Some Problems in Combinatorial Topology of Flag Complexes

by

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Declarations

This thesis is the author's own work or work based on joint research. The following parts of this work have been published, accepted for publication, or circulated as preprints.

Chapter	Reference	Joint with
Sections 2.2, 2.3, 2.4 and part of Section 2.5; Chapter 3	[2], J. Comb. Th. A	
part of Section 2.5	[6], Disc. Math.	J.A. Barmak
Sections 4.1, 4.3, 4.4, 4.5	[8], SoCG'12	J. Stacho
Chapter 5	[4], preprint	
Chapter 6	[5], preprint	
Chapter 7	[3], Isr. J. Math.	
Chapter 8	[7], preprint	J. Hladký

The thesis has not been submitted for a degree at any other university.

Abstract

In this work we study simplicial complexes associated to graphs and their homotopical and combinatorial properties. The main focus is on the family of flag complexes, which can be viewed as independence complexes and clique complexes of graphs.

In the first part we study independence complexes of graphs using two cofibre sequences corresponding to vertex and edge removals. We give applications to the connectivity of independence complexes of chordal graphs and to extremal problems in topology and we answer open questions about the homotopy types of those spaces for particular families of graphs. We also study the independence complex as a space of configurations of particles in the so-called hard-core models on various lattices.

We define, and investigate from an algorithmic perspective, a special family of combinatorially defined homology classes in independence complexes. This enables us to give algorithms as well as NP-hardness results for topological properties of some spaces. As a corollary we prove hardness of computing homology of simplicial complexes in general.

We also view flag complexes as clique complexes of graphs. That leads to the study of various properties of Vietoris-Rips complexes of graphs.

The last result is inspired by a problem in face enumeration. Using methods of extremal graph theory we classify flag triangulations of 3-manifolds with many edges. As a corollary we complete the classification of face vectors of flag simplicial homology 3-spheres.

Chapter 1

Introduction

This chapter contains an overview of the contents of the thesis. Detailed introductions to the particular topics can be found in the respective chapters. In Section 1.2 we collect the common notation and other basic prerequisites used throughout the work.

1.1 Overview

The objects we study in this work are *simplicial complexes*, combinatorial models of topological spaces. We are particularly interested in the subclass of simplicial complexes called *flag*. These are the maximal simplicial complexes with a given 1-skeleton or, equivalently, the complexes with no missing faces other than edges. They include such well-studied families as barycentric subdivisions of simplicial complexes, Vietoris-Rips complexes of metric spaces, order complexes of posets, chessboard and matching complexes and many others. Finite flag complexes model all homeomorphism types of finite CW-complexes.

The main feature of a flag complex is that it is completely determined by its 1-skeleton, which is a graph. Therefore there are two equivalent viewpoints of flag complexes: as clique complexes $\mathrm{Cl}(G)$ and independence complexes $I(G)$ of graphs G. Their faces correspond, respectively, to the cliques and independent sets (stable sets) of G. The perspective of independence complexes dominates in Chapters 2 – 6, while the clique-complex approach is followed in Chapters 7 and 8.

The tools which allow us to study independence complexes functorially are introduced in **Chapter 2**. They are based on two cofibre sequences which describe the behaviour of $I(G)$ under vertex and edge removals in G (Propositions 2.1 and

2.4). This unified approach allows us to present simpler proofs of a number of known results.

This chapter also includes many elementary examples and applications based on those techniques. In Section 2.5 we present a counterexample to the general version of the Aharoni-Berger-Ziv conjecture about the connectivity of independence complexes and we give a new simple proof that the conjecture holds for chordal graphs. In Sections 2.6 we look at an extremal problem of maximizing the Betti number of a flag complex with a given number of vertices.

Chapter 2 is based on the results of [2] and [6].

In **Chapter 3** we apply the splitting techniques of Chapter 2 to answer a question of Kozlov, who asked about the homotopy types of $I(C_n^r)$. Here C_n^r denotes the r-th power of the cycle C_n, that is a graph in which two vertices of the n-cycle are adjacent if they are no more than r steps away. Our main result, Theorem 3.1, gives a recursive formula for computing the homotopy type of $I(C_n^r)$. This chapter contains the core result of the paper [2].

Chapter 4 is about algorithms and complexity for homology calculations in flag complexes. We first set up a construction (Definition 4.1) which produces a non-zero homology class, called a cross-cycle, in $\widetilde{H}_*(I(G))$. In combinatorial terms it is determined by a specific type of induced matching in G, making it an appealing object from the point of view of algorithmic graph theory. In Theorem 4.7 we show that for chordal graphs G cross-cycles generate $\widetilde{H}_*(I(G))$. It suggests that some topological properties of $I(G)$ might be efficiently decidable for such graphs. Indeed, in Section 4.4 we sketch an algorithm which decides the contractibility of $I(G)$ for a chordal graph G in polynomial time.

This result has a hardness counterpart, Theorem 4.25, which is the NP-hardness of deciding whether a specified homology group of $I(G)$ is non-zero. As a corollary we obtain, in Theorem 4.27, that computing homology groups of arbitrary simplicial complexes given by a list of facets is NP-hard. It appears that this is the first proof of this fact, however obvious and well-known it would seem to be.

Chapter 4 contains the results of the paper [8].

Chapters 5 and 6 are inspired by the interpretation of the independence complex as a space of configurations in a hard-core interaction model on a graph. In this model we consider particles in the vertices of the graph with the restriction that two particles cannot occupy the same spot or two adjacent spots simultaneously, hence all possible configurations form precisely the independence complex of the graph. For a more detailed introduction to this framework, see Section 5.1.

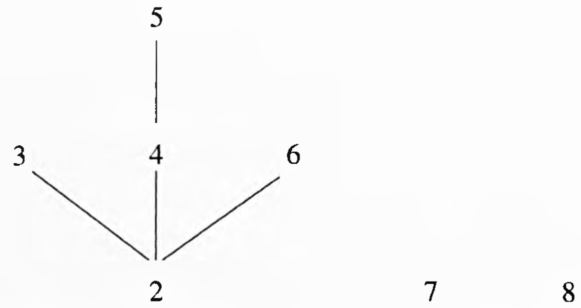


Figure 1.1: An approximate poset of dependencies between the chapters.

In **Chapter 5**, we investigate the phenomenon of *superfrustration*, which occurs when the total Betti number of $I(G)$ is exponential in the number of vertices of G as the size of G increases. We provide a proof of superfrustration for certain shapes of lattice graphs which are of interest in physics (Proposition 5.2). The main tool are the homology classes introduced in Chapter 4. In Section 5.4 we also improve the corresponding upper bounds on the Betti numbers of $I(G)$ for such lattice graphs. This chapter is based on the preprint [4].

In **Chapter 6** we restrict the hard-core model to the *hard-squares* model, which means studying independence complexes of rectangular grids. Those spaces are notoriously difficult to identify, but their Euler characteristics are more tractable and display interesting regularities.

In Theorem 6.1 we show how to lift some existing results about the periodicity of Euler characteristic to actual homotopy equivalences for some grids of small sizes. We use the techniques of Chapter 2. In the second part we derive generating functions for the Euler characteristic of a particular family of square grids, namely cylinders of even circumference. This complements some results by Jonsson. We find many regularities which lead to periodicity conjectures. In the last section one such conjecture is phrased in terms of a particularly simple combinatorial model.

The contents of this chapter appears in the preprint [5].

In **Chapter 7** we treat flag complexes as clique complexes of graphs. Every graph G has canonical homomorphisms $G \hookrightarrow G^r$ into its higher powers and we study the induced maps of clique complexes. Geometrically the spaces $\mathrm{Cl}(G^r)$ are the Vietoris-Rips complexes of the graph G treated as a metric space.

One of our main results, Theorem 7.9, shows the universality of those spaces, which means that for any r every simplicial complex is homotopy equivalent to one of the form $\mathrm{Cl}(G^r)$. The surprise factor of this result is that the graphs of the form

G^r for $r \geq 2$ are of quite special form, so one would expect (as the author first did), that their clique complexes would also be distinguished in some way. The proof of universality is an application of the nerve lemma together with the analysis of some shortest paths in iterated barycentric subdivisions.

In Section 7.5 we completely describe clique complexes of line graphs and related constructions. Section 7.6 ends this chapter with a calculation of $\mathrm{Cl}(C_n^r)$ which is, in a sense, dual to that of Chapter 3. It is an application of the recent technique of star clusters by Barmak.

The results of this chapter will appear in the paper [3].

Face enumeration is one of the main branches of combinatorial topology. In **Chapter 8** we study the face numbers of flag simplicial 3-spheres, and, more generally, flag 3-manifolds. The main result is Theorem 8.2. We show that a flag 3-manifold with high edge density is a join of two polygons, hence a 3-sphere. As a consequence we obtain an almost complete classification of face vectors of flag 3-spheres, as conjectured by Gal (Conjecture 8.1).

We use methods from another major branch of modern combinatorics: extremal graph theory. This is possible because the Dehn-Sommerville relations combined with an appropriate stable version of Turán's theorem allow us to conclude that the 1-skeleton of a very dense flag manifold is very similar to a complete bipartite graph. For a more detailed, but still not too technical description, see the beginning of Section 8.2. To the author's best knowledge it is the first application of such techniques in this area.

The contents of this chapter appears in the preprint [7].

1.2 Notation

In this section we establish the minimal amount of common notation and prerequisites required throughout this work. More specific definitions and results are introduced when they are first used.

Graphs

The reference for notions related to graph theory is [19].

We work with finite, undirected graphs without loops or multiple edges. We write $V(G)$ and $E(G)$, respectively, for the set of vertices and edges of G. Edges will be denoted by $e = (u, v)$ or simply $e = uv$. For a vertex v of G the neighbourhood is $N_G(v) = \{w : vw \in E(G)\}$ and the closed neighbourhood is $N_G[v] = N_G(v) \cup \{v\}$.

If there is no danger of ambiguity we will write just $N(v)$ and $N[v]$. If $e = (u, v)$ is any pair of vertices (not necessarily an edge), we write $N[e] = N[u] \cup N[v]$ for the closed neighbourhood of e. In general, if $X \subseteq V(G)$ is any subset then we define $N[X] = \bigcup_{x \in X} N[x]$.

For a subset $W \subseteq V(G)$ of the vertices let $G[W]$ denote the subgraph of G induced by W. We are going to write $G \setminus v$ and $G \setminus W$ instead of the more correct $G[V(G) \setminus \{v\}]$ and $G[V(G) \setminus W]$. The notation $G - e$ or $G \cup e$ means G with the edge e removed or added.

A subset $\sigma \subseteq V(G)$ is a *clique* if every two vertices in σ are adjacent. A subset $\sigma \subseteq V(G)$ is an *independent set* if every two vertices in σ are non-adjacent.

A vertex v of G is called *simplicial* if $G[N(v)]$ is a non-empty clique. This is a most unfortunate name, given all other meanings of the word "simplicial" in this thesis, but it is traditionally used in graph theory.

A graph G is called *discrete* if $E(G) = \emptyset$ and *empty* if $V(G) = \emptyset$. A graph is called a *cone* if there is a vertex v such that $N_G[v] = V(G)$. A set $X \subseteq V(G)$ is a *dominating set* if $N_G[X] = V(G)$.

The *girth* of a graph is the length of its shortest cycle or ∞ for a forest. The symbol $\overline{G}$ denotes the complement of G and $\sqcup$ is the disjoint union of graphs.

We write $\mathrm{dist}_G(u, v)$ for the length of the shortest path in G from u to v. For a graph G and integer $r \geq 0$ the r-th power G^r is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if their distance in G is at most r.

The symbols K_n, C_n, P_n denote, respectively, the complete graph, cycle and path with n vertices. They are understood to be the empty graph when $n \leq 0$.

Since chordal graphs play a role in a number of points in this work, we will briefly review their most important properties. For a deeper treatment the reader is referred to [19, Chapter 9].

A graph G is *chordal* if it has no induced cycle of length greater than 3. It is a very classical and well-studied class of graphs. One sees from the definition that the property of being chordal is hereditary, i.e. if G is chordal then so is $G \setminus v$ for any $v \in V(G)$. The following is a classical fact about chordal graphs (see [19, Thm. 9.21], originally [30]).

Fact 1.1. *A chordal graph has a simplicial vertex.*

By removing a simplicial vertex we obtain a smaller chordal graph, which again has a simplicial vertex etc. The order in which the vertices are removed is called the *perfect elimination ordering* of a chordal graph. As a consequence we get:

Fact 1.2. *A graph is chordal if and only if it has a perfect elimination order, i.e. the vertices of G can be arranged in a sequence $v_1, \dots, v_n$ such that the neighbourhood of v_i in $G[v_i, \dots, v_n]$ is a clique for $i = 1, \dots, n-1$.*

Simplicial topology

The references for simplicial complexes and homotopy theory are [72, 51] and an excellent concise introduction can be found in [96, Chapter 0.3].

A simplicial complex K will normally be thought of as either a collection of faces or a topological space (the geometric realization) without any change in notation. It will always be clear from the context which interpretation is used.

If K and L are two simplicial complexes with disjoint vertex sets then the *join* $K * L$ is a simplicial complex with vertex set $V(K) \cup V(L)$ whose faces are all unions $\tau \cup \sigma$ for $\tau \in K$, $\sigma \in L$. It is a standard fact that $S^k * S^l = S^{k+l+1}$ for spheres S^k, S^l with $k, l \geq -1$. The *cone* CK is the join of K with one point (the apex) and the *unreduced suspension* is $\Sigma K = S^0 * K$. The symbol $\sqcup$ is the disjoint union of complexes. By $K^{(n)}$ we denote the n-dimensional skeleton of K. Every graph can be treated as a 1-dimensional simplicial complex.

We also have induced subcomplexes. If W is a subset of the vertices of K then $K[W]$ denotes the subcomplex induced by W, i.e. the simplicial complex with vertices W whose faces are all the faces of K contained in W. If v is a vertex of any simplicial complex K then we define the link $\mathrm{lk}_K(v)$, star $\mathrm{st}_K(v)$ and deletion $K \setminus v$ in the usual way [72].

The reduced homology and cohomology groups of K, denoted $\widetilde{H}_*(K)$, $\widetilde{H}^*(K)$, are the homology groups of the augmented chain, resp. cochain complex of K. Unless otherwise indicated, we use either integer or rational coefficients and omit them from notation. We have $\widetilde{H}_i(\Sigma K) = \widetilde{H}_{i-1}(K)$. There is a standard bilinear pairing, denoted $\langle \cdot, \cdot \rangle$:

$$\langle \cdot, \cdot \rangle : \widetilde{H}^i(K) \otimes \widetilde{H}_i(K) \to \mathbb{Q}$$

given by evaluating cochains on chains. The i-th reduced Betti number is $\widetilde{\beta}_i(K) = \dim \widetilde{H}_i(K)$ and the *total Betti number* of K is

$$\widetilde{\beta}(K) = \sum_i \widetilde{\beta}_i(K).$$

It satisfies $\widetilde{\beta}(\Sigma K) = \widetilde{\beta}(K)$. We denote by $\widetilde{\chi}(K)$ the reduced Euler characteristic of K.

Note the empty simplicial complex $\emptyset$ which has no vertices and a unique face $\emptyset$. It satisfies $\Sigma\,\emptyset = S^0$, so it is good to think of it as S^{-1}. It has a single reduced homology group $\widetilde{H}_{-1}(\emptyset) = \mathbb{Q}$ and in particular $\widetilde{\beta}(\emptyset) = 1$. Of course for every non-empty space K we have $\widetilde{H}_i(K) = 0$ for $i < 0$.

We recall the basic language of cofibre sequences. For any continuous map $f : A \to X$ the *homotopy cofibre* (or mapping cone) is the space

$$C(f) = (X \sqcup (A \times [0,1]))/f(a) \sim (a,1),\ (a,0) \sim (a',0).$$

If $f : A \hookrightarrow X$ is a subcomplex inclusion then $C(f)$ is just X with a cone over A attached and it is homotopy equivalent to X/A. There is a cofibre (or Puppe) sequence

$$A \xrightarrow{f} X \hookrightarrow C(f) \to \Sigma\,A \xrightarrow{\Sigma f} \Sigma\,X \to \Sigma\,C(f) \to \Sigma^2\,A \to \cdots$$

with the property that every consecutive triple is, up to homotopy, a map followed by its mapping cone. Since the homotopy type of $C(f)$ depends only on the homotopy class of f, we get that if $f : A \to X$ is null-homotopic then $C(f) \simeq X \vee \Sigma\,A$. In particular, if A is contractible then $C(f) \simeq X$.

We write $\bigvee^k X$ for the wedge sum of k copies of a topological space X. In all applications the choice of basepoint for X will not influence the homotopy type of $\bigvee^k X$. The symbol $\equiv$ means homeomorphism and $\simeq$ stands for homotopy equivalence.

Finally Δ_n denotes the n-dimensional simplex (with $n+1$ vertices) and $\partial\Delta_n$ is its boundary.

Discrete Morse theory

We are going to use the following elementary language of discrete Morse theory to describe collapsing sequences (see [72, Chapter 11], [42, 43]).

Definition 1.3. *An* acyclic matching *in a simplicial complex K is a set $M \subseteq K \times K$ of pairs of faces such that*

- *if $(\sigma, \tau) \in M$ then σ is a codimension 1 face of τ,*
- *every face σ belongs to at most one element of M,*
- *there is no cycle*

$$\sigma_0, \tau_0, \sigma_1, \tau_1, \sigma_2, \ldots, \sigma_n, \tau_n, \sigma_0$$

such that $(\sigma_i, \tau_i) \in M$*,* σ_{i+1} *is a codimension* 1 *face of* τ_i *(where* $\sigma_{n+1} = \sigma_0$*), all* σ_i *are distinct and* $n \geq 1$*.*

The faces of K *which do not belong to any element of* M *are called* critical.

Fact 1.4. *If* K *is a simplicial complex with an acyclic matching whose set of critical faces is a non-empty simplicial subcomplex* L *then* K *simplicially collapses to* L*.*

Flag complexes

If G is a graph then $I(G)$ and $\mathrm{Cl}(G)$ denote, respectively, the *independence complex* and the *clique complex* of G. They both have $V(G)$ as vertex set and the faces are, respectively, the independent sets or the cliques in G. Clearly $I(G) = \mathrm{Cl}(\overline{G})$.

Let K be a simplicial complex with 1-skeleton $G = K^{(1)}$. All of the following conditions are equivalent and we say K is a *flag complex* if it satisfies any of them:

- $K = \mathrm{Cl}(G)$,
- $K = I(\overline{G})$,
- K is the (unique) maximal simplicial complex with 1-skeleton G,
- every minimal non-face of K has dimension 1.

If $G \sqcup H$ is the disjoint union of two graphs then its independence complex satisfies

$$I(G \sqcup H) = I(G) * I(H)$$

where $*$ is the simplicial join. In particular, if $I(G)$ is contractible then so is $I(G \sqcup H)$ for any H. If e is understood as the graph consisting of a single edge then $I(e) = S^0$ and $I(e \sqcup G) = S^0 * I(G) = \Sigma\, I(G)$ is the suspension of $I(G)$.

The flag counterpart of $\partial\Delta_n$ is the complex

$$O_n = \underbrace{S^0 * \cdots * S^0}_{n}$$

The flag simplicial complex O_n is the boundary of the n-dimensional cross-polytope. It has $2n$ vertices and it is homeomorphic to S^{n-1}.

If v is a vertex in a flag simplicial complex K then the link $\mathrm{lk}_K(v)$, star $\mathrm{st}_K(v)$ and deletion $K \setminus v$ are all induced subcomplexes and they are also all flag.

Chapter 2

Cofibre sequences of independence complexes and their applications

2.1 Introduction

In this chapter we study the complex $I(G)$ using the natural inclusions $I(G \setminus v) \hookrightarrow I(G)$ and $I(G) \hookrightarrow I(G - e)$ for a vertex v and an edge e. They fit into two cofibre sequences

$$I(G \setminus N[v]) \hookrightarrow I(G \setminus v) \hookrightarrow I(G) \to \Sigma\, I(G \setminus N[v]) \to \cdots\,, \tag{2.1}$$

$$\Sigma\, I(G \setminus N[e]) \hookrightarrow I(G) \hookrightarrow I(G - e) \to \Sigma^2\, I(G \setminus N[e]) \to \cdots\,. \tag{2.2}$$

Results based on various special instances of these sequences are scattered around in the literature, eg. [27, 32, 34, 35, 36, 70, 69, 78, 83]. For example the *fold lemma* of [35], which says that if $N(u) \subseteq N(v)$ then $I(G \setminus v)$ and $I(G)$ are homotopy equivalent, corresponds to the case where the first space in the cofibre sequence (2.1) is contractible. Another interesting situation occurs when the map $I(G \setminus N[v]) \hookrightarrow I(G \setminus v)$ is null-homotopic, as then the cofibre sequence splits and we have an equivalence $I(G) \simeq I(G \setminus v) \vee \Sigma\, I(G \setminus N[v])$. This happens, for example, when $N[u] \subseteq N[v]$ for some vertex u, as in [78].

In Section 2.2 we present a unified approach to results of this kind using (2.1) and (2.2). We also identify combinatorial situations in which the two cofibre sequences split and lead to exact results. Another splitting result of Mayer-Vietoris type is analyzed in Section 2.3. Section 2.4 contains some applications and examples.

In particular, we give quick proofs of some results of [35, 69, 78, 99]. In Section 2.5 we discuss the Aharoni-Berger-Ziv conjecture about the connectivity of independence complexes of chordal graphs. Section 2.6 contains an application of the cofibre sequence (2.2) to the problem of maximizing Betti numbers of flag complexes.

We emphasize that the functorial behaviour of the independence complex under vertex removals and (contravariantly) under edge removals is our key technique. In particular, all homotopy equivalences and splittings we derive are natural, that is induced by some morphisms of the underlying graphs.

2.2 Two cofibre sequences and their splitting

We start with vertex removals. Various parts of the next proposition are well-known.

Proposition 2.1. *There is always a cofibre sequence*

$$I(G \setminus N[v]) \hookrightarrow I(G \setminus v) \hookrightarrow I(G) \to \Sigma I(G \setminus N[v]) \to \cdots .$$

In particular

a) if $I(G \setminus N[v])$ is contractible then the natural inclusion $I(G \setminus v) \hookrightarrow I(G)$ is a homotopy equivalence,

b) if the map $I(G \setminus N[v]) \hookrightarrow I(G \setminus v)$ is null-homotopic then there is a splitting

$$I(G) \simeq I(G \setminus v) \vee \Sigma I(G \setminus N[v]).$$

Proof. Any independent set in G is either contained in $G \setminus v$ or it is the union of $\{v\}$ and some independent set in $G \setminus N[v]$, so we have a decomposition

$$I(G) = S \cup T$$

where

$$S = I(G \setminus v), \quad T = v * I(G \setminus N[v]) \simeq *, \quad S \cap T = I(G \setminus N[v]).$$

Therefore $I(G)$ is the homotopy cofibre of the inclusion $S \cap T \hookrightarrow S$. The statements a) and b) follow from the properties discussed in Section 1.2. □

The "generic combinatorial cases" of a) and b) are the following.

Theorem 2.2 ([35]). *If u, v are two distinct vertices with $N(u) \subseteq N(v)$ then there is a homotopy equivalence*

$$I(G) \simeq I(G \setminus v).$$

Theorem 2.3 ([78]). *If u, v are two distinct vertices with $N[u] \subseteq N[v]$ then there is a homotopy equivalence*

$$I(G) \simeq I(G \setminus v) \vee \Sigma I(G \setminus N[v]).$$

Proof of Theorems 2.2 and 2.3. If u is such that $N(u) \subseteq N(v)$ then the graph $G \setminus N[v]$ has u as an isolated vertex, hence the complex $I(G \setminus N[v])$ is contractible and Theorem 2.2 follows from part a) above. If, on the other hand, u is such that $N[u] \subseteq N[v]$ then the inclusion $I(G \setminus N[v]) \hookrightarrow I(G \setminus v)$ factors through the contractible space $u * I(G \setminus N[v])$, so Theorem 2.3 follows from part b). □

Note that the condition $N(u) \subseteq N(v)$ implies that u and v are not adjacent in G, while $N[u] \subseteq N[v]$ forces them to be adjacent.

A similar discussion applies to edges. If $e = (u, v)$ is an edge then $e \sqcup (G \setminus N[e])$ is the induced subgraph of G whose vertices are u, v and all the vertices of $G \setminus N[e]$. Then we have the following proposition.

Proposition 2.4. *There is always a cofibre sequence*

$$I(e \sqcup (G \setminus N[e])) \hookrightarrow I(G) \hookrightarrow I(G - e) \to \Sigma I(e \sqcup (G \setminus N[e])) \to \cdots.$$

In particular

a) if $I(G \setminus N[e])$ is contractible then the natural inclusion $I(G) \hookrightarrow I(G - e)$ is a homotopy equivalence,

b) if the map $I(e \sqcup (G \setminus N[e])) \hookrightarrow I(G)$ is null-homotopic then there is a splitting

$$I(G - e) \simeq I(G) \vee \Sigma^2 I(G \setminus N[e]).$$

Proof. The first statement is an observation of [83]: any independent set in $G - e$ is either independent in G or it contains both endpoints of e together with some independent set in $G \setminus N[e]$. This gives a decomposition

$$I(G - e) = K \cup L$$

where

$$K = I(G), \quad L = e * I(G \setminus N[e]) \simeq *, \quad K \cap L = I(e \sqcup (G \setminus N[e])).$$

Again, it means that $I(G-e)$ is homotopy equivalent to the homotopy cofibre of the inclusion $K \cap L \hookrightarrow K$. The statements a) and b) follow from the properties discussed in Section 1.2 and the fact that $I(e \sqcup (G \setminus N[e])) = \Sigma I(G \setminus N[e])$. $\square$

As before there are some useful special circumstances when conditions a) and b) can be verified at the combinatorial level.

Definition 2.5. *An edge $e = (u, v)$ in G is called* isolating *if the induced subgraph $G \setminus N[e]$ has an isolated vertex.*

Clearly part a) holds for isolating edges, i.e. the removal of an isolating edge does not change the homotopy type of the independence complex. Note that any such statement can also be used in the opposite direction, that is to say that the insertion of an edge which becomes isolating preserves the homotopy type.

The situations where part b) of Proposition 2.4 applies are more complicated.

Theorem 2.6. *Let $e = (u, v)$ be an edge in G. Suppose $T \subseteq G$ is an induced subgraph which contains the edge e and such that $I(T)$ is contractible and, moreover, for every $x \in T$ we have $N[x] \subseteq N[e]$. Then the inclusion $I(e \sqcup (G \setminus N[e])) \hookrightarrow I(G)$ is null-homotopic. Consequently, there is a splitting*

$$I(G-e) \simeq I(G) \vee \Sigma^2 I(G \setminus N[e]).$$

Proof. The inclusion $I(e \sqcup (G \setminus N[e])) \hookrightarrow I(G)$ factors through $I(G[V(T) \cup (V(G) \setminus N[e])])$. Since neither of the vertices $x \in V(T)$ has an edge to $V(G) \setminus N[e]$, the last graph is in fact $T \sqcup (G \setminus N[e])$, so its independence complex is a join where one of the factors is $I(T) \simeq *$. It means that our inclusion factors through a contractible space. $\square$

The simplest graph which can play the role of T in the last statement is the 4-vertex path P_4, hence we have the next corollary, which will be one of the main tools in Chapter 3.

Theorem 2.7. *Let $e = (u, v)$ be an edge in G. Suppose there are vertices $x, y \in N[e]$ such that $N[x] \cup N[y] \subseteq N[e]$ and the induced subgraph $G[x, y, u, v]$ is isomorphic to the 4-vertex path P_4. Then the inclusion $I(e \sqcup (G \setminus N[e])) \hookrightarrow I(G)$ is null-homotopic.*

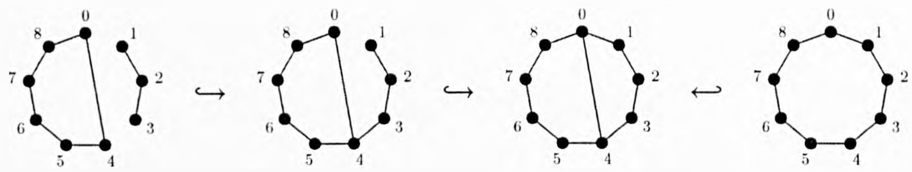


Figure 2.1: Graph inclusions in Example 2.8 which induce a zigzag of homotopy equivalences upon the application of $I(\cdot)$.

Consequently, there is a splitting

$$I(G-e) \simeq I(G) \vee \Sigma^2 I(G \setminus N[e]).$$

Example 2.8. We illustrate the applications of isolating edges by reproving the classical homotopy equivalence

$$I(C_n) \simeq \Sigma I(C_{n-3})$$

first proved by Kozlov [71]. We present the argument in detail as it is the prototype of the methods used in Section 3. See Fig.2.1.

Start with the cycle C_n with vertices labeled $0,\dots,n-1$. Let $C_n' = C_n \cup \{(0,4)\}$. The edge $(0,4)$ in C_n' is isolating because removing $N[0] \cup N[4]$ leaves 2 isolated. By Proposition 2.4.a) it means that extending C_n to C_n' preserves the homotopy type of the independence complex. More precisely, the induced inclusion

$$I(C_n') \hookrightarrow I(C_n)$$

is an equivalence. Now in C_n' the edge $(0,1)$ is isolating as removing $N[0] \cup N[1]$ isolates 3. We can delete $(0,1)$ without affecting the independence complex (up to homotopy). Then in $C_n' \setminus \{(0,1)\}$ the edge $(3,4)$ is isolating as removing $N[3] \cup N[4]$ isolates 1. Again, we can delete $(3,4)$. But the graph we finally obtained, $C_n' \setminus \{(0,1),(3,4)\}$, is a disjoint union of a path $1-2-3$ and C_{n-3} so its independence complex is homotopy equivalent to $S^0 * I(C_{n-3}) = \Sigma I(C_{n-3})$. We obtain a zigzag of equivalences

$$\Sigma I(C_{n-3}) \simeq I(C_n' \setminus \{(0,1),(3,4)\}) \xleftarrow{\simeq} I(C_n' \setminus \{(0,1)\}) \xleftarrow{\simeq} I(C_n') \xrightarrow{\simeq} I(C_n)$$

in which every map is induced functorially by some graph morphism.

Notation 2.9. From now on we are going to abbreviate such arguments by writing:

there is a sequence of isolating operations

$$\text{Add}(0,4;2),\ \text{Del}(0,1;3),\ \text{Del}(3,4;1)$$

which reads: add the edge $(0,4)$, where 2 is the vertex that certifies the isolating property, then remove $(0,1)$ for which 3 is the certificate etc. Note that such sequence of operations is indeed a *sequence*: they may no longer be isolating if performed in a different order. Every isolating sequence generates a zigzag of weak equivalences as in the example.

2.3 Mayer-Vietoris splitting

Combinatorial splittings can also be obtained from the following result.

Theorem 2.10. *Suppose $X, Y \subseteq V(G)$ are two vertex sets which satisfy the conditions:*

- $X \cup Y = V(G)$,
- *the independence complex of $G[X \cap Y]$ is contractible,*
- *every vertex in $X \setminus Y$ has an edge to every vertex of $Y \setminus X$.*

Then there is a splitting

$$I(G) \simeq I(G[X]) \vee I(G[Y])$$

which is natural in the sense that the inclusions $I(G[X]) \hookrightarrow I(G)$ and $I(G[Y]) \hookrightarrow I(G)$ induced by inclusions of $G[X]$ and $G[Y]$ in G are homotopic to the inclusions of the two wedge summands.

Proof. Let $K = I(G[X])$ and $L = I(G[Y])$. First let us check that $K \cup L = I(G)$. Suppose σ is an independent set in G and $\sigma \notin L$. Then σ must have a vertex v in $X \setminus Y$. The third condition implies that σ cannot have any vertices in $Y \setminus X$, therefore $\sigma \subseteq X$ which means $\sigma \in K$. That completes the verification.

Now $I(G)$ is the union $K \cup L$ of two subcomplexes such that $K \cap L = I(G[X \cap Y])$ is contractible. Then there is an equivalence $I(G) \simeq K \vee L$. □

We can use it to identify the graph inclusions corresponding to the two summands in Theorem 2.3 and Theorem 2.6.

Proof of Theorem 2.3 from Theorem 2.10. We use the previous theorem with $X = (V(G) \setminus N(v)) \cup \{u\}$ and $Y = V(G) \setminus \{v\}$. Clearly $X \cup Y = V(G)$. Since $X \cap Y = \{u\} \cup (V(G) \setminus N[v])$ and u does not have any edges to $V(G) \setminus N[v]$, the induced graph $G[X \cap Y]$ has u as an isolated vertex, so the complex $I(G[X \cap Y])$ is contractible. Finally $X \setminus Y = \{v\}$ and $Y \setminus X = N(v)$ so the third condition in Theorem 2.10 is automatically satisfied.

In the splitting obtained from Theorem 2.10 the complex $I(G[Y])$ is $I(G \setminus v)$. The graph in the second summand, $G[X]$, is the disjoint union of an edge $e = (u, v)$ with $G \setminus N[v]$. This is because neither v (by definition) nor u (by assumption) have edges to $V(G) \setminus N[v]$. It follows that $I(G[X]) = I(e) * I(G \setminus N[v]) = S^0 * I(G \setminus N[v]) = \Sigma\, I(G \setminus N[v])$ and the proof is complete. □

Proof of Theorem 2.6 from Theorem 2.10. Let $H = G - e$. First extend H to a bigger graph H_+ by adding an extra vertex w with edges to $N(u) \cup N(v)$. The inclusion $I(H) \hookrightarrow I(H_+)$ is an homotopy equivalence by Proposition 2.1.a) (because $H_+ \setminus N[w]$ contains isolated vertices u, v). In H_+ the operation $\mathrm{Add}(u, v; w)$ is isolating. Let G_+ denote the resulting graph. It contains G as $G_+ \setminus w$ and $I(G_+) \simeq I(G - e)$.

Set $X = (V(G) \setminus N[e]) \cup V(T) \cup \{w\}$ and $Y = V(G)$. Clearly $X \cup Y = V(G_+)$. Since $X \cap Y = V(T) \cup (V(G) \setminus N[e])$ and $V(T)$ has no edges to $V(G) \setminus N[e]$, the induced graph $G[X \cap Y]$ contains T as a connected component and $I(G[X \cap Y])$ is contractible. Finally $X \setminus Y = \{w\}$ and $Y \setminus X \subseteq N(w)$ so the third condition in Theorem 2.10 is automatically satisfied.

In the splitting of $I(G_+)$ obtained from Theorem 2.10 the complex $I(G_+[Y])$ is $I(G)$. The graph in the other summand, $G_+[X]$, is the disjoint union of $G_+[V(T) \cup \{w\}]$ with $G \setminus N[e]$. But $I(G_+[V(T) \cup \{w\}])$ consists of the contractible subspace $I(T)$ together with two edges wu and wv, so it is homotopy equivalent to S^1. It follows that $I(G_+[X]) \simeq S^1 * I(G \setminus N[e]) = \Sigma^2\, I(G \setminus N[e])$ and the proof is complete. □

2.4 Corollaries and examples

We start with a simple application of isolating edges to a known reduction result.

Lemma 2.11 ([27, 78, 13]). *Let G be a graph and $e = (x, y)$ an edge. If G' is obtained from G by replacing e with a path $x - u - v - w - y$ with 3 new vertices then $I(G') \simeq \Sigma\, I(G)$.*

Proof. There is a sequence of isolating operations in G':

$$\mathrm{Add}(x, y; v),\ \mathrm{Del}(x, u; w),\ \mathrm{Del}(y, w; u)$$

which results in the graph $\{(u, v), (v, w)\} \sqcup G$. $\square$

A more general result we can recover using isolating operations follows also from the main theorem of [13].

Lemma 2.12. *If v is a vertex of G of degree 2 with neighbours u, w which satisfy $N[u] \cap N[w] = \{v\}$ then $I(G)$ is homotopy equivalent to $\Sigma I(G')$ where G' is obtained from G by removing u, v, w and spanning a complete bipartite graph between vertices which belonged to $N[u]$ and those from $N[w]$.*

Proof. Denote $U = N[u] \setminus \{u, v\}$ and $W = N[w] \setminus \{w, v\}$. We can first perform all isolating insertions $\mathrm{Add}(x, y; v)$ for all pairs $x \in U$, $y \in W$ which had not already been an edge. Then we can perform isolating deletions $\mathrm{Del}(u, x; w)$ for $x \in U$ followed by $\mathrm{Del}(w, y; u)$ for $y \in W$. We end up with $\{(u, v), (v, w)\} \sqcup G'$ and conclude as before. $\square$

Let us mention two more specializations of Theorem 2.3.

Corollary 2.13. *If u is a vertex of degree 1 and v is its only neighbour then $I(G) \simeq \Sigma I(G \setminus N[v])$.*

Proof. The vertices u and v satisfy the assumptions of Theorem 2.3. Moreover $G \setminus v$ has u as an isolated vertex so $I(G \setminus v)$ is contractible. $\square$

Corollary 2.14 ([35, 36, 69]). *Let u be a vertex such that $N(u)$ is a clique. Then there is a homotopy equivalence*

$$I(G) \simeq \bigvee_{v \in N(u)} \Sigma I(G \setminus N[v]).$$

Proof. Let $v \in N(u)$ be any vertex. Then $N[u] \subseteq N[v]$ so $I(G)$ splits into $\Sigma I(G \setminus N[v])$ and $I(G \setminus v)$. Let $G' = G \setminus v$. In G' the neighbours of u again form a clique so by induction $I(G')$ splits as $\bigvee_{v' \in N_{G'}(u)} \Sigma I(G' \setminus N_{G'}[v'])$. However, since (v, v') is an edge in G for all $v' \in N_{G'}(u)$ we have $G' \setminus N_{G'}[v'] = G \setminus N[v']$ which together with the first summand gives the desired splitting. $\square$

Example 2.15. Suppose G is a connected graph with n vertices and m edges. Let G_3 denote the graph obtained from G by subdividing each edge into 3 parts. Let

e be any of the "middle" edges of G_3, that is edges connecting two subdividing vertices.

In $G_3 - e$ we have two vertices of degree 1 and we see that it can be reduced to the empty graph by successfully applying Corollary 2.13 n times, once for each vertex of G. In $G_3 \setminus N[e]$ the situation is similar, but this time we perform one reduction for each of the remaining $m-1$ edges. It means that $I(G_3 - e) \simeq \Sigma^n \emptyset = S^{n-1}$ and $I(G_3 \setminus N[e]) \simeq \Sigma^{m-1}\emptyset = S^{m-2}$ and the cofibre sequence (2.2) becomes

$$S^{m-1} \to I(G_3) \to S^{n-1} \to S^m \to \Sigma I(G_3) \to S^n \to S^{m+1} \to \cdots .$$

If G is not a tree then $m > n-1$ so the map $S^{n-1} \to S^m$ must be null-homotopic and we get $\Sigma I(G_3) \simeq S^n \vee S^m$. This almost recovers the result of Csorba [27] who proved that in fact $I(G_3) \simeq S^{n-1} \vee S^{m-1}$.

Example 2.16. Fix a field $\mathbf{k}$ and suppose G is a graph with n vertices. Consider the polynomial ring

$$R = \mathbf{k}[x_1, \ldots, x_n]$$

whose generators have degree 1 and correspond to the vertices of G. The ideal

$$R \supseteq I_G = \big(x_i x_j \ : \ x_i x_j \in E(G)\big)$$

generated by square-free monomials of degree two is called the *edge ideal* of G. It is a special case of the Stanley-Reisner ideal of a simplicial complex (in fact R/I_G is the Stanley-Reisner ring of $I(G)$). See [32] and the references therein for more information.

Let H_G be the graph defined by

$$\begin{aligned} V(H_G) =& \{v_0, v_1, v_2, v_3, v_4, v_5 \mid v \in V(G)\}, \\ E(H_G) =& \{u_0 v_0 \mid u, v \in V(G), uv \in E(G)\} \\ & \cup \ \{v_0v_1, v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_3v_5, v_4v_5 \mid v \in V(G)\}. \end{aligned}$$

In other words, H_G is obtained from G by attaching at each vertex a 5-vertex gadget as shown in Fig.2.2.

Note that the neighbourhood of the vertex v_2 is the edge v_1v_3, so the assumptions of Corollary 2.14 are satisfied and we can apply the corresponding splitting. When $N[v_1]$ is removed from H_G, we lose the vertex v_0 and gain an isolated edge v_4v_5. If, on the other hand, $N[v_3]$ is removed then the whole gadget attached at v_0 disappears but v_0 remains. Applying Corollary 2.14 consecutively to all the vertices

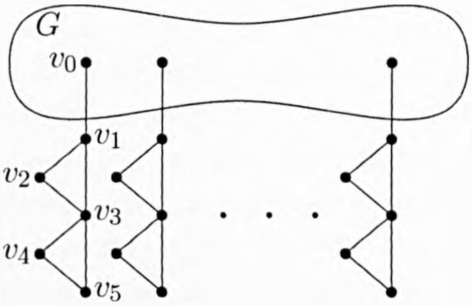


Figure 2.2: The Hochster graph H_G of a graph G.

v_2, for $v \in V(G)$, we therefore get a splitting

$$I(H_G) \simeq \bigvee_{W \subseteq V(G)} \Sigma^{2n-|W|} I(G[W]).$$

(In each summand W corresponds to the set of vertices v for which $N[v_3]$ was removed.)

It follows that the reduced Betti numbers (over $\mathbf{k}$) of $I(H_G)$ are

$$\begin{aligned}\widetilde{\beta}_i(I(H_G)) &= \sum_{W \subseteq V(G)} \widetilde{\beta}_{i-2n+|W|}(I(G[W])) \\ &= \sum_{W \subseteq V(G)} \widetilde{\beta}_{|W|-(2n-i-2)-2}(I(G[W])) \\ &= \dim_{\mathbf{k}} \operatorname{Tor}^R_{2n-i-2}(I_G; \mathbf{k})\end{aligned}$$

where the last equality is precisely the well-known Hochster's formula [96, Thm.4.8]. The numbers $\dim_{\mathbf{k}} \operatorname{Tor}^R_i(I_G; \mathbf{k})$ are called the algebraic Betti numbers of the edge ideal I_G and are subject of intensive study in commutative algebra. The above construction exhibits them as topological Betti numbers of $I(H_G)$ for a relatively small graph H_G.

2.5 Application: The Aharoni-Berger-Ziv conjecture

In [10], Aharoni, Berger and Ziv proposed a function ψ defined on graphs which is a lower bound for the connectivity of $I(G)$ and conjectured that this bound is optimal. No explicit proof of this bound is given in that article, although the corresponding bound for the homological connectivity follows immediately from a result of Meshulam [83, Claim 3.1]. Moreover, a homological version of the conjecture has been considered, as well as reformulations taking into account the existence of counterexamples in which the independence complex is simply-connected or not [11].

In this section we give an explicit proof of the fact that $\psi(G)$ is a lower bound for the connectivity of $I(G)$, we prove that the conjecture is true in the cases where $I(G)$ is not simply-connected or where $\psi(G) \leq 1$, we show that there exist counterexamples to the conjecture with $\psi(G) = 2$, and that there are counterexamples in which $\psi(G)$ and the connectivity of $I(G)$ take arbitrary values l, k with $3 \leq l < k$. We also provide a very short proof of the result of Kawamura [69] that the Aharoni-Berger-Ziv conjecture holds when G is chordal.

The connectivity $\text{conn}(X)$ of a topological space X is usually defined as follows: $\text{conn}(\emptyset) = -2$, $\text{conn}(X) = k$ if $\pi_i(X) = 0$ for every $0 \leq i \leq k$ and $\pi_{k+1}(X) \neq 0$, and $\text{conn}(X) = \infty$ if $\pi_i(X) = 0$ for every $i \geq 0$. The homological connectivity $\text{conn}_H(X)$ is defined in the same way replacing the homotopy groups $\pi_i(X)$ by the reduced homology groups with integer coefficients $\widetilde{H}_i(X)$. In this context, however, in order to keep the notation of [10], we will use the shifted versions

$$\eta(X) = \text{conn}(X) + 2, \quad \eta_H(X) = \text{conn}_H(X) + 2.$$

With this notation, X is non-empty when $\eta(X) \geq 1$, path-connected if $\eta(X) \geq 2$ and simply-connected when $\eta(X) \geq 3$. By the Hurewicz theorem, connectivity and homological connectivity coincide for simply-connected spaces, while in general $\eta(X) \leq \eta_H(X)$.

Consider the function ψ defined for all finite simple graphs G with values in $\{0, 1, \ldots, \infty\}$, as follows

$$\psi(G) = \begin{cases} 0 & \text{if } G = \emptyset \\ \infty & \text{if } G \neq \emptyset \text{ is discrete} \\ \max_{e \in E(G)} \big\{ \min\{\psi(G - e), \psi(G \setminus N[e]) + 1\} \big\} & \text{otherwise.} \end{cases}$$

Theorem 2.17. *For any graph G, $\psi(G) \leq \eta(I(G))$.*

Proof. We prove first that $\psi(G) \leq \eta_H(I(G))$. This part of the proof is implicit in [10]. The inequality is trivial for discrete graphs. Assume then that G is non-discrete and let $e \in E(G)$ be such that $\psi(G) = \min\{\psi(G - e), \psi(G \setminus e) + 1\}$. By induction $\psi(G - e) \leq \eta_H(I(G - e)) = \text{conn}_H(I(G - e)) + 2$ and $\psi(G \setminus e) \leq \eta_H(I(G \setminus e)) = \text{conn}_H(I(G \setminus e)) + 2$, and therefore $\widetilde{H}_i(I(G - e)) = 0$ for every $0 \leq i \leq \psi(G) - 2$ and $\widetilde{H}_i(I(G \setminus e)) = 0$ for every $0 \leq i \leq \psi(G) - 3$.

The cofibre sequence of Proposition 2.4 yields a long exact sequence of homology groups

$$\cdots \to \widetilde{H}_{i-1}(I(G \setminus N[e])) \to \widetilde{H}_i(I(G)) \to \widetilde{H}_i(I(G-e)) \to \widetilde{H}_{i-2}(I(G \setminus N[e])) \to \cdots .$$

We deduce then that $\widetilde{H}_i(I(G)) = 0$ for every $0 \leq i \leq \psi(G) - 2$ or, in other words, that $\psi(G) \leq \eta_H(I(G))$.

To prove the theorem it suffices to show that the condition $\psi(G) \geq 3$ implies that $I(G)$ is simply-connected. If G is discrete, $I(G)$ is a simplex. Otherwise, by definition of ψ, there exists an edge e such that

$$\psi(G-e) \geq 3 \quad \text{and} \quad \psi(G \setminus N[e]) \geq 2.$$

By induction $I(G-e)$ is simply-connected and since $\eta_H(I(G \setminus N[e])) \geq \psi(G \setminus N[e]) \geq 2$, the space $I(G \setminus N[e])$ is connected. The suspension $\Sigma\, I(G \setminus N[e])$ is then simply-connected and by van Kampen's theorem $\pi_1(I(G-e)) = \pi_1(I(G))$, hence $\pi_1(I(G)) = 0$. □

In [10, Conjecture 2.4] it was conjectured that $\psi(G) = \eta(I(G))$. This has been confirmed for some classes of graphs, e.g. chordal graphs [69], but, as we will show, it is not true in general. In view of Theorem 2.17 it is clear that the homological version of the conjecture, i.e. the equation $\psi(G) = \eta_H(I(G))$, does not hold in general since $\eta_H(I(G))$ can be strictly greater than $\eta(I(G))$. This follows from the existence of a finite connected complex K with non-trivial fundamental group but such that $H_1(K) = 0$ and the well-known fact that for every finite simplicial complex K there is a graph G with $I(G)$ homeomorphic to K, for instance the complement graph of the 1-skeleton of the barycentric subdivision of K.

Proposition 2.18. *Let G be a graph.*

a) If $\psi(G) \in \{0, 1\}$, then $\psi(G) = \eta(I(G))$.

b) If $I(G)$ is not simply-connected, then $\psi(G) = \eta(I(G))$.

Proof. It is easy to see that $\psi(G) = 0$ if and only if G is empty, so the only non-trivial case of a) is $\psi(G) = 1$.

Since the 1-skeleton of $I(G)$ is the complement $\overline{G}$ of G, we have that $\eta(I(G)) = 1$ if and only if $\overline{G}$ is disconnected. We will prove, by induction on the number of edges in G, that if $\psi(G) = 1$ then $\overline{G}$ is disconnected. By definition of ψ, G is non-discrete and for every edge e of G we have

$$\psi(G-e) = 1 \text{ or } G \setminus N[e] \text{ is empty.}$$

If there exists an edge $e \in G$ such that $\psi(G-e) = 1$ then, by induction, $\overline{G-e}$ is disconnected and therefore so is $\overline{G}$. It suffices then to consider the case when for every edge $e \in G$ the graph $G \setminus N[e]$ is empty. Translating this into a statement about complements we see that $\overline{G}$ has the following property:

for every pair of non-adjacent vertices x, y we have $N(x) \cap N(y) = \emptyset$.

It is easy to see that this property characterizes precisely the graphs in which every connected component is a clique. Since $\overline{G}$ is not a clique itself, it must be disconnected, as we wanted to show.

To prove b) note that if $I(G)$ is not simply-connected, then we have $\psi(G) \leq \eta(I(G)) \leq 2$ by Theorem 2.17, and the result follows from part a). □

We now prove that the conjecture is not true in general. The first argument we show is not constructive and reduces to the fact that it is algorithmically undecidable whether $\eta(I(G)) \geq 3$ or $\eta(I(G)) \leq 2$ for a given graph G, while $\psi(G)$ is a computable function of G. This argument is the idea of Jonathan Barmak.

Proposition 2.19. *There exists a graph G with $\psi(G) = 2$ and $\eta(I(G)) \geq 3$.*

Proof. The truth of the implication

$$\text{if } \psi(G) = 2 \text{ then } \eta(I(G)) = 2$$

together with Theorem 2.17 and Proposition 2.18 would provide an algorithm (Turing machine) capable of determining if a given finite simplicial complex K is simply-connected. The algorithm would just find a graph G with $I(G)$ homeomorphic to K and check if $\psi(G) \geq 3$. However it is known that there can be no such algorithm. It is a consequence of the non-existence of an algorithm to determine whether a group Γ given by a finite presentation is trivial or not [9, 91] and a construction that associates to each presentation of Γ a finite 2-dimensional complex with fundamental group isomorphic to Γ (see [49] for example). □

We will give more explicit counterexamples to the conjecture, all of them different from the one shown in Proposition 2.19. Their construction requires the next observation in which $G \sqcup H$ denotes the disjoint union of graphs G and H.

Lemma 2.20. *For any graphs G and H we have $\psi(G \sqcup H) = \psi(G) + \psi(H)$.*

Proof. The result holds when both G and H are discrete. The general case now follows by induction on the number of edges in $G \sqcup H$. For every $e \in E(G)$ we have

$(G \sqcup H) - e = (G - e) \sqcup H$ and $(G \sqcup H) \setminus N[e] = (G \setminus N[e]) \sqcup H$. If G is non-discrete, then by induction

$$\begin{aligned} &\max_{e \in E(G)} \left\{ \min\{\psi((G \sqcup H) - e), \psi((G \sqcup H) \setminus N[e]) + 1\}\right\} = \\ &= \max_{e \in E(G)} \left\{ \min\{\psi((G - e) \sqcup H), \psi((G \setminus N[e]) \sqcup H) + 1\}\right\} = \\ &= \max_{e \in E(G)} \left\{ \min\{\psi(G - e), \psi(G \setminus N[e]) + 1\}\right\} + \psi(H) = \\ &= \psi(G) + \psi(H). \end{aligned}$$

The same equation holds if H is non-discrete and the maximum is taken over the edges $e \in E(H)$. Then the result follows. □

The lemma also follows immediately from the interpretation of $\psi(G)$ as the maximal value achievable in a certain two-player game (see [10, p.257]).

Proposition 2.21. *For any $l, k \in \{3, 4, \ldots, \infty\}$ with $l \le k$ there exists a graph G such that $\psi(G) = l$ and $\eta(I(G)) = k$.*

Proof. Note that $\psi(e) = 1$. The case $l = \infty$ is trivial. Assume then that l is finite. Note that if G is such that $\psi(G) = l$ and $\eta(I(G)) = k \ge 3$, then $\psi(G \sqcup e) = \psi(G) + \psi(e) = l + 1$ by Lemma 2.20, and $\eta(I(G \sqcup e)) = \eta(\Sigma\, I(G)) = \eta_H(\Sigma\, I(G)) = \eta_H(I(G)) + 1 = \eta(I(G)) + 1 = k + 1$. Therefore, it suffices to prove the case $l = 3$.

Let K be an acyclic finite simplicial complex with non-trivial fundamental group, i.e. with the properties

$$\pi_1(K) \neq 0, \quad \tilde{H}_i(K) = 0 \text{ for all } i.$$

(Such K can be obtained for example by triangulating the two-dimensional CW-complex of [51, Example 2.38]). Note that the suspension ΣK is simply-connected and acyclic, hence contractible.

Assume first that k is finite. Since every finite simplicial complex can be realized, up to homeomorphism, as an independence complex of some graph, we can choose a graph H such that we have a homeomorphism

$$I(H) \equiv K \vee S^{k-2}.$$

Since $\eta(K \vee S^{k-2}) = 2$, we have $\psi(H) = 2$ by Proposition 2.18.

Let $G = H \sqcup e$. Then $I(G) = \Sigma\, I(H)$ is homotopy equivalent to $\Sigma\, K \vee S^{k-1}$, which in turn is homotopy equivalent to S^{k-1} since $\Sigma\, K$ is contractible. It follows that $\eta(I(G)) = k$. On the other hand $\psi(G) = \psi(H) + \psi(e) = 3$ by Lemma 2.20. Therefore G has the desired property.

For the remaining case $l = 3$, $k = \infty$, we consider a graph H such that $I(H) \equiv K$ and define $G = H \sqcup e$. Then $I(G) \equiv \Sigma\, K$ is contractible and $\psi(G) = 3$. □

Still, the study of the conjecture in special cases and for particular classes of graphs is an interesting problem and the bound provided by Theorem 2.17 can be useful even when it is not sharp. We are now going to use the methods introduced in the previous sections to analyze independence complexes of chordal graphs. In particular, we will show that chordal graphs satisfy the Aharoni-Berger-Ziv conjecture.

Before stating the next result recall that the *domination number* $\gamma(G)$ of G is the minimal cardinality of a *dominating set* in G, that is a subset $W \subseteq V(G)$ such that $N_G[W] = V(G)$. Recall from Section 1.2 that a graph is *chordal* if it does not have an induced cycle of length at least 4. The following was proved in [69], with the "wedge of spheres" part also following from earlier results.

Corollary 2.22. *Suppose G is a chordal graph.*

a) [100, 101, 32, 69] $I(G)$ is either contractible or homotopy equivalent to a wedge of spheres of dimension at least $\gamma(G) - 1$,

b) [69] G satisfies the Aharoni-Berger-Ziv conjecture.

Proof. a) The result is true for the empty graph (we assume $S^{-1} = \emptyset$, which is consistent with $\Sigma S^{-1} = S^0$) and for any discrete graph. Now suppose G has at least one edge. By a well-known characterization (see Fact 1.1 and [19, Thm. 9.21]) every chordal graph has a vertex u such that $N(u)$ is a clique. Choose any $v \in N(u)$. Then $N[u] \subseteq N[v]$, so by Theorem 2.3

$$I(G) \simeq I(G \setminus v) \vee \Sigma\, I(G \setminus N[v]).$$

Both graphs $G \setminus v$ and $G \setminus N[v]$ are chordal so by induction their independence complexes are either contractible or equivalent to wedges of spheres of dimension at least, respectively, $\gamma(G \setminus v) - 1$ and $\gamma(G \setminus N[v]) - 1$. Of course $\gamma(G) \leq \gamma(G \setminus N[v]) + 1$. Moreover, every dominating set in $G \setminus v$ is also dominating in G because to dominate

u it must contain a vertex in $N[u] \setminus \{v\} \subseteq N[v]$. It means that $\gamma(G) \leq \gamma(G \setminus v)$. It follows that each wedge summand of $I(G)$ is either contractible or a wedge of spheres of dimension at least $\min\{\gamma(G) - 1, (\gamma(G) - 2) + 1\} = \gamma(G) - 1$.

b) Let $f = (u, v)$. Then in the graph $G - f$ we have $N(u) \subseteq N(v)$, so by Theorem 2.2 the complex $I(G - f)$ is homotopy equivalent to $I(G \setminus v)$. The complex $I(G \setminus N[v])$ is clearly equal to $I(G \setminus N[f])$ as $N[u] \subseteq N[v]$ in G. The splitting of a) can thus be rewritten as

$$I(G) \simeq I(G - f) \vee \Sigma I(G \setminus N[f]).$$

The graph $G \setminus N[f]$ is chordal and a quick verification shows that the condition $N[u] \subseteq N[v]$ and the fact that $N(u)$ is a clique imply that also $G - f$ is chordal. By a) their independence complexes are wedges of spheres, so we have

$$\begin{aligned}\eta(I(G)) = \mathrm{conn}(I(G)) + 2 &= \min\{\mathrm{conn}(I(G - f)), \mathrm{conn}(I(G \setminus N[f])) + 1\} + 2 \\ &= \min\{\psi(G - f), \psi(G \setminus N[f]) + 1\} \\ &\leq \max_{e \in E(G)} \big\{ \min\{\psi(G - e), \psi(G \setminus N[e]) + 1\}\big\} = \psi(G)\end{aligned}$$

where the first equality follows from the splitting. Together with Theorem 2.17 this yields $\psi(G) = \eta(I(G))$. □

2.6 Application: Maximal Betti number of a flag complex

In this section we ask the following question. Suppose K is a simplicial complex with n vertices. How big can the reduced homology groups $\widetilde{H}_*(K)$ be?

Of course there is a trivial upper bound: since K has at most 2^n faces, it is homeomorphic to a CW-complex with at most 2^n cells, therefore $\dim \widetilde{H}_*(K) \leq 2^n$. For arbitrary K we can asymptotically almost achieve that value. The k-dimensional skeleton $\Delta_n^{(k)}$ of the n simplex is known to be homotopy equivalent to the wedge of $\binom{n}{k+1}$ spheres, and for $k \approx n/2$ that is roughly $\binom{n}{n/2} \approx \frac{2^n}{\sqrt{n}}$ (by [17] this is in fact optimal).

The problem is less trivial for flag complexes, and so the new question we ask is:

Question 2.23. *Given an n-vertex graph G, how big can the total reduced Betti number $\widetilde{\beta}(I(G)) = \dim \widetilde{H}_*(I(G))$ be?*

Example 2.24. Suppose n is divisible by 5 and let $G = (n/5) \cdot K_5$ be the disjoint union of $n/5$ copies of the complete graph K_5. The complex $I(G)$ is the join of $n/5$ copies of $I(K_5)$. The latter is the discrete space with 5 points, or, in other words, the wedge sum $\bigvee^4 S^0$. That means we have a homotopy equivalence $I(G) \simeq \bigvee^{4^{n/5}} S^{n/5-1}$, and therefore

$$|\widetilde{\chi}(I(G))| = \dim \widetilde{H}_*(I(G)) = 4^{n/5} \approx 1.32^n.$$

We could just as well take $G = (n/q) \cdot K_q$ for any $q \geq 2$ and get a wedge of $((q-1)^{1/q})^n$ spheres. The number $(q-1)^{1/q}$ is maximized for $q = 5$.

We prove that the graph of Example 2.24 is in fact extremal.

Theorem 2.25. *For any n-vertex graph G we have*

$$\dim \widetilde{H}_*(I(G); \mathbb{Q}) \leq (4^{1/5})^n.$$

There are several results that bound $|\widetilde{\chi}(I(G))|$ or $\dim \widetilde{H}_*(I(G))$ in terms of various parameters of G, see [37, 76, 77]. In [102] and [74] it is shown that the same upper bound holds for the Euler characteristic and total Betti number of the order complex of any n-element poset. Finally let us note that Theorem 2.25 follows easily from the results of [73].

It is convenient to make the following general statement.

Proposition 2.26. *Suppose $\beta(G)$ is any function that assigns to every graph G a non-negative integer and which satisfies three conditions:*

- $\beta(\emptyset) = 1$, *where $\emptyset$ denotes the empty graph with no vertices,*
- $\beta(G) = 0$ *if G has an isolated vertex,*
- $\beta(G) \leq \beta(G - e) + \beta(G \setminus N_G[e])$ *for any edge e of G.*

Given such a function define

$$\beta_n = \max_{G:\ |V(G)| \leq n} \beta(G).$$

Then we have

$$\beta_n \leq 4^{n/5}.$$

Our main theorem then follows from the next observation.

Proposition 2.27. *The function $G \mapsto \dim \widetilde{H}_*(I(G); \mathbb{Q})$ satisfies the assumptions of Proposition 2.26.*

Proof. The first condition follows because $I(\emptyset)$ is the empty space and that has a single reduced homology group in the augmentation degree -1. The second condition holds because in this case $I(G)$ is a contractible space. The third condition is a direct consequence of Proposition 2.4. $\square$

Remark 2.28. Another function which also satisfies Proposition 2.26 is $G \mapsto |\widetilde{\chi}(I(G))|$. It follows from the same argument.

It remains to prove Proposition 2.26 and we do so in a series of lemmas.

Lemma 2.29. *Let v be any vertex of a graph G and let $e_1, \dots, e_d$ be all the edges incident with v. Define $G_1 = G$ and*

$$G_i = G - \{e_1, \dots, e_{i-1}\}, \quad i = 2, \dots, d+1.$$

Then

$$\beta(G) \le \sum_{i=1}^{d} \beta(G_i \setminus N_{G_i}[e_i]).$$

Proof. Since $G_{i+1} = G_i - e_i$, this follows immediately by induction from the third condition for β:

$$\begin{aligned}
\beta(G) = \beta(G_1) \quad &\le \quad \beta(G_2) + \beta(G_1 \setminus N_{G_1}[e_1]) \le \\
&\le \quad \beta(G_3) + \beta(G_2 \setminus N_{G_2}[e_2]) + \beta(G_1 \setminus N_{G_1}[e_1]) \le \\
&\dots \\
&\le \quad \beta(G_{d+1}) + \sum_{i=1}^{d} \beta(G_i \setminus N_{G_i}[e_i]).
\end{aligned}$$

But G_{d+1} has v as an isolated vertex, so $\beta(G_{d+1}) = 0$. $\square$

Lemma 2.30. *If* $|V(G)| = n$ *and* $\mathrm{mindeg}(G) = d$ *then*

$$\beta(G) \le d \cdot \beta_{n-(d+1)}.$$

Proof. Let v be any vertex of degree d with incident edges $e_i = vu_i$, $i = 1, \dots, d$. In the notation of the previous lemma we have an inclusion of vertex sets

$$V(G_i \setminus N_{G_i}[e_i]) \subseteq V(G_i \setminus N_{G_i}[u_i]) = V(G \setminus N_G[u_i]).$$

Since $\deg_G(u_i) \ge d$ the last set has at most $n - (d+1)$ elements, therefore

$$\beta(G_i \setminus N_{G_i}[e_i]) \le \beta_{n-(d+1)}.$$

The result follows from the previous lemma. $\square$

Proof of Proposition 2.26. The result holds for $n = 0$: the only graph with 0 vertices is the empty graph $\emptyset$ with $\beta(\emptyset) = 1 = 4^{0/5}$.

Now suppose G is any graph with $n \geq 1$ vertices and let $d = \mathrm{mindeg}(G)$. Using the previous lemma and the induction hypothesis we get

$$\begin{aligned} \beta(G) &\leq d \cdot \beta_{n-(d+1)} \leq d \cdot 4^{(n-(d+1))/5} = \\ &= 4^{n/5} \cdot \frac{d}{4^{(d+1)/5}} \leq 4^{n/5}. \end{aligned}$$

The last inequality holds because the function $f(d) = \frac{d}{4^{(d+1)/5}}$ attains maximum (for integer values of d) when $d = 4$ and $f(4) = 1$. Since G was arbitrary that completes the proof. $\square$

Remark 2.31. A close inspection of the proof reveals that (when n is divisible by 5) the disjoint union of copies of K_5 is the only graph for which the total Betti number attains maximum. If n is not a multiple of 5 one must adjust the size of one or two copies of K_5. We omit the details which are analogous as in [102, 74].

Our formulation of the problem makes it natural to ask the same question with various restrictions on G. Consider the problem of maximizing the total Betti number $\dim \widetilde{H}_*(I(G))$ for bipartite graphs G.

Example 2.32. For $q \geq 1$ let $K_{q,q}$ be the complete bipartite graph with parts of equal size q and let $B_{2q} = K_{q,q} - M$ be the same graph with a perfect matching removed. Then the space $I(B_{2q})$ consists of two q-vertex simplices joined by q segments, hence it is homotopy equivalent to $\bigvee^{q-1} S^1$.

Now suppose n is divisible by $2q$ and let $G = (n/2q) \cdot B_{2q}$. As before, we obtain that $I(G)$ is homotopy equivalent to a wedge of $(q-1)^{n/2q} = ((q-1)^{1/2q})^n$ spheres and that this expression is maximized for $q = 5$. We therefore have a bipartite graph G with n vertices and total Betti number $(2^{1/5})^n \approx 1.15^n$.

Note that this graph is the so-called *bipartite double cover* of the graph of Example 2.24.

We conjecture that the graph from the previous example is extremal.

Conjecture 2.33. *For any n-vertex bipartite graph G we have*

$$\dim \widetilde{H}_*(I(G); \mathbb{Q}) \leq (2^{1/5})^n.$$

There is also another, equivalent formulation of this conjecture.

Conjecture 2.34. *If K is any simplicial complex (not necessarily flag) with a vertices and b maximal faces then*

$$\dim \widetilde{H}_*(K;\mathbb{Q}) \leq (2^{1/5})^{a+b}.$$

Proof of the equivalence of Conjectures 2.33 and 2.34. Consider a bipartite n-vertex graph G with parts U, W of sizes $|U| = a$, $|W| = b$, $a + b = n$. Let K_G be the simplicial complex with vertex set U and with maximal faces of the form

$$U \setminus N_G[w] \quad \text{for all } w \in W.$$

It is a known fact (see [13, Thm.3.7], [64, Sect. 3]) that $I(G) \simeq \Sigma K_G$, so the spaces $I(G)$ and K_G have the same total reduced Betti numbers. Since K_G has a vertices and at most b maximal faces, Conjecture 2.34 implies

$$\dim \widetilde{H}_*(I(G)) = \dim \widetilde{H}_*(K_G) \leq (2^{1/5})^{a+b} = (2^{1/5})^n.$$

Conversely, if K is any complex with a vertices and b maximal faces, we construct a bipartite graph G_K. It has one vertex for each vertex and for each maximal face of K and a vertex u is adjacent to a maximal face f if $u \notin f$. Now the previous homotopy equivalence translates into $I(G_K) \simeq \Sigma K$ and G_K has $a + b$ vertices so Conjecture 2.33 implies

$$\dim \widetilde{H}_*(K) = \dim \widetilde{H}_*(I(G_K)) \leq (2^{1/5})^{a+b}.$$

□

The methods of this chapter suffice to prove a partial result, weaker than the conjectured optimum. First consider, for every $d \geq 1$, the function

$$f_d(x) = x^{-(d+1)} + x^{-(d+2)} + \cdots + x^{-2d}$$

and let $1 \leq \alpha_d < 2$ be the unique solution to $f_d(\alpha_d) = 1$. Then one checks easily that $\alpha = \alpha_3 \approx 1.25$ is the largest of all the α_d and we have the next result.

Proposition 2.35. *For every triangle-free n-vertex graph G we have*

$$\dim \widetilde{H}_*(I(G);\mathbb{Q}) \leq \alpha^n.$$

Proof. Once again the proof works for any function β which satisfies the conditions of 2.26. Define

$$\beta'_n = \max_{\substack{G \text{ triangle-free} \\ |V(G)| \leq n}} \beta(G).$$

We follow the inductive argument of the previous section, but triangle-freeness gives a better estimate of the size of the removed neighbourhoods.

The result holds for $n = 0$. Now suppose G is a triangle-free graph with $n \geq 1$ vertices and $\mathrm{mindeg}(G) = d$. Let v be a vertex of degree d. In the graph G_i of Lemma 2.29 the endpoints of e_i have degrees at least $d - i + 1$ and d and their neighbourhoods are disjoint, so

$$|V(G_i \setminus N_{G_i}[e_i])| \leq n - (d - i + 1) - d = n - 2d + i - 1.$$

All graphs $G_i \setminus N_{G_i}[e_i]$ are triangle-free, so by Lemma 2.29 we get

$$\begin{aligned} \beta(G) &\leq \beta'_{n-2d} + \beta'_{n-(2d-1)} + \cdots + \beta'_{n-(d+1)} \leq \\ &\leq \alpha^{n-2d} + \alpha^{n-(2d-1)} + \cdots + \alpha^{n-(d+1)} = \alpha^n f_d(\alpha) \leq \alpha^n \end{aligned}$$

since $f_d(\alpha) \leq f_d(\alpha_d) = 1$. Since G was arbitrary that completes the proof. □

As a consequence Conjectures 2.33 and 2.34 hold with the constant $2^{1/5}$ replaced by α.

Chapter 3

Independence complexes and the powers of cycles

3.1 Introduction

This chapter describes a computation which was the initial motivation behind the study of the splitting conditions for the cofibre sequences of Chapter 2. We use the splitting results associated with the sequence (2.2) to calculate the homotopy types of independence complexes of a particular family of graphs, namely the powers C_n^r of cycles. Recall that D. Kozlov in [71] computed the homotopy types of $I(P_n)$ and $I(C_n)$, where P_n is the path and C_n is the cycle on n vertices. The answers are determined by the homotopy equivalences

$$I(P_n) \simeq \Sigma I(P_{n-3}), \quad I(C_n) \simeq \Sigma I(C_{n-3}).$$

An open question of [71] is to find similar statements for the complexes $I(P_n^r)$ and $I(C_n^r)$, $r \geq 2$. Here G^r denotes the *r-th distance power of G*, which is the graph with the same vertex set in which two vertices are adjacent if and only if their distance in G is *at most r*. Therefore C_n^r is the graph spanned by the vertices of the n-gon, with two vertices being adjacent if and only if they are at most r steps away along the perimeter of the n-gon. For P_n^r the n-gon is replaced with an n-vertex path.

The answer for $I(P_n^r)$ is given in [35] in the form of a recursive relation

$$I(P_n^r) \simeq \Sigma I(P_{n-(r+2)}^r) \vee \Sigma I(P_{n-(r+3)}^r) \vee \cdots \vee \Sigma I(P_{n-(2r+1)}^r), \quad n \geq r+1. \quad (3.1)$$

(Note that [71, 35] denote our $I(P_n^r)$, $I(C_n^r)$ by, respectively, $\mathcal{L}_n^{r+1}$, $\mathcal{C}_n^{r+1}$). Here we obtain a corresponding statement for $I(C_n^r)$, answering the question raised in

[71, 35].

Theorem 3.1. *For every $r \geq 1$ and $n \geq 5r+4$ there is a homotopy equivalence*

$$I(C_n^r) \simeq \Sigma^2 I(C_{n-(3r+3)}^r) \vee X_{n,r}$$

where $X_{n,r}$ is a space which splits, up to homotopy, into a wedge sum of complexes of the form $\Sigma^3 I(P_{n-a}^r)$ for various values of $4r+6 \leq a \leq 6r+3$.

The reader will see that the proof of the theorem gives an algorithmic way of enumerating all the wedge summands that go into $X_{n,r}$; there are asymptotically r^3 of them and we list them at the end of this chapter. For example, when $r=1$ we will have $I(C_n) \simeq \Sigma^2 I(C_{n-6})$ with $X_{n,1}$ being trivial, which agrees with Kozlov's recurrence. When $r=2$ the exact answer is

$$I(C_n^2) \simeq \Sigma^2 I(C_{n-9}^2) \vee \bigvee^4 \Sigma^3 I(P_{n-14}^2) \vee \bigvee^5 \Sigma^3 I(P_{n-15}^2),$$

and so on.

The idea of the proof is as follows. We extend C_n^r to another graph $\widetilde{C_n^r}$ on the same vertex set but with more edges. The new graph will have the property that $I(\widetilde{C_n^r}) \simeq \Sigma^2 I(C_{n-(3r+3)}^r)$ (Proposition 3.4). Since $\widetilde{C_n^r}$ is obtained from C_n^r by inserting new edges, we get a natural inclusion

$$\Sigma^2 I(C_{n-(3r+3)}^r) \simeq I(\widetilde{C_n^r}) \hookrightarrow I(C_n^r).$$

This is our guess for what the inclusion of the first wedge summand in Theorem 3.1 should be. We then need to show that, up to homotopy, the image of this inclusion indeed splits off. This is accomplished by analyzing the construction of $\widetilde{C_n^r}$ from C_n^r edge by edge and showing that every single edge insertion yields a splittable inclusion of independence complexes.

3.2 The proof of Theorem 3.1

We will use an obvious inductive consequence of Proposition 2.4.b), which we record below for convenience.

Lemma 3.2. *Suppose G is a graph and $e_1, \ldots, e_k$ is a sequence of edges which are* not *in G. Let $G_0 = G$ and let $G_i = G_{i-1} \cup e_i$ for $1 \leq i \leq k$. Suppose that for each*

$i = 1, \ldots, k$ *the inclusion*

$$I(e_i \sqcup (G_i \setminus N[e_i])) \hookrightarrow I(G_i)$$

is null-homotopic. Then there is a homotopy equivalence

$$I(G) \simeq I(G_k) \vee \bigvee_{i=1}^{k} \Sigma^2 I(G_i \setminus N[e_i]).$$

Proof. For every $i = 1, \ldots, k$ we have $G_i - e_i = G_{i-1}$, so Proposition 2.4.b) yields splittings $I(G_{i-1}) \simeq I(G_i) \vee \Sigma^2 I(G_i \setminus N[e_i])$, from which the result follows by induction. □

We now describe the construction of the graph $\widetilde{C_n^r}$. The vertices of an n-cycle are labeled with elements of $\mathbb{Z}/n$. We start with C_n^r and add new edges in the order described below (see Fig.3.1 and Fig.3.2).

- *First phase.* It consists of $r-1$ stages.
  - In stage s, where $1 \leq s \leq r-1$, we add two groups of edges:
    - first group: $(i, i+2r-s+2)$ for $i = 1, \ldots, r+s+1$,
    - second group: $(i, i+3r-s+3)$ for $i = 1, \ldots, s$.
- *Second phase.* Add all the edges of the form

$$(-x, 3r+3+y) \quad \text{for} \quad 0 \leq x \leq r-1,\ 1 \leq y \leq r,\ x+y \leq r.$$

Let T_{3r+3} denote the subgraph of $\widetilde{C_n^r}$ induced by the vertices $\{1, \ldots, 3r+3\}$ (this subgraph does not depend on n, see Fig.3.2). Also, let R_n^r be the remaining part of $\widetilde{C_n^r}$ i.e. the subgraph induced by $\{3r+4, \ldots, -1, 0\}$. Note that all the edges added in the first phase of the construction belong to T_{3r+3}, all the edges from the second phase are in R_n^r and the only edges between the two parts are those that were originally in C_n^r. The condition $n \geq 5r+4$ of Theorem 3.1 guarantees that all the edges added in the construction (esp. in the second phase) are indeed "new".

We start with some technical properties of $\widetilde{C_n^r}$ and T_{3r+3} which ultimately lead to the fact that $I(\widetilde{C_n^r})$ is a homotopical model for $\Sigma^2 I(C^r_{n-(3r+3)})$.

Lemma 3.3. *The graphs $\widetilde{C_n^r}$ and T_{3r+3} have the following properties:*

a) The graphs T_{3r+3} and $\widetilde{C_n^r}$ have an axis of symmetry, in the sense that there is an edge (i, j) if and only if there is an edge $(3r+4-i, 3r+4-j)$.

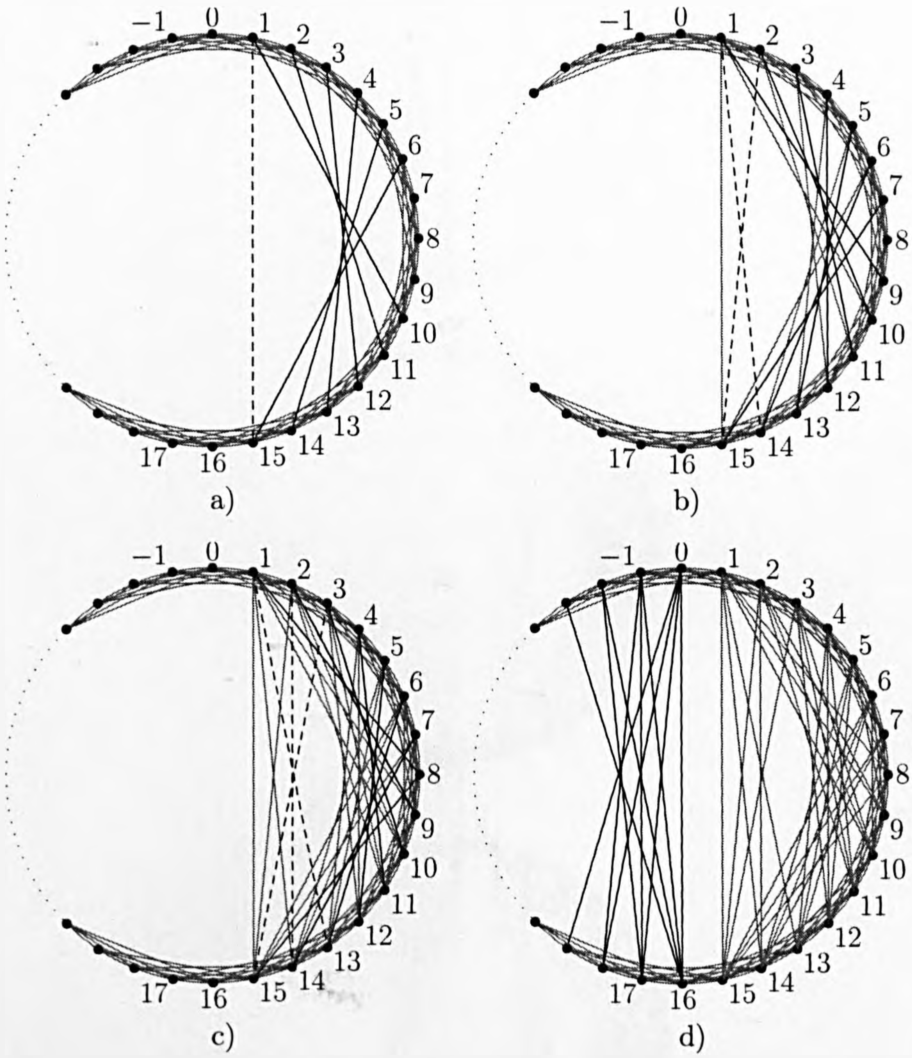


Figure 3.1: Construction of $\widetilde{C_n^r}$ for $r = 4$. Figures a),b),c) highlight edges added in stages $s = 1, 2, 3$, where the edges of the first group are solid and those of the second group are dashed. Figure d) highlights the edges of the second phase.

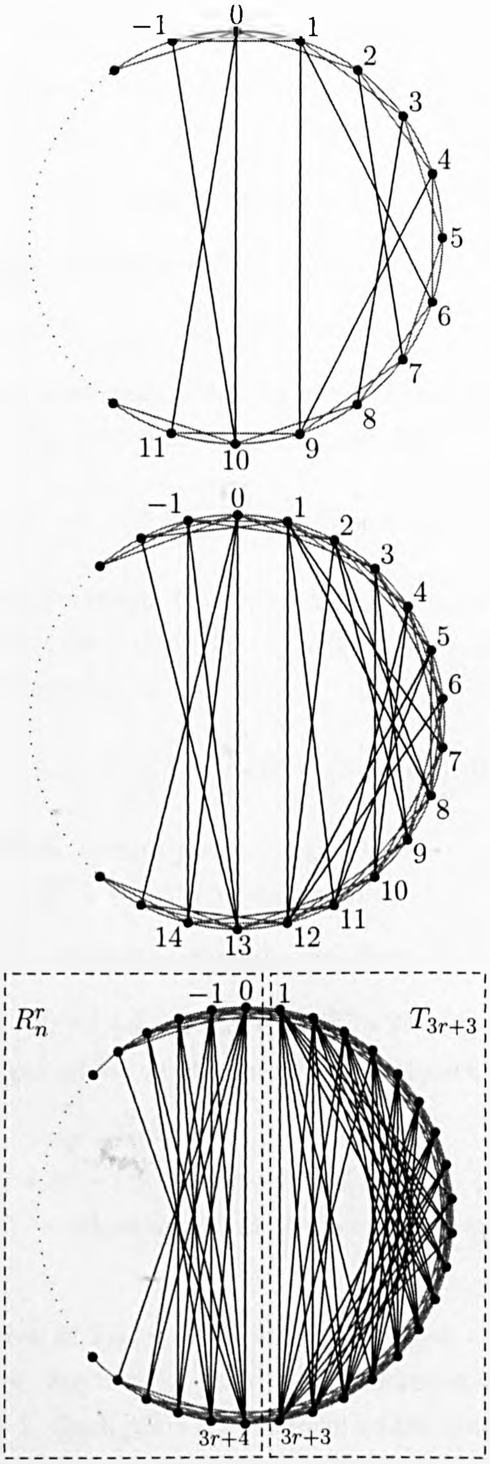


Figure 3.2: The graphs $\widetilde{C_n^r}$ for $r = 2, 3$ and a general decomposition shown for $r = 5$.

b) For any $1 \leq i \leq r$ the graph $T_{3r+3} \setminus N[i]$ is isomorphic to the path P_4 on the vertices $i + r + 1$, $i + r + 2$, $i + 2r + 2$, $i + 2r + 3$.

c) If i, j are two vertices of T_{3r+3} with $1 \leq j - i \leq 2r + 1$ then (i, j) is not an edge of T_{3r+3} if and only if $j = i + r + 1$ or $j = i + r + 2$.

d) For any $0 \leq k \leq r + 1$ we have $I(T_{3r+3}[k + 1, \ldots, k + 2r + 2]) \simeq *$.

e) There is a homotopy equivalence $I(T_{3r+3}) \simeq S^1$.

f) The graphs R_n^r and $C_{n-(3r+3)}^r$ are isomorphic.

Proof. **a)** The statement obviously holds for the original edges of C_n^r. If $(i, j) = (i, i + 2r - s + 2)$ is an edge added in the s-th stage then

$$(3r + 4 - j, 3r + 4 - i) = (r + s + 2 - i, (r + s + 2 - i) + 2r - s + 2)$$

was also added in the same stage as $1 \leq r+s+2-i \leq r+s+1$. A similar argument applies to the edges of the form $(i, i + 3r - s + 3)$. Every edge $(-x, 3r + 3 + y)$ of the second phase is mirrored by

$$(3r + 4 - (3r + 3 + y), 3r + 4 - (-x)) = (-(y - 1), 3r + 3 + (x + 1))$$

which was also added in the second phase.

b) Any vertex i with $1 \leq i \leq r$ is connected to

- all of $1, \ldots, i + r$ — using the original edges from C_n^r,
- vertices between $i+2r-1+2 = i+2r+1$ and $i+2r-(r-1)+2 = i+r+3$ (going backwards) — edges added in the first groups of each stage as $i \leq r + s + 1$ for all s,
- vertices between $i + 3r - i + 3 = 3r + 3$ and $i + 3r - (r - 1) + 3 = i + 2r + 4$ (going backwards) — edges added in the second groups of each stage s that satisfies $i \leq s$.

It means that the vertices of $T_{3r+3} \setminus N[i]$ are exactly $\{i + r + 1, i + r + 2, i + 2r + 2, i + 2r + 3\}$. Moreover, any two of them with difference other than 1 or r have difference $r + 1$ and $r + 2$. Such pairs do not form edges because the shortest edges added in the first phase span over a distance of at least $2r - (r - 1) + 2 = r + 3$. That means $T_{3r+3} \setminus N[i]$ is precisely a P_4.

c) As observed in b), there are no edges $(i, i + r + 1)$ and $(i, i + r + 2)$. If $j - i \leq r$ then (i, j) is an edge already in C_n^r. Now suppose that $r + 3 \leq j - i \leq 2r + 1$

and let $s = 2r + i - j + 2$. The constraints on i, j are equivalent to $1 \leq s \leq r - 1$ and the inequality $j \leq 3r + 3$ is equivalent to $i \leq r + s + 1$. It means that the edge $(i, i + 2r - s + 2) = (i, j)$ was added in stage s.

d) We will show that the complement of the graph $T_{3r+3}[k+1, \ldots, k+2r+2]$ is a path and then the result immediately follows. Part c) gives a complete description of edges in that complement. Vertices $k+r+1$ and $k+r+2$ have one incident edge each (to $k + 2r + 2$ and $k + 1$, respectively) and every vertex $k + i$ with $1 \leq i \leq r$ has edges to $k + i + r + 1$ and $k + i + r + 2$. This easily implies that the graph in question is the path

$$k + r + 1, k + 2r + 2, k + r, k + 2r + 1, k + r - 1, \ldots, k + 1, k + r + 2.$$

e) By part b) the complexes $I(T_{3r+3} \setminus N[i])$ are contractible and contained in $\{r + 1, \ldots, 3r + 3\}$ for all $1 \leq i \leq r$. By Proposition 2.1.a) we can therefore sequentially remove all those i from T_{3r+3} without affecting the homotopy type of the independence complex. That means

$$I(T_{3r+3}) \simeq I(T_{3r+3}[r + 1, \ldots, 3r + 3]).$$

Let $H = T_{3r+3}[r + 1, \ldots, 3r + 3]$. Using part d) with $k = r + 1$ we get that $I(H \setminus \{r + 1\})$ is contractible. Moreover the graph $H \setminus N[r + 1]$ is the 3-vertex path induced by $2r + 2, 2r + 3, 3r + 3$, so $I(H \setminus N[r + 1]) \simeq S^0$. The cofibration sequence of Proposition 2.1 now yields $I(H) \simeq \Sigma S^0 = S^1$.

f) This is obvious as the edges added in the second phase of the construction are exactly those needed to close the long power of a path $P^r_{n-(3r+3)}$ into the same power of a cycle. □

Proposition 3.4. *There is a homotopy equivalence*

$$I(\widetilde{C^r_n}) \simeq \Sigma^2 I(C^r_{n-(3r+3)}).$$

Proof. We will show that all edges that connect T_{3r+3} with R^r_n can be removed without changing the homotopy type of the independence complex, i.e. that the inclusion

$$I(\widetilde{C^r_n}) \hookrightarrow I(T_{3r+3} \sqcup R^r_n)$$

is a homotopy equivalence. Then the result follows from e) and f) of Lemma 3.3.

Because of the symmetry of Lemma 3.3.a) it suffices to consider the removal of edges of C^r_n which "go across 0". Every such edge is of the form $e = (x, -(r - y))$

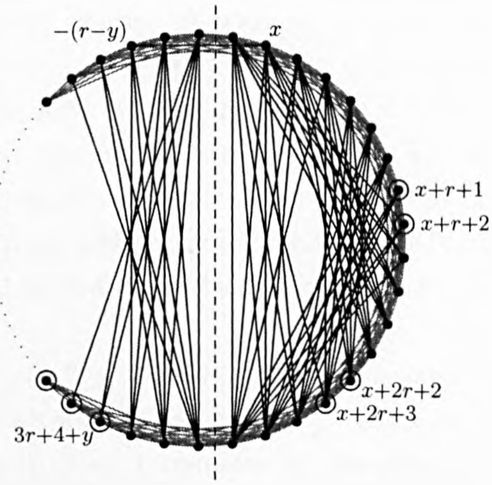


Figure 3.3: The proof of Proposition 3.4. The circled vertices are those that remain after removing $N[e]$ for $e = (x, -(r-y))$.

for $1 \leq x \leq y \leq r$ (see Fig.3.3). By Proposition 2.4.a) all we need to check is that the complex $I(\widetilde{C_n^r} \setminus N[e])$ is contractible. (To be precise, we need to know this not for $\widetilde{C_n^r}$ but for the intermediate graph we obtain after some edges of this form have already been removed. It is, however, easy to see that it will be exactly the same thing.)

By Lemma 3.3.b) the removal of $N[x]$ deletes all vertices in $\{1, \ldots, 3r+3\}$ except $x+r+1$, $x+r+2$, $x+2r+2$, $x+2r+3$. The removal of $N[-(r-y)]$ deletes (in particular) all of $0, \ldots, -r$ and $3r+4, \ldots, 3r+3+y$. The first vertex in R_n^r which remains is $3r+4+y$ and

$$(3r+4+y) - (x+2r+3) = r+1+(y-x) \geq r+1$$

so it is too far to be adjacent to the vertices which remain inside T_{3r+3}. It follows that $\widetilde{C_n^r} \setminus N[e]$ is a disjoint union of P_4 and some subgraph of R_n^r, hence its independence complex is contractible. This is what we needed to prove. □

We can now move on to the second part of the program outlined at the beginning of this section. This means proving:

Proposition 3.5. *The sequence of edges listed in the construction of $\widetilde{C_n^r}$ from C_n^r satisfies the assumptions of Lemma 3.2.*

Proof. We start from $G = C_n^r$ and expand it edge by edge.

Edges of the first phase. Suppose we are now in stage s, $1 \leq s \leq r-1$.

***i*-th edge of first group.** Suppose our current graph G includes all the edges up to the edge $e = (u, v) = (i, i+2r-s+2)$ of the first group in the s-th stage. We are going to use Theorem 2.7 with $x = i + 1$, $y = i + 2r - s + 1$, see Fig.3.4.a). The graph induced by $\{x, y, u, v\}$ has edges $e = uv$, ux and vy and no others because the differences between remaining pairs of vertices are at least $2r - s \geq r + 1$ and less than $2r - s + 2$, so those edges may potentially only be added in the first groups of future stages. It means that the induced graph is a P_4. It remains to check that $N[x] \cup N[y] \subseteq N[e]$.

Note that $2r - s + 2 \leq 2r + 1$ so the whole interval $\{u - r, \ldots, v + r\}$ is in $N[e] = N[u] \cup N[v]$ already in the graph C_n^r. It means that all the neighbours of x or y in C_n^r belong to $N[e]$. It remains to concentrate on the new adjacencies induced by the edges added previously in the construction. Consider first the vertex $x = i + 1$. It can have the following, previously added edges.

- $(x, j) = (i+1, (i+1)+2r-s'+2) = (i+1, i+2r-s'+3)$ for some $1 \leq s' \leq s$. Then $j - v = s - s' + 1 < r$, so $j \in N[v] \subseteq N[e]$.

- $(j, x) = ((i+1)-(2r-s'+2), i+1) = (i-2r+s'-1, i+1)$ for some $1 \leq s' \leq s$ (see Fig.3.4.a)). Let $s'' = s' + s - r$. The inequality $j \geq 1$ is equivalent to $2r - s' + 2 \leq i$. Together with the inequality $i \leq r + s + 1$, which holds because we are currently in stage s, they yield $s'' = s + s' - r \geq 1$. Clearly $s'' < s'$ so s'' is a valid number of a past stage. We also have

$$j = i - 2r + s' - 1 \leq r + s + 1 - 2r + s' - 1 = s' + s - r = s''$$

  which means that in stage s'' we added an edge of the second group

$$(j, j+3r-s''+3) = (j, i-2r+s'-1+3r-(s'+s-r)+3) = (j, i+2r-s+2) = (j, v)$$

  so $j \in N[v] \subseteq N[e]$ at the present stage, as required.

- $(x, j) = (i + 1, (i + 1) + 3r - s' + 3)$ for some $1 \leq s' < s$. If that edge was added in stage s', we must have had $i + 1 \leq s'$. Let $s'' = s' - 1$. Then $1 \leq i \leq s'' < s \leq r - 1$ so stage s'' existed and in that stage we added the edge

$$(i, i + 3r - s'' + 3) = (i, i + 3r - s' + 1 + 3) = (i, j)$$

  so $j \in N[u] \subseteq N[e]$.

- $(j, x) = ((i+1)-(3r-s'+3), i+1) = (i-3r+s'-2, i+1)$ for some $1 \leq s' < s$.

We must have $j \geq 1$ and $i \leq r+s+1$, so

$$1 \leq i - 3r + s' - 2 \leq r + s + 1 - 3r + s' - 2 = s + s' - 2r - 1 < 0,$$

which is a contradiction.

This completes the proof that $N[x] \subseteq N[e]$. Note that in this proof we only used the existence of edges from previous stages and never needed to refer to the edges added earlier in the same s-th stage. The part of the graph constructed up to the complete $(s-1)$ stages has the axis of symmetry of Lemma 3.3.a), therefore the same proof will work to show $N[y] \subseteq N[e]$. It means that $N[x] \cup N[y] \subseteq N[e]$ and the assumptions of Theorem 2.7 are satisfied.

i-th edge of the second group. Now suppose we are adding the edge $e = (u, v) = (i, i + 3r - s + 3)$ in the second group of stage s, and all the previous edges are already in the graph. We are going to use Theorem 2.7 with $x = i + 1$, $y = i + 3r - s + 2$, see Fig.3.4.b). The graph induced by $\{x, y, u, v\}$ has edges $e = uv$, ux and vy. There are no other edges because the remaining differences are smaller than the one between u and v, but at least $3r - s + 1 \geq 2r + 2$, so those edges may potentially only be added in the second groups of future stages. It means that the induced graph is a P_4. As before, to check $N[x] \subseteq N[e]$ we only need to restrict to those edges from x whose endpoints are not obviously covered by the neighbours of u and v from C_n^r. Those include:

- $(x, j) = (i + 1, (i + 1) + r)$, see Fig.3.4.b). Note that by construction we must have $i \leq s$ therefore $j \leq r + s + 1$, so in the first group of the present stage we added the edge

  $$(j, j + 2r - s + 2) = (j, i + 1 + r + 2r - s + 2) = (j, i + 3r - s + 3) = (j, v)$$

  so $j \in N[v] \subseteq N[e]$.

- $(x, j) = (i + 1, (i + 1) + 3r - s' + 3)$ for some $1 \leq s' \leq s$. If that edge was added in stage s', we must have had $i + 1 \leq s'$. Let $s'' = s' - 1$. Then $1 \leq i \leq s'' < s \leq r - 1$ so stage s'' existed and in that stage we added the edge

  $$(i, i + 3r - s'' + 3) = (i, i + 3r - s' + 1 + 3) = (i, j)$$

  so $j \in N[u] \subseteq N[e]$.

- $(x, j) = (i+1, (i+1)+2r-s'+2) = (i+1, i+2r-s'+3)$ for some $1 \leq s' \leq s$.

But then

$$v - j = (i + 3r - s + 3) - (i + 2r - s' + 3) = r - (s - s') \leq r$$

so $j \in N[v]$ already in C_n^r.

It proves that $N[x] \subseteq N[e]$ and $N[y] \subseteq N[e]$ follows from symmetry as before. Again, we invoke Theorem 2.7 to verify the assumption in Lemma 3.2.

Second phase. Assuming that the first phase is complete we are now going to add edges of the second phase. Here the order is irrelevant to the argument. Suppose we have already constructed some graph G, which includes all edges of the first phase (in particular, the whole T_{3r+3} is already there), and that we are now adding the edge (see Fig.3.4.c))

$$e = (-x, 3r + 3 + y), \quad \text{for} \quad 0 \leq x \leq r - 1,\ 1 \leq y \leq r,\ x + y \leq r.$$

Let

$$\begin{aligned} V &= \{-x\} \cup \{1, \ldots, -x + (3r + 2)\} \cup \{3r + 3 + y\} \\ W &= \{3r + 3 + y + (r + 1), \ldots, -x - (r + 1)\}. \end{aligned}$$

The inclusion $I(e \sqcup (G \setminus N[e])) \hookrightarrow I(G)$, which we need to show is null-homotopic, factors through $I(G[V \cup W])$. Indeed, W contains all the vertices of $(G \setminus N[e]) \cap R_n^r$. To see that V covers all of $(G \setminus N[e]) \cap T_{3r+3}$ note that the last vertex not in $N[3r + 3 + y]$ is $2r + 2 + y$, but

$$2r + 2 + y \leq 2r + 2 + (r - x) = -x + (3r + 2)$$

so $(G \setminus N[e]) \cap T_{3r+3} \subseteq V$.

There are no edges from V to W, so $I(G[V \cup W]) = I(G[V]) * I(G[W])$. We are going to show that

$$I(G[V]) \simeq *$$

and this gives the desired conclusion.

To analyze $G[V]$ first look at the vertex $-x$. We have

$$G[V] \setminus N[-x] = T_{3r+3}[-x + r + 1, \ldots, -x + 3r + 2]$$

and the independence complex of the last graph is contractible by Lemma 3.3.d).

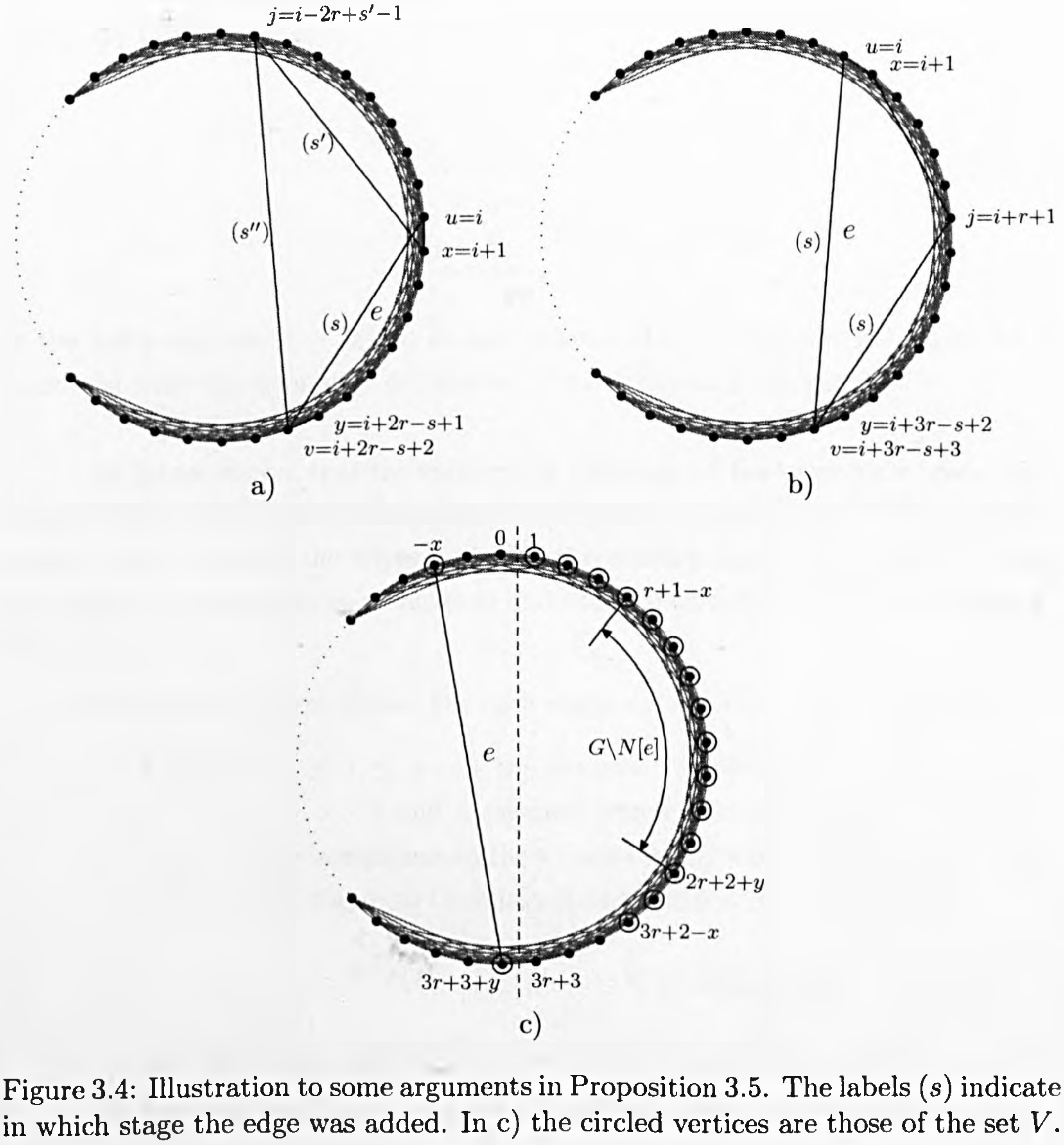


Figure 3.4: Illustration to some arguments in Proposition 3.5. The labels (s) indicate in which stage the edge was added. In c) the circled vertices are those of the set V.

Therefore, by Proposition 2.1.a) the removal of $-x$ preserves the homotopy type:

$$I(G[V]) \simeq I(G[V \setminus \{-x\}]).$$

But in the graph $G[V\setminus\{-x\}]$ the neighbourhood of $3r+3+y$ is $\{2r+3+y,\dots,3r+2-x\}$. All those vertices are between $2r+4$ an $3r+2$, so they form a clique already in C_n^r. By Corollary 2.14

$$\begin{aligned} I(G[V \setminus \{-x\}]) &\simeq \bigvee_{i=2r+3+y}^{3r+2-x} \Sigma\, I(G[V \setminus \{-x\}] \setminus N[i]) \\ &\simeq \bigvee_{i=2r+3+y}^{3r+2-x} \Sigma\, I(T_{3r+3} \setminus N[i]) \end{aligned}$$

In the last wedge sum $i \geq 2r+4$, so each summand is contractible by Lemma 3.3.b) combined with the symmetry of Lemma 3.3.a). That ends the proof. □

So far we proved that the splitting of Theorem 3.1 holds for some space $X_{n,r}$. Lemma 3.2 also provides a description of $X_{n,r}$ as a wedge sum of $\Sigma^2 I(G_i \setminus N[e_i])$ where e_i runs through the edges added in the construction of $\widetilde{C_n^r}$. We will briefly sketch how to identify those summands and this will complete the proof of Theorem 3.1.

- *First groups in first phase.* For each stage s if $e = (i, i+2r-s+2)$ then:
  - For every $1 \leq i \leq s-1$ the removal of $N[e]$ leaves only the vertex $v = i + 3r - s + 3$ and a segment isomorphic to $P^r_{n-4r+i-4}$ within R^r_n. The vertex v is adjacent to the $r-s+i$ initial vertices of the path power. They form a clique so Corollary 2.14 identifies $\Sigma^2 I(G \setminus N[e])$ as

    $$\Sigma^3 I(P^r_{n-5r+i-5}) \vee \cdots \vee \Sigma^3 I(P^r_{n-6r+s-4})$$

    and this is the contribution of each pair (i,s) with $1 \leq i < s \leq r-1$.
  - For every $r+3 \leq i \leq r+s+1$ the situation is symmetric, so we can just include the contribution of the previous part twice.
  - When $s \leq i \leq r+2$ then the vertices left after removing $N[e]$ form a $P^r_{n-4r+s-3}$. For every s there are $r+3-s$ suitable values of i, so the total contribution of this part for every s is

    $$\bigvee^{r+3-s} \Sigma^2 I(P^r_{n-4r+s-3}).$$

This can be expanded into third suspensions using (3.1).

- *Second groups in first phase.* For each stage s if $e = (i, i+3r-s+3)$ then the removal of $N[e]$ leaves a disjoint union of $P^r_{n-5r+s-4}$ with a clique of size $r-s$ induced by $\{i+r+2, \ldots, i+2r-s+1\}$. There are s edges in this group, so here stage s contributes

$$\bigvee^{s(r-s-1)} \Sigma^3 I(P^r_{n-5r+s-4})$$

(in particular when $s = r-1$ the clique has size 1 and the summand is contractible).

- *Second phase.* For an edge $e = (-x, 3r+3+y)$ its removal leaves a copy of $P^r_{n-5r-4-(x+y)}$ and a segment of T_{3r+3} induced by $\{r+1-x, \ldots, 2r+2+y\}$. The independence complex of the last piece equals

$$I(T_{3r+3}[y+1, \ldots, y+2r+2]) \setminus \{y+1, \ldots, y+(r-x-y)\}.$$

In the proof of Lemma 3.3.d) we saw that $I(T_{3r+3}[y+1, \ldots, y+2r+2])$ is homeomorphic to a path. The order of the vertices of that path implies that the removal of each of $y+1, \ldots, y+(r-x-y)$ increases the number of connected components by 1. Therefore the resulting space is homotopy equivalent to the wedge of $r-(x+y)$ copies of S^0. Since the possible values of $x+y$ are $t = 1, \ldots, r$ and value t is attained t times we get that the total contribution of the second phase is

$$\bigvee_{t=1}^{r} \bigvee^{t(r-t)} \Sigma^3 I(P^r_{n-5r-t-4})$$

(again, the summands for $r = t$ are trivial).

A tedious calculation, which will be omitted, allows to express the combination of all the contributions in the following form.

Corollary 3.6. *The space $X_{n,r}$ of Theorem 3.1 satisfies*

$$X_{n,r} \simeq \Sigma^3 \bigvee_{i=4r+6}^{6r+3} \bigvee^{k_i} I(P^r_{n-i})$$

where

$$k_i = \begin{cases} \frac{1}{2}(i-4r-5)(i-2r-2) & \text{for } i \leq 5r+4, \\ \frac{1}{2}(6r+4-i)(i-2r-1) & \text{for } i \geq 5r+5. \end{cases}$$

Remark 3.7. This work provides a natural recursive relation for $I(C_n^r)$, but does not say anything about the "initial conditions", that is the case when $n < 5r+4$. It is reasonable to expect that all those spaces are, up to homotopy, wedges of spheres. Other methods of computing the homotopy types of $I(C_n^r)$ were recently obtained in [98, 59], although the claims of [59] were later withdrawn.

Chapter 4

Induced matchings, cross-cycles and complexity

4.1 Introduction: Homology classes defined by matchings

There is a certain efficient construction that defines a class in the homology of a flag complex and, at the same time, delivers a proof that this class is non-trivial. We describe it now. Let S^0 be the zero-sphere, that is the simplicial complex consisting of two disjoint vertices. Recall from Section 1.2 that the k-fold join

$$O_k = \underbrace{S^0 * \cdots * S^0}_{k}$$

is a complex with $2k$ vertices, combinatorially equivalent to the boundary of the cross-polytope and homeomorphic to the $(k-1)$-sphere.

Definition 4.1. *A cross-cycle (of size k) in a flag simplicial complex K is an induced subcomplex of K isomorphic to O_k and such that it contains a maximal face of K.*

A cross-cycle is an embedded sphere $S^{k-1} \subseteq K$; hence, it defines a homology class in $\widetilde{H}_{k-1}(K)$. This class must be non-zero as its representing cycle contains a maximal face, and so it cannot be hit by a differential. The name cross-cycle refers to the fact that the sphere is isomorphic to the boundary of a cross-polytope.

Cross-cycles have been used to construct homology classes in a number of contexts [4, 27, 63, 95]. They appear as the main contribution to the homology of the clique complexes of random geometric graphs [67]. Moreover, they are the

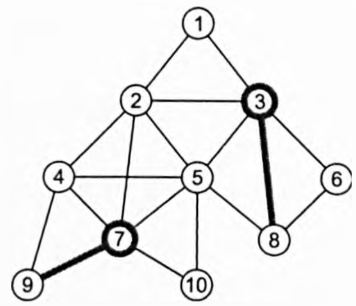


Figure 4.1: Example graph with an induced matching containing a maximal independent set.

minimal models, in the sense that for a flag complex K every non-zero homology class in $\widetilde{H}_{k-1}(K)$ must be supported on at least $2k$ vertices and, if the support size is exactly $2k$, then the class must be given by an embedded O_k, see [66]. This leads to our interest in their algorithmic properties.

Note that O_k is the independence complex of the disjoint union of k edges. This immediately leads to a characterization of cross-cycles in $I(G)$ in terms familiar to graph theorists.

A *matching* of size k is the disjoint union of k edges. An *induced matching* in a graph G is a matching which is an induced subgraph of G. Explicitly, it is a set of k edges of G such that any two vertices from distinct edges are non-adjacent in G.

Observation 4.2. If G is a graph then a cross-cycle of size k in $I(G)$ determines, and is determined, by an induced matching M of size k in G such that the vertex set of M contains a subset σ that is a maximal independent set of G. For simplicity we shall say *an induced matching containing a maximal independent set.*

Note that σ must contain exactly one vertex from each edge of M, and σ is a dominating set of G. For example, in the graph G in Fig. 4.1, $M = \{(7,9),(3,8)\}$ is an induced matching containing an independent set $\sigma = \{7,3\}$, and σ is a maximal independent set in G. This defines a cross-cycle and therefore a non-trivial homology class in $\widetilde{H}_1(I(G))$.

Note that as a clique complex, O_k is the clique complex of rather more involved object, namely the complete k-partite graph $K_{2,\dots,2}$.

Let us now be more specific. We will define a homology *and* a cohomology class associated to a pair (M,σ) consisting of an induced matching M containing a maximal independent set σ.

First of all, suppose that

$$M = \{(v_{1,1}, v_{1,2}), (v_{2,1}, v_{2,2}), \dots, (v_{k,1}, v_{k,2})\}$$

is any induced matching of size k in a graph G. Then we define a $(k-1)$-chain $\alpha_M \in C_{k-1}(I(G))$ by

$$\alpha_M = ([v_{1,1}] - [v_{1,2}]) \wedge ([v_{2,1}] - [v_{2,2}]) \wedge \cdots \wedge ([v_{k,1}] - [v_{k,2}]) \tag{4.1}$$
$$= \sum_{\epsilon_1,\dots,\epsilon_k \in \{1,2\}} (-1)^{\epsilon_1+\dots+\epsilon_k} [v_{1,\epsilon_1}, v_{2,\epsilon_2}, \dots, v_{k,\epsilon_k}]. \tag{4.2}$$

Since M is an induced matching, each $\{v_{1,\epsilon_1}, \dots, v_{k,\epsilon_k}\}$ is an independent set in G, and the definition makes sense. It is obvious that $\partial_{k-1}\alpha_M = 0$, therefore α_M is a cycle and it determines a homology class which we continue to denote $\alpha_M \in \widetilde{H}_{k-1}(I(G))$.

Note that if $i_M : O_k \hookrightarrow I(G)$ is the inclusion of the cross-cycle defined by M, then α_M is just the image of the fundamental class in $\widetilde{H}_{k-1}(O_k) = \widetilde{H}_{k-1}(S^{k-1})$ under the induced map in homology (up to a sign given by a choice of orientation).

If σ is an independent set of cardinality k in G then we denote by $\sigma^\vee \in C^{k-1}(I(G))$ the cochain which associates ± 1 to the two orientations of σ and 0 to all other simplices (again, this depends, up to sign, on the choice of orientation for σ). If σ is a maximal independent set (hence a maximal face in $I(G)$) then this cochain is in fact a cocycle, hence it determines an element of $\widetilde{H}^{k-1}(I(G))$ which we continue to denote $\sigma^\vee$.

Lemma 4.3. *If (M, σ) is an induced matching containing a maximal independent set then α_M and $\sigma^\vee$ are nonzero, non-torsion classes in $\widetilde{H}_*(I(G))$ and $\widetilde{H}^*(I(G))$, respectively. Moreover*

$$\langle \sigma^\vee, \alpha_M \rangle = \pm 1.$$

Proof. The last statement holds because $\sigma^\vee$ evaluates to ± 1 on exactly one of the simplices in the chain representation (4.2) of α_M. It immediately implies that both elements are nonzero and of infinite order. □

Unfortunately it is not always the case that $\widetilde{H}_{k-1}(I(G))$ is generated by the classes α_M over all induced matchings M of size k. For instance, for the cycle C_5 on 5 vertices, it is easy to see that $I(C_5) \equiv S^1$, but C_5 does not even have induced matchings of size two. However, we are going to see in Section 4.3 that the situation is better for chordal graphs.

4.2 Example: $\mathbb{Z}/n$-action on $\widetilde{H}_*(I(C_n))$

Consider the classical homotopy equivalences

$$I(C_{3k}) \simeq S^{k-1} \vee S^{k-1}, \quad I(C_{3k+1}) \simeq S^{k-1}, \quad I(C_{3k+2}) \simeq S^k$$

(see Example 7.23). The common feature of all methods of deriving results of this sort is that they require one to break the symmetry and forget the equivariant structure induced by the action of the automorphism group of the graph.

As a sample application of cross-cycles we will compute the action of $\mathbb{Z}/n$ on the homology of the above spaces. In fact it is this simple exercise (and its solution) that made the author independently "discover" cross-cycles and think of their other applications.

Let ρ denote the generator of the cyclic group $\mathbb{Z}/n$ acting on C_n by taking vertex i to $i+1$ (arithmetic modulo n) for $i = 0, \ldots, n-1$.

Theorem 4.4. *Let ρ_* denote the map induced by ρ on the unique nontrivial reduced homology group of $I(C_n)$. Then there is a basis of $\widetilde{H}_*(I(C_n))$ in which ρ_* is given by the matrix*

$$\rho_* = \begin{cases} (-1)^{n-1}\mathrm{id} & \text{for} \quad n \equiv 1,2 \pmod 3 \\ \begin{pmatrix} 0 & -1 \\ 1 & (-1)^n \end{pmatrix} & \text{for} \quad n \equiv 0 \pmod 3. \end{cases}$$

In other words, the representation of $\mathbb{Z}/n$ on $\widetilde{H}_*(I(C_n))$ factors through $\mathbb{Z}/\gcd(n,6)$.

The proof of the theorem depends on the value of n mod 3.

The action on $I(C_{3k})$.

Suppose $n = 3k$. In this case one can write explicit cross-cycles which generate homology. We have induced matchings

$$\begin{aligned} M_1 &= \{(1,2),(4,5),\ldots,(3k-2,3k-1)\}, \\ M_2 &= \{(2,3),(5,6),\ldots,(3k-1,3k)\}, \end{aligned}$$

with maximal independent sets, respectively:

$$\begin{aligned} \sigma_1 &= \{1,4,\ldots,3k-2\}, \\ \sigma_2 &= \{3,6,\ldots,3k\}. \end{aligned}$$

Their pairings are:

$$\langle\sigma_1^\vee, \alpha_{M_1}\rangle = 1, \quad \langle\sigma_1^\vee, \alpha_{M_2}\rangle = 0, \quad \langle\sigma_2^\vee, \alpha_{M_1}\rangle = 0, \quad \langle\sigma_2^\vee, \alpha_{M_2}\rangle = (-1)^k.$$

Since we know that $\widetilde{H}_{k-1}(I(C_{3k})) = \mathbb{Z}\oplus\mathbb{Z}$, the above equalities imply that $\{\alpha_{M_1}, \alpha_{M_2}\}$ is a basis of that group. Similarly $\{\sigma_1^\vee, \sigma_2^\vee\}$ is a basis of $\widetilde{H}^{k-1}(I(C_{3k}))$.

We can now calculate the action of ρ. Clearly $\rho_*\alpha_{M_1} = \alpha_{M_2}$. Moreover

$$\begin{aligned}
\langle\sigma_1^\vee, \rho_*\alpha_{M_2}\rangle &= \Big\langle\{1,4,\ldots,3k-2\}^\vee,\ ([3]-[4])\wedge([6]-[7])\wedge\cdots\wedge([3k]-[1])\Big\rangle \\
&= (-1)^{k-1}\cdot(-1)^k = -1 \\
\langle\sigma_2^\vee, \rho_*\alpha_{M_2}\rangle &= \Big\langle\{3,6,\ldots,3k\}^\vee,\ ([3]-[4])\wedge([6]-[7])\wedge\cdots\wedge([3k]-[1])\Big\rangle \\
&= 1
\end{aligned}$$

Therefore, if $\rho_*\alpha_{M_2} = x_1\alpha_{M_1} + x_2\alpha_{M_2}$, then

$$\begin{aligned}
x_1 &= \langle\sigma_1^\vee, \rho_*\alpha_{M_2}\rangle / \langle\sigma_1^\vee, \alpha_{M_1}\rangle = -1, \\
x_2 &= \langle\sigma_2^\vee, \rho_*\alpha_{M_2}\rangle / \langle\sigma_2^\vee, \alpha_{M_2}\rangle = (-1)^k = (-1)^n,
\end{aligned}$$

and that ends the calculation in this case.

Cycles of length not divisible by 3.

Cycles of length not divisible by 3 do not have induced matchings with maximal independent sets, but we can still avoid having to write an explicit generator by making a comparison with paths.

Lemma 4.5. *The independence complexes of paths satisfy:*

a) $I(P_{3k+1})$ *is contractible.*

b) The inclusion $i : I(P_{3k}) \hookrightarrow I(C_{3k+1})$*, induced by omitting one vertex of* C_{3k+1}*, is a homotopy equivalence.*

c) The inclusion $j : I(C_{3k+2}) \hookrightarrow I(P_{3k+2})$*, induced by omitting one edge of* C_{3k+2}*, is a homotopy equivalence.*

Proof. Part a) follows by a k-fold application of Corollary 2.13. Part b) follows from Proposition 2.1.a) because $I(C_{3k+1} \setminus N[v]) = I(P_{3(k-1)+1})$ is contractible. Part c) follows from Proposition 2.4.a) because $I(C_{3k+2} \setminus N[e]) = I(P_{3(k-1)+1})$ is contractible. □

Suppose $n = 3k+1$. Let $i : I(P_{3k}) \hookrightarrow I(C_{3k+1})$ be the homotopy equivalence obtained by removing vertex $3k+1$. Let (M, σ) be an induced matching with a maximal independent set in P_{3k} defined by

$$\begin{aligned} M &= \{(1,2),(4,5),\ldots,(3k-2,3k-1)\}, \\ \sigma &= \{2,5,\ldots,3k-1\}. \end{aligned}$$

Since the homology and cohomology groups in the interesting dimension are both $\mathbb{Z}$ the relation $\langle\sigma^\vee, \alpha_M\rangle = (-1)^k$ implies that α_M and $\sigma^\vee$ are generators of $\widetilde{H}_{k-1}(I(P_{3k}))$ and $\widetilde{H}^{k-1}(I(P_{3k}))$, respectively. If N is another induced matching in P_{3k} defined as $N = \{(2,3),(5,6),\ldots,(3k-1,3k)\}$ then $\langle\sigma^\vee, \alpha_N\rangle = 1$, hence $\alpha_N = (-1)^k\alpha_M$ in $\widetilde{H}_{k-1}(I(P_{3k}))$.

Since i is a homotopy equivalence, $i_*\alpha_M$ is a generator of $\widetilde{H}_{k-1}(I(C_{3k+1}))$. By construction $\rho_*(i_*\alpha_M) = i_*\alpha_N$ so we have

$$\rho_*(i_*\alpha_M) = i_*\alpha_N = (-1)^k i_*\alpha_M$$

which means that ρ acts on the group $\widetilde{H}_{k-1}(I(C_{3k+1})) = \mathbb{Z}$ as $(-1)^k = (-1)^{n-1}$.

Next suppose $n = 3k+2$. We essentially dualize the previous argument. Let $M = \{(1,2),(4,5),\ldots,(3k+1,3k+2)\}$ be an induced matching in P_{3k+2} with two maximal independent sets $\sigma_1 = \{2,5,\ldots,3k+2\}$ and $\sigma_2 = \{1,4,\ldots,3k+1\}$. Again, since $\langle\sigma_1^\vee, \alpha_M\rangle = (-1)^{k+1}$, and the groups in question are all $\mathbb{Z}$, we get that $\sigma_1^\vee$ and α_M are generators of $\widetilde{H}^k(I(P_{3k+2}))$ and $\widetilde{H}_k(I(P_{3k+2}))$. Since $\langle\sigma_2^\vee, \alpha_M\rangle = 1$ we get $\sigma_2^\vee = (-1)^{k+1}\sigma_1^\vee$ in $\widetilde{H}^k(I(P_{3k+2}))$.

Since j is a homotopy equivalence, $j^*\sigma_1^\vee$ is a generator of $\widetilde{H}^k(I(C_{3k+2}))$ and by construction $\rho^*(j^*\sigma_1^\vee) = j^*\sigma_2^\vee = (-1)^{k+1}j^*\sigma_1^\vee$. It means that ρ acts on cohomology, and hence also on homology, as $(-1)^{k+1} = (-1)^{n-1}$.

4.3 Homology of chordal graphs

Recall from Section 1.2 that a graph is *chordal* if every induced cycle has length three. By Corollary 2.22 the independence complex of a chordal graph is homotopy equivalent to a wedge of spheres. Moreover, every wedge of spheres arises, up to homotopy, as an independence complex of a chordal graph [69]. Induced matchings in chordal graphs are directly related to algebraic invariants of their edge ideals [48]. Another reason to study chordal graphs in this context is that for this family of graphs, cross-cycles detect all of the homology of the independence complex, as we now explain.

Let us recall the following result we proved in Section 2.4, and originally coming from [35].

Lemma 4.6 (Corollary 2.14). *If G is any graph and v is a simplicial vertex, then there is a homotopy equivalence*

$$I(G) \simeq \bigvee_{u \in N(v)} \Sigma I(G \setminus N[u]).$$

This gives a method of recursively computing the homotopy type of $I(G)$ when G is chordal. For example, for the graph in Fig. 4.1, choosing $v = 1$ yields

$$I(G) \simeq \Sigma I\big(G[6,8,9,10]\big) \vee \Sigma I\big(G[4,7,9,10]\big).$$

The first graph has isolated vertices, so its independence complex is a cone, hence contractible. In the other graph, $v = 10$ is a simplicial vertex of degree 1. We conclude that $I(G[4,7,9,10]) \simeq \Sigma I(\emptyset) = \Sigma S^{-1} = S^0$, and so $I(G) \simeq S^1$.

This naive method leads only to an exponential time algorithm computing the homotopy type of $I(G)$ for a chordal graph. However, it provides a connection to graph theory.

Theorem 4.7. *For a chordal graph G and any $k \geq 0$, the homology group $\widetilde{H}_{k-1}(I(G))$ is non-trivial if and only if G has an induced matching of size k containing a maximal independent set.*

Proof. The 'if' part follows from the discussion of cross-cycles in Section 4.1. To prove the 'only if' part we use induction on the size of G.

If $V(G) = \emptyset$, then $I(G) = S^{-1}$ and $\widetilde{H}_{-1}(S^{-1}) = \mathbb{Z}$. In this case the empty matching of size 0 satisfies the requirements.

Thus suppose that G is a chordal graph with at least one vertex and that $\widetilde{H}_{k-1}(I(G)) \neq 0$. Let v be any simplicial vertex of G (which exists by [30]). By Lemma 4.6 there is a splitting

$$\widetilde{H}_{k-1}(I(G)) = \bigoplus_{u \in N(v)} \widetilde{H}_{k-2}(I(G \setminus N[u])).$$

It follows that there exists a vertex $u \in N(v)$ such that $\widetilde{H}_{k-2}(I(G \setminus N[u])) \neq 0$. The graph $G \setminus N[u]$ is chordal, so by induction it has an induced matching M' of size $k-1$ containing a maximal independent set σ'.

Now define a new pair (M, σ) of size k in G by setting

$$M = M' \cup \{vu\}, \qquad \sigma = \sigma' \cup \{u\}.$$

We easily see that M is an induced matching in G and σ is an independent set which is maximal in G.

This completes the proof. □

Remark 4.8. A more careful analysis of this argument shows that, in fact, slightly more is true. We leave the proof to the reader, as we do not need the full strength of the next result for our algorithmic applications.

Proposition 4.9. *If G is chordal and $k \geq 0$ then the homology group $\widetilde{H}_{k-1}(I(G))$ is generated by the classes α_M as M runs through all induced matchings of size k containing maximal independent sets.*

4.4 Contractibility of $I(G)$ for a chordal graph G in polynomial time — outline

Theorem 4.7 establishes a strong connection between the homotopy type of $I(G)$ and the combinatorics of induced matchings of a chordal graph G. The results of this, and the next section, describe quite precisely how much of the homotopical information about $I(G)$ can be recovered from G in polynomial time.

Theorem 4.10. *There is an $O(|E(G)|^2)$ time algorithm that decides, for a chordal graph G, if G has an induced matching containing a maximal independent set.*

By Theorem 4.7 this result has an immediate topological consequence.

Theorem 4.11. *For a chordal graph G one can decide in polynomial time*

- *if $I(G)$ is contractible,*
- *if $I(G)$ is simply-connected.*

Note that for an arbitrary graph G the problems of deciding if $I(G)$ is simply-connected or contractible are both undecidable (resp. [49], [14]).

We will outline the proof of Theorem 4.10, and that means constructing a polynomial time algorithm checking if a chordal graph G has an induced matching M containing a maximal independent set σ. For simplicity, in this section we call such a pair (M, σ) a *solution* to G. The algorithm is due to Juraj Stacho and for its full details the reader is referred to [8].

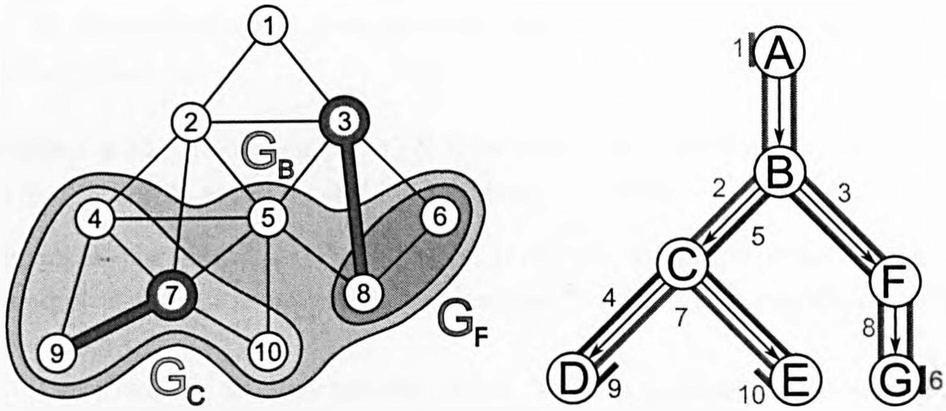


Figure 4.2: Example chordal graph G and its rooted tree model.

Below, we first describe necessary definitions and preliminary steps before we present the proof.

A *tree model* of a graph $G = (V, E)$ consists of a tree T, called *a host tree*, and a collection of subtrees of T, one for each vertex u of G, denoted by $\{T_u\}_{u \in V}$, with the property that $uv \in E$ if and only if $V(T_u) \cap V(T_v) \neq \emptyset$. For clarity, we shall use capital letters $X, Y, \ldots$ for the vertices of T and call them *nodes*. The following is a well-known fact [46].

Theorem 4.12. [46] *A graph is chordal if and only if it has a tree model.*

We remark that there is a linear time algorithm [92] to determine whether an input graph G is chordal, and if so, to construct a tree model of G. Thus, for the rest of this section, we shall assume that we have a fixed tree model of G and all subsequent considerations are always with respect to this model. We consider T rooted at some node, and we direct all edges of T away from the root (Fig. 4.2). For $X, Y \in V(T)$, we write $X \preceq Y$ if there is in T a directed path from Y to X. Observe that $\preceq$ is a partial order. We write $X \prec Y$ if $X \preceq Y$ and $X \neq Y$. If XY is an edge of T oriented from Y to X, we say that Y is the *parent* of X, and X is a *child* of Y. If $X \prec Y$, we say that Y is an *ancestor* of X, and X is a *descendant* of Y.

Notation 4.13. For $u \in V(G)$, $\mathsf{top}(u)$ denotes the maximum element of $V(T_u)$ with respect to $\preceq$.

The following is a simple consequence of the definition of a tree model.

Fact 4.14. *If* $uv \in E(G)$, *then* $\mathsf{top}(u) \succeq \mathsf{top}(v) \in V(T_u)$ *or* $\mathsf{top}(v) \succeq \mathsf{top}(u) \in V(T_v)$.

To decide whether a solution to G exists, it suffices to consider particular type of solutions.

Definition 4.15. *A solution (M, σ) to G is* canonical *if every edge uv in M is such that $u \in \sigma$ if $\mathsf{top}(u) \succ \mathsf{top}(v)$ and $v \in \sigma$ if $\mathsf{top}(u) \prec \mathsf{top}(v)$.*

For example, the solution $(\{(7,9),(3,8)\},\{7,3\})$ to the graph presented (with its tree model) in Fig.4.2 is canonical, because $\mathsf{top}(7) = C \succ D = \mathsf{top}(9)$ and $\mathsf{top}(3) = A \succ F = \mathsf{top}(8)$.

The following is an important result which is ultimately a consequence of Lemma 4.6.

Lemma 4.16. *If there exists a solution to G, then there exist a canonical solution to G of the same size.*

Our algorithm is based on dynamic programming on T that tries to find canonical solutions for subgraphs of G and then combines these solutions to obtain a solution to G if one exists. In particular, we focus on subgraphs induced by vertices whose subtrees lie completely below some node of T.

Notation 4.17. For $X \in V(T)$, G_X denotes the subgraph of G induced on all vertices v with $\mathsf{top}(v) \preceq X$.

If X is the root of T, then $G_X = G$. We distinguish the following special type of solutions to G_X.

Definition 4.18. *A solution (M, σ) to G_X is* rooted *if there exists $u \in \sigma$ such that $\mathsf{top}(u) = X$.*

Notation 4.19. $\mathcal{S}$ denotes the set of all nodes $X \in V(T)$ such that there exists a solution to G_X.

Notation 4.20. $\mathcal{R}$ denotes the set of all nodes $X \in V(T)$ such that there exists a rooted canonical solution to G_X.

Let us explain the notation and our strategy using the example in Fig. 4.2. We want to know whether the root of T is in $\mathcal{S}$. To find out, we recursively find the nodes that admit a rooted solution (the set $\mathcal{R}$). For example, $C \in \mathcal{R}$ since the subgraph G_C has a solution $(\{(7,9)\},\{7\})$. On the other hand, $B \notin \mathcal{R}$ since for any choice of an edge $(5,v)$ in G_B, the graph $G_B \setminus (N[5] \cup N[v])$ has an isolated vertex, and thus no solution. Still, G_B has a non-rooted solution combined from the rooted solutions of the disjoint subgraphs G_C and G_F.

Finally for $G = G_A$, we try the edge $(3, 8)$ and see that $G_A \setminus (N[3] \cup N[8]) = G_C$. We then check that the rooted solution to G_C together with the edge $(3, 8)$ indeed form a solution to G_A. This implies that $A \in \mathcal{S}$ as we wanted.

We now state, without proof, three technical lemmas that explain how we can compute the sets $\mathcal{R}$ and $\mathcal{S}$. Lemma 4.22 explains how to obtain a solution to G_X (if one exists) by combining rooted canonical solutions for descendants of X, and Lemma 4.23 provides a way to determine if a rooted solution exists to G_X. The proofs appear in [8].

Lemma 4.21. *Let $X \in V(T)$ and suppose that there exists a canonical solution (M, σ) to G_X. Then*

(i) if $Y = \mathsf{top}(u)$ for some $u \in \sigma$, then $Y \in \mathcal{R}$, and

(ii) if $Y \preceq X$ and $Y \notin \bigcup_{u \in \sigma} V(T_u)$, then $Y \in \mathcal{S}$.

Lemma 4.22. *Let $X \in V(T)$. Define $\mathcal{R}_X = \{Y \mid Y \preceq X,\ Y \in \mathcal{R}\}$ and let $\mathcal{R}_X^*$ denote the set of maximal elements of $\mathcal{R}_X$ with respect to $\preceq$. For each $Y \in \mathcal{R}_X^*$, let (M_Y, σ_Y) be a rooted canonical solution to G_Y, and define*

$$M^* = \bigcup_{Y \in \mathcal{R}_X^*} M_Y \qquad \sigma^* = \bigcup_{Y \in \mathcal{R}_X^*} \sigma_Y$$

Then, if there exists a solution to G_X, then (M^*, σ^*) is a canonical solution to G_X.

Lemma 4.23. *Let $Z \in V(T)$, and let $uv \in E(G)$ be an edge such that $\mathsf{top}(u) = Z$ and $\mathsf{top}(v) \preceq Z$.*

Define $\mathcal{L}_{uv} = \{Y \mid Y \preceq Z,\ Y \notin V(T_u) \cup V(T_v)\}$ and let $\mathcal{L}_{uv}^*$ be the set of maximal elements of $\mathcal{L}_{uv}$ with respect to $\preceq$.

For each $X \in \mathcal{L}_{uv}^*$, define $\mathcal{R}_X^*$ just like in Lemma 4.22, and for each $Y \in \mathcal{R}_X^*$, let (M_Y, σ_Y) be a rooted canonical solution to G_Y. Finally, define

$$M_{uv}^* = \{uv\} \cup \bigcup_{X \in \mathcal{L}_{uv}^*} \bigcup_{Y \in \mathcal{R}_X^*} M_Y \qquad \sigma_{uv}^* = \{u\} \cup \bigcup_{X \in \mathcal{L}_{uv}^*} \bigcup_{Y \in \mathcal{R}_X^*} \sigma_Y$$

Then, if there exists a canonical solution (M, σ) to G_Z such that $uv \in M$ and $u \in \sigma$, then $(M_{uv}^*, \sigma_{uv}^*)$ is a rooted canonical solution to G_Z.

With these observations we can now present our algorithm.

Proof of Theorem 4.10. We proceed in two steps. In the first phase, we process all nodes X of T from leaves to the root and decide if $X \in \mathcal{R}$. For every node X for which the answer is 'yes', we also store (for later use) a rooted canonical solution to G_X.

In the second phase, we check if the root of T is in $\mathcal{S}$. This determines whether or not there is a solution to G.

The recursive step of the first phase works as follows. Suppose we are at a node $Z \in V(T)$ and that all descendants of Z have already been processed. We try every edge $uv \in E(G)$ such that $\mathsf{top}(u) = Z$ and $\mathsf{top}(v) \preceq Z$. We construct the sets M^*_{uv} and σ^*_{uv} as described in Lemma 4.23. Note that we are able to construct these sets because all descendants of Z have already been processed. Then we test if σ^*_{uv} is a dominating set of G_Z. If so, we declare that there exists a rooted canonical solution $(M^*_{uv}, \sigma^*_{uv})$ to G_Z. If we fail for every possible choice of uv, then we declare that there is no rooted canonical solution to G_Z. The correctness of this procedure is guaranteed by Lemma 4.23.

We then proceed with the second phase and test if there exists a canonical solution to G. To do so we construct the sets M^* and σ^* as described in Lemma 4.22 for X = root of T and test if σ^* is a dominating set of G. If so, we declare that G has a solution given by (M^*, σ^*). If not, we declare that no solution for G exists. The correctness of this step is guaranteed by Lemma 4.22.

To analyze the running time, denote $n = |V(G)|$ and $m = |E(G)|$. Each time we process a node Z, we test a subset of edges of G_Z and each edge uv of G is tested this way exactly once during the whole run of the algorithm. To construct M^*_{uv} and σ^*_{uv} it suffices to search through the descendants of Z in T. This clearly takes at most $O(n)$ time, since both the sets M^*_{uv} and σ^*_{uv} have no more than n elements and also T contains at most n nodes. Afterwards, we test if σ^*_{uv} is a dominating set of G_Z. This can be done directly in time $O(m)$ by exploring the neighbourhood of every vertex in $V(G_Z) \setminus \sigma^*_{uv}$. The same applies to the construction and testing of M^* and σ^* in the second phase. Thus, altogether, the total complexity is $O(m^2)$ which concludes the proof. □

4.5 Hardness of computing homology

The polynomial time algorithms from the previous section have the following hardness counterparts.

Theorem 4.24. *Given a chordal graph G and an integer k, it is NP-complete to decide if G has an induced matching of size k containing a maximal independent set.*

Theorem 4.25. *Given a chordal graph G and integer k, it is NP-complete to decide if the group $\widetilde{H}_{k-1}(I(G))$ is non-trivial.*

Of course Theorem 4.25 is an immediate consequence of Theorem 4.24 (via Theorem 4.7), so it suffices to prove Theorem 4.24. This is the goal of this section. At the end we derive a general statement, Theorem 4.27, about NP-hardness of computing homology groups of arbitrary simplicial complexes.

We start with another problem used in our reductions.

Lemma 4.26. *It is NP-complete to decide, for a given chordal graph G and integer k, whether or not G has a maximal independent set of size exactly k.*

Proof. We construct a reduction from the following problem: given a graph H and an integer ℓ, decide if H has a dominating set of size ℓ. This problem is NP-complete since the corresponding minimization problem (minimum size dominating set) is NP-hard (cf. [45, Prob.GT2]).

Consider an instance to this problem, a graph $H = (V, E)$ and an integer ℓ. Construct the following graph G:

$$
\begin{aligned}
V(G) =& \{v_0, v_1, \ldots, v_6 \mid v \in V\}, \\
E(G) =& \{u_0v_0 \mid u, v \in V,\ u \neq v\} \cup \{u_0v_1 \mid uv \in E\} \cup \\
& \{v_0v_1, v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_6 \mid v \in V\}.
\end{aligned}
$$

Let $V_0 = \{v_0 \mid v \in V\}$. Note that V_0 is a clique of G. From this, it is not difficult to see that G is a chordal graph.

We show that the following two conditions are equivalent:

- H has a dominating set of cardinality ℓ,
- G has a maximal independent set of cardinality $4|V| - 2\ell$.

This will yield the polynomial-time reduction and the proof.

First, if $D \subseteq V$ is a dominating set in H with $|D| = \ell$, then

$$\{v_1, v_6 \mid v \in D\} \cup \{v_2, v_3, v_4, v_5 \mid v \notin D\}$$

is a maximal independent set in G of size

$$2\ell + 4(|V| - \ell) = 4|V| - 2\ell.$$

For the converse, let $D \subseteq V(G)$ be a maximal independent set in G of size $|D| = 4|V| - 2\ell$. For any vertex v, we have

- if $v_1 \in D$, then $v_6 \in D$ and $v_2, v_3, v_4, v_5 \notin D$,
- if $v_1 \notin D$, then $v_3, v_4, v_5 \in D$ and exactly one of v_2, v_6 is in D.

This yields:

$$|D \cap \{v_1, \ldots, v_6\}| = \begin{cases} 2 & \text{if } v_1 \in D, \\ 4 & \text{if } v_1 \notin D. \end{cases} \tag{4.3}$$

Now, let $D' = \{v \in V \mid v_1 \in D\}$. By (4.3) we can conclude

$$|D| = |D \setminus V_0| + |D \cap V_0| = 2|D'| + 4(|V| - |D'|) + |D \cap V_0|$$

Since V_0 is a clique of G, and D is an independent set of G, we conclude $|D \cap V_0| \leq 1$. However, $|D \cap V_0| = 1$ implies that the cardinality of D is odd, which is not the case.

We must conclude $D \cap V_0 = \emptyset$, and thus, $4|V| - 2\ell = |D| = 2|D'| + 4(|V| - |D'|)$. From this, we obtain $|D'| = \ell$. Moreover, D' is a dominating set of H, as otherwise some vertex $v_0 \in V_0$ could be used to enlarge D to a bigger independent set in G. That completes the proof. □

Proof of Theorem 4.24. We perform a reduction from the exact cardinality independent dominating set problem in chordal graphs (Lemma 4.26). Consider a chordal graph $G = (V, E)$ and an integer k. We construct a graph G' from G by substituting an edge for every vertex of G. Namely:

$$\begin{aligned} V(G') &= \{v_1, v_2 \mid v \in V\}, \\ E(G') &= \{v_1v_2 \mid v \in V\} \cup \{u_1v_1, u_1v_2, u_2v_1, u_2v_2 \mid uv \in E\}. \end{aligned}$$

Note that G' is a chordal graph, since chordal graphs are closed under the operation of replacing a vertex by a clique.

If $D \subseteq V$ is a maximal independent set in G then

$$M' = \{v_1v_2 \mid v \in D\}, \quad D' = \{v_1 \mid v \in D\}$$

is an induced matching containing a maximal independent set in G' of the same cardinality as D.

Conversely, if (M', D') is a solution to G', then by setting

$$D = \{v \in V \mid v_1 \in D' \text{ or } v_2 \in D'\}$$

we obtain a maximal independent set of G of the same size.

This concludes the proof of Theorem 4.24. $\square$

Our constructions have a more general consequence for the hardness of calculating homology groups of arbitrary flag complexes.

Theorem 4.27. *The following problems are NP-hard.*

- *Given the* 1*-skeleton of a flag complex* K*, and an integer* k*, decide whether* $\widetilde{H}_{k-1}(K) = 0$*.*
- *Given any simplicial complex* K*, represented by the list of maximal faces, and an integer* k*, decide whether* $\widetilde{H}_{k-1}(K) = 0$*.*

Proof. The first statement follows directly from Theorem 4.25. To prove the second statement we describe a reduction from the NP-complete problem of Theorem 4.25. Suppose we have a chordal graph G with n vertices. Let K_G be the simplicial complex with vertices $V(G)$ whose maximal faces are the "complements" of edges in G, that is

$$\begin{aligned} \sigma \in K_G &\iff G[\overline{\sigma}] \text{ contains an edge} \\ &\iff \overline{\sigma} \text{ is not independent in } G \end{aligned}$$

where $\overline{\sigma}$ denotes $V(G) \setminus \sigma$.

Note that K_G is the Alexander dual [18] of $I(G)$ and that the list of the maximal faces of K_G has size polynomial in the size of G. Next, we have

$$\widetilde{H}_k(I(G)) = \widetilde{H}^k(I(G)) = \widetilde{H}_{n-k-3}(K_G)$$

where the second equality is Alexander duality [18, Thm. 1.1] and the first holds because $I(G)$ has the homotopy type of a wedge of spheres. This reduces the problem of Theorem 4.25 to the problem of computing the homology of K_G. $\square$

Let us make a few comments about the last theorem. Its second part answers [68, Problem 33]. Theorem 4.27 is not surprising, as there are no known methods of calculating $\widetilde{H}_k(K)$ without enumerating in some way or other the (exponentially many) k-faces. There is much research on practical algorithms and their performance (see [103] and the references therein). However, the author is not aware of any previous proof that the problems are in fact hard. See [93] for a recent result about the hardness of computing the Euler characteristic.

Open Problem 4.28. Note that our hardness results do not address the problem of finding *minimum* or *maximum* size cross-cycles in the case of chordal graphs. By Theorem 4.7 these are equivalent to calculating the connectivity and homological dimension of $I(G)$ for a chordal graph G.

We suspect that these two problems are also hard, but we were not able to prove this.

Chapter 5

Superfrustration in some lattices

5.1 Introduction

The purpose of this, and the next chapter, is to investigate some topological questions arising from the study of the so-called hard-core models on grids. It is motivated by a recent collection of papers [60, 61, 21, 37, 99, 62], which in turn were motivated by some combinatorial questions in statistical physics [41]. The hard-core model itself is very classical and has been studied from different points of view in physics, topology and combinatorics.

Suppose one has a finite graph L, which in applications is usually a periodic lattice with some boundary conditions. Square, triangular or hexagonal grids are the most notable examples. The vertices of the graph can be occupied by particles, such as fermions, which satisfy the *hard-core* restriction: two adjacent vertices cannot be occupied simultaneously. A configuration of particles which satisfies this assumption is therefore precisely an independent set in the graph L.

There is a close connection between the simplicial and topological invariants of $I(L)$ and certain characteristics of the corresponding lattice model which are of interest to physicists. It is beyond the scope of this thesis to discuss this relationship in detail; we refer to [55] and we limit ourselves to presenting just the most basic dictionary:

the partition function of L	the f-polynomial of $I(L)$,
the Witten index of L	minus the reduced Euler characteristic $-\widetilde{\chi}(I(L))$,
the number of zero energy ground states	the dimension of $\widetilde{H}_*(I(L);\mathbb{Q})$.

There has been some very successful work calculating the Witten index [60, 61, 21, 37, 99, 62, 40], homology groups [37, 33, 41, 55, 63, 56, 54] or indeed the complete homotopy type of the independence complex [21, 99, 53, 56] for various lattices.

In this chapter we focus on the large-scale picture. Computer simulations of van Eerten [33] indicate that for some types of lattices, as their size increases, the number of ground states grows exponentially with the number of vertices, that is

$$\dim \widetilde{H}_*(I(L)) \sim a^{v(L)}$$

for some constant a depending on the type of the lattice, where $v(L)$ denotes the number of vertices in a graph L. This situation is called *superfrustration* and has interesting physical implications, see [54]. Engström [37] developed a general method of computing *upper* bounds for the constant a. For the lattices of [33] it gives bounds very close to the values predicted in [33].

This chapter has two main parts. In the first one we present a method which can be used to construct exponentially many linearly independent homology classes in $I(L)$ for graphs L of certain type. The homology classes in question are given by the cross-cycles introduced in Chapter 4. That proves superfrustration of certain lattices and we give examples based on modifications of the triangular lattice. In the second part we prove a generalization of the main result of [37], which can sometimes give better upper bounds.

Both methods work particularly nicely with one type of lattice studied in [33, 37]: the *hexagonal dimer*, also known as the *Kagome lattice* (see Fig.5.1). Under suitable divisibility conditions on the height and width we will prove that a graph $\mathbb{H}$ of that type satisfies:

$$1.02^{v(\mathbb{H})} \approx (2^{1/36})^{v(\mathbb{H})} \leq \dim \widetilde{H}_*(I(\mathbb{H})) \leq (14^{1/36} \cdot 2^{1/6})^{v(\mathbb{H})} \approx 1.21^{v(\mathbb{H})}.$$

We will prove the lower bound in Section 5.2 and the upper bound in Section 5.4. The previous upper bound of [37] was $2^{1/3} \approx 1.26$ and the experimental approximation by [33] is 1.25 ± 0.1.

Our technique for lower bounds produces slightly more than just homology classes: we obtain a large wedge of spheres that splits off. For instance, for a suitable lattice $\mathbb{H}$ of Kagome type, this reads as a homotopy equivalence

$$I(\mathbb{H}) \simeq \Big(\bigvee^{(2^{1/36})^{v(\mathbb{H})}} S^{2v(\mathbb{H})/9-1} \Big) \vee X$$

for some space X. This type of result is proved in Section 5.3.

Remark 5.1. Recall the absolute upper bound (Theorem 2.17): for any graph G we have

$$\dim \widetilde{H}_*(I(G)) \leq (2^{2/5})^{v(G)} \approx 1.32^{v(G)}.$$

5.2 Hexagonal dimer and related grids

The *hexagonal dimer* or the *Kagome lattice* is the lattice obtained from the triangular lattice by erasing every other line in each direction in the way shown in Fig.5.1. It is invariant under the translations by $(2,0)$ and $(0,\sqrt{3})$. Let $\mathbb{H}_{n,m}$ denote the quotient of that lattice by the action of the translation group generated by the vectors $n\cdot(2,0)$ and $m\cdot(0,\sqrt{3})$. It has $v(\mathbb{H}_{n,m}) = 3nm$ vertices and if one additionally assumes that $6|n$ and $4|m$ then it can be tiled with large hexagons in the way shown in Fig.5.1.

In fact there is no harm forgetting about n and m. Let $\mathbb{H}$ be any quotient of the hexagonal dimer lattice (i.e. its finite portion with cyclic boundary conditions) with the property that it can be covered in this way by the large hexagons. Then the number of those hexagons is always $v(\mathbb{H})/36$ and we have the next result.

Proposition 5.2. *For a hexagonal dimer lattice $\mathbb{H}$ which admits a tiling as in Fig.5.1 we have*

$$\widetilde{\beta}(I(\mathbb{H})) \geq (2^{1/36})^{v(\mathbb{H})}.$$

Proof. We are going to use the notation introduced in Section 4.1 in relation to cross-cycles.

Let $k = v(\mathbb{H})/36$ be the number of large hexagons in the tiling of $\mathbb{H}$. We are going to construct 2^k linearly independent elements in $\widetilde{H}_{8k-1}(I(\mathbb{H}))$. That proves the claim since then $\widetilde{\beta}(I(\mathbb{H})) \geq 2^k = 2^{v(\mathbb{H})/36}$.

Consider the tiles A and B of Fig.5.1, where they are shown with an induced matching (thick edges) and a maximal independent set (thick vertices). Note that for any placement of A and B in place of the large hexagons in the grid we obtain a valid induced matching containing a maximal independent set in $\mathbb{H}$. Indeed, there are no edges directly between the tiles, so the matching is induced. Moreover, each vertex located between the tiles is adjacent to one of the thick vertices on the outer cycle of a tile. We therefore have 2^k homology (and cohomology) classes in $I(\mathbb{H})$.

To prove linear independence we need some notation. Suppose that the big hexagons in the grid are labeled $1,\ldots,k$ in some order. For any sequence

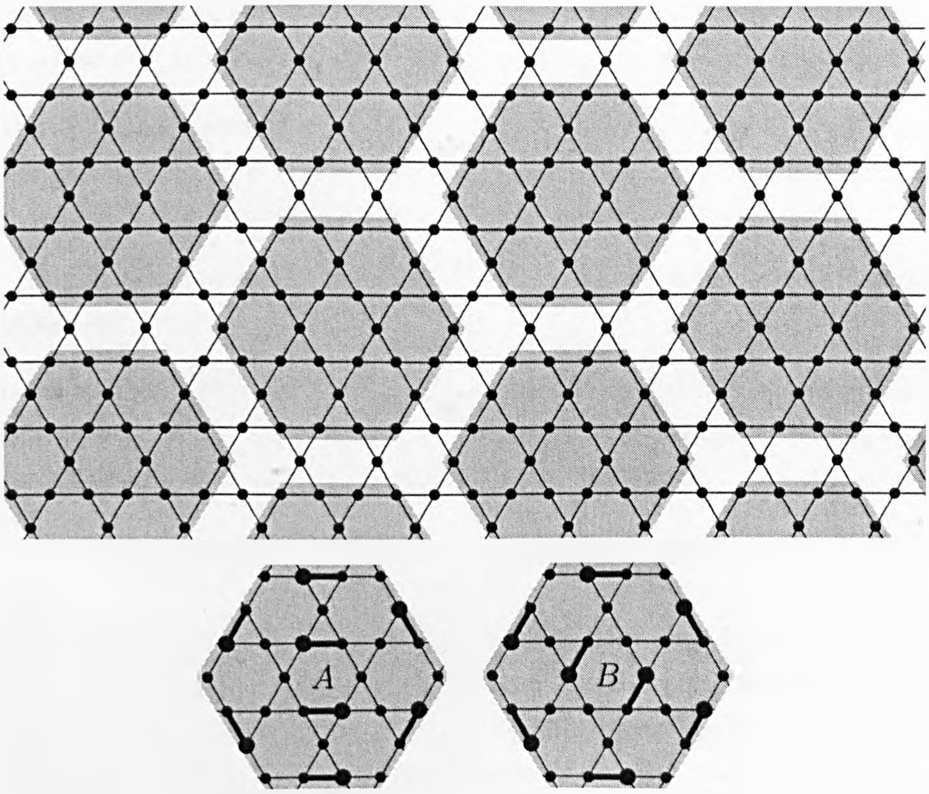


Figure 5.1: The hexagonal dimer lattice, its tiling with large hexagons and two types of tiles.

$s = (s_1, \ldots, s_k)$ of letters A, B let $M(s)$ and $\sigma(s)$ denote the matching and the independent set obtained in the above construction by placing the tile of type s_i in the i-th spot for $i = 1, \ldots, k$. Then for any two sequences s and t we have

$$\langle \sigma(t)^\vee, \alpha_{M(s)} \rangle = \begin{cases} 0 & \text{if } t_i = B \text{ and } s_i = A \text{ for some } i \\ \pm 1 & \text{otherwise} \end{cases}$$

Consider a $2^k \times 2^k$ matrix with rows and columns indexed by sequences in $\{A, B\}^k$ in lexicographical order, where the entry in column t and row s is $\langle \sigma(t)^\vee, \alpha_{M(s)} \rangle$. By the last observation that matrix has ± 1 on the diagonal and 0 above the diagonal (where $t >_{LEX} s$), so it is of full rank. Therefore all $\alpha_{M(s)}$ are linearly independent. □

Remark 5.3. In general, if $\alpha_1, \ldots, \alpha_k \in \widetilde{H}_i(X)$ and $\gamma_1, \ldots, \gamma_l \in \widetilde{H}^i(X)$ then the i-th Betti number $\widetilde{\beta}_i(X)$ is at least as big as the rank of the $l \times k$ matrix with entries $\langle \gamma_s, \alpha_t \rangle$ for $1 \le s \le l$, $1 \le t \le k$.

Similar results can be obtained for some other lattices derived from the triangular lattice.

Proposition 5.4. *For $d = 3, 4$ let $\mathbf{\Delta}_d$ be any finite quotient of the lattice obtained by removing every d-th line in each direction from the triangular lattice, such that the new lattice admits a tiling as in Fig.5.2. Then we have*

$$\widetilde{\beta}(I(\mathbf{\Delta}_3)) \ge (2^{1/8})^{v(\mathbf{\Delta}_3)}, \qquad \widetilde{\beta}(I(\mathbf{\Delta}_4)) \ge (2^{1/45})^{v(\mathbf{\Delta}_4)}.$$

Proof. The proof is similar: $\mathbf{\Delta}_3$ has $k = v(\mathbf{\Delta}_3)/8$ tiles and $\mathbf{\Delta}_4$ has $k = v(\mathbf{\Delta}_4)/45$ of them. In each case we construct 2^k induced matchings containing maximal independent sets using the tiles A, B of appropriate type (Fig.5.2) and prove linear independence of the resulting homology classes as before. □

5.3 Homotopical splittings

We will now show for completeness that induced matchings containing maximal independent sets in G correspond to sphere wedge summands in the homotopy type of $I(G)$. This is stronger than just saying that they define a nonzero homology class. The results of this section are not needed in the previous calculations, where analyzing the cohomology-homology pairing is sufficient (and easier).

Lemma 5.5. *Let (M, σ) be an induced matching of size k containing a maximal independent set in G. Denote by $j : I(M) \hookrightarrow I(G)$ the embedding induced by the*

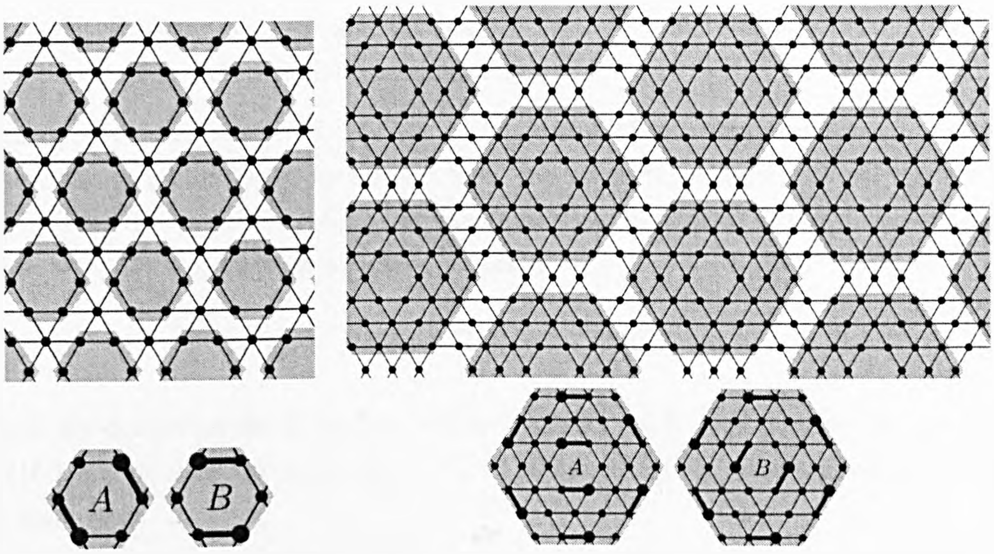


Figure 5.2: Two other lattices derived from the triangular lattice and their sets of tiles.

inclusion $i : M \hookrightarrow G$. Then there is a homotopy equivalence

$$I(G) \simeq I(M) \vee C(j)$$

where $C(j)$ is the homotopy cofibre of j.

Proof. Since σ is a maximal face of $I(G)$ we can remove it and form the simplicial complex $I(G) \setminus \sigma$ whose geometric realization is obtained by removing the interior of σ. Then we have a cofibre sequence

$$\partial\sigma \hookrightarrow I(G) \setminus \sigma \hookrightarrow I(G) \to \Sigma\,(\partial\sigma) \to \cdots$$

i.e. $I(G)$ is the homotopy cofibre of the inclusion $\partial\sigma \hookrightarrow (I(G)\setminus\sigma)$. But this inclusion is null-homotopic since it factors through the space $I(M) \setminus \sigma$, homeomorphic to the disk D^{k-1}. As a consequence

$$I(G) \simeq (I(G) \setminus \sigma) \vee \Sigma\,(\partial\sigma) = (I(G) \setminus \sigma) \vee \Sigma\, S^{k-2} = (I(G) \setminus \sigma) \vee S^{k-1}.$$

Now the fact that $I(M) \setminus \sigma$ is a contractible subcomplex of $I(G) \setminus \sigma$ implies that

$$I(G) \setminus \sigma \simeq (I(G) \setminus \sigma)/(I(M) \setminus \sigma) = I(G)/I(M) \simeq C(j).$$

Together with $S^{k-1} = I(M)$ this ends the proof. $\square$

To analyze our situation further we need to understand homotopy cofibres of maps between independence complexes.

Lemma 5.6. *Suppose $i : H \hookrightarrow G$ is an inclusion of an induced subgraph. Let $C(i)$ denote the graph obtained by adding to G a new vertex adjacent to all the vertices of $V(G) \setminus V(H)$. Then $I(C(i))$ is the homotopy cofibre of the induced inclusion $I(H) \hookrightarrow I(G)$, i.e. there is a cofibre sequence*

$$I(H) \hookrightarrow I(G) \hookrightarrow I(C(i)) \to \Sigma I(H) \to \cdots .$$

Proof. By definition the homotopy cofibre of $I(H) \hookrightarrow I(G)$ is obtained by attaching to $I(G)$ a cone over the subspace $I(H)$. The space $I(C(i))$ is obtained in precisely the same way. □

Now we can prove the main result of this section.

Proposition 5.7. *Suppose (M_i, σ_i), for $i = 1, \dots, k$, is a sequence of induced matchings containing maximal independent sets in a graph G, such that for every $i < j$ we have*

$$\sigma_j \setminus V(M_i) \neq \emptyset. \tag{*}$$

Then there is a homotopy equivalence

$$I(G) \simeq \Big(\bigvee_{i=1}^{k} I(M_i) \Big) \vee X$$

for some space X.

Proof. Let $i_1 : M_1 \hookrightarrow G$ be the inclusion of the first matching. Then, by Lemmas 5.5 and 5.6 we have a splitting

$$I(G) \simeq I(M_1) \vee I(C(i_1)).$$

If $k = 1$ then we are done. Otherwise note that the pairs (M_j, σ_j) for $2 \leq j \leq k$ define a sequence of induced matchings containing maximal independent sets in the graph $C(i_1)$. Indeed, to build $C(i_1)$ we did not add any edges within G itself, so the matchings are still induced. Moreover, by (*) every set σ_j contains a vertex which is not in $V(M_1)$, hence it is adjacent to the new vertex of $C(i_1)$, which means σ_j is a dominating set in $C(i_1)$. The condition (*) still holds, so by induction

$$I(C(i_1)) \simeq \bigvee_{j=2}^{k} I(M_j) \vee X$$

and that completes the proof. $\square$

Remark 5.8. The last result applies to all the situations of the previous section, once one orders the matchings in the lexicographical order of their defining $\{A, B\}$-words. In particular we get a splitting of $I(\mathbb{H})$ which includes a wedge sum of exponentially many spheres.

5.4 Upper bounds

In this section we improve, for the hexagonal dimer grids $\mathbb{H}$, the upper bound of [37]. We develop a more general result, which is modeled entirely on the technique of [37] with just two small improvements. Firstly, it avoids discrete Morse theory and relies just on the Betti numbers and homotopy. Secondly, it allows arbitrary graphs in a place where [37] requires a forest (see Cor.5.12).

We first need some extra notation. Suppose K is a simplicial complex with a fixed splitting of the vertex set into two disjoint subsets U and W ($V(K) = U \sqcup W$). For every simplex $\sigma \in K[U]$ we denote by $\mathrm{lk}_W\sigma$ the subcomplex of $K[W]$ consisting of those simplices $\tau \in K[W]$ for which $\tau \sqcup \sigma \in K$. By $\mathrm{st}_W\sigma$ we denote the subcomplex of K consisting of those $\tau \in K$ for which $\tau \cap U \subseteq \sigma$ and $\tau \cap W \in \mathrm{lk}_W\sigma$. Clearly $\mathrm{st}_W\sigma = (\mathrm{lk}_W\sigma) * \sigma$.

Example 5.9. If $\sigma = \emptyset$ then we always have $\mathrm{lk}_W\emptyset = \mathrm{st}_W\emptyset = K[W]$. If $U = \{v\}$ then the complexes $\mathrm{lk}_W v$ and $\mathrm{st}_W v$ coincide with the usual link and star of v in K. The star $\mathrm{st}_W\sigma$ is always contractible when $\sigma \neq \emptyset$.

Lemma 5.10. *Suppose K is a simplicial complex with a vertex partition $U \sqcup W$ as above and such that for every $\sigma \in K[U]$ we have $\widetilde{\beta}(\mathrm{lk}_W\sigma) \leq B$. Then*

$$\widetilde{\beta}(K) \leq B \cdot |K[U]|.$$

Proof. Denote $D = |K[U]|$. Fix any ordering $\emptyset = \sigma_0, \sigma_1, \ldots, \sigma_{D-1}$ of the simplices in $K[U]$ such that every simplex is preceded by all its faces. For $0 \leq l < D$ define subcomplexes

$$F_l = \bigcup_{i=0}^{l} \mathrm{st}_W\sigma_i.$$

Then $K[W] = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{D-1} = K$ is an increasing, exhaustive filtration of K with quotients:

$$F_l/F_{l-1} = \mathrm{st}_W\sigma_l/(\mathrm{st}_W\sigma_l \cap F_{l-1}) \simeq \Sigma^{\dim\sigma_l+1}\,\mathrm{lk}_W\sigma_l.$$

Since each of those quotients has total Betti number at most B, the E_1 page of the homology spectral sequence associated with this filtration has dimension at most $B \cdot D$. The spectral sequence converges to $\widetilde{H}_*(K)$, hence the result. $\square$

Lemma 5.11. *Suppose $U \subseteq V(G)$ is a vertex set with the property: For every simplex $\sigma \in I(G[U])$ the total Betti number of*

$$I(G \setminus (U \cup N[\sigma]))$$

is at most B. Then

$$\widetilde{\beta}(I(G)) \leq B \cdot |I(G[U])| \leq B \cdot 2^{|U|}.$$

Proof. Consider $K = I(G)$ with vertex set partitioned into U and $W = V(G) \setminus U$. Then for every simplex $\sigma \in I(G[U])$ we have precisely

$$\mathrm{lk}_W \sigma = I(G \setminus (U \cup N[\sigma])),$$

so Lemma 5.10 applies. $\square$

Corollary 5.12 ([37]). *If $U \subseteq V(G)$ is a vertex set such that $G \setminus U$ is a forest then $\widetilde{\beta}(I(G)) \leq |I(G[U])|$.*

Proof. If $G \setminus U$ is a forest then so is $G \setminus (U \cup N[\sigma])$ for every σ. Since by [34] the independence complex of a forest is either contractible or homotopy equivalent to a sphere (possibly S^{-1}), we can apply Lemma 5.11 with $B = 1$. $\square$

Fact 5.13. *For any two topological spaces X and Y*

$$\widetilde{\beta}(X * Y) = \widetilde{\beta}(X)\widetilde{\beta}(Y).$$

Proof. This follows from the formula for the reduced homology of the join (eg.[84, Lemma 2.1]), which for rational coefficients reduces to

$$\widetilde{H}_k(X * Y) = \bigoplus_{\substack{i,j \geq -1 \\ i+j=k-1}} \widetilde{H}_i(X) \otimes \widetilde{H}_j(Y), \qquad k \geq -1.$$

$\square$

Proposition 5.14. *If $\mathbb{H}$ is any hexagonal dimer lattice which can be tiled as in Fig.5.1 then*

$$\widetilde{\beta}(\mathbb{H}) \leq (14^{1/36} \cdot 2^{1/6})^{v(\mathbb{H})}.$$

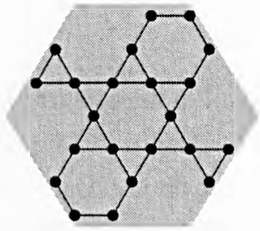


Figure 5.3: The induced subgraph G of the 30-vertex tile with $\widetilde{\beta}(I(G)) = 14$.

Proof. Recall that $\mathbb{H}$ contains $k = v(\mathbb{H})/36$ large hexagons. Let U consist of those vertices of $\mathbb{H}$ which are not covered by the tiles. Then $|U| = 6k = v(\mathbb{H})/6$ and the graph $\mathbb{H} \setminus U$ is a disjoint union of k hexagonal tiles with 30 vertices each. The graph $\mathbb{H}[U]$ has no edges.

Now suppose that σ is any subset of U. Then the graph $\mathbb{H} \setminus (U \cup N[\sigma])$ is a disjoint union of k graphs, each of which is an induced subgraph of the 30-vertex tile. More precisely, it is an induced subgraph obtained by removing the neighbourhood of some subset of the vertices of U which surround that tile. There are $2^{12} = 4096$ graphs that can arise in this way, with only 217 isomorphism classes [81], and the homology of their independence complexes can be easily calculated by a computer [47]. It turns out that for each of those graphs the total Betti number is at most 14 (the graph which attains maximum is shown in Fig.5.4). Using Fact 5.13 we get that the total Betti number of $\mathbb{H} \setminus (U \cup N[\sigma])$ is at most $B = 14^k$.

Lemma 5.11 now gives the conclusion:

$$\widetilde{\beta}(I(\mathbb{H})) \leq 14^k \cdot 2^{|U|} = 14^{v(\mathbb{H})/36} \cdot 2^{v(\mathbb{H})/6}.$$

□

Remark 5.15. If the dimensions of the grid $\mathbb{H}$ do not allow it to be tightly tiled with the large hexagons then Prop.5.14 still holds asymptotically. This is because one can pack $\mathbb{H}$ with hexagons leaving just a region of size proportional to the perimeter of the grid. The wasted vertices can then be added to the set U.

Remark 5.16. Using exactly the same technique one proves the following counterparts to the lower bounds of 5.4:

$$\widetilde{\beta}(I(\mathbf{\Delta}_3)) \leq (2^{3/8})^{v(\mathbf{\Delta}_3)}, \qquad \widetilde{\beta}(I(\mathbf{\Delta}_4)) \leq (10^{1/45} \cdot 2^{1/5})^{v(\mathbf{\Delta}_4)}.$$

Concluding remarks

It would be most interesting to prove similar lower bounds for the classical triangular and hexagonal lattice, for which superfrustration is also predicted [33]. In those cases, however, the present methods do not seem to work, and it is not clear if exponentially many homology classes should be given by embedded spheres or if more complicated constructions are necessary. Recent results in this direction include [56], where "long and thin" triangular lattices of size $c \times n$ for small constants $2 \leq c \leq 7$ are investigated. Also, the constructions of Jonsson [63] can be adapted to show that for the hexagonal grids of suitable sizes $c \times n$ with fixed c the number of ground states is exponential in n.

More generally, it would also be interesting to know what aspect of regularity is responsible for superfrustration in arbitrary lattices.

Chapter 6

Hard squares on cylinders revisited

6.1 Introduction

In this chapter we continue the study of the hard-core models in the special case of *hard squares.* For an introduction to the hard-core models and their characteristics see the beginning of Chapter 5. In this chapter we will mostly be able to trace one parameter of the model, namely the Witten index, or the reduced Euler characteristic of the independence complex of the underlying graph. In rare cases we will also be able to determine the homotopy type.

We continue the research line of Jonsson who studied the spaces $I(G)$ for the square grids with various boundary conditions. The free square grid is $G = P_m \times P_n$, its cylindrical version is $G = P_m \times C_n$ and the toroidal one is $G = C_m \times C_n$ where P_k is the path and C_k is the cycle with k vertices. Here by $G \times H$ we mean the graph with vertex set $V(G) \times V(H)$ and with edges $(g, h) - (g', h)$ for $gg' \in E(G)$ and $(g, h) - (g, h')$ for $hh' \in E(H)$. In other words, $P_m \times P_n$, $P_m \times C_n$ and $C_m \times C_n$ are square grids in the, respectively, rectangle, cylinder and torus.

In the first part we show natural recursive dependencies in all three models. We concentrate mainly on cylinders.

Theorem 6.1. *We have the following homotopy equivalences in the cylindrical case:*

a) $I(P_1 \times C_n) \simeq \Sigma\, I(P_1 \times C_{n-3})$,

b) $I(P_2 \times C_n) \simeq \Sigma^2\, I(P_2 \times C_{n-4})$,

c) $I(P_3 \times C_n) \simeq \Sigma^6\, I(P_3 \times C_{n-8})$, *d)* $I(P_m \times C_3) \simeq \Sigma^2 I(P_{m-3} \times C_3)$,

e) $I(P_m \times C_5) \simeq \Sigma^2 I(P_{m-2} \times C_5)$,

f) $I(P_m \times C_7) \simeq \Sigma^6 I(P_{m-4} \times C_7)$.

Here a) is a classical result of Kozlov [71], while d) and e) also follow from [99] where those spaces were identified with spheres by means of explicit Morse matchings. The results b), c) and f) are new and were independently proved by a different method in [57]. Note that a), b), c) are 'dual' to, respectively, d), e) and f) in the light of Thapper's conjecture [99, Conj. 3.1] (also [57, Conj. 1.9]) that $I(P_m \times C_{2n+1}) \simeq I(P_n \times C_{2m+1})$. If one assumes the conjecture holds, then a), b) and c) are equivalent to, respectively, d), e) and f). Theorem 6.1 together with an easy verification of initial conditions imply the conjecture for $m \leq 3$. The statements d), e) and f) take the periodicity of Euler characteristic, proved by Jonsson [60], to the level of homotopy type.

It is an interesting question whether the results of Theorem 6.1 can be extended further, the next obvious step being $I(P_4 \times C_n)$ or $I(P_m \times C_9)$. K. Iriye [58] suggested that there is an equivalence $I(P_4 \times C_{2k+1}) \simeq I(P_k \times C_9)$ with both spaces being, up to homotopy, wedges of spheres with the number of wedge summands growing to infinity as $k \to \infty$. That would mean no recursive relation as simple as those in Theorem 6.1 is possible for $I(P_m \times C_9)$ nor $I(P_4 \times C_n)$. However, K. Iriye has subsequently withdrawn his claim, so the question remains open.

Let us also mention that a completely analogous method proves the following.

Proposition 6.2. *We have the following homotopy equivalences in the free and toroidal cases:*

- $I(P_1 \times P_n) \simeq \Sigma I(P_1 \times P_{n-3})$,
- $I(P_2 \times P_n) \simeq \Sigma I(P_2 \times P_{n-2})$,
- $I(P_3 \times P_n) \simeq \Sigma^3 I(P_3 \times P_{n-4})$,
- $I(C_3 \times C_n) \simeq \Sigma^2 I(C_3 \times C_{n-3})$.

All those results are proved in Section 6.2.

In the second part of this work we aim to provide a method of recursively calculating the Euler characteristic in the cylindrical case $P_m \times C_n$ when the circumference n is even. Since it is customary to use the Witten index in this context, we will adopt the same approach and define for any space X

$$Z(X) = -\tilde{\chi}(X)$$

where $\tilde{\chi}(X)$ is the reduced Euler characteristic. Then $Z(X) = 0$ for a contractible X, $Z(\Sigma X) = -Z(X)$ for any finite simplicial complex X, and $Z(S^k) = (-1)^{k-1}$. The value

$$Z(G) := Z(I(G))$$

is what is usually called the Witten index of the underlying grid model.

Table 1 in Section 6.7 contains some initial values of $Z(P_m \times C_n)$ arranged so that m labels the rows and n labels the columns of the table. Let

$$f_n(t) = \sum_{m=0}^{\infty} Z(P_m \times C_n)t^m$$

be the generating function of the sequence in the n-th column. By an ingenious matching Jonsson [60] computed the numbers $Z(P_m \times C_{2n+1})$ for odd circumferences and found that for each fixed n they are either constantly 1 or periodically repeating $1, 1, -2, 1, 1, -2, \ldots$. Precisely

$$\begin{aligned} f_{6n+1}(t) = f_{6n-1}(t) &= \frac{1}{1-t}, \\ f_{6n+3}(t) &= \frac{1-2t+t^2}{1-t^3}. \end{aligned}$$

The behaviour of $Z(P_m \times C_{2n})$ is an open problem of that work, which we tackle here. Also, recently Braun notes that some problems faced in [22] are reminiscent of the difficulty of determining the homotopy types of the spaces $I(P_m \times C_{2n})$.

Our understanding of the functions $f_{2n}(t)$ comes in three stages of increasing difficulty.

Theorem 6.3. *Each $f_{2n}(t)$ is a rational function, such that all zeroes of its denominator are complex roots of unity.*

Our method also provides an algorithm to calculate $f_{2n}(t)$, see Section 6.7. This already implies that for each fixed n the sequence $a_m = Z(P_m \times C_{2n})$ has polynomial growth. However, we can probably be more explicit:

Conjecture 6.4. *For every $n \geq 0$ we have*

$$f_{4n+2}(t) = \frac{h_{4n+2}(t)}{(1+t^2) \cdot \left[(1-t^{8n-2})(1-t^{8n-8})(1-t^{8n-14}) \cdots (1-t^{2n+4})\right]},$$

$$f_{4n}(t) = \frac{h_{4n}(t)}{(1-t^2)\cdot\left[(1-t^{8n-6})(1-t^{8n-12})(1-t^{8n-18})\cdots(1-t^{2n+6})\right]}.$$

for some polynomials h_n.

In the denominators the exponents decrease by 6.

Conjecture 6.4 is in fact still just the tip of an iceberg, because $p_n(t)$ turn out to have lots of common factors with the denominators. This leads to the grande finale:

Conjecture 6.5. *After the reduction of common factors:*

- *$f_{4n+2}(t)$ can be written as a quotient whose denominator has no multiple zeroes. Consequently, for any fixed n, the sequence $a_m = Z(P_m \times C_{4n+2})$ is periodic.*

- *$f_{4n}(t)$ can be written as a quotient whose denominator has only double zeroes. Consequently, for any fixed n, the sequence $a_m = Z(P_m \times C_{4n})$ has linear growth.*

A direct computation shows that the first part holds for a number of initial cases, with periods given by the table:

$4n+2$	2	6	10	14	18	22
period	4	12	56	880	360	276640

In Section 6.3 we prepare our main tool for the proof of Theorem 6.3: *patterns* and their μ-invariants. A small example of how they work is presented in detail in Section 6.4. The proof of Theorem 6.3 then appears in Section 6.5. In Section 6.6 we describe a completely independent combinatorial object, the *necklace graph*, which is a simplified model of interactions between patterns. It has some conjectural properties, esp. Conjecture 6.24, whose verification would prove Conjecture 6.4. Section 6.6, up to and including Conjecture 6.24, can be read without any knowledge of any other part of this work. As for Conjecture 6.5, it seems unlikely that the methods of this chapter will be sufficient to prove it.

To avoid confusion we remark that our results are in a sense orthogonal to some questions raised by Jonsson, who asked if the sequence in each *row* of Table 1 (Section 6.7) is periodic. That question is equivalent to asking if the eigenvalues of certain transfer matrices are complex roots of unity, and this work is not about them.

6.2 Proofs of Theorem 6.1 and Proposition 6.2

Before embarking on the proofs we recall some results of Chapter 2 and their simple consequences.

Lemma 6.6. *For any vertex v and edge e of G we have*

$$\begin{aligned} Z(G) &= Z(G \setminus v) - Z(G \setminus N[v]), \\ Z(G) &= Z(G - e) - Z(G \setminus N[e]). \end{aligned}$$

Lemma 6.7. *We have the following implications*

a) (Fold lemma, [35]) If $N(u) \subseteq N(v)$ then $I(G) \simeq I(G \setminus v)$.

b) If u is a vertex of degree 1 and v is its only neighbour then $I(G) \simeq \Sigma I(G \setminus N[v])$.

c) If u and v are two adjacent vertices of degree 2 which belong to a 4-cycle in G together with two other vertices x and y then $I(G) \simeq \Sigma I(G \setminus \{u, v, x, y\})$ (see Fig.6.1).

d) If G is a graph that contains any of the configurations shown in Fig.6.2, then $I(G)$ is contractible.

Proof. Part a) is Theorem 2.2 and part b) is Corollary 2.13. In c) one can first remove x without affecting the homotopy type (because $N(u) \subseteq N(x)$), and then apply part b) at v. Finally d) follows because a single operation of type described in b) or c) leaves a graph with an isolated vertex. $\square$

For simplicity we will use the following language. A vertex v of G is called *removable* if the inclusion $I(G \setminus v) \hookrightarrow I(G)$ is a homotopy equivalence. We call the graph $G \setminus N[v]$ the *residue graph* of v in G. By Proposition 2.1.a) if the residue graph of v has a contractible independence complex then v is removable.

If e is an edge of G then we say e is *removable* if the inclusion $I(G) \hookrightarrow I(G-e)$ is a homotopy equivalence. If e is not an edge of G then e is *insertable* if the inclusion $I(G \cup e) \hookrightarrow I(G)$ is a homotopy equivalence or, equivalently, if e is removable from $G \cup e$. In both cases we call the graph $G \setminus N[e]$ the *residue graph* of e in G. By Proposition 2.4.a) if the residue graph of e has a contractible independence complex then e is removable or insertable, accordingly.

We identify the vertices of P_m with $\{1, \ldots, m\}$ and the vertices of C_n with $\mathbb{Z}/n = \{0, \ldots, n-1\}$. Product graphs have vertices indexed by pairs. To deal with the degenerate cases it is convenient to assume that $C_2 = P_2$, that C_1 is a single

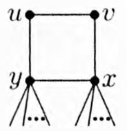


Figure 6.1: A configuration which can be removed at the cost of a suspension (Lemma 6.7.c).

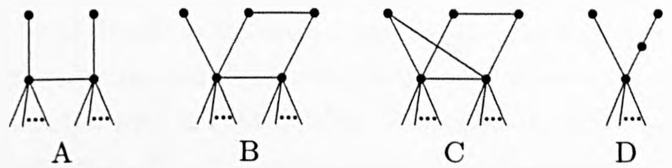


Figure 6.2: Four types of configurations which force contractibility of the independence complex (Lemma 6.7.d).

vertex with a loop and that $C_0 = P_0$ are empty graphs. This convention forces all spaces $I(C_1)$, $I(P_0)$ and $I(C_0)$ to be the empty space $\emptyset = S^{-1}$.

Proof of 6.1.a). This was proved in Example 2.8. $\square$

Proof of 6.1.b). In $P_2 \times C_n$ the edge $e_1 = \{(1,0),(1,5)\}$ is insertable, because its residue graph contains a configuration of type A (see Lemma 6.7.d), namely with vertices $(2,1)$ and $(2,4)$ having degree one. For the same reason the edge $e_2 = \{(2,0),(2,5)\}$ is insertable. Now in the graph $(P_2 \times C_n) \cup \{e_1, e_2\}$ the edges

$$f_1 = \{(1,0),(1,1)\}, f_2 = \{(2,0),(2,1)\}, f_3 = \{(1,4),(1,5)\}, f_4 = \{(2,4),(2,5)\}$$

are all sequentially removable, because the residue graph in each case contains a configuration of type B. Therefore

$$\begin{aligned} I(P_2 \times C_n) &\simeq I((P_2 \times C_n) \cup \{e_1, e_2\} - \{f_1, f_2, f_3, f_4\}) = \\ &= I(P_2 \times C_{n-4} \sqcup P_2 \times P_4) = I(P_2 \times C_{n-4}) * I(P_2 \times P_4) \simeq \\ &\simeq I(P_2 \times C_{n-4}) * S^1 = \Sigma^2 I(P_2 \times C_{n-4}) \end{aligned}$$

where $I(P_2 \times P_4)$ can be found by direct calculation or from Proposition 6.2. $\square$

Remark 6.8. All the proofs in this section will follow the same pattern, that is to split the graph into two parts. One of those parts will be small, i.e. of some fixed size, and its independence complex will always have the homotopy type of a single sphere. Every time we need to use a result of this kind about a graph of small, fixed size, we will just quote the answer, leaving the verification to the reader (homology calculations with [47] were used).

Proof of 6.1.c). We follow the strategy of b). First we need to show that the edges $e_1 = \{(1,0),(1,9)\}$, $e_2 = \{(2,0),(2,9)\}$, $e_3 = \{(3,0),(3,9)\}$ are insertable.

For e_1 the residue graph is shown in Fig.6.3.a. To prove its independence complex is contractible we describe a sequence of operations that either preserve the homotopy type or throw in an extra suspension. The sequence will end with a graph whose independence complex is contractible for some obvious reason, so also the complex we started with is contractible. The operations are as follows: remove $(3,2)$ (by 6.7.a with $u = (2,1)$); remove $(2,3)$ (by 6.7.a with $u = (1,2)$); remove $N[(3,4)]$ (by 6.7.b with $u = (3,3)$); remove $(2,6)$ (by 6.7.a with $u = (1,7)$); remove $N[(1,5)]$ (by 6.7.b with $u = (2,5)$). In the last graph there is a configuration of type A on the vertices $(3,6)$ and $(1,7)$.

The same argument works for e_3. For e_2 the residue graph has a connected component shown in Fig.6.3.b. The modifications this time are: remove $N[(1,2)]$, $N[(1,7)]$, $N[(3,2)]$ and $N[(3,7)]$ for reasons of 6.7.b. The graph that remains contains a configuration of type A.

Next it remains to check that in $(P_3 \times C_n) \cup \{e_1, e_2, e_3\}$ the edges $\{(0,i),(1,i)\}$ and $\{(8,i),(9,i)\}$ are removable for $i = 1,2,3$. The types of residue graphs one must consider are quite similar and the arguments for the contractibility of their independence complexes are exact copies of those for e_1, e_2, e_3 above. We leave them as an exercise to the reader.

We can thus conclude as before

$$I(P_3 \times C_n) \simeq I(P_3 \times C_{n-8}) * I(P_3 \times P_8) \simeq I(P_3 \times C_{n-8}) * S^5 = \Sigma^6 I(P_3 \times C_{n-8}).$$

□

Proof of 6.1.d). In $P_m \times C_3$ each edge $\{(3,i),(4,i)\}$ is removable for $i = 0,1,2$ because each residue graph contains a configuration of type C. As before, this implies an equivalence

$$I(P_m \times C_3) \simeq I(P_{m-3} \times C_3) * I(P_3 \times C_3) \simeq I(P_{m-3} \times C_3) * S^1 = \Sigma^2 I(P_{m-3} \times C_3)$$

□

Proof of 6.1.e). In $P_m \times C_5$ each edge $\{(2,i),(3,i)\}$ is removable for $i = 0,1,2,3,4$ because each residue graph contains a configuration of type A with vertices $(1, i-1)$

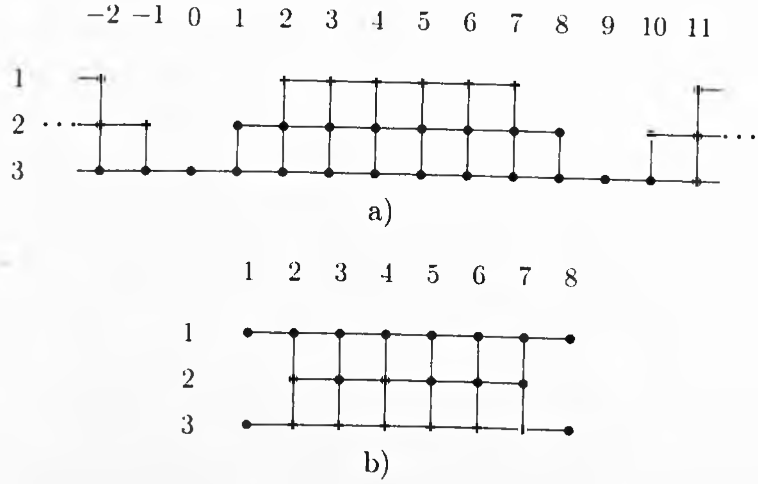


Figure 6.3: Two residue graphs for edges insertable into $P_3 \times C_n$.

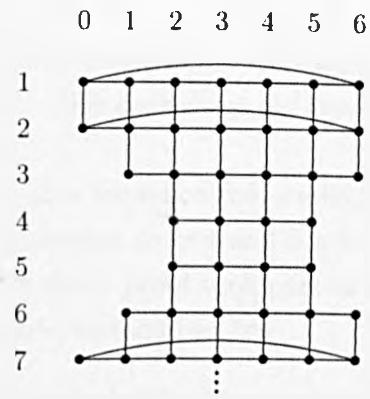


Figure 6.4: The residue graph for edges removable from $P_m \times C_7$.

and $(1, i+1)$ having degree 1. Again, this means

$$I(P_m \times C_5) \simeq I(P_{m-2} \times C_5) * I(P_2 \times C_5) \simeq I(P_{m-2} \times C_5) * S^1 = \Sigma^2 I(P_{m-2} \times C_5)$$

□

Proof of 6.1.f). We want to show that the edges $e_i = \{(4, i), (5, i)\}$, $i = 0, \ldots, 6$ are sequentially removable. Then the result will follow as before:

$$I(P_m \times C_7) \simeq I(P_{m-4} \times C_7) * I(P_4 \times C_7) \simeq I(P_{m-4} \times C_7) * S^5 = \Sigma^6 I(P_{m-4} \times C_7).$$

The residue graph of e_0 is the graph G from Fig.6.4. We need to show that the independence complex of that graph is contractible. To do this, we will first show that each of the vertices $(4, j)$, $j = 2, 3, 4, 5$, is removable in G. By symmetry, it suffices to consider $(4, 2)$ and $(4, 3)$.

Consider first the vertex $(4, 2)$ and its residue graph $G \setminus N[(4, 2)]$. It can be transformed in the following steps: remove $N[(2, 1)]$ (by 6.7.b with $u = (3, 1)$); remove $N[(1, 6)]$ (by 6.7.b with $u = (1, 0)$); remove $N[(1, 3)]$ (by 6.7.b with $u = (1, 2)$); remove $N[(3, 4)]$ (by 6.7.b with $u = (3, 3)$). In the final graph $(2, 5)$ is isolated.

Now we prove the residue graph $G \setminus N[(4, 3)]$ has a contractible independence complex. Decompose it as follows: remove $(3, 1), (3, 2), (2, 1), (2, 2)$ (by 6.7.c); remove $(1, 6)$ (by 6.7.a with $u = (2, 0)$); remove $(2, 5)$ (by 6.7.a with $u = (3, 6)$); remove $N[(1, 4)]$ (by 6.7.b with $u = (1, 5)$). In the final graph $(2, 3)$ is isolated.

Since all the vertices in row 4 of G are removable, $I(G)$ is homotopy equivalent to the join $I(G[1, 2, 3]) * I(G[5, \ldots])$, where $G[\ldots]$ means the subgraph of G spanned by the numbered rows. But a direct calculation shows that $I(G[1, 2, 3])$ is contractible, hence so is $I(G)$. This ends the proof that the edge $e_0 = \{(4, 0), (5, 0)\}$ of $P_m \times C_7$ was removable.

For all other edges e_i in a sequence the residue graph will look exactly like G with possibly some edges between rows 4 and 5 missing. This has no impact on contractibility since all of the above proof took part in rows $1, 2, 3$ of Fig.6.4. That means that all e_i are removable, as required. □

Proof of Proposition 6.2. We just sketch the arguments and the reader can check the details. Part a) is a result of [71] and also follows from Lemma 6.7.b). Part b) follows directly from Lemma 6.7.c).

For part c), each edge $\{(i, 4), (i, 5)\}$, $i = 1, 2, 3$ of $P_3 \times P_n$ is removable because their residue graphs either contain, or can easily be reduced to contain, a configuration of type C or D. Then the graph splits into two components and we conclude as usually.

In d) we first show that each edge $\{(i, 0), (i, 4)\}$, $i = 0, 1, 2$ is insertable into $C_3 \times C_n$ because the residue graph contains a configuration of type C. Then in the enlarged graph the obvious edges which must be removed to obtain a disjoint union $C_3 \times C_{n-3} \sqcup C_3 \times P_3$ are indeed removable, again because of a type C configuration in their residue graphs. We conclude as always. □

6.3 Cylinders with even circumference: Patterns

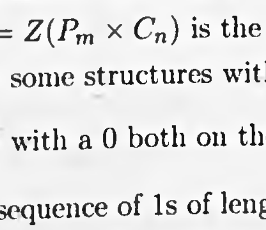
To prove the results about cylinders of even circumference we will need quite a lot of notation. On the plus side, once all the objects are properly defined, the proofs will follow in a fairly straightforward way. It is perhaps instructive to read this and the following sections simultaneously. The next section contains a working example of what is going on for $n = 6$. From now on n, the length of the cycle, is a fixed even integer which will not appear in the notation.

A pattern $\mathcal{P}$ is a matrix of size $2 \times n$ with 0/1 entries and such that if $\mathcal{P}(1,i) = 1$ then $\mathcal{P}(2,i) = 1$, i.e. below a 1 in the first row there is always another 1 in the second row. An example of a pattern is

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

We also call n the length of the pattern. The rows of a pattern are indexed by 1 and 2, while the columns are indexed with $0, \ldots, n-1$, as the vertices of C_n. Given i we say $\mathcal{P}(1,i)$ is 'above' $\mathcal{P}(2,i)$ and $\mathcal{P}(2,i)$ is 'below' $\mathcal{P}(1,i)$. We identify a pattern with patterns obtained by a cyclic shift or by a reflection, since they will define isomorphic graphs (see below). Also the words 'left', 'right' and 'adjacent' are understood in the cyclic sense.

Given a pattern $\mathcal{P}$ define $G(\mathcal{P}; m)$ as the induced subgraph of $P_m \times C_n$ obtained by removing those vertices $(1,i)$ and $(2,i)$ for which $\mathcal{P}(1,i) = 0$, resp. $\mathcal{P}(2,i) = 0$. This amounts to applying a 'bit mask' defined by $\mathcal{P}$ to the first two rows of $P_m \times C_n$. The graph $G(\mathcal{P}; m)$ for the pattern $\mathcal{P}$ above is:



Define the simplified notation $Z(\mathcal{P}; m) := Z(G(\mathcal{P}; m))$. If $\mathcal{I}$ denotes the all-ones pattern then $Z(\mathcal{I}; m) = Z(P_m \times C_n)$ is the value we are interested in.

We now need names for some structures within a row:

- **A singleton** is a single 1 with a 0 both on the left and on the right.
- **A block** is a contiguous sequence of 1s of length at least 3, which is bounded by a 0 both on the left and on the right.
- **A run** is a sequence of blocks and singletons separated by single 0s.

- A **nice run** is a run in which every block has length exactly 3.

Example 6.9. The sequence 01010111010111010 is a nice run. The sequence 01110111101010 is a run, but it is not a nice run. The sequences 01011010 and 0101001110 are not runs.

A pattern is **reducible** if the only occurrences of 1 in the first row are singletons. Otherwise we call it **irreducible**. In particular, a pattern whose first row contains only 0s is reducible.

We now need three types of pattern transformations, which we denote V (for vertex), N (for neighbourhood) and R (for reduction). The first two can be performed on any pattern. Suppose $\mathcal{P}$ is a pattern and i is an index such that $\mathcal{P}(1,i) = 1$.

- An **operation of type** V sets $\mathcal{P}(1,i) = 0$. We denote the resulting pattern $\mathcal{P}^{V,i}$.
- An **operation of type** N sets $\mathcal{P}(1,i-1) = \mathcal{P}(1,i) = \mathcal{P}(1,i+1) = \mathcal{P}(2,i) = 0$. We denote the resulting pattern $\mathcal{P}^{N,i}$.

All other entries of the pattern remain unchanged. If it is clear what index i is used we will abbreviate the notation to $\mathcal{P}^V$ and $\mathcal{P}^N$.

Lemma 6.10. *For any pattern $\mathcal{P}$ and index i such that $\mathcal{P}(1,i) = 1$ we have*

$$Z(\mathcal{P};m) = Z(\mathcal{P}^{V,i};m) - Z(\mathcal{P}^{N,i};m), \quad m \geq 2.$$

Proof. This is exactly the first equality of Lemma 6.6 for $G = G(\mathcal{P};m)$. □

The third **operation, of type** R, can be applied to a reducible pattern $\mathcal{P}$ and works as follows. First, temporarily extend $\mathcal{P}$ with a new, third row, filled with ones. Now for every index i such that $\mathcal{P}(1,i) = 1$ (note that no two such i are adjacent) make an assignment

$$\mathcal{P}(1,i) = \mathcal{P}(2,i-1) = \mathcal{P}(2,i) = \mathcal{P}(2,i+1) = \mathcal{P}(3,i) = 0.$$

Having done this for all such i remove the first row (which is now all zeroes) and let $\mathcal{P}^R$ be the pattern formed by the second and third row.

Lemma 6.11. *Suppose $\mathcal{P}$ is a reducible pattern and k is the number of ones in its first row. Then*

$$Z(\mathcal{P};m) = (-1)^k Z(\mathcal{P}^R;m-1), \quad m \geq 2.$$

Proof. It follows from a k-fold application of Lemma 6.7.b). $\square$

We now describe a class of patterns which arise when one applies the above operations in some specific way to the all-ones pattern. A pattern $\mathcal{P}$ is called **proper** if it satisfies the following conditions:

- The whole second row is either a run or it consists of only 1s.
- If the second row has only 1s then the first row is a nice run.
- Above every singleton 1 in the second row there is a 0 in the first row.
- Above every block of length 3 in the second row there are 0s in the first row.
- If B is any block in the second row of length at least 4 then above B there is a nice run R, subject to the conditions:
  - if the leftmost group of 1s in R is a block (of length 3) then the leftmost 1 of that block is located exactly above the 3rd position of B,
  - if the leftmost group of 1s in R is a singleton then it is located exactly above the 2nd or 3rd position of B.

  By symmetry the same rules apply to the rightmost end of B and R.

Note that a proper pattern does not contain the sequences 0110 nor 1001 in any row. Also note that the first row of any proper pattern can only contain singletons and blocks of length 3, and no other groups of 1s.

Example 6.12. Here are some proper patterns:

$$\mathcal{A}=\begin{pmatrix}0&1&0&1&0&0&0&0&0&0\\1&1&1&1&1&0&1&1&1&0\end{pmatrix}\quad \mathcal{B}=\begin{pmatrix}1&0&1&1&1&0&1&1&1&0\\1&1&1&1&1&1&1&1&1&1\end{pmatrix}$$

$$\mathcal{C}=\begin{pmatrix}0&0&1&1&1&0&0&0&0&0\\1&1&1&1&1&1&1&0&1&0\end{pmatrix}\quad \mathcal{D}=\begin{pmatrix}0&0&1&0&1&1&1&0&0&0\\1&1&1&1&1&1&1&1&1&0\end{pmatrix}$$

A pattern is called **initial** if it is obtained from the all-ones pattern $\mathcal{I}$ by performing, for each even index $i = 0, 2, 4, \ldots, n-2$ one of the operations of type V or N. It means there should be $2^{n/2}$ initial patterns, but some of them can be identified via cyclic shift or reflection. One can easily see that every initial pattern is reducible. Moreover, by a repeated application of Lemma 6.10 we get that $Z(P_m \times C_n)$ is a linear combination of the numbers $Z(\mathcal{P}; m)$ for initial patterns $\mathcal{P}$. More importantly, we have:

Lemma 6.13. *Every initial pattern is proper.*

Proof. If the operations we perform in positions $i = 0, 2, \dots, n-2$ are all V or all N then we get one of the patterns described in Lemma 6.16. If we perform N in points i, $i+2$ we get a singleton in position $i+1$ of the second row with a 0 above it. For a choice of NVN in i, $i+2$, $i+4$ we get a block of length 3 in the second row with 0s above. Finally for a longer segment $NV \cdots VN$ the outcome is a block with a nice run of singletons starting and ending above the 3rd position in the block. We can never get two adjacent 0s in the second row, so it is a run. Here is a summary of the possible outcomes:

$$\begin{array}{c}
\begin{array}{ccccccccccccccccccccc}
 & & \mathbf{N} & & V & & V & & V & & V & & \mathbf{N} & & V & & \mathbf{N} & & \mathbf{N} & &
\end{array} \\
\begin{pmatrix}
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots
\end{pmatrix}
\end{array}$$

where the labels V, N indicate which operation was applied at a position. $\square$

Now we introduce the main tool: an invariant which splits proper patterns into classes which can be analyzed recursively.

Definition 6.14. *For any proper pattern $\mathcal{P}$ we define the μ-invariant as follows:*

$$\begin{aligned} \mu(\mathcal{P}) \;=\; & (\text{ number of blocks in the first row of } \mathcal{P} \text{ }) + \\ & (\text{ number of blocks in the second row of } \mathcal{P} \text{ }). \end{aligned}$$

Example 6.15. All the proper patterns in Example 6.12 have μ-invariant 2.

Lemma 6.16. *For every proper pattern $\mathcal{P}$ we have $0 \leq \mu(\mathcal{P}) \leq n/4$. The only patterns with $\mu(\mathcal{P}) = 0$ are*

$$\mathcal{P}_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Proof. If $\mu(\mathcal{P}) = 0$ then $\mathcal{P}$ has no blocks in either row. If the second row is all-ones then the first row must be a nice run with no blocks, so $\mathcal{P} = \mathcal{P}_1$. Otherwise the second row is an alternating 0/1 but then the first row cannot have a 1 anywhere, so $\mathcal{P} = \mathcal{P}_2$.

Now consider the following map. To every block in the second row we associate its two rightmost points, its leftmost point and the immediate left neighbour of the leftmost point. To every block in the first row we associate its three points and the point immediately left. This way every block which contributes to $\mu(\mathcal{P})$ is given 4 points, and those sets are disjoint for different blocks. The only non-obvious part of the last claim follows since a block in the first row does not have any point over the two outermost points of the block below it. That ends the proof of the upper bound. $\square$

Next come the crucial observations about operations on proper patterns and their μ-invariants.

Proposition 6.17. *Suppose $\mathcal{P}$ is a proper, irreducible pattern and let i be the index of the middle element of some block in the first row. Then $\mathcal{P}^{V,i}$ and $\mathcal{P}^{N,i}$ are proper and*

$$\mu(\mathcal{P}^{V,i}) = \mu(\mathcal{P}) - 1, \quad \mu(\mathcal{P}^{N,i}) = \mu(\mathcal{P}).$$

Proof. There are two cases, depending on whether the block of length 3 centered at i is, or is not, the outermost group of 1s in its run. If it is the outermost one then it starts over the 3rd element of the block below it. The two possible situations are depicted below.

$$\begin{pmatrix} \cdots & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots \\ \cdots & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \end{pmatrix}, \quad \begin{pmatrix} \cdots & 1 & 0 & 1 & 1 & 1 & 0 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \end{pmatrix}.$$

In $\mathcal{P}^V$ the second row is the same as in $\mathcal{P}$. Above the current block we still have a nice run and its outermost 1s are in the same positions. That means $\mathcal{P}^V$ is still proper. The number of blocks in the first row dropped by one, so $\mu(\mathcal{P}^V) = \mu(\mathcal{P}) - 1$.

The proof for $\mathcal{P}^N$ depends on the two cases. In the first case an operation of type N splits the block in the second row creating a new block of size 3 with 0s above it. In the second block that comes out of the splitting the first two 1s have 0s above them, so whatever run there was in $\mathcal{P}$ it is still there and starts in an allowed position. That means we get a proper pattern. One block was removed and one split into two, so μ does not change.

In the second case the situation is similar. We increase the number of blocks in the second row by one while removing one block from the first row. The two outermost positions in the new block(s) have 0s above them, so the nice runs which remain above them start in correct positions. Again $\mathcal{P}^N$ is proper. $\square$

Proposition 6.18. *If $\mathcal{P}$ is a proper, reducible pattern then $\mathcal{P}^R$ is proper and*

$$\mu(\mathcal{P}^R) = \mu(\mathcal{P}).$$

Proof. Consider first the case when $\mathcal{P}$ has only 0s in the first row. Then the second row is a run with blocks only of size 3 (because a longer block would require something above it). This means $\mathcal{P}^R$ has a full second row with a nice run in the first row. Such pattern is proper and $\mu(\mathcal{P}^R) = \mu(\mathcal{P})$ as we count the same blocks.

Now we move to the case when $\mathcal{P}$ has at least one 1 in the first row. Note that $\mathcal{P}(1,i) = 1$ if and only if $\mathcal{P}^R(2,i) = 0$. That last condition implies that the second row of $\mathcal{P}^R$ is a run (as the first row of $\mathcal{P}$ does not contain the sequences 11 nor 1001).

If $\mathcal{P}^R(2,i)$ is a singleton 1 then in $\mathcal{P}$ we must have had $\mathcal{P}(1,i-1) = \mathcal{P}(1,i+1) = 1$ but then $\mathcal{P}(2,i)$ was erased by the operation R and therefore $\mathcal{P}^R(1,i) = 0$. This proves $\mathcal{P}^R$ has zeroes above singletones of the second row.

Now consider any block B in the second row of $\mathcal{P}^R$ and assume without loss of generality that it occupies positions $1,\dots,l$, hence $\mathcal{P}(1,0) = \mathcal{P}(1,l+1) = 1$, $\mathcal{P}^R(2,0) = \mathcal{P}^R(2,l+1) = 0$ and $\mathcal{P}(1,i) = 0$ for all $1 \le i \le l$. It means that the situation in $\mathcal{P}$ must have looked like one of these (up to symmetry):

$$\begin{array}{c} \begin{array}{ccccccccccc} & 0 & & & & & & & & l+1 & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \\ \begin{array}{ccccccccccc} & 0 & B & B & B & & B & B & B & 0 & \end{array} \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccccc} & 0 & & & & & & & & l+1 & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \\ \begin{array}{ccccccccccc} & 0 & B & B & B & & B & B & B & 0 & \end{array} \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccccc} & 0 & & & & & & & & l+1 & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \\ \begin{array}{ccccccccccc} & 0 & B & B & B & & B & B & B & 0 & \end{array} \end{array}$$

The letters B indicate where the block B will stretch in what will become the future second row of $\mathcal{P}^R$. The 1s in $\mathcal{P}(1,0)$ and $\mathcal{P}(1,l+1)$ must be located above the 2nd or 3rd element of a block. The part of the second row in $\mathcal{P}$ denoted by $\cdots$ is a run with no 1s above, so it must be a nice run. It follows that in $\mathcal{P}^R$ above B we will get a nice run and by checking the three cases we see that the run starts above the 2nd or 3rd element of B, and it only starts above the 2nd element if it has a singleton there.

There is just one way in which we can obtain a block B of size 3:

$$\begin{array}{ccccc} 0 & & & & l+1 \\ \hline \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} 0 & & & & l+1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \\ \begin{array}{ccccc} 0 & B & B & B & 0 \end{array} \end{array}$$

and then the operation R will erase everything above that block. This completes the check that the pattern $\mathcal{P}^R$ is proper.

It remains to compute $\mu(\mathcal{P}^R)$. Our previous discussion implies that:

- every block of size 3 in the second row of $\mathcal{P}$ becomes a block in the first row of $\mathcal{P}^R$,
- every two consecutive blocks longer than 3 yield, between them, a block B in the second row of $\mathcal{P}^R$,

and every block in $\mathcal{P}^R$ arises in this way. It means that every block in $\mathcal{P}$ contributes one to the count of blocks in $\mathcal{P}^R$ (in either first or second row). That proves $\mu(\mathcal{P}^R) = \mu(\mathcal{P})$. $\square$

6.4 An example: $n = 6$

First of all the value $Z(P_m \times C_6) = Z(\mathcal{I}; m)$ for the all-ones pattern $\mathcal{I}$ splits into a linear combination of Z-values for the following patterns.

$$\mathcal{A} = \begin{array}{c} \begin{array}{cccccc} V & & V & & V & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \qquad \mathcal{B} = \begin{array}{c} \begin{array}{cccccc} V & & V & & N & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$\mathcal{C} = \begin{array}{c} \begin{array}{cccccc} V & & N & & N & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \qquad \mathcal{D} = \begin{array}{c} \begin{array}{cccccc} N & & N & & N & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

The labels V, N indicate which operation was applied to the particular position $i = 0, 2, 4$. Any other pattern we get is isomorphic to one of these, and Lemma 6.10 unfolds recursively into:

$$Z(\mathcal{I}; m) = Z(\mathcal{A}; m) - 3Z(\mathcal{B}; m) + 3Z(\mathcal{C}; m) - Z(\mathcal{D}; m). \tag{6.1}$$

We also see that (Definition 6.14):

$$\mu(\mathcal{A}) = 0 + 0 = 0,\ \mu(\mathcal{B}) = 0 + 1 = 1,\ \mu(\mathcal{C}) = 0 + 1 = 1,\ \mu(\mathcal{D}) = 0 + 0 = 0.$$

All the patterns we have now are reducible. The first obvious reductions are

$$\mathcal{A}^R = \mathcal{D}, \quad \mathcal{D}^R = \mathcal{A}$$

and by Lemma 6.11 they lead to

$$Z(\mathcal{D}; m) = Z(\mathcal{A}; m-1), \quad Z(\mathcal{A}; m) = -Z(\mathcal{D}; m-1) = -Z(\mathcal{A}; m-2).$$

It means that given the initial conditions for $Z(\mathcal{A}; m)$ and $Z(\mathcal{D}; m)$ we have now completely determined those sequences and their generating functions.

Note that $\mathcal{A}$ and $\mathcal{D}$ had μ-invariant 0. Now we move on to the patterns with the next μ-invariant value 1. We can reduce $\mathcal{B}$ (even three times) and $\mathcal{C}$ and apply Lemma 6.11:

$$\mathcal{B}^{RRR} = \mathcal{E},\ \mathcal{C}^R = \mathcal{E}; \quad Z(\mathcal{B}; m) = -Z(\mathcal{E}; m-3),\ Z(\mathcal{C}; m) = Z(\mathcal{E}; m-1)$$

where

$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Still $\mu(\mathcal{E}) = 1$. Now we can apply a V, N-type splitting in the middle of the length 3 block in the first row of $\mathcal{E}$, as in Proposition 6.17. We have by Lemma 6.10:

$$\mathcal{E}^V = \mathcal{A},\ \mathcal{E}^N = \mathcal{B}; \qquad Z(\mathcal{E}; m) = Z(\mathcal{A}; m) - Z(\mathcal{B}; m) = Z(\mathcal{A}; m) + Z(\mathcal{E}; m-3)$$

where $\mu(\mathcal{A}) = 0$, so the sequence $Z(\mathcal{A}; m)$ is already known.

This recursively determines all the sequences and it is a matter of a mechanical calculation to derive their generating functions (some care must be given to the initial conditions). We can also check periodicities directly. The sequences with μ-invariant 0 are 4-periodic:

$$Z(\mathcal{A}; m) = -Z(\mathcal{A}; m-2) = Z(\mathcal{A}; m-4)$$

and those with μ-invariant 1 are 12-periodic:

$$Z(\mathcal{E}; m) = Z(\mathcal{A}; m) + Z(\mathcal{A}; m-3) + Z(\mathcal{A}; m-6) + Z(\mathcal{A}; m-9) + Z(\mathcal{E}; m-12)$$

$$= Z(\mathcal{E}; m-12)$$

since $Z(\mathcal{A}; m) = -Z(\mathcal{A}; m-6)$. By (6.1) this means 12-periodicity of $Z(P_m \times C_6)$.

6.5 Proof of Theorem 6.3

Everything is now ready to prove Theorem 6.3. We are going to deduce it from a refined version given below. Throughout this section an even number n is still fixed. For any pattern $\mathcal{P}$ of length n define the generating function

$$f_{\mathcal{P}}(t) = \sum_{m=0}^{\infty} Z(\mathcal{P}; m)t^m. \tag{6.2}$$

Proposition 6.19. *For any proper pattern $\mathcal{P}$ the function $f_{\mathcal{P}}(t)$ is a rational function such that all zeroes of its denominator are complex roots of unity.*

The sequence $Z(P_m \times C_n)$ is a linear combination of sequences $Z(\mathcal{P}; m)$ for initial patterns $\mathcal{P}$ (modulo initial conditions). Every initial pattern is proper (Lemma 6.13), so Theorem 6.3 follows. It remains to prove Proposition 6.19, and this is done along the lines of the example in Section 6.4.

Proof. We will prove the statement by induction on the μ-invariant of $\mathcal{P}$. If $\mu(\mathcal{P}) = 0$, then $\mathcal{P}$ is one of the patterns from Lemma 6.16. Each of them satisfies $\mathcal{P}^{RR} = \mathcal{P}$ and by Lemma 6.11:

$$Z(\mathcal{P}; m) = (-1)^{n/2} Z(\mathcal{P}; m-2)$$

hence $f_{\mathcal{P}}(t)$ has the form

$$\frac{a+bt}{1-(-1)^{n/2}t^2}.$$

Now consider any fixed value $\mu > 0$ of the μ-invariant and suppose the result was proved for all proper patterns with smaller μ-invariants. Consider the directed graph whose vertices are all proper patterns with that invariant μ. For any reducible $\mathcal{P}$ there is an edge $\mathcal{P} \to \mathcal{P}^R$ and for any irreducible $\mathcal{P}$ there is an edge $\mathcal{P} \to \mathcal{P}^N$ for some (only one) choice of N-type operation in the middle of a block. Since the graph is finite and the outdegree of each vertex is 1, it consists of directed cycles with some attached trees pointing towards the cycles.

If $\mathcal{P}$ is a vertex on one of the cycles, then by moving along the cycle and performing the operations prescribed by the edges we will get back to $\mathcal{P}$ and obtain

(Lemmas 6.10, 6.11 and Propositions 6.17,6.18) a recursive equation of the form

$$Z(\mathcal{P};m) = \pm Z(\mathcal{P};m-a) + \sum_{\mathcal{R}} \pm Z(\mathcal{R};m-b_{\mathcal{R}}) \tag{6.3}$$

where $a > 0$, $b_{\mathcal{R}} \geq 0$ and $\mathcal{R}$ runs through some proper patterns with invariant $\mu - 1$ (Proposition 6.17). Now the result follows by induction (the generating function $f_{\mathcal{P}}(t)$ will add an extra factor $1 \pm t^a$ to the denominator coming from the combination of functions $f_{\mathcal{R}}(t)$).

If $\mathcal{P}$ is not on a cycle then it has a path to a cycle and the result follows in the same way. □

Remark 6.20. It is also clear that in order to prove Conjecture 6.4 one must have better control over the cycle lengths in the directed graph appearing in the proof. We will construct a more accessible model for this in the next section, see Theorem 6.25.

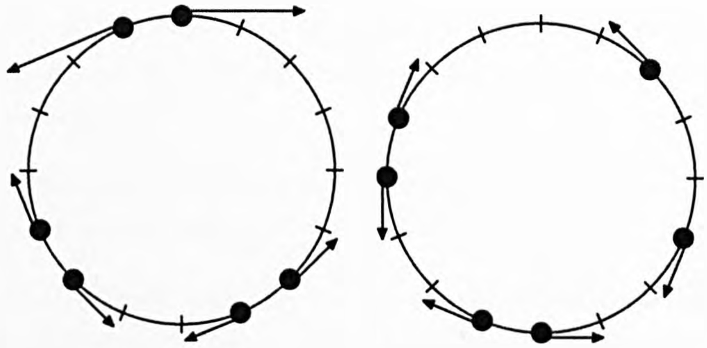
6.6 Necklaces

In this section we describe an appealing combinatorial model which encodes the reducibility relation between patterns. As before n is an even positive integer and k is any positive integer.

We define a ***(k, n)-necklace***. It is a collection of $2k$ points (*stones*) distributed along the circumference of a circle of length n, together with an assignment of a number from $\{-2, -1, 1, 2\}$ to each of the stones. We call these numbers *stone vectors* and we think of them as actual short vectors attached to the stones and tangent to the circle. The vector points 1 or 2 units clockwise (positive value) or anti-clockwise (negative value) from each stone and we say a stone *faces* the direction of its vector. See Fig.6.5 for an example worth more than a thousand words.

During a *jump* a stone moves 1 or 2 units along the circle in the direction and distance prescribed by its vector. The configuration of stones and vectors is subject to the following conditions:

- consecutive stones face in opposite directions,
- if two consecutive stones face away from each other then their distance is an odd integer,
- if two consecutive stones face towards each other then their distance *plus* the lengths of their vectors is an odd integer,



a) b)

Figure 6.5: Two sample $(3,16)$-necklaces. Each has 6 stones. The arrows are stone vectors of length 1 (shorter) or 2 (longer). The necklace in b) is the image under T of the necklace in a).

- if two consecutive stones face towards each other then their distance is at least 3; moreover if their distance is exactly 3 then their vectors have length 1.

The last two conditions can be conveniently rephrased as follows: if two stones facing towards each other simultaneously jump then after the jump their distance will be an odd integer and they will not land in the same point nor jump over one another.

We identify (k,n)-necklaces which differ by an isometry of the circle. Clearly the number of (k,n)-necklaces is finite.

Next we describe a **necklace transformation** T which takes a (k,n)-necklace and performs the following operations:

- (JUMP) all stones jump as dictated by their vectors,
- (TURN) all stone vectors change according to the rule

$$-2 \to 1, \quad -1 \to 2, \quad 1 \to -2, \quad 2 \to -1,$$

  i.e. both direction and length are switched to the other option,
- (FIX) if any two stones find themselves in distance 3 facing each other and any of their vectors has length 2, then adjust the offending vectors by reducing their length to 1.

An example of N and TN is shown in Fig.6.5. It is easy to check that if N is a (k,n)-necklace then so is TN.

Definition 6.21. *Define* $\mathrm{Neck}(k,n)$ *to be the directed graph whose vertices are all the isomorphism classes of* (k,n)*-necklaces and such that for each* (k,n)*-necklace* N *there is a directed edge* $N \to TN$.

Lemma 6.22. *The graph* $\mathrm{Neck}(k,n)$ *is nonempty if and only if* $1 \leq k \leq n/4$ *and it is always a disjoint union of directed cycles.*

Proof. To each stone whose vector faces clockwise we associate the open arc segment of length 2 from that stone in the direction of its vector. To each stone whose vector faces counter-clockwise we associate the open arc segment of length 2 of which this stone is the midpoint. The segments associated to different stones are disjoint hence $2k \cdot 2 \leq n$, as required (compare the proof of Lemma 6.16).

The out-degree of every vertex in $\mathrm{Neck}(k,n)$ is 1, so it suffices to show that the in-degree is at least 1. Given a (k,n)-necklace N let T^{-1} be the following operations:

- for each stone which *does not* face towards another stone in distance 3, change the stone vector according to the rule

$$-2 \to -1, \quad -1 \to -2, \quad 1 \to 2, \quad 2 \to 1,$$

- jump with all the stones,
- change all stone vectors according to the rule

$$-2 \to 2, \quad -1 \to 1, \quad 1 \to -1, \quad 2 \to -2.$$

One easily checks that $T^{-1}N$ is a (k,n)-necklace and that $TT^{-1}N = T^{-1}TN = N$. $\square$

Some boundary cases of $\mathrm{Neck}(k,n)$ are easy to work out.

Lemma 6.23. *For any even* n *the graph* $\mathrm{Neck}(1,n)$ *is a cycle of length* $n-3$. *For any* k *the graph* $\mathrm{Neck}(k,4k)$ *is a single vertex with a loop. For any* k *the graph* $\mathrm{Neck}(k,4k+2)$ *is a cycle of length* $k+2$ *and* $\lfloor k/2 \rfloor$ *isolated vertices with loops.*

Proof. A $(1,n)$-necklace is determined by a choice of an odd number $3 \leq d \leq n-1$ (the length of the arc along which the two stones face each other) and a choice of $\epsilon \in \{1,2\}$ (the length of the vectors at both stones which must be the same due to the parity constraints), with the restriction that if $d=3$ then $\epsilon = 1$. If we denote the resulting necklace $A^{\epsilon}_{n,d}$ then

$$TA^1_{n,n-1} = A^1_{n,3}, \quad TA^1_{n,d} = A^2_{n,n-d+2} \ (3 \leq d \leq n-3),$$
$$TA^2_{n,d} = A^1_{n,n-d+4} \ (5 \leq d \leq n-1)$$

and it is easy to check that they assemble into a $(n-3)$-cycle.

An argument as in Lemma 6.22 shows that there is just one $(k, 4k)$-necklace N, with distances between stones alternating between 3 (stones facing each other) and 1 (stones facing away) and all vectors of length 1. It satisfies $TN = N$.

The analysis of the last case again requires the enumeration of all possible cases and we leave it to the interested reader. □

The following is our main conjecture about $\mathrm{Neck}(k, n)$.

Conjecture 6.24. *The length of every cycle in the graph* $\mathrm{Neck}(k, n)$ *divides* $n - 3k$. *In other words, for every* (k, n)*-necklace* N *we have* $T^{n-3k}N = N$.

This conjecture was experimentally verified for all even $n \leq 36$, see Table 3 in Section 6.7.

It is now time to explain what necklaces have to do with patterns and what Conjecture 6.24 has to do with Conjecture 6.4.

Intuitively, (k, n)-necklaces are meant to correspond to reducible proper patterns $\mathcal{P}$ of length n and $\mu(\mathcal{P}) = k$. The operation T mimics the reduction $\mathcal{P} \to \mathcal{P}^R$, although the details of this correspondence are a bit more complicated (see proof of Theorem 6.25). The lengths of cycles in the necklace graph $\mathrm{Neck}(k, n)$ determine the constants a in the recursive equations (6.3) and therefore also the exponents in the denominators of $f_n(t)$ (Conjecture 6.4). A precise statement is the following.

Theorem 6.25. *Let* $g_{i,n}$ *be the any common multiple of the lengths of all cycles in the graph* $\mathrm{Neck}(i, n)$. *Suppose* $\mathcal{P}$ *is a proper pattern of even length* n *and* $\mu(\mathcal{P}) = k$. *Then the generating function* $f_{\mathcal{P}}(t)$ *(see (6.2)) is of the form*

$$f_{\mathcal{P}}(t) = \frac{h_{\mathcal{P}}(t)}{1 - (-1)^{n/2}t^2} \cdot \prod_{i=1}^{k} \frac{1}{1 - t^{2g_{i,n}}}$$

for some polynomial $h_{\mathcal{P}}(t)$.

Before proving this result first observe:

Theorem 6.26. *Conjecture 6.24 implies Conjecture 6.4.*

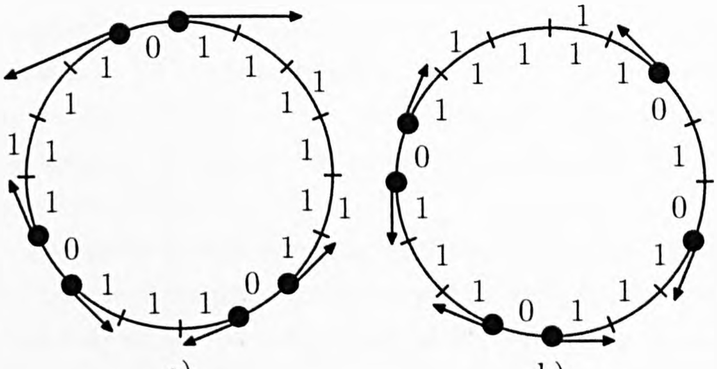


Figure 6.6: The correspondence between necklaces and patterns. If N is the necklace then the numbers inside the circle form the second row of the pattern UN and the numbers outside form its first row (only 1's are shown in the first row, the remaining entries are 0's). The second row has a block between each pair of stones facing each other. Over each block of length greater than 3 the number of outermost 0's in the first row equals the length of the stone vector.

Proof. The sequence $Z(P_m \times C_n)$ is a linear combination of sequences $Z(\mathcal{P}; m)$ for proper (in fact initial) patterns $\mathcal{P}$ of length n. Theorem 6.25 therefore implies that

$$f_n(t) = \frac{h_n(t)}{1-(-1)^{n/2}t^2} \cdot \prod_{i=1}^{\lfloor n/4 \rfloor} \frac{1}{1-t^{2g_{i,n}}}$$

for some polynomial $h_n(t)$. If Conjecture 6.24 is true then we can take $g_{i,n} = n - 3i$, thus obtaining the statement of Conjecture 6.4. $\square$

Proof of Theorem 6.25. For $k \geq 1$ and even n let $\mathrm{Prop}(k,n)$ denote the set of proper patterns $\mathcal{P}$ of length n and such that $\mu(\mathcal{P}) = k$. Moreover, let $\mathrm{Prop}_0(k,n) \subseteq \mathrm{Prop}(k,n)$ consist of patterns which do not have any block in row 1. These are exactly the reducible patterns.

There is a map

$$S : \mathrm{Prop}(k,n) \to \mathrm{Prop}_0(k,n)$$

defined as follows. If $\mathcal{P} \in \mathrm{Prop}(k,n)$ then all blocks in the first row of $\mathcal{P}$ have length 3. We apply the operation of type N in the middle of every such block and define $S\mathcal{P}$ to be the resulting pattern. It has no blocks in the first row and $\mu(S\mathcal{P}) = \mu(\mathcal{P})$ by Proposition 6.17.

Next we define a map

$$Q : \mathrm{Prop}_0(k,n) \to \mathrm{Neck}(k,n).$$

Note that a pattern $\mathcal{P} \in \mathrm{Prop}_0(k,n)$ is determined by the positions of blocks in the second row and, for every endpoint of a block, the information whether the outermost 1 in the first row (if any) is located above the 2nd or 3rd position of the block. This already determines the whole run above a block (because it is an alternating run of 0's and 1's).

Now we transcribe it into a necklace $Q\mathcal{P}$ as follows (see Fig.6.6). Label the unit intervals of a circle of length n with the symbols from the second row of $\mathcal{P}$. For every block place two stones bounding that block and facing towards each other. The lengths of the vectors at those stones are determined by the rule:

- if the stone bounds a block of length 3 then the length of its vector is 1,
- otherwise the length of a stone vector is the number of outermost 0's in the first row of $\mathcal{P}$ over the edge of the block bounded by the stone.

If two stones face away from each other then "between them" the second row of $\mathcal{P}$ contains a run $0101\cdots10$ of odd length. If two stones face towards each other then the length of the block between them is either 3 or it is the odd length of $101\cdots01$ plus p_1+p_2 where p_i are their stone vector lengths. It verifies that $Q\mathcal{P}$ is a (k,n)-necklace.

The map Q is a bijection and we let

$$U : \mathrm{Neck}(k,n) \to \mathrm{Prop}_0(k,n)$$

be its inverse. More specifically, the second row of UN is obtained by placing a block of 1's between every pair of stones that face each other and an alternating run $010\cdots10$ between stones facing away. In the first row of UN, over each block, we place either 0's (if the block has length 3) or an alternating sequence $101\cdots01$ leaving out as many outermost positions as dictated by the stone vector lengths. We fill the remaining positions in the first row with 0's. The construction is feasible thanks to the parity conditions satisfied by N.

All the maps are defined in such a way that for every (k,n)-necklace N we have

$$TN = QS((UN)^R) \quad \text{or equivalently} \quad UTN = S((UN)^R). \tag{6.4}$$

To see this consider how the reduction operation $(\cdot)^R$ and the map S change the neighbourhood of an endpoint of a block in the second row of UN. The argument is very similar to the proof of Proposition 6.18 and the details are left to the reader.

Now we complete the proof by induction on $\mu(\mathcal{P})$. The case $\mu(\mathcal{P})=0$ was dealt with in the proof in Section 6.5. Now suppose $k=\mu(\mathcal{P})\geq 1$. If $\mathcal{P} \in \mathrm{Prop}(k,n)$

is any pattern then by Propositions 6.10 and 6.17 we have an equation

$$Z(\mathcal{P};m) = \pm Z(S\mathcal{P};m) + \sum_{\mathcal{R}} \pm Z(\mathcal{R};m)$$

for some patterns $\mathcal{R}$ which satisfy $\mu(\mathcal{R}) = k-1$. That means it suffices to prove the result for patterns in $\mathrm{Prop}_0(k,n)$. Every such pattern is of the form UN for some (k,n)-necklace N. Equation (6.4) and Propositions 6.10, 6.11 and 6.17 lead to an equation

$$\begin{aligned} Z(UN;m) &= \pm Z((UN)^R;m-1) \\ &= \pm Z(S((UN)^R);m-1) + \sum_{\mathcal{R}} \pm Z(\mathcal{R};m-1) \\ &= \pm Z(UTN;m-1) + \sum_{\mathcal{R}} \pm Z(\mathcal{R};m-1) \end{aligned}$$

with $\mathcal{R}$ as before. Now a $g_{k,n}$-fold iterated application of this argument for each of $N, TN, \ldots, T^{g_{k,n}}N = N$ produces an equation

$$Z(UN;m) = \pm Z(UN;m-g_{k,n}) + \sum_{\mathcal{R}} \pm Z(\mathcal{R};m-b_{\mathcal{R}})$$

and its double application allows to avoid the problem of the unknown sign, that is we obtain

$$Z(UN;m) = Z(UN;m-2g_{k,n}) + \sum_{\mathcal{R}'} \pm Z(\mathcal{R}';m-b_{\mathcal{R}'}).$$

It follows that the generating function of $Z(UN;m)$ can be expressed using combinations of the same rational functions which appeared in the generating functions for patterns of μ-invariant $k-1$ together with $1/(1-t^{2g_{k,n}})$. That completes the proof. □

6.7 Tables

Table 1.

Some values of $Z(P_m \times C_n)$:

$m \setminus n$	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	-1	-2	-1	1	2	1	-1	-2	-1	1	2	1	-1
2	-1	1	3	1	-1	1	3	1	-1	1	3	1	-1
3	1	1	-3	1	1	1	5	1	1	1	-3	1	1
4	1	-2	5	1	4	1	5	-2	1	1	8	1	1
5	-1	1	-5	1	-1	1	3	1	9	1	-5	1	-1
6	-1	1	7	1	1	1	7	1	-1	1	7	1	13
7	1	-2	-7	1	4	1	1	-2	1	1	8	1	1
8	1	1	9	1	1	1	1	1	1	1	9	1	1
9	-1	1	-9	1	-1	1	-1	1	-11	1	-9	1	13
10	-1	-2	11	1	2	1	3	-2	-1	1	14	1	-1
11	1	1	-11	1	1	1	-3	1	1	1	-11	1	15
12	1	1	13	1	1	1	5	1	11	1	13	1	1

Table 2.

Some initial generating functions $f_n(t)$ for even n are given below in reduced form. By $\Phi_k(t)$ we denote the k-th cyclotomic polynomial ($\Phi_1(t) = t - 1$, $\Phi_2(t) = t + 1$).

$$\begin{aligned}
f_2(t) &= \frac{-(t-1)}{\Phi_4(t)} \\
f_4(t) &= \frac{-(t^2+1)}{\Phi_1(t)\Phi_2(t)^2} \\
f_6(t) &= \frac{-(t^4+2t^3+2t+1)}{\Phi_1(t)\Phi_3(t)\Phi_4(t)} \\
f_8(t) &= \frac{-(t^6-t^5+2t^4+6t^3+2t^2-t+1)}{\Phi_1(t)\Phi_2(t)^2\Phi_{10}(t)} \\
f_{10}(t) &= \frac{q_{10}(t)}{\Phi_1(t)\Phi_4(t)\Phi_7(t)\Phi_8(t)} \\
f_{12}(t) &= \frac{q_{12}(t)}{\Phi_1(t)\Phi_2(t)^2\Phi_3(t)\Phi_6(t)^2\Phi_{18}(t)} \\
f_{14}(t) &= \frac{q_{14}(t)}{\Phi_1(t)\Phi_4(t)\Phi_5(t)\Phi_{11}(t)\Phi_{16}(t)} \\
f_{16}(t) &= \frac{q_{16}(t)}{\Phi_1(t)\Phi_2(t)^2\Phi_{10}(t)\Phi_{14}(t)\Phi_{26}(t)} \\
f_{18}(t) &= \frac{q_{18}(t)}{\Phi_1(t)\Phi_3(t)\Phi_4(t)\Phi_5(t)\Phi_8(t)\Phi_9(t)\Phi_{12}(t)\Phi_{15}(t)\Phi_{24}(t)} \\
f_{20}(t) &= \frac{q_{20}(t)}{\Phi_1(t)\Phi_2(t)^2\Phi_4(t)\Phi_7(t)\Phi_{14}(t)\Phi_{22}(t)\Phi_{34}(t)}
\end{aligned}$$

$$f_{22}(t) = \frac{q_{22}(t)}{\Phi_1(t)\Phi_4(t)\Phi_7(t)\Phi_{13}(t)\Phi_{19}(t)\Phi_{20}(t)\Phi_{32}(t)}$$

The longer numerators are as follows.

$$\begin{aligned}
q_{10}(t) &= -(t^{12} - t^{11} + t^8 + 9t^7 + 9t^5 + t^4 - t + 1) \\
q_{12}(t) &= -(t^{14} + 2t^{13} + 3t^{12} - 3t^{11} + 8t^{10} - 5t^9 + 6t^8 \\
&\quad +6t^7 + 6t^6 - 5t^5 + 8t^4 - 3t^3 + 3t^2 + 2t + 1) \\
q_{14}(t) &= -(t^{24} - t^{19} + 14t^{18} + 14t^{17} + 29t^{16} + 42t^{15} + 42t^{14} + 55t^{13} + 56t^{12} \\
&\quad +55t^{11} + 42t^{10} + 42t^9 + 29t^8 + 14t^7 + 14t^6 - t^5 + 1) \\
q_{16}(t) &= -(t^{28} - 2t^{27} + 5t^{26} + 5t^{24} - 2t^{23} + 9t^{22} - 7t^{21} + 39t^{20} - 37t^{19} \\
&\quad +44t^{18} - 25t^{17} + 30t^{16} - 24t^{15} + 26t^{14} - 24t^{13} + 30t^{12} \\
&\quad -25t^{11} + 44t^{10} - 37t^9 + 39t^8 - 7t^7 + 9t^6 - 2t^5 + 5t^4 + 5t^2 - 2t + 1) \\
q_{18}(t) &= -(t^{38} + 2t^{37} - t^{36} + 2t^{35} + 6t^{34} - 2t^{33} + 2t^{32} + 30t^{31} - 2t^{30} \\
&\quad +t^{29} + 70t^{28} - t^{27} + t^{26} + 92t^{25} - t^{24} + t^{23} + 130t^{22} - t^{21} \\
&\quad +168t^{19} - t^{17} + 130t^{16} + t^{15} - t^{14} + 92t^{13} + t^{12} - t^{11} + 70t^{10} \\
&\quad +t^9 - 2t^8 + 30t^7 + 2t^6 - 2t^5 + 6t^4 + 2t^3 - t^2 + 2t + 1) \\
q_{20}(t) &= -(t^{46} - 2t^{45} + 6t^{44} - 10t^{43} + 19t^{42} - 18t^{41} + 34t^{40} - 40t^{39} \\
&\quad +64t^{38} - 28t^{37} + 60t^{36} - 31t^{35} + 120t^{34} - 96t^{33} + 189t^{32} \\
&\quad -147t^{31} + 240t^{30} - 195t^{29} + 283t^{28} - 230t^{27} + 258t^{26} \\
&\quad -193t^{25} + 218t^{24} - 208t^{23} + 218t^{22} - 193t^{21} + 258t^{20} \\
&\quad -230t^{19} + 283t^{18} - 195t^{17} + 240t^{16} - 147t^{15} + 189t^{14} \\
&\quad -96t^{13} + 120t^{12} - 31t^{11} + 60t^{10} - 28t^9 + 64t^8 - 40t^7 \\
&\quad +34t^6 - 18t^5 + 19t^4 - 10t^3 + 6t^2 - 2t + 1) \\
q_{22}(t) &= -(t^{62} + t^{61} + t^{58} + t^{57} - t^{55} + 22t^{54} + \cdots\cdots + 22t^8 - t^7 + t^5 + t^4 + t + 1)
\end{aligned}$$

Table 3.

The decomposition of the directed graph $\mathrm{Neck}(k,n)$ into a disjoint union of cycles. Here l^p stands for p copies of the cycle C_l.

$n \setminus k$	1	2	3	4	5	6	7
4	1^1						
6	3^1						
8	5^1	1^1					
10	7^1	$1^1 4^1$					
12	9^1	$2^1 3^2 6^1$	1^1				
14	11^1	$2^3 8^3$	$1^1 5^1$				
16	13^1	$2^5 5^3 10^3$	7^4	1^1			
18	15^1	$1^1 2^9 12^6$	$3^4 9^9$	$1^2 6^1$			
20	17^1	$2^{14} 7^4 14^6$	11^{22}	$1^1 4^3 8^5$	1^1		
22	19^1	$2^{22} 16^{10}$	13^{42}	$2^1 5^6 10^{21}$	$1^2 7^1$		
24	21^1	$2^{30} 9^5 18^{10}$	$3^1 5^{10} 15^{70}$	$2^5 3^{10} 4^3 6^9 12^{63}$	$1^1 3^1 9^9$	1^1	
26	23^1	$1^1 2^{42} 20^{15}$	17^{120}	$2^{15} 7^{28} 14^{165}$	11^{49}	$1^3 8^1$	
28	25^1	$2^{55} 11^6 22^{15}$	19^{186}	$2^{43} 8^{43} 16^{378}$	13^{194}	$1^3 5^4 10^{11}$	1^1
30	27^1	$2^{73} 24^{21}$	$7^{19} 21^{270}$	$2^{99} 3^6 6^4 9^{90} 18^{765}$	$3^{26} 5^{31} 15^{613}$	$1^1 3^5 4^{10} 6^{13} 12^{80}$	$1^3 9^1$
32	29^1	$2^{91} 13^7 26^{21}$	23^{396}	$2^{217} 5^{43} 10^{83} 20^{1480}$	17^{1750}	$2^1 7^{52} 14^{435}$	$1^5 11^{17}$
34	31^1	$1^1 2^{115} 28^{28}$	$5^1 25^{551}$	$2^{429} 11^{242} 22^{2600}$	19^{4334}	$2^7 4^{22} 8^{124} 16^{1791}$	$1^1 13^{155}$
36	33^1	$2^{140} 15^8 30^{28}$	$9^{31} 27^{738}$	$1^1 2^{809} 8^{12} 12^{216} 24^{4483}$	$1^1 3^{18} 7^{247} 21^{9693}$	$2^{31} 3^{83} 6^{76} 9^{316} 18^{6100}$	$1^1 3^{18} 5^{31} 15^{970}$

Chapter 7

Vietoris-Rips complexes of graphs

7.1 Introduction

In this chapter we investigate the behaviour of clique complexes of powers of graphs.

Recall that if G is a graph and r is a non-negative integer then the *r-th power* or *r-th distance power* of G, denoted G^r, is a new graph with the same vertex set in which two vertices are adjacent if and only if their distance in G is *at most* r. Any graph G gives rise to a sequence of graph inclusions

$$G \hookrightarrow G^2 \hookrightarrow G^3 \hookrightarrow \cdots \tag{7.1}$$

which eventually stabilizes (at the complete graph if G is connected).

Recall that for a graph G the *clique complex* $\mathrm{Cl}(G)$ is a simplicial complex whose vertices are the vertices of G and the simplices are the *cliques* (complete subgraphs) in G. Clearly Cl is a functor from graphs to simplicial complexes and we have inclusions

$$\mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2) \hookrightarrow \mathrm{Cl}(G^3) \hookrightarrow \cdots \tag{7.2}$$

which, for a connected graph G, stabilize at the full simplex. In a geometer's language $\mathrm{Cl}(G^r)$ is precisely the Vietoris-Rips complex whose faces are subsets of diameter at most r in the discrete metric space $V(G)$ with the shortest path distance.

Note that not every graph is of the form G^r for $r \geq 2$ (in fact the recognition of graph squares [85] and arbitrary graph powers [1] is NP-hard), so we may ask about interesting properties of the spaces $\mathrm{Cl}(G^r)$ and of the inclusions $\mathrm{Cl}(G^r) \hookrightarrow \mathrm{Cl}(G^{r+1})$.

For example, if $G = C_7$ is the 7-cycle then $\mathrm{Cl}(C_7^2)$ has maximal faces of the form $\{i, i+1, i+2\}$ (mod 7). It is homeomorphic to the Möbius strip and it collapses to its subcomplex $\mathrm{Cl}(C_7) \equiv S^1$. If, on the other hand, $G = C_6$, then the complex $\mathrm{Cl}(C_6^2)$ is the boundary of the octahedron, homeomorphic to S^2, and the sequence (7.2) is, up to homotopy, $S^1 \to S^2 \to * \to \cdots$.

Let us outline the structure of this chapter. Section 7.2 contains some preliminary results, in particular on powers of graphs with no short cycles. In Section 7.3 we restrict to graph squares ($r = 2$) and prove topological and combinatorial conditions which guarantee that the inclusion $\mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2)$ is a homotopy equivalence.

In Section 7.4 we discuss *universality* of $\mathrm{Cl}(G^r)$, proving that for any r every finite complex can be realized as $\mathrm{Cl}(G^r)$ up to homotopy. Contrary to the case $r = 1$, for higher r not every space has a realization as $\mathrm{Cl}(G^r)$ up to homeomorphism. Our method is based on some results of [31] and the analysis of shortest paths in iterated barycentric subdivisions.

Section 7.5 provides a complete description of the clique complexes of the total graph and the line graph of G.

In the last part, Section 7.6, we calculate the homotopy types of $\mathrm{Cl}(G^r)$ in the first nontrivial case, that is for the cycles $G = C_n$. A quick preview of those can be found in Section 7.7. They turn out to be obtained from a small number of initial cases by an action of a double suspension operator Σ^2. To see this we run the theory of star clusters of [13] on the independence complexes of the complements $\overline{C_n^r}$, the circular complete graphs.

7.2 Preliminaries

Fact 7.1. *For any connected graph G the map of fundamental groups*

$$\pi_1(\mathrm{Cl}(G)) \to \pi_1(\mathrm{Cl}(G^r))$$

induced by the inclusion $G \hookrightarrow G^r$ is surjective.

Proof. It suffices to prove that $\pi_1(\mathrm{Cl}(G^{r-1})) \to \pi_1(\mathrm{Cl}(G^r))$ is surjective for $r \geq 2$. Consider a based path α in $\mathrm{Cl}(G^r)$. By cellular approximation we can assume it lies in the 1-skeleton and is piecewise linear. If $e = uv \in E(G^r) \setminus E(G^{r-1})$ then there is a vertex w such that $uw, wv \in E(G^{r-1})$. Then $\{u, w, v\}$ is a face of $\mathrm{Cl}(G^r)$ and any segment of α going along uv can be continuously deformed to go along uwv without moving the endpoints. Performing this operation for every segment in

$E(G^r) \setminus E(G^{r-1})$ we obtain a based path homotopic to α which lies in $\mathrm{Cl}(G^{r-1})$. $\square$

One situation when $\mathrm{Cl}(G^r)$ is homotopy equivalent to (in fact, collapses to) $\mathrm{Cl}(G)$ is when r is not too large compared to the girth of G (the *girth* of a graph is the length of its shortest cycle or ∞ for a forest). Of course as soon as G is triangle-free $\mathrm{Cl}(G) \equiv G$ is 1-dimensional.

Proposition 7.2. *Let $r \geq 1$. If G is a graph of girth at least $3r+1$ then for every $2 \leq k \leq r$ the complex $\mathrm{Cl}(G^k)$ collapses to $\mathrm{Cl}(G^{k-1})$. In particular $\mathrm{Cl}(G^r)$ collapses to its subcomplex $\mathrm{Cl}(G) \equiv G$.*

Proof. Let $\mathcal{E} = E(G^k) \setminus E(G^{k-1})$ be the set of "new" edges in G^k and let $\mathcal{F} \subseteq \mathrm{Cl}(G^k)$ be the set of faces which contain at least one edge of $\mathcal{E}$. We have $\mathrm{Cl}(G^{k-1}) = \mathrm{Cl}(G^k) \setminus \mathcal{F}$. If $\mathcal{E} = \emptyset$ there is noting to do, so assume $\mathcal{E} \neq \emptyset$.

The nonexistence of cycles of length $3k$ or less in G has the following consequences. First, every maximal clique σ in G^k corresponds to a subtree of diameter k in G. Second, every edge in $\mathcal{E}$ (hence also every face in $\mathcal{F}$) belongs to a *unique* maximal face of $\mathrm{Cl}(G^k)$. To see the second statement let $e = uv \in \mathcal{E}$ and suppose x, y are two vertices such that xuv and yuv are both faces of $\mathrm{Cl}(G^k)$. Denote by α the shortest path in G from u to v. By the first observation there is a vertex $x' \in \alpha$ such that the shortest paths from x to u and v join the path α at x'. Similarly, there is a $y' \in \alpha$ with the same property for y and we may assume w.l.o.g. that the order along α is $u - x' - y' - v$. Then

$$\begin{aligned} \mathrm{dist}_G(x,y) &= \mathrm{dist}_G(x,x') + \mathrm{dist}_G(x',y') + \mathrm{dist}_G(y',y) = \\ &= \mathrm{dist}_G(x,v) + \mathrm{dist}_G(y,u) - \mathrm{dist}_G(u,v) \leq k + k - k = k \end{aligned}$$

which proves the claim.

Let σ be some maximal face of $\mathrm{Cl}(G^k)$ and $v \in \sigma$ any fixed vertex whose distance in G to all vertices of $G[\sigma]$ is strictly less than k (for example the centre of the tree $G[\sigma]$). We define an acyclic matching M_σ on σ by taking all the pairs

$$(f, f \cup \{v\})$$

for all faces $f \in \mathcal{F}$ such that $f \subseteq \sigma \setminus \{v\}$. Since no edge of $\mathcal{E}$ which lies in σ has v as its endpoint, every face of $\mathcal{F}$ contained in σ is of the form f or $f \cup \{v\}$ above. It follows that M_σ matches all faces of σ which are in $\mathcal{F}$ (and only those).

Let $M = \bigcup_\sigma M_\sigma$ be the union of those matchings over all maximal faces σ. By the previous remarks it is well-defined, acyclic and its critical faces form the subcomplex $\mathrm{Cl}(G^{k-1})$. This ends the proof. $\square$

The girth bound of $3r+1$ is optimal, because $\mathrm{Cl}(C_{3r}^r) \simeq \bigvee^{r-1} S^2$ by the results of Section 7.6.

One standard way of analyzing the homotopy type of $\mathrm{Cl}(G)$ is via the notions of folds and dismantlability. If $u, v \in V(G)$ are distinct vertices such that $N_G[u] \subseteq N_G[v]$ then we say *G folds onto $G \setminus u$*. A graph G is *dismantlable* if there exists a sequence of folds from G to a single vertex. It is a classical fact that a fold preserves the homotopy type of the clique complex and, in fact, induces a collapsing of $\mathrm{Cl}(G)$ onto $\mathrm{Cl}(G \setminus u)$, so the clique complex of a dismantlable graph is collapsible (see for example [20, Lemma 2.2]). In this context we have the following simple result.

Lemma 7.3. *If G is dismantlable then so is G^r for any $r \geq 1$.*

Proof. We use induction on $|V(G)|$. Let u be a vertex such that G folds onto $G \setminus u$ and $G \setminus u$ is dismantlable. Let v be a vertex which satisfies $N_G[u] \subseteq N_G[v]$. First note that

$$(G \setminus u)^r = G^r \setminus u.$$

Indeed, the inclusion $\subseteq$ is obvious. For $\supseteq$ note that any occurrence of u in a path can be replaced with v or removed without increasing the length of the path.

The graph $G^r \setminus u = (G \setminus u)^r$ is dismantlable by induction. Moreover $N_{G^r}[u] \subseteq N_{G^r}[v]$. It follows that G^r folds onto $G^r \setminus u$ and the dismantlability of G^r is proved. $\square$

Both 7.2 and 7.3 imply the following.

Corollary 7.4. *For every tree T and any integer r the complex $\mathrm{Cl}(T^r)$ is collapsible (and, in particular, contractible).*

7.3 Stability

In this section we only consider graph squares ($r = 2$). We describe more general criteria which guarantee that the inclusion $\mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2)$ is a homotopy equivalence.

Note that for any vertex v of G the set $N_G[v]$ forms a clique in G^2.

Theorem 7.5. *Suppose G satisfies the following condition:*

- *Every clique in G^2 is contained in a set of the form $N_G[v]$ for some vertex v.*

Then the inclusion $i : \mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2)$ is a homotopy equivalence.

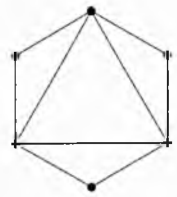


Figure 7.1: The 3-sun graph S_3.

Proof. By passing to connected components we can assume G is connected. We use the following local criterion of [80, Thm. 6] (see also [79, Cor. 1.4]):

- Suppose $p : X \to Y$ is a continuous map and Y has an open cover $\mathcal{U} = \{U_\alpha\}$ such that if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$. If for every $U \in \mathcal{U}$ the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a weak equivalence then so is p.

Since we are working with finite simplicial complexes we can just as well replace open sets with closed subcomplexes (by taking a small open neighbourhood of a subcomplex) and weak equivalences with homotopy equivalences (by Whitehead's theorem).

For each $v \in V(G)$ let $U_v = \mathrm{Cl}(G^2)[N_G[v]]$. Each of U_v is a simplex in $\mathrm{Cl}(G^2)$. By assumption we have $\mathrm{Cl}(G^2) = \bigcup_{v\in V(G)} U_v$ and we can take a cover $\mathcal{U}$ of $\mathrm{Cl}(G^2)$ consisting of all intersections of the sets U_v.

If $U = U_{v_1} \cap \cdots \cap U_{v_k}$ is non-empty then it is an intersection of faces of $\mathrm{Cl}(G^2)$, hence it is contractible. It remains to show that $i^{-1}(U)$ is also contractible. Let $X = \bigcap_{i=1}^k N_G[v_i]$ be the set of vertices spanning U. Since i is a subcomplex inclusion, we have $i^{-1}(U) = \mathrm{Cl}(G)[X]$. Because in G every vertex of X is in distance at most 1 from each of v_i, the set $X \cup \{v_1, \ldots, v_k\}$ forms a clique in G^2. Our assumption then gives a vertex v such that

$$X \cup \{v_1, \ldots, v_k\} \subseteq N_G[v].$$

In particular $v \in N_G[v_i]$ for each $i = 1, \ldots, k$, so $v \in X$. Moreover, since $X \subseteq N_G[v]$, the vertex v is adjacent in G to every other element of X, i.e. $G[X]$ is a cone with apex v. It implies that $\mathrm{Cl}(G)[X] = \mathrm{Cl}(G[X])$ is a simplicial cone with apex v, hence it is contractible. This completes the proof. $\square$

There is a more direct combinatorial condition which guarantees that the assumption of Theorem 7.5 is satisfied. Recall that we say G is *H-free* if G does not have an *induced* subgraph isomorphic to H. If $H_1, \ldots, H_k$ is a sequence of graphs then G is $(H_1, \ldots, H_k)$-free if it does not have any of the H_i as induced subgraphs.

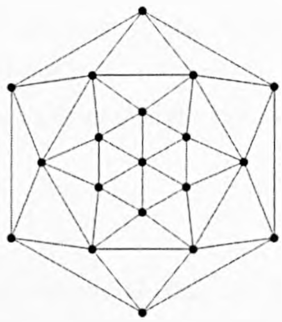


Figure 7.2: A graph G with $\mathrm{Cl}(G) \simeq *$ and $H_2(\mathrm{Cl}(G^2)) \neq 0$.

Consider the graph of Fig.7.1, usually denoted S_3 and called 3-sun.

Theorem 7.6. *If G is (C_4, C_5, C_6, S_3)-free then G satisfies the condition in Theorem 7.5*

Proof. Suppose, on the contrary, that $K = \{v_1, \ldots, v_k\}$ is the smallest clique in G^2 which is not contained in any set $N_G[v]$. Then $k \geq 3$ and there exist $w_1, \ldots, w_k$ such that $K \setminus v_i \subseteq N_G[w_i]$. The vertices $w_1, \ldots, w_k$ are pairwise distinct (as $w_i = w_j$ would mean $K \subseteq N_G[w_i]$) and there is no edge $w_i v_i$ in G for any i (same reason). It means that we have

$$\begin{cases} N_G[w_1] \cap \{v_1, v_2, v_3\} = \{v_2, v_3\} \\ N_G[w_2] \cap \{v_1, v_2, v_3\} = \{v_3, v_1\} \\ N_G[w_3] \cap \{v_1, v_2, v_3\} = \{v_1, v_2\} \\ v_1 \neq v_2 \neq v_3 \neq v_1, \quad w_1 \neq w_2 \neq w_3 \neq w_1 \end{cases} \tag{*}$$

By Theorem 3 of [23] a graph is (C_4, C_5, C_6, S_3)-free if and only if it does not have a configuration satisfying (*) (which, using the notation of [23], is saying that the *neighbourhood hypergraph* of G is triangle-free). That ends the proof. $\square$

Remark 7.7. For an arbitrary graph G one might at least hope that the inclusion $\mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2)$ stabilizes the homotopy type, for example by increasing the connectivity of the space. This is not the case. For example, let G be the graph of Fig.7.2 and let V denote the set of vertices of the outermost 6-cycle. Then $\mathrm{Cl}(G)$ is contractible while one can check by a direct calculation that $H_2(\mathrm{Cl}(G^2)) = \mathbb{Z} \oplus \mathbb{Z}$ where one of the generators is represented by the subcomplex $\mathrm{Cl}(G^2)[V]$, homeomorphic to S^2.

Remark 7.8. The converse of Theorem 7.6 is false as can be seen by taking any graph G which is a cone and has one of the forbidden induced subgraphs. The converse of Theorem 7.5 is also false and the counterexample is the 3-sun S_3. Indeed, $\mathrm{Cl}(S_3) \simeq \mathrm{Cl}(S_3^2) \simeq *$ and $S_3^2 = K_6$ is complete but S_3 itself is not a cone.

7.4 Universality

It is a known fact that any finite simplicial complex K is homeomorphic to $\mathrm{Cl}(G)$ for some graph G. One can take G to be the 1-skeleton of the barycentric subdivision of K.

Clearly clique complexes of higher graph powers cannot represent all homeomorphism types. For instance, if $\mathrm{Cl}(G^2)$ is a connected space of dimension two then every vertex of G must have degree at most 2. It means G must be a path or cycle and a direct check narrows the possible two-dimensional homeomorphism types of $\mathrm{Cl}(G^2)$ to D^2, S^2, the Möbius strip and $D^1 \times S^1$, where D^n is the n-dimensional disk.

It is, however, true that arbitrary graph powers realize all homotopy types.

Theorem 7.9. *For every finite simplicial complex K and integer $r \geq 1$ there exists a graph G such that $\mathrm{Cl}(G^r)$ is homotopy equivalent to K.*

In fact there is an explicit candidate for G. Given a finite complex K and $s \geq 0$ let $\mathrm{bd}^s K$ denote its s-th iterated barycentric subdivision and let the graph G_s be its 1-skeleton:

$$G_s = (\mathrm{bd}^s K)^{(1)}$$

(from now on we will suppress the complex K from notation). Replacing, if needed, K with its subdivision we can assume $K = \mathrm{Cl}(G_0)$ and then for every $s \geq 0$ we have $\mathrm{bd}^s K = \mathrm{Cl}(G_s)$. Then we have the following result.

Theorem 7.10. *For any finite simplicial complex K and $1 \leq r < 2^{s-2}$*

$$\mathrm{Cl}((G_s)^r) \simeq K.$$

The proof strategy resembles that of [31]. For any vertex v of the original complex K let $B_{s,v}$ and $S_{s,v}$ denote the vertex sets in G_s defined as

$$\begin{aligned} B_{s,v} &= \{w : \mathrm{dist}_{G_s}(v,w) < 2^s\}, \\ S_{s,v} &= \{w : \mathrm{dist}_{G_s}(v,w) = 2^s\}. \end{aligned}$$

The letters B and S stand for the open Ball and the Sphere of radius 2^s around v in G_s. In the geometric realization the vertices of $B_{s,v}$ belong to the open star

$\mathrm{st}_K(v) \setminus \mathrm{lk}_K(v)$ of v in K while the vertices of $S_{s,v}$ lie in the link $\mathrm{lk}_K(v)$. Note that $B_{s,v_1} \cap \dots \cap B_{s,v_k}$ is nonempty if and only if $\{v_1, \dots, v_k\}$ is a face of K.

The main technical result we use is proved in [31, 3.7,3.8].

Proposition 7.11 ([31]). *For any face $\{v_1, \dots, v_k\}$ of K the graph*

$$G_s[B_{s,v_1} \cap \dots \cap B_{s,v_k}]$$

is dismantlable.

Now consider an integer $r < 2^{s-2}$. We intend to prove that $\mathrm{Cl}(G_s^r) \simeq K$ using the nerve lemma [72, 15.21]. Define subcomplexes of $\mathrm{Cl}(G_s^r)$ by

$$X_{s,v} = \mathrm{Cl}((G_s[B_{s,v}])^r) \subseteq \mathrm{Cl}(G_s^r) \tag{7.3}$$

for the vertices v of K. The reader should be warned that the subcomplex $X_{s,v}$ is *not* induced; in particular it should not be confused with $\mathrm{Cl}(G_s^r[B_{s,v}])$, which is usually bigger.

Proposition 7.12. *The family of subcomplexes $X_{s,v}$ is a covering of* $\mathrm{Cl}(G_s^r)$. *The nerve of this covering is K.*

Proof. The second statement is obvious since the vertex set of $X_{s,v_1} \cap \dots \cap X_{s,v_k}$ is $B_{s,v_1} \cap \dots \cap B_{s,v_k}$ and this is nonempty only for a face $\{v_1, \dots, v_k\}$ of K.

Let us prove the first statement. Suppose σ is a clique in G_s^r. Fix any $w \in \sigma$. There exists a vertex v of K such that

$$\mathrm{dist}_{G_s}(v, w) \leq 2^{s-1}.$$

Fix also that v. Now any vertex $w' \in \sigma$ satisfies

$$\begin{aligned} \mathrm{dist}_{G_s}(w', v) &\leq \mathrm{dist}_{G_s}(w', w) + \mathrm{dist}_{G_s}(w, v) \leq \\ &\leq r + 2^{s-1} < 2^{s-2} + 2^{s-1} \leq 2^s - 1. \end{aligned}$$

Therefore $\sigma \subseteq B_{s,v}$.

Now we want to show that for any two vertices $w', w'' \in \sigma$ the shortest path from w' to w'' in G_s lies in $G_s[B_{s,v}]$. Indeed, if z is any vertex on that path then

$$\begin{aligned} \mathrm{dist}_{G_s}(z, v) &\leq \mathrm{dist}_{G_s}(z, w') + \mathrm{dist}_{G_s}(w', v) \leq \\ &\leq r + (r + 2^{s-1}) < 2 \cdot 2^{s-2} + 2^{s-1} = 2^s. \end{aligned}$$

so $z \in B_{s,v}$. Since σ is a set of diameter at most r in G_s and the shortest paths between its vertices lie in $B_{s,v}$ it follows that σ is a set of diameter at most r in $G_s[B_{s,v}]$. It means that $\sigma \in X_{s,v}$. □

The point here was that the whole clique σ was located at least r steps away from $S_{s,v}$, so the path in G_s could not take the advantage of any shortcut outside $B_{s,v}$.

Proposition 7.12 and the nerve lemma [72, 15.21] imply Theorem 7.10 as soon as we prove that the nonempty intersections $X_{s,v_1} \cap \cdots \cap X_{s,v_k}$ are contractible. This is arranged for by the following lemma.

Proposition 7.13. *For any vertices $v_1, \ldots, v_k$ of K we have*

$$X_{s,v_1} \cap \cdots \cap X_{s,v_k} = \mathrm{Cl}((G_s[B_{s,v_1} \cap \cdots \cap B_{s,v_k}])^r)$$

Proof. We can restrict to the case when $\{v_1, \ldots, v_k\}$ is a face of K, otherwise the intersections are empty. By the definition of $X_{s,v}$ what we need to prove is

$$\mathrm{Cl}((G_s[B_{s,v_1}])^r) \cap \cdots \cap \mathrm{Cl}((G_s[B_{s,v_k}])^r) = \mathrm{Cl}((G_s[B_{s,v_1} \cap \cdots \cap B_{s,v_k}])^r).$$

The inclusion $\supseteq$ is obvious, so we need to prove $\subseteq$. It is equivalent to the statement

$\mathcal{D}(k)$: If $u, w \in B_{s,v_1} \cap \cdots \cap B_{s,v_k}$ are vertices such that

$$\mathrm{dist}_{G_s[B_{s,v_1}]}(u,w) \le r, \ \ldots, \ \mathrm{dist}_{G_s[B_{s,v_k}]}(u,w) \le r$$

then

$$\mathrm{dist}_{G_s[B_{s,v_1} \cap \cdots \cap B_{s,v_k}]}(u,w) \le r.$$

We prove it by induction on k. Clearly $\mathcal{D}(1)$ holds. Now suppose $k \ge 2$. By induction there is a path α from u to w in $G_s[B_{s,v_1} \cap \cdots \cap B_{s,v_{k-1}}]$ of length at most r. Denote by β the path in $G_s[B_{s,v_k}]$ from u to w of length at most r. If α lies completely in B_{s,v_k} or β lies in $B_{s,v_1} \cap \cdots \cap B_{s,v_{k-1}}$ then $\mathcal{D}(k)$ follows. If none of those two cases holds then α passes through some point $p \in S_{s,v_k}$ and β passes through some $q \in S_{s,v_1} \cup \cdots \cup S_{s,v_{k-1}}$. Assume without loss of generality that $q \in S_{s,v_1}$. Then $p \overset{\alpha}{\text{---}} u \overset{\beta}{\text{---}} q$ is a path in G_s of length at most $2r < 2 \cdot 2^{s-2} < 2^s$ which connects $p \in B_{s,v_1} \cap S_{s,v_k}$ with $q \in S_{s,v_1} \cap B_{s,v_k}$ and this whole path lies in $B_{s,v_1} \cup B_{s,v_k}$ (because $\alpha \subseteq G_s[B_{s,v_1}]$ and $\beta \subseteq G_s[B_{s,v_k}]$). The existence of such path, however, is excluded by the next lemma and this contradiction ends the inductive step. □

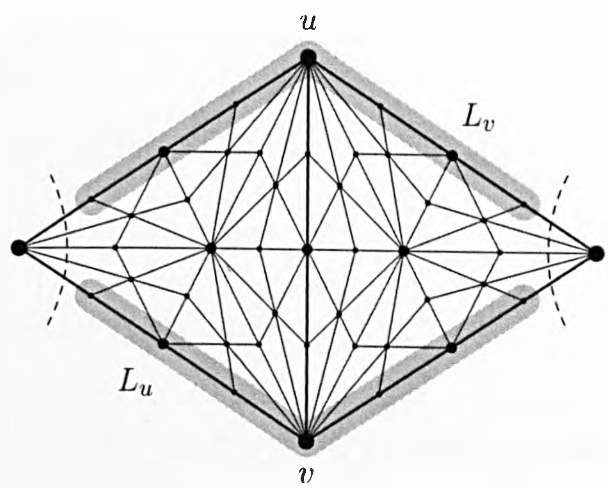


Figure 7.3: An example with $s = 2$. The shaded sets $L_u = S_{s,u} \cap B_{s,v}$ and $L_v = B_{s,u} \cap S_{s,v}$ contain vertices of $B_{s,u} \cup B_{s,v}$ in distance 2^s from, respectively, u and v. The distance between these two sets within $B_{s,u} \cup B_{s,v}$ is also 2^s, although their distance in G_s is only 2.

We are left with the last technical lemma whose intuitive meaning is the following. Suppose σ, τ are two faces of the same simplex in K. Suppose we look at the s-th barycentric subdivision of K and the paths in its 1-skeleton. Then the points of σ are very far apart from the points of τ if one is not allowed to go through $\sigma \cap \tau$.

Using the standard notation

$$\text{dist}_G(X, Y) = \min\{\text{dist}_G(x, y) : x \in X, y \in Y\}$$

for $X, Y \subseteq V(G)$ we can express this idea as follows (see Fig.7.3).

Lemma 7.14. *For any two adjacent vertices u, v of the original complex K*

$$\text{dist}_{G_s[B_{s,u} \cup B_{s,v}]}(B_{s,u} \cap S_{s,v}, S_{s,u} \cap B_{s,v}) = 2^s.$$

Proof. A *partial labeling* l of a graph G is an assignment of a real number $l(v)$ to some of the vertices of G. If $X \subseteq V(G)$ we write $l(X)$ for the set of labels assigned to the vertices in X, with $l(X) = \emptyset$ if the value of $l(v)$ is undefined for all $v \in X$.

We will construct partial labelings l_s of G_s for $s \geq 0$ with the properties:

a) The set of vertices for which l_s is defined is $B_{s,u} \cup B_{s,v}$.

b) For every simplex $\sigma \in \mathrm{Cl}(G_s)$ the vertices of σ are assigned at most two different labels.

c) For every edge $xy \in E(G_s)$ such that both $l_s(x)$ and $l_s(y)$ are defined we have

$$|l_s(x) - l_s(y)| \in \{0, \frac{1}{2^s}\}.$$

d) $l_s(B_{s,u} \cap S_{s,v}) = \{0\}$, $l_s(S_{s,u} \cap B_{s,v}) = \{1\}$.

The partial labeling l_0 is defined by $l_0(u) = 0$, $l_0(v) = 1$ and undefined otherwise. Suppose l_{s-1} has been defined. Every vertex $x \in V(G_s)$ represents a face $\tau \in \mathrm{Cl}(G_{s-1})$ and we set

$$l_s(x) = \begin{cases} a & \text{if } l_{s-1}(\tau) = \{a\} \\ (a+b)/2 & \text{if } l_{s-1}(\tau) = \{a, b\} \\ \text{undefined} & \text{if } l_{s-1}(\tau) = \emptyset. \end{cases}$$

This is well-defined since l_{s-1} satisfies b). Note that if $x \in V(G_{s-1})$ then $l_s(x) = l_{s-1}(x)$.

To prove that l_s satisfies a) recall that $B_{s-1,u} \cup B_{s-1,v}$ are the vertices of G_{s-1} located in the union of the open stars of u and v in K. Therefore a vertex x of G_s receives a label from l_s if and only if it represents a face of $\mathrm{Cl}(G_{s-1})$ which intersects that union of open stars. Such a vertex x itself lies in that union, therefore in $B_{s,u} \cup B_{s,v}$. To prove d) note that if x is a vertex of $B_{s,u} \cap S_{s,v}$ then x lies in the link $\mathrm{lk}_K v$, hence it represents a face of $\mathrm{Cl}(G_{s-1})$ contained in that link. By induction all vertices of that face are l_{s-1}-labeled 0 or unlabeled hence $l_s(x) = 0$ (since $x \in B_{s,u}$ it cannot remain unlabeled). This and a symmetric argument for $S_{s,u} \cap B_{s,v}$ proves d).

If $\tau \in \mathrm{Cl}(G_{s-1})$ is a simplex with $l_{s-1}(\tau) = \{a\}$ then every vertex $x \in G_s$ which subdivides a face of τ will receive l_s-label a or no label at all and therefore b), c) still hold for the simplices and edges of $\mathrm{Cl}(G_s)$ contained within τ. Now suppose that $l_{s-1}(\tau) = \{a, b\}$. Note that no simplex of $\mathrm{Cl}(G_s)$ subdividing τ contains vertices x, y with $l_s(x) = a$ and $l_s(y) = b$. Indeed, if τ_1 and τ_2 are the faces of τ in $\mathrm{Cl}(G_{s-1})$ represented by x and y respectively, then $\tau_1 \cap l_{s-1}^{-1}(a) \neq \emptyset$, $\tau_1 \cap l_{s-1}^{-1}(b) = \emptyset$ and vice versa for τ_2. But then neither $\tau_1 \subseteq \tau_2$ nor $\tau_2 \subseteq \tau_1$ hence xy is not an edge in $\mathrm{bd}(\tau)$. Eventually we conclude that for every simplex of $\mathrm{Cl}(G_s)$ subdividing τ the set of l_s-labels is either empty, or a singleton or one of $\{a, \frac{a+b}{2}\}$, $\{b, \frac{a+b}{2}\}$. This, together with the induction hypothesis, proves b) and c) in this case.

The existence of the partial labeling l_s completes the proof of the lemma:

every path from $B_{s,u} \cap S_{s,v}$ to $S_{s,u} \cap B_{s,v}$ in $G_s[B_{s,u} \cup B_{s,v}]$ passes through l_s-labeled vertices (by a)). In each step the label changes by at most $\frac{1}{2^s}$ (by c)) while the total change is 1 (by d)). It means that the path requires at least 2^s steps. Of course there exists a path (e.g. the subdivision of the edge uv) of length exactly 2^s.

□

For a convenient reference let us summarize the proof of Theorem 7.10.

Proof of Theorem 7.10. Fix $1 \le r < 2^{s-2}$. Consider the subcomplexes $X_{s,v}$ of $\mathrm{Cl}(G_s^r)$ defined in (7.3). By Proposition 7.12 they form a covering of $\mathrm{Cl}(G_s^r)$ with nerve K. By Proposition 7.13 every nonempty intersection of the X_{s,v_i} is of the form

$$\mathrm{Cl}((G_s[B_{s,v_1} \cap \cdots \cap B_{s,v_k}])^r).$$

Every such complex is contractible because Proposition 7.11 and Lemma 7.3 imply that the graph $(G_s[B_{s,v_1} \cap \cdots \cap B_{s,v_k}])^r$ is dismantlable. The equivalence $\mathrm{Cl}(G_s^r) \simeq K$ now follows from the nerve lemma [72, 15.21]. □

Remark 7.15. The purpose of [31] was to prove that for any complex K and any connected, non-discrete graph T there exists a graph G with a homotopy equivalence

$$\mathrm{Cl}(G^T) \simeq K$$

where $(-)^T$ denotes the *exponential graph* functor, the right adjoint to the categorical product $- \times T$ of graphs (see [72, 18.18]). The idea was to use the graph G_s and its subgraphs $G_s[B_{s,v}]$ to form a covering of G_s^T (for s depending on the diameter of T). Despite these similarities the author does not see a direct way to compare (up to homotopy) the complexes $\mathrm{Cl}(G^r)$ of distance graph powers with any of the complexes $\mathrm{Cl}(G^T)$.

7.5 Line graphs and edge subdivisions

Let $S(G)$ denote the graph obtained from G by subdividing every edge with one vertex. The graph $T(G) = S(G)^2$ is often called the *total graph* of G. Recall that the *line graph* $L(G)$ of G is the incidence graph of the edges of G.

Write $V(S(G))$ as $\mathcal{V} \cup \mathcal{E}$ where $\mathcal{V}$ is the set of original vertices of G and $\mathcal{E}$ is the set of subdividing vertices, one for each edge. Then we have isomorphisms

$$T(G)[\mathcal{V}] = G, \quad T(G)[\mathcal{E}] = L(G)$$

and we see that the inclusions

$$\mathrm{Cl}(L(G)) = \mathrm{Cl}(T(G))[\mathcal{E}] \hookrightarrow \mathrm{Cl}(T(G)) \hookleftarrow \mathrm{Cl}(T(G))[\mathcal{V}] = \mathrm{Cl}(G)$$

make $\mathrm{Cl}(T(G))$ a subcomplex of the join $\mathrm{Cl}(G) * \mathrm{Cl}(L(G))$.

Denote by $\mathrm{t}(G)$ the number of triangles in G. Then we have the following result.

Theorem 7.16. *For any graph G there is a homotopy equivalence*

$$\mathrm{Cl}(T(G)) \simeq \mathrm{Cl}(G) \vee \bigvee^{\mathrm{t}(G)} S^2.$$

This is another way in which a given complex can be represented as a clique complex of a graph square up to homotopy and up to a number of 2-spheres. As a byproduct of the proof method we also obtain the next result. Recall that $K^{(2)}$ denotes the 2-dimensional skeleton of K.

Theorem 7.17. *For any non-discrete, connected graph G*

$$\mathrm{Cl}(L(G)) \simeq \mathrm{Cl}(G)^{(2)}.$$

Both theorems depend on a simple classification.

Lemma 7.18. *Every maximal face in $\mathrm{Cl}(T(G))$ is of one of the following forms:*

a) a maximal face of dimension at least 2 in $\mathrm{Cl}(G)$,

b) $\{v, e, w\}$ where v, w are vertices of G and $e = vw$,

c) $\{v, e_1, \dots, e_k\}$ where e_i are the edges incident with a vertex v of G,

d) $\{e_1, e_2, e_3\}$ where e_1, e_2, e_3 are edges forming a triangle in G.

Proof. Let σ be a maximal face in $\mathrm{Cl}(T(G))$. If σ contains at least three vertices of $\mathcal{V}$ then those vertices form a clique in G and no edge is incident with all of them, so it cannot be extended by a vertex of $\mathcal{E}$. If σ contains precisely two vertices v, w of $\mathcal{V}$, then $e = vw$ is the only vertex of $\mathcal{E}$ adjacent to both of them. If $|\sigma \cap \mathcal{V}| = \{v\}$ then σ must be of the form c). Eventually if $\sigma = \{e_1, \dots, e_k\} \in \mathrm{Cl}(L(G))$ then not all of e_i are incident with a common vertex, but every two e_i, e_j have a common vertex. This easily implies $k = 3$. $\square$

Proof of Theorem 7.16. Consider the subcomplex $K \subseteq \mathrm{Cl}(T(G))$ consisting of all faces of $\mathrm{Cl}(T(G))$ which are not of the form $\{e_1, e_2, e_3\}$ for some three edges forming a triangle in G. Consider a matching on K defined as follows

- the faces of $\mathrm{Cl}(G)$ are unmatched,
- for each edge $e = uv$ the faces $\{e\}$, $\{v, e\}$, $\{e, u\}$ and $\{v, e, u\}$ are unmatched,
- for every face $\sigma \in K \cap \mathrm{Cl}(L(G))$ of dimension at least one there exists a unique vertex $v \in G$ such that $\sigma \cup \{v\}$ is a face of K (that vertex is the common end of the edges of σ). In such case match σ with $\sigma \cup \{v\}$.

This is clearly an acyclic matching on K in the sense of Definition 1.3. Its critical faces form the subcomplex

$$K' = \mathrm{Cl}(G) \cup \{\{e\}, \{v, e\}, \{e, u\}, \{v, e, u\} \text{ for } e = uv \in \mathcal{E}\}.$$

This K' easily collapses to $\mathrm{Cl}(G)$, therefore also K collapses to $\mathrm{Cl}(G)$.

Now $\mathrm{Cl}(T(G))$ arises from K by attaching the $\mathrm{t}(G)$ cells $\{e_1, e_2, e_3\}$ for all triangles $\{v_1, v_2, v_3\}$ of G. The attaching map of every such cell is homotopic in K to the boundary of the face $\{v_1, v_2, v_3\}$ of $\mathrm{Cl}(G)$, therefore it is null-homotopic. It follows that $\mathrm{Cl}(T(G)) \simeq K \vee \bigvee^{\mathrm{t}(G)} S^2 \simeq \mathrm{Cl}(G) \vee \bigvee^{\mathrm{t}(G)} S^2$. $\square$

Proof of Theorem 7.17. Let K be the subcomplex of $\mathrm{Cl}(T(G))$ consisting of all faces σ such that $|\sigma \cap \mathcal{V}| \leq 1$. Then K is the union of $\mathrm{Cl}(L(G))$ and simplices of the form 7.18.c) for every $v \in \mathcal{V}$. For each v the link $\mathrm{lk}_K(v) \subseteq \mathrm{Cl}(L(G))$ is contractible (because it is a simplex) hence the removal of v from K does not change the homotopy type. It means that $K \simeq \mathrm{Cl}(L(G))$.

Let $K' \subseteq K$ be obtained from K by removing the maximal faces $\{e_1, e_2, e_3\}$ corresponding to triangles of G. Then K' is collapsible to the graph $S(G)$ by an acyclic matching argument identical to that used in 7.16, pairing σ with $\sigma \cup \{v\}$ for any set σ of at least two elements of $\mathcal{E}$ and their common endpoint v. Note that $S(G)$ and G are homeomorphic as spaces.

Now K is recovered from K' by attaching a 2-face $\{e_1, e_2, e_3\}$ for every triangle t of G. The attaching map is homotopic in K' to the inclusion of $S(t)$ in $S(G)$. It follows that K is homotopy equivalent to G with a 2-cell attached along every triangle. This is precisely $\mathrm{Cl}(G)^{(2)}$. It follows that

$$\mathrm{Cl}(L(G)) \simeq K \simeq K' \cup \coprod^{t(G)} \Delta^2 \Big/_{\sim} \simeq \mathrm{Cl}(G)^{(2)}.$$

□

Example 7.19. The *stable Kneser graph* $SG_{n,k}$ is a graph whose vertices are the n-element subsets of $\{1, \dots, k+2n\}$ which do not contain two consecutive (in the cyclic sense) elements. One of the goals of [22] is to calculate the homotopy types of the independence complexes $I(SG_{2,k})$. Since the complex $I(SG_{2,k})$ is exactly $\mathrm{Cl}(L(\overline{C_{k+4}}))$, Theorem 7.17 identifies it, up to homotopy, with $I(C_{k+4})^{(2)}$. This explains why these space are homotopically at most two-dimensional, as stated in [22, Thm.1.4].

Remark 7.20. From the two theorems of this section we immediately recover the result of [75, Cor. 5.4], which is that the spaces $\mathrm{Cl}(G)$, $\mathrm{Cl}(T(G))$ and $\mathrm{Cl}(L(G))$ have isomorphic fundamental groups.

7.6 Clique complexes of powers of cycles

In this section we determine the homotopy types of the clique complexes of the graphs C_n^r, i.e. the powers of cycles. It follows from Proposition 7.2 that for $1 \leq r \leq \frac{n-1}{3}$ the complex $\mathrm{Cl}(C_n^r)$ collapses to $\mathrm{Cl}(C_n) \simeq S^1$. On the other hand, for $r \geq \lfloor\frac{n}{2}\rfloor$ the complex $\mathrm{Cl}(C_n^r) = \mathrm{Cl}(K_n)$ is contractible. The intermediate values for some small pairs n, r are shown in Section 7.7. The purpose of this section is to exhibit a systematic pattern in that table. It turns out to be best expressed in terms of the independence complexes of the complements of C_n^r. These results may also be interesting on their own right as one way of generalizing the calculation of Kozlov [71] of the homotopy types of $I(C_n)$.

For any pair of non-negative integers n, k *of opposite parity* and with $1 \leq k \leq n-1$ let $T_{n,k}$ denote the graph obtained by connecting every vertex of the regular n-gon with the k "most opposite" vertices. The notion of "most opposite" is well defined if n and k have opposite parity. For example, $T_{n,1}$ is the disjoint union of $\frac{n}{2}$ edges and examples of $T_{n,2}$ and $T_{n,3}$ are shown in Fig.7.4a and Fig.7.5a. To describe these graphs we are also going to use another parameter $r = r(n,k) = \frac{n-k-1}{2}$. Of course

$$T_{n,k} = \overline{C_n^r}.$$

The graphs $T_{n,k}$ are called *circular complete graphs* and form a subclass of *circulant graphs.* The usual notation for $T_{n,k} = \overline{C_n^r}$ is

$$K_{n/r+1} \quad \text{or} \quad C_n(r+1, \dots, \lfloor\frac{n}{2}\rfloor)$$

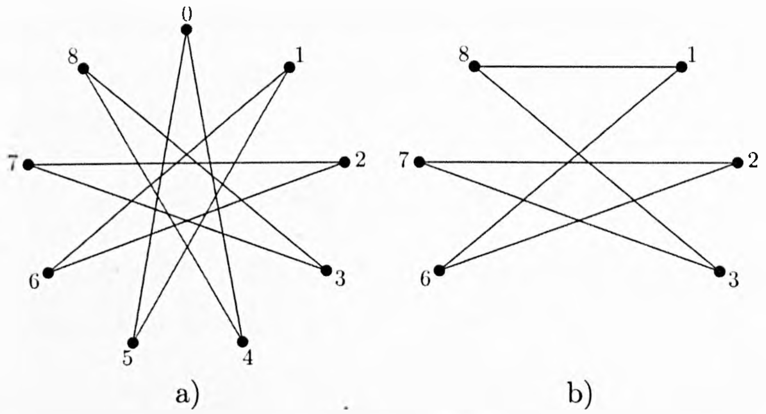


Figure 7.4: a) $T_{9,2} = \overline{C_9^3}$. b) $S_{9,2}$.

but we will keep using the notation $T_{n,k}$ which is more intuitive for this application. For information about circulant and circular complete graphs and their independent sets see e.g. [12, 24, 52, 87].

We will identify the vertices of $T_{n,k}$ with $\mathbb{Z}/n$. Under this identification each vertex i is connected to the vertices in the set

$$N_{T_{n,k}}(i) = \{i + r + 1, \dots, i + r + k\} \mod n. \tag{7.4}$$

We are also going to need another auxiliary graph $S_{n,k}$. It is the induced subgraph of $T_{n,k}$ on the vertex set

$$V = \{1, \dots, r\} \cup \{-1, \dots, -r\} \tag{7.5}$$

equipped *additionally* with the edges $(-i, j)$ for all pairs $i, j \in \{1, \dots, k-1\}$ such that $i + j \leq k$. For examples of $S_{n,k}$ see Fig.7.4b and Fig.7.5b.

The main results that lead to the calculation of $\mathrm{Cl}(C_n^r)$ are the following propositions. Recall that Σ denotes the unreduced suspension.

Proposition 7.21. *If $n \geq 3k - 1$ then*

$$I(T_{n,k}) \simeq \Sigma I(S_{n,k}).$$

Proposition 7.22. *If $n \geq 3k + 3$ then*

$$I(S_{n,k}) \simeq \Sigma I(T_{n-2(k+1),k}).$$

Example 7.23. Consider the special case $k = 2$. We have $T_{n,2} = C_n$ for every odd $n \geq 3$ and $S_{n,2} = C_{n-3}$ for every odd $n \geq 7$. The previous two propositions thus combine to the statement

$$I(C_m) \simeq \Sigma I(C_{m-3}) \qquad \text{for all } m \geq 6.$$

Moreover $I(C_3) \equiv S^0 \vee S^0$, $I(C_4) \simeq S^0$ and $I(C_5) = \mathrm{Cl}(\overline{C_5}) = \mathrm{Cl}(C_5) \simeq S^1$ so it follows by induction that for all $m \geq 1$

$$I(C_{3m}) \simeq S^{m-1} \vee S^{m-1}, \quad I(C_{3m+1}) \simeq S^{m-1}, \quad I(C_{3m+2}) \simeq S^m.$$

This was first established by Kozlov [71] and then reproved in a number of ways.

Corollary 7.24. *If* $n \geq 3k+3$ *then*

$$I(T_{n,k}) \simeq \Sigma^2 I(T_{n-2(k+1),k}).$$

Corollary 7.25. *For any* $1 \leq k \leq n-1$*, with* k *and* n *of opposite parity, we have*

$$I(T_{n,k}) \simeq \begin{cases} \bigvee^k S^{2l} & \text{if } n = (2l+1)(k+1) \\ S^{2l+1} & \text{if } (2l+1) < \frac{n}{k+1} < (2l+3) \end{cases} \quad \text{for some } l \geq 0.$$

Proof. First we establish the result when $k+1 \leq n \leq 3k+2$. If $n = k+1$ then

$$I(T_{k+1,k}) = I(K_{k+1}) \equiv \bigvee^k S^0.$$

Now suppose that $k+2 \leq n \leq 3k+2$. These inequalities imply that $r \geq 1$, $3r+1 \leq n$ and $1 < \frac{n}{k+1} < 3$ so $l = 0$. Since $n \geq 4$ by Proposition 7.2 we get

$$I(T_{n,k}) = \mathrm{Cl}(C_n^r) \simeq S^1 = S^{2l+1}$$

as required. For $n \geq 3k+3$ the result follows by induction using Corollary 7.24 because every increase of n by $2(k+1)$ adds a double suspension to the homotopy type. □

These results can be transformed into statements about $\mathrm{Cl}(C_n^r)$ by a straightforward calculation. Corollary 7.24 translates into:

Corollary 7.26. *For any* $\frac{n}{3} \leq r < \frac{n}{2}$

$$\mathrm{Cl}(C_n^r) \simeq \Sigma^2 \mathrm{Cl}(C_{4r-n}^{3r-n}) = \Sigma^2 \mathrm{Cl}(C_{n-2\cdot(n-2r)}^{r-1\cdot(n-2r)}).$$

It follows that in the (n,r)-chart of the complexes $\mathrm{Cl}(C_n^r)$ (see Section 7.7) the double suspension operator Σ^2 acts always along the lines of slope $(2,1)$. The translation of Corollary 7.25 is:

Corollary 7.27. *For any $n \geq 3$ and $0 \leq r < \frac{n}{2}$ we have*

$$\mathrm{Cl}(C_n^r) \simeq \begin{cases} \bigvee^{n-2r-1} S^{2l} & \text{if } r = \frac{l}{2l+1}n \\ S^{2l+1} & \text{if } \frac{l}{2l+1}n < r < \frac{l+1}{2l+3}n \end{cases} \quad \text{for some } l \geq 0.$$

Remark 7.28. The relevant value of l for each pair (n,r) is given by

$$l = \lfloor \frac{r}{n-2r} \rfloor$$

It remains to prove Propositions 7.21 and 7.22. Our tool to analyze the homotopy types of $I(T_{n,k})$ and $I(S_{n,k})$ are the *star clusters* introduced by J.Barmak [13]. Let us recall the main result of that work.

Theorem 7.29 (Barmak, [13]). *Suppose v is a non-isolated vertex of G which does not belong to any triangle. Let K be the subcomplex of $I(G)$ defined as*

$$K = \mathrm{st}(v) \cap \bigcup_{w \in N_G(v)} \mathrm{st}(w) \tag{7.6}$$

where all stars are taken in $I(G)$. Then there is a homotopy equivalence $I(G) \simeq \Sigma K$.

In the proofs of Propositions 7.21 and 7.22 we are going to choose a vertex v as in the theorem and identify the subcomplex K with the independence complex of some graph using the following technical lemma.

Lemma 7.30. *Let $v_1, v_2, \ldots, v_{2d}$ be a sequence of (not necessarily distinct) vertices of G such that every consecutive $d+1$ vertices $v_i, v_{i+1}, \ldots, v_{i+d}$ are pairwise distinct for $i = 1, \ldots, d$.*

Let L be the subcomplex of $I(G)$ consisting of those faces σ which satisfy the condition

$$\{v_s, v_{s+1}, \ldots, v_{s+d-1}\} \cap \sigma = \emptyset \text{ for some } s \in \{1, 2, \ldots, d+1\}. \tag{7.7}$$

Then L is isomorphic with the complex $I(H)$, where H is a graph obtained from G by adding the edges

$$(v_i, v_j) \text{ for all } 1 \leq i \leq d,\ d+1 \leq j \leq 2d, \text{ such that } j - i \leq d.$$

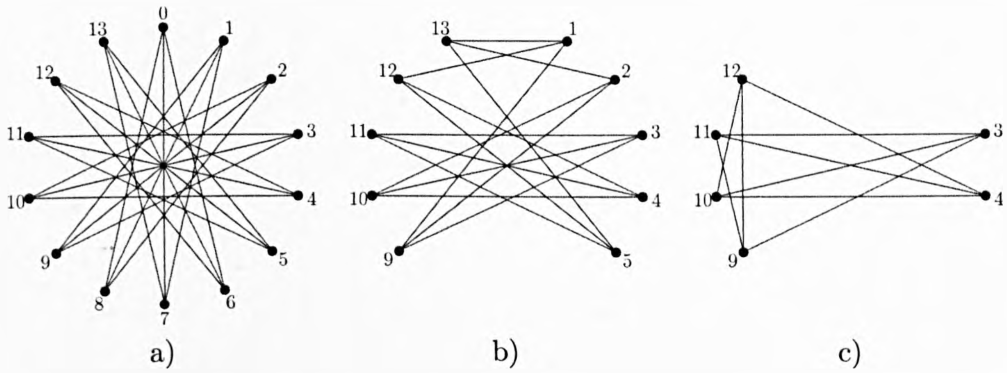


Figure 7.5: a) $T_{14,3} = \overline{C_{14}^5}$. b) $S_{14,3}$. c) A graph H obtained in the proof of Prop.7.22, isomorphic to $T_{6,3}$.

Proof. Let σ be a face of $I(G)$ which satisfies (7.7) for some s. Since every pair (v_i, v_j) with $1 \le i \le d$, $d+1 \le j \le 2d$ and $j - i \le d$ has at least one of its elements in $\{v_s, v_{s+1}, \dots, v_{s+d-1}\}$, the face σ cannot contain both elements v_i and v_j simultaneously. It means that σ determines an independent set in H. Conversely, if σ is a face of $I(H)$ then define s by the formula

$$s = 1 + \max\{1 \le i \le d : v_i \in \sigma\} \tag{7.8}$$

(where $\max \emptyset = 0$). One easily checks that s and σ satisfy (7.7). $\square$

Proof of Proposition 7.21. First note that the assumption $n \ge 3k - 1$ is equivalent with $r \ge k - 1$. Consider the vertex 0 of $T_{n,k}$. Its neighbours in $T_{n,k}$ are the vertices of

$$N(0) = \{r + 1, r + 2, \dots, r + k\}.$$

No two of those vertices are adjacent because their distances along the circle are at most $k - 1 \le r$, so 0 is not in any triangle. By Theorem 7.29 $I(T_{n,k}) \simeq \Sigma K$ where K is the subcomplex of $I(T_{n,k})$ given by

$$K = \mathrm{st}(0) \cap \bigcup_{w \in N(0)} \mathrm{st}(w).$$

Note that K is in fact a subcomplex of $I(T_{n,k}[V])$, where $V = \{1, \dots, r\} \cup \{-1, \dots, -r\}$ is the set of vertices non-adjacent to 0 in $T_{n,k}$. The complex K consists precisely of those independent sets σ in $T_{n,k}[V]$ for which there exists a vertex $w \in N(0)$ such that $\sigma \cup \{w\}$ is an independent set in $T_{n,k}$ or, in other words, such that $\sigma \cap N(w) = \emptyset$.

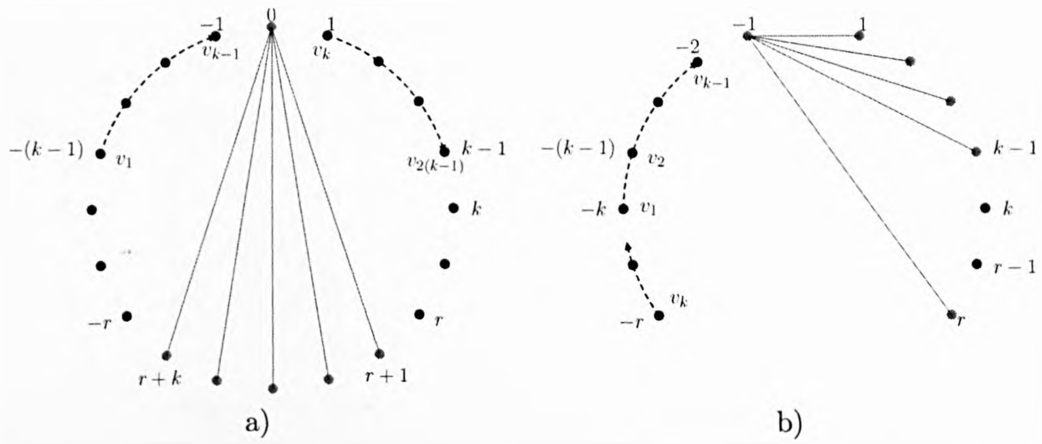


Figure 7.6: Schematics for the proofs of 7.21 and 7.22. The shaded vertices are in $N_G[v]$ and will be removed. Additional edges will be added between vertices marked with dashed lines.

If $w = r + j$ for $1 \leq j \leq k$ then (see also Fig.7.6a)

$$N(r+j) \cap V = \{-(k-j), \dots, -1\} \cup \{1, \dots, j-1\}.$$

It follows that σ is a face of K if and only if it is an independent set of $T_{n,k}[V]$ such that

$$\{-(k-j), \dots, -1, 1, \dots, j-1\} \cap \sigma = \emptyset \text{ for some } j \in \{1, 2, \dots, k\}.$$

We can now apply Lemma 7.30 with $d = k-1$, $G = T_{n,k}[V]$ and $(v_1, \dots, v_{2d}) = (-(k-1), \dots, -1, 1, \dots, k-1)$, where all the vertices in the last sequence are distinct. The graph H obtained in the lemma is $S_{n,k}$ because the additional edges are precisely $(-(k-1), 1)$, $(-(k-2), 1)$, $(-(k-2), 2)$, etc., as in the definition of $S_{n,k}$. Therefore $I(T_{n,k}) \simeq \Sigma K = \Sigma I(H) = \Sigma I(S_{n,k})$. $\square$

Proof of Proposition 7.22. The assumption $n \geq 3k+3$ is equivalent with $r \geq k+1$. We apply the same strategy as before with respect to the vertex (-1). Its neighbours in $S_{n,k}$ are

$$N(-1) = \{1, \dots, k-1\} \cup \{r\}.$$

No two of these vertices are adjacent, so (-1) is not in any triangle. Exactly as before we obtain that $I(S_{n,k}) \simeq \Sigma K$ where K is a subcomplex of $I(S_{n,k}[V])$ where

$$V = \{k, \dots, r-1\} \cup \{-r, \dots, -2\}$$

(see Fig.7.6b) is the set of vertices of $S_{n,k}$ non-adjacent to (-1). Note that both sets in the above union are nonempty. The complex K consists of those faces σ of $I(S_{n,k}[V])$ for which there exists a $w \in N(-1)$ such that $\sigma \cap N(w) = \emptyset$. Note that

$$\begin{aligned}
N(r) \cap V &= \{-k, \ldots, -2\} \\
N(1) \cap V &= \{-(k-1), \ldots, -2\} \cup \{-r\} \\
N(2) \cap V &= \{-(k-2), \ldots, -2\} \cup \{-r, -r+1\} \\
&\cdots \\
N(k-1) \cap V &= \{-r, -r+1, \ldots, -r+(k-2)\}.
\end{aligned}$$

In the sequence

$$S = (-k, \ldots, -2, -r, \ldots, -r+(k-2))$$

of length $2(k-1)$ every k consecutive vertices are pairwise distinct. Because of the cyclic behaviour it is enough to check this for the subsequence $(-k, \ldots, -2, -r)$, where it boils down to the inequality $-r < -k$ which follows from $r \geq k+1$.

By Lemma 7.30 the complex K is therefore homotopy equivalent to $I(H)$, where H arises from $S_{n,k}[V]$ by adding the edges $(-(k-i), -r+j)$ for all $0 \leq j \leq i \leq k-2$. It remains to identify this graph H with $T_{n-2(k+1),k}$. This can be best seen geometrically (cf. Fig.7.5c, Fig.7.6b). The graph H differs from $T_{n,k}$ by the removal of $2(k+1)$ vertices $\{-1, 0, \ldots, k-1\} \cup \{r, r+1, \ldots, r+k\}$ which form two "gaps" of length $k+1$ each. Note that the vertices not in S are not affected at all by the construction, so their neighbourhoods in $T_{n,k}$ and H coincide. The vertices in S are located at most $k-1$ steps from the boundaries of the gaps and for them the missing connections are provided by the extra edges in H, so that the neighbours of each vertex of S form a contiguous block of length k in the cyclic ordering of vertices in H inherited from $T_{n,k}$ and we again have a circular complete graph $T_{n-2(k+1),k}$. This identification completes the proof of the proposition: $I(S_{n,k}) \simeq \Sigma K \equiv \Sigma I(H) \equiv \Sigma I(T_{n-2(k+1),k})$. □

7.7 Appendix: Examples of clique complexes of cycle powers

The table presents the homotopy types of some initial clique complexes $\mathrm{Cl}(C_n^r)$. The entries below the shaded area are all S^1 by Proposition 7.2 and the entries above it are all $*$ (a contractible space). The arrows show the action of the double suspension operator Σ^2 of Corollary 7.26.

$r=$	0	1	2	3	4	5	6	7	8
C_3	$\bigvee^2 S^0$	$*$	$*$	$*$	$*$				
C_4	$\bigvee^3 S^0$	S^1	$*$	$*$	$*$				
C_5	$\bigvee^4 S^0$	S^1	$*$	$*$	$*$				
C_6	$\bigvee^5 S^0$	S^1	S^2	$*$	$*$	$\cdots$			
C_7	$\bigvee^6 S^0$	S^1	S^1	$*$	$*$				
C_8	$\bigvee^7 S^0$	S^1	S^1	S^3	$*$				
C_9	$\bigvee^8 S^0$	S^1	S^1	$\bigvee^2 S^2$	$*$				
C_{10}	$\bigvee^9 S^0$	S^1	S^1	S^1	S^4				
C_{11}	$\bigvee^{10} S^0$				S^3				
C_{12}	$\bigvee^{11} S^0$		$\cdots$		$\bigvee^3 S^2$	S^5			
C_{13}	$\bigvee^{12} S^0$					S^3			
C_{14}	$\bigvee^{13} S^0$					S^3	S^6		
C_{15}	$\bigvee^{14} S^0$					$\bigvee^4 S^2$	$\bigvee^2 S^4$		
C_{16}	$\bigvee^{15} S^0$						S^3	S^7	
C_{17}	$\bigvee^{16} S^0$						S^3	S^5	
C_{18}	$\bigvee^{17} S^0$						$\bigvee^5 S^2$	S^3	S^8
C_{19}	$\bigvee^{18} S^0$							S^3	S^5
C_{20}	$\bigvee^{19} S^0$							S^3	$\bigvee^3 S^4$
C_{21}	$\bigvee^{20} S^0$							$\bigvee^6 S^2$	S^3
C_{22}	$\bigvee^{21} S^0$								S^3
C_{23}	$\bigvee^{22} S^0$								S^3

Chapter 8

Dense flag triangulations of 3-manifolds via extremal graph theory

8.1 Introduction

One of the trends in enumerative combinatorics is to classify face numbers of various families of simplicial complexes. In this chapter we study flag triangulations of closed 3-manifolds with sufficiently many vertices and high edge density. As a consequence we confirm, for sufficiently large number of vertices, a conjecture of Gal regarding face vectors of flag triangulations of generalized homology 3-spheres.

If K is a finite simplicial complex and $\sigma \in K$ is a face we denote by $|\sigma|$ its number of vertices and by $\dim \sigma = |\sigma| - 1$ its dimension. The dimension of K, $\dim K$, is the maximum over all $\sigma \in K$ of $\dim \sigma$.

The *f-vector* of a simplicial complex K of dimension d is the sequence

$$(f_{-1}, f_0, \dots, f_d) \tag{8.1}$$

where f_i is the number of faces of dimension i. Always $f_{-1} = 1$. The *h-vector* of K is the sequence

$$(h_0, \dots, h_{d+1}) \tag{8.2}$$

determined by the equation

$$\sum_{i=0}^{d+1} h_i x^{d+1-i} = \sum_{i=-1}^{d} f_i (x-1)^{d-i}. \tag{8.3}$$

(Note that we consistently use d for the dimension of K, rather than the cardinality of its largest face, hence the indices and exponents in most formulae are shifted by 1 compared to what the usually look like). Of course the f-vector and the h-vector determine one another and carry the same information, but the h-vector often enjoys better combinatorial properties; the Dehn-Sommerville equation (8.4) below being one example. Always $h_0 = 1$.

Next we introduce the class of Gorenstein* and Eulerian complexes. The reader not interested in this level of generality can equally well think about simplicial complexes which triangulate a standard sphere

A simplicial complex K of dimension d is a *generalized homology sphere* (or *Gorenstein* complex*) if for every face $\sigma \in K$ the homology of $\mathrm{lk}_K\sigma$ is the same as the homology of a sphere of dimension $d - |\sigma|$. In particular, when $\sigma = \emptyset$, this means that K itself has the homology of a d-sphere. We are going to use the short name 'd-GHS'. A simplicial complex K of dimension d is *Eulerian* if for every face $\sigma \in K$ the Euler characteristic of $\mathrm{lk}_K\sigma$ is the same as of a sphere of dimension $d - |\sigma|$.

Both conditions on the links are relaxations of the conditions that hold within an actual triangulated manifold, in which case the links are homeomorphic to the appropriate spheres. Any triangulation of the standard d-sphere is a d-GHS and every d-GHS is Eulerian. By Poincaré duality the Euler characteristic of an odd-dimensional closed manifold is 0, hence every such manifold is Eulerian. (A closed manifold means a compact manifold without boundary).

Any Eulerian complex of dimension d satisfies the classical *Dehn-Sommerville equations*

$$h_i = h_{d+1-i} \tag{8.4}$$

and, following Gal [44], one can encode the coefficients h_i in a shorter, integer-valued γ-*vector*

$$(\gamma_0, \ldots, \gamma_{\lfloor \frac{d+1}{2} \rfloor}) \tag{8.5}$$

determined by the equation

$$\sum_{i=0}^{d+1} h_i x^i = \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \gamma_i x^i (x+1)^{d+1-2i}. \tag{8.6}$$

We always have $\gamma_0 = 1$.

The classification of h- (or f-, γ-) vectors of simplicial spheres and generalized homology spheres is predicted by the celebrated g-conjecture of McMullen [82]. In this work we pick up a research line started by Gal, who investigated these invariants for the restricted family of flag complexes.

For flag generalized homology spheres the γ-vector is the most efficient and interesting parameter. The major conjecture of Gal [44, Conj. 2.1.7], which states that the γ-vector of a flag d-GHS is non-negative, is known to hold for $d \leq 4$ [44, Cor.2.2.3]. For any flag $(2d-1)$-GHS this conjecture is a strengthening of the famous Charney-Davis conjecture [25]. On the other hand, Gal's conjecture itself has a stronger version which states that the γ-vector of a flag d-GHS is an f-vector of some flag complex [88]. See [89] and references therein for progress in that area.

Following Murai and Nevo [86], let Λ_d denote the set of all γ-vectors of flag d-GHSs. When $d = 1, 2$ then the $(k+4)$-gon or its join with the two-point sphere S^0 are simplicial d-spheres with γ-vector $(1, k)$ for any integer $k \geq 0$, and by the previous discussion these exhaust Λ_1 and Λ_2, i.e., we have

$$\Lambda_1 = \Lambda_2 = \{(1, k) \in \mathbb{Z}^2 \ : \ k \geq 0\} \ .$$

Gal [44, Conj. 3.1.7] proved that $\gamma_2 \leq \gamma_1^2/4$ must hold for any γ-vector $(1, \gamma_1, \gamma_2)$ in Λ_3 or Λ_4 and a simple join construction [86, Thm. 5.1.ii] shows that this is tight in dimension 4, that is

$$\Lambda_4 \ = \ \left\{(1, \gamma_1, \gamma_2) \in \mathbb{Z}^3 \ : \ \gamma_2 \leq \frac{\gamma_1^2}{4}, \quad \gamma_1, \gamma_2 \geq 0\right\}.$$

Going back to dimension 3, Gal [44, Thm. 3.2.1] showed that

$$\begin{aligned} \Lambda_3 \ \supseteq \ &\left\{(1, \gamma_1, \gamma_2) \in \mathbb{Z}^3 \ : \ \gamma_2 \leq \frac{(\gamma_1 - 1)^2}{4}, \quad \gamma_1, \gamma_2 \geq 0\right\} \ \cup \\ &\cup \ \{(1, k+l, kl) \in \mathbb{Z}^3 \ : \ k, l \geq 0\}. \end{aligned} \tag{8.7}$$

The elements of the first set can be realized as γ-vectors of some appropriate iterated edge subdivisions of the boundary of the cross-polytope. The elements of the second kind are the γ-vectors of a join of a $(k+4)$-gon with an $(l+4)$-gon.

Gal then conjectures that the inclusion (8.7) is in fact an equality. Since the γ-vector of a flag 3-GHS is non-negative, the stronger version of that conjecture is the following (see [44, Con. 3.2.2] or [86, Conj. 5.2]).

Conjecture 8.1. *If $(1, \gamma_1, \gamma_2)$ is the γ-vector of a flag 3-GHS K and $\gamma_2 > \frac{(\gamma_1-1)^2}{4}$ then K is a join of two polygons.*

Also, note that the two constructions which show the inclusion (8.7) are flag triangulations of the 3-sphere. Thus – if true – Conjecture 8.1 provides a characterization of γ-vectors (or f-vectors) of flag triangulations of the 3-sphere. Even this special case is open. The conjecture was verified for order complexes of posets [86].

To make the following discussion more concrete, suppose that K is an Eulerian complex of dimension 3 with face numbers $(1, f_0, f_1, f_2, f_3)$. Then the Dehn-Sommerville relations translate into

$$f_2 = 2(f_1 - f_0), \quad f_3 = f_1 - f_0. \tag{8.8}$$

Moreover, we find

$$\gamma_1 = f_0 - 8, \quad \gamma_2 = f_1 - 5f_0 + 16 \tag{8.9}$$

and the conditions $(\gamma_1 - 1)^2/4 < \gamma_2 \leq \gamma_1^2/4$ are equivalent to

$$\frac{1}{4}(f_0^2 + 2f_0 + 17) < f_1 \leq \frac{1}{4}f_0^2 + f_0. \tag{8.10}$$

Our results

Below is the main result of the chapter. It determines the structure of closed flag 3-manifolds which have many edges.

Theorem 8.2. *There exists a number n_0 such that the following holds. If M is a closed flag 3-manifold with $f_0 \geq n_0$ vertices, f_1 edges, and such that*

$$f_1 > \frac{1}{4}(f_0^2 + 2f_0 + 17)$$

then M is a join of two polygons (and, in particular, it is homeomorphic to S^3).

Theorem 8.2 resolves Conjecture 8.1 in positive for large enough complexes because every 3-GHS is a closed manifold (see Remark 8.9). In other words, the inclusion (8.7) is an equality up to, perhaps, a finite number of elements.

Below, we prepare tools for our proof of Theorem 8.2. We shall reduce Theorem 8.2 to a certain statement in extremal graph theory (Theorem 8.6).

In order to make the notation more transparent, in this chapter we write N_v, instead of the usual $N(v)$, for the open neighborhood of a vertex v in a graph.

Definition 8.3. *If G is a graph and σ is a clique in G then define the* link of σ *in G as*

$$\mathrm{lk}_G\sigma = G\left[\bigcap_{v\in\sigma} N_v\right].$$

That is, $\mathrm{lk}_G\sigma$ is the subgraph of G induced by the vertices which are not in σ, but are adjacent to every vertex of σ.

Definition 8.3 is designed so that it is compatible with the topological notion of link in flag complexes. For each flag complex K we have $\mathrm{lk}_{K^{(1)}}\sigma = (\mathrm{lk}_K\sigma)^{(1)}$, where on the left-hand side we use the link of Definition 8.3 and on the right-hand side the link is understood in the simplicial sense.

Let us define the class of graphs which arise in our setting.

Definition 8.4. *A graph G with n vertices and m edges is* fascinating *if it satisfies the following conditions*

a) G contains exactly $2(m-n)$ triangles.

b) For every edge e in G the link $\mathrm{lk}_G e$ is a cycle of length at least 4.

c) For every triangle t in G the link $\mathrm{lk}_G t$ is the discrete graph with 2 vertices and no edges.

d) For every vertex v in G the link $\mathrm{lk}_G v$ is a connected, planar graph whose every face (including the unbounded one) is a triangle. In particular – by Kuratowski's Theorem – it does not contain the complete bipartite graph $K_{3,3}$ as a subgraph.

Further, $\mathrm{lk}_G v$ contains at least 6 vertices.

Our reduction is based on the next observation.

Lemma 8.5. *If M is a closed flag 3-manifold then the 1-skeleton of M is fascinating.*

Proof. Let $G = M^{(1)}$. Condition a) follows since M is Eulerian, and so it satisfies (8.8). Parts b)-d) are consequences of the fact that $\mathrm{lk}_M t$, $\mathrm{lk}_M e$, $\mathrm{lk}_M v$ are flag triangulations of, respectively, S^0, S^1 and S^2. The last statement in d) follows from the known fact that a flag triangulation of S^j requires at least $2(j+1)$ vertices.

□

The *graph join* of graphs G and H, which we will denote $G * H$, is the disjoint union of G and H together with all the edges between $V(G)$ and $V(H)$. For any simplicial complexes K and L we have $(K * L)^{(1)} = K^{(1)} * L^{(1)}$, where on the left-hand side we use the simplicial join.

Lemma 8.5 now means that Theorem 8.2 is a consequence of the following result.

Theorem 8.6. *There exists a number n_0 such that the following holds. Suppose G is a fascinating graph with $n \geq n_0$ vertices, m edges and $m > \frac{1}{4}(n^2+2n+17)$. Then G is a join of two cycles.*

The rest of the chapter is concerned with the proof of this theorem. The strategy is outlined in the next section.

Remark 8.7. Along the way we will also see that the result is tight in the following sense: There exists flag 3-spheres with arbitrarily large f_0 and with exactly

$$f_1 = \frac{1}{4}(f_0^2 + 2f_0 + 17)$$

edges which are not a join of two cycles. Moreover, we will classify those boundary cases: Any fascinating graph G with $n \geq n_0$ vertices and exactly $m = \frac{1}{4}(n^2+2n+17)$ edges is one of the graphs in Figure 8.2.

Remark 8.8. Theorem 8.2 implies that for $f_0 \geq n_0$ every closed flag 3-manifold satisfies $f_1 \leq \frac{1}{4}f_0^2 + f_0$ (or, equivalently, $\gamma_2 \leq \frac{1}{4}\gamma_1^2$). This result in fact holds for *all* values of f_0 by the same proof that works for 3-GHSs in [44].

Remark 8.9. In dimensions $d = 0, 1, 2$ the classes of (flag) d-spheres and d-GHS coincide and in dimension $d = 3$ every 3-GHS is a closed, connected manifold. To see this, first note that it is an easy consequence of the definition that if L is a d-GHS and $\sigma \in L$ then $\mathrm{lk}_L\sigma$ is a $(d-|\sigma|)$-GHS. Now the only 0-complex with the homology of S^0 is S^0 itself. In a 1-GHS all vertex links are the two-point space, so a 1-GHS is a disjoint union of polygons, of which only a single polygon has the homology of S^1. In a 2-GHS the link of every vertex is an S^1, so a 2-GHS is a closed 2-manifold, and the classification of 2-manifolds shows only S^2 has the correct homology. Finally in a 3-GHS all vertex links are S^2, so a 3-GHS is a closed manifold.

8.2 Proof of Theorem 8.6

The main idea behind our approach is that G has a lot of edges (more than $n^2/4$), but relatively few triangles – just $\Theta(n^2)$. Graphs with this edge density must have many more triangles, namely $\Theta(n^3)$, unless they look very "similar", in some sense, to the complete bipartite graph $K_{n/2,n/2}$. This phenomenon is called *supersaturation* and is one of the basic principles of extremal (hyper)graph theory with fundamental applications to areas like additive combinatorics or property testing in computer science. In our setting the additional properties of G coming from Definition 8.4 can be used to refine the similarity to $K_{n/2,n/2}$ to determine the structure of G

exactly. This is a relatively standard approach in Extremal Graph Theory, called the *Stability method*, and introduced by Simonovits [94]. However, our proof is somewhat more complex than most of the applications of the Stability method to problems in extremal graph theory. Indeed, in these problems one usually tries to determine exactly the structure of a unique extremal graph while here we are dealing with joins of two cycles, i.e., graphs with somewhat looser structure.

Here is a more detailed outline of the proof. Mantel's Theorem (which is a special case of Turán's Theorem) asserts that the complete balanced bipartite graph $K_{\lfloor h/2\rfloor,\lceil h/2\rceil}$ is the unique maximizer of the number edges among all triangle-free graphs on h vertices. Note that this graph has $\lfloor h^2/4\rfloor$ edges. The graph $K_{\lfloor h/2\rfloor.\lceil h/2\rceil}$ is *stable* for this extremal problem in the following sense: if H is a graph on the same vertex with at least $h^2/4$ edges and contains only $o(h^3)$ triangles, it must be "very similar" (the precise meaning appears in Theorem 8.11) to $K_{\lfloor h/2\rfloor.\lceil h/2\rceil}$. These conditions are satisfied for the fascinating graph G of Theorem 8.6. By exploiting other properties of G we will be able to show that G is close to being a join of two cycles in the sense of the next definition.

Definition 8.10. *A fascinating graph G is called t-*joinlike *if there is a partition $V(G) = C_1 \sqcup C_2 \sqcup X$ where*

- *the graphs $G[C_i]$ are cycles,*
- *there are edges $e_i \in G[C_i]$ such that $\mathrm{lk}_G e_i = G[C_{3-i}]$,*
- *$|X| = t$.*

The vertices of X are called exceptional.

Note that a 0-joinlike graph is a join of two cycles. At the end of this Section we will establish that G must be t-joinlike for $t = 0$, 1 or 2 with some extra conditions satisfied by the exceptional vertices.

Observe that the balanced join of two cycles of lengths $\approx \frac{n}{2}$ has $\approx \frac{n^2}{4} + n$ edges (and joins of cycles of unbalanced lengths have even less edges), so our graph G is only allowed to "lose" $\approx \frac{n}{2}$ edges with respect to that number before it violates the bound of Theorem 8.6. In many cases, however, we will be able to show that a 2-joinlike graph loses a lot more just by counting the edges missing in the sparse planar links of exceptional vertices (Definition 8.4d)).

This leaves us with just a handful of possible scenarios considered in Section 8.4. Those are the difficult ones, in the sense that the graphs G approach, and in fact even reach, the bound $m = \frac{1}{4}(n^2 + 2n + 17)$. That means we can no longer use rough estimates. We then have to examine the structure of G more closely. This is the part where the examples advertised in Remark 8.7 show up.

Let $e(H) = |E(H)|$ and we write $e(H[A,B])$, (resp. $\overline{e}(H[A,B])$) for the number of edges (resp. non-edges) crossing between two disjoint vertex sets $A, B \subseteq V(H)$.

Let us now state the Supersaturation Theorem of Erdős and Simonovits [38], tailored to our needs. As said above, this version of the Supersaturation Theorem gives an approximate structure in graphs with edge density at least $\frac{1}{2}$ which contain subcubically many triangles in the order of the graph.

Theorem 8.11. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. Let H be an h-vertex graph with at least $h^2/4$ edges, containing at most δh^3 triangles. Then there exists a partition $V(H) = A_1 \sqcup A_2$, with $\big||A_1| - |A_2|\big| \leq 1$, such that*

$$e(H[A_1]) + e(H[A_2]) + \overline{e}(H[A_1, A_2]) \leq \varepsilon h^2 . \tag{8.11}$$

Let $0 < \gamma \ll 1$, $\alpha < \gamma/1000$ and $\varepsilon < \alpha\gamma$ be fixed. Let δ be given by Theorem 8.11 for input parameter ε. Let n_0 be sufficiently large. Suppose that G is the graph as in Theorem 8.6. Definition 8.4a) gives us that G has $2(e(G) - n) < n^2 < \delta n^3$ triangles. Therefore, Theorem 8.11 applies with parameters δ and ε. Let $A_1 \sqcup A_2$ be the partition of $V(G)$ from Theorem 8.11.

Let us fix additional notation. Given a vertex v and a set of vertices X we write

$$\deg(v, X) = |N_v \cap X|.$$

Define the following vertex sets for $i = 1, 2$:

$$\begin{aligned}
B_i &= \{v \in A_i : \deg(v, A_{3-i}) \geq \frac{n}{2} - \gamma n\}, \\
W_i &= \{v \in A_i \setminus B_i : \deg(v, B_i) \geq \frac{n}{2} - \gamma n\}, \\
X_i &= (A_i \setminus B_i) \setminus W_i.
\end{aligned}$$

Claim 8.12. *We have $|A_i \setminus B_i| \leq \alpha n$ for $i = 1, 2$. In particular $|W_i|, |X_i| < \alpha n$ and $|B_i| \geq \frac{n}{2} - \alpha n$.*

Proof. By definition every vertex of $A_i \setminus B_i$ has at least γn non-edges to A_{3-i}. If we had $|A_i \setminus B_i| > \alpha n$ then

$$\overline{e}(G[A_1, A_2]) \geq |A_i \setminus B_i| \cdot \gamma n \geq \alpha\gamma n^2 > \varepsilon n^2$$

contrary to the choice of A_1 and A_2. $\square$

Now define the partition $V(G) = S_1 \sqcup S_2 \sqcup X$ as follows

$$\begin{aligned} S_i &= B_i \cup W_{3-i}, \\ X &= X_1 \cup X_2. \end{aligned}$$

Observe that $\frac{n}{2} - \alpha n \le |S_i| \le \frac{n}{2} + \alpha n$ and $|X| \le 2\alpha n$. Denote $x = |X|$.

Claim 8.13. *For $i = 1, 2$ and for every vertex $v \in S_i$ we have* $\deg(v, S_{3-i}) \ge \frac{n}{2} - 2\gamma n$.

Proof. If $v \in B_i$ then v has at least $\frac{n}{2} - \gamma n$ neighbors in A_{3-i} and by Claim 8.12 at least $\frac{n}{2} - 2\gamma n$ of them hit B_{3-i}. If $v \in W_{3-i}$ then v has at least $\frac{n}{2} - \gamma n$ neighbors in B_{3-i}. $\square$

Claim 8.14. *For $i = 1, 2$ and for every vertex $v \in S_i$ we have* $\deg(v, S_i) \le 2$. *Consequently, $e(G[S_1]) + e(G[S_2]) \le n$. Moreover, $G[S_i]$ is triangle-free.*

Proof. Suppose a vertex $v \in S_i$ has three neighbors $u_1, u_2, u_3 \in S_i$. By Claim 8.13 we have

$$|N_v \cap N_{u_1} \cap N_{u_2} \cap N_{u_3} \cap S_{3-i}| \ge \frac{n}{2} - 13\gamma n \ge 3\,.$$

This implies that $\mathrm{lk}_G v$ contains a copy of $K_{3,3}$ (with u_1, u_2, u_3 on one side and the other being in S_{3-i}), which is a contradiction.

The proof of the last statement is similar: if t is a triangle in $G[S_i]$ then $\mathrm{lk}_G t$ contains most of S_{3-i}, so G fails Definition 8.4c). $\square$

Claim 8.15. *If $v \in X$ then* $\deg(v, S_i) \le \frac{n}{2} - \frac{2}{3}\gamma n$ *for $i = 1, 2$.*

Proof. By definition every vertex $v \in X$ satisfies $\deg(v, B_i) \le \frac{n}{2} - \gamma n$ for $i = 1, 2$. Therefore

$$\deg(v, S_i) \le \deg(v, B_i) + |W_{3-i}| \le \frac{n}{2} - \gamma n + \alpha n \le \frac{n}{2} - \frac{2}{3}\gamma n.$$

$\square$

We call a vertex $v \in X$ *poor* if $\deg(v, S_1) \ge 3$ and $\deg(v, S_2) \ge 3$. Let $P \subseteq X$ be the set of poor vertices. Choose a partition $X \setminus P = T_1 \sqcup T_2$ such that the vertices $v \in T_i$ satisfy $\deg(v, S_i) \le 2$ for $i = 1, 2$. Let $p = |P|$.

Claim 8.16. *If $v \in X \setminus P$ then* $\deg(v, S_1 \cup S_2) \le \frac{n}{2} - \frac{1}{2}\gamma n$.

Proof. This is obvious from Claim 8.15. $\square$

Claim 8.17. *If $v \in P$ then* $\deg(v, S_i) \le 12\gamma n$ *for* $i = 1, 2$.

Proof. Suppose the contrary. Let $u_1, u_2, u_3 \in N_v \cap S_1$ be three different vertices. By Claim 8.13 the set $N_{u_1} \cap N_{u_2} \cap N_{u_3} \cap S_2$ has at least $\frac{n}{2} - 10\gamma n$ vertices, therefore N_v hits at least γn of them. In particular $G[N_v]$ contains a $K_{3,3}$, a contradiction. □

We can now plug in the bounds from the claims above to count the number of edges in G to obtain the following estimation

$$\begin{aligned}\frac{1}{4}n^2 + \frac{1}{2}n + \frac{17}{4} < e(G) &\le e(G[S_1, S_2]) + e(G[S_1]) + e(G[S_2]) + e(G[P, S_1 \cup S_2]) + \\ &\quad + e(G[X \setminus P, S_1 \cup S_2]) + \binom{|X|}{2} \\ &\le \left(\frac{n-x}{2}\right)^2 + n + 24p\gamma n + (x-p)\left(\frac{n}{2} - \frac{1}{2}\gamma n\right) + \frac{x^2}{2}\end{aligned} \tag{8.12}$$

which is equivalent to

$$x\left(\frac{\gamma n}{2} - \frac{3}{4}x\right) + \frac{pn}{2}(1 - 49\gamma) + \frac{17}{4} < \frac{n}{2}.$$

Since $x \le 2\alpha n < \frac{1}{3}\gamma n$, we have $\frac{\gamma n}{2} - \frac{3}{4}x > \frac{\gamma n}{4}$, and the last inequality implies

$$\frac{x\gamma n}{4} + \frac{pn}{2}(1 - 49\gamma) + \frac{17}{4} < \frac{n}{2}. \tag{8.13}$$

It follows that

$$x < \frac{2}{\gamma}\,, \text{ and} \tag{8.14}$$

$$p < \frac{1}{1 - 49\gamma} < 1.5\,. \tag{8.15}$$

In particular we can only have $p = 0$ or $p = 1$.

Let $K_i = S_i \cup T_i$ for $i = 1, 2$. Note that

$$\frac{n}{2} - \alpha n \le |K_i| \le \frac{n}{2} + \alpha n + x \le \frac{n}{2} + 2\alpha n.$$

Let $b = \overline{e}(G[K_1, K_2])$ be the number of missing edges between K_1 and K_2. The following bound follows directly from Claim 8.14, the definition of T_i and (8.14).

Claim 8.18. *For each $v \in K_i$ we have that* $\deg(v, K_i) \le |T_i| + 2 \le x + 2 \le \frac{4}{\gamma}$.

Claim 8.19. *For $i = 1, 2$ and each set $Y \subseteq S_i$, $|Y| \le \frac{n}{8}$ we have that $G[S_i \setminus Y]$ contains at least one edge. In particular $G[S_i]$ contains at least one edge.*

Proof. Suppose the claim does not hold for example for $i = 1$ and some set $Y \subseteq S_1$. Let t_i be the number of triangles in G with at least two vertices in K_i.

Let $v \in T_2$ be a fixed vertex. By Claim 8.18 inside K_2 there are at most $\deg(v, K_2)^2 \le 16/\gamma^2$ triangles touching v. We further see that there are at most $\deg(v, K_2)|K_1| \le 4n/\gamma$ triangles through v with two vertices in K_2 and one vertex in K_1. Summing over all $v \in T_2$ we get that the number of triangles touching T_2 with at least two vertices in K_2 is at most $|T_2| \times (\frac{16}{\gamma^2} + \frac{4n}{\gamma}) \le \frac{17n}{\gamma^2}$.

To bound t_2 it only remains to add triangles whose two vertices are in S_2 and the third is in K_1 (by Claim 8.14 there are no triangles entirely inside S_2). By Claim 8.14 we have

$$e(G[S_2]) \le |S_2| \le \frac{11n}{20} . \tag{8.16}$$

Since each edge in S_2 can be extended in at most $|K_1| \le \frac{11n}{20}$ ways to such a triangle we get that

$$t_2 \le \frac{17n}{\gamma^2} + \frac{11n}{20} \cdot \frac{11n}{20} \le \frac{122n^2}{400} .$$

To bound the number t_1 of triangles with at least two vertices inside K_1 we proceed similarly, except that the fact $e(G[S_1 \setminus Y]) = 0$ allows us to strengthen the counterpart of (8.16) to $e(G[S_1]) \le 2|Y| \le \frac{n}{4}$. Consequently,

$$t_1 \le \frac{17n}{\gamma^2} + \frac{n}{4} \cdot \frac{11n}{20} \le \frac{3n^2}{20} .$$

We get that the number of triangles is $t_1 + t_2 < 0.46n^2 < 2(m - n)$, a contradiction to Definition 8.4a). $\square$

Next, we claim that there are no poor vertices.

Claim 8.20. *We have $p = 0$.*

Proof. Suppose that $p = 1$ and let $P = \{q\}$. Employing Claims 8.14 and 8.16 we get

$$e(G[K_1 \cup K_2]) \le \left(\frac{n-1}{2}\right)^2 - b + n + \binom{|T_1|}{2} + \binom{|T_2|}{2} \overset{(8.14)}{\le} \left(\frac{n-1}{2}\right)^2 - b + n + \frac{4}{\gamma^2} .$$

By Claim 8.17 we then have the following estimate

$$\frac{1}{4}n^2 + \frac{1}{2}n + \frac{17}{4} < e(G) \le \left(\frac{n-1}{2}\right)^2 - b + n + 25\gamma n. \tag{8.17}$$

This implies
$$b \leq 25\gamma n. \tag{8.18}$$
Consider any edge $e \in G[S_1]$. The link $\mathrm{lk}_G e$ is a cycle C which contains, by Claim 8.13, at least $\frac{n}{2} - 6\gamma n$ vertices of S_2 and, by Claim 8.14, does not pass through S_1. The number of vertices in which C can exit S_2 is bounded from above by $2(x+1)$. Eliminating the vertices of C which are adjacent (in the graph G) to T_2 (at most $2x$) or to q (at most $12\gamma n$ by Claim 8.17) we find that $G[S_2]$ contains at least $\frac{1}{2}(\frac{n}{2} - 30\gamma n)$ vertex-disjoint edges $e' = u'v'$ which satisfy $V(\mathrm{lk}_G e') \subseteq K_1$.

We claim that for at least one such edge $e' = u'v'$ we have $K_1 \subseteq N_{u'} \cap N_{v'}$. Indeed, each edge e' for which this does not hold is incident with at least one non-edge in $G[K_1, K_2]$, and thus otherwise we would get at least $\frac{1}{2}(\frac{n}{2} - 30\gamma n)$ non-edges in $G[K_1, K_2]$, a contradiction to (8.18).

Let us fix an edge e' as above. We now have that $\mathrm{lk}_G e' = G[K_1]$ and therefore $G[K_1]$ is a cycle. A symmetric argument starting with an appropriate edge $e'' \in G[K_1]$ for which $\mathrm{lk}_G e'' = G[K_2]$ shows that $G[K_2]$ is a cycle as well.

We now see that G, with the decomposition $V(G) = K_1 \sqcup K_2 \sqcup \{q\}$, is 1-joinlike in the sense of Definition 8.10. We shall however later in Proposition 8.32 show that this leads to a contradiction. $\square$

For the remaining part we can therefore assume $P = \emptyset$. Our short-term goal for now is to prove that G is 0-, 1- or 2-joinlike. The same way we derived (8.17) we get that
$$\frac{1}{4}n^2 + \frac{1}{2}n + \frac{17}{4} < e(G) \leq \left(\frac{n}{2}\right)^2 - b + n + \frac{4}{\gamma^2}.$$
This implies
$$b < \frac{n}{2} + \frac{4}{\gamma^2} - \frac{17}{4} < 0.51n. \tag{8.19}$$

Let E_i be the set containing T_i and all the neighbors in S_i of the vertices in T_i. By definition of T_i we have $|E_i| \leq 3x$. Note that $K_i \setminus E_i = S_i \setminus E_i$ and for any vertex $v \in K_i \setminus E_i$ we have $\deg(v, K_i) \leq 2$.

Fix two edges $e_1 \in G[S_1 \setminus E_1]$ and $e_2 \in G[S_2 \setminus E_2]$; such edges exist by Claim 8.19. For each $i = 1, 2$ the link $\mathrm{lk}_G e_{3-i}$ lies in K_i and its intersection with $K_i \setminus E_i$ is a collection of at most $3x$ paths of total length at least $\frac{n}{2} - 6\gamma n$ by Claim 8.13, or a sole cycle. Define a *segment* in $G[K_i]$ as a maximal connected sub-path (or a cycle) of $\mathrm{lk}_G e_{3-i}$ which lies in $K_i \setminus E_i$. (Note that our definition of segments is with respect to fixed edges e_1 and e_2.) There are at most $3x \leq 6/\gamma$ segments in K_i. A segment is called *long* if it has at least αn vertices and *short* otherwise. The total length of short segments in K_i is at most $\frac{6}{\gamma} \cdot \alpha n < 0.09n$, hence the total length of long segments in each K_i is at least $0.4n$.

Claim 8.21. *Let R_1 and R_2 be two segments in K_1 and K_2, respectively. If for some vertices $x_1 \in R_1$, $x_2 \in R_2$ we have $x_1x_2 \in E(G)$ then $G[R_1, R_2]$ is complete bipartite.*

Proof. If x_1', x_1'' are the neighbours of x_1 in K_1 and x_2', x_2'' are the neighbours of x_2 in K_2, then the link $\mathrm{lk}_G x_1x_2$ is a cycle contained in $\{x_1', x_1'', x_2', x_2''\}$, hence, by Definition 8.4b) it must pass through all those vertices. Therefore $x_1x_2', x_1x_2'', x_2x_1', x_2x_1'' \in E(G)$. By successively repeating the same argument for the newly forced edges we prove the claim. $\square$

Claim 8.22. *If R_1 and R_2 are two long segments in K_1 and K_2 respectively then $G[R_1, R_2]$ is complete bipartite.*

Proof. If not then, by Claim 8.21, the bipartite graph $G[R_1, R_2]$ does not contain any edges. Then

$$\overline{e}(G[K_1, K_2]) \geq \overline{e}(G[R_1, R_2]) = |R_1| \cdot |R_2| \geq \frac{\alpha^2 n^2}{2},$$

a contradiction to (8.19). $\square$

Let L_1, and L_2 be the vertex sets of all the long segments in K_1 and K_2, respectively. By Claim 8.22 the graph $G[L_1, L_2]$ is complete bipartite. For $i = 1, 2$ choose edges $\widetilde{e}_i \in G[L_i]$ which minimize the quantity

$$|K_{3-i} \setminus V(\mathrm{lk}_G \widetilde{e}_i)| \tag{8.20}$$

and let $C_i \subseteq K_i$ be the vertex set of the cycle $\mathrm{lk}_G \widetilde{e}_{3-i}$.

Claim 8.23. *We have $|K_1 \setminus C_1| + |K_2 \setminus C_2| \leq 2$.*

Proof. Let $d_i = |K_i \setminus C_i|$. By the optimality of the choice of $\widetilde{e}_i$ we get that the link of every edge in $G[L_i]$ misses at least d_{3-i} vertices of K_{3-i}. Since $G[L_1, L_2]$ is complete bipartite by Claim 8.22, those missing edges must contribute to $\overline{e}(G[L_i, K_{3-i} \setminus L_{3-i}])$. Recall that $G[L_i]$ is a collection of at most $3x \leq 6/\gamma$ vertex-disjoint paths (or a cycle) of total length at least $0.4n$. We get

$$\overline{e}(G[L_i, K_{3-i} \setminus L_{3-i}]) \geq \frac{d_{3-i}}{2}(|L_i| - 3x) \geq d_{3-i} \cdot 0.19 \cdot n.$$

The two sets of missing edges we count this way for $i = 1, 2$ are disjoint. Therefore, using (8.19)

$$0.51n > b \geq \overline{e}(G[L_1, K_2 \setminus L_2]) + \overline{e}(G[L_2, K_1 \setminus L_1]) \geq 0.19n(d_1 + d_2)$$

which implies $d_1 + d_2 < 2.7$. That ends the proof. $\square$

The graphs $G[C_1]$, $G[C_2]$ are cycles and the minimizing edges $\widetilde{e}_i \in L_i \subseteq C_i$ satisfy $\mathrm{lk}_G \widetilde{e}_i = G[C_{3-i}]$. Together with Claim 8.23 it shows that G is t-joinlike for $t \leq 2$. If $t = 0$ then we are done. The case $t = 1$ leads to a contradiction as shown in Proposition 8.32 below. We can therefore assume that $t = 2$ and call the two exceptional vertices q and q'. We can assume without loss of generality that either

$$K_1 \setminus C_1 = \{q\}, \quad K_2 \setminus C_2 = \{q'\}, \tag{8.21}$$

or

$$K_1 \setminus C_1 = \{q, q'\}, \quad K_2 \setminus C_2 = \emptyset. \tag{8.22}$$

Define the following quantities for $i = 1, 2$,

$$\begin{aligned} d_i(q) = \deg(q, C_i) \quad &\text{and} \quad d_i(q') = \deg(q', C_i)\ , \\ e_i(q) = e(G[N_q \cap C_i]) \quad &\text{and} \quad e_i(q') = e(G[N_{q'} \cap C_i])\ . \end{aligned}$$

If any of the numbers $d_1(q), d_1(q'), d_2(q), d_2(q')$ is at most 2, then the result follows from Proposition 8.39 below. We will therefore assume that

$$\min\{d_1(q), d_1(q'), d_2(q), d_2(q')\} \geq 3\ .$$

The proof under this assumption splits into the two cases (8.21) and (8.22) and is presented in the next section.

8.3 Two exceptional vertices of large degrees

In this section we show that each of the cases (8.21) and (8.22) from the previous section leads to a contradiction. We use the same notation.

We are going to exploit the fact that the graphs $\mathrm{lk}_G q$ and $\mathrm{lk}_G q'$ are planar. Recall that Euler's formula implies an h-vertex planar graph can have at most $3h - 6$ edges. So, planar graphs are sparse, and a substantial number of edges must be missing between C_1 and C_2. A careful edge counting will lead to a contradiction.

We start with an auxiliary claim.

Claim 8.24. *We have an inequality*

$$\overline{e}(G[N_q \cap C_1, N_q \cap C_2]) \geq d_1(q)d_2(q) - 3d_1(q) - 3d_2(q) + e_1(q) + e_2(q) + 6 \ .$$

An analogous inequality holds for q'.

Proof. The graph $G[N_q \cap (C_1 \cup C_2)]$ is a planar graph with $d_1(q) + d_2(q)$ vertices and

$$d_1(q)d_2(q) - \overline{e}(G[N_q \cap C_1, N_q \cap C_2]) + e_1(q) + e_2(q)$$

edges. The claim now follows from Euler's formula. □

From previous estimates we have $\frac{n}{2} - 2\alpha n \leq |C_i| \leq \frac{n}{2} + 2\alpha n$. The next easy statement records the fact that if q is adjacent to most of C_i then $\mathrm{lk}_G q$ also contains most of the edges from $G[C_i]$.

Claim 8.25. *Suppose $\beta \geq 4\alpha$. If $d_i(q) \geq \frac{n}{2}(1-\beta)$ then $e_i(q) \geq \frac{n}{2}(1-5\beta)$. The same holds for q'.*

Proof. Since $|C_i| \leq \frac{n}{2} + 2\alpha n$ the set N_q misses at most

$$\frac{n}{2} + 2\alpha n - \frac{n}{2}(1-\beta) = n\left(\frac{1}{2}\beta + 2\alpha\right) \leq \beta n$$

vertices of C_i. Recall that $G[C_i]$ is a cycle. It follows that at most $2\beta n$ edges of $G[C_i]$ are not in $\mathrm{lk}_G q$. Hence

$$e_i(q) \geq \frac{n}{2} - 2\alpha n - 2\beta n = \frac{n}{2}(1 - 4\alpha - 4\beta) \geq \frac{n}{2}(1-5\beta).$$

□

8.3.1 The case (8.21).

By Claim 8.18 we have $d_1(q), d_2(q') \leq \frac{4}{\gamma}$. Therefore

$$\overline{e}(G[C_1, C_2]) \geq \overline{e}(G[N_q \cap C_1, N_q \cap C_2]) + \overline{e}(G[N_{q'} \cap C_1, N_{q'} \cap C_2]) - \frac{16}{\gamma^2} \ .$$

The inequality

$$\frac{1}{4}(n^2 + 2n + 17) < e(G) \leq \left(\frac{n-2}{2}\right)^2 + n + \deg(q) + \deg(q') - \overline{e}(G[C_1, C_2])$$

$$\leq \frac{n^2}{4} + \deg(q) + \deg(q')$$
$$- \bar{e}(G[N_q \cap C_1, N_q \cap C_2]) - \bar{e}(G[N_{q'} \cap C_1, N_{q'} \cap C_2]) + \frac{16}{\gamma^2}$$

together with Claim 8.24 and $e_1(q), e_2(q') \leq \frac{4}{\gamma}$ gives

$$\frac{1}{2}n + (d_1(q) - 4)(d_2(q) - 4) + (d_1(q') - 4)(d_2(q') - 4) + e_2(q) + e_1(q') \leq O(1) , \quad (8.23)$$

where $O(1)$ denotes some universal constant. Observe that if $d_1(q) \geq 4$ then the inequalities $d_1(q) \leq \frac{4}{\gamma}$ and $d_2(q) \geq 3$ imply $(d_1(q) - 4)(d_2(q) - 4) \geq -\frac{4}{\gamma}$. A similar observation holds for q'. Therefore, if $d_1(q), d_2(q') \geq 4$ then we get a contradiction because then the left-hand side of (8.23) is at least $\frac{1}{2}n - \frac{8}{\gamma}$.

Let us then assume that $d_1(q) = 3$. Then the inequality (8.23) becomes

$$\frac{1}{2}n + (d_1(q') - 4)(d_2(q') - 4) + e_2(q) + e_1(q') \leq d_2(q) + O(1) . \quad (8.24)$$

If $d_2(q') \geq 4$ then $(d_1(q') - 4)(d_2(q') - 4) \geq -\frac{4}{\gamma}$, and therefore (8.24) implies $d_2(q) \geq 0.49n$. By Claim 8.25 we have $e_2(q) \geq 0.45n$ and plugging this back into (8.24) we get $d_2(q) \geq \frac{1}{2}n + 0.45n - O(1) \geq 0.94n$, which is a contradiction with $d_2(q) \leq |C_2| \leq 0.51n$.

We are now left with the case when $d_1(q) = d_2(q') = 3$ and (8.24) reduces to

$$\frac{1}{2}n + e_2(q) + e_1(q') \leq d_2(q) + d_1(q') + O(1) . \quad (8.25)$$

We now need the following claim.

Claim 8.26. *If $v \in C_2$ is an isolated vertex of the graph $G[N_q \cap C_2]$ then $vq' \in E(G)$.*

Proof. The cycle $\mathrm{lk}_G qv$ is contained in $(N_q \cap C_1) \cup \{q'\}$ and since $d_1(q) = 3$, the latter set has 4 vertices. By Definition 8.4b) $\mathrm{lk}_G qv$ must pass through all of them and in particular $q' \in N_v$. □

Because $d_2(q') = 3$ the claim implies that $G[N_q \cap C_2]$ can have at most 3 isolated vertices and therefore $e_2(q) \geq \frac{1}{2}(d_2(q) - 3)$. By symmetry we get $e_1(q') \geq \frac{1}{2}(d_1(q') - 3)$ and (8.25) implies

$$n \leq d_1(q') + d_2(q) + O(1) . \quad (8.26)$$

It follows that $d_1(q'), d_2(q) \geq 0.48n$ but then, by Claim 8.25, $e_1(q'), e_2(q) \geq 0.4n$ and going back to the inequality (8.25) gives a contradiction.

8.3.2 The case (8.22).

This time we have $d_1(q), d_1(q') \leq \frac{4}{\gamma}$. The missing edges in $G[N_q \cap C_1, N_q \cap C_2]$ and $G[N_{q'} \cap C_1, N_{q'} \cap C_2]$ can have a significant overlap, so we begin by using just the contribution of one of them to obtain a bound. We have

$$\frac{1}{4}(n^2+2n+17) < e(G) \leq \left(\frac{n-2}{2}\right)^2 + n + \deg(q) + \deg(q') - \bar{e}(G[N_q \cap C_1, N_q \cap C_2]) ,$$

and plugging in the bound from Claim 8.24 we obtain

$$\frac{1}{2}n + (d_1(q) - 4)(d_2(q) - 4) + e_2(q) \leq d_2(q') + O(1) . \tag{8.27}$$

In the same way we obtain a symmetric version with q and q' interchanged:

$$\frac{1}{2}n + (d_1(q') - 4)(d_2(q') - 4) + e_2(q') \leq d_2(q) + O(1) . \tag{8.28}$$

Now suppose that $d_1(q) \geq 4$. Then $(d_1(q)-4)(d_2(q)-4) \geq -\frac{4}{\gamma}$, and so (8.27) implies $d_2(q') \geq 0.49n$. Therefore, $e_2(q') \geq 0.45n$ by Claim 8.25. Then the inequality (8.28) can be rewritten as

$$\begin{aligned} d_2(q) &\geq \frac{1}{2}n + (d_1(q') - 4)(d_2(q') - 4) + e_2(q') - O(1) \\ &\geq 0.94n + (d_1(q') - 4)(d_2(q') - 4) . \end{aligned}$$

This inequality can only be satisfied if the last product is negative, which implies $d_1(q') = 3$. Using $d_2(q') \leq 0.51n$ we further obtain

$$d_2(q) \geq 0.94n - 0.51n = 0.43n .$$

By Claim 8.25 we get $e_2(q) \geq 0.15n$, but then (8.27) gives

$$d_2(q') \geq \frac{1}{2}n + 0.15n - O(1) \geq 0.64n ,$$

which is a contradiction.

By symmetry we also arrive at a contradiction assuming that $d_1(q') \geq 4$. It means we must have $d_1(q) = d_1(q') = 3$.

We have that $|(N_q \cup N_{q'}) \cap C_1| \leq 6$. Consequently, there are only a finite number of possibilities for the graph $G[(N_q \cup N_{q'}) \cap C_1]$. We will first show that the actual possibilities for $G[(N_q \cup N_{q'}) \cap C_1]$ are even more limited. Call a vertex $v \in C_1$ *free* if $v \notin N_q \cup N_{q'}$, a *q-vertex* if $v \in N_q \setminus N_{q'}$, a *$q'$-vertex* if $v \in N_{q'} \setminus N_q$, a

qq'-*vertex* if $v \in N_q \cap N_{q'}$ and a *boundary* vertex if v belongs to an edge $e \in G[C_1]$ such that $\mathrm{lk}_G e \cap \{q, q'\} = \emptyset$. Observe that each free vertex is also boundary.

Claim 8.27. *The vertices in C_1 have the following properties:*

a) if $v \in C_1$ is boundary then $C_2 \subseteq N_v$,

b) if $v \in C_1$ is a q-vertex then at least one of its neighbors in C_1 is in N_q,

b') if $v \in C_1$ is a q'-vertex then at least one of its neighbors in C_1 is in $N_{q'}$,

c) if $v \in C_1$ is a qq'-vertex then at least one of its neighbors in C_1 is in $N_q \cup N_{q'}$,

d) if $e_1, e_2 \in G[C_1]$ are two vertex-disjoint edges, such that $\mathrm{lk}_G e_1$ contains q but not q' and $\mathrm{lk}_G e_2$ contains q' but not q, then in at least one of those edges both endpoints are non-boundary,

e) if v is a q-vertex and w is a q'-vertex then $vw \notin E(G[C_1])$.

Proof. a) Consider any edge $e \in G[C_1]$ such that $v \in e$ and $\mathrm{lk}_G e \cap \{q, q'\} = \emptyset$. Then $\mathrm{lk}_G e = G[C_2]$, so in particular $C_2 \subseteq N_v$.

b) Let $v', v'' \in C_1$ be the neighbors of v. If none of v', v'' is in N_q then all three of v, v', v'' are boundary, so by a) all are adjacent to the whole C_2. Pick any vertex $w \in N_q \cap C_2$ and let w', w'' be its neighbors in C_2. Then the link $\mathrm{lk}_G vw$ contains the cycle $w'v'w''v''$ and the vertex q, which is impossible. By symmetry we also get b').

c) The proof is the same as b).

d) Suppose the contrary. Let $e_1 = xx'$, $e_2 = yy'$ where x' and y' are boundary vertices. By a) $C_2 \subseteq N_{x'}, N_{y'}$, therefore

$$\mathrm{lk}_G(e_1) = \{q\} \cup (N_x \cap C_2), \quad \mathrm{lk}_G(e_2) = \{q'\} \cup (N_y \cap C_2).$$

It follows that $N_x \cap C_2$ is a path within C_2 and q is adjacent only to the endpoints of that path. The same argument for y and q' shows that $N_y \cap C_2$ is a path with q' adjacent only to the endpoints of that path. It follows that, except for up to 4 special vertices, every vertex in C_2 is missing an edge to either q or x and it is missing an edge to either q' or y. Since x, y, q, q' are four different vertices this yields at least $2(|C_2| - 4) \approx n$ missing edges from K_2 to K_1, contradicting (8.19).

e) Suppose vw is an edge. Then v and w are both boundary. Let $v'vww'$ be the path of length three on the cycle $G[C_1]$. By b) and b') we have $v' \in N_q$ and $w' \in N_{q'}$. Then the edges vv' and ww' contradict d). □

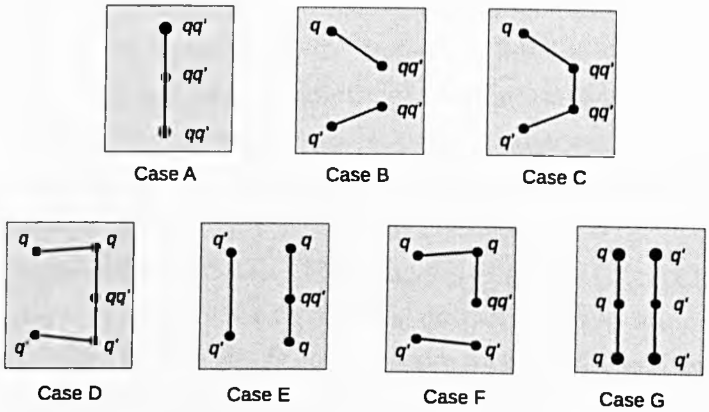


Figure 8.1: Seven possibilities of the graph $G[(N_q \cup N_{q'}) \cap C_1]$ with the types of the vertices (types q-, q'-, and qq'-).

It turns out that Claim 8.27 provides enough information to restrict $G[(N_q \cup N_{q'}) \cap C_1]$ to just one possibility.

Claim 8.28. *We have $N_q \cap C_1 = N_{q'} \cap C_1 = \{v_1, v_2, v_3\}$ where v_1, v_2, v_3 are three consecutive vertices in C_1.*

Proof. Claim 8.27 gives us that $G[(N_q \cup N_{q'}) \cap C_1]$ is a graph with no cycle, in which every vertex has degree 1 or 2, and there is no edge from a q-vertex to a q'-vertex. By considering the possible number of qq'-vertices (3, 2, 1 or 0) and then their degrees, we obtain seven graphs which satisfy the above property, up to exchanging q and q'. They are shown in Figure 8.1. The graphs B–G have a pair of edges which violates Claim 8.27d). That leaves us only with Case A. □

As all the vertices in C_1 except v_2 are boundary, we have by Claim 8.27a) that $C_2 \subseteq N_v$ for each $v \in C_1 \setminus \{v_2\}$.

Claim 8.29. *There is no edge $e \in G[C_2]$ with $q, q' \in \mathrm{lk}_G e$.*

Proof. If e was such an edge then v_1 would be a vertex of degree 3 in $\mathrm{lk}_G e$. □

Claim 8.30. *We have $|N_q \cap N_{v_2} \cap C_2| \leq 2$ and $|N_{q'} \cap N_{v_2} \cap C_2| \leq 2$.*

Proof. Any 3 vertices in $N_q \cap N_{v_2} \cap C_2$ together with $\{v_1, v_2, v_3\}$ would form a $K_{3,3}$ in $\mathrm{lk}_G q$, contradicting Definition 8.4d). □

To complete the proof we consider two cases. First suppose $qq' \in E(G)$. Then, we have $|N_q \cap N_{q'} \cap C_2| \leq 2$. Indeed, otherwise v_1 would be a vertex of degree at least 3 in $\mathrm{lk}_G qq'$, a contradiction to Definition 8.4b). It follows that every vertex of C_2, except for at most 6 special ones, is adjacent to at most one element of $\{q, q', v_2\}$, and then there at least $2(|C_2| - 6) \approx n$ edges missing from K_2 to K_1. This contradicts (8.19).

Now suppose $qq' \notin E(G)$. Then $\mathrm{lk}_G qv_3 = \{v_2\} \cup (N_q \cap C_2)$ and $\mathrm{lk}_G q'v_3 = \{v_2\} \cup (N_{q'} \cap C_2)$. It means that $G[N_q \cap C_2]$ and $G[N_{q'} \cap C_2]$ are paths – say P and P' – within C_2. By Claim 8.29, P and P' share at most the endvertices. Moreover, the interior vertices of P and P' are not adjacent to v_2. Consequently, every vertex in C_2, except for at most 4 special vertices, is adjacent to at most one element of $\{q, q', v_2\}$. Again, the total number of missing edges from K_2 to K_1 is at least $2(|C_2| - 4) \approx n$, contradicting (8.19).

This ends the consideration of the case (8.22), thereby completing the proof of Theorem 8.6.

8.4 Exact results

In the proof of Theorem 8.6 we used, as black-boxes, two results about the sparseness of certain 1- and 2-joinlike graphs — Propositions 8.32 and 8.39. They will be proved in this section. Unlike previously, when we were free to count edges with an accuracy of $\Theta(n)$, in this part we will need to determine the precise structure of some fascinating graphs and count their edges exactly.

In this section G means any fascinating graph, which will always be 1- or 2-joinlike, with C_1, C_2 referring to the cycles from Definition 8.10 and with exceptional vertices called q and q'. We will frequently use the observation that if q is an exceptional vertex of a t-joinlike graph G then $C_i \setminus N_q \neq \emptyset$ for $i = 1, 2$.

Proposition 8.31. *If G is 1-joinlike and q is the exceptional vertex then* $\deg(q, C_i) \geq 3$ *for* $i = 1, 2$.

Proof. Suppose that $\deg(q, C_1) \leq 2$. If $\deg(q, C_2) = 0$ then $\mathrm{lk}_G q$ contains at most 2 vertices, so G fails Definition 8.4d). Otherwise let $x \in N_q \cap C_2$ be any vertex with at least one neighbor in $C_2 \setminus N_q$. We see that $\mathrm{lk}_G qx$ contains at most 3 vertices, which is a contradiction. □

Proposition 8.32. *If G is 1-joinlike then* $e(G) \leq \frac{1}{4}(n^2+2n+17)$, *where* $n = |V(G)|$.

Proof. Let q be the exceptional vertex. We will say that a vertex $v \in C_i$ is a *q-vertex* if $qv \in E(G)$, a *free vertex* otherwise and a *boundary vertex* if it is a q-vertex adjacent to a free vertex.

We refer to C_1 and C_2 as "sides".

Claim 8.33. *If $v \in C_i$ is free or boundary then $C_{3-i} \subseteq N_v$.*

Proof. Indeed, v belongs to an edge $e \in G[C_i]$ with $q \notin \mathrm{lk}_G e$ and therefore with $\mathrm{lk}_G e = G[C_{3-i}]$. That means $C_{3-i} \subseteq N_v$. □

By Proposition 8.31 and because $N_q \cap C_i \neq C_i$ for $i = 1, 2$, there are at least three q-vertices and at least two boundary vertices on each side. If there were 3 boundary vertices in, say, C_1, then the graph formed by those 3 vertices in C_1 and any 3 neighbors of q in C_2 would form, by Claim 8.33, a $K_{3,3}$ in $\mathrm{lk}_G q$, which is impossible. That implies there are exactly 2 boundary vertices on each side. In other words each $N_q \cap C_i$ induces a path inside C_i of some length $a_i \geq 3$ for $i = 1, 2$.

If $u \in C_1$ and $w \in C_2$ are q-vertices which are not boundary and $uw \in E(G)$ then by Claim 8.33 there is a $K_{3,3}$ in $\mathrm{lk}_G q$ formed by u, w and the 2 boundary vertices on each side. This means $uw \notin E(G)$ for such u, w.

We now know the exact structure of G and we can compute its number of edges. Denoting $c_i = |C_i|$ and using $n = c_1 + c_2 + 1$ we have

$$\begin{aligned} e(G) &= c_1c_2 + c_1 + c_2 + a_1 + a_2 - (a_1 - 2)(a_2 - 2) \\ &= \frac{1}{4}(n^2 + 2n + 17) - \frac{1}{4}(c_1 - c_2)^2 - (a_1 - 3)(a_2 - 3) \leq \frac{1}{4}(n^2 + 2n + 17). \end{aligned}$$

□

The second part of the analysis in this section deals with 2-joinlike graphs. We start off by a counterpart of Proposition 8.31.

Proposition 8.34. *If G is 2-joinlike and q is any exceptional vertex then* $\deg(q, C_i) \geq$ 2 *for $i = 1, 2$.*

Proof. Suppose that $\deg(q, C_1) \leq 1$. If $\deg(q, C_2) = 0$ then $\mathrm{lk}_G q$ contains at most 2 vertices, so G fails Definition 8.4d). Otherwise let $x \in N_q \cap C_2$ be any vertex with at least one neighbor in $C_2 \setminus N_q$. We see that $\mathrm{lk}_G qx$ contains at most 3 vertices, which is a contradiction. □

We shall later need the following simple inequality.

Lemma 8.35. *If $n = k + l + 2$ then*

$$kl + 2k + l + 6 \leq \frac{1}{4}(n^2 + 2n + 17).$$

Proof. One checks that

$$kl + 2k + l + 6 = \frac{1}{4}(n^2 + 2n + 17) - \frac{1}{4}(l - k + 1)^2.$$

□

Proposition 8.39 below is a combination of a case distinction captured by Proposition 8.36 and Proposition 8.37.

Proposition 8.36. *If G is 2-joinlike with exceptional vertices $\{q, q'\}$ such that $\deg(q, C_1) = 2$ and the two vertices of $N_q \cap C_1$ are adjacent, then $e(G) \leq \frac{1}{4}(n^2 + 2n + 17)$, where $n = |V(G)|$.*

Proof. Let $N_q \cap C_1 = \{u, v\}$. Let $x, x' \in C_2$ be neighbors such that $qx \in E(G)$, $qx' \notin E(G)$ and let y be the other neighbor of x in C_2 (their existence is guaranteed by Proposition 8.34 and the fact that $N_q \cap C_2 \neq C_2$). Then $\mathrm{lk}_G qx \subseteq \{u, v, q', y\}$, and since $uv \in E(G)$ we can assume that $\mathrm{lk}_G qx$ is the cycle $vuyq'$ (this is the unique possibility up to the order of u, v). In particular $qq', q'v \in E(G)$ and $q'u \notin E(G)$.

If $u' \neq v$ is the other neighbor of u in C_1 then $\mathrm{lk}_G uu'$ contains neither q nor q', so it must be all of C_2. In particular $C_2 \subseteq N_u$. It means that $\mathrm{lk}_G uq = \{v\} \cup (N_q \cap C_2)$, so $N_q \cap C_2$ is a path of length at least 3 within C_2, whose both endpoints, call them v_1, v_2, are connected to v, while the interior vertices of the path are not connected to v. Let $a = |N_q \cap C_2|$ be the length of this path.

The link of every edge in $G[N_q \cap C_2]$ contains u and q, so to be a cycle it must also contain q'. It follows that $N_{q'} \cap C_2 \supseteq N_q \cap C_2$.

Let $t \neq u$ be the other neighbor of v in C_1. Consider the link $\mathrm{lk}_G q'v$. It contains the path $v_1 q v_2$. If $t \notin \mathrm{lk}_G q'v$ then this link must contain, apart from v_1, q and v_2, all the vertices in $C_2 \setminus N_q$. However, that would imply $C_2 \setminus N_q \subseteq N_{q'}$. Put together with the previously established $N_{q'} \cap C_2 \supseteq N_q \cap C_2$ we would get $C_2 \subseteq N_{q'}$, a contradiction. This means that $t \in \mathrm{lk}_G q'v$, i.e. $q't \in E(G)$.

Consider any vertex $x \in (C_1 \cap N_{q'}) \setminus \{u, v\}$ which has at least one neighbor $\bar{x}$ in $C_1 \setminus N_{q'}$. By Proposition 8.34 and the fact that $q'u \notin E(G)$ such a vertex must exist. The link $\mathrm{lk}_G x\bar{x}$ is a cycle which does not touch $C_1 \cup \{q, q'\}$. Consequently, $\mathrm{lk}_G x\bar{x} = G[C_2]$, and in particular, $C_2 \subseteq N_x$. The link $\mathrm{lk}_G xq'$ consists of one vertex in C_1 and of the whole $N_{q'} \cap C_2$. We get that $N_{q'} \cap C_2$ is a path within C_2, containing

$N_q \cap C_2$. Let w_1, w_2 be the endpoints and let $b = |N_{q'} \cap C_2|$. Assume that v_1 is between w_1 and v_2 on this path (perhaps $w_1 = v_1$ or $w_2 = v_2$).

For every edge e in $G[(C_2 \setminus N_{q'}) \cup \{w_1, w_2\}]$ we have $\mathrm{lk}_G e = G[C_1]$. As $C_2 \cap N_{q'}$ induces a path with endvertices w_1 and w_2 and $G[C_2]$ is a cycle, we must have that $G[(C_2 \setminus N_{q'}) \cup \{w_1, w_2\}]$ is a path, in particular this graph contains no isolated vertices. It follows that for every vertex $x \in (C_2 \setminus N_{q'}) \cup \{w_1, w_2\}$ we have $C_1 \subseteq N_x$. Now consider the link $\mathrm{lk}_G q'v$. It contains the vertices q, t, v_1, v_2, w_1, w_2, with paths $v_1 q v_2$ and $w_1 t w_2$. This is only possible if v is adjacent to all of $(N_{q'} \setminus N_q) \cap C_2$ while t is not adjacent to any vertex of $(((N_{q'} \setminus N_q) \cap C_2) \cup \{v_1, v_2\}) \setminus \{w_1, w_2\}$.

Let $|C_1| = k, |C_2| = l$, with $n = k + l + 2$. The remaining part of the proof splits into two cases. First we assume that t is non-adjacent to all of $(N_q \cap C_2) \setminus \{v_1, v_2\}$. In that case t is non-adjacent to $b - 2$ vertices of C_2, v is non-adjacent to $a - 2$ vertices and using a bound $\deg(q', C_1) \leq k - 1$ we get

$$\begin{aligned} e(G) &\leq kl + k + l + (a+2) + (b + k - 1) + 1 - (a-2) - (b-2) \\ &= kl + 2k + l + 6 , \end{aligned}$$

so the conclusion follows from Lemma 8.35.

Next suppose that t has a neighbor y in $(N_q \cap C_2) \setminus \{v_1, v_2\}$ and let $s \neq v$ be the other neighbor of t in C_1. The link $\mathrm{lk}_G q't$ contains v, w_1, w_2, y and possibly s with edges $w_1 v w_2$, and apart from v and s it is contained in $N_{q'} \cap C_2$. Any cycle with that property must contain an edge $e \in G[N_q \cap C_2]$ and it follows that there exists an edge $e \in G[N_t \cap N_q \cap C_2]$. But $\mathrm{lk}_G e$ is a cycle passing through $uqq't$ and not through v, therefore necessarily going through all of $C_1 \setminus \{v\}$. In particular $N_{q'} \cap C_1 = \{v, t\}$ and so $s \notin \mathrm{lk}_G q't$. It means that $\mathrm{lk}_G q't = \{v\} \cup (N_{q'} \cap C_2)$ which, by the restrictions on N_t, implies $v_1 = w_1$, $v_2 = w_2$, $a = b$ and $C_2 \subseteq N_t$. This determines the graph G and we obtain

$$\begin{aligned} e(G) &= kl + k + l + (a+2) + (a+2) + 1 - (a-2) \\ &= kl + k + l + a + 7 \\ &= \frac{1}{4}(n^2 + 2n + 17) - \frac{1}{4}(k - l + 1)^2 - (l - 1 - a) \leq \frac{1}{4}(n^2 + 2n + 17) \end{aligned}$$

because $a \leq l - 1$. $\square$

Proposition 8.37. *If G is 2-joinlike with exceptional vertices $\{q, q'\}$ such that $\deg(q, C_1) = 2$ and the two vertices of $N_q \cap C_1$ are not adjacent, then $e(G) \leq \frac{1}{4}(n^2 + 2n + 17)$ where $n = |V(G)|$.*

Proof. The proof uses similar techniques as the proof of Proposition 8.36. Set $N_q \cap C_1 = \{u, v\}$.

Let $x \in C_2$ be any vertex with $qx \in E(G)$ and such that x has a neighbor $x' \in C_2$ with $qx' \notin E(G)$. Let y be the other neighbor of x in C_2. We have $\mathrm{lk}_G qx \subseteq \{u, v, q', y\}$, with u and v being independent. It follows that $\mathrm{lk}_G qx$ is the cycle $uq'vy$, in particular $q'u, q'v, ux, vx, q'x \in E(G)$ and $qq' \in E(G)$.

It follows that the number of vertices $x \in C_2$ with the property described in the previous paragraph is at most 2. Indeed, we proved that every such vertex is adjacent to u, v, q', and the claim follows since $\mathrm{lk}_G q$ is $K_{3,3}$-free. It means that $N_q \cap C_2$ is a path within C_2 of length $a = |N_q \cap C_2| \geq 3$. Moreover, if $v_1, v_2 \in C_2$ are the endpoints of that path then $q'v_j, uv_j, vv_j \in E(G)$ for $j = 1, 2$.

The link $\mathrm{lk}_G qu$ contains q', v_1, v_2 and no vertex in C_1, so it must be $\{q'\} \cup (N_q \cap C_2)$. That, and the same argument for $\mathrm{lk}_G qv$ mean that $N_q \cap C_2 \subseteq N_u, N_v$ and that q' is non-adjacent to vertices in $(N_q \cap C_2) \setminus \{v_1, v_2\}$.

We will now prove the following claim.

Claim 8.38. *Suppose $x \in C_1 \setminus \{u, v\}$ and $y \in (N_q \cap C_2) \setminus \{v_1, v_2\}$. Let x', x'' be the neighbors of x in C_1, and let y', y'' be the neighbors of y in C_2. If $xy \in E(G)$ then $xy', xy'', x'y, x''y \in E(G)$.*

Proof. The link $\mathrm{lk}_G xy$ contains neither q nor q'. Hence it must be contained in $\{x', x'', y', y''\}$, and it follows that these 4 vertices must form a 4-cycle with x and y adjacent to all of them. $\square$

The vertices u, v divide $G[C_1]$ into two paths which we call P_1, P_2, so that there is a partition $C_1 = P_1 \sqcup P_2 \sqcup \{u, v\}$. We also write $\overline{P_j} = P_j \cup \{u, v\}$ for $j = 1, 2$ for the "closures" of those paths. Claim 8.38 implies that for $j = 1, 2$ the bipartite graph $G[P_j, (N_q \cap C_2) \setminus \{v_1, v_2\}]$ is either edgeless or complete bipartite. Suppose first that both of these graphs are complete. Take any edge e in $G[N_q \cap C_2]$. As $a \geq 3$, such an edge exists. The above then gives that $\mathrm{lk}_G e$ contains all of C_1, and q, a contradiction. Suppose next that both of these graphs are empty. Taking any edge e in $G[N_q \cap C_2]$ we observe that $\mathrm{lk}_G e$ spans at most three vertices $\{q, u, v\}$, again a contradiction. We can therefore assume that $G[\overline{P_1}, N_q \cap C_2]$ is complete bipartite and $G[P_2, (N_q \cap C_2) \setminus \{v_1, v_2\}]$ has no edges.

For every edge $f \in G[\overline{P_2}]$ the link $\mathrm{lk}_G f$ misses q and $N_q \setminus \{v_1, v_2\}$ hence it must contain q'. We therefore have that

$$\overline{P_2} \subseteq N_{q'} \,. \tag{8.29}$$

The rest of the proof depends on whether $N_{q'} \cap P_1$ is empty.

First suppose that q' is adjacent to some vertex of P_1. Recalling that $N_{q'} \cap C_1 \neq C_1$ and combining this with (8.29) we have $N_{q'} \cap P_1 \neq P_1$. We can find $t \in P_1$ with neighbors $t', t'' \in \overline{P_1}$ such that $tq' \in E(G)$ and $t'q' \notin E(G)$. Since $\mathrm{lk}_G tt'$ contains neither q nor q' it must be all of C_2 hence $C_2 \subseteq N_t$. We then have $\mathrm{lk}_G q't = G[\{t''\} \cup (N_{q'} \cap C_2)]$, so $N_{q'} \cap C_2$ induces a path within C_2 and t'' is not adjacent to its internal vertices. Since $v_1, v_2 \in N_{q'} \cap C_2$ we obtain that $N_{q'} \cap C_2 = (C_2 \setminus N_q) \cup \{v_1, v_2\}$.

Let $|C_1| = k, |C_2| = l$. Subtracting the edges we lose from P_2 to $(N_q \cap C_2) \setminus \{v_1, v_2\}$ and from $t'' \in \overline{P_1}$ to $C_2 \setminus N_q$ and using $\deg(q', C_1) \leq k-1$, $|P_2| \geq 1$ and $a \geq 3$ we get

$$\begin{aligned} e(G) &\leq kl + k + l + (a+2) + (l - a + 2 + k - 1) + 1 - |P_2|(a-2) - (l-a) \\ &\leq kl + 2k + l + 6. \end{aligned}$$

Next consider the case $N_{q'} \cap P_1 = \emptyset$. By the usual argument we have $C_2 \subseteq N_u, N_v$. Let $s \in P_2$ be the neighbor of v. Then $\mathrm{lk}_G q'v = \{s, q\} \cup (N_{q'} \cap C_2)$ and it contains the edges $v_1 q v_2$. It follows that there are vertices $w_1, w_2 \in C_2$ such that $N_{q'} \cap C_2$ has two parts, stretching from v_1 to w_1 and from v_2 to w_2 (possibly $w_1 = v_1$ or $w_2 = v_2$). Moreover, looking at $\mathrm{lk}_G q'v$ we see that $sw_1, sw_2 \in E(G)$ but s is not adjacent to the vertices in $(N_{q'} \cap C_2) \setminus \{w_1, w_2\}$.

Let $b = |N_{q'} \cap C_2|$. Counting the missing edges from P_2 to $(N_q \cap C_2) \setminus \{v_1, v_2\}$ and the disjoint set of missing edges from s to $(N_{q'} \cap C_2) \setminus \{w_1, w_2\}$ we have:

$$\begin{aligned} e(G) &\leq kl + k + l + (a+2) + (b + k - 1) + 1 - |P_2|(a-2) - (b-2) \\ &\leq kl + 2k + l + 6. \end{aligned}$$

An application of Lemma 8.35 completes the proof. □

Putting the above propositions together we get the main result of this section concerning 2-joinlike graphs.

Proposition 8.39. *If G is 2-joinlike with exceptional vertices $\{q, q'\}$ and $\deg(q, C_1) \leq 2$ then $e(G) \leq \frac{1}{4}(n^2 + 2n + 17)$ where $n = |V(G)|$.*

Proof. This is a consequence of Propositions 8.34, 8.36 and 8.37. □

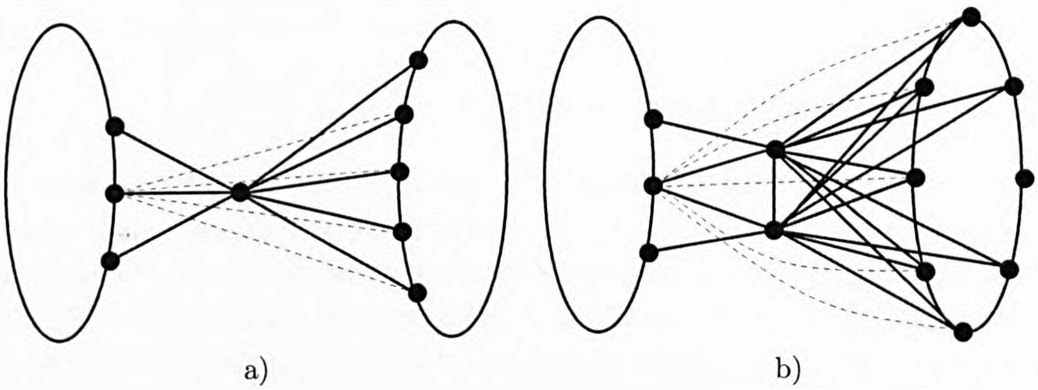


Figure 8.2: The 1-skeleta of two triangulations of S^3 with $f_1 = \frac{1}{4}(f_0^2 + 2f_0 + 17)$. Starting from the join of two cycles remove the dashed edges and add the exceptional point(s) with the solid edges. In a) $|C_1| = |C_2|$ and $\deg(q, C_1) = 3$. In b) $|C_2| = |C_1| + 1$, $\deg(q, C_1) = \deg(q', C_1) = 2$ and $\deg(q, C_2) = \deg(q', C_2) = |C_2| - 1$.

8.5 Closing remarks

A careful analysis of the proofs in Section 8.4 reveals two families of fascinating graphs which satisfy the equality $m = \frac{1}{4}(n^2 + 2n + 17)$ for $n \geq n_0$. They appear in Proposition 8.32 and Proposition 8.36, see Figure 8.2. This proves the claim made in Remark 8.7; we omit the details.

It is natural to try to generalize Theorem 8.2 to higher dimensions.

Conjecture 8.40. *For every $s \geq 2$ there exists a number $n_0 = n_0(s)$ such that the following holds. If M is a closed flag $(2s-1)$-manifold or a flag $(2s-1)$-GHS with $f_0 \geq n_0$ vertices and f_1 edges then*

$$f_1 \leq f_0^2 \cdot \frac{s-1}{2s} + f_0. \tag{8.30}$$

Moreover, if M satisfies

$$f_1 > f_0^2 \cdot \frac{s-1}{2s} + f_0 \cdot \frac{s-1}{s} + \frac{7s+3}{2s} \tag{8.31}$$

then M is a join of s polygons, in particular it is homeomorphic to S^{2s-1}.

The maximal value in (8.30) is achieved by the balanced join of s cycles of lengths f_0/s. The expression in (8.31) is the number of edges in the single edge-subdivision of such a join.

Let us sketch how one might prove this conjecture (the details will appear elsewhere). Fix $s \geq 2$ and denote $n = f_0$. First of all, M is Eulerian and the "middle" Dehn-Sommerville equation $h_{s-1} = h_{s+1}$ can be rewritten in the form

$$f_s = sf_{s-1} + a_2 f_{s-2} + \cdots + a_s f_0$$

for some coefficients a_i depending only on s. It follows that the number of $(s+1)$-cliques in the 1-skeleton $G = M^{(1)}$ is only $O(n^s)$. However, the number of edges in G is above the Turán bound for a complete, balanced s-partite graph, which is the maximizer of the number of edges among K_{s+1}-free graphs. By an application of the stability method we get that G looks very similar to $K_{\ell,\ell,\ldots,\ell}$, where $\ell = n/s$. Next, as in the case of fascinating graphs, we see that in G the link of every $(2s-1-j)$-clique is a triangulation of S^j for $j = 0, 1, 2$ (or for all $0 \leq j \leq 2s-2$ if M is a manifold) and one can try to exploit those conditions to rigidify the structure of G.

Finally, we suggest that the inequality $\gamma_2 \geq 0$, conjectured by Gal for flag spheres, expressed in terms of the face numbers, ought to hold for any flag manifold. This would be the flag analogue of the Lower Bound Theorem [15].

Conjecture 8.41. *If M is a flag d-manifold, then*

$$f_1 \geq (2d-1)f_0 - 2(d^2-1).$$

This is known to hold for a flag 3-GHS, where it is equivalent to the 3-dimensional Charney-Davis conjecture, proved in [29].

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Bibliography

[1] A. Adamaszek, M. Adamaszek, *Large-girth roots of graphs*, SIAM J. Discrete Math. Volume 24, Issue 4, pp. 1501-1514 (2010)

[2] M. Adamaszek, *Splittings of independence complexes and the powers of cycles*, J. Combin. Theory Ser. A, 119 (2012), 1031-1047

[3] M. Adamaszek, *Clique complexes and Graph powers*, Israel J. Math., to appear

[4] M. Adamaszek, *Special cycles in independence complexes and superfrustration in some lattices*, `arxiv/1109.4914`

[5] M. Adamaszek, *Hard squares on cylinders revisited*, `arxiv/1202.1655`

[6] M. Adamaszek, J.A. Barmak, *On a lower bound for the connectivity of the independence complex of a graph*, Discrete Mathematics 311(21): 2566-2569 (2011)

[7] M. Adamaszek, J. Hladký, *Dense flag triangulations of 3-manifolds via extremal graph theory*, `arxiv/1205.4060`

[8] M. Adamaszek, J. Stacho, *Algorithmic complexity of finding cross-cycles in flag complexes*, Proceedings of 28th Symposium on Computational Geometry (SoCG 2012)

[9] S.I. Adjan, *The algorithmic unsolvability of problems concerning certain properties of groups* (in Russian), Dokl. Akad. Nauk SSSR 103 (1955), 533-535.

[10] R. Aharoni, E. Berger, R. Ziv, *Independent systems of representatives in weighted graphs.* Combinatorica 27(3) (2007), 253-267.

[11] R. Aharoni, personal communication.

[12] C. Bachoc, A. Pêcher, A. Thiéry, *On the theta number of powers of cycle graphs*, `arxiv/1103.0444`

[13] J.A. Barmak, *Star clusters in independence complexes of graphs*, `arxiv/1007.0418`

[14] J.A. Barmak, private communication of a proof by M. Tancer, April 2012

[15] D. Barnette, *Graph theorems for manifolds*, Israel J. Math. 16 (1973), 62-72

[16] A. Björner, *Topological methods*, in "Handbook of Combinatorics" (eds. R. Graham, M. Grötschel and L. Lovász), North-Holland, Amsterdam, 1995, pp. 1819–1872

[17] A. Björner, G. Kalai, *An extended Euler-Poincare theorem*, Acta Math. 161 (1988), 279-303

[18] A. Björner, M. Tancer, *Note: Combinatorial Alexander Duality – A Short and Elementary Proof*, Discrete & Computational Geometry 42(4): 586-593 (2009)

[19] J.A. Bondy, U.S.R. Murty, *Graph theory*, Graduate Texts in Mathematics, 244, Springer, New York, 2008

[20] R. Boulet, E. Fieux, B. Jouve, *Simplicial simple-homotopy of flag complexes in terms of graphs*, European Journal of Combinatorics Volume 31, Issue 1, January 2010, p. 161-176

[21] M. Bousquet-Melou, S. Linusson, E. Nevo, *On the independence complex of square grids*, Journal of Algebraic Combinatorics, Vol. 27, Number 4 (2008), 423-450

[22] B. Braun, *Independence Complexes of Stable Kneser Graphs*, Electronic Journal of Combinatorics, P118, Vol. 18(1), 2011

[23] A.E. Brouwer, P. Duchet, A. Schrijver, *Graphs whose neighbourhoods have no special cycles*, Disc. Math. 47 (1983) 177-182

[24] J. Brown, R. Hoshino, *Independence polynomials of circulants with an application to music*, Disc. Math. 309 (2009) 2292-2304

[25] R. Charney, M. Davis, *The Euler characteristic of a non-positively curved, piecewise Euclidean manifold*, Pacific J. Math., 171 (1995), 117-137

[26] P. Csorba, *Homotopy types of box complexes*, Combinatorica 27(6): 669–682 (2007)

[27] P. Csorba, *Subdivision yields Alexander duality on independence complexes*, Electronic J. Comb. 16(2) (2009), #R11

[28] M. Davis, *Unsolvable problems*, In: Handbook of mathematical logic, North-Holland (1977), 567-594.

[29] M. Davis, B. Okun, *Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups*, Geom. Topol. 5 (2001), 7-74

[30] G. A. Dirac, *On rigid circuit graphs*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 25 (1961) 71-76

[31] A. Dochtermann, *The universality of* Hom *complexes of graphs*, Combinatorica 29 (4) (2009), 433-448

[32] A. Dochtermann, A. Engström, *Algebraic properties of edge ideals via combinatorial topology*, Electron. J. Combin., Volume 16(2), 2009

[33] H. van Eerten, *Extensive ground state entropy in supersymmetric lattice models*, J. Math. Phys. 46, 123302 (2005)

[34] R. Ehrenborg, G. Hetyei, *The topology of the independence complex*, Eur. J. Comb. 27(6): 906-923 (2006)

[35] A. Engström, *Complexes of directed trees and independence complexes*, Discrete Mathematics 309(10): 3299-3309 (2009)

[36] A. Engström, *Independence complexes of claw-free graphs*, Eur. J. Comb. 29(1): 234–241 (2008)

[37] A. Engström, *Upper bounds on the Witten index for supersymmetric lattice models by discrete Morse theory*, Eur. J. Comb. 30(2): 429-438 (2009)

[38] P. Erdős, M. Simonovits, *Supersaturated graphs and hypergraphs*, Combinatorica 3 (1983), 181-192

[39] M. Farber, *Independent domination in chordal graphs*, Operations Research Letters 1 (1982) 134-138

[40] P. Fendley, K. Schoutens, *Exact Results for Strongly Correlated Fermions in* $2+1$ *Dimensions*, Phys. Rev. Lett. 95, 046403 (2005)

[41] P. Fendley, K. Schoutens, W. van Eerten, *Hard squares with negative activity*, J. Phys. A, 38(2): 315-322, 2005

[42] R. Forman, *Morse Theory for Cell Complexes*, Adv. Math. 134 (1998), 90-145

[43] R. Forman, *A user's guide to discrete Morse theory*, Sém. Lothar. Combin. 48 (2002)

[44] Ś. R. Gal, *Real root conjecture fails for five- and higher-dimensional spheres*, Discrete Comput. Geom. 34 (2005), 269–284

[45] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman, 1979

[46] F. Gavril, *The intersection graphs of subtrees in trees are exactly the chordal graphs*, Journal of Combinatorial Theory B 16 (1974), 47-56

[47] E. Gawrilow, M. Joswig. *polymake: a framework for analyzing convex polytopes*, Polytopes — combinatorics and computation (Oberwolfach, 1997), 43–73, DMV Sem., 29, Birkhäuser, Basel, 2000

[48] H.T. Hà, A. Van Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin. 27 (2008), no. 2, 215-245,

[49] W. Haken, *Connections between topological and group theoretical decision problems*, In: Boone, Cannonito and Lyndon (1973), 427-441.

[50] K.A. Hardie, J.J.C. Vermeulen, P.J. Witbooi, *A nontrivial pairing of finite T_0 spaces*, Topology Appl. 125 (2002), no. 3, 533-542

[51] A. Hatcher, *Algebraic Topology*, Camb. Univ. Press, 2002.

[52] R. Hoshino, *Independence polynomials of circulant graphs*, PhD. Thesis, Dalhouse University, 2007

[53] L. Huijse, J. Halverson, P. Fendley, K. Schoutens, *Charge Frustration and Quantum Criticality for Strongly Correlated Fermions*, Phys. Rev. Lett. 101, 146406 (2008)

[54] L. Huijse, K. Schoutens, *Superfrustration of charge degrees of freedom*, The European Physical Journal B, Volume 64, Numbers 3-4, 543-550

[55] L. Huijse, K. Schoutens, *Supersymmetry, lattice fermions, independence complexes and cohomology theory*, Adv. Theor. Math. Phys., 14.2 (2010), 643-694

[56] L. Huijse, D. Mehta, N. Moran, K. Schoutens, J. Vala, *Supersymmetric lattice fermions on the triangular lattice: superfrustration and criticality*, `arxiv/1112.3314`

[57] K. Iriye, *On the homotopy types of the independence complexes of grid graphs with cylindrical identification*, Kyoto J. Math., to appear

[58] K. Iriye, private communication, April 2012

[59] D. Jojic, *Shellability of complexes of directed trees*, `arxiv/1109.4475`

[60] J. Jonsson, *Hard Squares with Negative Activity on Cylinders with Odd Circumference*, Electr. J. Comb. 16(2) (2009), #R5

[61] J. Jonnson, *Hard Squares with Negative Activity and Rhombus Tilings of the Plane*, Electr. J. Comb. 13(1) (2006), #R67

[62] J. Jonsson, *Hard squares on grids with diagonal boundary conditions*, preprint, 2008

[63] J. Jonsson, *Certain Homology Cycles of the Independence Complex of Grids*, Discrete & Computational Geometry 43(4): 927-950 (2010)

[64] J. Jonsson *On the topology of independence complexes of triangle-free graphs*, unpublished manuscript, `http://www.math.kth.se/~jakobj/doc/preprints/indbip.pdf`

[65] M. Joswig, F.H. Lutz, *One-point suspensions and wreath products of polytopes and spheres*, J. Combin. Theory Ser. A 110 (2005), no. 2, 193-216

[66] M. Kahle, *Topology of random clique complexes*, Discrete Math. 309(6): 1658-1671 (2009)

[67] M. Kahle, E. Meckes, *Limit theorems for Betti numbers of random simplicial complexes*, `arxiv/1009.4130`

[68] V. Kaibel, M. E. Pfetsch, *Some Algorithmic Problems in Polytope Theory* in Algebra, Geometry, and Software Systems 2003: 23-47

[69] K. Kawamura, *Independence complexes of chordal graphs*, Discrete Math. 310 (2010), 2204-2211

[70] K. Kawamura, *Homotopy types of independence complexes of forests*, Contr. to Disc. Math. 5 (2010), 67–75

[71] D. Kozlov, *Complexes of directed trees*, J. Combin. Theory A 88 (1) (1999), 112-122

[72] D. Kozlov, *Combinatorial Algebraic Topology*, Algorithms and Computation in Mathematics, Vol. 21, Springer-Verlag Berlin Heidelberg 2008

[73] D. Kozlov, *Convex hulls of f- and β-vectors*, Discrete Comput. Geom. 18 (1997), no. 4, 421-431

[74] D. Kozlov, *Maximizing the Möbius function of a poset and the sum of Betti numbers of the order complex*, Combinatorica 19 (4) (1999) 533-548

[75] F. Larrión, M.A. Pizaña, R. Villarroel-Flores, *The fundamental group of the clique graph*, European Journal of Combinatorics, Volume 30, Issue 1, January 2009, pp. 288-294

[76] V. E. Levit, E. Mandrescu, *A simple proof of an inequality connecting the alternating number of independent sets and the decycling number*, Discrete Mathematics, Volume 311, Issue 13, 6 July 2011, Pages 1204-1206

[77] V. E. Levit, E. Mandrescu, *The independence polynomial of a graph at -1*, `arxiv/0904.4819`

[78] M. Marietti, D. Testa, *Cores of simplicial complexes*, Disc. Comput. Geom. 40 (2008), 444–468

[79] J.P. May, *Weak equivalences and quasifibrations*, In: R.Piccinini, ed., Groups of self-equivalences and related topics, Lecture Notes in Mathematics Vol. 1425, Springer-Verlag, 1990, 91-101

[80] M.C. McCord, *Singular homology groups and homotopy groups of finite topological spaces*, Duke Math. J. Volume 33, Number 3 (1966), 465-474

[81] B.D. McKay, The *Nauty* graph automorphism package, `http://cs.anu.edu.au/~bdm/nauty/`

[82] P. McMullen, *The numbers of faces of simplicial polytopes*, Israel J. Math. 9 (1971), 559-570

[83] R. Meshulam, *Domination numbers and homology*, J. Combin. Theory Ser. A 102 (2003), 321-330

[84] J. Milnor, *Construction of universal bundles, II*, Annals of Math., Vol.63, No.3, May 1956, pp. 430-436

[85] R. Motwani, M. Sudan, *Computing Roots of Graphs is Hard*, Discrete Applied Mathematics 54(1): 81-88 (1994)

[86] S. Murai, E. Nevo, *The flag f-vectors of Gorenstein* order complexes of dimension* 3, `arXiv/1108.0572`

[87] M. Muzychuk, *A solution of the isomorphism problem for circulant graphs*, Proc. London Math. Soc. (3) 88 (2004) 1-41

[88] E. Nevo, T. K. Petersen, *On γ-vectors satisfying the Kruskal-Katona inequalities*, Discrete Comput. Geom. 5 (2011), 503–521

[89] E. Nevo, T. K. Petersen, B. E. Tenner, *The γ-vector of a barycentric subdivision*, J. Comb. Theory Ser. A 118 (2011), 1346-1380

[90] The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`

[91] M.O. Rabin, *Recursive unsolvability of group theoretic problems*, Ann. of Math. 67 (1958), 172-194.

[92] D. J. Rose, R. E. Tarjan, G. S. Leuker, *Algorithmic aspects of vertex elimination on graphs*, SIAM Journal on Computing 5 (1976) 266-283

[93] B. H. Roune, E. Sáenz de Cabezón, *Complexity and Algorithms for Euler Characteristic of Simplicial Complexes*, `arxiv/1112.4523`

[94] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), 279-319, Academic Press, New York, 1966

[95] J. Shareshian, M. L. Wachs, *Torsion in the matching complex and chessboard complex*, Adv. Math. 212 (2007), no. 2, 525-570

[96] R. P. Stanley, *Combinatorics and commutative algebra*, Progress in Mathematics 41, Birkhauser, 1983

[97] R. P. Stanley, *The number of faces of a simplicial convex polytope*, Adv. Math. 35 (1980), 236-238

[98] D. Testa, private communication.

[99] J. Thapper, *Independence complexes of cylinders constructed from square and hexagonal grid graphs*, Lic. Thesis, `arxiv/0812.1165`

[100] A. Van Tuyl, R. H. Villarreal, *Shellable graphs and sequentially Cohen-Macaulay bipartite graphs*, J. Combin. Theory Ser. A 115 (2008), no. 5, 799–814

[101] R. Woodroofe, *Vertex decomposable graphs and obstructions to shellability*, Proc. Amer. Math. Soc. 137 (2009), No. 10, 3235–3246

[102] G. M. Ziegler, *Posets with maximal Möbius function*, J. Combin. Theory Ser. A, 56 (1991), 203-222

[103] A. Zomorodian, *The tidy set: a minimal simplicial set for computing homology of clique complexes*, Symposium on Computational Geometry 2010, 257-266