also a covering.

Proof

It follows from I.1.43 that if $\tilde{x} \in \prod_\alpha \tilde{X}_\alpha$,

$$N(\tilde{x}) \cong \prod_\alpha N(\tilde{p}_\alpha(\tilde{x})) ,$$

$$N(\prod_\alpha f_\alpha(\tilde{x})) \cong \prod_\alpha N(p_\alpha(\prod_\beta f_\beta(\tilde{x}))) = \prod_\alpha N(f_\alpha(\tilde{p}_\alpha(\tilde{x}))) ,$$

where $\tilde{p}_\beta : \prod_\alpha \tilde{X}_\alpha \to \tilde{X}_\beta$, $p_\beta : \prod_\alpha X_\alpha \to X_\beta$ are the projections. Moreover the square

$$\begin{array}{ccc} N(\tilde{x}) & \cong & \prod_\alpha N(\tilde{p}_\alpha(\tilde{x})) \\ N(\prod_\alpha f_\alpha(\tilde{x}))|\prod_\alpha f_\alpha|N(\tilde{x}) & & \prod_\alpha \{N(f_\alpha(\tilde{p}_\alpha(\tilde{x})))|f_\alpha|N(\tilde{p}_\alpha(\tilde{x}))\} \\ N(\prod_\alpha f_\alpha(\tilde{x})) & \cong & \prod_\alpha N(f_\alpha(\tilde{p}_\alpha(\tilde{x}))) \end{array}$$

commutes. Now by a universal argument using I.1.41 the product of fuzziomorphisms is a fuzziomorphism, thus since the f_α are local fuzziomorphism so is $\prod_\alpha f_\alpha$, hence by 7.1 $\prod_\alpha f_\alpha$ is a covering. ■

7.5. Proposition

If $f : X \to Y$, $g : Y \to Z$ are fuzmaps,

f and $g \circ f$ are coverings $\iff$ f and g are coverings

$\iff$ $g \circ f$ and g are coverings.

Proof

For $x \in X$, consider
$$f' = N(f(x))|f|N(x)$$
$$g' = N(g \circ f(x))|g|N(f(x)$$
$$(g \circ f)' = N(g \circ f(x))|g \circ f|N(x) \ .$$

Evidently $(g \circ f)' = g' \circ f'$.

Categorically,

f' and $g' \circ f'$ are fuzziomorphisms $\iff$ f' and g' are fuzziomorphisms

$\iff$ $g' \circ f'$ and g' are fuzziomorphisms.

The result follows by 7.1. ■

Note. The analogues of both 7.4 and 7.5 are untrue in general in topology. (See, e.g., [25] 2.2.8 and 2.2.9.)

7.6. Definitions

7.60. A fuzmap $f: \tilde{X} \to X$ has unique path lifting (UPLift) if for paths ω, ω' in $\tilde{X}$ such that $\omega(0) = \omega'(0)$,
$$f \circ \omega = f \circ \omega' \Rightarrow \omega = \omega' \ .$$

7.61. A fuzmap $f : \tilde{X} \to X$ has the homotopy lifting property (the HoLP) if for any commutative square

$$\begin{array}{ccccc} y & & (Y,\sigma) & \xrightarrow{g} & (\tilde{X},\tilde{\tau}) \\ \rightsquigarrow & & \downarrow j & & \downarrow f \\ (y,0) & & Y\times[0,m] & \xrightarrow[H]{} & (X,\tau) \end{array}$$

of fuzmaps there exists a fuzmap $\tilde{H} : Y\times[0,m] \to \tilde{X}$ such that $\tilde{H}j = g$, $f\tilde{H} = H$.

7.62. If $f : X \to Y$ is a fuzmap, its <u>characteristic group</u> $\pi_x(f)$ <u>at</u> $x \in X$ is the subgroup $\pi f(\pi X, \{x\})$ of the object group $(\pi Y, \{f(x)\})$. If $f : (X,x_0) \to (Y,y_0)$ is a pointed fuzmap, its <u>characteristic group</u> $\pi_1(f)$ is $\pi_{x_0}(f)$.
Notice that if X is simply connected, $\pi_1(f)$ must be trivial.

7.7. Proposition

A fuzmap $f : \tilde{X} \to X$ is a covering if and only if it has UPLift and the HoLP.

Proof

A) Suppose f is a covering. Then

(i) f has UPLift, for if $\omega, \omega' : [0,m] \to X$ have $\omega(0) = \omega'(0)$, $f \circ \omega = f \circ \omega'$, inductively

$$\omega(n) = \omega'(n) \Rightarrow \{\omega(n+1), \omega'(n+1)\} \subset N(\omega(n)) \quad (\omega, \omega' \text{ fuzmic.})$$
$$\Rightarrow \omega(n+1) = \omega'(n+1) \qquad (f(N(\omega(n)) \text{ monic.})$$

(ii) Inductively f has the HoLP for general m if it has it for $m = 1$.
Now if

$$f_{\tilde{x}} = (N(f(\tilde{x}))|f|N(\tilde{x}))^{\leftarrow}$$

define $\tilde{H}(y,i) = f_{g(y)}(H(y),i)$, for $i \in \{0,1\}$.
Then $\tilde{H}j = g$ and $f\tilde{H} = H$, and $\tilde{H}$ is fuzmic since, for $\{i,i'\} \subset \{0,1\}$,

$$(y,i)\ \sigma^{\approx}(y',i') \Rightarrow (\{y,y'\} \times \{0,1\}) \subseteq \sigma^{\approx} \quad (\text{def}^n \text{ I.1.43})$$
$$\Rightarrow H(y,i) \quad H(y,i') \in N(H(y,0)) \cap N(H(y',0)) ,$$

whatever i,i' are, since H is fuzmic.

But $f_{g(y)} = f_{g(y')}$ on $N(H(y,0)) \cap N(H(y',0))$, since

$$N(H(y,0)) \cap N(H(y',0))|f|N(g(y)) \cap N(g(y'))$$

is there a two-sided inverse for both.

$\therefore\ \tilde{H}(y,i) = f_{g(y)}(H(y,i)) = f_{g(y')}(H(y,i))$

But $f_{g(y')}(H(y,i)) \,\tilde{\tau}\, f_{g(y')}(H(y',i'))$, since $f_{g}(y')$ is fuzmic.

Thus $\tilde{H}(y,i) \,\tilde{\tau}\, \tilde{H}(y',i')$, and $\tilde{H}$ is fuzmic.

B) Suppose f has UPLift and the HoLP.

For $\tilde{x} \in \tilde{X}$, $f(\tilde{x}) = x$, define $f_{\tilde{x}}$ (notation of A(ii)) as follows:

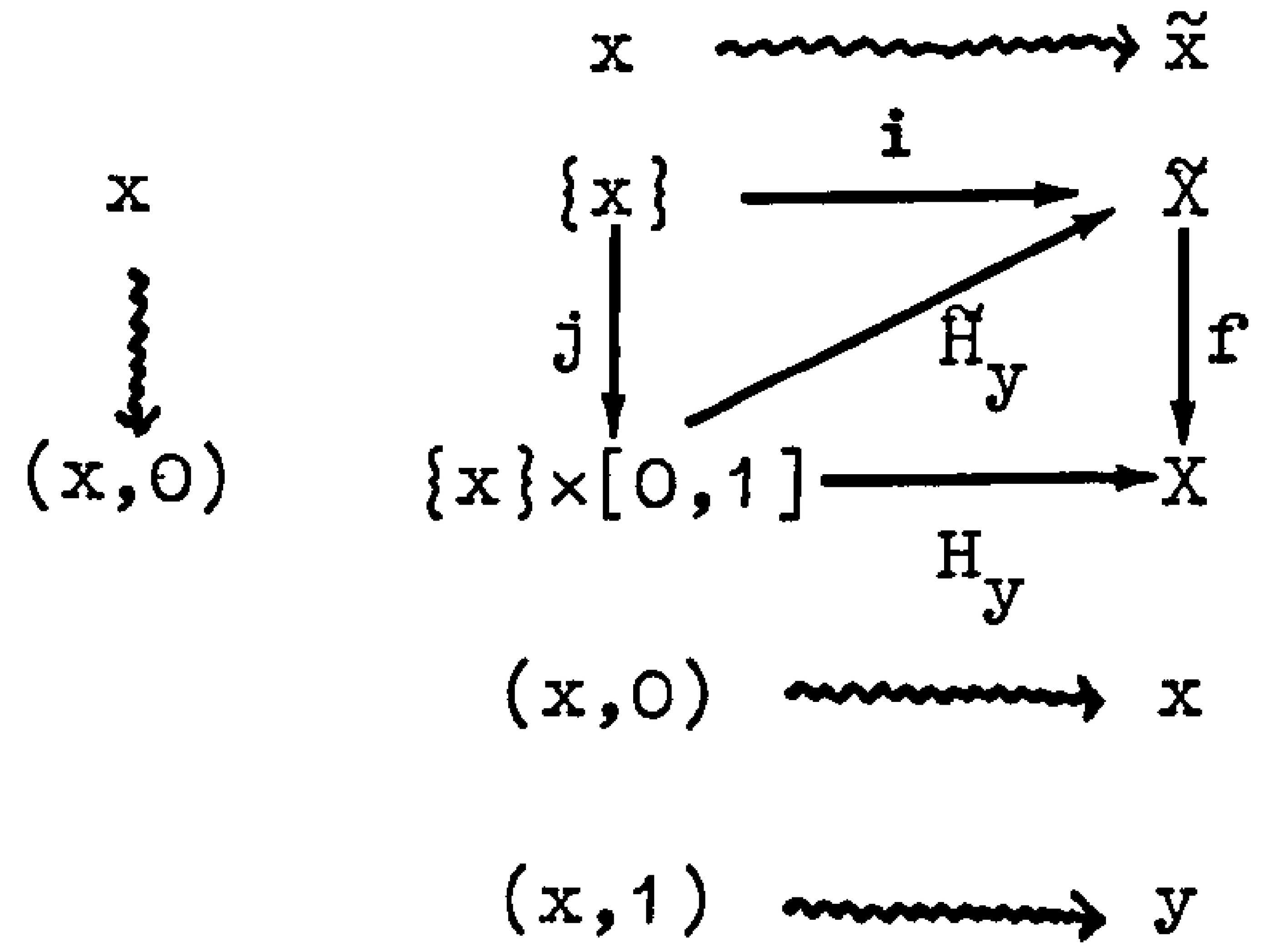
If $y \tau x$, there are fuzziomorphisms

$\varphi_y : (\{x,y\},\tau) \cong \{x\}\times[0,1]$ and $\psi_x : [0,1] \cong \{x\}\times[0,1]$.

$x \rightsquigarrow (x,0)$

$y \rightsquigarrow (x,1)$.

By the HoLP, $\exists\ \tilde{H}_y$ making



commute. Moreover $\tilde{H}_y$ is unique, for if also

$f\tilde{H}'_y = H_y$, $\tilde{H}'_y\, j = i$

then $f\tilde{H}'_y\psi_x = H_y\psi_x = f\tilde{H}_y\psi_x$

so $\tilde{H}'_y\psi_x = \tilde{H}_y\psi_x$ by UPLift

hence $\tilde{H}'_y = \tilde{H}_y$ since ψ_x is a fuzziomorphism.

Thus $f_{\tilde{x}}(y) = \tilde{H}_y\varphi_y(y)$

is well defined, and evidently a set-theoretic two-sided inverse for $N(x)|f|N(\tilde{x})$. It is fuzmic since if $y,y' \in N(x)$ have $y \tau y'$, the homotopy of length one between

$\omega : [0,1] \to X$ and $\omega' : [0,1] \to O$

$0 \rightsquigarrow x$ $\quad$ $0 \rightsquigarrow x$

$1 \rightsquigarrow y$ $\quad$ $1 \rightsquigarrow y'$

lifts by the HoLP to a homotopy of length one between

$\tilde{\omega} : [0,1] \to \tilde{X}$ (evidently a lift of ω , and

$0 \rightsquigarrow \tilde{x}$ fuzmic since $\tilde{H}_y$ is)

$1 \rightsquigarrow f_{\tilde{X}}(y)$

and a path which by UPLift can only be

$\tilde{\omega}' : [0,1] \to \tilde{X}$

$0 \rightsquigarrow \tilde{x}$

$1 \rightsquigarrow f_{\tilde{X}}(y')$

and the existence of such a homotopy is precisely equivalent to the indistinguishability of $f_{\tilde{X}}(y)$ and $f_{\tilde{X}}(y')$.

Hence $f_{\tilde{X}}$ is a fuzziomorphism, and therefore f is a covering. ∎

7.71. Corollary

For a covering $f : \tilde{X} \to X$, with $\tilde{x} \in \tilde{X}$, $f(\tilde{x}) = x$, any paths ω, ω' beginning at x lift to paths $\tilde{\omega}, \tilde{\omega}'$ beginning at $\tilde{x}$, and

$$\omega = \omega' \iff \tilde{\omega} = \tilde{\omega}'$$

$$\omega \equiv \omega' \iff \tilde{\omega} = \tilde{\omega}'$$

∎

7.72. Corollary

For a covering $f : \tilde{X} \to X$ and a connected fuzzy space Z , if $g,h : Z \to \tilde{X}$ have $f\circ g = f\circ h$ and $g(z) = h(z)$ for some $z \in X$, then $f = g$.

∎

7.73. Corollary

If the fuzmap $f : \tilde{X} \to X$ is a covering, $\pi f : \pi\tilde{X} \to \pi X$ is a covering morphism of groupoids.

Proof

7.71 imples that for any $\tilde{x} \in X$, πf is bijective between equivalence classes of paths beginning at $\tilde{x}$ and at $f(\tilde{x})$. But this is precisely the definition of covering morphism. (See [5], p.295.) ■

7.74. Corollary

For a covering $f : \tilde{X} \to X$ with X connected, if $x,y \in X$ the sets $f^{\leftarrow}(x)$, $f^{\leftarrow}(y)$ have the same cardinality. (If this is n, say, f is called an n-fold covering.)

Proof

The map f is $\mathrm{Ob}(\Pi f)$. The result follows by 7.73 and [5], 9.2.2. ■

7.75. Corollary

A covering $f : (\tilde{X},\tilde{x}_0) \to (X,x_0)$ is n-fold if and only if the index of $\pi_1(f)$ in $\pi_1(X)$ is n.

Proof

Apply 7.74 and [5], 9.4.2, Corollary (2).■

7.80. Definition

If (X,τ) is a fuzzy space and $p : G \to \pi X$ is a groupoid covering morphism, define the lifted fuzzy $p^{\dagger}(\tau)$ on $\tilde{X} = \mathrm{Ob}(G)$ as follows:

If $(x,y) \in \tau$, denote $(\{x,y\},\tau)$ by $[x,y]$.
Now $\pi[x,y]$ is connected and simply connected, thus, by [5] 9.3.3. if $p(\tilde{x}) = x$ $i : \pi[x,y] \subset \Pi X$ lifts uniquely to a morphism $\tilde{i} : \pi[x,y] \to G$ such that $\tilde{i}(x) = \tilde{x}$. If $\tilde{i}(y) = \tilde{y}$, say that $\{\tilde{x},\tilde{y}\}$ <u>lifts</u> $\{x,y\}$. Now define

$$p^{\dagger}(\tau) = \{(\tilde{x},\tilde{y}) \mid \{\tilde{x},\tilde{y}\} \text{ lifts } \{x,y\}, \text{ where } (x,y) \in \tau\}.$$

Evidently $p^{\dagger}(\tau)$ is reflexive and symmetric, and hence a fuzzy, and

$$Ob(p) : (\tilde{X}, p^{\dagger}(\tau)) \to (X,\tau)$$

is fuzmic.

7.9. Proposition

If (X,τ) is a fuzzy space

$$\Phi : [f : \tilde{X} \to X] \leadsto [\pi f : \pi\tilde{X} \to \Pi X]$$

$$\left[\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & \tilde{X}' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}\right] \leadsto \left[\begin{array}{ccc} \pi X & \xrightarrow{\pi g} & \pi X' \\ & {\scriptstyle \pi f}\searrow \quad \swarrow{\scriptstyle \pi f'} & \\ & \pi X & \end{array}\right]$$

is an equivalence between the category $\mathcal{FC}_X$ of coverings of X and the category $\mathcal{GC}_{\pi X}$ of covering morphisms of πX, with inverse

$$\Psi : [p : G \to \pi X] \leadsto [Ob(p) : (Ob(G), p^{\dagger}(\tau)) \to (X,\tau)]$$

$$\left[\begin{array}{ccc} G & \xrightarrow{r} & G' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \pi X & \end{array}\right] \leadsto \left[\begin{array}{ccc} (Ob(G), p^{\dagger}(\tau)) & \xrightarrow{Ob(r)} & (Ob(G), q^{\dagger}(\tau)) \\ & {\scriptstyle Ob(p)}\searrow \quad \swarrow{\scriptstyle Ob(q)} & \\ & X & \end{array}\right]$$

Proof

(i) If $f : \tilde{X} \to X$ is a covering in $\mathfrak{F}$uz,

$\Phi(f) = \pi f$ is a covering morphism by 7.73, and for

$\tilde{x}_1 \neq \tilde{x}_2$ (trivial otherwise),

$$\begin{aligned}
(\tilde{x}_1,\tilde{x}_2) \in \tilde{\tau} \Leftrightarrow\ & \pi[\tilde{x}_1,\tilde{x}_2]\ ,\ \pi[f(\tilde{x}_1),f(\tilde{x}_2)] \text{ connected} \\
& \text{and simply connected and } f(\tilde{x}_1) \neq f(\tilde{x}_2) \\
\Leftrightarrow\ & \pi[f(\tilde{x}_1),f(\tilde{x}_2)] \to \pi X \text{ lifts } \pi[f(\tilde{x}_1),f(\tilde{x}_2)] \subset \pi X\ , \\
& f(\tilde{x}_i) \rightsquigarrow \tilde{x}_i \qquad \text{and } f(\tilde{x}_1)\ \tau\ f(\tilde{x}_2) \\
& f(\tilde{x}_i) \to f(\tilde{x}_{i'}) \rightsquigarrow \tilde{x}_i \to \tilde{x}_{i'} \\
\Leftrightarrow\ & (\tilde{x}_1,\tilde{x}_2) \in (\pi f)^{\dagger}(\tau). \qquad (7.80.)
\end{aligned}$$

Hence $(\pi f)^{\dagger}(\tau) = \tilde{\tau}$, and thus

$$\Psi\Phi\left[\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & \tilde{X}' \\ & {\scriptstyle f}\searrow \ \swarrow{\scriptstyle f'} & \\ & (X,\tau) & \end{array}\right] = \left[\begin{array}{ccc} (\tilde{X},(\pi f)^{\dagger}(\tau)) & \xrightarrow{Ob(\pi g)} & (\tilde{X}',(\pi f')^{\dagger}(\tau)) \\ & {\scriptstyle Ob(\pi f)}\searrow \ \swarrow{\scriptstyle Ob(\pi f')} & \\ & (X,\tau) & \end{array}\right] = \left[\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & \tilde{X}' \\ & {\scriptstyle f}\searrow \ \swarrow{\scriptstyle f'} & \\ & (X,\tau) & \end{array}\right]$$

Hence $\Psi\Phi$ = Identity on $\mathfrak{F}\mathfrak{C}_X$

(ii) If $p : G \to \pi X$ is a covering morphism, for any

$\tilde{x} \in \tilde{X} = Ob(G)$ the groupoid $\pi(N(p(\tilde{x}))$ is simply

connected, so that there is a unique lift

$\tilde{i}_{\tilde{x}} : \pi N(p(x)) \to G$ of $i_{\tilde{x}} : \pi N(p(\tilde{x})) \subset \pi X$ taking

$p(\tilde{x})$ to $\tilde{x}$, monic on objects since $i_{\tilde{x}}$ is, and

$(\tilde{y},\tilde{x}) \in p^{\dagger}(\tau) \Leftrightarrow \tilde{y} \in \tilde{i}_{\tilde{x}}(N(p(\tilde{x})))$.

Thus $Ob(\tilde{i}_{\tilde{x}})$ is a set-theoretic inverse to

$N(p(\tilde{x}))|Ob(p)|N(\tilde{x})$, and is fuzmic since

$$\begin{aligned}
y,y' \in N(p(\tilde{x}))\ ,\ y\ \tau\ y' \Rightarrow\ & \{\tilde{i}_{\tilde{x}}(y),\tilde{i}_{\tilde{x}}(y')\} \text{ lifts } \{y,y'\} \\
\Rightarrow\ & (\tilde{i}_{\tilde{x}}(y),\tilde{i}_{\tilde{x}}(y')) \in p^{\dagger}(\tau)\ .
\end{aligned}$$

Hence $\Psi(p)$ is a covering morphism.

Thus also by (i) $\Phi\Psi(p)$ is a covering morphism, so if $\varphi_{\tilde{x}}$, $\psi_{\tilde{x}}$ are the bijections defined by p and $\pi(\mathrm{Ob}(p))$ respectively between those elements of G and $\pi\tilde{X}$ respectively beginning at $\tilde{x}$ and those elements of πX beginning at $p(\tilde{x})$, and I is the function taking an element of a groupoid to its initial object, define

$$\theta(G) : G \to \pi(\tilde{X}, p^{\dagger}(\tau))$$
$$\tilde{x} \rightsquigarrow \tilde{x} \quad \text{on objects}$$
$$e \rightsquigarrow \overleftarrow{\psi}_{I(e)}\varphi_{I(e)}(e) \quad \text{on elements.}$$

$\theta(G)$ is obviously a natural bijection. It is also a homomorphism since if $e : \tilde{x} \to \tilde{y}$, $e' : \tilde{y} \to \tilde{z}$,

$$\begin{aligned} \theta(G)(e'e) &= \overleftarrow{\psi}_{\tilde{x}}\varphi_{\tilde{x}}(e'e) \\ &= \overleftarrow{\psi}_{\tilde{x}}(\varphi_{\tilde{y}}(e')\ \varphi_{\tilde{x}}(e)) \quad \text{(by [5] 9.3.1.)} \\ &= \overleftarrow{\psi}_{\tilde{y}}(\varphi_{\tilde{y}}(e'))\ \overleftarrow{\psi}_{\tilde{x}}(\varphi_{\tilde{x}}(e)) \quad \text{(by [5] 9.3.1.)} \\ &= \theta(G)(e')\ \theta(G)(e) . \end{aligned}$$

Hence $\theta(G)$ is an isomorphism, so that if

$$\Theta : [G \underset{p}{\to} \pi X] \rightsquigarrow \left[\begin{array}{ccc} G & \xrightarrow{\theta(G)} & \pi(\mathrm{Ob}(G), p^{\dagger}(\tau)) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi(\mathrm{Ob}(p))} & \\ & \pi X & \end{array}\right],$$

Θ is a natural equivalence between the identity on $\mathcal{CG}_{\pi X}$ and $\Phi\Psi$. ■

7.91. Corollary

For a covering $f : \tilde{X} \to X$ in $\mathcal{F}uz$, if a fuzmap $g : Z \to X$ lifts to a fuzmap $\tilde{g} : Z \to \tilde{X}$ then $\pi g : \pi Z \to \pi X$ lifts to $\pi\tilde{g} : \pi Z \to \pi\tilde{X}$ in $\mathcal{G}pd$, and if πg lifts to $\widetilde{\pi g}$ in $\mathcal{G}pd$, g lifts to $\mathrm{Ob}(\widetilde{\pi g})$ in $\mathcal{F}uz$.

Proof

If g lifts to $\tilde{g}$, πg lifts to $\widetilde{\pi g}$ since π is a functor. If πg lifts to $\widetilde{\pi g}$, g obviously lifts in set to $Ob(\widetilde{\pi g})$, and for $z \in Z$,

$$Ob(\pi g)(N(z)) = g(N(z)) \subseteq N(g(z)) = N(\pi g(z)) \quad (g \text{ fuzmic})$$

$$\text{i.e. } Ob(f)\ Ob(\widetilde{\pi g})\ N(z) \subseteq N(\pi g(z))$$

$$Ob(\widetilde{\pi g})\ N(z) \subseteq \tilde{1}_{\widetilde{\pi g}(z)}\ N(\pi g(z))$$

$$= N(\widetilde{\pi g}(z))$$

Thus g lifts in $\mathcal{F}uz$ to $Ob(\widetilde{\pi g})$.■

7.92. Corollary

If $p : (\tilde{X},\tilde{x}_0) \to (X,x_0)$ and $q : (\tilde{Y},\tilde{y}_0) \to (X,x_0)$ are coverings in $\mathcal{F}uz_*$ with $\tilde{X}$, $\tilde{Y}$ connected, then $\pi_1(p) \subset \pi_1(q)$ if and only if there exists a pointed fuzmap $r : (\tilde{X},\tilde{x}_0) \to (\tilde{Y},\tilde{y}_0)$ such that $p = qr$. Such an r is unique, is a covering, and if $\pi_1(q) = \pi_1(p)$ is a fuzziomorphism.

Proof

Apply [5] 9.3.3. Corollary (1).■

7.93. Corollary

Every fuzzy space X has a universal cover; that is, a covering $f : \tilde{X} \to X$ such that $\tilde{X}$ is simply connected and if $g : Y \to X$ is also a covering there exists a covering $\tilde{g} : \tilde{X} \to Y$ such that $g\tilde{g} = f$. If g is also a universal cover, $\tilde{g}$ is a fuzziomorphism.

Proof

Apply 7.92 and [5] 9.4.2. Corollary (1).■

8. Homology Axioms

8.0. Definition

A homology theory $\{H_*, \partial_*\}$ in fuzzy geometry consists of

(a) a covariant functor $H_* : \mathcal{F}uzz \to \underline{\mathcal{G}}$, with values of degree 0 on morphisms, and

(b) a natural transformation ∂_* of degree (-1) from H_* to the functor $H_* \circ I$, where I is the functor

$$\mathcal{F}uzz \to \mathcal{F}uzz ,$$
$$(X,A) \rightsquigarrow (A,\emptyset)$$
$$f \rightsquigarrow f|A ,$$

which satisfy the following axioms:-

(i) Homotopy Axiom

If $f_0 \simeq f_1 : (X,A) \to (Y,B)$ then

$$H_*(f_0) = H_*(f_1) : H_*(X,A) \to H_*(Y,B) .$$

(ii) Exactness Axiom

For any pair (X,A) with inclusion maps

$$i : A \to X$$
$$j : (X,\emptyset) \to (X,A)$$

there is an exact sequence

$$\ldots \xrightarrow{\partial_{n+1}(X,A)} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\partial_n(X,A)} \ldots$$

(iii) Excision Axiom

For any pair (X,A), if $U \subseteq A$ is such that ~~$d(X,A)$~~ $d(X \setminus A, U) > 1$ (see I.6.02 (a)), then the excision map

$$e : (X \setminus U , A \setminus U) \subset (X,A)$$

induces an isomorphism of graded abelian groups

$$H_*(e) : H_*(X \setminus U , A \setminus U) \cong H_*(X,A) .$$

(iv) Dimension Axiom

On the full subcategory $\mathcal{B} \subset \mathcal{F}uz \subset \mathcal{F}uzz$ of one-point fuzzy spaces, there is a natural equivalence of H_* with the constant functor

$$O : \begin{cases} B \rightsquigarrow \{\delta_{0n} Z\}_{n \in Z} \\ (f : B \to Q) \rightsquigarrow Id(\{\delta_{0n} Z\}_{n \in Z}) \end{cases}$$

8.1. Definition

A cohomology theory $\{H^*, \partial^*\}$ in fuzzy geometry consists of

(a) a contravariant functor $H^* : \mathcal{F}uzz \to \underline{\mathcal{G}}$, with values of degree 0 on morphisms, and

(b) a natural transformation ∂^* of degree (+1) from the functor $H^* \circ I$ to H^*, where I is the functor defined in 8.0 (b),

which satisfy the following axioms:-

(i) Homotopy Axiom

If $f_0 \simeq f_1 : (X,A) \to (Y,B)$, then

$$H^*(f_0) = H^*(f_1) : H^*(Y,B) \to H^*(X,A) .$$

(ii) Exactness Axiom

For any pair (X,A) with inclusion maps

$$i : A \to X$$
$$j : (X,\emptyset) \to (X,A)$$

there is an exact sequence

$$\cdots \xrightarrow{\partial_{n-1}(X,A)} H^n(X,A) \xrightarrow{H^n(j)} H^n(X) \xrightarrow{H^n(i)} H^n(A) \xrightarrow{\partial_n(X,A)} \cdots$$

(iii) Excision Axiom

For any pair (X,A) , if $U \subset A$ is such that $d_\tau(X \setminus A, U) > 1$, then the excision map

$$e : (X \setminus U , A \setminus U) \subset (X,A)$$

induces an isomorphism of graded abelian groups

$$H^*(e) : H^*(X,A) \cong H^*(X \setminus U , A \setminus U) .$$

(iv) Dimension Axiom

On the full subcategory $\mathcal{B} \subset \mathcal{F}uz \subset \mathcal{F}uzz$ of one-point fuzzy spaces, there is a natural equivalence of H^* with the constant functor $\bigcirc$ defined in 8.0 (iv).

9. Simplicial and Cubical Functors

9.0. Definitions (Complexes)

9.00. The standard n-simplex, $[n]$, is the fuzzy space $(\{0,1,\ldots,n\},\iota)$.

9.01. The standard n-cube $[1]^n$, for $n > 0$, is the fuzzy space $[1] \times [1] \times \ldots \times [1]$ (n times).

9.02. The standard 0-cube $[1]^0$, is the fuzzy space $[0]$.

9.03. A Vietoris n-simplex of a fuzzy space (X,τ) is a fuzmap $[n] \to (X,\tau)$.

9.04. A Vietoris n-cube of a fuzzy space (X,τ) is a fuzmap $[1]^n \to (X,\tau)$.

9.05. The Vietoris simplicial complex, $S(X,\tau)$ or SX , of a fuzzy space (X,τ) is defined by

$S_nX = \{s \mid s \text{ a Vietoris n-simplex}\}$

with face operators

$$F_{jn} : S_nX \to S_{n-1}X \qquad \text{where} \quad i_{jn} : [n] \to [n+1] \ , j=0,\ldots,n+1,$$

$$s \mapsto s \circ i_{j(n-1)} \qquad\qquad x \mapsto \begin{cases} x & , x<j \\ x+1 & , x \geq j \end{cases}$$

degeneracy operators

$D_{jn} : S_nX \to S_{n+1}X$ where $d_{jn} : [n] \to [n-1]$, $j=0,\dots,n-1$,

$s \rightsquigarrow s \circ d_{j(n+1)}$ $\quad x \rightsquigarrow \begin{cases} x & ,x \leq j \\ x-1 & ,x > j \end{cases}$

and permutation operators

$T_{jn} : S_nX \to S_nX$ where $t_{jn} : [n] \to [n]$, $j=0,\dots,n-1$,

$s \rightsquigarrow s \circ t_{jn}$ $\quad x \rightsquigarrow (x + \delta_{jx} - \delta_{(j+1)x})$.

9.06. The Vietoris cubical complex, $C(X,\tau)$ or CX ,

of a fuzzy space (X,τ) is defined by

$C_nX = \{c \mid c$ a Vietoris n-cube of $X\}$

with face operators

$F^k_{jn} : C_nX \to C_{n-1}X$ where $i^k_{jn} : [1]^n \to [1]^{n+1}$, $k=0,1$, $j=1,\dots,n+1$

$c \rightsquigarrow c \circ i^k_{j(n-1)}$ $\quad (x_1,\dots,x_n) \rightsquigarrow (x_1,\dots,x_{j-1},k,x_j,\dots,x_n)$

and degeneracy operators

$D_{jn} : C_nX \to C_{n+1}X$ where $d_{jn} : [1]^n \to [1]^{n-1}$, $j=1,\dots,n$

$c \rightsquigarrow c \circ d_{j(n+1)}$ $\quad (x_1,\dots,x_n) \rightsquigarrow (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$.

9.07. If $f : X \to Y$ is fuzmic, define

$$S_n(f) : S_n(X) \to S_n(Y)$$
$$s \rightsquigarrow f \circ s \ .$$

$S = \{S_n\}_{n \in Z}$ is then clearly a functor.

9.08. If $f : X \to Y$ is fuzmic, define

$$C_n(f) : C_n(X) \to C_n(Y)$$
$$c \rightsquigarrow f \circ c$$

$C = \{C_n\}_{n \in Z}$ is then clearly a functor.

9.1. Definitions (chain complexes)

9.10. The Vietoris simplicial chain complex $\mathcal{F}^s(X)$ of a fuzzy space X consists of

$$\mathcal{F}^s_n(X) = \begin{cases} \text{Free abelian group on } S_n(X), & n \in \mathbb{N} \\ \text{Trivial abelian group } 0, & n < 0 \end{cases}$$

with boundary operators

$$\delta_n : \mathcal{F}^s_n(X) \to \mathcal{F}^s_{n-1}(X)$$

$$s \rightsquigarrow 0, \quad n \leqslant 0$$

$$\sum_{j=0}^{n} (-1)^j \overline{F_{jn}}(s), \ n > 0$$

where $\overline{F}_{jn} = F_{jn}$ on generators.

It is immediate that $\delta_n \delta_{n+1} = 0 : \mathcal{F}^s_{n+1}(X) \to \mathcal{F}^s_{n-1}(X)$, so that $\mathcal{F}^s(X) = \{\mathcal{F}^s_n(X), \delta_n\}_{n \in Z}$ is indeed a chain complex.

If $f : X \to Y$ is fuzmic, define $\mathcal{F}^s_n(f) : \mathcal{F}^s_n(X) \to \mathcal{F}^s_n(Y)$ by $\mathcal{F}^s_n(f) = S_n(f)$ on generators.

$\mathcal{F}^s$ is then clearly a functor.

9.11. The Vietoris cubical chain complex $\mathcal{F}^c(X)$ of a fuzzy space consists of

$$\mathcal{F}^c_n(X) = \begin{cases} \text{Free abelian group on } C_n(X), & n \in \mathbb{N} \\ \text{Trivial abelian group } 0, & n < 0 \end{cases}$$

with boundary operators

$$\delta_n : \mathcal{F}^c_n(X) \to \mathcal{F}^c_{n-1}(X)$$

$$c \rightsquigarrow 0, \quad n \leqslant 0$$

$$\sum_{\substack{j=0 \\ k=0,1}}^{n} (-1)^{j+k} \overline{F}^k_{jn}(c), \quad n > 0$$

where $\overline{F}^k_{jn} = F^k_{jn}$ on generators.

It is immediate that $\delta_n \delta_{n+1} = 0 : \mathcal{F}^c_{n+1}(X) \to \mathcal{F}^c_{n-1}(X)$, so that $\mathcal{F}^c(X) = \{\mathcal{F}^c_n(X)\}_{n \in Z}$ is indeed a chain complex.

If $f : X \to Y$ is a fuzmap, define $\mathcal{F}^c_n(f) : \mathcal{F}^c_n(X) \to \mathcal{F}^c_n(Y)$ by $\mathcal{F}^c_n(f) = C_n(f)$ on generators. $\mathcal{F}^c$ is then clearly a functor.

<u>9.12.</u> Now in $\mathcal{F}^s(X)$, if $s \in \mathcal{F}^s_n(X)$,

$$\begin{aligned}
\delta_{n+1}\overline{D}_{ln}(s) &= \sum_{j=0}^{n}(-1)^j(\overline{F}_{j(n+1)}\overline{D}_{ln}(s)) \\
&= \sum_{j=0}^{l-2}(-1)^j(\overline{D}_{(l-1)(n-1)}\overline{F}_{jn}(s)) \\
&\quad + \sum_{j=l+1}^{n}(\overline{D}_{l(n-1)}\overline{F}_{(j-1)n}(s)) \\
&= \overline{D}_{(l-1)(n-1)}\sum_{j=0}^{l-2}(-1)^j\overline{F}_{jn}(s) \\
&\quad + \overline{D}_{l(n-1)}\sum_{j=l+1}^{n}(-1)^j\overline{F}_{(j-1)n}(s)
\end{aligned}$$

and

$$\begin{aligned}
\delta_n(1+\overline{T}_{ln})(s) &= \sum_{j=0}^{n}(-1)^j\overline{F}_{jn}(s) + \sum_{j=0}^{n}(-1)^j\overline{F}_{jn}\overline{T}_{ln}(s) \\
&= \sum_{j=0}^{n}(-1)^j\overline{F}_{jn}(s) + \sum_{j=0}^{l-1}(-1)^j\overline{T}_{(l-1)(n-1)}\overline{F}_{jn}(s) \\
&\quad + \sum_{j=l,l+1}(-1)^{j+1}\overline{F}_{jn}(s) \\
&\quad + \sum_{j=l+2}^{n}(-1)^j\overline{T}_{l(n-1)}\overline{F}_{jn}(s) \\
&= (1+\overline{T}_{(l-1)(n-1)})\sum_{j=0}^{l-1}(-1)^j\overline{F}_{jn}(s) \\
&\quad + (1+\overline{T}_{l(n-1)})\sum_{j=l+2}^{n}(-1)^j\overline{F}_{jn}(s)
\end{aligned}$$

where $\overline{D}_{ln} = D_{ln}$, $\overline{T}_{ln} = T_{ln}$ on generators.

Thus if we set

$$\mathcal{D}_n^s X = \sum_{i=0}^{n-1} (\overline{D}_{i(n-1)}(\mathcal{J}_{n-1}^s X) + (1 + \overline{T}_{in})(\mathcal{J}_n^s X))$$

we may restrict and corestrict the boundary operators δ_n to homomorphisms $\mathcal{D}_n^s X \to \mathcal{D}_{n-1}^s X$, giving the <u>degenerate Vietoris simplicial chain complex</u> $\mathcal{D}^s X$ of X.

<u>9.13.</u> The inclusion $\mathcal{D}^s X \subset \mathcal{J}^s X$ is clearly natural with respect to restrictions of chain maps, so that $\mathcal{D}^s$ with these restricted maps as its values on morphisms is a functor, and the induced maps

$$\frac{\mathcal{J}^s X}{\mathcal{D}^s X} \to \frac{\mathcal{J}^s Y}{\mathcal{D}^s Y}$$

are well defined and functorial. Hence with the maps so induced as its values on morphisms, the <u>alternating normalised Vietoris simplicial chain complex</u> $\mathcal{A}$, where $\mathcal{A}_n X = \dfrac{\mathcal{J}_n^s X}{\mathcal{D}_n^s X}$, is a functor.

<u>9.14.</u> $\mathcal{A}_n X$ is generated by the permutation classes of the non-degenerate n-simplices, i.e. those that are injective. These classes will be called the <u>cells</u> of $\mathcal{A}_n X$.

<u>9.15.</u> Similarly in $\mathcal{J}^c X$, if $c \in \mathcal{J}_n^c X$,

$$\begin{aligned}\delta_{n+1}\overline{D}_{ln}(c) &= \sum_{\substack{j=1\\k=0,1}}^{n} (-1)^{j+k}(\overline{F}^k_{j(n+1)}\overline{D}_{ln}(c)) \\ &= \sum_{\substack{j=1\\k=0,1}}^{l} (-1)^{j+k}(\overline{D}_{(l-1)(n-1)}\overline{F}^k_{jn}(c)) \\ &\quad + \sum_{\substack{j=l+1\\k=0,1}}^{n} (-1)^{j+k}(\overline{D}_{l(n-1)}\overline{F}^k_{(j+1)n}(c))\end{aligned}$$

$$= \overline{D}_{(l-1)(n-1)} \sum_{\substack{j=1 \\ k=0,1}}^{l} (-1)^{j+k} \overline{F}_{jn}(c)$$

$$+ \overline{D}_{l(n-1)} \sum_{\substack{j=l+1 \\ k=0,1}}^{n} (-1)^{j+k} \overline{F}_{(j+1)n}(c)$$

where $\overline{D}_{ln} = D_{ln}$ on generators.

Thus if we set

$$\mathcal{D}_n^c X = \sum_{i=1}^{n} \overline{D}_{i(n-1)}(\mathcal{F}_{n-1}^c X)$$

we may restrict and corestrict the boundary operators δ_n to homomorphisms $\mathcal{D}_n^c X \to \mathcal{D}_{n-1}^c$, which we shall also denote by δ_n, giving the

<u>degenerate Vietoris cubical chain complex</u> $\mathcal{D}^c X$ of X.

<u>9.16.</u> The inclusion $\mathcal{D}^c X \subset \mathcal{F}^c X$ is clearly natural with respect to restrictions of chain maps, so that $\mathcal{D}$ with these restricted maps as its values on morphisms is a functor, and the induced maps

$$\frac{\mathcal{F}^c X}{\mathcal{D}^c X} \to \frac{\mathcal{F}^c Y}{\mathcal{D}^c Y}$$

are well defined and functorial. Hence with the maps so induced as its values on morphisms, the normalised Vietoris cubical chain complex $\mathcal{C}$, where $\mathcal{C}_n X = \dfrac{\mathcal{F}_n^c X}{\mathcal{D}_n^c X}$, is a functor.

<u>9.17.</u> $\mathcal{C}_n X$ is generated by the classes of the non-degenerate cubes, i.e. those not factoring through d_{jn} for any j. These classes will be called <u>cells</u> of $\mathcal{C}_n X$.

9.18. If $i : A \to X$ is an inclusion of fuzzy spaces, define the relative alternating normalised Vietoris simplicial chain complex $\mathcal{A}(X,A)$ of the pair (X,A) to be the chain complex $\mathcal{A}(X)/\mathcal{A}(i)(\mathcal{A}(A))$, and

9.19. Define the relative normalised Vietoris cubical chain complex $\mathcal{C}(X,A)$ of the pair (X,A) to be the chain complex $\mathcal{C}(X)/\mathcal{C}(i)(\mathcal{C}(A))$.

9.2. Definitions (homology)

If $f : (X,A) \to (Y,B)$ is a map of pairs, it is evident that the induced maps

$$\mathcal{A}(f) : \mathcal{A}(X)/\mathcal{A}(i_A)(\mathcal{A}(A)) \to \mathcal{A}(Y)/\mathcal{A}(i_B)(\mathcal{A}(B))$$

$$\mathcal{C}(f) : \mathcal{C}(X)/\mathcal{C}(i_A)(\mathcal{C}(A)) \to \mathcal{C}(Y)/\mathcal{C}(i_B)(\mathcal{C}(B))$$

are well defined and make $\mathcal{A}$, $\mathcal{C}$ functors $\mathcal{F}uzz \to \mathcal{D}$.

Thus we have short exact sequences

$$E^s(X,A) : 0 \to \mathcal{A}A \xrightarrow{\mathcal{A}(i)} \mathcal{A}X \xrightarrow{\mathcal{A}(j)} \mathcal{A}(X,A) \to 0$$

$$E^c(X,A) : 0 \to \mathcal{C}A \xrightarrow{\mathcal{C}(i)} \mathcal{C}X \xrightarrow{\mathcal{C}(j)} \mathcal{C}(X,A) \to 0$$

and splitting maps

$$\mathcal{A}X \to \mathcal{A}A \ , \ \mathcal{C}X \to \mathcal{C}A$$

well defined on generators by

$$cls(s) \rightsquigarrow \begin{cases} cls(s) & , s \in SA \\ 0 & , s \in SX \backslash SA \end{cases} \quad \text{and } cls(c) \rightsquigarrow \begin{cases} cls(c) & , c \in CA \\ 0 & , c \in CX \backslash CA \end{cases}$$

respectively, functorial covariantly on $\mathcal{F}uzz$.

Hence if $H_* : \mathcal{D} \to \mathcal{G}$, $H^* : \mathcal{D} \to \mathcal{G}$ are the usual algebraic homology and cohomology groups respectively, and $\partial_*(E)$, $\partial^*(E)$ the connecting homomorphisms defined for the homology and cohomology respectively of a split short exact sequence E of chain complexes (cf., e.g., [19] pp 44-48) we may define the following:

<u>9.21.</u> (Alternating simplicial homology)

$$H_*^s(X,A) = H_*(\mathcal{A}(X,A))$$
$$H_*^s(f) = H_*(\mathcal{A}(f))$$
$$\partial_*^s(X,A) = \partial_*(E^s(X,A))$$

<u>9.22.</u> (Alternating simplicial cohomology)

$$H_s^*(X,A) = H^*(\mathcal{A}(X,A))$$
$$H_s^*(f) = H^*(\mathcal{A}(f))$$
$$\partial_s^*(X,A) = \partial^*(E^s(X,A))$$

<u>9.23.</u> (Cubical homology)

$$H_*^c(X,A) = H_*(\mathcal{C}(X,A))$$
$$H_*^c(f) = H_*(\mathcal{C}(f))$$
$$\partial_*^c(X,A) = \partial_*(E^c(X,A))$$

<u>9.24.</u> (Cubical cohomology)

$$H_c^*(X,A) = H^*(\mathcal{C}(X,A))$$
$$H_c^*(f) = H^*(\mathcal{C}(f))$$
$$\partial_c^*(X,A) = \partial^*(E^c(X,A)) .$$

<u>9.30. Definition</u>

Two maps $f_0 , f_1 : SX \to SY$ are <u>s-contiguous</u> if for each $n \in \mathcal{N}$, $s \in S_nX$, $\exists F_n(s) \in S_{n+m}(Y)$ s.t. $f_j(s) = F_n(s) \circ i_j$, $j = 0,1$, for some injective $i_0 , i_1 : [n] \to [n+m]$.

<u>9.31. Definition</u>

Two maps $f_0 , f_1 : CX \to CY$ are <u>c-contiguous</u> if for each $n \in \mathcal{N}$, $c \in C_nX$, $\exists F_n(c) \in C_{n+1}Y$ s.t. $f_j(c) = F_n(c) \circ i_{nn}^j$, $j = 1,2$. (cf. 9.06).

9.4. Proposition

If f_0 , f_1 : $(X,A) \to (Y,B)$ induce s-contiguous maps $SX \to SY$ then

$$H_*^s(f_0) = H_*^s(f_1) : H_*^s(X,A) \to H_*^s(Y,B)$$

and $$H_s^*(f_0) = H_s^*(f_1) : H_s^*(X,A) \to H_s^*(Y,B) .$$

Proof

$\mathcal{F}^s(f_0)$, $\mathcal{F}^s(f_1)$ are chain homotopic (by, e.g., [25] p.171, Th^m 9).

Thus by the naturality of alternatisation, normalisation and relativisation $\mathcal{A}(f_0)$, $\mathcal{A}(f_1)$ are chain homotopic.

Therefore $H_*(\mathcal{A}(f_0)) = H_*(\mathcal{A}(f_1))$

i.e. $H_*^s(f_0) = H_*^s(f_1)$

and $H^*(\mathcal{A}(f_0)) = H^*(\mathcal{A}(f_1))$

i.e. $H_s^*(f_0) = H_s^*(f_1)$. ∎

9.5. Proposition

If f_0 , f_1 : $(X,A) \to (Y,B)$ induce c-contiguous maps $CX \to CY$ then

$$H_*^c(f_0) = H_*^c(f_1) : H_*^c(X,A) \to H_*^c(Y,B)$$

and $$H_c^*(f_0) = H_c^*(f_1) : H_c^*(X,A) \to H_c^*(Y,B) .$$

Proof

Let $F_n : C_nX \to C_{n+1}Y$ be as in Def^n 9.31, and define $\bar{F}_n : \mathcal{F}_n^cX \to \mathcal{F}_{n+1}^cY$ by $\begin{cases} \bar{F}_n = (-1)^n F_n \text{ on generators, } n \geq 0 \\ \bar{F}_n = 0 \text{ , } n < 0 \text{ .} \end{cases}$

Then $$\bar{F}^k_{j(n+1)}\bar{F}_n(c) = \left.\begin{cases} (-1)^n f_k \circ c, j=n+1, k=0,1 \\ -\bar{F}_{n-1}\bar{F}^k_{jn}(c), j=1,\ldots,n, k=0,1 \end{cases}\right\} \text{ on generators.}$$

so $(\delta_{n+1}\overline{F}_n + \overline{F}_{n-1}\delta_n)(c) = \sum_{\substack{j=1\\k=0,2}}^{n} (-1)^{j+k}\overline{F}^k_{j(n+1)}\overline{F}_n(c)$

$$+ \overline{F}_{n-1} \sum_{\substack{j=1\\k=0,1}}^{n} (-1)^{j+k}\overline{F}^k_{jn}(c)$$

$$= (f_1 \circ c) - (f_0 \circ c) + \sum_{\substack{j=1\\k=0,1}}^{n} (-1)^{j+k+1}\overline{F}_{n-1}\overline{F}^k_{jn}(c) + \overline{F}_{n-1} \sum_{\substack{j=1\\k=0,1}}^{n} (-1)^{j+k}\overline{F}^k_{jn}(c)$$

$= \mathcal{F}^c(f_1)(c) - \mathcal{F}^c(f_0)(c)$ on generators.

Thus $\delta_{n+1}\overline{F}_n + \overline{F}_{n-1}\delta_n = \mathcal{F}^c(f_1) - \mathcal{F}^c(f_0)$

Hence $\overline{F} = \{\overline{F}_n\}_{n\in Z}$ is a chain homotopy $\mathcal{F}^c(f_1) \to \mathcal{F}^c(f_2)$

Therefore by the naturality of normalisation and relativisation, $\mathscr{C}(f_0)$ and $\mathscr{C}(f_1)$ are chain homotopic.

Therefore $H_*(\mathscr{C}(f_0)) = H_*(\mathscr{C}(f_1))$

i.e. $H^c_*(f_0) = H^c_*(f_1)$

and $H^*(\mathscr{C}(f_0)) = H^*(\mathscr{C}(f_1))$

i.e. $H^*_c(f_0) = H^*_c(f_1)$. ∎

9.6. Proposition

$\{H^s_*, \partial^s_*\}$ is a homology theory.

Proof

(a) $H^s_* = H_* \circ \mathcal{A}$, and is thus a functor.

(b) E^s : $\mathcal{F}uzz \to$ (category S of short exact sequences in $\mathcal{Q}$) is a functor, and if we define

$$J : S \to \mathcal{Q} \qquad \text{and} \qquad K : S \to \mathcal{Q}$$

$$(A' \to A \to A'') \rightsquigarrow A' \qquad (A' \to A \to A'') \rightsquigarrow A''$$

$$f \rightsquigarrow f|A' \qquad f \rightsquigarrow f|A''$$

Then ∂_* is a natural transformation,

$H_* \circ K \to H_* \circ J$,

of degree (-1) .

Hence $\partial_*^S = \partial_* \circ E^S$ is a natural transformation

$H_*^S = H_* \circ K \circ E^S \to H_* \circ J \circ E^S = H_*^S \circ I$

of degree (-1) .

(i) Homotopy Axiom:

If $f_0 \simeq f_1 : (X,A,\tau) \to (Y,B,\sigma)$, $\exists$ a path $F : f_0 \to f_1$ in the function space $(Y,B)^{(X,A)}$ such that $f_1 = F(m)$, say.

For each $i = 0 ,\ldots, m-1$,

$$F(i)\sigma^{\tau} F(i+1)$$

so if $s \in S_n X$

$j,k \in [n] \Rightarrow s(j) \ \tau \ s(k) \Rightarrow F(i)(s(j)) \ \sigma \ F(i+1)(s(k))$

Thus $F_n(s) : [2n+1] \to Y$

$$j \rightsquigarrow \begin{cases} F(i)(s(j)) & , j \leq n \\ F(i+1)(s(j-n-1)) & , j > n \end{cases}$$

is fuzmic, and so $F_n(s) \in S_{2n+1}Y$.

But if we take

$$i_k : [n] \to [2n+1] \ , \ k = 0,1$$
$$j \rightsquigarrow j - (n+1)\delta_{0k}$$

we have

$$(s(F(i)))(s) = F_n(s) \circ i_0$$

and $(s(F(i+1)))(s) = F_n(s) \circ i_1$.

Thus $s(F(i)), s(F(i+1))$ are contiguous for all i .

Thus $H_*^S(F(i)) = H_*^S(F(i+1))$ for all i (by 9.4).

Hence $H_*^S(f_0) = H_*^S(F(0)) = H_*^S(F(m)) = H_*^S(f_1)$.

(ii) Exactness Axiom:

The sequence

$$\cdots \xrightarrow{\partial^s_{q+1}(X,A)} H^s_q(A) \xrightarrow{H^s_q(i)} H^s_q(X) \xrightarrow{H^s_q(j)} H^s_q(X,A) \xrightarrow{\partial^s_q(X,A)} \cdots$$

is precisely the sequence

$$\cdots \xrightarrow{\partial_{q+1}(E^s(X,A))} H_q(\mathcal{A}A) \xrightarrow{H_q(\mathcal{A}(i))} H_q(\mathcal{A}X) \xrightarrow{H_q(\mathcal{A}(j))} H_{q}(\mathcal{A}(X,A)) \xrightarrow{\partial_q(E^s(X,A))} \cdots$$

which is exact by [19] p.45, Th^m4.1.

(iii) Excision Axiom:

For (X,A) , $U \subset A$ s.t. $d(X \setminus A, U) > 1$, consider $s \in S_nX$ s.t. $s(i) \in U$, for some $i \in [n]$.

We have then $s([n]) \subset A$, for if $\exists\, j \in [n]$ s.t. $s(j) \in X \setminus A$, since $i \iota j$ and s fuzmic we have $s(i) \tau s(j)$, contradicting $d(X \setminus A, U) > 1$.

i.e. $\quad s' \in S_nX \setminus S_nA \Rightarrow s'([n]) \cap U = \emptyset$

Thus $\quad S_nX \setminus S_nA \subset S_n(X \setminus U)$.

Hence $\quad S_nX \subset S_n(X \setminus U) \cup S_nA$.

But trivially $\quad S_nX \supset S_n(X \setminus U) \cup S_nA$.

Hence $\quad S_nX = S_n(X \setminus U) \cup S_nA$.

Also

$$\begin{aligned} s \in S_n(A \setminus U) &\Leftrightarrow s([n]) \subseteq A\,,\ s([n]) \cap U = \emptyset \\ &\Leftrightarrow s \in S_nA\,,\ s \in S_n(X \setminus U) \\ &\Leftrightarrow s \in S_n(X \setminus U) \cap S_nA\,. \end{aligned}$$

Hence $\quad S_n(A \setminus U) = S_n(X \setminus U) \cap S_nA$.

So the induced map

$s(e) : (s(X \setminus U), s(A \setminus U)) \to (SX, SA)$

is the inclusion

$(s(X \setminus U), s(X \setminus U) \cap SA) \subset (s(X \setminus U) \cup SA, SA)$

and hence by [ ] p.165, Th^m3.6,

$\mathcal{F}^s(e) : \mathcal{F}^c(X \setminus U)/\mathcal{F}^c(A \setminus U) \to \mathcal{F}^c(X)/\mathcal{F}^c(A)$

is an isomorphism, and by the naturality of alternatisation and normalisation

$$\mathcal{A}(e) : \mathcal{A}(X \setminus U , A \setminus U) \to \mathcal{A}(X,A)$$

is also an isomorphism. Therefore

$H_*^s(e) = H_*(\mathcal{A}(e)) : H_*^s(X \setminus U , A \setminus U) \to H_*^s(X,A)$

is an isomorphism, since H_* is a functor.

(iv) Dimension Axiom:

If P is a one-point fuzzy space and $s \in S_n(P)$, either $n = 0$, or s is not injective and hence $s \in \mathcal{D}_n^s(P)$. Thus for $n \neq 0$ there are no cells in $\mathcal{A}_n(P)$, so that $\mathcal{A}_n(P) = 0$.

There is a unique singular 0-simplex s_0, which is non-degenerate, and there are no T_{j0} 's, so that

$$\mathcal{F}_0^s(P) \cong Z$$
$$\mathcal{D}_0^s(P) \cong 0$$

and hence $\mathcal{A}_0(P) \cong Z$.

Thus
$$\begin{aligned} H_*^s(P) &= H_*(\mathcal{A}(P)) \\ &= H_*(\{\delta_{0n} Z\}_{n \in Z}) \\ &\cong \{\delta_{0n} Z\}_{n \in Z} \\ &= \bigcirc(P) . \end{aligned}$$

All morphisms in $\mathcal{B}$ are isomorphisms specified completely by their end-points, hence the isomorphism is natural. Thus we have a natural equivalence $H_*^s \to \bigcirc$ on $\mathcal{B}$. ■

By precisely similar proofs:

<u>9.7. Proposition</u>

$\{H_s^*, \partial_s^*\}$ is a cohomology theory.■

<u>9.8. Proposition</u>

$\{H_*^c, \partial_*^c\}$ is a homology theory.■

<u>9.9. Proposition</u>

$\{H_c^*, \partial_c^*\}$ is a cohomology theory.■

Note

H_*^S is far from being the only choice possible of a simplicial homology functor. For example the Dowker homology [7] of the fuzzy as a relation makes an n-simplex a set of (n+1) points indistinguishable from a common point, and this is the definition preferred by Zeeman. The axioms 8.0 and 8.1 hold for the resulting theory if "$d(X \setminus A,U) > 1$" in the excision axioms is replaced by "$d(X \setminus A,U) > 2$" . However the uniqueness in 11.31 and 12.32 is lost, since the theory arising from an n-simplex being a set of (n+1) points any two of which are indistinguishable from a common point (not necessarily the same for different pairs) also satisfies the same modified axioms, and gives different homology groups on, for example, $[0,4]/\{0,4\}$. This latter theory is of course simply H_*^S applied to the fuzzy obtained by 'doubling' the fuzzy on the space given. If one defines

$$x \ \tau^n \ x' \Leftrightarrow d(x,x') \leqslant n$$

a sequence of fuzzy spaces may be obtained in this manner, and hence a sequence of graded abelian groups results from applying either H_*^S or the Dowker homology. Zeeman [34] has a class of tolerances on the collection of such sequences of graded groups, and the result that if there is a tolerance homeomorphism (an embedding $f : X \to Y$ s.t. $N(f(X)) = Y$) between X and Y then their sequences of Dowker homologies are within tolerance, if X and Y are comfortably large.

(The large size is essential: for all (X,τ), the embedding

$(X,\tau) \subset (X \cup \{p\}, \tau \cup (X \times \{p\}) \cup (\{p\} \times X)) = Y$

is a tolerance homeomorphism, and Y is contractible and has trivial homology in any theory. But its internal diameter is only 2 hops.) This result is the refinement, replacing the theorem that a tolerance homeomorphism induces an isomorphism in homology, referred to in [13], p.151.

10. Axiomatic Homology

10.00. Definition

A proper triad $(X;X_1,X_2)$ of fuzzy spaces consists of a fuzzy space X and two subspaces X_1, X_2 such that $d(X_1 \setminus X_2, X_2 \setminus X_1) > 1$.

If $X = X_1 \cup X_2$, then $(X;X_1;X_2)$ is a Mayer-Vietoris triad.

With the obvious maps, proper triads form the category $\mathcal{F}uzzz$.

10.1 Proposition

For a proper triad $(X;X_1,X_2)$ the diagram

$$H_*(X_1,X_1 \cap X_2) \to H_*(X_1 \cup X_2, X_1 \cap X_2) \leftarrow H_*(X_2,X_1 \cap X_2)$$

is a direct sum, and the diagram

$$H^*(X_1,X_1 \cap X_2) \leftarrow H^*(X_1 \cup X_2,X_1 \cap X_2) \to H^*(X_2,X_1 \cap X_2)$$

is a direct product, in any homology and cohomology theories $\{H_*,\partial_*\}$, $\{H^*,\partial^*\}$, where the maps in each diagram are those induced by inclusions.

10.2. Proposition

For a proper triad $(X;X_1,X_2)$ there are exact sequences

$$\cdots \to H_q(X_1,X_1\cap X_2) \to H_q(X,X_2) \to H_q(X,X_1\cup X_2) \to H_{q-1}(X_1,X_1\cap X_2) \to \cdots$$

$$\cdots \leftarrow H^q(X_1,X_1\cap X_2) \leftarrow H^q(X,X_2) \leftarrow H^q(X,X_1\cup X_2) \leftarrow H^{q-1}(X_1,X_1\cap X_2) \leftarrow \cdots$$

and for a Mayer-Vietoris triad $(X;X_1,X_2)$ there are exact sequences

$$\cdots \to H_q(X_1\cap X_2) \to H_q(X_1) \oplus H_q(X_2) \to H_q(X) \to H_{q-1}(X_1\cap X_2) \to \cdots$$

$$\cdots \leftarrow H^q(X_1\cap X_2) \leftarrow H^q(X_1) \oplus H_q(X_2) \leftarrow H^q(X) \leftarrow H_{q-1}(X_1\cap X_2) \leftarrow \cdots$$

Proofs of 10.1, 10.2:

The inclusions

$$(X_1,X_1 \cap X_2) \to (X_1 \cup X_2,X_2)$$
$$(X_2,X_1 \cap X_2) \to (X_1 \cup X_2,X_1)$$

induce isomorphisms in homology and cohomology by the excision axiom. The results follow by the same algebraic manipulation as in the topological case. ■ ■

10.3. Definitions

10.31. For any fuzzy space X define the suspension SX of X to be the fuzzy space

$$[0,2] \times X/_{\{0,2\} \times X} .$$

10.32. Iteratively, the n-fold suspension S^nX of X of X is $\underbrace{S(S(\ldots S(}_{n \text{ times}}X)\ldots))$,

10.33. The one-point cones C_0X , $C_2 \subset SX$ on X are the fuzzy spaces $[0,1] \times X/_{\{0\} \times X}$ and $[1,2] \times X/_{\{2\} \times X}$ respectively.

Evidently $C_0X \cong C_2X$, and they are contractible.

10.34. Define the fuzzy space $S^0 = (\{0,2\},\delta)$, and inductively $S^n = S^n(S^0)$.

Evidently $n(S^m) = 2(m+1)$.

10.4. Proposition

For all fuzzy spaces X ,

$H_q(SX) \cong H_{q-1}(X)$, $H^q(SX) \cong H^{q-1}(X)$,

and hence by iteration

$H_q(S^nX) \cong H_{q-n}(X)$, $H^q(S^nX) \cong H^{q-n}(X)$.

Proof

As in topology, using the Mayer-Vietoris triad $(SX;C_0X,C_1X)$. ■

10.41. Corollary.

$H_q(S^n) \cong \delta_{nq}Z \oplus \delta_{oq}Z \cong H^q(S^n)$.■

10.42. Note

These propositions, in particular 10.41, have been included as examples of the manner in which homology and cohomology measure similar things ('number of n-dimensional holes') to those they measure in topology, and may be computed in similar ways. However, the 'non-stretching' character of $\mathcal{F}uz$ again gives rise to differences: there are precisely eight essential maps $S^1 \to S^1$, no two of which are homotopic, so that the representation of cohomology by maps into an Eilenberg-MacLane space cannot be set up. Similarly ΩX and SX are in no sense dual.

10.5. Proposition

For a fuzzy space X with subspaces $X_1, \ldots, X_r, A$, such that

$$X_1 \cup X_2 \cup \ldots \cup X_r \cup A = X$$
$$d(X_i \setminus A, X_j \setminus A) > 1 \ , \ i \neq j$$
$$d(X_i \setminus A \ , \ A \setminus X_i) > 1 \ , \ \forall i$$

the sets of maps

$$\{H_*(X_i, X_i \cap A) \to H_*(X,A)\}$$
$$\{H^*(X,A) \to H^*(X_i, X_i \cap A)\}$$

induced by inclusions form a direct sum and product respectively, for any homology and cohomology theories $\{H_*, \partial_*\}$, $\{H^*, \partial^*\}$.

Proof

The results follow inductively from the excision axiom and 10.1, 10.2 by the same argument as in [10] the analogous results (Theorems III.2.3 and III.2.3.c) follow from Theorems III.2.1 and I.14.2, III.2.1.c and I.14.2.c respectively. ■

10.6. Proposition

For any fuzzy pair (X,A) , if $B \subseteq X$ has $N(B)$ contractible and either $N(B) \cap A$ contractible or $A \cap B$ empty then the identification map

$$\eta: (X,A) \to (X/_B \ , \ A/_{B \cap A})$$

induces an isomorphism in homology for any theory $\{H_*, \partial_*\}$.

Proof

Consider first the case $A = \emptyset$.

We have a commutative diagram

$$\begin{array}{ccc}
H_n(X) & \xrightarrow{H_n(\eta)} & H_n({}^{X}/_{B}) \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j'_*} \\
H_n(X, N(B)) & \xrightarrow{H_n(\bar{\eta})} & H_n({}^{X}/_{B} \, , \, {}^{N(B)}/_{B}) \\
\uparrow{\scriptstyle e_*} & & \uparrow{\scriptstyle e'_*} \\
H_n(X \setminus B, N(B) \setminus B) & \xrightarrow{H_n(\bar{\bar{\eta}})} & H_n({}^{X}/_{B} \setminus \{[B]\} \, , \, {}^{N(B)}/_{B} \setminus \{[B]\})
\end{array}$$

where $\bar{\eta}$, $\bar{\bar{\eta}}$ are defined by η , and the vertical maps are induced by inclusions.

Now:

$N(B)$ is contractible, so j_* is an isomorphism.

${N(B)}/_{B}$ is a one-point cone on $N(B) \setminus B$ and is hence contractible, so j'_* is an isomorphism.

e_* , e'_* are induced by excisions satisfying the conditions of the excision axiom, so e_* , e'_* are isomorphisms.

$\bar{\bar{\eta}}$ is a fuzziomorphism so $H_n(\bar{\bar{\eta}})$ is an isomorphism.

Hence $H_n(\eta)$ is an isomorphism.

The general result follows by application of the 5 Lemma to the map of homology exact sequences induced by η . ∎

Dually:-

10.7. Proposition

For any fuzzy pair (X,A) , if $B \subset X$ has $N(B)$ contractible and either $N(B) \cap A$ contractible or $A \cap B = \emptyset$ then the identification map

$$\eta : (X,A) \to (X/B\, ,\, A/B \cap A)$$

induces an isomorphism in cohomology for any theory $\{H^*, \partial^*\}$. ∎

11. Natural Equivalences and Finitude.

11.0. Proposition

There is a natural equivalence

$$\{H_*^S, \partial_*^S\} \cong \{H_*^C, \partial_*^C\}$$

on the category $\mathcal{F}uzz$.

Proof

If $A \subset (X,\tau)$, define the discriminator $D(A)$ of A in X by

$$D(A) = \{x | x \in X\, ,\ x \,\tau\, a\ \forall\, a \in A\} .$$

Now if c is a cell of $\mathscr{C}(X)$, there is a unique non-degenerate Vietoris cube c' of x s.t. $c' \in C$. Define $|c|$ = image (c') .

If s is a cell of $\mathcal{A}(X)$, and s' , s'' are injective Vietoris simplices of X such that $s' \in S$, $s'' \in S$, then $\mathrm{im}(s') = \mathrm{im}(s'')$, so that we may unambiguously define

$$|s| = \mathrm{im}(s')$$

where s' is any injective singular simplex of X such that $s' \in S$.

Define

$\mathcal{F}X = \{0\} \cup \{s \otimes c \mid s \text{ a cell of } \mathcal{A}X,\ c \text{ a cell of } \mathcal{C}X,\ |s|\times|c| \subseteq \tau\}$

where $\mathcal{F}X \subset \mathcal{A}(X) \otimes \mathcal{C}(X)$

(i) Clearly

$s \otimes c \in \mathcal{F}X$, s' a face of s, c' a face of $c \Rightarrow s' \otimes c' \in \mathcal{F}X$.

(ii) If $f : (X,\tau) \to (Y,\sigma)$ is a fuzmap

$s\otimes c \in \mathcal{F}X \Rightarrow |s|\times|c| \subseteq \tau$

$\Rightarrow (f,f)(|s|\times|c|) \subset \sigma$

$\Rightarrow |\mathcal{A}(f)(s)|\times|\mathcal{C}(f)(c)| \subseteq \sigma$, or $f_o s'$ or $f_o c'$ (s',c' as above) is degenerate, so that $\mathcal{A}(f)(s)$ or $\mathcal{C}(f)(c)$ is not a cell.

$\Rightarrow |\mathcal{A}(f)(s)|\times|\mathcal{C}(f)(c)| \subset \sigma$, or $\mathcal{A}(f)(s) = 0$, or $\mathcal{C}(f)(c) = 0$

$\Rightarrow (\mathcal{A}(f)(s)) \otimes (\mathcal{C}(f)(c)) \in \mathcal{F}Y$.

(iii) If s is a cell of $\mathcal{A}X$, consider

$\mathcal{F}(s) = \{c \in \mathcal{C}X \mid s \otimes c \in \mathcal{F}X\}$

$= \{0\} \cup \{c \mid \tau \supseteq |s|\times|c|\}$

$= \{0\} \cup \{c \mid c \text{ a cell of } \mathcal{C}(D|s|,\tau)\}$

Therefore $\mathcal{F}(s)$ generates $\mathcal{C}(D|s|,\tau)$. But $(D|s|,\tau)$ is a one-point cone on $D|s| \setminus \{x\}$ for any $x \in |s|$, and hence contractible. Therefore since $H_*^C = H_* \circ \mathcal{C}$ is a homology theory $\mathcal{C}(D|s|,\tau)$ is acyclic.

Similarly if c is a cell of $\mathcal{C}X$, then

$\mathcal{F}(c) = \{s \in \mathcal{A}X \mid s\otimes c \in \mathcal{F}X\}$

generates $\mathcal{A}(D|s|,\tau)$, which is acyclic by a similar argument.

Hence by (iii), (ii), (i) $\mathcal{F}$ is an acyclic functorial facing relation, in the sense of [30], and by Lemma 3 of that paper there is a natural equivalence

$$\{H_*^S, \partial_*^S\} \cong \{H_*^C, \partial_*^C\}$$

on the category $\mathcal{F}uzz$. ■

11.01. Corollary

There is a natural equivalence

$$\{H_s^*, \partial_s^*\} \cong \{H_c^*, \partial_c^*\}$$

on the category $\mathcal{F}uzz$.

Proof

This also follows from the facing relation $\mathcal{F}$, by the dual of the lemma used above. ■

11.10. Definition

For any fuzzy space (X,τ) define the simplicial space KX of X as follows:

Let $K_nX = \{\{x_o,\ldots,x_n\} | x_i \tau x_j \ \forall\ i,j \in \{0,\ldots,n\}, x_i=x_j \Rightarrow i=j\}$

Then $KX = (\bigcup_{n=0}^{\infty} K_nX, \sigma)$, where

$$(k,k') \in \sigma \Leftrightarrow k \subset k' \text{ or } k' \subset k.$$

If $f : X \to Y$ is a fuzmap, define

$$Kf : KX \to KY$$

$$\{x_o,\ldots,x_n\} \rightsquigarrow \{f(x_o),\ldots,f(x_n)\}$$

(Note that $Kf(K_nX) \not\subset K_nY$ in general.)

K is then a functor

$$\mathcal{F}uz \to \mathcal{F}uz$$

and extends canonically to a functor

$$\mathcal{F}uzz \to \mathcal{F}uzz \ .$$

Define similarly

$$K^qX = (\bigcup_{n=0}^{q} K_nX, \sigma) \subset KX$$

$$K^q(X,A) \subset K(X,A) = (KX,KA)$$

$$K^qf : K^qX \to K^qY \ .$$

11.2. Proposition

If (X,A,τ) is a finite fuzzy pair, and X is given a total ordering (not necessarily a total fuzzy order, as in I.7.00), the map

$$q : (KX,KA) \to (X,A)$$
$$k \leadsto \min(k)$$

is fuzmic and induces an isomorphism in homology, independent of the order on X, for any theory $\{H_*, \partial_*\}$.

Proof

If $k \,\sigma\, k'$, either $k \subset k'$ or $k' \subset k$.

Say $k \subset k'$, then $\min(k) \in k'$, so $\min(k) \,\tau \min(k')$ since $\{\min(k), \min(k')\} \subset k'$, i.e. $q(k) \,\tau\, q(k')$.

Thus q is fuzmic.

If $X = \{x_1,\ldots,x_m\}$ is the ordering on X, define

$$K^jX = KX/\rho_j \quad , \quad K^jA = KA/\rho_j \cap (KA \times KA)$$

where $\rho_j = \{(k,k') \mid k=k', \text{ or } \min(k) = \min(k') \in \{x_1,\ldots,x_j\}\}$, for $j = 1,\ldots,m$,

and $\rho_0 = \{(k,k') \mid k = k'\}$.

Evidently ρ_j is an equivalence relation, $j = 0,\ldots,m$, so that K^jX, K^jA are well defined, and

$$K^mX \cong X \ , \ K^mA \cong A \ ,$$
$$K^0X \cong KX \ , K^0A \cong KA \ .$$

Denote the ρ_j-equivalence class of $k \in KX$ by $[k]_j$, for $j = 0,\ldots,m$, and define

$$\eta_j : K^{j-1}X \to K^jX \ , \text{ for } j = 1,\ldots,m \ .$$
$$[k]_{j-1} \quad [k]_j$$

Then if $B_jX = \{k \in KX | \min(k) = x_j\} \subset KX$,
$B_jX \subset K^{j-1}X$ naturally,

and η_j is the identification map
$K^{j-1}X \to K^{j-1}X/_{B_jX} = K^jX$.

Now in $K^{j-1}X$,

$N(B_j)$ is the set-theoretic disjoint union of

B_j ,

$B_j' = \{x_i | x_i \ \tau \ x_j \ , \ i < j\}$, and

$B_j'' = \{k | x_j < \min(k) \ , \ k \subset k' \text{ for some } k' \in B_j\}$,

though not the disjoint union in $\mathcal{F}uz$.

Define $f_j : N(B_j) \to N(B_j)$ to be the identity on B_j and B_j' , and the map

$$k \leadsto k \cup \{x_j\}$$

on B_j'' . Then f_j is fuzmic and

$$1_{N(B_j)} \ \tau^\tau \ f_j \ , \ f_j \ \tau^\tau \ C_j \ ,$$

where

$$C_j : N(B_j) \to \{x_j\} \ ,$$

so that $N(B_j)$ is contractible.

If $x_j \in A$, then similarly

$$1_{N(B_j) \cap A} \ \tau^\tau \ (f_j|A) \ , \ (f_j|A) \ \tau^\tau \ (C_j|A)$$

so that $N(B_j) \cap A$ is also contractible.

If $x_j \notin A$, $B_j \cap K^{j-1}A = \emptyset$.

Thus by 10.6, for any j ,

$H_*(\eta_j) : H_*(K^{j-1}X \ , \ K^{j-1}A) \to H_*(K^jX \ , \ K^jA)$

is an isomorphism. Thus

$H_*(q) = H_*(\eta_m\eta_{m-1}\cdots\eta_1) = H_*(\eta_m)H_*(\eta_{m-1}) \ \cdots \ H_*(\eta_1)$

is an isomorphism.

Suppose a different total order on X is taken, defining a different identification map

$$q' + (KX, KA, \sigma) \to (X, A, \tau) .$$

Then

$$(k,k') \in \sigma \Rightarrow k \subset k' \text{ or } k' \subset k$$
$$\Rightarrow \{q(k), q'(k')\} \subset k' \text{ or } \{q(k), q'(k')\} \subset k ,$$
since always $q(k) \in k$, $q'(k') \in k'$,
$$\Rightarrow q(k) \ \tau \ q'(k') \text{ by the definition of } KX .$$

Thus $q \ \tau^{\sigma} \ q'$, so that $q \simeq q'$ and by the homotopy axiom

$$H_*(q') = H_*(q) .$$

Hence the isomorphism

$$H_*(q) : H_*(K(X,A)) \to H_*(X,A)$$

is independent of the choice of order on X . ■

11.21. Corollary

There is a natural equivalence

$$\{H_* \circ K , \partial_* \circ K\} \cong \{H_*, \partial_*\}$$

for any homology theory $\{H_*, \partial_*\}$, on the category of finite fuzzy pairs. ■

Dually:-

11.3. Proposition

If (X,A,τ) is a finite fuzzy pair, and X is given a total ordering, the map

$$q : (KX, KA) \to (K, A)$$
$$k \rightsquigarrow \min(k)$$

induces an isomorphism in cohomology, independent of the order on X , for any theory $\{H^*, \partial^*\}$.■

11.31. Corollary

There is a natural equivalence

$$\{H^* \circ K \ , \ \partial^* \circ K\} \cong \{H^*, \partial^*\}$$

for any cohomology theory $\{H^*, \partial^*\}$, on the category of finite fuzzy pairs. ■

11.4. Proposition

If $\{H_*, \partial_*\}$, $\{H_*^!, \partial_*^!\}$ are homology theories, there is a natural equivalence

$$\{H_* \circ K \ , \ \partial_* \circ K\} \cong \{H_*^! \circ K \ , \ \partial_*^! \circ K\}$$

on the category of finite fuzzy pairs.

Proof

The argument of [10] III establishes the proposition if:-

(i) Theorem III.2.3 is replaced by 10.5 above,

(ii) the q-simplex s^q of section 3 is defined to be

$$K(\{x_{i_o}, \ldots, x_{i_q}\}, \iota) \ ,$$

$\dot{s}^q$ to be

$$s^q \setminus \{\{x_{i_o}, \ldots, x_{i_q}\}\} \ ,$$

s^{q-1} to be

$$K(\{x_{i_o}, \ldots, x_{i_q}\} \setminus \{y\}, \iota) \text{ where } y = x_{i_j}, \text{ some } j,$$

and c^{q-1} to be

$$\dot{s}^q \setminus \{\{x_{i_o}, \ldots, x_{i_q}\} \setminus \{y\}\} \ , \text{ and}$$

(iii) (X,A) , $|K|, |L|, |K^q|, |L^q|$ and $|K^q \cup L|$ are replaced by

(KX, KA), KX, KA, K^qX, K^qA and $K^qX \cup KA$

respectively throughout.

No simplicial approximations, of course, become involved in Section 10. With these modifications, the proof may be adopted almost word for word, and in view of its length it would seem a work of supererogation to include it in detail here. ■

11.41. Corollary

If $\{H_*, \partial_*\}$, $\{H'_*, \partial'_*\}$ are homology theories, there is a natural equivalence

$$\{H_*, \partial_*\} \cong \{H'_*, \partial'_*\}$$

on the category of finite fuzzy pairs. ■

Dually, we have:-

11.5. Proposition

If $\{H^*, \partial^*\}$, $\{\bar{H}^*, \bar{\partial}^*\}$ are cohomology theories, there is a natural equivalence

$$\{H^* \circ K \, , \, \partial^* \circ K\} \cong \{\bar{H}^* \circ K \, , \, \bar{\partial}^* \circ K\}$$

on the category of finite fuzzy pairs. ■

11.51. Corollary

If $\{H^*, \partial^*\}$, $\{\bar{H}^*, \bar{\partial}^*\}$ are cohomology theories, there is a natural equivalence

$$\{H^*, \partial^*\} \cong \{\bar{H}^*, \bar{\partial}^*\}$$

on the category of finite fuzzy pairs. ■

11.6. Proposition

For any finite fuzzy pair (X,A) there exists $m \in \mathbb{N}$ such that

$$p \geqslant m \Rightarrow H_p(X,A) = H^p(X,A) = 0$$

in any homology and cohomology theories.

Proof

Take $m = n(X,A)$.

There can be no injective p-simplices for $p \geqslant m$, hence no p-cells, hence $\mathcal{A}_p(X,A) = 0$, $p \geqslant m$.

Thus the homology and cohomology of $\mathcal{A}(X,A)$ vanish for $p \geqslant m$, so the simplicial, and thus by 11.41 and 11.51 all, homology and cohomology theories vanish on (X,A) for $p \geqslant m$. ■

11.61. Remark

A lower limit than that in the above proof is evidently possible - plausibly S^n is the minimal fuzzy space with n-homology, in which case we would have $m = \frac{1}{2}(n(X,A) - 1)$ as the general bound - but seems unlikely to have useful applications. The result is of interest less in terms of the particular bound than as a point emphasising the finite nature of this part of the theory.

11.70. Notation

In the light of Propositions 11.0 - 11.5, we shall denote both $\{H_*^s, \partial_*^s\}$ and $\{H_*^c, \partial_*^c\}$ by $\{H_*, \partial_*\}$, $\{H_s^*, \partial_s^*\}$ and $\{H_c^*, \partial_c^*\}$ by $\{H^*, \partial^*\}$ in general, using a distinguishing affix only if we wish to relate something else to homology by way of some specific construction of the homology groups.

11.8. Remark

Uniqueness fails in general (for example, as in topology with proper homotopies required in the homotopy axiom, on spaces with unbounded components). However, the philosophy of this paper is that the chief interest lies in the finite case, since finite fuzzy spaces include all those physically presented. (On these grounds one is tempted to denote this subcategory by $\mathcal{F}yz$.) Infinite cases have therefore largely been included, as for instance in 7.4, 11.0, when they are natural extensions of the finite case, and no special techniques have been evolved to consider them, since if infinities are dominant one might as well be doing topology.

This approach of assuming that finite fuzzy spaces are the heart of the subject is not, however, the only possible one: far from rejection of the infinite, the great cause for commendation of tolerance homeomorphisms in [32] is the passage they allow between finite and infinite description.

If this direction is followed, an important class of objects will clearly be the <u>compact</u> fuzzy spaces; those spaces X with a finite subset Y such that $N(Y) = X$. These have nice properties: their components must have finite internal diameter, and the fuzmic image of a compact space is compact. Moreover, though I have not worked out the details of the devices necessary to prove it I make the following conjecture:-

<u>11.9. Conjecture</u>

If $\{H_*, \partial_*\}$, $\{H'_*, \partial'_*\}$ are homology theories, and $\{H^*, \partial^*\}$, $\{\breve{H}^*, \breve{\partial}^*\}$ cohomology theories, there are natural equivalences

$$\{H_*, \partial_*\} \cong \{H'_*, \partial'_*\}$$
$$\{H^*, \partial^*\} \cong \{\breve{H}^*, \breve{\partial}^*\}$$

on the category of compact fuzzy pairs. (Since a subspace of a compact space need not be compact - any space embeds in a compact one-point cone on it - this will mean the category of pairs (X,A) with X , A both compact.)

⍰

12. Approximate Fixed Point Theorem

12.0. Proposition

If (X,τ) is a contractible finite fuzzy space, for any fuzmap $f : (X,\tau) \to (X,\tau)$ there exists a set $\{x_o,\ldots,x_{p-1}\} \subset X$ of mutually indistinguishable points such that $f(x_i) = x_{i+1}$, $i \in Z_p$.

Proof

Since (X,τ) is contractible, $f \simeq Id(X)$, and thus

$$\begin{aligned} H^s_*(f) &= H^s_*(Id(X)) \\ &= Id(H^s_*(X)) \\ &= Id(\{\delta_{on} Z\}_{n\in Z}) \end{aligned}$$

Hence $\sum_{n=0}^{\infty} (-1)^n Tr(H^s_n(f)) = 1$.

Thus by the Hopf Trace Theorem ([15] p.166)

$$\sum_{n=0}^{\infty} (-1)^n Tr(\mathcal{F}^s_n(f)) = 1 .$$

Thus for some $n \in \mathcal{N}$,

$$Tr(\mathcal{F}^s_n(f)) \neq 0 .$$

Thus considering generators, for some cell $s \in \mathcal{F}^s_n(X)$,

$$\mathcal{F}^s_n(f)(s) = ms + r \text{ , } r \text{ independent of } s .$$

By construction, all maps $\mathcal{F}^s_n(f)$ must take generators to generators, thus

$$\mathcal{F}^s_n(f)(s) = \pm s .$$

Thus $f(|s|) = |s|$. (Notation of 11.0)

Choose $x_o \in |s|$, and define

$$x_1 = f(x_o)$$

$$x_{i+1} = f(x_i) \text{ inductively.}$$

Since $n(|s|) = n+1$, $x_p = x_o$ for some $p \leqslant n$. Choose the first such p . Then $x_o,\ldots,x_{p-1}$ are mutually indistinguishable since they are all in $|s|$, and $f(x_i) = x_{i+1}$, $i \in Z_p$, as required. ∎

12.1. Corollary

If $D(\ )$ is the discriminator introduced in 11.0, we have $A = \{x_0, \ldots, x_{p-1}\} \subset D(A)$, $f(D(A)) \subset D(A)$.

12.2. Remark

The applications of the topological Brouwer Fixed Point Theorem have been generally to show that something stays put; that is, to show the existence of stable states. (For a slightly questionable but highly illustrative example, see [32], Theorem 4 and its corollary.) Theorem 12.0 is precisely the fuzzy analogue one would expect: there may be nothing that stays put, but there is something which varies undetectably if at all, since all its positions are mutually indistinguishable. This is typical of the replacement in fuzzy of Platonically ideal and untestable statements by limited and physically meaningful ones, just as a physically-given function can be found to be not fuzmic, whereas proving it discontinuous would require an infinity of infinitely-accurate observations.

The comments of Remark 11.8 on finiteness apply equally here. With reasonable conviction, but the suspicion that the proof is substantially harder than that of 11.9, I make the following conjecture:-

12.3. Conjecture

If (X,τ) is a compact contractible fuzzy space, for any fuzmap $f : (X,\tau) \to (X,\tau)$ there exists a set $A \subset X$ of mutually indistinguishable points such that

$$f(A) \subseteq A .$$

?

13. The Hurewicz Map

13.0. Definitions

13.00. For $n > 0$, define the map

$$c_{i_1 i_2 \dots i_n} : [1]^n \to \overset{n}{\prod^*} \mathcal{N}$$

$$(x_1, \dots, x_n) \rightsquigarrow [x_1 + i_1, \dots, x_n + i_n]$$

13.01. If $f : (\overset{n}{\prod^*} \mathcal{N}, *) \to (X, x_o)$ is bounded by m , define

$$\varphi(f) = \sum_{i_n=0}^{m-1} \cdots \sum_{i_2=0}^{m-1} \sum_{i_1=0}^{m-1} (\overline{f \circ c_{i_1 i_2 \dots i_n}}) \in \mathcal{F}_n^c(X) .$$

Then

$$\delta_n(\varphi(f)) = \delta_n\left(\sum_{i_n=0}^{m-1} \cdots \sum_{i_o=0}^{m-1} (\overline{f \circ c_{i_1 \dots i_n}}) \right)$$

$$= \sum_{\substack{j=0 \\ k=0,1}}^{n} (-1)^{j+k} \sum_{i_n=0}^{m-1} \cdots \sum_{i_1=0}^{m-1} \overline{F}^k_{jn} (\overline{f \circ c_{i_1 \dots i_n}})$$

$$= \sum_{j=0}^{n} (-1)^j \left(\sum_{i_n=0}^{m-1} \cdots \sum_{i_o=0}^{m-1} \overline{F}^0_{jn} (\overline{f \circ c_{i_1 \dots i_n}}) - \sum_{i_n=0}^{m-1} \cdots \sum_{i_o=0}^{m-1} \overline{F}^1_{jn} (\overline{f \circ c_{i_1 \dots i_n}}) \right)$$

$$= \sum_{j=0}^{n} (-1)^j \left[\sum_{i_n=0}^{m-1} \cdots \sum_{i_{n+1}=0}^{m-1} \sum_{i_{j-1}=0}^{m-1} \cdots \sum_{i_o=0}^{m-1} (\overline{F}^0_{jn} (\overline{f \circ c_{i_1 \dots i_{j-1} 0 i_{j+1} \dots i_n}}) - \overline{F}^1_{jn} (\overline{f \circ c_{i_1 \dots i_{j-1} (m-1) i_{j+1} \dots i_n}})) \right]$$

$$= \sum_{j=0}^{n} (-1)^j \left[\sum_{i_n=0}^{m-1} \cdots \sum_{i_{j+1}=0}^{m-1} \sum_{i_{j-1}=0}^{m-1} \cdots \sum_{i_o=0}^{m-1} ((\overline{[1]^{n-1} \to \{x_o\}}) - (\overline{[1]^{n-1} \to \{x_o\}})) \right]$$

$$= 0 .$$

Thus if N is the quotient map

$$\mathcal{F}_n^c X \to \mathcal{C}X$$

we have also

$$\delta_n(N(\varphi(f))) = 0$$

so that

$$\mathrm{im}(N \circ \varphi) \subset \ker(\delta_n)$$

Thus if K is the quotient map

$$\ker(\delta_n) \to \ker(\delta_n)/_{\mathrm{im}(\delta_{n+1})} = H_n^c(X)$$

the composite

$K \circ N \circ \varphi : \{f : (\overset{n}{\boxtimes}\mathcal{N},*) \to (X,x_o) | f \text{ bounded}\} \to H_n^c(X)$

is defined (modulo the usual abuse of rigour over choice of bound - c.f. 4.11.).

<u>13.02.</u> By the obvious arguments, if f, $g : (\overset{n}{\boxtimes}\mathcal{N},*) \to (X,x_o)$ are homotopic (not necessarily relative to *), then $K \circ N \circ \varphi(f) = K \circ N \circ \varphi(g)$, so that we get an induced <u>Hurewicz map</u>

$$\pi_n(X,x_o) \cong [\overset{n}{\boxtimes}\mathcal{N},X]_*^b \overset{\theta_n}{\to} H_n(X,x_o)$$

which commutes with the action of $\pi_1(X,x_o)$ on $\pi_n(X,x_o)$ [i.e. the restriction to $(p_n')^{\leftarrow}(1) = \tilde{\pi}_n(X,x_o) = \pi_n(X,x_o)$ of the inner automorphism action of $i_n'(\pi_1(X,x_o)) \cong \pi_1(X,x_o)$ on $(\pi^n X,\{x_o\})$; cf. 5.00, 5.2.].

$N \circ \varphi$ carries composition of paths in

$\Omega^n(X,x_o) \cong \{f : (\overset{n}{\boxtimes}\mathcal{N},*) \to (X,x_o) | f \text{ bounded}\}$

to addition in $\mathcal{C}_n X$, hence θ_n is a homomorphism, factoring through a homomorphism θ_n' from the reduced homotopy groups $\pi_n'(X,x_o)$; the groups $\pi_n(X,x_o)$ factored out by the action of $\pi_1(X,x_o)$.

By analogy with topology:-

13.1. Conjecture

For any fuzzy space (X,τ) , if $m \in \mathcal{N}$ is the least number such that $\pi_m(X,\tau) \neq 0$, the Hurewicz map gives an isomorphism

$$\theta_n' : \pi_n'(X,\tau) \to H_n^c(X,\tau)$$

for all $n \leqslant m$.

?

13.11. Corollary.

For any fuzzy space (X,τ) , the first cubical homology group is the fundamental group abelianised.

The proof of this is a substantial exercise, since the usual methods in topology (a) use a standard n-cube as an n-ball and stretch it, taking without loss of generality each $f \in [f] \in \pi_n(X)$ to be a single singular n-cube, and (b) involve the homology of the standard n-cube relative to the union of its faces. Neither (a) nor (b) is possible in $\mathcal{F}uz$. The devices and formulae necessary to replace them probably possess sufficient intricacy to make the homotopy on p.58 appear trivial. Now the homology functors have difference-geometric applications but the higher homotopy groups have been included here only as a natural extension of the fundamental group material, since with the approach used it required little extra machinery to set them up. The fundamental group classifies covering spaces (cf. §7) and may be expected to crop up, as in topology, in a variety of other applications. The higher homotopy groups however are of interest mainly in homotopy theory, and for reasons explored in I.8.3. this field looks to be much less fruitful in fuzzy geometry than in topology. Moreover the prospect of applications of them to coordinate-free difference equations, my chief interest among aspects for development, appears extremely remote.

The proof of 13.1 is therefore left as an exercise for the interested reader.

14. Fuzzy Fibration Theory

This section like the last is a sketch only, for similar reasons. It is included for its interesting differences from topology.

14.0. Definitions

A fuzmap $f : E \to B$ is:

14.00) a fibre bundle if for all $x \in B$

$$f^{\leftarrow}(N(x)) \cong N(x) \times F_x$$

for some fuzzy space F_x ,

14.01) a strict fibration if it has the HoLP (7.61), and

14.02) a fuzzy fibration if $\pi f : \pi E \to \pi B$ is a fibration of groupoids (i.e. if for any point x in E , πf restricted to the elements of πE beginning at x and corestricted to those beginning at $f(x)$ is surjective. cf. [5] p.302).

14.03 In each of 14.00 - 14.02, the fibre over $x \in B$ is $f^{\leftarrow}(x)$. In 14.00, $p^{\leftarrow}(x) \cong F_x$.

14.1. Properties

14.10. Implications and examples.

By straightforward reasoning,

14.00 ⇒ 14.01 ⇒ 14.02.

Evidently a covering is a fibre bundle, with F_x a discrete set, of constant cardinality for $x \in B$ if B is connected.

If $E(Y,y_o)$ is the subspace of PY consisting of paths beginning at y_o, and $p(\omega)$ is the point at which a path ω ends, then

$$p : E(Y,y_o) \to Y$$

is a strict fibration with fibre $\Omega(Y,y_o)$ over y_o.

In topology 14.01 and 14.02 are not essentially different if B is Hausdorff and paracompact, but in $\mathfrak{F}$uz an interesting example is given by the appended exhibit H. ('The fuzzy Hopf map'.)

14.11. Remark on H.

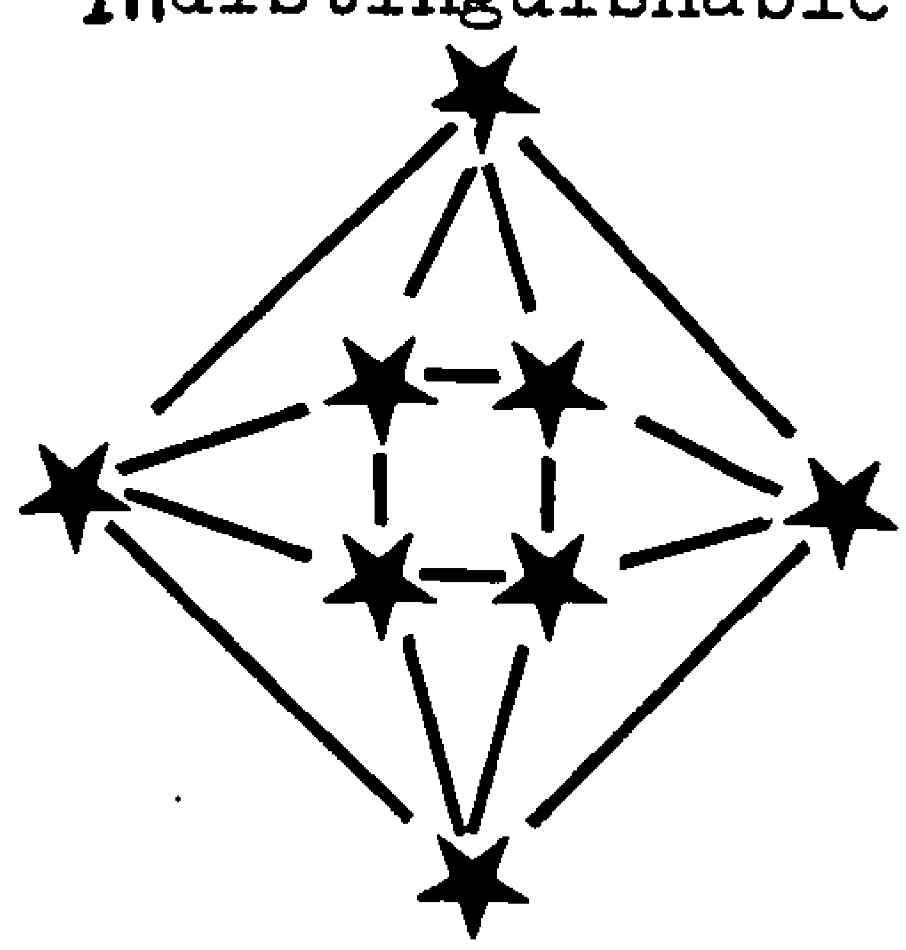
Note that this is not a model in the merely illustrative and suggestive sense in which for example models of the regular polyhedra are, but just as much and as little a mathematical object as an equation is. The conventions for written mathematics may be replaced by the following (not different in logical/philosophical character):

The points of the fuzzy spaces are the woolly balls. A ball is within fuzzy of itself always, and within fuzzy of another if connected to it by a wire or thread without an arrow attached. Points in the upper space are related by H to points in the lower space by sharing the same colour.

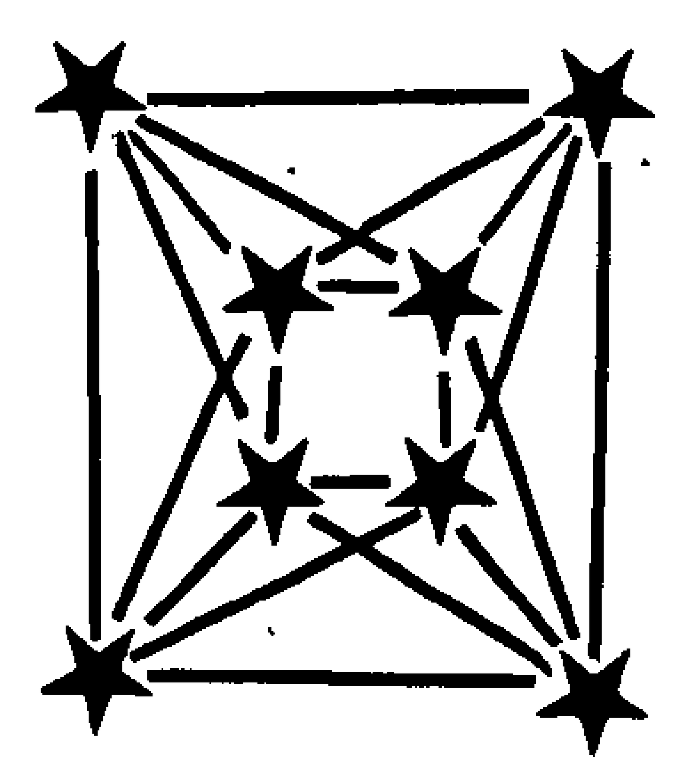
The relation H is then functional, and a fuzmap. It is also incomparably more perspicuous than any written formula with the same logical structure, though admittedly more awkward to reproduce.†
Notice that the lower space is fuzziomorphic to S^2 (10.34). We shall therefore refer to it as such. Denote the upper space by E .

<u>14.12.</u> Properties of H .

The fibre of H over any point in S^2 is fuzziomorphic to S^1 . However, if $x , y \in S^2$ are indistinguishable, $H^{\leftarrow}\{x,y\}$ is of the form



, and not



, as it would be if H had had the local product structure required by 14.00. Moreover if $\tilde{g}$ is the identity on the green fibre G , then there is a homotopy of length one between $H \circ \tilde{g}$ and the map $f : G \to S^2$,

$x \rightsquigarrow$ yellow ball

but consideration of definitions I.3.01, I.8.0(ii) shows that there is no map $\tilde{f} : G \to E$ such that $H \circ \tilde{f} = f$ and $\tilde{f} \simeq \tilde{g}$ (so that 14.01 does not hold either). However this failure does not occur with

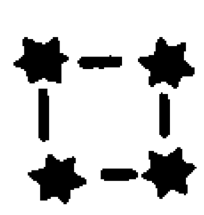
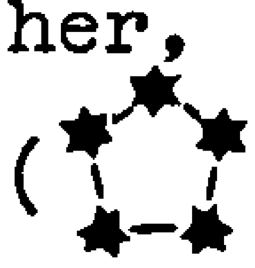
---

† For this reason it is unfortunately not possible to include a copy of H with the duplicated copies of this thesis.

respect to bounded maps $\mathcal{N} \to E$, though even with respect to these H does not have the HoLP , since while a homotopy can always be covered, it may have to be by a homotopy of greater length. Thus we have just enough to establish 14.02, and no more.

Hence 14.01, 14.02 are significantly different in $\mathcal{F}uz$. Moreover the weaker case, as exemplified by H , is important, as will appear in 14.14.

<u>14.13.</u> Fibres

Similarly to the topological case, if B is connected the fibres of a fibre bundle must be fuzziomorphic, and the fibres of a strict fibration must be of the same homotopy type. (This latter equivalence is stronger in $\mathcal{F}uz$ than in topology; cf. I.8.3.) The fibres of a fuzzy fibration must have the same homotopy groups, but if for instance one fibre, F say, is fuzziomorphic to S^1 () , and another, F' , is fuzziomorphic to $(Z_5, a-b \in \{1,0,4\})$ () , there is no essential fuzmap $F \to F'$, and a fortiori no homotopy equivalence.

<u>14.14.</u> Exact sequence

If $f : E \to B$ is a fuzzy fibration with fibre F over some B , the algebraic property 14.02 suffices to establish the homotopy exact sequence

§ $$\dots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \dots$$

for any consistent choice of base points for F , E and B . Thus for instance since a covering $f : \tilde{X} \to X$ has discrete fibre we have $\pi_n(\tilde{X}) \cong \pi_n(X)$ for $n > 1$.

Also, since by 10.41 we have

$$H_q(S^2) \cong \delta_{2q}Z \oplus \delta_{0q}Z$$

and, by a similar proof,

$$H_q(E) \cong \delta_{3q}Z \oplus \delta_{0q}Z$$

for E as in 14.11, 14.12, by 13.1 and the exactness of § we get

$$\pi_3(S^2) \cong Z ,$$

as in topology.

Thus the higher homotopy groups show promise of being quite as unpleasantly complicated in fuzzy as in topology, without the compensation of the CW complex type of application.

I do not intend to pursue that promise.

III. Difference Geometry

All the definitions in this chapter will refer to an arbitrary standard field $\mathcal{R}$, which will not be mentioned in the terms defined; thus we refer, e.g., to tensors not $\mathcal{R}$-tensors, though the symmetric/skew-symmetric distinction (2.04.) for example loses meaning if $\mathcal{R}$ is Z_2 . Uncountability and limits being irrelevant, $\mathcal{R}$ may perhaps best be thought of as the real or complex algebraic numbers. I do not consider here the structures involved when $\mathcal{R}$ has a fuzzy also.

1. Tangent bundle

1.0. Motivation

In topology tangent bundles exists only for very special topological spaces - manifolds - with moreover an added differential structure; an optional extra in the same fashion as a metric (c.f. I.6.01) and as capable of non-isomorphic options (for example there exist exotic spheres). Just as with a metric, however, any fuzzy space has an intrinsic difference structure, and corresponding tangent bundle, implicit in its definition.

Among the numerous ways of defining the tangent bundle of a differentiable manifold is the following: at a point, consider all differentiable paths through it, take classes corresponding to the equivalence relations 'going in the same direction at the same speed' (=having the same first differential), put a vector space structure on the result to get the tangent space at the point, and stick the tangent spaces together to get the tangent bundle. Now a fuzzy path has been defined (I.§5) as a succession of hops from point to indistinguishable point; the 'first difference' at a point corresponding to the first differential, is precisely the 'next hop'. And the set to which possible next hops may be taken is precisely the fuzzy neighbourhood of the point, which is thus isomorphic to the analogous 'tangent space' to the differential one constructed as above. Thus we define:-

1.1. Definitions

Denote by p_1, p_2 the projections $X \times X \to X$ to first and second factors respectively.

1.10. For a fuzzy space (X, τ) the tangent bundle is the composite map

$$t : TX \subset X \times X \xrightarrow{p_1} X$$

where $TX \doteq (\tau, \tau \cdot \tau)$.

1.11. The tangent space T_xX or $(TX)_x$ to X at $x \in X$ is the fuzzy space $t^{\leftarrow}(x)$.

1.12. The exponential map

$$e_x : T_xX \to X$$

at $x \in X$ is given by the composite

$$T_xX \subset X \times X \xrightarrow{p_2} X .$$

Evidently each e_x is a fuzziomorphism.

Since for any $y \in TX$, $e_x(y)$ is defined only if $x = t(y)$, there need be no ambiguity if we sometimes drop the suffix where $t(y)$ is fixed by context, and refer merely to $e(y)$.

1.2. Remark

The fuzzy tangent bundle is not in general a fibre bundle in the sense of II.14.00 (and nor are the tensor bundles of §2), since if it were the fuzzy neighbourhood of any two points would be fuzziomorphic, making the space rather like a flawless crystal. This is not possible in reasonable generality: for example it would mean that the only '2-spherical' objects to be allowed would be only a few hops across if finite at all.

For similar reasons, the tangent space at a point cannot in general be given a vector space structure: it would than have to be isomorphic to a product of copies of the field it was over, again giving overall a crystal-type space.

If however the concept of dimension as rank is abstracted from vector spaces, and the additive and scalar-multiplicative structure (which makes the existence of some points require the existence of others, to be their sums etc.) neglected, a highly significant object remains; a matroid. Invented by Whitney in the 1930's ([28]), matroids have since been studied in combinatorics as global objects only, with the interest mainly lying in the way they generalise graphs and in classification of the matroids with n points, for each n. As a local structure they provide exactly the weakening of the concept of vector space appropriate to a fuzzy tangent space (c.f. §5).

1.20. Definition

If $f : X \to Y$ is a fuzmap, the first difference Δf of f is the map

$$TY|(f,f)|TX .$$

Inductively, the nth difference $\Delta^n f$ of f is the map $\Delta(\Delta^{n-1}f)$, where $\Delta^1 f = \Delta f$.

2. Tensor bundles

2.0. Definitions

For any fuzzy space X, and r, $s \in \mathcal{N}$:-

2.00. Define the contravariant r-tensor bundle

$$t^r : T^r X \to X$$

of X as follows:

$$T^r X = X^{[r]} \quad \text{(c.f. Def}^n\text{s I.3.01., II. 9.00.)}$$

$$t^r(\tilde{x} : [r] \to X) = \tilde{x}(0) .$$

The contravariant r-tensor space $(T^r X)_x$ of X at $x \in X$ is the fuzzy space $(t^r)^{\leftarrow}(x)$.

Evidently $T^1 X \cong TX$.

2.01. Define the covariant s-tensor space $(T_s X)_x$ of X at $x \in X$ to be the set of functions

$$(T^s X)_x \to \mathcal{R} .$$

Define the covariant s-tensor bundle

$$t_s : T_s X \to X$$

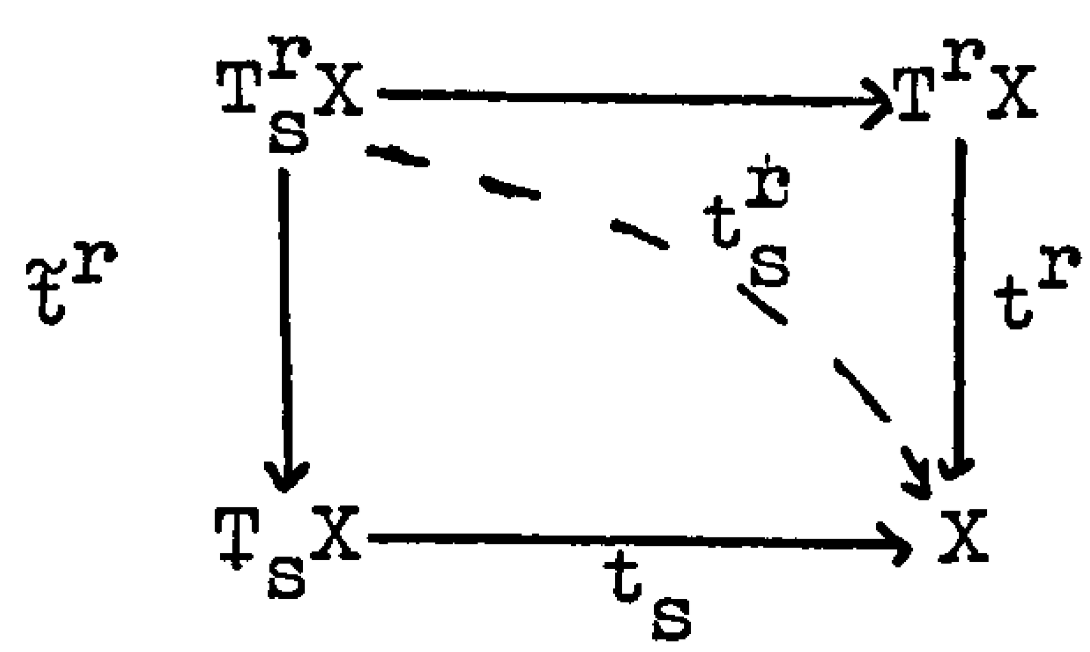
of X by $(t_s)^{\leftarrow}(x) = (T_s X)_x$.

Consider $T_s X$, $(T_s X)_x$ as fuzzy spaces with the fuzzy induced by t_s.

2.02. Define the mixed (r,s)-tensor bundle

$$t^r_s : T^r_s X \to X$$

by the pull-back



where t^r, t_s are as in 2.00. and 2.01. Define the mixed (r,s)-tensor space $(T^r_s)_x$ of X at $x \in X$ to be fuzzy space $(t^r_s)^{\leftarrow}(x)$.

2.03. An (r,s)-tensor field on X is a set-theoretic section of

$$t^r_s : T^r_s X \to X.$$

A tensor field will be assumed fuzmic (as a map $X \to T^r_s X$) unless otherwise stated. Denote the set of fuzmic (r,s)-tensor fields by $\tilde{T}^r_s X$, the set of all (r,s)-tensor fields by $\approx{T}^r_s X$. (For $r = 0$, we have $\tilde{T}^o_s X = \approx{T}^o_s X$.)

2.04. An (r,s)-tensor field F on X is (with $\tilde{t}^r$ as in the diagram in 2.02) symmetric if for all $\tilde{x} \in T^s X$ and permutations σ of $[s]$ we have

$$\tilde{t}^r \circ F(\tilde{x} \circ \sigma) = \tilde{t}^r \circ F(\tilde{x}) ,$$

skew-symmetric if for all such x, σ we have

$$\tilde{t}^r \circ F(\tilde{x} \circ \sigma) = (\mathrm{sgn}(\sigma)).(\tilde{t}^r \circ F(\tilde{x})) .$$

2.05. If $f : X \to Y$ is a fuzmap, define

$$T^r(f) : T^r X \to T^r Y$$
$$\tilde{x} \rightsquigarrow f \circ \tilde{x}$$

and

$$T^r_o(f) : T^r_o X \to T^r_o Y$$

induced by $T^r(f)$.

2.06. If $f : X \to Y$ is a fuzmap, define

$$\tilde{T}^o_s(f) : \tilde{T}^o_s Y \to \tilde{T}^o_s X$$

and $\approx{T}_s(f) : \approx{T}_s Y \to \approx{T}_s X$

by $F \rightsquigarrow [x \rightsquigarrow F(f(x)) \circ T^r(f)]$

2.07. The maps defined in 2.05. evidently commute with t^r, t^r_o, so that Def^{n}s 2.05., 2.06. make the contravariant tensor bundles covariant functors and the collections of covariant tensor fields contravariant functors. It is hard to know whether this produces greater confusion than would reversing the usage of contra - and covariance traditional with differential tensor fields.

As it is possible that this material may eventually be of relevance to physics I have decided to keep to the traditional practice.

2.08. If $f : X \to Y$ is a fuzziomorphism, we have induced isomorphisms for all $r, s \in \mathcal{N}$,

$$T^r_s(f) : T^r_s X \to T^r_s Y$$

$$\tilde{T}^r_s(f) : \tilde{T}^r_s Y \to \tilde{T}^r_s X$$

defined in the obvious way, and coinciding with the maps defined in 2.05., 2.06. on their domains of definitions.

2.09. If $f : X \to X$ is a fuzmap, an (r,s)-tensor field F on X is <u>invariant by</u> f if $\tilde{T}^r_s(f)$ is defined and $\tilde{T}^r_s(f)\,F = F$.

3. Vector fields

3.0. Definitions

3.00. Note that (unlike the differential case) we do not have $T_0^1X \cong TX$, since an element of T_0^1X is an element of $T^1X \cong TX$ associated with an element of $\mathscr{R}$. By $i : x \rightsquigarrow (x,1)$, however, we may embed TX in T_0^1X, and if $p : (x,\alpha) \rightsquigarrow x$ for any $\alpha \in \mathscr{R}$, we have $p \circ i$ as the identity on TX and $i \circ p$ indistinguishable from the identity on T_0^1X, so that the difference is not highly significant. We shall thus identify a section F of the tangent bundle with the (1,0)-tensor field $i \circ F$ it defines, and refer to either as a vector field or difference field.

If we wish to refer specifically to a section of TX rather than T_0^1X we shall call it a geometric vector field.

Denote $\tilde{T}_0^1X$ by $\mathscr{D}X$, (and $\tilde{T}_0^1X$ by $\tilde{\mathscr{D}}X$). Denote the trivial vector field $x \rightsquigarrow ([1] \rightarrow \{x\} \subset X)$ by o.

3.01. For all fuzzy spaces X we have

$$t^\circ : T^\circ X \cong X ,$$

so that a (0,0)-tensor field may be considered as precisely a function $\psi : X \rightarrow \mathscr{R}$. Such a function will be called a scalar field on X.

Denote $\tilde{T}_0^\circ X$, with the ring structure of pointwise addition and multiplication, by ψX. If $f : X \rightarrow Y$ is a fuzmap, denote the ring homomorphism

$$\tilde{T}_0^\circ (f) : \psi Y \rightarrow \psi X$$

by Ψf.

3.02. Remark

Note that as the elements of $(T_s X)_x$ are functions with range $\mathcal{R}$, we have $(T^o_s X)_x$ $(\cong (T_s X)_x)$ and $\tilde{T}^o_s X$ $(\cong \tilde{T}_s X)$ as vector spaces over $\mathcal{R}$. However in the absence of a vector space structure on the contravariant tensors, the rank of $(T_s X)_x$ is simply the cardinality of $(T^s X)_x \cong (N(x))^s$.

3.03. The exponential map

$$e_x : T_x X \to X \qquad (1.12.)$$

extends naturally to a map, also denoted by e_x or e and called the exponential map,

$$T^1_o X \to T^o_o X .$$

3.04. We have $T^1_o X \subset TX \times \mathcal{R}$, $T^o_o X \cong X \times \mathcal{R}$ naturally, and we shall denote by $p_{\mathcal{R}}$ both the maps

$$T^1_o X \to \mathcal{R} , \qquad T^o_o X \to \mathcal{R}$$

induced by the projections.

3.1. Definitions

3.10. If $\psi \in \Psi X$, $D \in \mathcal{D}X$, the difference of ψ with respect to D is the scalar field $D\psi$, where

$$D\psi(x) = p_{\mathcal{R}}(Dx).(\psi(e(Dx)) - \psi(x)) .$$

Evidently the operation of D on ΨX is linear. In contrast to the differential case, we have, for $\psi, \psi' \in \Psi X$, $D \in \mathcal{D}X$,

$$D(\psi.\psi') = D\psi.\psi' + \psi.D\psi' + \frac{D\psi.D\psi'}{p_{\mathcal{R}}(Dx)}$$

by a brief computation.

(The vanishing in the limit of the final term on the right is of course the fundamental dodge of the differential calculus. It took some two centuries to justify;

Newton's argument, for example, was admirably torn to shreds by Bishop Berkeley. ([2],p.16).)

3.11. If D,D' operate on ΨX (in particular, if $D,D' \in \mathcal{D}X$) define the Lie bracket

$$[D,D'] : \Psi X \to \Psi X$$
$$\psi \rightsquigarrow (D(D'\psi) - D'(D\psi))$$

In the absence of a vector space structure on T_xX , and hence on $\mathcal{D}X$, we do not have for $D,D' \in \mathcal{D}X$ their Lie bracket corresponding in general to differencing with respect to another field on X . We do have the Jacobi identity:

$$[D,[D',D'']] + [D',[D'',D]] + [D'',[D,D']] = 0$$

and the skew-symmetry conditions:

$$[D,D'] + [D',D] = 0 \ , \ [D,D] = 0 \ .$$

(independent over a general field $\mathcal{R}$ in the absence of a vector field structure on $\mathcal{D}X$, and hence of any linearity condition.)

3.12. If $[D,D'] = 0$, we say D and D' commute.

3.13. For a geometric vector field on (X,τ) , the local transformation φ_D induced by D is the map

$$\varphi_D : X \to X \ .$$
$$x \rightsquigarrow e_x(Dx)$$

Now for all $x \in X$, $D \in \mathcal{D}X$ we have $d(x,\varphi_0(x)) \leqslant 1$, and given a set-theoretic map $\varphi : X \to X$ such that $d(x,\varphi(x)) \leqslant 1$ for all $x \in X$, then if we define

$$D_\varphi : X \to \dot{T}X$$
$$x \rightsquigarrow (e_x)^{\leftarrow}(\varphi(x))$$

we have $D_\varphi \in \mathcal{D}X$ and

$$\varphi_{D_\varphi} = \varphi \ , \ D_{\varphi_D} = D \ .$$

Moreover,

$$\varphi_D \text{ is a fuzmap} \Leftrightarrow [x\tau y \Rightarrow \varphi_D(x)\,\tau\,\varphi_D(y)]$$
$$\Leftrightarrow [x\tau y \Rightarrow e_x(Dx)\,\tau\,e_y(Dy)]$$
$$\Leftrightarrow [x\tau y \Rightarrow (Dx)\tau.\tau(Dy)]$$
$$\Leftrightarrow D \text{ is a fuzmic vector field.}$$

Thus the set ΦX (resp $\tilde{\Phi}X$) of set-theoretic (fuzmic) maps $\varphi : X \to X$ such that $d(x,\varphi(x)) \leqslant 1$ is isomorphic to the set $\mathfrak{D}X$ ($\tilde{\mathfrak{D}}X$) of (fuzmic) vector fields on X .

3.14. A vector field D on X is reversible if φ_D is a fuzziomorphism. Define the reversal of D , denoted by $D_{\leftarrow}$, to be $D_{(\varphi_D)^{\leftarrow}}$.

Recalling that a fuzziomorphism must be an isometry (I.6.01), reversibility of a vector field is thus here a highly special condition, as it is for physical processes, rather than a universal one, as it is for mathematical physics. Indeed, on a given fuzzy space there may be no reversible vector fields at all.

(Referring to [31]p.248, and entering into the spirit of it, one may argue that Theorem 4, on the existence of stable thoughts, falls in the absence of a proof that all positions in the thought cube are physiologically possible brain states, but the 'next thought' function $t \rightsquigarrow t'$ stands. One may suppose it fuzmic (more relevant in this context than the continuity appealed to in the proof of the theorem) and the paragraph above shows it to be highly unlikely that the associated vector field is reversible. If not, the 'next thought' function must be metric-decreasing.

The long term implication is the inevitability of senility and death.)

3.15. If a geometric vector field $D \in \mathcal{D}X$ has φ_D a surjection, we will say also that D is surjective. In this case the map

$$\Psi(\varphi_D) : \Psi X \to \Psi X$$

is a transforming operator on ΨX in the sense of [35], making $R(X,D) \underset{\text{def}^n}{=} (\Psi X, \Psi(\varphi_D))$ a difference ring. If D is reversible, $R(X,D)$ is inversive.

If $f : X \to Y$ is a fuzmap, and D_X, D_Y are surjective geometric vector fields on X,Y such that

$$T^1(f)(D_X(X)) \subset D_Y(Y)$$

then $\Psi f : \Psi Y \to \Psi X$

is a homomorphism of difference rings.

If D', D' are surjective geometric vector fields on X, $R(X,[D,D']) \underset{\text{def}^n}{=} (\Psi X, 1 + [D,D'])$ is also a difference ring if

$$(\Psi(\varphi_D \varphi_{D'}) - \Psi(\varphi_{D'} \varphi_D))\psi \neq \psi \quad \forall\, \psi \in \Psi X .$$

These observations suggest that the structures in the differential case involving the fact, false here (3.10.), that vector fields define derivations, may be replaced in this context by difference-algebraic structures. I hope to develop this approach, but time prevents me from doing so here.

3.2. Proposition

Two geometric vector fields D and D' on X commute (Def^n 3.12.) if and only if $\varphi_D \circ \varphi_{D'} = \varphi_{D'} \circ \varphi_D$.

Proof

If $\varphi_D \circ \varphi_{D'} = \varphi_{D'} \circ \varphi_D$, we have, for any $\psi \in \Psi X$,

$$\begin{aligned} DD'\psi(x) &= D(\psi(\varphi_{D'}(x)) - \psi(x)) \\ &= \psi(\varphi_{D'}(\varphi_D(x)) - \psi(\varphi_{D'}(x)) - \psi(\varphi_D(x)) + \psi(x) \\ &= \psi(\varphi_D(\varphi_{D'}(x)) - \psi(\varphi_D(x)) - \psi(\varphi_{D'}(x)) + \psi(x) \\ &= D'(\psi(\varphi_D(x)) - \psi(x)) \\ &= D'D\psi(x) \ , \end{aligned}$$

so that

$$[D,D'] = 0 \ .$$

If we have $[D,D'] = 0$, then $DD'\psi = D'D\psi$ for all $\psi \in \Psi X$, and in particular for each $\delta_{xy} : X \to \mathcal{R}$, $x \in X$. The result follows. ■

3.21. Remark

The classical partial difference operators $\Delta_x, \Delta_y, \Delta_z$ ([3]) constitute a commuting set of reversible vector fields on the fuzzy space $(Z \times Z \times Z, \approx \cdot \approx \cdot \approx)$. (This motivates the alternative term 'difference field'.)

3.30. Definition

For a vector field D on a fuzzy space X, define the Lie derivative with respect to D , for any $s \in \mathcal{N}$, by

$$L_D : \tilde{T}_s X \to \tilde{T}_s X$$

$$F \leadsto (F - T_s(\varphi_D)) \qquad \text{(cf. 2.06.)}$$

Evidently F is invariant by φ_D (2.09.) if and only if $L_D(F) = 0$.

4. Difference Forms

4.0. Definitions

4.00. A difference n-form on a fuzzy space X is a skew-symmetric $(0,n)$-tensor field on X.

Denote the set of n-forms on X, with the ring structure of pointwise addition and multiplication, by $\Lambda_n X$.

4.01. The difference $d\omega$ of an n-form ω on X is the (n+1)-form defined by

$$T^{n+1}X \to \mathcal{R}$$

$$\tilde{x} \rightsquigarrow \sum_{j=1}^{n+1} (-1)^j \omega(\tilde{x}(0))(x \circ i_{jn}) + \omega(\tilde{x}(1))(x \circ i_{0n})$$

where $i_{jn} : [n] \to [n+1]$

$$x \rightsquigarrow \begin{cases} x, & x < j \\ x+1, & x \geq j \end{cases}$$

4.02. The n-form ω is closed if $d\omega = 0$, exact if $\omega = d\omega'$ for some (n-1)-form $d\omega'$. By a brief computation $d(d\omega) = 0$ always, so that all exact forms are closed.

4.1. Remark

By 4.0. we have $\Lambda X = \{\Lambda_n X\}_{n \in Z}$ a cochain complex. By analogy with the differential case, the cohomology of this complex might be called the De Rham cohomology of the fuzzy space (X,τ). Inspection of the definitions, however, reveals that ΛX is identical with the dual of the chain complex $\mathcal{A}X$ of II.9.13., so that the De Rham cohomology of (X,τ) is precisely its alternating simplicial cohomology (II.9.22.) with coefficients in $\mathcal{R}$. (Since we have chosen a standard $\mathcal{R}$, we will take all homology and cohomology with coefficients in $\mathscr{R}$, and suppress it in the notation.)

Thus the fuzzy analogue of De Rham's Theorem is true, but not a theorem.

5. Fuzzy Manifolds

(I regret that Definition 5.10. is not wholly local, and I suspect redundancy in Definition 5.21., but I have been unable as yet to remedy these defects.)

5.0. Definitions

5.00. A blob of a fuzzy space (X,τ) is a subset $B \subset X$ such that $(B,\tau) \cong (B,\iota)$.

5.01. A fuzzy space is locally finite if every blob of it is finite.

5.02. A local matroid structure (abbrev. l.m.s.) on a locally finite fuzzy space X is defined by a function ρ (called a rank function) from the set $\mathscr{B}X$ of blobs of X to $\mathscr{N}$ satisfying R1) $0 \leqslant \rho(B) < |B|$, where $|\ \ |$ denotes cardinality, R2) $B' \subset B \Rightarrow \rho(B') \leqslant \rho(B)$

$$\text{R3)}\ \rho(B \cap B') + \rho(B \cup B') \leqslant \rho(B) + \rho(B') + 1 ,$$

if $B \cup B'$ is a blob.

5.03. For fuzzy spaces X,X' with local matroid structures ρ,ρ' , a fuzzmap $f : X \to X'$ is flat if

$$\rho'(\{f(x_1),\ldots,f(x_n)\}) \leqslant \rho(\{x_1,\ldots,x_n\})$$

for all blobs $\{x_1,\ldots,x_n\} \subset X$. If equality always holds, f is proper.

5.04. If X' is a subspace of the fuzzy space with l.m.s. (X,τ,ρ), X' is taken to have the l.m.s. $\rho|\mathscr{B}(X')$, since $\mathcal{B}(X') \subset \mathcal{B}(X)$ naturally. With this l.m.s. on X', also denoted by ρ, the embedding

$$(X',\tau,\rho) \subset (X,\tau,\rho)$$

is proper.

5.05. An n-form ω on a fuzzy space with l.m.s. (X,τ,ρ) is compatible with ρ if

F1) $\rho(\tilde{x}[n]) < n \Rightarrow \omega(\tilde{x}) = 0 \quad \forall\, \tilde{x} \in T^nX$

F2) $\rho(\tilde{x}[n+1]) < n \Rightarrow d\omega(\tilde{x}) = 0 \quad \forall\, \tilde{x} \in T^{n+1}X$

Evidently the set of such forms determines a sub-cochain complex $\underset{\rho}{\Lambda}X$ of ΛX.

5.06. Dually, if a Vietoris n-simplex s of X has $\rho(s[n]) = n$ it is regular. The set of regular simplices defines a sub-simplicial complex $\underset{\rho}{S}X$ of SX, and hence $\underset{\rho}{\mathcal{A}}X$ of $\mathcal{A}X$, since it follows from R1 and R3 by induction thatt all faces of a regular simplex are regular.

5.1. Definitions

5.10. A fuzzy space with l.m.s. (M,τ,ρ) is a fuzzy n-manifold if:

M1) $\rho(B) = n$

M2) $H_i(M,M\backslash B) \cong \delta_{in}Z$ or 0

$H_n(M\backslash B) \cong 0$

(M1, M2) for all maximal blobs $B \subset M$.

M3) The inclusion $\underset{\rho}{\Lambda}M \subset \Lambda M$ induces an isomorphism in cohomology.

Forms on a manifold will be assumed to be compatible with the l.m.s. unless otherwise stated.

5.11. A subspace of a fuzzy n-manifold (M,τ,ρ) or (M,ρ)

submanifold of dimension m (/ codimension $(n-m)$) if it is an m-manifold under the restriction of ρ .

5.12. If an n-manifold M has $H_n(M,M\backslash B) \cong Z$ for all maximal blobs $B \subset M$, M is without boundary (abbrev. wob.)

5.13. If an n-manifold M has an (n-1)-submanifold wob ∂M , such that, where j is the inclusion $\partial M \subset M$, the adjunction space $M_j \sqcup_j M = \overline{M}$ is an n-manifold wob, $(M,\partial M)$ is a manifold with boundary ∂M (abbrev. wib.).

5.2. Definitions

5.20. For an n-manifold $M, H^i(M) \cong H^i(\Lambda_p(M)) = 0$, $i > n$. By the universal coefficient theorem, $H_i(M) = 0$, $i > n$. By the exactness of

$$H_n(M\backslash B) \to H_n(M) \to H_n(M,M\backslash B)$$

we have $H_n(M) \cong 0$ or Z , with Z only possible if M is wob. If $H_n(M) \cong Z$, M is orientable, and an orientation of M is a choice σ of generator for $H_n(M)$.

5.21. For an n-manifold wib $(M,\partial M)$, by the exactness of the Mayer-Vietoris sequence (II.10.2)

$$\underset{0}{H_n(\partial M)} \to H_n(M) \oplus H_n(M) \to \underset{0 \text{ or } Z}{H_n(\overline{M})}$$

we have $H_n(M) = 0$, since Z has no subgroups of the form $G \oplus G$. By the exactness of

$$H_n(M) \to H_n(M,\partial M) \to H_{n-1}(\partial M)$$

we have $H_n(\partial M) \cong Z$ or 0 .

Define $(M,\partial M)$ to be orientable if $H_n(M,\partial M) \cong Z$ and $H_{n-1}(M)$ is torsion-free, and an orientation of $(M,\partial M)$

to be a choice σ of generator for $H_n(M,\partial M)$.

By the exactness of

$$\begin{array}{ccccccc} H_n(M) & \to & H_n(M,\partial M) & \overset{\partial}{\to} & H_{n-1}(\partial M) & \to & H_{n-1}(M) \\ \| & & \| & & \| & & \\ 0 & & Z & & 0 \text{ or } Z & & \text{torsion-free} \end{array}$$

∂ is an isomorphism, so that ∂M is orientable and $\partial(\sigma)$ is an orientation, the induced orientation, of ∂M , which we shall always choose for $(M,\partial M)$ an oriented manifold wib.

6. Summation on Manifolds

6.0. Definitions

6.00. If ω is an n-form on an oriented n-manifold wob M , with orientation σ , the sum

$$\sum_M \omega$$

of ω over M is defined to be $\omega(t)$, for any $t \in \mathcal{A}_n(M)$ such that the homology class of t is σ . This is well defined, since if $t' \in \mathcal{A}_n(M)$ also has homology class σ , we have

$$\begin{aligned} \omega(t) - \omega(t') &= \omega(t-t') \\ &= \omega(\delta s) , \quad s \in \mathcal{A}_{n+1}M \quad (t,t' \text{ homologous}) \\ &= d\omega(s) \\ &= 0 \quad (5.05.\ F2;\ 5.10.) \end{aligned}$$

6.01. If ω is an n-form on an oriented n-manifold wib M , with orientation σ , the sum

$$\sum_{M} \omega$$

of ω over M is defined to be $\omega(r)$, for any $r \in \mathcal{A}_n(M)$ such that the homology class of $j^*(r)$ is σ , where j_* is the quotient map

$$\mathcal{A}_n(M) \to \frac{\mathcal{A}_n(M)}{\mathcal{A}_n(\partial M)} .$$

This is well defined, since if $r' \in \mathcal{A}_n(M)$ also has homology class σ , we have

$$\begin{aligned} \omega(r) - \omega(r') &= \omega(r-r') \\ &= \omega(\delta s+q), \; s \in \mathcal{A}_{n+1}(M), \; q \in \mathcal{A}_n(\partial M) \\ &= d\omega(s) + \omega(q) \\ &= 0 \qquad (5.05.\ F1,F2;\ 5.10.) \end{aligned}$$

7. Stokes' Theorem

7.0. Proposition

If ω is an $(n-1)$-form on an oriented n-manifold wib $(M,\partial M)$, then

$$\sum_{M} d\omega = \sum_{\partial M} \omega$$

Proof

Let $\sigma \in H_n(M,\partial M)$ be the orientation class of $M,\partial M$, and take $r \in \mathcal{A}_n(M)$, $t \in \mathcal{A}_{n-1}(\partial M)$ such that

$$\text{cls}(j_*(r)) = \sigma \ , \text{cls}(t) = \partial(\sigma) .$$

We have the usual map between short exact sequences:

$$\begin{array}{ccccccccc} 0 \to & \mathcal{A}_n(\partial M) & \to & \mathcal{A}_n(M) & \xrightarrow{j_*} & \mathcal{A}_n(M,\partial M) & \to 0 \\ & & & \delta & & & \\ 0 \to & \mathcal{A}_{n-1}(\partial M) & \xrightarrow[i_*]{} & \mathcal{A}_{n-1}(M) & \to & \mathcal{A}_{n-1}(M,\partial M) & \to 0 \end{array}$$

Now since

$$\partial(\mathrm{cls}(j_*(r)) = \mathrm{cls}(t)$$

we have

$$\delta(r) = i_*(t)$$

by the definition of the connecting homomorphism ∂.

Thus
$$\sum_M d\omega = d\omega(r)$$
$$= \omega\delta(r)$$
$$= \omega i_*(t)$$
$$= \omega(t)$$
$$= \sum_{\partial M} \omega$$

■

8. Transversality

The geometrical idea of transversality is evidently relevant here, but the formal treatment of it is awkward, and the rigidity of the fuzzy category makes for example the idea of a 'small displacement' of a submanifold, useful in defining general position, not always possible.

I do not know how best to handle it, but (i) & (ii) of the following definition are independent and certainly necessary:

8.00. Definition

An r-submanifold M^r and s-submanifold M^s of a fuzzy n-manifold M are <u>transversal</u> if for any maximal blob B of M meeting M^r, M^s in maximal blobs,

(i) B is spanned as a matroid by $(B \cap M^r) \cup (B \cap M^r)$.

(That is, every point in B is dependent on $(B \cap M^r) \cup (B \cap M^s)$; $\text{rank}(B \cap M^r) \cup (B \cap M^s)) = \text{rank}(B) = n$.)

(ii) $H_i(N(B) \cap M^r, N(B) \backslash N(M^s)) \cong \delta_{i(m-s)} Z$

$H_i(N(B) \cap M^s, N(B) \backslash N(M^r)) \cong \delta_{i(m-r)} Z$.

By application of the inequality R3 of 5.02. we have:

8.1. Proposition

If submanifolds M^r, M^s of M, as above, are transversal and meet in a submanifold M', we have

$$\dim(M') \leqslant r + s - n . \blacksquare$$

The '<' possibility is not removable while the manifolds are required to 'meet' in the strong sense of have M' as their intersection. For example, two 2-submanifolds of a 3-manifold can be transversal and yet meet only in a single point, if elsewhere they interpenetrate like nets without knots in common. Intuitively they must clearly meet in some sense 'along a line'. It seems plausible that the appropriate sense is the following:

8.2. Conjecture

If submanifolds M^r, M^s of M, as above are transversal and $N(M^r) \cap N(M^s) \not\subseteq \emptyset$, there eexists an $(r + s - n)$-submanifold I of M such that

$$I \subset N(M_1) \cap N(M_2)$$

and $$1 \geq \delta(N(I), N(M_1) \cap N(M_2))$$ (cf. I.6.02.(d)) .

⍰

IV. Discrete Potential Theory

As in Chapter III, we shall work over a standard field $\mathcal{R}$, and suppress reference to it. The work of this chapter could be carried out over any field of characteristic zero, but for convenience we shall suppose $\mathcal{R}$ to be a subfield of the reals. We do not require completeness.

All fuzzy spaces in this chapter will be assumed finite, whether or not this is explicitly stated.

1. The discrete Laplacian

1.0. Definitions

1.00. For a fuzzy space X, denote the set of cells of $\mathcal{A}_n X$ (cf. II.9.14) by $A_n X$. Then for each $p \in \mathcal{N}$, on the vector space $\Lambda_p X$ over $\mathcal{R}$ of p-forms on X define the inner product

$$\langle \omega, \mu \rangle_X = \sum_{a \in A_p X} \omega(a)\mu(a) .$$

Evidently this is indeed an inner product.

We may write simply $\langle \omega, \mu \rangle$ for $\langle \omega, \mu \rangle_X$ where this is unambiguous, and similarly omit the suffix X on objects, such as the following, defined with the aid of $\langle \ , \ \rangle_X$.

1.01. For a fuzzy space X, define the codifference

$$d^*_X : \Lambda_{p+1} X \to \Lambda_p X$$

by

D1) $$\langle d^*_X \omega, \mu \rangle_X = \langle \omega, d\mu \rangle_X \quad \forall \mu \in \Lambda_p X .$$

This is well defined: it is unique since for any $a \in A_p X$ we may take μ defined on generators by $\mu = \delta_{ab}$, so that

D2) $$d_X^*\omega(a) = \langle d_X^*\omega, \delta_{ab}\rangle_X = \langle \omega, d(\delta_{ab})\rangle$$

and it exists because any $\mu \in \Lambda_p X$ has

$$\mu = \sum_{a \in A_p X} \mu(a)\delta_{ab} ,$$

so that if $d_X^*\omega(a)$ is defined for each $a \in A_p X$ by D2, $d_X^*\omega$ satisfies D1.

<u>1.02.</u> Define the Laplacian L_X on ΛX by

$$L_X = (d + d_X^*)^2 = d_X^* d + d d_X^* .$$

<u>1.03.</u> A p-form ω on X is <u>harmonic on</u> a subset Y of X if

$$L_X\omega | \Lambda_p Y \equiv 0 .$$

L_X being linear, the p-forms on X harmonic on Y forms a vector space which we shall denote by $\mathcal{H}_p^Y(X)$. We shall denote $\mathcal{H}_p^X(X)$ by $\mathcal{H}_p(X)$, and call its elements simply <u>harmonic</u>.

<u>1.1. Proposition</u>

L_X is a chain map. (And hence the Laplacian of an exact form is exact, the Laplacian of a closed form is closed, and the difference of a harmonic form is harmonic.)

<u>Proof</u>

$$\begin{aligned} dL_X &= d(d_X^* d + d d_X^*) \\ &= d d_X^* d \\ &= (d d_X^* + d_X^* d)d \\ &= L_X d . \end{aligned}$$

■

<u>1.2. Proposition</u>

L_X is a self-adjoint linear operator on $\Lambda_p X$.

Proof

$$\begin{aligned}\langle L\omega,\mu\rangle &= \langle d^*d\omega + dd^*\omega,\mu\rangle \\ &= \langle d^*d\omega,\mu\rangle + \langle dd^*\omega,\mu\rangle \\ &= \langle d\omega,d\mu\rangle + \langle d^*\omega,d^*\mu\rangle \\ &= \langle \omega,d^*d\mu\rangle + \langle \omega,dd^*\mu\rangle \\ &= \langle \omega,L\mu\rangle\end{aligned}$$

■

1.21. Corollary

For any p , $\Lambda_p X$ has an orthonormal basis $\omega_1,\ldots,\omega_n$ of eigenforms of L_X ; i.e. forms such that

$$\langle \omega_i,\omega_j\rangle = \delta_{ij}$$
$$L_X(\omega_i) = \lambda_i\omega_i \ , \quad \text{for some} \quad \lambda_i \in \mathcal{R} \ .$$

Proof

Apply [16] p.266, Theorem 20. ■

1.3. Proposition

For a fuzzy space X and $\varphi \in \Psi X = \Lambda_o X$ (cf. III.3.01) we have

$$L\varphi(x) = \sum_{z\tau x} (\varphi(x) - \varphi(z)) \ .$$

Proof

$$\begin{aligned}L\varphi(x) &= (d^*d + dd^*)\varphi(x) \\ &= d^*d\varphi(x) && (\Lambda_{-1}X = 0) \\ &= \langle d(\delta_{xy}),d\varphi\rangle && (\text{cf. } 1.01) \\ &= \sum_{A_1X} d(\delta_{xy})(a)d\varphi(a) \\ &= \sum_{a\in T_xX} (-1)d\varphi(a) && (\text{cf. III.1.11, III.2.01}) \\ &= \sum_{z\tau x} (-1)d\varphi(e_x^{\leftarrow}(z)) && (\text{cf. III.1.12}) \\ &= \sum_{z\tau x} (\varphi(x) - \varphi(z)) \ .\end{aligned}$$

■

1.31. Corollary (Maximum Principle)

If a function $\varphi \in \Psi X$ is harmonic on a connected subspace of a fuzzy space X, it is either constant on $N(Y)$ or attains its maximum and minimum only on $X \setminus Y$.

Proof

If $y \in Y$ and $\varphi(y)$ is a maximum,

$\varphi(x) \leqslant \varphi(y) \quad \forall x \in N(y)$.

Thus

$$L_X\varphi(y) = \sum_{x\tau y} (\varphi(y) - \varphi(x)) = 0 \Leftrightarrow \varphi(x) = \varphi(y) \quad \forall x \in N(y)$$

But therefore $\varphi(x)$ is also maximal, for all $x \in N(y)$.

Hence by connectivity

$$\varphi(x) = \varphi(y) , \forall x \in N(y) .$$

■

1.4. Remark

The definitions used here are the most natural in this approach, and correspond to those used in harmonic analysis, but they give rise to differences from tradition that it would be as well to mention. The 'codifference' used here corresponds to the negative of the physicists' 'divergence' operator, while the 'difference' corresponds exactly to the 'gradient'. L is thus '-div.grad' on functions instead of 'div.grad'. This difference in sign apart, L corresponds exactly to the physicists' ∇^2 ; on scalars this is clear. On vectors I have sometimes found the Laplacian defined by treating each component as a scalar and applying the scalar Laplacian. The possibility is not raised that this might not define a vector aperation (nor indeed does it in curvilinear coordinates) but such misfortune is in fact ruled out for (x,y,z)-coordinates by the proof of the identity

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{h}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla} . \underset{\sim}{h}) - \nabla^2 \underset{\sim}{h} ,$$

which establishes ∇^2 as the difference of two vector operations and hence itself a vector operation. Now after sorting out the confusions arising from the duality isomorphisms on a Riemannian 3-manifold (such as our alleged physical space) between all of

the tangent space

its dual

the (2,0)-tensor space

and _its_ dual,

by which it is sustomary in physics to identify the lot, it emerges that $\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{h})$ corresponds to $d^*d\underset{\sim}{h}$. Recalling that 'div' corresponds to $(-d^*)$, so that $\underset{\sim}{\nabla}(\underset{\sim}{\nabla} . \underset{\sim}{h})$ becomes $(-dd^*\underset{\sim}{h})$, we find that the Laplacian defined by 1.02 corresponds here also with that of the physicists - with, again, a difference in sign.

The sign for the scalar Laplacian corresponding to the usage in physics is universal in the literature concerning discrete potential theory on lattices. In contrast to the remark in III.2.07 I make no apology here for departure from traditional usage. Differences in sign are endemic in physics anyway: there is no consensus, for example, on whether to choose a positive or a negative signature for a Minkowski metric, and it is not customary to state which one is being used. (It is generally necessary for a reader to deduce the author's choice from internal evidence.) In the face of such confusion, one can do more than resolve to use the sign convention most natural in a given formalism, and to state it clearly.

2. Hodge's Theorem

2.0. Proposition

For all fuzzy spaces X and $p \in Z$,

$$\Lambda_p X = d(\Lambda_{p-1}X) \oplus d^*(\Lambda_{p+1}X) \oplus \mathcal{H}_p X .$$

Proof

For convenience and perspicuity, we shall suppress reference to X, and attach grading indices to the difference operator and its dual.

We use the following notations from linear algebra:

V^* is the dual space of V.

If $f : V \to W$ is a linear map, its transpose is

$$f^t : W^* \to V^*$$
$$g \leadsto g \circ f .$$

If $U \subseteq V$, $W \subseteq V^*$,

$$U^\perp = \{v \in V | \langle v,u \rangle = 0 , \forall u \in U\}$$
$$N(U) = \{f \in V^* | f(U) = 0\}$$
$$N(W) = \{v \in V | f(v) = 0 , \forall f \in W\} .$$

We have a (non-commutative) diagram

$$\begin{array}{ccccc}
\Lambda_{p-1} & \xrightarrow{d_{p-1}} & \Lambda_p & \xrightarrow{d_p} & \Lambda_{p+1} \\
\downarrow i_{p-1} & & \downarrow i_p & & \downarrow i_{p+1} \\
\Lambda^*_{p-1} & \xleftarrow{d^t_{p-1}} & \Lambda^*_p & \xleftarrow{d^t_p} & \Lambda^*_{p+1}
\end{array}$$

such that

$$d_j^* = i_j^{\leftarrow} d_j^t i_{j+1} \ , \text{ for } \ j = p, p-1 \ ,$$

where i_j is the isomorphism

$$\omega \rightsquigarrow [\mu \rightsquigarrow \langle \omega, \mu \rangle] \ , \text{ for } j = p-1, p, p+1 \ .$$

Now,

$$\begin{aligned} d_p^t i_{p+1}(\Lambda_{p+1}) &= d_p^t(\Lambda_{p+1}^*) \\ &= N(\ker(d_p)) \ . \quad ([\quad], \text{ p.103 Th}^m 20(ii)) \end{aligned}$$

Therefore

$$\begin{aligned} d_p^*(\Lambda_{p+1}) &= i_p^{\leftarrow} d_p^t i_{p+1}(\Lambda_{p+1}) \\ &= i_p^{\leftarrow}(N(\ker(d_p)) \\ &= (\ker(d_p))^{\perp} \ . \end{aligned}$$

Similarly

$$\begin{aligned} d_{p-1}(\Lambda_{p-1}) &= N(\ker(d_{p-1}^t)) \\ &= (i_p^{\leftarrow}(\ker(d_{p-1}^t)))^{\perp} \\ &= (\ker(d_{p-1} i_p))^{\perp} \\ &= (\ker(i_{p-1}^{\leftarrow} d_{p-1}^t i_p))^{\perp} \quad (i_{p-1}^{\leftarrow} \text{ monic}) \\ &= (\ker(d_{p-1}^*))^{\perp} \end{aligned}$$

Hence

$$\begin{aligned} d_{p-1}^*(((\ker(d_p))^{\perp}) &= d_{p-1}^* d_p^*(\Lambda_{p+1}) \\ &= 0 \end{aligned}$$

and we have

$$d_p^*(\Lambda_{p+1}) = (\ker(d_p))^{\perp} \subseteq \ker(d_{p-1}^*) = (d_{p-1}(\Lambda_{p-1}))^{\perp} \ .$$

Hence

$$d_{p-1}(\Lambda_{p-1}) + d_p^*(\Lambda_{p+1}) = d_{p-1}(\Lambda_{p-1}) \oplus d_p^*(\Lambda_{p+1})$$

Moreover

$$\begin{aligned}(d_{p-1}(\Lambda_{p-1}) \oplus d_p^*(\Lambda_{p+1}))^{\perp} &= (d_{p-1}(\Lambda_{p-1}))^{\perp} \cap (d_p^*(\Lambda_{p+1}))^{\perp} \\ &= (\ker(d_{p-1}^*)) \cap (\ker(d_p)) \\ &\subseteq (\ker(d_{p-1}d_{p-1}^*)) \cap (\ker(d_p^*d_p)) \\ &\subseteq (\ker(d_{p-1}d_{p-1}^* + d_p^*d_p)) \\ &= \mathcal{H}_p X .\end{aligned}$$

and conversely

$$\begin{aligned}\omega \in \mathcal{H}_p X &\Rightarrow L\omega = 0 \\ &\Rightarrow 0 = \langle L\omega, \omega\rangle \\ &\qquad = \langle dd^*\omega + d^*d\omega, \omega\rangle \\ &\qquad = \langle dd^*\omega, \omega\rangle + \langle d^*d\omega, \omega\rangle \\ &\qquad = \langle d^*\omega, d^*\omega\rangle + \langle d\omega, d\omega\rangle \\ &\Rightarrow \langle d\omega, d\omega\rangle = 0 = \langle d^*\omega, d^*\omega\rangle \\ &\qquad\qquad (\langle,\rangle \text{ positive}) \\ &\Rightarrow \quad d\omega = 0 = d^*\omega \quad (\langle,\rangle \text{ definite}) \\ &\Rightarrow \omega \in (\ker(d^*)) \cap (\ker(d)) \\ &\Rightarrow \omega \in (d_{p-1}(\Lambda_{p-1}) \oplus d_p^*(\Lambda_{p+1}))^{\perp}\end{aligned}$$

Therefore $\mathcal{H}_p X = (d_{p-1}(\Lambda_{p-1}) \oplus d_p^*(\Lambda_{p+1}))^{\perp}$

Therefore $\Lambda_p X = d_{p-1}(\Lambda_{p-1}) \oplus d_p^*(\Lambda_{p+1}) \oplus \mathcal{H}_p X$.

■

<u>2.01. Corollary</u>

$$\mathcal{H}_p(X) = \Lambda_p(X) \cap \ker(d) \cap \ker(d^*) .$$

■

2.02. Corollary

For any p-form ω on X, the generalised Poisson equation

$$L\mu = \omega$$

has a solution if and only if

$$\langle \omega, \eta \rangle = 0$$

for every harmonic p-form η, and hence if and only if

$$\omega = d\omega' + d^*\omega'' .$$

Proof

Since L is self-adjoint,

$$L(\Lambda_p(X)) = (\ker L)^\perp$$

So

$$L(\Lambda_p(X)) = (\mathcal{H}_p X)^\perp$$

$$= d(\Lambda_{p-1}(X)) \oplus d^*(\Lambda_{p+1}(X)) .$$

$\blacksquare$

2.03. Corollary

For any p-form ω on X, if $L\omega$ is closed it is exact.

Proof

$$L(\Lambda_p(X)) = d(\Lambda_{p-1}(X)) \oplus d^*(\Lambda_{p+1}(X)) \quad \text{as in 2.02.}$$

$$= d(\Lambda_{p-1}(X)) \oplus (\ker(d))^\perp .$$

$\blacksquare$

(2.03. could also be proved directly after Proposition 1.1., since $(-1)^p d_p^*$ defines a chain homotopy $(-1)^p L \simeq 0$.)

2.04. Corollary

$$\mathcal{H}_p(X) \cong H^p(X) \qquad \text{(c.f. III.4.1.)}$$

and in each cohomology class there is a unique harmonic form.

Proof

$$\ker(d_p) = (d^*(\Lambda_{p+1}(X)))^\perp$$

$$= d(\Lambda_{p-1}(X)) \oplus \mathcal{H}_p(X) .$$

Therefore

$$H^p(X) = \frac{\ker(d_p)}{d(\Lambda_{p-1}(X))}$$

$$\cong \mathcal{H}_p(X) \ .$$

■

2.05. Corollary

If a fuzzy space X is contractible, the only harmonic forms on X are the constant functions. ■

3. Dirichlet's Principle

3.00. Definition

Define the Dirichlet form on $\Lambda_p(X)$ by

$$D_X[\omega,\mu] = \langle d\omega, d\mu\rangle_X + \langle d_X^*\omega, d_X^*\mu\rangle$$

and the Dirichlet sum to be the associated quadratic form on $\Lambda_p(X)$

$$D_X[\omega] = D_X[\omega,\omega] .$$

Evidently D is a positive indefinite symmetric bilinear form.

3.1. Proposition

If ω is an eigen form of L_X, with eigenvalue λ,

$$\frac{D_X[\omega]}{\langle\omega,\omega\rangle} = \lambda .$$

Proof

$$\begin{aligned} D[\omega] &= \langle d\omega, d\omega\rangle + \langle d^*\omega, d^*\omega\rangle \\ &= \langle\omega, d^*d\omega\rangle + \langle\omega, dd^*\omega\rangle \\ &= \langle\omega, d^*d\omega + dd^*\omega\rangle \\ &= \langle\omega, L\omega\rangle \\ &= \langle\omega, \lambda\omega\rangle \\ &= \lambda\langle\omega,\omega\rangle . \end{aligned}$$

■

3.2. Definitions

If (X,Y) is a fuzzy pair, define:

3.20. The interior boundary $I_X(Y)$ of Y in X is the set $Y \cap N(X\setminus Y)$.

3.21. The exterior boundary $E_X(Y)$ of Y in X is the set $N(Y) \cap (X\setminus Y)$.

3.22. The border $B_X(Y)$ of Y in X is the subset $(Y\times(X\backslash Y))\cap\tau$ of $X\times X$.

3.23. The interior $\mathring{Y}$ of Y with respect to X is the set $Y\backslash I^X(Y)$.

3.3. Proposition (Dirichlet's Principle)

For a fuzzy space X and an arbitrary† subset B of X satisfying $B\cap X'\neq\emptyset$ for all components X' of X, and any function $g\in\Psi B$, there is a unique function φ on X among the set $\Psi_g X$ of those agreeing with g on B for which $D_X[\varphi]$ attains a minimum, which is also the unique function in $\Psi_g X$ harmonic on $X\backslash B$. The values of $\varphi/X\backslash B$ depend only on those of $g|I_X(B)$.

Proof

Without loss of generality we assume X connected.

Let $X\backslash B = Y$.

(i) For some $\psi\in\Psi X$, $y\in Y$, suppose $L\psi(y) = c$. Then

$$\sum_{x\tau y}(\psi(y)-\psi(x)) = c \qquad (\text{Prop}^n.\ 1.3.)$$

Define $\psi'\in\Psi X$ by

$$\psi' = \psi - \frac{c}{n}.\delta_{xy}\ , \quad \text{where } n=n(N(y)\backslash\{y\})\ .$$

$$\text{Then } L\psi'(y) = \sum_{\substack{x\tau y\\ x\neq y}}\left(\left(\psi(y)-\frac{c}{n}\right)-\psi(x)\right)$$

$$= \sum_{x\tau y}(\psi(y)-\psi(x)) - c$$

$$= 0\ ,$$

†In the continuous case B must satisfy strong restrictions; if for example it has isolated points the result is false. It is generally required that B be a piece-wise smooth boundary to X. No such conditions are necessary here.

and

$$
\begin{aligned}
D[\psi]-D[\psi'] &= \langle d\psi,d\psi\rangle-\langle d\psi',d\psi'\rangle \\
&= \langle L\psi,\psi\rangle-\langle L\psi',\psi'\rangle \\
&= \langle L\psi,\psi-\psi'\rangle+\langle L\psi-L\psi',\psi\rangle-\langle L\psi-L\psi',\psi-\psi'\rangle \\
&= \langle L\psi,\psi-\psi'\rangle+\langle \psi-\psi',L\psi\rangle-\langle L\psi,\psi-\psi'\rangle+ L\psi',\psi-\psi'\rangle \\
&= \langle L\psi,\psi-\psi'\rangle+\langle L\psi',\psi-\psi'\rangle \\
&= \langle L\psi,\tfrac{c}{n}\psi_{xy}\rangle \\
&= \frac{c^2}{n} .
\end{aligned}
$$

Thus if D attains a minimum at ψ , $c = 0$, i.e.

$$L\psi(y) = 0 .$$

But this holds for all $y \in Y$, hence

$$L\psi|Y \equiv 0 ,$$

and ψ is thus harmonic on Y .

(ii) Suppose $\varphi \in \Psi_g X$ is harmonic on Y , and $\psi \in \Psi_g X$ also. Then $\psi = \varphi + \eta$, say, where $\eta|B \equiv 0$, and

$$
\begin{aligned}
D[\psi] &= D[\varphi]+2D[\varphi,\eta]+D[\eta] \\
&= D[\varphi]+2\langle d\varphi,d\eta\rangle+D[\eta] \\
&= D[\varphi]+2\langle L\varphi,\eta\rangle+D[\eta] \\
&= D[\varphi]+D[\eta]
\end{aligned}
$$

Therefore

$$D[\psi] \geqslant D[\varphi] \quad (D \text{ positive})$$

and

$$
\begin{aligned}
D[\psi] = D[\varphi] &\Leftrightarrow D[\eta] = 0 \\
&\Leftrightarrow \langle d\eta,d\eta\rangle = 0 \\
&\Leftrightarrow d\eta \equiv 0 \quad (\langle,\rangle \text{ positive definite}) \\
&\Leftrightarrow \eta \text{ is constant } (X \text{ connected}) \\
&\Leftrightarrow \eta \equiv 0 \quad (\eta|B \equiv 0) \\
&\Leftrightarrow \psi = \varphi
\end{aligned}
$$

Thus if $\varphi \in \Psi_g X$ is harmonic on Y, D attains a unique minimum on $\Psi_g X$ at φ.

(iii) The conditions

$$\mathcal{D} = \begin{cases} L\varphi | Y \equiv 0 \\ \varphi | B = g \end{cases}$$

determine a system of $n = n(Y)$ inhomogeneous linear equations in n variables when for each equation

$$L\varphi(y) = 0$$

we put the values of $\varphi | Y$ on the left hand side and those of g on the right. Thus we have a linear operator $[L]$ on $\mathcal{R}^n$ an equation

$$\tilde{\mathcal{D}} : \quad [L] \begin{bmatrix} \varphi(y_1) \\ \vdots \\ \varphi(y_n) \end{bmatrix} = \bar{g} = \begin{bmatrix} \sum_{b \in N(y_1) \setminus Y} g(b) \\ \vdots \\ \sum_{b \in N(y_n) \setminus Y} g(b) \end{bmatrix}$$

and any solution $\tilde{\varphi}$ of $\tilde{\mathcal{D}}$ determines a solution of $\mathcal{D}$ if we extend $\tilde{\varphi}$ by g. Thus the problems are equivalent, and the nature of their solutions depends only on $\bar{g}$. It follows that it depends only on $g | \dot{E}(Y) = g | \dot{I}(B)$ as asserted, and that if $\bar{g} = 0$ we may suppose $g \equiv 0$ without altering the solution of the problems. Now by the maximum principle (1.31.) the unique solution if $g \equiv 0$ is $\varphi \equiv 0$, so that $[L]$ has a null kernel and is hence invertible. Therefore $\mathcal{D}$ has a unique solution, determined by $[L]^{\leftarrow}(\bar{g})$. ■

3.31. Remarks

3.310. Part (ii) of the above replaces the argument in the original proof on $Z \times Z \times Z$ for the existence of a minimum, which was essentially a compactness argument ([6],p.40), and thus avoids the requirements that $\mathcal{R}$ be complete.

3.111. Part (iii) is perhaps not the most elegant manner of proving the existence of a solution; that would probably be by way of a relative version of 2.04., which moreover would be of use in proving higher-order analogues of this and other results. However its physicality, in the use it makes of the Laplacian's vanishing as meaning that the value at a point is equal to the local average, is highly attractive. The connection with the balancing of tensions in a soap film, or of electrical potential in a conductor, is far more apparent in this approach.

3.312. The eigenvalue problem for L (c.f. Corollary 1.21. and Proposition 3.1.) can also be related to a minimisation problem (c.f. [6] p.41.)

3.32. Corollary

If D attains a minimum on $\Psi_g X$ at φ, that minimum satisfies

$$D[\varphi] = \langle g, L_X \varphi \rangle_B$$

Proof

$$\begin{aligned} D[\varphi] &= \langle d\varphi, d\varphi \rangle_X \\ &= \langle \varphi, L_X \varphi \rangle_X \\ &= \langle g, L\,\varphi \rangle_B \text{ , since } \begin{cases} L\,\varphi \equiv 0 & \text{on } X \setminus B \\ \varphi = g \text{ on } B . \end{cases} \end{aligned}$$

■

3.34. Corollary

Let φ be harmonic on $X\backslash B$ and take specified values on B. Then for any point $x \in X\backslash B$,

$$\varphi(x) = \sum_{b \in I(B)} C_{x,b}\varphi(b) .$$

where the coefficients $C_{x,b}$ are independent of φ.

Proof

If the values on B are specified by g, $[\varphi(x_i)]$ is precisely $[L]^{\leftarrow}(\overline{g})$.

The statement follows. ■

Note

Though the above result is essentially a part of the proof here of the main theorem, it is worth drawing attention to the fact that it does so by giving it special mention in this way, since it is proved in [8], p.243 (together with an inequality on the coefficients) as a separate theorem for the special case of a sphere in $Z\times Z\times Z$, in such a way that its algebraic nature is not so readily apparent.

For a given $b \in I(B)$, the coefficients $C_{x,b}$ as a function of x define of course precisely the harmonic function corresponding to the 'boundary conditions' $g = \delta_{bc}$.

4. The Poisson equation

4.0. Proposition

For a fuzzy space X and boundary conditions as in Proposition 3.3, and any function $\psi \in \Psi(X \setminus B)$, there exists a unique function $\varphi \in \Psi_g X$ satisfying

$$L_X \varphi | X \setminus B = \psi$$

Proof

Again we have a system of $n = n(X \setminus B)$ equations, giving this time an affine map $\bar{L} : \mathscr{R}^n \to \mathscr{R}^n$, when we put the values of g on the right hand side. Then if

$$\Psi : \mathscr{R}^n \to \mathscr{R}^n$$
$$\varphi \rightsquigarrow \varphi - \psi$$

is translation by ψ, we have

$$\bar{L} = \bar{\psi}[L]$$

where $[L]$ is as in 3.3.(iii). Hence since $\bar{\psi}$ and $[L]$ are invertible, so is $\bar{L}$, and $\bar{L}^{\leftarrow}(\bar{g})$ gives the unique solution required. ∎

4.01. Corollary

If $x \in B \setminus X$, we may define a Green's function g_x on X by

$$Lg_x(y) = \delta_{xy} \quad , \quad y \in B \setminus X$$
$$g_x(y) = 0 \quad , \quad y \in B \,.$$ ∎

4.02. Corollary

If $x, y \in B \setminus X$ have $x \tau y$, we may define a dipole potential $g_{x,y}$ on X by

$$Lg_{x,y}(z) = \delta_{xz} - \delta_{yz} \quad , \quad z \in B \setminus X$$
$$g_{x,y}(z) = 0 \quad , \quad z \in B \,.$$ ∎

5. The Divergence Theorem

5.0. Proposition

For any 1-form (i.e. skew-symmetric covariant 1-tensor field, or 'covector field') on a fuzzy space X , and for any subset Y of X , we have

$$\sum_Y d_X^* f = -\sum_{\substack{(y,x)\in \\ B_X(Y)}} f(e_y^{\leftarrow}(x)) \ .$$

Proof

We introduce the notation

$$d_{X,Y}^* = d_X^* - d_Y^*$$

By application of 1.01. (D2) we have

$$d_{X,Y}^* f(x) = -\sum_{y\in N(x)\setminus Y} f(e_x^{\leftarrow}(y))$$

$$= 0 \quad \text{if} \quad x \in \mathring{Y}$$

on convector fields. Then

$$\sum_Y d_X^* f = \langle 1, d_X^* f\rangle_Y$$

$$= \langle 1, d_Y^* f\rangle_Y + \langle 1, d_{X,Y}^* f\rangle_Y$$

$$= \langle d(1), f\rangle_Y + \sum_{y\in Y} (1.(-\sum_{x\in N(y)\setminus Y} f(e_y^{\leftarrow}(x)))$$

$$= -\sum_{B_X(Y)} f(e_y^{\leftarrow}(x)) \ .$$

■

5.01. Corollary

For any function $\varphi \in \Psi X$ on a fuzzy space X , and any subset Y of X , we have

$$\sum_Y L_X \varphi = \sum_{\substack{(y,x)\in \\ B_X(A)}} (\varphi(y) - \varphi(x)) \ .$$

Proof

$$\sum_Y L_X \varphi = \sum_Y d_X^*(d\varphi)$$

$$= -\sum_{B_X(Y)} d\varphi(e_y^{\leftarrow}(x))$$

$$= \sum_{B_X(Y)} (\varphi(y) - \varphi(x)) \ .$$

■

Note

The twin preoccupations of the literature on the discrete potential theory of integer-point lattices since - but honorably excepting - [6] being computation of tables of numbers like $\left(-\frac{6508}{3465\pi}\right)$ [9] and the establishing of difference analogues to 19th-century differential results, one might have expected to find in it the Divergence Theorem. However in the absence of the concept of covector field, or of anything of non-zero order, this was not possible.

6. Green's Formula

6.0. Proposition

For any pair, φ , ψ of scalar functions on a fuzzy space X , and any subset Y of X , we have

$$\sum_{y\in Y}(\varphi(y)L_X\,\psi(y) - L_X\varphi(y)\psi(y)) = \sum_{(y,x)\in B_X(Y)}(\varphi(x)\psi(y) - \varphi(y)\psi(x)) .$$

Proof

Using $d^*_{X,Y}$ as in the proof of 5.0 ,

$$\sum_Y(\varphi(y)L_X\psi(y)-L_X\varphi(y)\psi(y)$$

$$= \langle\varphi,L_X\psi\rangle_Y - \langle L_X\varphi,\psi\rangle_Y$$

$$==\langle\varphi,d^*_Yd\psi\rangle_Y + \langle\varphi,d^*_{X,Y}d\psi\rangle_Y - \langle d^*_Yd\varphi,\psi\rangle_Y - \langle d^*_{X,Y}d\varphi,\psi\rangle_Y$$

$$= \langle d\varphi,d\psi\rangle_Y + \langle\varphi,d^*_{X,Y}d\psi\rangle_{I_X(Y)} - \langle d\varphi,d\psi\rangle_Y - \langle d^*_{X,Y}d\varphi,\psi\rangle_{I_X(Y)}$$

$$= \sum_{y\in I_X(Y)}[\varphi(y)(-\sum_{x\in N(y)\setminus Y}d\psi(e^{\leftarrow}_y(x)))] - \sum_{y\in I_X(Y)}[(-\sum_{x\in N(y)\setminus Y}d\varphi(e^{\leftarrow}_y(x)))(\psi(y))]$$

$$= \sum_{(y,x)\in B_X(Y)}(d\varphi(e^{\leftarrow}_y(x))\psi(y) - \varphi(y)d\psi(e^{\leftarrow}_y(x)))$$

$$= \sum_{B_X(Y)}((\varphi(x) - \varphi(y))\psi(y) - \varphi(y)(\psi(x) - \psi(y)))$$

$$= \sum_{B_X(Y)}(\varphi(x)\psi(y) - \varphi(y)\psi(x)) .$$

■

7. Polyharmonic functions

7.00. Definition

A function φ on a fuzzy space X is p-harmonic on a subset Y of X, for $p \in \mathcal{N}$, if

$$L_X^p \varphi | Y \equiv 0 .$$

7.1. Proposition

For a fuzzy space X and an arbitrary subset B of X satisfying $B \cap X' \neq \emptyset$ for all components X' of X, and any function $g \in \Psi B$, there is a unique function in $\Psi_g X$ 2^n-harmonic on $X \setminus B$, for any $n \in \mathcal{N}$.

Proof

Let $X \setminus B = Y$

We have

$$L_X^{2^0} \varphi | Y \equiv 0 \, , \quad \varphi | B \equiv 0 \Rightarrow \varphi \equiv 0$$

by the maximum principle (1.31).

Suppose

$$L_X^{2^n} | Y \equiv 0 \, , \quad \varphi | B \equiv 0 \Rightarrow \varphi \equiv 0 \, ,$$

and

$$L_X^{2p} | Y \equiv 0 \, , \quad \varphi | B \equiv 0 \, , \text{ where } \; p = 2^n .$$

Then we have

$$\begin{aligned} \langle L_X^p \varphi, L_X^p \varphi \rangle_Y &= \langle L_X^p \varphi, L_X^p \varphi \rangle_X - \langle L_X^p \varphi, L_X^p \varphi \rangle_B \\ &= \langle L_X^{2p} \varphi, \varphi \rangle_X - \langle L_X^p \varphi, L_X^p \varphi \rangle_B \\ &= - \langle L_X^p \varphi, L_X^p \varphi \rangle_B \end{aligned}$$

Therefore

$$\langle L_X^p \varphi, L_X^p \varphi \rangle_Y = 0 \qquad (\langle \, , \, \rangle_Y \text{ and } \langle \, , \, \rangle_B \text{ positive})$$

Therefore

$$L_X^p \varphi \equiv 0 \qquad (\langle \, , \, \rangle_Y \text{ definite})$$

Therefore

$$\varphi \equiv 0 \qquad (\text{induction hypothesis})$$

.

The result follows by the same argument as in part (iii) of the proof of Dirichlet's Principle. ■

In precisely the way we proved Proposition 4.0 we may here prove the corresponding analogue, and thus similarly define higher-order Green's functions. I have included this result to further illustrate the economy of argument provided by the formalism of §1, as against the notation traditional with lattice functions. It would seem unlikely that it cannot be extended to all $p \in \mathcal{N}$, rather than just the powers of 2.

8. Random walks on fuzzy spaces.

It is shown in [6], for the special case of a bounded subspace of $Z \times Z$, that if a particle starts at a non-boundary point P and wanders at random from point to neighbouring point, the probability of its passing at some time through a non-boundary point Q without having passed through any boundary point is precisely the value at Q of the Green's function g_P. It is fairly clear that this can be proved also for a general finite fuzzy space, as with Dirichlet's Principle and other non-numerical lattice results, such as Green's Formula. As with Dirichlet's Principle, however, the proof in [6] involves a compactness argument and hence requires that $\mathcal{R}$ be the real numbers. When the result shows that the probability is rational, this does not seem reasonable, and one would hope to be able to avoid it (cf. Remark 3.3.30).

Alternatively, if completeness is essential the natural context for this discussion is that of passage to the continuous case as a limit. (For this purpose the isotropic character of a general fuzzy space would seem to give considerable advantages as against an n-dimensional integer lattice, in which a particle can travel only in 2n directions.) Either of these approaches requires considerable analysis, for which time is presently lacking. I hope to consider them at length in a later paper.

V. Relations with Continuous Mathematics

1. Analogy and connection

The analogies and failures of analogy between fuzzy geometry and topology - general, algebraic and differential - have been a subsidiary theme throughout this paper. Where so much analogy is possible, it is natural to ask what, if any, direct connections are possible. Since of recent years I have been diminishingly fearful of attack by a maddened horned sphere or of being trussed up by a wild knot, and increasingly in agreement that "any use of differential equations presupposes the existence of a three-dimensional mathematical ether which is every bit as pernicious as the physical ether of the last century" [31], I have not usually ordered my work in terms of this question. But just as in the time of Galileo and Kepler, with its theologically-oriented intellectual climate, astronomers automatically considered the relation of their work to the limiting cause, it is impossible now totally to ignore the limiting case.

One obvious approach is to formalise the analogy: establish categorical propositions from which the corresponding theorems in both fuzzy and topology flow. (A high proportion of the category theory that has been established with topology as its ultimate motivation involves the assumption of the Homotopy Extension Property at a very deep level, and is consequently not directly applicable to $\mathfrak{F}$uz: cf I.8.3.)
Another, aspects of which are discussed in §4 and §5, is to consider the case as a limit of fuzzy problems, or as in §3 to look at a single fuzzy approximation of a particular infinitistic mathematical construct.

In the latter approach the infinitistic object cannot be recovered from the fuzzy one, except as a limiting object of a sequence starting from the fuzzy case. However a more direct transfer from $\mathcal{F}uz$ to $\mathcal{T}op$ is possible (though elaborate questions arise in reversing the direction of this transfer) which would seem for instance to be related to the work of Luce. ([18], and cf. I.7.01). It is discussed in the following section.

2. Tolerable topology

2.00. Motivation

If we take the set R of real numbers with a fuzzy of the natural kind (say, $x\tau y \Leftrightarrow |x-y|<1$), the set of fuzzy neighbourhoods of points of R form a sub-basis for the usual topology: every open set in R is a union of finite intersections of open intervals of length 2, and every such union is open. In a similar fashion from a natural tolerance (reverting to the alternative term (I.1.00.) due to Zeeman, for reasons of euphony that will become apparent) and so indeed for the usual topology on R^n and on any closed manifold. Moreover if with the Euclidean norm we give R^n the tolerance

$$x\tau y \Leftrightarrow \|x-y\| < 1$$

not merely do we get the usual topology from it but the hop distance (I.6.00.) between two points is precisely their Euclidean distance rounded up to the next integer. The relation and resemblance between the two is thus very strong: for example, Pythagoras's Theorem is true, up to the fuzzy considered as a limit to fineness of measurement, for the fuzzy space as for the Euclidean one. (This is perhaps why Euclidean geometry has been so successful in describing physical space. "The measurement of distances is relatively simple; the measurement of proximities has never been possible." [27].)

By analogy with metrisability, therefore, we make the following definition:

2.01. Definition

A topological space (X,θ) is tolerable if there exists a tolerance or fuzzy τ on X such that the set $\{N(x) | x \in X\}$ of fuzzy neighbourhoods serves as a sub-basis for the set θ of open sets of X. If a space is not tolerable, it is intolerable.

Tolerability is not implied by metrisability, nor need a subspace of a tolerable space be tolerable, as witness the following counter-example:

2.1. Proposition

The topological space J consisting of the points $\{\frac{1}{n} | n \in \mathcal{N}\} \cup \{0\}$ with the topology induced by inclusion in the space of real numbers (a homeomorph of the one-point compactification of the integers) is intolerable.

Proof

If J is tolerable, consider it as a tolerance space with an appropriate tolerance.

The symbol N( ) throughout this proof will refer to taking the fuzzy neighbourhood, not a topological neighbourhood.

Let $X = N(0)$,

$Y = J \backslash X$,

$Z = X \backslash \{0\}$.

Since X is a set of the sub-basis it is open and is hence a topological neighbourhood of 0 . Y is therefore finite.

Evidently $N(Y) \subseteq J \backslash \{0\}$.

(i) If $N(Y) \neq J\setminus\{0\}$,

consider $x \in (J\setminus\{0\})\setminus N(Y)$.

Since no fuzzy neighbourhood of a point in Y includes x , all fuzzy neighbourhoods that do include x are of points in X , and hence by symmetry also include 0 . Thus $\{x\}$ cannot be expressed as a finite intersection of fuzzy neighbourhoods, hence (being a singleton) it cannot be expressed as a union of such. It is therefore not open, contrary to hypothesis.

(ii) If $N(Y) = J\setminus\{0\}$,

$$N(Y) \cap X = Z .$$

Since Z is infinite, and the set 2^Y of subsets of Y is finite, at least one of the sets

$$\tilde{V} = \cap\{N(y) | y \in V\} , V \in 2^Y$$

must be infinite.

Choose

$$z < \min(\cup\{\tilde{V} | V \in 2^Y, \tilde{V} \text{ finite}\})$$

in Z . Then any finite intersection of fuzzy neighbourhoods of points of X contains a topological neighbourhood of 0 , since they are all open sets containing 0 , and any finite intersection of fuzzy neighbourhoods of points of $J\setminus X = Y$ that contains z contains also an infinite set of points with 0 as an accumulation point. It follows that any finite intersection of fuzzy neighbourhoods of points of J that contains z contains also an infinite set of other points. Hence $\{z\}$ cannot be expressed as a finite intersection of fuzzy neighbourhoods, hence it cannot be expressed as a union of such. It is therefore not open, contrary to hypothesis. ■

Topologists consider a large number of properties which from time to time they will indicate by 'nice'. Among these are for instance connected, locally simply-connected, finite-dimensional, contractible, completely regular, uniform, normal, totally ordered, compact, Hausdorff and metrisable (with a variety of implications between them), but in the face of all of these together I make the following

2.2. Conjecture

The unit interval is intolerable.

▢

I do not have a proof of this, but in the course of seeking one I have established that if the unit interval can be tolerated the tolerance involved must be very peculiar, in contrast to the reasonableness in the case of R^n, discussed above.

The obstacle, as with $\{1/\ \}\cup\{0\}$, is the asymmetry at the end-points. This might suggest the intolerability in general of manifolds-with-boundary, but this is false: the unit n-ball or half-R^n, $n > 1$ may be tolerated with as natural a tolerance as R^n itself. The characterisation of tolerance spaces thus presents problems of a quite different flavour to those of the other characterisation questions of general topology.

Even when a space is tolerable, the toleration need not be in any reasonable sense unique. For example, S^1 may be tolerated by taking

$$e^{i\theta}\tau e^{i\psi} \Leftrightarrow |\theta-\psi| < \varepsilon$$

for a variety of values of ε , and if $\varepsilon = \frac{\pi}{2}$, for instance, the fuzzy space involved has the homology to be expected of an S^1 . But S^1 may equally well - by means of a somewhat larger value for ε - be exhibited in terms of a fuzzy space which is homologically not an S^1 . For example, with $\varepsilon = \frac{3\pi}{4}$ we get 2-homology but no 1-homology.

A related approach is suggested by the observation that taking the distinction fuzzy (cf. I.6.02.(f)) on the set of subsets of a fuzzy space X makes X a proximity space, but this I have not investigated.

3. Approximating Riemannian Manifolds

For any Riemannian manifold M we may take a locally finite Lebesgue covering C such that each set of C is the image of a convex open set by the exponential map at each of its points, select a point $\tilde{c}$ from each $c \in C$ and choose a finite subset $\mathcal{B}_c$ of the tangent space $T_{\tilde{c}}M$ at $\tilde{c}$ such that $\mathcal{B}_c \cap \overleftarrow{e_{\tilde{c}}}(C \cap C^1 \cap \ldots \cap c^i)$ spans $T_{\tilde{c}}M$ for each non-empty intersection of c with other sets of the cover. (This is possible since each $\overleftarrow{e_{\tilde{c}}}(C \cap C' \cap \ldots \cap c^i)$ is open in $T_{\tilde{c}}M$ and hence spans.) Then if

$$X = \bigcup_{c \in C} e_{\tilde{c}}(\mathcal{B}_c)$$

we may make X a fuzzy space by specifying

$$x \tau y \Leftrightarrow \{x,y\} \subseteq c \in C$$

and define a local matroid structure (III.5.02.) on X as follows:

If $B = \{x_1, \dots x_n\}$ is a blob of X, each pair x_i, x_j of points in B are contained in at least one element of C. Hence they are contained in the image of the exponential map at each of them and we have a well defined choice of a geodesic path between them. If gB is the set of points of all such paths between pairs in B, choose x_i in B and define

$$\rho(B) = \text{vector space rank of } (\overleftarrow{e}_{x_i}(gB)) .$$

This is well defined, for suppose we had chosen $x_{i'} \in B$, $i' \neq i$. Both e_{x_i} and $e_{x_{i'}}$ carry straight segments to geodesics, and $x_i, x_{i'}$ are each contained in the image of the exponential map at the other. Hence $\overleftarrow{e}_{x_{i'}} \circ e_{x_i}$ defines a homeomorphism between convex open subsets of $T_{x_i}M$ and $T_{x_{i'}}M$ which preserves straight segments, and thus extends to a projective map which is an isomorphism and therefore preserves dimension.

It may then be verified that the fuzzy space X with this local matroid structure is a fuzzy manifold, with the same cohomology as M, so that examples of fuzzy manifolds are easily multiplied by anyone who believes in differential geometry and requires motivation for the definition in III§5. Moreover the Lebesgue number of the covering may be made as small as desired, so that we have as fine an approximate version of the Riemannian manifold as we wish. This gives the most general setting for the question of the convergence of difference equations and their solutions to the differential case, of which more in the next section.

4. Difference schemes

In contrast to the general concept of fuzzy space, which has received only the freehand attention of Poincare and Zeeman, and the qualitative theory of partial difference equations as objects in their own right, on which discussion is scanty, the questions concerning the convergence of the solution of a difference equation on a lattice of points with mesh $h \to 0$ to the solution of a differential equation is the object of a substantial literature. (For instance, for a considerable body of theory and a discussion of the literature, see [13].)

The orientation throughout, however, is alien to the spirit of the present work; while I find fuzzy spaces more convincing for descriptive purposes than the notion of space which leads to the Banach-Tarski paradox, I am interested in the possibility of using infinitistic techniques to prove theorems about fuzzy spaces, in the manner that the Hardy-Littlewood number theory uses analysis to establish properties of the things we count beans with. But in the literature the question is never "how well does that differential equation provide us with a manipulative technique for handling this difference equation?" but "how well does that difference equation provide us with a computational technique for handling approximately this differential equation?", with subsidiary questions such as "how much extra error do we get from so much error in the initial/boundary conditions?" and "how many steps will we need on the machine?".

The restriction to lattice functions in the literature, together with the direction in which the question of connection is supposed important, makes application to the type of question of interest here problematic. However, for those who believe in differential equations but are sufficiently on the pure-mathematical side of the great division to prefer the objects of discussion to be coordinate-free, the previous section indicates a possible route by which at least the questions of convergence and its uniformity can be approached in a more general manner. Moreover in the light of Chapters III and IV these questions may now be considered for objects of higher order than the lattice functions investigated heretofore.

5. Constructive analysis

In §3 and §4 a differential object was taken as given, together with all the usual structures on it, and the question discussed of the relationship of these structures, and problems within them, to limits of corresponding problems and structures on fuzzy spaces embedded in the given object.

However, not every mathematician will take an uncountable object as given so readily, and various methods of working with far more restrictive concepts of 'proof', 'number', etc. have been evolved. In [26], first the real line is considered as a sequence of finite strings of points and then (p.53) a "basic structure" is defined to be a sequence of fuzzy spaces $\{A_n\}$ such that A_n is a subset of A_{n+1}, and if $a \tau b$ in A_n, in A_{n+1} $N(a) \cap N(b) \not\subseteq \emptyset$. (The hop distance is introduced at each level, and the product of basic structures corresponds to the product in Fuz(I.1.4.) taken at each level.) Thus the differential/analytic object itself is conceived as a kind of 'limit' of fuzzy spaces, just as a real number can be indentified with a Cauchy sequence converging to it, the operations on it should emerge naturally as a similar 'limit' of fuzzy space operations. It seems very possible that material of the type in this paper combined with the approach of [26] may yield an effective approach both to constructive topology and to the handling of fuzzy space questions by way of limits. Moreover the full-bloodedly 'infinite as potential rather than actual' or as 'limit of finite' nature of this attack might make it a better tool for the investigation of convergence questions than the current assumption that the infinite object is fundamental.

I hope, with John Staples, to explore these possibilities.

VI. Problems and Possibilities

Apart from the conjectures and growth-points already presented by the latter half of this paper, some of them minor queries and some of them major research areas, there are a number of points of interest that deserve study, and some of them are discussed here. Since their nature is so various, I do not expect to be able to make all of them the subject of my own research in the immediate future.

1. Fuzzy knot theory

With definitions of the obvious type, three points are immediately clear:

(i) There are no wild fuzzy knots.

(ii) All fuzziomorphisms being isometries (cf. 6.01) the topological definition of a knot cannot be usefully transferred to $\mathcal{F}$uz, and equivalence of knots must be defined here in terms of isotopies.

(iii) Since the topological results on tame knots can successfully be checked with string, their infinitistic presentation is spurious, and they can reasonably be expected to hold for fuzzy knots.

Question 1

Develop a theory of fuzzy knots.

Question 2

Having learnt how to set up knot theory in terms of isotopies, see whether the techniques involved are useful in the topological case.

2. Compact Spaces

In II.11.8 a definition of compact fuzzy spaces is made, and II.11.9 and II.12.2 are conjectures concerning them. It is clear that if a tolerable topological space (V §2) is compact so also is a fuzzy space by which it is tolerated (though the converse is false) so that this definition will be of significance also in the theory of tolerability.

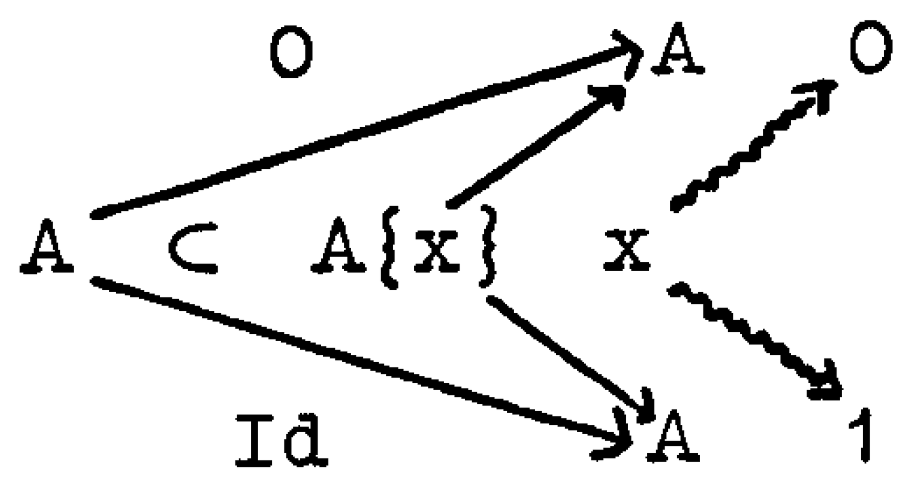
Question 3

Develop the theory of compact fuzzy spaces, prove or disprove II.11.9 and II.12.2, find out to what extent the other finite results of this paper can be extended to the compact case, and investigate the relationship with tolerable compact topological spaces.

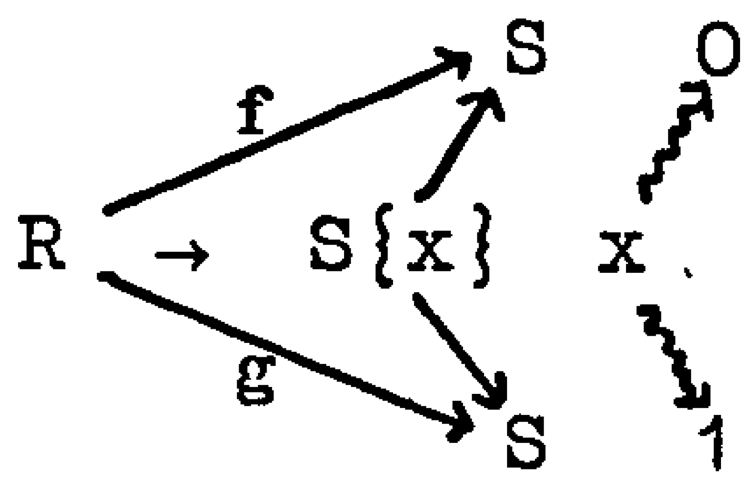
3. Algebraic K-theory

This paper will have served a useful purpose if it communicates the idea that a set with a symmetric reflexive relation - a fuzzy space - is a geometric object in a very strong sense. (The rigidity involved in the necessity that an isomorphism be an isometry illustrates this admirably.) It is thus appropriate whenever a symmetric reflexive relation arises to consider it from this point of view, as well as the ways one was previously looking at the area in which it has cropped up. This is well exemplified by recent work in algebraic K-theory. (For convenience of exposition I shall suppose all rings to have a unit, though the work to which I refer operates more generally by adjoining a unit and later discarding it.)

Karoubi and Villamayor [17] define a Banach ring A to be contractible if there is a commutative diagram



where A{x} is the ring of absolutely convergent series in A . This is extended by Gersten [12] to the definition that two ring homomorphisms f, g : R → S are simply homotopic if there is a commutative diagram



This relation however is reflexive and symmetric but intransitive, and homotopy in general is therefore taken as the equivalence relation generated by it. But this is precisely to look at a fuzzy space only in terms of its O-connectivity; surely losing a great deal of information, just as one would by ignoring the higher connectivities of a topological space. It can be of importance, for example, to know whether a particular space of functions is contractible. It is natural, therefore, to pose

<u>Question 4</u>

Apply fuzzy geometry to algebraic K-theory.

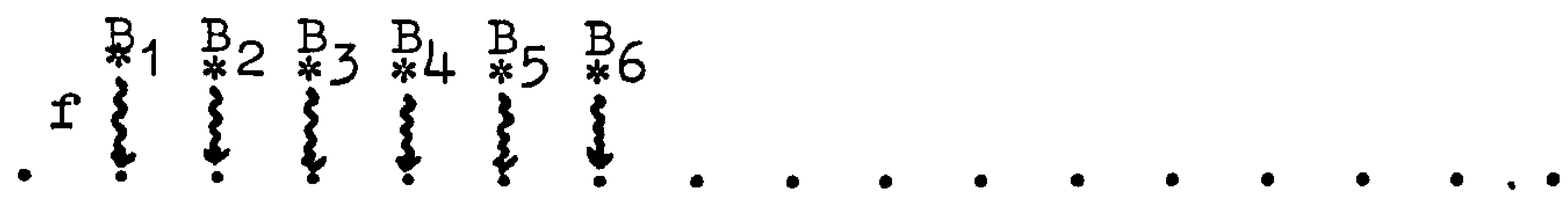
4. Dynamics

The theory of Chapter IV suggests a programme of developing a more general theory of elliptic difference operators, and it is my intention to follow this up. However, although elliptic differential operators are important in physics they are essentially concerned with space rather than space-time; electro- and magnetostatics, and steady situations generally, rather than wave phenomena. If, as at present seems possible, this material is to be of major relevance to physics, it will be necessary also to consider the hyperbolic case. This has been done with lattice functions, but the generalisation to the coordinate-free setting of fuzzy geometry presents considerable difficulty, which I hope to overcome though at present I am still groping for the appropriate insights. Among the reasons for devoting effort to this end are that by eliminating infinitism from physics one will necessarily eliminate the infinities with which the subject is currently plagued, that it is philosophically more wholesome to describe quantized, approximate events as occurring in a quantized, approximate setting, and the possibility of a further gain as follows:

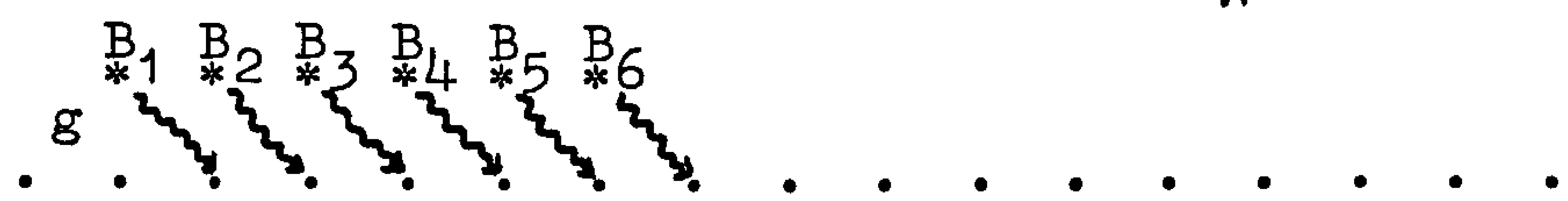
The present theory of quantum mechanics is only very uneasily related to the theory of relativity. A great deal of work is done by assuming a Newtonian space-time and then applying a 'relativistic correction' to the result. Such an approach does not lead to very coherent theory. It is possible, and illuminating, to treat magnetic forces as a relativistic correction to electrostatic ones [11], but this is no substitute for a unified theory of electromagnetism. Now it seems highly probable that a properly formulated fuzzy theory of the wave operator, and, by extension, of for example Maxwell's Laws and the Schrödinger equation must be essentially relativistic if it is possible at all. This is partly because in the absence of a local vector space structure the formulation of the theory can only be coordinate-free (though in numerical computation it may be convenient to assume some regularity of structure), and if space-time is described in a coordinate-free way with a finite light-speed it is not easy to be non-relativistic, but there is also a further consideration. Though it is suggestive only, it arises so naturally that it can hardly be a mere curiosity.

Allowing now space and time to be fully distinct again, as in Newtonian mechanics, but with fuzzy and thus difference structures rather than differential, one looks for a Newtonian description of motion. The motion of a 'point mass' or 'particle' will be described by a map from time T into space S, which must clearly in this setting be fuzmic if we are not to permit the particle to teleport. Hence by I.6.01 the map must be metric-decreasing; thus we have on

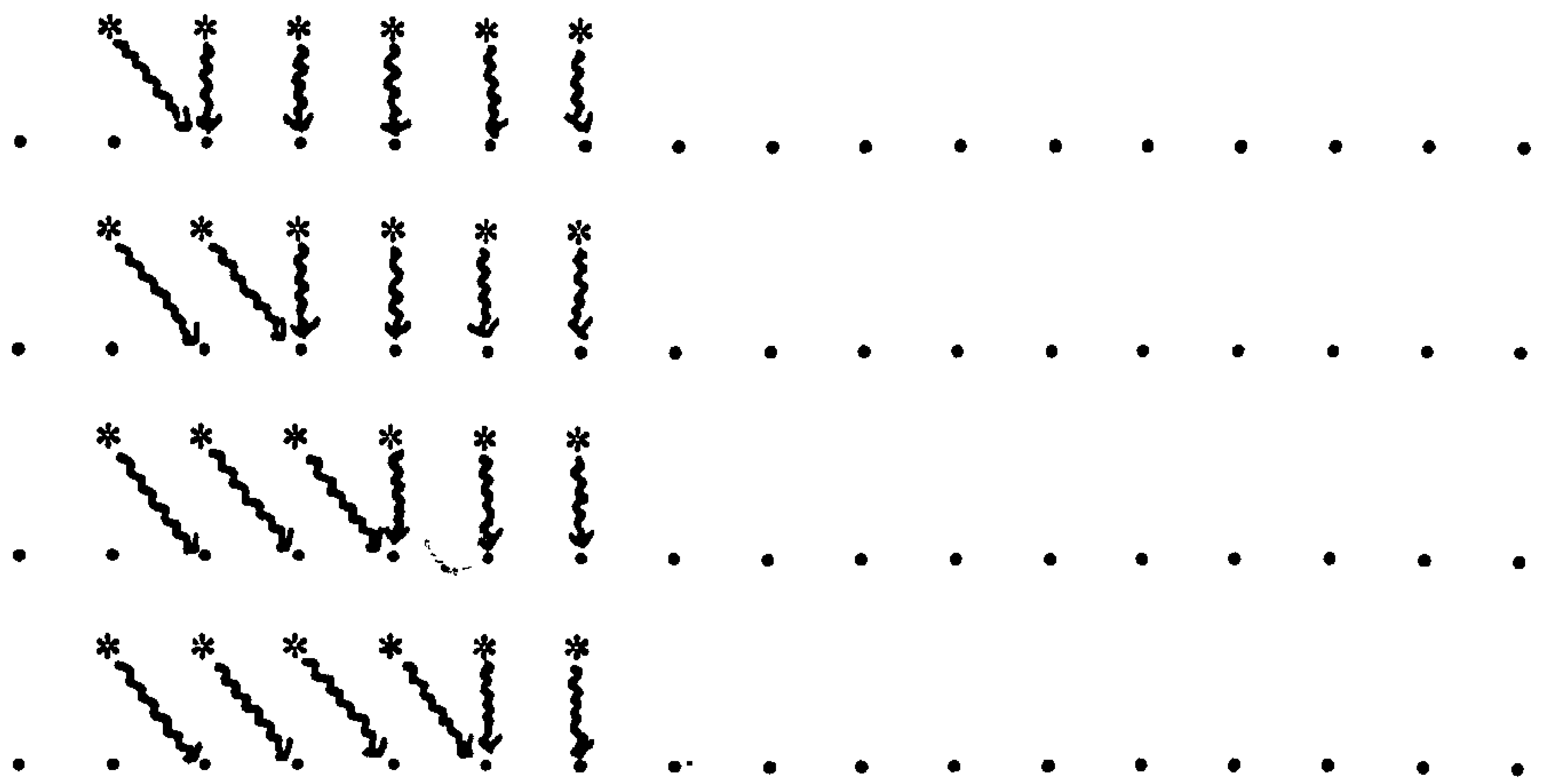
Newtonian assumptions a limiting velocity. Moreover, if we consider the motion of a body B of larger size, a description of the motion must involve a map from T to the function space S^B of positions of B. But in this case a further phenomenon appears, which can most conveniently be described in the case of one dimension each for space and time, with each a fuzziomorph of the integers with the fuzzy "$x\ \tau\ y \Leftrightarrow |x-y| \leqslant 1$", and a body five hops long. (However, nothing in the following reasoning restricts us to this simplified case.) Now if the position of B at time t is represented by



its position at time (t+1) cannot be that represented by

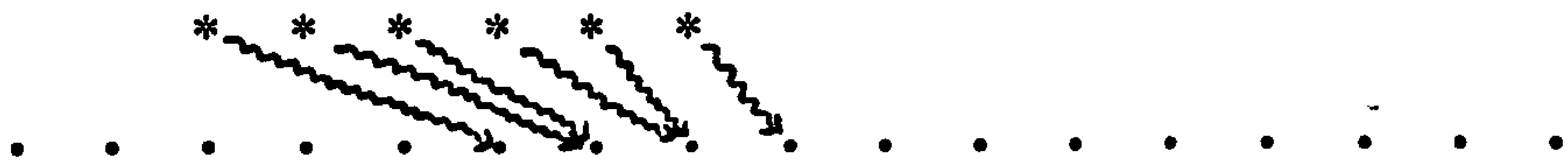


since $(f,g) \notin \tau^\tau$ (cf. I.3.01) because for example $(f(B_5)\ ,\ g(B_6)) \notin \tau$. The position g is connected to f , but only by a path via



and

which is six hops along. Thus B is moving at only a fraction of the limiting velocity, and while in motion the space it occupies is one hop shorter. Faster motion will involve positions of the form



with more than one wrinkle passing through at once (and more shortening) and only by way of positions of the form

can it travel at the limiting velocity. (Notice also that inertia is involved; though it is possible to move B one hop of space in less than six hops of time, this cannot be done from a 'standing start' like f as a result of the time needed to accelerate.) In sum, we have

(a) There exists a limiting velocity.

(b) Motion is essentially a wave phenomenon: rigid body motion is impossible.

(c) Wave phenomena themselves, such as the compression waves above of which the motion of B was composed, travel - unlike B itself - at the limiting velocity. Thus since light is a wave phenomen it may be expected to travel at the limiting velocity; put the other way round, the limit is precisely the speed of light.

(d) Faster motion in any direction involves contraction in that direction until zero length is reached at the speed of light.

These observations do not amount to an adequate physical theory, for a variety of reasons, but they do show that the attempt to frame even Newtonian physics in fuzzy-geometric terms leads to relativistic conclusions. Against such a background it would seem implausible that a properly framed theory of the wave operator and its application in physics in these terms should be other than a synthesis of quantum mechanics and relativity if it works at all. Hence I have considerable interest in

Question 5

Set up the theory of hyperbolic operators on fuzzy spaces and apply it to the development of a finitistic quantum mechanics.

5. Biology

As mentioned in the Introduction, both Poincaré and Zeeman introduced fuzzy spaces in the context of perception and dimension. Zeeman developed this approach to a very general theory of the workings of the brain. The result met with very heavy opposition from biologists, on often innumerate grounds; for example it is mathematically innocuous (whether or not it is significant) to point out the rates of firing of 10^{10} neurones constitute a 10^{10}-dimensional phase space, but this was sometimes understood as an assertion that the brain has ten billion physical dimensions! The theory does however contain difficulties that it is not obvious how to rectify. The first paper in which it is presented [31] ends with a number of questions concerning this biological application of fuzzy spaces, such as the devising of experiments to create 7-dimensional vision†, and finally a sixth question: "Develop the appropriate structures on tolerance spaces to express the laws of physics." Since this paper contains a possible contribution to an answer to this question, it would perhaps be pleasantly symmetrical to end with

Question 6

(i) Develop the theory of Zeeman to a more fully satisfactory description of the brain.

(ii) Convince the biologists.

---

† Surely all a topologist needs is 5? Then he can see whether the Poincaré conjecture is true!

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