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Eigenforms of Half-Integral Weight

by

Soma Purkait

Thesis

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Contents

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Declarations

Chapter 2 of this thesis summarizes background results found in the literature and is not my work, except for the presentation. Chapters 3, 4, 5 are entirely my own work, except where explicitly indicated.

Abstract

Let k be an odd integer and N a positive integer such that $4 \mid N$. Let χ be a Dirichlet character modulo N. Shimura decomposes the space of half-integral weight forms $S_{k/2}(N,\chi)$ as

$$S_{k/2}(N,\chi) = S_0(N,\chi) \oplus \bigoplus_{\phi} S_{k/2}(N,\chi,\phi)$$

where ϕ runs through the newforms of weight $k-1$ and level dividing $N/2$ and character χ^2; $S_{k/2}(N,\chi,\phi)$ is the subspace of forms that are Shimura-equivalent to ϕ; and $S_0(N,\chi)$ is the subspace generated by single-variable theta-series. We give an explicit algorithm for computing this decomposition.

Once we have the decomposition, we can explore Waldspurger's theorem expressing the critical values of the L-functions of twists of an elliptic curve in terms of the coefficients of modular forms of half-integral weight. Following Tunnell, this often allows us to give a criterion for the n-th twist of an elliptic curve to have positive rank in terms of the number of representations of certain integers by certain ternary quadratic forms.

Chapter 1

Introduction

1.1 Overview of previous work

In 1983 J. B. Tunnell gave a remarkable solution to the *congruent number problem*, assuming the celebrated Birch and Swinnerton-Dyer Conjecture. This ancient Diophantine question asks for the classification of congruent numbers, those positive integers which are the areas of right-angled triangles whose sides are rational numbers.

Let n be a square-free positive integer. It is relatively easy to show that n is a congruent number if and only if the elliptic curve

$$E_n \;:\; Y^2 = X^3 - n^2 X$$

has infinitely many rational points. If E_n has infinitely many rational points, then by a theorem of Coates and Wiles (which is a special case of the Birch and Swinnerton-Dyer Conjecture), $\mathrm{L}(E_n, 1) = 0$, where L is the L-function of the elliptic curve E_n. If we assume the Birch and Swinnerton-Dyer Conjecture, then the reverse implication holds: if $\mathrm{L}(E_n, 1) = 0$ then E_n has infinitely many rational points.

Note here that E_n is the quadratic twist of the elliptic curve

$$E_1 \;:\; Y^2 = X^3 - X,$$

by n. Tunnell [42] proved the following theorem.

Theorem 1.1.1 (Tunnell). *If n is a square-free odd positive integer that is a congruent number, then*

$$\#\{x, y, z \in \mathbb{Z} \mid n = 2x^2 + y^2 + 32z^2\} = \frac{1}{2}\#\{x, y, z \in \mathbb{Z} \mid n = 2x^2 + y^2 + 8z^2\}.$$

If n is a square-free even positive integer that is a congruent number then,

$$\#\{x, y, z \in \mathbb{Z} \mid \frac{n}{2} = 4x^2 + y^2 + 32z^2\} = \frac{1}{2}\#\{x, y, z \in \mathbb{Z} \mid \frac{n}{2} = 4x^2 + y^2 + 8z^2\}.$$

If the weak Birch and Swinnerton-Dyer Conjecture is assumed for E_n, then, conversely, these equalities imply that n is a congruent number.

The proof of Tunnell's Theorem comprises of two main steps. The first step is to explicitly construct certain cusp forms of weight 3/2 which are "Shimura-equivalent" to the newform of weight 2 corresponding to the elliptic curve E_1 via the Modularity Theorem. The second is to apply Waldspurger's Theorem 4.3.4 to these cusp forms; this relates the critical value of the L-function of a modular form of even integral weight to the square of the coefficients of the q-expansion of a corresponding form (again via Shimura-equivalence) of half-integral weight.

Given an elliptic curve $E/\mathbb{Q}$, one can ask similar questions:

- Which of the quadratic twists of E have infinitely many rational points?
- Is there a nice formula for the critical value of the L-function as in the case of the congruent number curve?

To be able to answer such questions, we would like to explicitly construct the half-integral weight forms corresponding via Shimura-equivalence to the elliptic curve E. It is well-known by the results of Flicker (Theorem 4.3.1) and Vigneras (Theorem 4.3.2) that such a half-integral weight forms exist, although there is no indication of their levels.

One of the methods to construct cusp forms of weight 3/2 is as in the paper of Tunnell. Let $M_{1/2}(N_1, \psi_1)$ be the space of modular forms of weight 1/2, level N_1 and character ψ_1, and let $S_1(N_2, \psi_2)$ be the space of cusp forms of weight 1, level N_2 and character ψ_2. Then

$$M_{1/2}(N_1, \psi_1) \otimes S_1(N_2, \psi_2) \subset S_{3/2}(N, \psi_1.\psi_2.\chi_{-1})$$

where $N = \operatorname{lcm}(N_1, N_2)$. A basis for the space $M_{1/2}(N, \psi)$ is given by Serre and Stark (see Theorem 2.3.4). Also, due to Deligne and Serre [15], there is one-to-one correspondence between newforms in $S_1(N, \psi)$ and certain two-dimensional Galois representations of the absolute Galois group $G_{\mathbb{Q}}$. For more details, see for example, [2].

Tunnell in fact used this idea and constructed a unique normalized newform g of weight 1, level 128 and character $\chi_{-2} := \left(\frac{-2}{\cdot}\right)$, having q-expansion

$$g = \sum_{m,n \in \mathbb{Z}} (-1)^n q^{(4m+1)^2 + 8n^2}.$$

For an integer t it is known that $\theta_t = \sum_{-\infty}^{\infty} q^{tm^2}$ is a modular form of weight $1/2$, level $4t$ and character $\chi_t := \left(\frac{t}{\cdot}\right)$. Thus,

$$g\theta_2 \in S_{3/2}(128, \chi_{\text{triv}}), \qquad g\theta_4 \in S_{3/2}(128, \chi_2).$$

Moreover, it turns out that $g\theta_2$ and $g\theta_4$ are Shimura-equivalent to the newform corresponding to E_1. Let $g\theta_2 = \sum a_n q^n$ and $g\theta_4 = \sum b_n q^n$. Tunnell showed that if d is an odd positive square-free integer, then

$$\mathrm{L}(E_d, 1) = a_d^2 \cdot \frac{\Omega}{4\sqrt{d}}, \qquad \mathrm{L}(E_{2d}, 1) = b_d^2 \cdot \frac{\Omega}{2\sqrt{2d}}$$

where Ω denotes the real period of E_1 given by

$$\Omega = \int_1^{\infty} \frac{dx}{(x^3 - x)^{1/2}} = 2.62205\ldots.$$

In particular, $\mathrm{L}(E_d, 1) = 0$ if and only if $a_d = 0$ for d odd, and if and only if $b_{\frac{d}{2}} = 0$ for d even. The Birch and Swinnnerton-Dyer Conjecture now implies that d (respectively $2d$) is congruent number if and only if $a_d = 0$ (respectively $b_d = 0$).

In general, however, it is not known, given an elliptic curve $E/\mathbb{Q}$, that one can always construct corresponding modular forms of weight $3/2$ by the above method. For example, in [3], Basmaji considers the elliptic curve $53A$ given by

$$E \,:\, Y^2 + XY + Y = X^3 - X^2.$$

He examines the space of cusp forms of weight 3/2 and level up to $2^4 \cdot 53$, using the above method. It turns out that there is no linear combination of theta-series, obtained from multiplying theta-series of weight 1/2 with forms of weight 1, that gives an eigenform corresponding to E. However there exists an eigenform corresponding to E in $S_{3/2}(\Gamma_0(2^4 \cdot 53))$, namely,

$$F_E(z) = q - 2q^4 - 2q^9 + q^{13} - q^{16} - q^{17} - 2q^{24} - q^{25} + q^{28} + q^{29} + 4q^{36} + 5q^{37} + O(q^{40}).$$

Another possible way to construct such cusp forms of weight 3/2 is using positive-definite ternary quadratic forms. Each positive definite ternary quadratic form of level N, gives rise to a theta-series of weight 3/2 and level N. It is possible to obtain part of the space $S_{3/2}(N)$ this way, but not the whole space in general. In particular, the cusp form of weight 3/2 corresponding to the elliptic curve that we are interested in, might not arise from ternary quadratic forms. For example, let E be the curve $121D$, given by Weierstrass equation

$$E \,:\, Y^2 + Y = X^3 - X^2 - 7X + 10.$$

Bungert in [7] examined the spaces of theta-series of positive-definite ternary forms up to level $2^4 \cdot 121$, and showed that in these spaces, no cusp form of weight 3/2 exists which corresponds to E. Bungert however constructed such a cusp form of weight 3/2 using a two dimensional Galois representation as mentioned above. We will be discussing more about such cusp forms which come from ternary quadratic forms in the later chapters.

On the other hand, Kohnen in his paper [24] looks into a suitable subspace of the space of half-integral weight cusp forms for which the Shimura correspondence turns out to be an isomorphism of Hecke modules. For N a positive odd square-free integer, and λ a positive integer Kohnen defines what is called the *Kohnen plus space* $S^+_{\lambda+\frac{1}{2}}(4N)$, as follows:

$$S^+_{\lambda+\frac{1}{2}}(4N) := \{g(z) = \sum_{n=1}^{\infty} b_n q^n \in S_{\lambda+\frac{1}{2}}(4N) \text{ such that } b_n = 0 \text{ for } (-1)^\lambda n \equiv 2, 3 \pmod 4\}.$$

It is shown by Kohnen that this subspace of cusp forms is invariant under the action of the Hecke operators T_{p^2} for all primes p coprime to $4N$. Kohnen develops a theory of newforms for this subspace analogous to Atkin-Lehner's theory in the integral case and proves the strong 'multiplicity-one theorem' for ${S^+_{\lambda+\frac{1}{2}}}^{new}(4N)$ in this case. It is to be noted that the multiplicity-one theorem does not hold for a general level N. Kohnen proved the following remarkable theorem.

Theorem 1.1.2 (Kohnen)**.** *For N odd and square-free, there is an isomorphism between ${S^+_{\lambda+\frac{1}{2}}}^{new}(4N)$ and $S^{new}_{2\lambda}(N)$ as Hecke modules.*

The isomorphism is given by finite linear combination of Shimura correspondences.

In the later papers Kohnen and Zagier [26] proved the following formula for level 4 which was later generalized by Kohnen [25] to odd square-free level N:

Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S^{new}_{2\lambda}(N)$ be a newform of odd square-free level N and let $g(z) = \sum_{n=1}^{\infty} b_n q^n \in {S^+_{\lambda+\frac{1}{2}}}^{new}(4N)$ be the corresponding form under the above isomorphism. Let D be a fundamental discriminant such that $(-1)^\lambda D > 0$ and $(N, D) = 1$. Then,

$$\frac{\mathrm{L}(f, D, \lambda)}{\langle f, f\rangle} = \frac{(b_{|D|})^2}{\langle g, g\rangle} \frac{\pi^\lambda}{2^{\nu_1(N)}(\lambda-1)!|D|^{\lambda-1/2}},$$

$\nu_1(N)$ denotes for number of different prime divisors of N.

Kohnen's work is based on explicit relations involving traces of Hecke operators. This work has been generalized by Ueda [43] to a general odd level and recently Sakata [31] has given generalizations for the Kohnen-Zagier formula for such levels (with weights $\lambda \geqq 2$).

1.2 This Thesis

This thesis attempts to answer the questions raised in the previous section. We summarize the results step by step as follows:

1. Given a newform of even integral weight k, we give an algorithm to find the space of forms of weight $k + 1/2$ which are "Shimura equivalent" to

the newform. In particular, this leads to an algorithm for computing an eigenbasis for a space of half-integral weight forms under the action of Hecke operators T_{p^2} with p not dividing the level.

2. We simplify Waldspurger's Theorem in the case where the half-integral weight forms correspond to newforms with trivial character, and develop results that allow us to apply it.

3. We give examples of Tunnell-like formulae for $\mathrm{L}(E_n, 1)$ in terms of ternary quadratic forms, for certain rational elliptic curves E and certain families of twists E_n.

Chapter 2 of this thesis introduces basic definitions and parts of the theory of modular forms that we require in the rest of the thesis. Chapter 3 consists of several results which finally lead to our algorithm for computing the space of Shimura equivalent forms. In the process we also prove certain interesting theorems which we will be using in the later chapters. In Chapter 4 we discuss Waldspurger's Theorem in detail and simplify it for our use. We present some examples of elliptic curves for which we use our algorithm and Waldspurger's Theorem to give some explicit formulae for the critical values of L-functions of the quadratic twists. Finally in the last chapter we discuss the relation between modular forms and quadratic forms and we conclude with examples of Tunnell-like formulae in terms of ternary quadratic forms. In the Appendix, we give a table for the dimension of $S_{3/2}(N)$ and some of its subspaces, for $N \leq 2000$.

What follows is an example of the results we develop in the thesis; it is in fact Example 5.3.1 given in Chapter 5. Let E be an elliptic curve of conductor 50 given by

$$E : Y^2 + XY + Y = X^3 + X^2 - 3X + 1.$$

Let $Q_1, \ldots, Q_4$ be the following positive-definite ternary quadratic forms,

$$Q_1 = 25x^2 + 25y^2 + z^2, \qquad Q_2 = 14x^2 + 9y^2 + 6z^2 + 4yz + 6xz + 2xy$$

$$Q_3 = 25x^2 + 13y^2 + 2z^2 + 2yz, \qquad Q_4 = 17x^2 + 17y^2 + 3z^2 - 2yz - 2xz + 16xy.$$

Let n be positive square-free number such that $5 \nmid n$. Then,

$$\mathrm{L}(E_{-n}, 1) = \frac{\mathrm{L}(E_{-1}, 1)}{\sqrt{n}} \cdot c_n^2,$$

where

$$c_n = \sum_{i=1}^{4} \frac{(-1)^{i-1}}{2} \cdot \#\{(x, y, z) : Q_i(x, y, z) = n\}.$$

Chapter 2

Background

2.1 Congruence Subgroups

All the material in this section is standard, and can be found in any book on modular forms; for example [16], [28], [23].

Let R be any commutative ring with unity. We denote by $\mathrm{GL}_2(R)$, the group of 2×2 matrices with entries in R and determinant an invertible element of R. By $\mathrm{SL}_2(R)$ we denote the subgroup of $\mathrm{GL}_2(R)$ consisting of matrices with determinant 1. We will be generally interested in these groups when the ring $R = \mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$ and in those cases $\mathrm{GL}_2^+(R)$ stands for the subgroup of $\mathrm{GL}_2(R)$ consisting of matrices with positive determinant.

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\overline{\mathbb{C}}$ by the Möbius transformation, i.e., given $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{C}$,

$$Az := \frac{az+b}{cz+d}, \qquad A\infty := \frac{a}{c}.$$

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the complex upper half-plane. Then $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ since $\mathrm{Im}(Az) = |cz+d|^{-2}\det(A)\mathrm{Im}(z)$ and one can easily prove that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive.

In what follows, we will be interested in the group $\mathrm{SL}_2(\mathbb{Z})$, also known as the full modular group, and some of its special subgroups:

Definition 2.1.1. *Let N be a positive integer. Then*

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N,\ b \equiv c \equiv 0 \pmod N \right\},$$
$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N,\ c \equiv 0 \pmod N \right\},$$
$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}.$$

Definition 2.1.2. *A subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup of level N if it contains $\Gamma(N)$ for some positive integer N. Thus $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$ are congruence subgroups of level N.*

Note that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ and that $\Gamma(N') \subset \Gamma(N)$ whenever $N \mid N'$.

Proposition 2.1.3. *Let N be a positive integer. Then*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right),$$
$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right).$$

Proof. See either [23, Exercise III.1.7] or [16, Page 14]. □

It is easy to see that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the set $\mathbb{Q} \cup \{\infty\}$.

Definition 2.1.4. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Define a cusp of Γ to be an equivalence class of $\mathbb{Q} \cup \{\infty\}$ under the action of Γ on $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$.*

2.2 Modular Forms of Integral Weight

We continue reviewing standard material on modular forms.

Let k be a positive integer. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on the set of complex valued function on $\mathbb{H}$ as follows. Let $f : \mathbb{H} \to \mathbb{C}$ and $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$, then

$$f|[\gamma]_k(z) := \det(\gamma)^{k/2} j(\alpha, z)^{-k} f(\gamma z)$$

is a function on $\mathbb{H}$, where $j(\gamma, z) = cz + d$.

Let Γ be a congruence subgroup of level N.

Definition 2.2.1. *A modular form of weight k for Γ is a holomorphic function on $\mathbb{H}$ which satisfies*

(i) $f|[\gamma]_k = f$ *for all* $\gamma \in \Gamma$, *and*

(ii) If $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$, *then* $f|[\gamma_0]_k(z)$ *has a Fourier expansion of the form* $\sum_{n=0}^{\infty} a_n {q_N}^n$ *where* $q_N := e^{2\pi i z/N}$.

The condition (ii) is interpreted as holomorphicity of f at all the cusps of Γ. We call a modular form a cusp form if it vanishes at all the cusps of Γ, i.e., in (ii) above, $a_0 = 0$ for all $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$.

We denote by $M_k(\Gamma)$ and $S_k(\Gamma)$ respectively, the space of modular forms and the space of cusp forms of weight k for level Γ.

If $\Gamma \subset \Gamma'$ then clearly $M_k(\Gamma') \subset M_k(\Gamma)$. Also, note that since $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ belongs to $\Gamma_0(N)$ and $\Gamma_1(N)$ for any N, if f belongs to either $M_k(\Gamma_0(N))$ or $M_k(\Gamma_1(N))$ then f has a Fourier expansion at ∞ given by $f(z) = \sum_{n=0}^{\infty} a_n q^n$ where $q = e^{2\pi i z}$. Since $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \Gamma_0(N)$, there are no nonzero modular forms of odd weight k for $\Gamma_0(N)$.

Let χ be a Dirichlet character modulo N. We denote $M_k(N, \chi)$ and $S_k(N, \chi)$ to be the respective subspaces of $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$ consisting of $f(z)$ such that $f|[\gamma]_k = \chi(d) f$ for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. If χ is a trivial character then we denote $M_k(N, \chi)$ and $S_k(N, \chi)$ simply by $M_k(N)$ and $S_k(N)$.

From now on we will be only interested in the congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$. We now give definition for Hecke operators on the space of modular forms in terms of double cosets.

Definition 2.2.2. *Let G be any group and Γ and Γ' be two subgroups of G. We say that Γ and Γ' are commensurable if*

$$[\Gamma : \Gamma \cap \Gamma'] < \infty \qquad \textit{and} \qquad [\Gamma' : \Gamma \cap \Gamma'] < \infty.$$

Definition 2.2.3. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ such that $\Gamma_0(N)$ and $\alpha^{-1}\Gamma_0(N)\alpha$ are commensurable. Let n be a positive integer. Then for any $f \in M_k(N)$ we have the following linear operators.*

(i)

$$f|[\Gamma_0(N)\alpha\Gamma_0(N)]_k(z) := \det(\alpha)^{k/2-1}\sum_{\nu=1}^{d} f|[\alpha_\nu]_k(z),$$

where $\Gamma_0(N)\alpha\Gamma_0(N) = \bigsqcup_{\nu=1}^{d}\Gamma_0(N)\alpha_\nu$.

(ii)

$$T_n(f) := \sum f|[\Gamma_0(N)\alpha\Gamma_0(N)]_k,$$

where the sum is over all $\alpha = \left[\begin{smallmatrix} l & 0 \\ 0 & m \end{smallmatrix}\right]$ *with* l, m *positive integers,* $l \mid m$, $(l, N) = 1$ *and* $lm = n$.

(iii) If $(n, N) = 1$, *then*

$$T_{(n,n)}(f) := f|[\Gamma_0(N)\left[\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}\right]\Gamma_0(N)]_k.$$

The operators T_n *and* $T_{(n,n)}$ *are called the Hecke operators.*

The Hecke operators so defined preserve the cusp forms and one can similarly define the Hecke operators on the space of modular forms with characters. The following proposition lists the important properties of the Hecke operators.

Proposition 2.2.4. *(a) If* $(m, n) = 1$, *then* $T_{mn} = T_m T_n$.

(b) If p *is a prime dividing* N, *then* $T_{p^e} = {T_p}^e$ *for any positive integer* e.

(c) If p *is a prime such that* $(p, N) = 1$, *then for any positive integer* e, $T_{p^{e+1}} = T_p T_{p^e} - pT_{(p,p)}T_{p^{e-1}}$ *where for* $f \in M_k(N, \chi)$ *the action of* $T_{(p,p)}$ *can be explicitly expressed as* $T_{(p,p)}(f) = p^{k-2}\chi(p)f$.

Proof. See [28, Lemma 4.5.7] and [28, Pages 142-143]. $\square$

Hence the Hecke operators form an algebra over $\mathbb{Z}$ generated by T_p, $T_{(p,p)}$ and T_q where p, q varies over primes with $p \nmid N$ and $q \mid N$. We can write the action of Hecke operators in terms of q-expansions.

Proposition 2.2.5. *Let* f *be a modular form in* $M_k(N, \chi)$ *with* q*-expansion* $f(z) = \sum_{n=0}^{\infty} a_n q^n$. *Then* $T_p(f)(z) = \sum_{n=0}^{\infty} b_n q^n$ *where,*

$$b_n = a_{pn} + \chi(p)p^{k-1}a_{n/p}.$$

Here we take $a_{n/p} = 0$ if $p \nmid n$.

Proof. See [28, Lemma 4.5.14]. $\square$

A modular form $f(z) \in M_k(N, \chi)$ is called a *Hecke eigenform* if for every positive integer m there exists $\lambda_m \in \mathbb{C}$ with $T_m(f) = \lambda_m f$.

Proposition 2.2.6. *Let $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_k(N, \chi)$ be a Hecke eigenform as above. Then,*

(i) If $f(z)$ is non constant, then $a_1 \neq 0$.

(ii) If $f(z)$ is a normalised cusp form, that is, $a_1 = 1$, then $a_m = \lambda_m$ for all m and $a_{mn} = a_m a_n$ whenever $(m, n) = 1$.

(iii) If $a_0 \neq 0$, then $\lambda_m = \sum_{d|m} \chi(d) d^{k-1}$.

Proof. See [30, Proposition 2.6] or [28, Theorem 4.5.16]. $\square$

Definition 2.2.7. *Let f and g be cusp forms in $S_k(N, \chi)$. Then their Petersson inner product $\langle f, g \rangle$ is defined as*

$$\langle f, g \rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]} \int_{\Gamma_1(N)\backslash\mathbb{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}, \qquad z = x + iy.$$

It is well-known that the Petersson inner product is well-defined and induces a Hermitian scalar product on the space $S_k(N, \chi)$; for details see [28, Page 44]. With respect to this inner product, if $\alpha_n = \sqrt{\chi(n)}$ and $(n, N) = 1$ then the operators $\alpha_n T_n$ are Hermitian:

Proposition 2.2.8.

$$\langle \alpha_n T_n f, g \rangle = \langle f, \alpha_n T_n g \rangle \qquad \text{if } (n, N) = 1.$$

Proof. See [28, Theorem 4.5.4]. $\square$

Thus, $S_k(N, \chi)$ has a basis consisting of eigenforms under all Hecke operators T_n with $(n, N) = 1$.

There are several other important operators on the space of integral weight modular forms.

Let $p \mid N$ be a prime and $Q_p = p^l$. The *Atkin-Lehner operator* $|[W_{Q_p}]_k$ on $M_k(N)$ is defined by any matrix of the form

$$W_{Q_p} := \begin{bmatrix} Q_p\alpha & \beta \\ N\gamma & Q_p\delta \end{bmatrix} \in M_2(\mathbb{Z}), \qquad \alpha, \beta, \gamma, \delta \in \mathbb{Z}$$

with determinant Q_p; different choices of α, β, γ and δ do not affect the action of W_{Q_p} on $M_k(N)$.

The *Fricke involution* $|[W_N]_k$ is defined by $W_N := \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]$. It is to be noted that $|[W_{Q_p}]_k$, $|[W_N]_k$ are involutions on $M_k(N)$ and commute with the Hecke operators T_n for $(n, N) = 1$ (see [30, Proposition 2.21]).

Further, we define V-operator and U-operator. Let d be a positive integer and $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_k(N, \chi)$. Then,

$$V(d)f(z) := \sum_{n=0}^{\infty} a_n q^{dn} \in M_k(Nd, \chi),$$
$$U(d)f(z) := \sum_{n=0}^{\infty} a_{dn} q^n \in M_k(N, \chi) \text{ if } d \mid N, \text{ else } \in M_k(Nd, \chi).$$

It is clear that if f is a cusp form then $V(d)f$ and $U(d)f$ both vanish at infinity. In fact, more is true: both $V(d)f$ and $U(d)f$ are cusp forms ([30, Proposition 2.22]). It is easy to verify that T_p commutes with the operator $U(d)$ and for p coprime to d, T_p commutes with $V(d)$.

Another important notion of modular forms we will be considering in the later sections is that of a twist with a Dirichlet character.

Definition 2.2.9. *Let $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_k(N, \chi)$. If ψ is a Dirichlet character, then the ψ-twist of f is defined by*

$$f_\psi(z) = \sum_{n=0}^{\infty} \psi(n) a_n q^n.$$

Proposition 2.2.10. *Let f be as above and ψ be a Dirichlet character of conductor m, then*

$$f_\psi(z) = \sum_{n=0}^{\infty} \psi(n) a_n q^n \in M_k(Nm^2, \chi\psi^2).$$

Moreover, if f is a cusp form then so is f_ψ.

Remark. Note that here, f_ψ does not have to be in the new subspace at level Nm^2. However, if we suppose $(N, m) = 1$ and that f is a newform of level N, then that would be true.

Proof of Proposition 2.2.10. See Proposition 17 in [23, Chapter III] for the proof. □

For more details on this twisting operator, see for example Theorem 4.2.2.

Let us now recall the theory of newforms. Define the space of oldforms $S_k^{\mathrm{old}}(N)$ in $S_k(N)$ by

$$S_k^{\mathrm{old}}(N) := \bigoplus_{\substack{M|N \\ 1\le M<N}} \bigoplus_{d|(N/M)} V(d)(S_k(M)).$$

The new subspace, $S_k^{\mathrm{new}}(N)$, is defined to be the orthogonal complement of $S_k^{\mathrm{old}}(N)$ in $S_k(N)$ with respect to the Petersson inner product. Note that these spaces are preserved under T_n for $(n, N) = 1$.

Definition 2.2.11. *An element of $S_k^{new}(N)$ is called a newform if it is a normalised eigenform under all Hecke operators T_n and the Atkin-Lehner involutions $|[W_{Q_p}]_k$ for $p \mid N$ and $|[W_N]_k$.*

We have the following theorem on newforms.

Theorem 2.2.12. *(Atkin-Lehner) Let $f(z) = \sum_{n=0}^{\infty} a_n q^n \in S_k^{new}(N)$ be a newform. Then,*

(i) $T_n(f) = a_n f$ for all n.

(ii) If p is a prime such that $\mathrm{ord}_p(N) \ge 2$, then $a_p = 0$.

(iii) If $p \mid N$ with $\mathrm{ord}_p(N) = 1$, then $a_p = -\omega_p p^{k/2-1}$, where $\omega_p \in \{\pm 1\}$ is such that $f|[W_{Q_p}]_k = \omega_p f$.

Proof. See either [23, Theorem 2.27] or [28, Theorem 4.6.17] for the proof. □

It is to be noted that a similar theorem holds for newforms with characters (see [28, Theorem 4.6.17]); in particular the statement (i) of the above theorem is true if f is a newform in $S_k^{\mathrm{new}}(N,\chi)$.

It is a well-known result that, if $f \in S_k^{\mathrm{new}}(N)$ is a newform then the coefficients a_n of f belong to the ring of integers $\mathcal{O}_K$ for some number field K [16, Page 234]. Moreover, from the above theorem it is clear that the coefficients a_n are totally real, since they are the eigenvalues of Hermitian operators.

We will be using the following proposition which can be deduced as a corollary to the "multiplicity-one" theorem [28, Theorem 4.6.19] on newforms in the later sections.

Proposition 2.2.13. *Let f be a common eigenfunction $f \in S_k(N,\chi)$ of T_n with eigenvalues a_n for all n prime to N. Then there uniquely exist a divisor M of N satisfying $\mathrm{Cond}(\chi) \mid M$ and a newform $g \in S_k^{\mathrm{new}}(M,\chi)$ such that $T_n(g) = a_n g$ for all n prime to N, and f can be written as a linear combination*

$$f = \sum_{d|(N/M)} \alpha_d V_d(g).$$

Proof. This is Corollary 4.6.20 in [28]. □

We will conclude this section by stating the following result due to Sturm [40]. We start with a definition.

Definition 2.2.14. *Fix a number field F and let $\mathcal{O}_F$ be the ring of integers of F and λ be a prime ideal of $\mathcal{O}_F$. Suppose $f(z) = \sum_{n\geq 0} a_n q^n$ is a formal power series with coefficients in $\mathcal{O}_F$. Then we define $\mathrm{ord}_\lambda(f)$ to be*

$$\mathrm{ord}_\lambda(f) := \inf\{n : a_n \notin \lambda\}.$$

If $a_n \in \lambda$ for all n, then we let $\mathrm{ord}_\lambda(f) := \infty$.

It is easy to see that $\mathrm{ord}_\lambda(f_1 f_2) = \mathrm{ord}_\lambda(f_1) + \mathrm{ord}_\lambda(f_2)$.

Theorem 2.2.15. *(Sturm) Let Γ be a congruence subgroup and k be a positive integer. Let f, $g \in M_k(\Gamma)$ such that f and g have coefficients in $\mathcal{O}_F$, the ring* *of integers of a number field F. Let λ be a prime ideal of $\mathcal{O}_F$. If*

$$\operatorname{ord}_\lambda(f-g) > \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma],$$

then $\operatorname{ord}_\lambda(f-g) = \infty$, *i.e.*, $f \equiv g \pmod{\lambda}$.

Proof. See [40, Page 276]. □

2.3 Half-Integral Weight Modular Forms

In this section we summarize standard material on modular forms of half-integral weight found in Shimura's paper [36], supplemented by material from the papers of Serre and Stark [35] and Cohen and Oesterlé [12].

2.3.1 Definitions

Before getting into the definition of half-integral weight forms, we first define the standard Kronecker symbol $\left(\frac{c}{d}\right)$ and ϵ_d for $c, d \in \mathbb{Z}$ with $d \neq 0$:

(i) $\left(\frac{c}{d}\right) = 0$ if $(c, d) \neq 1$.

(ii) If d is an odd prime, then $\left(\frac{c}{d}\right)$ is the usual Legendre symbol.

(iii) If $d > 0$, the map $c \mapsto \left(\frac{c}{d}\right)$ is a character modulo d.

(iv) For $c \neq 0$, the map $d \mapsto \left(\frac{c}{d}\right)$ is a character of conductor equal to the modulus of the discriminant of the field $\mathbb{Q}(\sqrt{c})/\mathbb{Q}$. We denote this character by χ_c.

(v) $\left(\frac{c}{-1}\right) = 1$ or -1 according as $c > 0$ or $c < 0$ and, $\left(\frac{0}{\pm 1}\right) = 1$.

(vi) $\left(\frac{-1}{d}\right) = (-1)^{(d-1)/2}$ for all positive or negative odd integers d.

(vii) For odd d, $\epsilon_d = 1$ or $\sqrt{-1}$ according as $d \equiv 1$ or $3 \pmod 4$.

Also, for $z \in \mathbb{C}$, we shall take $\sqrt{z}$ to be the branch of the square root having argument in $(-\pi/2, \pi/2]$.

Let G be the group consisting of all ordered pairs $(\alpha, \phi(z))$, where $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\phi(z)$ is a holomorphic function on $\mathbb{H}$ satisfying

$$\phi(z)^2 = t\frac{cz+d}{\sqrt{\det\alpha}}$$

for some $t \in \{\pm 1\}$, with the group law defined by

$$(\alpha, \phi(z)) \cdot (\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)).$$

Let $P : G \to \mathrm{GL}_2^+(\mathbb{Q})$ be the homomorphism given by the projection map onto the first coordinate. The group G acts on the space of complex valued functions on $\mathbb{H}$ by $f|[\xi]_{k/2}(z) := f(\alpha z)\phi(z)^{-k}$, where $\xi = (\alpha, \phi(z)) \in G$ and $f : \mathbb{H} \to \mathbb{C}$.

Let N be a positive integer with $4 \mid N$. Then for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$ define

$$j(\gamma, z) := \left(\frac{c}{d}\right)\epsilon_d^{-1}\sqrt{cz+d}, \qquad \Delta_0(N) := \{\widetilde{\gamma} := (\gamma, j(\gamma, z)) | \gamma \in \Gamma_0(N)\}.$$

Then $\Delta_0(N)$ is a subgroup of G. The map $L : \Gamma_0(4) \to G$ given by $\gamma \mapsto \widetilde{\gamma}$ defines an isomorphism onto $\Delta_0(4)$. Thus $P|_{\Delta_0(4)} : \Delta_0(4) \to \Gamma_0(4)$ and $L : \Gamma_0(4) \to \Delta_0(4)$ are inverse of each other. Denote by $\Delta_1(N)$ and $\Delta(N)$ respectively the images of $\Gamma_1(N)$ and $\Gamma(N)$.

Definition 2.3.1. *Let k, N be positive integers with k odd and $4 \mid N$.A holomorphic function f on $\mathbb{H}$ is a modular form of weight $k/2$ for $\Delta_1(N)$ if f satisfies $f|[\widetilde{\gamma}]_{k/2} = f$ for all $\gamma \in \Gamma_1(N)$ and is holomorphic at all the cusps of $\Gamma_1(N)$. As before, f is called a cusp form if it vanishes on all cusps. We denote such a space of modular forms by $M_{k/2}(\Gamma_1(N))$ and the subspace of cusp forms by $S_{k/2}(\Gamma_1(N))$. Let χ be a Dirichlet character modulo N. Then $M_{k/2}(N, \chi)$ (respectively $S_{k/2}(N, \chi)$) is the subspace of $M_{k/2}(\Gamma_1(N))$ (respectively $S_{k/2}(\Gamma_1(N))$) consisting of all elements f such that $f|[\widetilde{\gamma}]_{k/2} = \chi(d)f$ for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$.*

For the precise meaning of 'holomorphicity at cusps' in the above definition, please refer to [36, Page 444].

It is clear that the space $M_{k/2}(N,\chi) = 0$ if χ is an odd character, that is, $\chi(-1) = -1$. Henceforth we will be assuming χ to be an even character. If χ is a trivial character, we write $M_{k/2}(N,\chi)$ and $S_{k/2}(N,\chi)$ simply by $M_{k/2}(N)$ and $S_{k/2}(N)$.

It is to be noted that since $(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], 1) \in \Delta_1(N)$, a modular form $f \in M_{k/2}(\Gamma_1(N))$ has a Fourier expansion of the form $f(z) = \sum_{n=0}^{\infty} a_n q^n$ where $q = e^{2\pi i z}$.

The theta-functions provides us with a large class of examples of half-integral weight modular forms. We are interested in *theta-functions of one variable* (also known as *theta-forms*).

Definition 2.3.2. *Let ν be either 0 or 1. Let ψ be a Dirichlet character such that $\psi(-1) = (-1)^{\nu}$. Then we define*

$$\Theta(\psi,\nu,z) := \sum_{n=-\infty}^{\infty} \psi(n) n^{\nu} q^{n^2}, \tag{2.1}$$

where 0^0 is taken to be 1.

Theorem 2.3.3. *(Shimura) Let ψ be a Dirichlet character with conductor r_ψ.*

(i) If ψ is even then $\Theta(\psi,0,z) \in M_{1/2}(4r_\psi^2,\psi)$.

(ii) If ψ is odd then $\Theta(\psi,1,z) \in S_{3/2}(4r_\psi^2,\psi\cdot\chi_{-1})$.

Proof. See [36, Section 2]. □

Serre and Stark [35] proved in fact that every modular form of weight $1/2$ can be written as a linear combination of theta-functions with $\nu = 0$.

Theorem 2.3.4. *(Serre and Stark) Let $4 \mid N$ and χ be an even Dirichlet character modulo N. Let $\Omega(N,\chi)$ be the set of pairs (ψ,t) with $t \in \mathbb{N}$ and ψ an even primitive Dirichlet character with conductor r_ψ satisfying*

$$i)\ 4r_\psi^2 t \mid N, \qquad ii)\ \chi(n) = \psi(n)\left(\frac{t}{n}\right) \text{ for } n \in \mathbb{Z} \text{ coprime to } N.$$

Then the theta-functions $\Theta(\psi,0,tz)$ with $(\psi,t) \in \Omega(N,\chi)$ form a basis of the space $M_{1/2}(N,\chi)$. Moreover, let $\Omega_e(N,\chi)$ be the subset of pairs (ψ,t) in

$\Omega(N,\chi)$ with ψ a square of some character, of conductor r_ψ if r_ψ is odd, and $2r_\psi$ if r_ψ is even. Let $\Omega_c(N,\chi) = \Omega(N,\chi) - \Omega_e(N,\chi)$. Then $\Theta(\psi, 0, tz)$ with $(\psi, t) \in \Omega_c(N,\chi)$ form a basis for $S_{1/2}(N,\chi)$.

Proof. See [35, Section 2] for the statements and [35, Sections 6,7] for the proofs. □

We will see later that there are many modular forms other than theta-functions for weights $\geq 3/2$.

2.3.2 Dimension Formulae

In this section we briefly state dimension formulae for $S_{k/2}(N,\chi)$ due to Cohen and Oesterlé [12], for odd k. The above theorem of Serre and Stark gives explicit bases in the case $k = 1$. Thus we restrict to k odd ≥ 3. As usual $4 \mid N$ and $\chi(-1) = 1$. Let f be the conductor of χ. Write

$$N = \prod p^{r_p}, \qquad f = \prod p^{s_p}.$$

Write

$$\lambda_p = \begin{cases} p^{r_p/2} + p^{r_p/2-1} & \text{if } 2s_p \leq r_p \text{ and } r_p \text{ is even} \\ 2p^{(r_p-1)/2} & \text{if } 2s_p \leq r_p \text{ and } r_p \text{ is odd} \\ 2p^{r_p - s_p} & \text{if } 2s_p > r_p. \end{cases}$$

The formulae involve another parameter ζ which we now define. If $r_2 \geq 4$ we let $\zeta = \lambda_2$; if $r_2 = 3$ we let $\zeta = 3$. As $4 \mid N$, the only case left is $r_2 = 2$. Suppose $r_2 = 2$. Let (C) be the following condition:

(C) there is a prime $p \equiv 3 \pmod 4$ such that $p \mid N$ with either r_p odd or $0 < r_p < 2s_p$.

If (C) holds then we let $\zeta = 2$. Suppose (C) does not hold. Let

$$\zeta = \begin{cases} 3/2 & \text{if } s_2 = 0 \text{ and } k \equiv 1 \pmod 4 \\ 5/2 & \text{if } s_2 = 2 \text{ and } k \equiv 1 \pmod 4 \\ 5/2 & \text{if } s_2 = 0 \text{ and } k \equiv 3 \pmod 4 \\ 3/2 & \text{if } s_2 = 2 \text{ and } k \equiv 3 \pmod 4. \end{cases}$$

Theorem 2.3.5. *(Cohen and Oesterlé [12, Théorème 2]) With notation as above,*

$$\dim S_{k/2}(N,\chi) - \dim M_{2-k/2}(N,\chi) = \frac{k-2}{24} N \prod_{p|N}(1+1/p) - \frac{\zeta}{2} \prod_{p|N, p\neq 2} \lambda_p.$$

Here we take $M_{2-k/2}(N,\chi) = 0$ for $k \geq 5$.

2.3.3 Operators

As in the case of integral weight modular forms we have several operators that act on the spaces $M_{k/2}(N,\chi)$ and $S_{k/2}(N,\chi)$.

We will start with the Hecke operators which are defined again in terms of double cosets. Let ξ be an element of G such that $\Delta_1(N)$ and $\xi^{-1}\Delta_1(N)\xi$ are commensurable. Define an operator $|[\Delta_1(N)\xi\Delta_1(N)]_{k/2}$ on $M_{k/2}(\Gamma_1(N))$ by

$$f|[\Delta_1(N)\xi\Delta_1(N)]_{k/2} = \det(\xi)^{k/4-1} \sum_\nu f|[\xi_\nu]_{k/2}$$

where $\Delta_1(N)\xi\Delta_1(N) = \bigcup_\nu \Delta_1(N)\xi_\nu$.

Now suppose m is a positive integer and $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right]$, $\xi = (\alpha, m^{1/4})$. Then the Hecke operator T_m is defined as the restriction of $|[\Delta_1(N)\xi\Delta_1(N)]_{k/2}$ to $M_{k/2}(N,\chi)$. It is to be noted that by [36, Proposition 1.0], if m is not a square and $(m,N) = 1$ then $|[\Delta_1(N)\xi\Delta_1(N)]_{k/2}$ is the zero operator. So we assume that $m = n^2$ for a positive integer n. We write the Hecke operator T_{n^2} as

$$T_{n^2}(f) := n^{\frac{k}{2}-2} \sum_\nu \chi(a_\nu) f|[\xi_\nu]_{k/2},$$

where ξ_ν are the right coset representatives of $\Delta_0(N)$ in $\Delta_0(N)\xi\Delta_0(N)$ such that $P(\xi_\nu) = \left[\begin{smallmatrix} a_\nu & * \\ * & * \end{smallmatrix}\right]$. We have the following theorem.

Theorem 2.3.6. *(Shimura) Let* $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_{k/2}(N, \chi)$. *Then* $T_{p^2}(f)(z) = \sum_{n=0}^{\infty} b_n q^n$ *where,*

$$b_n = a_{p^2 n} + \chi(p) \left(\frac{-1}{p} \right)^{\lambda} \left(\frac{n}{p} \right) p^{\lambda - 1} a_n + \chi(p^2) p^{k-2} a_{n/p^2},$$

and $\lambda = (k-1)/2$ *and* $a_{n/p^2} = 0$ *whenever* $p^2 \nmid n$.

Proof. See [36, Theorem 1.7]. □

As in the integral weight case, if $(m, n) = 1$, then $T_{m^2 n^2} = T_{m^2} T_{n^2}$; in particular the Hecke operators T_{m^2} and T_{n^2} commute (see [36, Proposition 1.6] for details). The operators T_{p^2} with p prime generate the Hecke algebra. Moreover, as before we can define a Petersson inner product on the space $S_{k/2}(N, \chi)$ and with respect to this inner product $\overline{\chi(p)} T_{p^2}$ are Hermitian whenever $(p, N) = 1$. Hence $S_{k/2}(N, \chi)$ has a basis of eigenforms under all Hecke operators T_{p^2} with $(p, N) = 1$.

Example 2.3.7. Just as in the integral case, it is not true that the space of cusp forms has a basis of eigenfunctions under all Hecke operators. We computed the action of T_4 on $S_{3/2}(N)$ for all N up to 180. We found that T_4 is not diagonalizable for $N = 160$ only.

`MAGMA` gives the following basis for the space $S_{3/2}(160)$:

$$\begin{aligned}
f_1 &= q - q^9 - q^{25} - 2q^{41} + 3q^{49} + O(q^{60}) \\
f_2 &= q^2 - q^{10} - q^{18} + 2q^{22} - 2q^{30} - 2q^{38} + q^{50} + 2q^{58} + O(q^{60}) \\
f_3 &= q^4 - q^{20} - 2q^{24} - q^{36} + 2q^{40} + 2q^{56} + O(q^{60}) \\
f_4 &= q^5 - 2q^{21} - 3q^{45} + O(q^{60}) \\
f_5 &= q^6 - q^{10} - q^{14} + q^{30} + 2q^{34} - q^{46} - 2q^{54} + O(q^{60}) \\
f_6 &= q^7 - q^{15} - q^{23} + q^{47} + O(q^{60}).
\end{aligned}$$

We find that $T_4(f_i) = 0$ for $i = 1, 2, 4, 5, 6$ and

$$T_4(f_3) = f_1 - f_4 - 2f_5.$$

Let M be the 6×6 matrix representing the action of T_4 with respect to the basis $f_1, \ldots, f_6$. Then M has eigenvalue 0 with multiplicity 6. If T_4 is diagonalizable, then $T_4 = 0$. Since this is not the case, we see that it is not diagonalizable.

Further, we can define V-operators and U-operator as in the integral weight case and we have the following proposition.

Proposition 2.3.8. *Let $f(z) \in M_{k/2}(N, \chi)$. Let d be a positive integer.*

(i) $V(d)(f) \in M_{k/2}(Nd, \left(\frac{4d}{\cdot}\right) \chi)$.

(ii) If $d \mid N$, $U(d)(f) \in M_{k/2}(N, \left(\frac{4d}{\cdot}\right) \chi)$.

Moreover in above cases $V(d)$ and $U(d)$ take cusp forms to cusp forms.

Proof. See [30, Proposition 3.7]. □

One can verify as in the integral weight case that T_{p^2} commutes with the operator $U(d)$ and for p coprime to d, T_{p^2} commutes with $V(d)$.

2.3.4 Shimura's Correspondence

We will conclude this section by presenting a fundamental theorem of Shimura [36] which connects the arithmetic of half-integral weight cusp forms and even integer weight modular forms.

Theorem 2.3.9. *(Shimura) Let N and k be positive integers such that $4 \mid N$ and $k \geq 3$. Let $\lambda = (k-1)/2$. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N, \chi)$. Let t be a square-free integer and let ψ_t be the Dirichlet character modulo tN defined by*

$$\psi_t(m) = \chi(m) \left(\frac{-1}{m}\right)^{\lambda} \left(\frac{t}{m}\right).$$

Let $A_t(n)$ be the complex numbers defined by

$$\sum_{n=1}^{\infty} A_t(n) n^{-s} = \left(\sum_{i=1}^{\infty} \psi_t(i) i^{\lambda-1-s}\right) \left(\sum_{j=1}^{\infty} a_{tj^2} j^{-s}\right). \tag{2.2}$$

Let $\mathrm{Sh}_t(f)(z) = \sum_{n=1}^{\infty} A_t(n) q^n$. Then $\mathrm{Sh}_t(f) \in M_{k-1}(N/2, \chi^2)$. If $k \geq 5$ then $\mathrm{Sh}_t(f)$ is a cusp form. Further if $k = 3$ then $\mathrm{Sh}_t(f)$ is a cusp form if f is in *the orthogonal complement of $S_0(N,\chi)$, the subspace of $S_{3/2}(N,\chi)$ spanned by single variable theta-functions.*

The formulation we used of Shimura's Theorem is one found in Ono's book [30, Theorem 3.14]. Please refer to section 3.1 for the explicit definition of $S_0(N,\chi)$.

The $\mathrm{Sh}_t(f)$ is called the *Shimura lift* of f corresponding to t. In the later chapters we will discuss deeper properties of Shimura lifts and several results surrounding them.

2.4 Algorithms for Computing Half-Integral Weight Modular Forms

As far as we know, the only algorithm found in the literature for computing a basis for the space of half-integral weight modular forms is given in Basmaji's thesis [3]. Basmaji's algorithm is for modular forms of half-integral weight and level divisible by 16. However the computer algebra system `MAGMA` [5] computes bases for spaces of half-integral weight modular forms of general level. By reading the relevant part of the `MAGMA` source code written by Steve Donnelly and William Stein, we have been able to write down the algorithm it is relying on, which is a variant of Basmaji's, and to verify its correctness.

Let $k > 1$ be an odd integer and $N \in \mathbb{N}$ such that $16 \mid N$. Let χ be a Dirichlet character modulo N. Basmaji in his thesis gives the following algorithm for computing a basis for $S_{k/2}(N,\chi)$. The idea of the algorithm is to use theta-series. Let

$$\Theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2},$$

$$\Theta_1(z) := \frac{1}{2} \sum_{\substack{n=-\infty \\ n\equiv 1 (\mathrm{mod} 2)}}^{\infty} q^{n^2} = \sum_{\substack{n=1 \\ n\equiv 1 (\mathrm{mod} 2)}}^{\infty} q^{n^2}$$

where $q = e^{2\pi i z}$.

From the work of Serre and Stark [35] we know that $\Theta \in M_{1/2}(4,\chi_{\mathrm{triv}})$ and $\Theta_1 \in M_{1/2}(16,\chi_{\mathrm{triv}})$ where χ_{triv} stands for the identity character; this is proved independently in Basmaji's thesis. Let χ_{-1} be the nontrivial Dirichlet character modulo 4 and

$$S = S_{\frac{k+1}{2}}\left(N,\ \chi \cdot \chi_{-1}^{\frac{k+1}{2}}\right).$$

Basmaji defines the following embedding,

$$\varphi : S_{k/2}(N, \chi) \to S \times S, \qquad\qquad f \mapsto (f\Theta, f\Theta_1),$$

proving that $f\Theta$ and $f\Theta_1$ do indeed belong to S. Let U be the subspace of $S \times S$ consisting of elements (f_1, f_2) such that

$$f_1 \cdot \Theta_1 = f_2 \cdot \Theta \tag{2.3}$$

holds. Then U is isomorphic to $S_{k/2}(N, \chi)$ via the map

$$(f_1, f_2) \mapsto f_1/\Theta\ (= f_2/\Theta_1).$$

There are standard methods for computing a basis for a space of modular forms of integral weight; see for example [39]. Thus one can start with a given basis for S and form a system of linear equations in terms of the coefficients of q-expansions of the basis elements and solve for (f_1, f_2) in the equation (2.3), thereby recovering a basis for $S_{k/2}(N, \chi)$.

It is to be noted that the hypothesis $16 \mid N$ is only used to show that $f\Theta_1$ belongs to the space S and so it seems possible to drop this hypothesis by working with other theta-series. This is precisely what is done in the `MAGMA` implementation for general level N. Suppose $4 \mid N$ and $16 \nmid N$. Let

$$\Theta_2(z) := \Theta(2z) = 1 + 2\sum_{n=1}^{\infty} q^{2n^2} \in M_{1/2}(8, \chi_8)$$

where $\chi_8 = \left(\frac{8}{\cdot}\right)$ is the Dirichlet character modulo 8. Let $N' = \mathrm{lcm}(N, 8)$. Let S be as before and

$$S' = S_{\frac{k+1}{2}}\left(N',\ \chi \cdot \chi_8 \cdot \chi_{-1}^{\frac{k+1}{2}}\right).$$

Then we have an embedding as above given by

$$\varphi : S_{k/2}(N, \chi) \to S \times S'$$

$$f \mapsto (f\Theta, f\Theta_2).$$

Lemma 2.4.1. *If $f \in S_{k/2}(N, \chi)$ then $f\Theta_2 \in S'$.*

We shortly prove Lemma 2.4.1. Let U' be the subspace of $S \times S'$ consisting of elements (g_1, g_2) such that

$$g_1\Theta_2 = g_2\Theta.$$

As before this gives a system of linear equations that we can solve and recover a basis for $S_{k/2}(N, \chi)$.

Proof of Lemma 2.4.1. Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N')$. Then

$$\begin{aligned}
(f\Theta_2)(\gamma z) &= f(\gamma z)\Theta_2(\gamma z) \\
&= \chi(d)\chi_8(d)j(\gamma, z)^{k+1}f(z)\Theta_2(z) \\
&= (\chi \cdot \chi_8)(d)(j(\gamma, z)^2)^{(k+1)/2}f(z)\Theta_2(z) \\
&= (\chi \cdot \chi_8)(d)(\epsilon_d^{-2}(cz + d))^{(k+1)/2}f(z)\Theta_2(z) \\
&= (\chi \cdot \chi_8 \cdot \chi_{-1}^{(k+1)/2})(d)(cz + d)^{(k+1)/2}(f\Theta_2)(z).
\end{aligned}$$

Note that $f\Theta_2$ is holomorphic on $\mathbb{H}$ as so are f and Θ_2. We want to show that $f\Theta_2$ is holomorphic at the cusps. Let $s \in \mathbb{Q} \cup \{\infty\}$ be any cusp. Then $s = \alpha\infty$ for some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Following the definitions one can easily show that

$$(f\Theta_2)(z)|_{[\alpha]_{(k+1)/2}} = \kappa_\alpha \cdot f(z)|_{[\alpha]_{k/2}}\Theta_2(z)|_{[\alpha]_{1/2}}.$$

where κ_α is a fourth root of unity. Now the result follows since f is a cusp form. $\square$

2.5 Automorphic Representations

Let F be a number field and $\mathbb{A}_F$ be its ring of adeles. In this section we will recall the theory of automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. We follow the standard material as presented in Bump's book [6].

Definition 2.5.1. *Let G be a locally compact abelian group. Then, by a* quasicharacter *of G we mean a continuous homomorphism $\chi : G \to \mathbb{C}^\times$. If $|\chi(g)| = 1$ for all $g \in G$, then χ is called a character. In particular, we say that a character χ_ν of $F_\nu^\times$ is* unramified *if it is trivial on the unit group $\mathcal{O}_\nu^\times$. Here F_ν is the completion of F at the place ν of F and $\mathcal{O}_\nu$ is the ring of integers of F_ν.*

Note that an unramified character of $F_\nu^\times$ is determined by its value on any uniformizer. In our subsequent work we will be only interested in the case of $\mathrm{GL}_n(\mathbb{A}_F)$ where $F = \mathbb{Q}$ and $n \leq 2$.

If $n = 1$, an automorphic representation of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ is indeed simply a Hecke character, i.e., a continuous homomorphism $\chi : \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ and it corresponds to a primitive Dirichlet character. This follows from Tate's thesis [9, Chapter XV] and we will discuss this in more detail in Section 4.1. We will henceforth assume that $n = 2$ and we will see that one can associate automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ to classical Hecke eigenforms.

Before going into the definition of automorphic representations, we first recall the theory of admissible representations of $G = \mathrm{GL}_2(\mathfrak{f})$ where $\mathfrak{f}$ is a non-Archimedean local field (that is a finite extension of $\mathbb{Q}_p$ for some finite prime p) with ring of integers $\mathfrak{o}$. Please refer to either [6, Chapter IV] or [14] for the details of what follows.

A representation of G on a complex vector space V is *smooth* if the stabilizer of any vector in V is an open subgroup of G; it is *admissible* if it is smooth and for every open subgroup U of G the space V^U of vectors stabilized by U is finite dimensional. We will be interested in irreducible admissible representations.

Let χ_1 and χ_2 be quasicharacters of $\mathfrak{f}^\times$. Let $\mathcal{B}(\chi_1, \chi_2)$ be the space of all smooth (i.e, locally constant) functions $f : G \to \mathbb{C}$ which satisfy the following identity

$$f\left(\begin{bmatrix} y_1 & x \\ 0 & y_2 \end{bmatrix} g\right) = \left|\frac{y_1}{y_2}\right|^{1/2} \chi_1(y_1)\chi_2(y_2)f(g).$$

Here $|\cdot|$ is the usual norm character of $\mathfrak{f}^\times$, which takes $y \in \mathfrak{f}^\times$ to $q^{-\mathrm{ord}_p(y)}$ where q is the cardinality of the residue field. Then G acts on $\mathcal{B}(\chi_1, \chi_2)$ by right translation, i.e., $(gf)(g') = f(g'g)$ and the resulting representation can be shown to be an admissible representation of G. Further, if we assume that $\chi_1\chi_2^{-1}$ is not equal to either of the quasicharacters $|\cdot|$ or $|\cdot|^{-1}$, then $\mathcal{B}(\chi_1, \chi_2)$ is irreducible (see [6, Theorem 4.5.1]) and in this case, the isomorphism classes of the $\mathcal{B}(\chi_1, \chi_2)$ are called the principal series representations; the isomorphism class of $\mathcal{B}(\chi_1, \chi_2)$ is denoted by $\pi(\chi_1, \chi_2)$.

When $\chi_1\chi_2^{-1}$ is equal to $|\cdot|^{\pm 1}$, the representation $\mathcal{B}(\chi_1, \chi_2)$ has two composition factors in its Jordan-Hölder series, a 1-dimensional factor and an infinite dimensional factor. Precisely, say $\chi_1\chi_2^{-1} = |\cdot|$ and write $\chi_1 = \chi|\cdot|^{1/2}$ and $\chi_2 = \chi|\cdot|^{-1/2}$. Then $\mathcal{B}(\chi_1, \chi_2)$ has a unique irreducible subrepresentation $\mathrm{St}_2(\chi)$ which is infinite dimensional. The quotient $\mathcal{B}(\chi_1, \chi_2)/\mathrm{St}_2(\chi)$ is 1-dimensional and G acts on it through the character $g \mapsto \chi(\det g)$. Write St_2 in place of $\mathrm{St}_2(\chi)$ when χ is the trivial character. The representation St_2 is called the Steinberg representation. One has $\mathrm{St}_2(\chi) = \mathrm{St}_2 \otimes \chi$.

An irreducible admissible representation (π, V) of G is called supercuspidal if associated "Jacquet module" $J(V)$ is zero. We have the following classification of the irreducible admissible representations of G which can be gleaned from Bump's book [6]; the formulation we use is that of [14].

Theorem 2.5.2. *Let (π, V) be an irreducible admissible representation of G. If V is finite dimensional then it is 1-dimensional and there exists a quasicharacter χ of $\mathfrak{f}^\times$ such that $\pi(g)v = \chi(det(g))v$ for all $g \in G$ and $v \in V$. Otherwise, (π, V) is equivalent to one and only one of the following:*

(i) An irreducible principal series representation $\pi(\chi_1, \chi_2)$ with $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$.

(ii) A twist $St_2 \otimes \chi$ of the Steinberg representation St_2.

(iii) A supercuspidal representation.

Proof. See [6, Section 4.5, 4.6, 4.7] for a complete proof. □

Definition 2.5.3. *An irreducible admissible representation (π, V) of G is called* spherical *(or* unramified*) if it has a vector which is invariant under the maximal compact subgroup $K = \mathrm{GL}_2(\mathfrak{o})$.*

It is well-known (see [6, Theorem 4.6.4]) that (π, V) is spherical if and only if either it is a 1-dimensional representation given by $g \mapsto \chi(\det(g))$ for some unramified quasicharacter χ of $\mathfrak{f}^\times$, or it is a principal series of the form $\pi(\chi_1, \chi_2)$ with χ_1 and χ_2 unramified quasicharacters of $\mathfrak{f}^\times$.

We will now define an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let ω be a Hecke character. Let $\mathrm{L}^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F), \omega)$ be the space of all functions $f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ that are measurable with respect to the Haar measure dg and satisfy

$$f(\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right] g) = \omega(z) f(g), \qquad z \in \mathbb{A}_F^\times,$$

$$f(\gamma g) = f(g), \qquad \gamma \in \mathrm{GL}_2(F),$$

and that are square integrable modulo centre $Z_{\mathbb{A}_F}$ (the group of scalar matrices with entries in $\mathbb{A}_F^\times$):

$$\int_{Z_{\mathbb{A}_F} \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} |f(g)|^2 dg < \infty.$$

Let $\mathrm{L}_0^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F), \omega)$ be the closed subspace (cusp forms) satisfying the *cuspidal* condition, that is,

$$\int_{F\backslash \mathbb{A}_F} f(\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] g) dx = 0$$

for almost all $g \in \mathrm{GL}_2(\mathbb{A}_F)$. The group $\mathrm{GL}_2(\mathbb{A}_F)$ acts on this L^2 space by right translation; this representation is called right regular representation and is denoted by ρ. The space of cusp forms (L_0^2 subspace) is invariant under this representation and decomposes into an infinite direct sum of irreducible invariant subspaces. If (π, V) is a representation of $\mathrm{GL}_2(\mathbb{A}_F)$ that is isomorphic to the representation on one of these invariant subspaces, then we say that (π, V) is an *automorphic cuspidal* representation with central character ω.

Let $\mathfrak{g}_\infty = \prod_{\nu \in S_\infty} \mathfrak{gl}_2(F_\nu)$, where S_∞ is the set of Archimedean places of F and $\mathfrak{gl}_2(F_\nu)$ is the Lie algebra of $\mathrm{GL}_2(F_\nu)$, i.e., the set of 2×2 matrices over F_ν . Let $K = \prod_\nu K_\nu$ where $K_\nu = \mathrm{GL}_2(\mathcal{O}_\nu)$ if ν is non-Archimedean, $K_\nu = O(2)$ if ν is a real and $K_\nu = U(2)$ if ν is a complex; note that $O(2)$ and $U(2)$ are respectively orthogonal group and unitary group of 2×2 matrices.

It turns out that if (π, V) is an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ then on the space of K-finite vectors in V one can write $\pi = \otimes'_\nu \pi_\nu$ where $\otimes'$ represents a *restricted tensor product*; here for each Archimedean place ν of F, π_ν is an irreducible admissible $(\mathfrak{g}_\infty, K_\nu)$-module and for each non-Archimedean place ν, π_ν is an irreducible admissible representation of $\mathrm{GL}_2(F_\nu)$. It is to be noted that π_ν is spherical for almost all ν, which allows us to define the restricted tensor product. For details see [6, Theorem 3.3.2, Theorem 3.3.3, Theorem 3.3.4].

Assume now $F = \mathbb{Q}$. Let $f \in S_k(N, \chi)$ be such that f is an eigenfunction for all Hecke operators T_p with $p \nmid N$. One can associate to χ a Hecke character ω as remarked earlier. Let $\omega = \prod_p \omega_p$. By the strong approximation theorem [6, Theorem 3.3.1], it follows that any element $g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ can be written as $g = \gamma g_\infty k_0$ where $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})$ and $k_0 \in K_0(N)$; here $K_0(N) = \prod_{p<\infty} K_0(N)_p$, where if $p \mid N$ then $K_0(N)_p$ is the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of the form $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ where $c \equiv 0 \pmod N$ in $\mathbb{Z}_p$ and for primes $p \nmid N$, $K_0(N)_p = \mathrm{GL}_2(\mathbb{Z}_p)$. Let Ω be the character of $K_0(N)$ given by $\Omega(\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]) = \prod_{p|N} \omega_p(\delta_p)$.

Then the adelization of f is the function $\phi_f : \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ defined by $\phi_f(g) := f|[g_\infty]_k(i) \cdot \Omega(k_0)$. Since f is a cusp form, ϕ_f satisfies several properties and in fact it turns out that ϕ_f is an *automorphic* form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ (see [6, Page 343] for details). We have the following theorem; the formulation is as in [21, Page 93].

Theorem 2.5.4. *Let π_f be restriction of the right regular representation ρ of $\mathrm{GL}_2(\mathbb{A}_Q)$ on the subspace V_f of $\mathrm{L}^2_0(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_Q), \omega)$ spanned by $\{\rho(g)\phi_f : g \in \mathrm{GL}_2(\mathbb{A}_Q)\}$. Then π_f is irreducible and hence an automorphic cuspidal representation with central character ω.*

Chapter 3

Shimura's Correspondence

Shimura's Correspondence relates certain cusp forms of half-integral weight to modular forms of integral weight. In this chapter we give a precise statement of this correspondence and use it to study eigenfunctions and what is known as the Shimura decomposition.

Let k be an odd integer ≥ 3 and N a positive integer such that $4 \mid N$. Let χ be an even Dirichlet character modulo N. As we saw in the previous chapter, $S_{k/2}(N, \chi)$ can contain single-variable theta-series for $k = 3$. We shall denote by $S_0(N, \chi)$ the subspace generated by single-variable theta-series. If $k \geq 5$ then $S_0(N, \chi) = 0$, but this is often not the case for $k = 3$.

The interesting part of the space $S_{k/2}(N, \chi)$ is the orthogonal complement of $S_0(N, \chi)$ with respect to the Petersson inner product, denoted by $S^{\perp}_{k/2}(N, \chi)$. It is cusp forms belonging to this subspace that feature in Shimura's decomposition. To compute the dimension of $S^{\perp}_{k/2}(N, \chi)$ we need to know the dimension of $S_0(N, \chi)$. A generating set for this is given in several references, e.g. Shimura's paper [36]. We show that this generating set is in fact a basis of eigenfunctions, although we have not found this result anywhere in the literature.

As we will see in this chapter, Shimura decomposes the space $S^{\perp}_{k/2}(N, \chi)$ as

$$S^{\perp}_{k/2}(N, \chi) = \bigoplus_{\phi} S_{k/2}(N, \chi, \phi)$$

where ϕ runs through the newforms of weight $k-1$ and level dividing $N/2$ and character χ^2; $S_{k/2}(N, \chi, \phi)$ is the subspace of forms that are Shimura-equivalent to ϕ. We give an explicit algorithm for computing this decomposition. This decomposition will be crucial for our efforts later on to express the critical values of L-functions of twists of elliptic curves in terms of coefficients of modular forms of weight 3/2.

3.1 The Space $S_0(N, \chi)$

Let N be a natural number such that $4 \mid N$. Let χ be an even Dirichlet character of modulus N.

Let ψ be a primitive odd Dirichlet character of conductor r_ψ and

$$h_\psi(z) := \frac{1}{2}\Theta(\psi, 1, z) = \sum_{m=1}^{\infty} \psi(m) m q^{m^2}.$$

Recall, by Theorem 2.3.3 that $h_\psi \in S_{3/2}(4r_\psi^2, (\frac{-1}{\cdot})\psi)$. Consider the operator $V(t)$ (see section 2.2). By definition,

$$V(t)(h_\psi)(z) = \sum_{m=1}^{\infty} \psi(m) m q^{tm^2} \in S_{3/2}\left(4r_\psi^2 t, \left(\frac{-4t}{\cdot}\right)\psi\right).$$

Following Shimura [36], we define the space $S_0(N, \chi)$ to be a subspace of $S_{3/2}(N, \chi)$ spanned by

$$S = \{\ V(t)(h_\psi) :\ 4r_\psi^2 t \mid N \text{ and } \psi \text{ is a primitive odd character of conductor } r_\psi \text{ such that } \chi = \left(\frac{-4t}{\cdot}\right)\psi\ \}.$$

The purpose of this section is to prove the following theorem.

Theorem 3.1.1. *The set S constitutes a basis of eigenforms for $S_0(N, \chi)$. In particular, the dimension of $S_0(N, \chi)$ is simply $\#S$.*

To prove the theorem we shall need a series of lemmas.

Lemma 3.1.2. *$V(t)h_\psi$ is an eigenform for the Hecke operators T_{p^2} for all* *primes p. Indeed,*

$$T_{p^2}V(t)h_\psi = \begin{cases} \psi(p)(1+p)V(t)h_\psi & \text{if } p \nmid 2t \\ \psi(p)pV(t)h_\psi & \text{if } p \mid 2t. \end{cases}$$

Proof. Let us write $V(t)h_\psi(z) = \sum_{n=1}^{\infty} a_n q^n$. Thus

$$a_n = \begin{cases} \psi(m)m & \text{if } n = tm^2 \\ 0 & \text{otherwise.} \end{cases}$$

Let p be any prime. Write $T_{p^2}V(t)h_\psi = \sum_{n=1}^{\infty} b_n q^n$. Then by Theorem 2.3.6,

$$b_n = a_{p^2n} + \left(\frac{4tn}{p}\right)\psi(p)a_n + \left(\frac{-4t}{p}\right)^2 \psi(p)^2 p a_{n/p^2}.$$

If n/t is not the square of an integer, then $b_n = 0$. Write $n = tm^2$. If $p \mid 2t$, then $b_n = a_{p^2n} = a_{tp^2m^2} = \psi(pm)pm$. This completes the proof when $p \mid 2t$. Suppose $p \nmid 2t$. Then

$$\begin{aligned} b_n &= a_{tp^2m^2} + \left(\frac{4t^2m^2}{p}\right)\psi(p)a_{tm^2} + \left(\frac{-4t}{p}\right)^2 \psi(p)^2 p a_{tm^2/p^2} \\ &= a_{tp^2m^2} + \left(\frac{m^2}{p}\right)\psi(p)a_{tm^2} + \psi(p)^2 p a_{tm^2/p^2} \\ &= \begin{cases} a_{tp^2m^2} + \left(\frac{m^2}{p}\right)\psi(p)a_{tm^2} & \text{if } p \nmid m \\ a_{tp^2m^2} + \psi^2(p)pa_{tm^2/p^2} & \text{if } p \mid m \end{cases} \\ &= \psi(pm)pm + \psi(pm)m \\ &= (1+p)\psi(p)a_{tm^2}. \end{aligned}$$

Hence the lemma follows. □

Lemma 3.1.3. *Let ψ be a Dirichlet character modulo r. Let ψ' be a Dirichlet character modulo R. Let N be a natural number such that $r \mid R \mid N$ and $\psi(n) = \psi'(n)$ for all n with $(n, N) = 1$. If ψ' is primitive character modulo R, then $R = r$ and $\psi' = \psi$.*

Proof. Let $R = \prod_{i=1}^{k} p_i^{\alpha_i}$ and $N = \prod_{i=1}^{k} p_i^{\beta_i} \cdot \prod_{j=1}^{l} q_j^{\gamma_j}$ where $p_1, \ldots, p_k, q_1, \ldots, q_l$ are distinct primes, and $\beta_i \geq \alpha_i$. Let $(n, R) = 1$. Then by Chinese Remainder

Theorem there exists an m such that

$$m \equiv \begin{cases} n \pmod{\prod_{i=1}^{k} p_i^{\beta_i}} \\ 1 \pmod{\prod_{j=1}^{l} q_j^{\gamma_j}}. \end{cases}$$

So $m \equiv n \pmod{R}$ and $(m, N) = 1$. Hence we have,

$$\psi(n) = \psi(m) = \psi'(m) = \psi'(n).$$

Thus ψ' is induced by ψ. Since ψ' is a primitive character modulo R we get $R = r$ and $\psi' = \psi$. □

We have following easy corollary to the above lemma.

Corollary 3.1.4. *Let ψ_1 and ψ_2 be primitive Dirichlet characters modulo r_1 and r_2 respectively, and suppose $r_1 \mid N$, $r_2 \mid N$. Let χ be a Dirichlet character modulo N such that $\psi_1(n) = \psi_2(n) = \chi(n)$ for all n such that $(n, N) = 1$. Then $r_1 = r_2$ and $\psi_1 = \psi_2$.*

Proof. Let the conductor of χ be r and ψ be the primitive Dirichlet character modulo r which induces χ. Then $r \mid r_1$ and $r \mid r_2$. Hence the result follows from the lemma. □

Proof of Theorem 3.1.1. We will prove the theorem by showing that the elements of the set S are linearly independent. Let $S = \{\, V(t_i)(h_{\psi_i}) : 1 \le i \le k \,\}$. We claim that t_i's are all distinct. Suppose not. Then there exists i, j such that $t_i = t_j$. We know that $\chi = (\frac{-4t_i}{\cdot})\psi_i = (\frac{-4t_j}{\cdot})\psi_j$. Thus, $\psi_i(n) = \psi_j(n)$ for all $(n, N) = 1$. Since ψ_i and ψ_j are primitive, we can apply Corollary 3.1.4 to get that $\psi_i = \psi_j$ and that $V(t_i)(h_{\psi_i}) = V(t_j)(h_{\psi_j})$. Hence the claim follows. We can assume that $t_1 < t_2 < \cdots < t_k$.

Now let α_i for $1 \le i \le k$ be such that

$$\alpha_1 V(t_1)(h_{\psi_1}) + \alpha_2 V(t_2)(h_{\psi_2}) + \cdots + \alpha_k V(t_k)(h_{\psi_k}) = 0.$$

By the above equation and the q-expansion of $V(t_i)(h_{\psi_i})$, it follows that

$$\text{coefficient of } q^{t_1} = \alpha_1 \psi_1(1) = 0.$$

Hence $\alpha_1 = 0$. Repeating the same argument with $t_2, t_3, \ldots, t_k$, we get that $\alpha_2 = \alpha_3 = \cdots = \alpha_k = 0$. Thus we are done. $\square$

Remark. In the literature (see [30]), $S_0(N, \chi)$ is referred to as the space spanned by single variable theta-functions. Kohnen states in [25] that the "space of theta-functions" is zero for square-free level and arbitrary character, and also for cube-free level and trivial character. Kohnen does not give a proof. We prove this statement in the following easy proposition.

Proposition 3.1.5. *(Kohnen) Suppose either of the following holds:*

1. *$N/4$ is square-free, or*
2. *$N/4$ is cube-free and χ is a trivial character.*

Then $S_0(N, \chi) = 0$.

Proof. In the case $N/4$ is square-free, it is clear that the set $S = \emptyset$. Let $N/4$ be cube-free and χ be a trivial character. Hence for any $V(t)h_\psi \in S$ we have $\left(\frac{-4t}{n}\right)\psi(n) = 1$ for all $(n, N) = 1$. That is, for all such n, $\psi(n) = \left(\frac{-t}{n}\right)$. It is to be noted that the character $\left(\frac{-t}{\cdot}\right)$ is a primitive character modulo $4t$ or t depending on the value of $t \pmod 4$ and hence using Corollary 3.1.4 we get that $r_\psi = 4t$ or $r_\psi = t$ respectively. However, $N = 4r_\psi^2 t$. This contradicts the assumption that $N/4$ is cube-free. Thus, in this case the set $S = \emptyset$. $\square$

Note. Recall that for $k \geq 5$, we defined $S_0(N, \chi) = 0$. In the upcoming sections we will use the following notation:

$$S_{k/2}^{\perp}(N, \chi) := S_0(N, \chi)^{\perp};$$

in words, the orthogonal complement to $S_0(N, \chi)$ with respect to the Petersson inner-product. Thus, for $k \geq 5$,

$$S_{k/2}^{\perp}(N, \chi) = S_{k/2}(N, \chi).$$

3.2 Shimura Lifts

For this section fix positive integers k, N with $k \geq 3$ odd and $4 \mid N$. Let χ be an even Dirichlet character of modulus N. Let $N' = N/2$. We recall Shimura's Theorem.

Theorem 3.2.1. *(Shimura) Let* $\lambda = (k-1)/2$. *Let* $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N, \chi)$. *Let t be a square-free integer and let ψ_t be the Dirichlet character modulo tN defined by*

$$\psi_t(m) = \chi(m) \left(\frac{-1}{m}\right)^{\lambda} \left(\frac{t}{m}\right).$$

Let $A_t(n)$ be the complex numbers defined by

$$\sum_{n=1}^{\infty} A_t(n) n^{-s} = \left(\sum_{i=1}^{\infty} \psi_t(i) i^{\lambda-1-s}\right) \left(\sum_{j=1}^{\infty} a_{tj^2} j^{-s}\right). \tag{3.1}$$

Let $\mathrm{Sh}_t(f)(z) = \sum_{n=1}^{\infty} A_t(n) q^n$. *Then*

(i) $\mathrm{Sh}_t(f) \in M_{k-1}(N', \chi^2)$.

(ii) If $k \geq 5$ *then* $\mathrm{Sh}_t(f)$ *is a cusp form.*

(iii) If $k = 3$ *and* $f \in S_{3/2}^{\perp}(N, \chi)$ *then* $\mathrm{Sh}_t(f)$ *is a cusp form.*

(iv) Suppose f is an eigenform for T_{p^2} for all primes p and let $T_{p^2} f = \lambda_p f$. Then $\sum_{n=1}^{\infty} A_0(n) q^n \in M_{k-1}(N', \chi^2)$ *where $A_0(n)$ is defined by*

$$\sum_{n=1}^{\infty} A_0(n) n^{-s} = \prod_p (1 - \lambda_p p^{-s} + \chi(p)^2 p^{k-2-2s})^{-1}. \tag{3.2}$$

In fact if $a_t \neq 0$ *then* $\mathrm{Sh}_t(f)/a_t = \sum_{n=1}^{\infty} A_0(n) q^n$.

Proof. For (i), (ii) and (iv) see [36, Section 3, Main Theorem, Corollary], for the rest see [30, Theorem 3.14]. In particular, the fact that $N' = N/2$ was proved by Niwa [29, Section 3]. $\square$

The following is clear from Equation(3.1).

Lemma 3.2.2. *The Shimura lift* Sh_t *is linear.*

Lemma 3.2.3. *If* $\mathrm{Sh}_t(f) = 0$ *for all positive square-free integers* t *then* $f = 0$.

Proof. By Equation (3.1) we know that $a_{tj^2} = 0$ for all positive square-free integers t and all positive integers j. Then $a_n = 0$ for all n. $\square$

In Ono's book [30, Chapter 3, Corollary 3.16] and several other places [24] we find the following result stated without proof.

Proposition 3.2.4. *Suppose* $f \in S_{k/2}(N, \chi)$. *Let* t *be a square-free positive integer. If* $p \nmid 4tN$ *is a prime then*

$$\mathrm{Sh}_t(T_{p^2} f) = T_p \, \mathrm{Sh}_t(f).$$

Here T_{p^2} is the Hecke operator on $S_{k/2}(N, \chi)$ and T_p is the Hecke operator on $M_{k-1}(N', \chi^2)$. We will denote by $\mathbb{T}_{k/2}$ and $\mathbb{T}_{k-1}$ the Hecke algebras over $\mathbb{Z}$ acting on the space $M_{k/2}(N, \chi)$ and $M_{k-1}(N', \chi^2)$ respectively.

For what follows we shall need the following strengthening of this result.

Proposition 3.2.5. *Suppose* $f \in S_{k/2}(N, \chi)$ *and* t *a square-free positive integer. If* p *is a prime then*

$$\mathrm{Sh}_t(T_{p^2} f) = T_p \, \mathrm{Sh}_t(f).$$

We do not know why the above references impose the condition $p \nmid tN$. We shall give a careful proof that does not use this assumption.

Proof of Proposition 3.2.5. The proof uses the explicit formulae for Hecke operators in terms of q-expansions. As in Shimura's Theorem above, write $f(z) = \sum_{n=1}^{\infty} a_n q^n$. Fix t to be a positive square-free integer. To simplify notation, we shall write A_n for $A_t(n)$. Thus we have the relation

$$\sum_{n=1}^{\infty} A_n n^{-s} = \left(\sum_{i=1}^{\infty} \psi_t(i) i^{\lambda-1-s} \right) \left(\sum_{j=1}^{\infty} a_{tj^2} j^{-s} \right).$$

We may rewrite this as

$$A_n = \sum_{ij=n} \psi_t(i) i^{\lambda-1} a_{tj^2}. \tag{3.3}$$

Let

$$T_{p^2}(f)(z) = \sum_{n=1}^{\infty} b_n q^n.$$

Then using Theorem 2.3.6 we get,

$$b_n = a_{p^2 n} + \psi_1(p) \left(\frac{n}{p}\right) p^{\lambda-1} a_n + \chi^2(p) p^{k-2} a_{n/p^2}. \tag{3.4}$$

The reader will recall that if n/p^2 is not an integer then we take $a_{n/p^2} = 0$.

Let $g = \mathrm{Sh}_t(f)(z) = \sum_{n=1}^{\infty} A_n q^n$. Write

$$T_p(g)(z) = \sum_{n=1}^{\infty} B_n q^n.$$

Let

$$\mathrm{Sh}_t(T_{p^2} f)(z) = \sum_{n=1}^{\infty} C_n q^n.$$

To prove the proposition, it is enough to show that $B_n = C_n$ for all n. We shall do this by direct calculation, expressing both B_n and C_n in terms of the a_i.

Since $g(z) = \sum A_n q^n \in M_{k-1}(N', \chi^2)$ and $T_p(g)(z) = \sum B_n q^n$ we know by Proposition 2.2.5 that

$$B_n = A_{pn} + \chi^2(p) p^{k-2} A_{n/p}.$$

Substituting from (3.3) we have

$$B_n = \sum_{ij=pn} \psi_t(i) i^{\lambda-1} a_{tj^2} + \sum_{ij=n/p} \chi^2(p) \psi_t(i) p^{k-2} i^{\lambda-1} a_{tj^2}; \tag{3.5}$$

here the second sum is understood to vanish if $p \nmid n$.

Recall $T_{p^2} f(z) = \sum b_n q^n$ and $\mathrm{Sh}_t(T_{p^2} f)(z) = \sum C_n q^n$. Hence by (3.3) we have

$$C_n = \sum_{ij=n} \psi_t(i) i^{\lambda-1} b_{tj^2}.$$

Using (3.4) we obtain

$$C_n = \sum_{ij=n} \psi_t(i) i^{\lambda-1} \left(a_{p^2tj^2} + \psi_1(p) \left(\frac{tj^2}{p} \right) p^{\lambda-1} a_{tj^2} + \chi^2(p) p^{k-2} a_{tj^2/p^2} \right).$$

Note that $\psi_1(p)\left(\frac{tj^2}{p}\right) = \psi_t(p)\left(\frac{j^2}{p}\right)$. So we can rewrite C_n as

$$C_n = \sum_{ij=n} \psi_t(i) i^{\lambda-1} \left(a_{p^2tj^2} + \psi_t(p) \left(\frac{j^2}{p} \right) p^{\lambda-1} a_{tj^2} + \chi^2(p) p^{k-2} a_{tj^2/p^2} \right). \quad (3.6)$$

Note that the Legendre symbol here is 1 unless of course $p \mid j$ in which case it is 0. Moreover $a_{tj^2/p^2} = 0$ whenever $p \nmid j$; this is because t is square-free.

We consider the following two cases.

Case $p \nmid n$. In this case the formulae for B_n and C_n simplify as follows.

$$\begin{aligned} B_n &= \sum_{ij=pn} \psi_t(i) i^{\lambda-1} a_{tj^2} \\ &= \sum_{ij=n} \psi_t(pi)(pi)^{\lambda-1} a_{tj^2} + \psi_t(i) i^{\lambda-1} a_{tp^2j^2} \\ &= \sum_{ij=n} \psi_t(i) i^{\lambda-1} (a_{tp^2j^2} + \psi_t(p) p^{\lambda-1} a_{tj^2}) \\ &= C_n. \end{aligned}$$

Case $p \mid n$. Write $n = p^r m$ where $r \geq 1$ and $p \nmid m$. We rewrite (3.5) as follows.

$$B_n = \sum_{j \mid p^{r+1}m} \psi_t(p^{r+1}m/j)(p^{r+1}m/j)^{\lambda-1} a_{tj^2} \\ + \sum_{j \mid p^{r-1}m} \chi^2(p) \psi_t(p^{r-1}m/j) p^{k-2} (p^{r-1}m/j)^{\lambda-1} a_{tj^2}.$$

This maybe re-expressed as $B_n = B_n^{(1)} + B_n^{(2)}$ where

$$B_n^{(1)} = \sum_{u=0}^{r+1} \sum_{k \mid m} \psi_t(p^{r+1-u}m/k)(p^{r+1-u}m/k)^{\lambda-1} a_{tp^{2u}k^2}$$

and

$$B_n^{(2)} = \sum_{u=0}^{r-1} \sum_{k|m} \chi^2(p)\psi_t(p^{r-1-u}m/k)p^{k-2}(p^{r-1-u}m/k)^{\lambda-1}a_{tp^{2u}k^2}.$$

Moreover, we can rewrite (3.6) as follows.

$$C_n = \sum_{j|p^r m} \psi_t(p^r m/j)(p^r m/j)^{\lambda-1} \left(a_{p^2tj^2} + \psi_t(p)\left(\frac{j^2}{p}\right)p^{\lambda-1}a_{tj^2} + \chi^2(p)p^{k-2}a_{tj^2/p^2} \right).$$

Thus we can write $C_n = C_n^{(1)} + C_n^{(2)} + C_n^{(3)}$ where

$$C_n^{(1)} = \sum_{u=0}^{r} \sum_{k|m} \psi_t(p^{r-u}m/k)(p^{r-u}m/k)^{\lambda-1}a_{tp^{2u+2}k^2},$$

and

$$C_n^{(2)} = \sum_{k|m} \psi_t(p^{r+1}m/k)(p^{r+1}m/k)^{\lambda-1}a_{tk^2},$$

and

$$C_n^{(3)} = \sum_{u=1}^{r} \sum_{k|m} \chi^2(p)\psi_t(p^{r-u}m/k)(p^{r-u}m/k)^{\lambda-1}p^{k-2}a_{tp^{2u-2}k^2}.$$

It is clear that $B_n^{(2)} = C_n^{(3)}$, and also that $B_n^{(1)} = C_n^{(1)} + C_n^{(2)}$; here $C_n^{(2)}$ corresponds to the $u = 0$ terms in $B_n^{(1)}$. Thus $B_n = C_n$ completing the proof. $\square$

3.3 Recursion Formula for the Hecke Operators $T_{p^{2l}}$

We keep the notation as in the previous section. Let l be a positive integer and p be a prime. In this section we are interested in the action of the Hecke operator $T_{p^{2l}}$ on the space $M_{k/2}(N, \chi)$. In the case $p \mid N$ we have the following easy lemma.

Lemma 3.3.1. *Let l be a positive integer and p be a prime dividing N. Let t be a square-free positive integer. Then*

(i) $T_{p^{2l}} = (T_{p^2})^l$.

(ii) $\mathrm{Sh}_t(T_{p^{2l}}f) = T_{p^l}(\mathrm{Sh}_t(f))$ *for* $f \in S_{k/2}(N,\chi)$.

In the above statements $T_{p^{2l}} \in \mathbb{T}_{k/2}$ *and* $T_{p^l} \in \mathbb{T}_{k-1}$.

Proof. Let $f = \sum_{n=0}^{\infty} a_n q^n \in M_{k/2}(N,\chi)$. It follows using [36, Proposition 1.5] that $T_{p^{2l}}(f) = \sum_{n=1}^{\infty} a_{np^{2l}} q^n$. Now part (i) follows using Theorem 2.3.6. Part (ii) follows by using Proposition 3.2.5 and part (b) of Proposition 2.2.4 since $p \mid N'$. $\square$

We will assume that $p \nmid N$ for the rest of this section. The main aim of this section is to prove the following result.

Theorem 3.3.2. *Let* $p \nmid N$ *be a prime and* $l \geq 2$ *be a positive integer. Then the following identity of the Hecke operators holds in* $\mathbb{T}_{k/2}$*:*

$$T_{p^{2l+2}} = T_{p^2}T_{p^{2l}} - \chi(p^2)p^{k-2}T_{p^{2l-2}}.$$

It is to be noted that for $l = 1$ the above relation does not hold. One can check directly that in $\mathbb{T}_{k/2}$,

$$T_{p^4} = (T_{p^2})^2 - \chi(p^2)(p^{k-3} + p^{k-2}).$$

We need the following lemma on Gauss sums which can be easily deduced from [28, Lemma 3.1.3]:

Lemma 3.3.3. *Let* p *be a prime and* n, α *be a given positive integer. Then*

(i) $\sum_{m=0}^{p^\alpha-1} \left(\frac{m}{p}\right) e^{\frac{2\pi i m n}{p^\alpha}} = \begin{cases} 0 & \text{if } p^{\alpha-1} \nmid n \\ p^{\alpha-1}\left(\frac{n'}{p}\right)\epsilon_p\sqrt{p} & \text{if } n = p^{\alpha-1}n'. \end{cases}$

(ii) $\sum_{m=0}^{p^\alpha-1} e^{\frac{2\pi i m n}{p^\alpha}} = \begin{cases} 0 & p^\alpha \nmid n \\ p^\alpha & p^\alpha \mid n. \end{cases}$

Proof of Theorem 3.3.2. Let $f \in M_{k/2}(N,\chi)$. Let $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ 0 & p^{2l} \end{smallmatrix}\right]$, $\xi = (\alpha, p^{l/2})$. Using [28, Lemma 4.5.6] we know that

$$\Gamma_0(N)\alpha\Gamma_0(N) = \bigcup_{\nu,m} \Gamma_0\alpha_{\nu,m}, \qquad \alpha_{\nu,m} = \begin{bmatrix} p^{2l-\nu} & m \\ 0 & p^\nu \end{bmatrix}$$

where $0 \leq \nu \leq 2l$, $0 \leq m < p^\nu$ and $\gcd(m, p^\nu, p^{2l-\nu}) = 1$. Let G be the group defined in Subsection 2.3.1. Let $\xi_{\nu,m} \in G$ be given by

$$\xi_{\nu,m} = \begin{cases} \left(\alpha_{\nu,m},\ p^{\frac{-2l+2\nu}{4}} \epsilon_p^{-1} \left(\frac{-m}{p}\right)\right) & \text{if } \nu \text{ is odd} \\ (\alpha_{\nu,m},\ p^{\frac{-2l+2\nu}{4}}) & \text{if } \nu \text{ is even.} \end{cases}$$

One can verify that $\xi_{\nu,m}$ with ν and m varying as above form a set of right coset representatives of $\Delta_0(N)$ in $\Delta_0(N)\xi\Delta_0(N)$ (see [36, Proposition 1.1]). Then we know by definition of $T_{p^{2l}}$ (see Subsection 2.3.3) that

$$T_{p^{2l}} f = (p^{2l})^{\frac{k}{4}-1} \left(A_0 + A_{2l} + \sum_{\nu=1}^{2l-1} A_\nu \right), \tag{3.7}$$

where

$$A_\nu = \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu - 1} \chi(p^{2l-\nu}) f|[\xi_{\nu,m}]_{k/2},\ A_{2l} = \sum_{m=0}^{p^{2l}-1} f|[\xi_{2l,m}]_{k/2},\ A_0 = \chi(p^{2l}) f|[\xi_{0,0}]_{k/2}.$$

Applying T_{p^2} to Equation (3.7) we obtain

$$\begin{aligned} T_{p^2} T_{p^{2l}} f &= (p^{2l})^{\frac{k}{4}-1} \left(\sum_{\nu=1}^{2l-1} T_{p^2} A_\nu + T_{p^2} A_{2l} + T_{p^2} A_0 \right) \\ &= (p^{2l+2})^{\frac{k}{4}-1} \left(\sum_{\nu=1}^{2l-1} B_\nu + B_{2l} + B_0 \right), \end{aligned} \tag{3.8}$$

where for ν with $0 \leq \nu \leq 2l-2$ we have

$$\begin{aligned} B_\nu = \chi(p^{2l-\nu+2}) &\sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu - 1} f|[([\begin{smallmatrix} p^{2l-\nu+2} & m \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2l+2\nu-2}{4}} r_{\nu,m})]_{k/2} \\ + \chi(p^{2l-\nu+1}) &\sum_{m'=1}^{p-1} \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu - 1} f|[([\begin{smallmatrix} p^{2l-\nu+1} & p^{2l-\nu}m'+mp \\ 0 & p^{\nu+1} \end{smallmatrix}],\ p^{\frac{-2l+2\nu}{4}} s_{\nu,m,m'})]_{k/2} \\ + \chi(p^{2l-\nu}) &\sum_{m'=0}^{p^2-1} \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu - 1} f|[([\begin{smallmatrix} p^{2l-\nu} & p^{2l-\nu}m'+mp^2 \\ 0 & p^{\nu+2} \end{smallmatrix}],\ p^{\frac{-2l+2\nu+2}{4}} r_{\nu,m})]_{k/2}, \end{aligned}$$

where

$$r_{\nu,m} = \begin{cases} \epsilon_p^{-1}\left(\frac{-m}{p}\right) & \nu \text{ odd} \\ 1 & \nu \text{ even} \end{cases}, \qquad s_{\nu,m,m'} = \begin{cases} \epsilon_p^{-2}\left(\frac{mm'}{p}\right) & \nu \text{ odd} \\ \epsilon_p^{-1}\left(\frac{-m'}{p}\right) & \nu \text{ even,} \end{cases}$$

and B_{2l} has the same expression as above with $\nu = 2l$ but without any coprimality condition on m, that is, we do not have $(m, p) = 1$ in the above terms while writing the expression for B_{2l}.

We express $T_{p^{2l+2}}f$ as in Equation (3.7) and compare it with Equation (3.8). Ruling out some of the terms using Euclidean algorithm and rewriting the action of matrices (we will give an example of the working later) we obtain

$$(T_{p^{2l+2}} - T_{p^2}T_{p^{2l}})(f) = -(p^{2l+2})^{\frac{k}{4}-1}\left(S_0 + S_{2l} + \sum_{\nu=1}^{2l-1}(D_\nu + E_\nu)\right) \tag{3.9}$$

where

$$S_0 = \sum_{m'=0}^{p^2-1} \chi(p^{2l}) f|[([\begin{smallmatrix} p^{2l} & p^{2l}m' \\ 0 & p^2 \end{smallmatrix}],\ p^{\frac{-l+1}{2}})]_{k/2}$$

$$S_{2l} = \sum_{\substack{m=0 \\ (m,p)\neq 1}}^{p^{2l}-1} \chi(p^2) f|[([\begin{smallmatrix} p^2 & m \\ 0 & p^{2l} \end{smallmatrix}],\ p^{\frac{l-1}{2}})]_{k/2}$$

$$D_\nu = \chi(p^{2l-\nu}) \sum_{m'=0}^{p^2-1} \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu-1} f|[([\begin{smallmatrix} p^{2l-\nu} & p^{2l-\nu}m'+mp^2 \\ 0 & p^{\nu+2} \end{smallmatrix}],\ p^{\frac{-2l+2\nu+2}{4}} r_{\nu,m})]_{k/2}$$

$$E_\nu = \chi(p^{2l-\nu+1}) \sum_{m'=1}^{p-1} \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu-1} f|[([\begin{smallmatrix} p^{2l-\nu+1} & p^{2l-\nu}m'+mp \\ 0 & p^{\nu+1} \end{smallmatrix}],\ p^{\frac{-2l+2\nu}{4}} s_{\nu,m,m'})]_{k/2}.$$

Further

$$\chi(p^2)p^{k-2}T_{p^{2l-2}}f = p^2(p^{2l+2})^{\frac{k}{4}-1}\left(\sum_{\nu=1}^{2l-3} C_\nu + C_{2l-2} + C_0\right), \tag{3.10}$$

where for ν with $0 \leq \nu \leq 2l-3$ we have

$$C_\nu = \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu-1} \chi(p^{2l-\nu}) f|[([\begin{smallmatrix} p^{2l-\nu-2} & m \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2l+2\nu+2}{4}} r_{\nu,m})]_{k/2}$$

and C_{2l-2} has the same expression as above with $\nu = 2l-2$ but without the condition $(m,p) = 1$ in the above sum. We first claim that the following relations hold:

(i) $D_\nu = p^2 C_\nu$ for $1 \leq \nu \leq 2l-3$, and $S_0 = p^2 C_0$.

(ii) $E_\nu = 0$ for $1 \leq \nu \leq 2l-2$.

We will only show the computation for part (ii) for case ν odd. The rest of the claim follows by similar method. Fix an odd ν with $1 \leq \nu \leq 2l-3$. Fix $1 \leq m' \leq p-1$. Then for each m with $0 \leq m \leq p^\nu - 1$ there exist unique a and b with $0 \leq b \leq p^\nu - 1$ such that $m + p^{2l-\nu-1}m' = ap^\nu + b$. Moreover $m \equiv b \pmod p$. Hence

$$(m,p) = 1 \iff (b,p) = 1, \qquad \left(\frac{-m}{p}\right) = \left(\frac{-b}{p}\right).$$

We can rewrite E_ν as

$$\begin{aligned}
E_\nu &= \chi(p^{2l-\nu+1}) \sum_{m'=1}^{p-1} \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu-1} f\left(\frac{p^{2l-\nu+1}z + p^{2l-\nu}m' + mp}{p^{\nu+1}}\right)\left(p^{\frac{-2l+2\nu}{4}} \epsilon_p^{-2} \left(\frac{mm'}{p}\right)\right)^{-k} \\
&= \chi(p^{2l-\nu+1})\epsilon_p^k \sum_{m'=1}^{p-1} \left(\frac{-m'}{p}\right) \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu-1} f\left|\left[\left([\begin{smallmatrix} p^{2l-\nu} & p^{2l-\nu-1}m'+m \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2l+2\nu}{4}} \epsilon_p^{-1} \left(\frac{-m}{p}\right)\right)\right]_{k/2}\right. \\
&= \chi(p^{2l-\nu+1})\epsilon_p^k \sum_{m'=1}^{p-1} \left(\frac{-m'}{p}\right) \sum_{\substack{b=0 \\ (b,p)=1}}^{p^\nu-1} f\left|\left[\left([\begin{smallmatrix} p^{2l-\nu} & b \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2l+2\nu}{4}} \epsilon_p^{-1} \left(\frac{-b}{p}\right)\right)\right]_{k/2}\right. \\
&= 0.
\end{aligned}$$

The second last equality follows since as elements of G we have

$$\left([\begin{smallmatrix} p^{2l-\nu} & p^{2l-\nu-1}m'+m \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2l+2\nu}{4}} \epsilon_p^{-1} \left(\frac{-m}{p}\right)\right) = ([\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}], 1) \cdot \left([\begin{smallmatrix} p^{2l-\nu} & b \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2l+2\nu}{4}} \epsilon_p^{-1} \left(\frac{-b}{p}\right)\right).$$

By working out similarly as above one can further see that

$$p^2C_{2l-2} - D_{2l-2} = \chi(p^2)\sum_{m'=0}^{p^2-1}\sum_{\substack{m=0\\(m,p)\neq 1}}^{p^{2l-2}-1} f|[([\begin{smallmatrix} p^2 & \frac{p^2m'+mp^2}{p^{2l}} \\ 0 & \end{smallmatrix}],\ p^{\frac{l-1}{2}})]_{k/2} =: F_{2l-2}.$$

Thus to prove the theorem we are left to show that

$$F_{2l-2} - S_{2l} - E_{2l-1} - D_{2l-1} = 0.$$

We claim that $D_{2l-1} = 0$ and $F_{2l-2} - S_{2l} - E_{2l-1} = 0$ which proves the theorem.

We first show that $D_{2l-1} = 0$. Let $f(z) = \sum_{n=0}^{\infty} a_n e(nz)$ where $e(nz) = e^{2\pi inz}$. Rewriting D_{2l-1} in terms of coefficients a_n we obtain

$$\begin{aligned}
D_{2l-1} &= \chi(p)p^{\frac{-lk}{2}}\epsilon_p^k\left(\frac{-1}{p}\right)\sum_{m'=0}^{p^2-1}\sum_{\substack{m=0\\(m,p)=1}}^{p^{2l-1}-1}\sum_{n=0}^{\infty} a_n e\left(\frac{npz+npm'+nmp^2}{p^{2l+1}}\right)\left(\frac{m}{p}\right)\\
&= \chi(p)p^{\frac{-lk}{2}}\epsilon_p^k\left(\frac{-1}{p}\right)\sum_{n=0}^{\infty} a_n e\left(\frac{nz}{p^{2l}}\right)\sum_{m'=0}^{p^2-1} e\left(\frac{nm'}{p^{2l}}\right)\sum_{m=0}^{p^{2l-1}-1} e\left(\frac{nm}{p^{2l-1}}\right)\left(\frac{m}{p}\right)\\
&= \chi(p)p^{\frac{-lk+4l-3}{2}}\epsilon_p^{k+1}\left(\frac{-1}{p}\right)\sum_{\substack{n=0\\p^{2l-2}|n}}^{\infty} a_n e\left(\frac{nz}{p^{2l}}\right)\left(\frac{n/p^{2l-2}}{p}\right)\sum_{m'=0}^{p^2-1} e\left(\frac{nm'/p^{2l-2}}{p^2}\right)\\
&= 0,
\end{aligned}$$

where last two equalities follows using Lemma 3.3.3 on Gauss sums. In order to prove the final claim we again use the coefficients method as above to obtain

$$F_{2l-2} - S_{2l} = \chi(p^2)p^{\frac{(-l+1)k+4l-2}{2}}\sum_{\substack{n=0\\p^{2l-2}\|n}}^{\infty} a_n e\left(\frac{nz}{p^{2l-2}}\right),$$

$$E_{2l-1} = \chi(p^2)p^{\frac{(-l+1)k+4l-2}{2}}\epsilon_p^{2k+2}\sum_{\substack{n=0\\p^{2l-2}\|n}}^{\infty} a_n e\left(\frac{nz}{p^{2l-2}}\right).$$

Now $\epsilon_p^{2k+2} = 1$ since $2k+2 \equiv 0 \pmod 4$. Hence we are done. $\square$

Corollary 3.3.4. *Let $p \nmid N$ be a prime and $l \geq 2$. Let $f \in S_{k/2}(N, \chi)$. Then*

$$\mathrm{Sh}_t(T_{p^{2l}} f) = (T_{p^l} - \chi(p^2)p^{k-3}T_{p^{l-2}})(\mathrm{Sh}_t(f)),$$

where as before $T_{p^{2l}} \in \mathbb{T}_{k/2}$ and T_{p^l}, $T_{p^{l-2}} \in \mathbb{T}_{k-1}$.

Proof. We use induction on l. Recall from part (c) of Proposition 2.2.4 that for prime $p \nmid N$, we have

$$T_{p^{e+1}}(\mathrm{Sh}_t f) = (T_p T_{p^e} - \chi(p^2)p^{k-2}T_{p^{e-1}})(\mathrm{Sh}_t f). \tag{3.11}$$

As we remarked earlier, for $l = 2$ we have the following relation in $\mathbb{T}_{k/2}$:

$$T_{p^4} = (T_{p^2})^2 - \chi(p^2)(p^{k-3} + p^{k-2}).$$

Hence we get

$$\begin{aligned}
\mathrm{Sh}_t(T_{p^4} f) &= \mathrm{Sh}_t((T_{p^2})^2 f) - \chi(p^2)(p^{k-3} + p^{k-2})(\mathrm{Sh}_t f)\\
&= ((T_p)^2 - \chi(p^2)p^{k-2})(\mathrm{Sh}_t f) - \chi(p^2)p^{k-3}(\mathrm{Sh}_t f)\\
&= (T_{p^2} - \chi(p^2)p^{k-3})(\mathrm{Sh}_t f).
\end{aligned}$$

Assume the statement holds for all $l \leq e$. Then

$$\begin{aligned}
&\mathrm{Sh}_t(T_{p^{2e+2}} f) = \mathrm{Sh}_t(T_{p^2} T_{p^{2e}} f) - \chi(p^2)p^{k-2}\,\mathrm{Sh}_t(T_{p^{2e-2}} f)\\
&= T_p(\mathrm{Sh}_t(T_{p^{2e}} f) - \chi(p^2)p^{k-2}\,\mathrm{Sh}_t(T_{p^{2e-2}} f)\\
&= (T_p T_{p^e} - \chi(p^2)(p^{k-3}T_p T_{p^{e-2}} + p^{k-2}T_{p^{e-1}}) + \chi(p^4)p^{2k-5}T_{p^{e-3}})(\mathrm{Sh}_t f)\\
&= (T_{p^{e+1}} - \chi(p^2)p^{k-3}(T_{p^{e-1}} + \chi(p^2)p^{k-2}T_{p^{e-3}}) + \chi(p^4)p^{2k-5}T_{p^{e-3}})(\mathrm{Sh}_t f)\\
&= (T_{p^{e+1}} - \chi(p^2)p^{k-3}T_{p^{e-1}})(\mathrm{Sh}_t f).
\end{aligned}$$

The first equality uses Theorem 3.3.2, third equality follows by using inductive hypothesis for $l = e$ and $l = e - 1$, the others follow by using Equation (3.11). $\square$

We also prove the following proposition, independently of the proof of Theorem 3.3.2.

Proposition 3.3.5. *Let $p \nmid N$ be a prime and l be a positive integer. For* *positive integers r such that $1 \leq r \leq \lfloor \frac{l}{2} \rfloor$ we give the following recursive construction of sequences $A_{r,l}(m)$ and $B_{r,l}(m)$:*

$$A_{1,l}(m) = 1, \qquad A_{r,l}(m) = A_{r-1,l}(m) - \binom{l-2(r-1)}{m-(r-1)} A_{r-1,l}(r-1);$$

$$B_{1,l}(m) = \binom{l}{m} - 1, \qquad B_{r,l}(m) = B_{r-1,l}(m) - \binom{l-2(r-1)}{m-(r-1)} B_{r-1,l}(r-1).$$

Let $\alpha_{r,l} = A_{r,l}(r)$ and $\beta_{r,l} = B_{r,l}(r)$. Then the following relation holds between operators in $\mathbb{T}_{k/2}$:

$$T_{p^{2l}} = (T_{p^2})^l - \sum_{r=1}^{\lfloor \frac{l}{2} \rfloor} \chi(p^{2r})(\alpha_{r,l} p^{r(k-2)-1} + \beta_{r,l} p^{r(k-2)})(T_{p^2})^{l-2r}.$$

Proof. Let $f = \sum_{n=0}^{\infty} a(n) q^n \in M_{k/2}(N, \chi)$. Our strategy will be to compare the nth coefficient of action of the above operators on f on both sides. Substituting the q-expansion of f in Equation (3.7) and using Lemma 3.3.3 on Gauss sums we obtain

$$T_{p^{2l}} f = I_0 + I_{2l} + \sum_{\substack{\nu=1 \\ \nu \text{odd}}}^{2l-1} I_\nu^{\text{odd}} + \sum_{\substack{\nu=1 \\ \nu \text{even}}}^{2l-1} I_\nu^{\text{even}}$$

where

$$I_0 = \chi(p^{2l}) p^{(k-2)l} \sum_{n=0}^{\infty} a(n/p^{2l}) q^n, \qquad I_{2l} = \sum_{n=0}^{\infty} a(np^{2l}) q^n$$

$$I_\nu^{\text{odd}} = \chi(p^{2l} - \nu) p^{(\frac{k}{2}-1)(2l-\nu)-\frac{1}{2}} \epsilon_p^{k+1} \left(\frac{-1}{p}\right) \sum_{\substack{n=0 \\ p^{2l-\nu-1} \mid n}}^{\infty} a(n/p^{2l-2\nu}) \left(\frac{n/p^{2l-\nu-1}}{p}\right) q^n$$

$$I_\nu^{\text{even}} = \chi(p^{2l} - \nu) p^{(\frac{k}{2}-1)(2l-\nu)-1} \Big(\sum_{\substack{n=0 \\ p^{2l-\nu} \mid n}}^{\infty} a(n/p^{2l-2\nu})(p-1) q^n - \sum_{\substack{n=0 \\ p^{2l-\nu-1} \parallel n}}^{\infty} a(n/p^{2l-2\nu}) q^n \Big).$$

Let n be a positive integer with $p^{2(l-1)} \mid n$. We can write the n-th coefficient of $T_{p^2}^l f$ as

$$a(np^{2l}) + \sum_{m=1}^{l-1} \binom{l}{m} \chi(p^{2m}) p^{(k-2)m} a(np^{2l-4m}) +$$
$$\chi(p^{2l-1}) \left(\frac{-1}{p}\right)^{\frac{k-1}{2}} \left(\frac{n/p^{2l-2}}{p}\right) p^{\frac{k-3}{2}+(k-2)(l-1)} a(n/p^{2l-2}) + \chi(p^{2l}) p^{(k-2)l} a(n/p^{2l}).$$

Thus the n-th coefficient of $T_{p^2}^l f - T_{p^{2l}} f$ is

$$\sum_{m=1}^{l-1} \left(\binom{l}{m} - 1\right) \chi(p^{2m}) p^{(k-2)m} a(np^{2l-4m}) + \sum_{m=1}^{l-1} \chi(p^{2m}) p^{(k-2)m-1} a(np^{2l-4m}).$$

We want to subtract a suitable multiple of $T_{p^2}^{l-2} f$ from the above so as to remove the terms involving $a(np^{2l-4})$ and $a(np^{4-2l})$, thereby reducing the number of terms in the above sum. Indeed we obtain that the n-th coefficient of $(T_{p^2}^l - T_{p^{2l}} - \chi(p^2)(p^{k-3} + (l-1)p^{k-2}) T_{p^2}^{l-2}) f$ is

$$\sum_{m=2}^{l-2} \left(1 - \binom{l-2}{m-1}\right) \chi(p^{2m}) p^{(k-2)m-1} a(np^{2l-4m}) +$$
$$\sum_{m=2}^{l-2} \left(\binom{l}{m} - 1 - (l-1)\binom{l-2}{m-1}\right) \chi(p^{2m}) p^{(k-2)m} a(np^{2l-4m}).$$

We iterate this process of subtracting suitable multiples of $T_{p^2}^{l-2r} f$ which leads us to the recursive formulae for $\alpha_{r,l}$ and $\beta_{r,l}$. $\square$

We obtain the following combinatorial result as a corollary of Theorem 3.3.2 and Proposition 3.3.5

Corollary 3.3.6. *Keeping the notation as in the previous proposition we get the following combinatorial identities for* $2 \leq r \leq \lfloor \frac{l}{2} \rfloor - 1$*:*

$$\alpha_{r-1,l-2} + \alpha_{r,l} - \alpha_{r,l-1} = 0, \qquad \beta_{r-1,l-2} + \beta_{r,l} - \beta_{r,l-1} = 0.$$

Proof. Let $p \nmid N$ be any prime. We substitute the formula for $T_{p^{2l}}$ given by Proposition 3.3.5 in the identity of Theorem 3.3.2,

$$T_{p^{2l+2}} - T_{p^2} T_{p^{2l}} + \chi(p^2) p^{k-2} T_{p^{2l-2}} = 0$$

to obtain

$$
\begin{aligned}
&- \sum_{r=2}^{\lfloor \frac{l}{2} \rfloor} \chi(p^{2r})(\alpha_{r,l} p^{r(k-2)-1} + \beta_{r,l} p^{r(k-2)})(T_{p^2})^{l-2r} \\
&+ \sum_{r=2}^{\lfloor \frac{l-1}{2} \rfloor} \chi(p^{2r})(\alpha_{r,l-1} p^{r(k-2)-1} + \beta_{r,l-1} p^{r(k-2)})(T_{p^2})^{l-2r} \\
&- \sum_{r=2}^{\lfloor \frac{l-2}{2} \rfloor + 1} \chi(p^{2r})(\alpha_{r-1,l-2} p^{r(k-2)-1} + \beta_{r-1,l-2} p^{r(k-2)})(T_{p^2})^{l-2r} = 0.
\end{aligned}
$$

It is clear, with fixed l and varying r, that the operators $(T_{p^2})^{l-2r}$ are linearly independent elements of $\mathbb{T}_{k/2}$ and hence

$$-\alpha_{r,l} + \alpha_{r,l-1} - \alpha_{r-1,l-2} + (\beta_{r,l} + \beta_{r,l-1} - \beta_{r-1,l-2})p = 0.$$

Since this holds for any prime p with $p \nmid N$ the above corollary follows.

□

3.4 Eigenforms in Half-Integral Weight

In the integral weight case, one way of computing the simultaneous cuspidal eigenspaces under the action of all the Hecke operators is to repeatedly split the new space using Hecke operators until the simultaneous eigenspaces are 1-dimensional. This works in the integral weight case because of the multiplicity-one theorem, which asserts that simultaneous eigenspaces are indeed 1-dimensional. The analogue of the multiplicity-one theorem in the half-integral weight case is false. The following two examples illustrate what can happen.

3.4.1 Two Examples

Example 3.4.1. In this example, we compute an eigenbasis for the space $S_{3/2}(44)$. Using `MAGMA` we obtain the following basis for this space

$$f_1(z) = q - q^4 - q^5 + q^{12} - 2q^{14} + 2q^{15} + O(q^{20})$$
$$f_2(z) = q^3 - q^4 - q^{11} - q^{12} + q^{15} + 2q^{16} + O(q^{20}).$$

We also find using 3.1.5 that the space $S_0(44)$ is zero-dimensional, hence f_1 and f_2 is a basis for $S^{\perp}_{3/2}(44)$. We compute

$$T_{3^2}(f_1) = -f_1,\ T_{5^2}(f_1) = f_1,\ T_{7^2}(f_1) = -2f_1,\ T_{11^2}(f_1) = f_1,$$

and

$$T_{3^2}(f_2) = -f_2,\ T_{5^2}(f_2) = f_2,\ T_{7^2}(f_2) = -2f_2,\ T_{11^2}(f_2) = f_2.$$

To compute an eigenbasis for $S_{3/2}(44)$ we note that

$$T_{2^2}(f_1)(z) = -q + q^3 + q^5 - q^{11} - 2q^{12} + 2q^{14} - q^{15} + 2q^{16} + O(q^{20})$$
$$T_{2^2}(f_2)(z) = -q - q^3 + 2q^4 + q^5 + q^{11} + 2q^{14} - 3q^{15} - 2q^{16} + O(q^{20}).$$

Thus

$$T_{2^2}(f_1) = -f_1 + f_2, \qquad T_{2^2}(f_2) = -f_1 - f_2.$$

By diagonalizing the matrix of T_{2^2} with respect to the basis f_1, f_2 we find that an eigenbasis is

$$h_1 = -f_1 + if_2, \qquad h_2 = -f_1 - if_2,$$

and

$$T_{2^2}(h_1) = (-1+i)h_1, \qquad T_{2^2}(h_2) = (-1-i)h_2.$$

Since these eigenspaces are 1-dimensional it is impossible to split them further and so h_1, h_2 is a simultaneous eigenbasis for all the Hecke operators. Let us check our computation against Shimura's correspondence (Theorem 3.2.1). We take h_1 and construct its Shimura lift $g(z) = \sum_{i=1}^{\infty} b_i q^i \in S_2(22)$. For each prime p let $T_{p^2}(h_1) = \lambda_p(h_1)$. Then we can recover the b_i from the following recipe from (3.2):

1. $b_1 = 1$,

2. $b_p = \lambda_p$ for all primes p,

3. $b_{p^v} = \lambda_p b_{p^{v-1}} - \chi(p)p^{k-2}b_{p^{v-2}}$ for $v \geq 2$,

4. $b_{mn} = b_m b_n$ if m, n are relatively prime.

In our case $\chi = \chi_{\text{triv}}$ is the trivial character of conductor 44 and so $\chi(p) = 1$ for all primes except $\chi(2) = \chi(11) = 0$. Moreover our $k = 3$.

We find

$$g(z) = q + (-1+i)q^2 - q^3 - 2iq^4 + q^5 + (1-i)q^6 - 2q^7 + (2+2i)q^8 - 4q^9 + O(q^{10}).$$

Using `MAGMA` we computed the following basis for $S_2(22)$:

$$g_1(z) = q - q^3 - 2q^4 + q^5 - 2q^7 + 4q^8 - 2q^9 + q^{11} + O(q^{12})$$
$$g_2(z) = q^2 - 2q^4 - q^6 + 2q^8 + q^{10} + O(q^{12}).$$

We observe that $g = g_1 + (i-1)g_2$ up to the coefficient of q^{11}, which is consistent with Shimura's correspondence.

Example 3.4.2. `MAGMA` gives the following basis for $S_{3/2}(72)$:

$$\begin{aligned} f_1 &= q - 2q^{10} - 2q^{13} + 4q^{22} - q^{25} + 2q^{34} + 4q^{37} - 4q^{46} - 3q^{49} + O(q^{50}) \\ f_2 &= q^2 - q^5 - 2q^{14} + q^{17} + 3q^{29} - q^{41} + O(q^{50}). \end{aligned} \tag{3.12}$$

Here $S_0(72) = 0$ and so $S_{3/2}^{\perp}(72) = S_{3/2}(72)$. Using the formula for the action of Hecke operators in Theorem 2.3.6, we computed the action of Hecke operators T_{p^2} for all primes $p \leq 50$; here we needed to work with cusp expansions with precision of $O(q^{5000})$. We found that f_1 and f_2 are eigenfunctions for each of these T_{p^2} with the same eigenvalue. Thus it seems the whole space $S_{3/2}(72)$ is a simultaneous eigenspace for all the Hecke operators, although we have not yet proved this.

It is to be noted that $S_{3/2}(24) = S_{3/2}(36) = 0$. Thus $S_{3/2}(72)$ is made up entirely of the new subspace and still seems not to satisfy a multiplicity-one result.

3.4.2 Generators for the Hecke Action

Theorem 3.4.3. *Let k, N be positive integers with $k \geq 3$ odd, and $4 \mid N$. Let χ be a Dirichlet character modulo N. Let $N' = N/2$. Let $\mathbb{T}$ be the restriction of Hecke algebra $\mathbb{T}_{k-1}$ to $S_{k-1}(N', \chi^2)$ and suppose $\mathbb{T}$ is generated as a $\mathbb{Z}$-module by the Hecke operators T_i for $i \leq r$. Then the Hecke operators T_{i^2} for $i \leq r$ generate the restriction of Hecke algebra $\mathbb{T}_{k/2}$ to $S_{k/2}^{\perp}(N, \chi)$ as a $\mathbb{Z}[\zeta_{\varphi(N)}]$-module. In particular, $f \in S_{k/2}^{\perp}(N, \chi)$ is an eigenform for all Hecke operators if and only if it is an eigenform for T_{i^2} for $i \leq r$.*

Proof. Let n be a positive integer with prime factorization $n = p_1^{n_1} p_2^{n_2} \cdots p_s^{n_s}$. Let $f \in S_{k/2}^{\perp}(N, \chi)$. Let t be a square-free positive integer. Using Theorem 3.3.2 or Proposition 3.3.5, for any prime p and a positive integer l we can express the action of $T_{p^{2l}}$ as

$$T_{p^{2l}} = \sum_{j=0}^{l} \gamma_j T_{p^2}^{j}, \qquad \gamma_j \in \mathbb{Z}[\zeta_{\varphi(N)}]. \tag{3.13}$$

Note that in the above expression $\gamma_l = 1$ and hence the Hecke operators $T_{p^2}^{j}$ with $1 \leq j \leq l$ generates the same $\mathbb{Z}[\zeta_{\varphi(N)}]$-module as do the Hecke operators $T_{p^{2j}}$ with $1 \leq j \leq l$. Thus we have

$$\begin{aligned}
\mathrm{Sh}_t(T_{n^2} f) &= \mathrm{Sh}_t(T_{p_1^{2n_1}} T_{p_2^{2n_2}} \cdots T_{p_s^{2n_s}} f) \\
&= \mathrm{Sh}_t\left(\left(\sum_{j_1=0}^{n_1} \gamma_{j_1} T_{p_1^2}^{j_1}\right) \cdots \left(\sum_{j_s=0}^{n_s} \gamma_{j_s} T_{p_s^2}^{j_s}\right) f\right) \\
&= \left(\sum_{j_1=0}^{n_1} \gamma_{j_1} T_{p_1}^{j_1}\right) \cdots \left(\sum_{j_s=0}^{n_s} \gamma_{j_s} T_{p_s}^{j_s}\right) (\mathrm{Sh}_t\, f) \\
&= \sum_{i=1}^{r} \delta_i T_i (\mathrm{Sh}_t\, f),
\end{aligned} \tag{3.14}$$

where the last equality follows since the T_i, with $1 \leq i \leq r$, generate $\mathbb{T}$ as a $\mathbb{Z}$-module, while the second last equality follows by Proposition 3.2.5.

Recall from Proposition 2.2.4, for any prime q and a positive integer l, the action of a Hecke operator T_{q^l} on $S_{k-1}(N', \chi^2)$ can be expressed as

$$T_{q^l} = \sum_{j=0}^{l} \alpha_j T_q^j, \qquad \alpha_j \in \mathbb{Z}[\zeta_{\varphi(N')}] \subset \mathbb{Z}[\zeta_{\varphi(N)}].$$

Let $1 \leq i \leq r$ have prime factorization $i = q_1^{m_1} q_2^{m_2} \cdots q_v^{m_v}$. Then each term $T_i(\mathrm{Sh}_t\, f)$ in Equation (3.14) can be written as

$$\begin{aligned} T_i(\mathrm{Sh}_t\, f) &= T_{q_1^{m_1}} T_{q_2^{m_2}} \cdots T_{q_v^{m_v}} (\mathrm{Sh}_t\, f) \\ &= \left(\sum_{j_1=0}^{m_1} \alpha_{j_1} T_{q_1}^{j_1} \right) \cdots \left(\sum_{j_v=0}^{m_v} \alpha_{j_v} T_{q_v}^{j_v} \right) (\mathrm{Sh}_t\, f) \\ &= \mathrm{Sh}_t \left(\left(\sum_{j_1=0}^{m_1} \alpha_{j_1} T_{q_1^2}^{j_1} \right) \cdots \left(\sum_{j_v=0}^{m_v} \alpha_{j_v} T_{q_v^2}^{j_v} \right) f \right) \\ &= \mathrm{Sh}_t \left(\left(\sum_{j_1=0}^{m_1} \beta_{j_1} T_{q_1^{2j_1}} \right) \cdots \left(\sum_{j_v=0}^{m_v} \beta_{j_v} T_{q_v^{2j_v}} \right) f \right) \\ &= \mathrm{Sh}_t \left(\sum_{j=1}^{i} A_j T_{j^2} f \right), \end{aligned} \tag{3.15}$$

where $A_j \in \mathbb{Z}[\zeta_{\varphi(N)}]$. In the above equalities we repeatedly use Proposition 3.2.5 and Equation (3.13). For the second last equality we use the remark below Equation (3.13). Now using Equations (3.14) and (3.15) we get

$$\mathrm{Sh}_t(T_{n^2} f) = \mathrm{Sh}_t \left(\sum_{i=1}^{r} B_i T_{i^2} f \right), \qquad B_i \in \mathbb{Z}[\zeta_{\varphi(N)}].$$

Since this is true for all positive square-free integers t, using Lemma 3.2.3 we deduce that

$$T_{n^2} f = \sum_{i=1}^{r} B_i T_{i^2} f.$$

Hence T_{i^2}, $i \leq r$ generate the restriction of $\mathbb{T}_{k/2}$ to $S_{k/2}^{\perp}(N, \chi)$ as a $\mathbb{Z}[\zeta_{\varphi(N)}]$-module. $\square$

We shall need the following theorem which is a consequence of Sturm's bound [40].

Theorem 3.4.4. *(Stein [39, Theorem 9.23]) Suppose* Γ *is a congruence sub-*

group that contains $\Gamma_1(N)$. Let

$$r = \frac{km}{12} - \frac{m-1}{N}, \qquad m = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma].$$

Then the Hecke algebra

$$\mathbb{T} = \mathbb{Z}[\dots, T_n, \dots] \subset \mathrm{End}(S_k(\Gamma))$$

is generated as a $\mathbb{Z}$-module by the Hecke operators T_n for $n \le r$.

From Theorem 3.4.4 we deduce the following.

Corollary 3.4.5. *Let k, N be positive integers with $k \ge 3$ odd, and $4 \mid N$. Let χ be a Dirichlet character modulo N. Let $N' = N/2$.*

$$m = N'^2 \prod_{p \mid N'} \left(1 - \frac{1}{p^2}\right), \qquad R = \frac{(k-1)m}{12} - \frac{m-1}{N'}.$$

Then T_{i^2} for $i \le R$ generate the restriction of $\mathbb{T}_{k/2}$ to $S^{\perp}_{k/2}(N, \chi)$ as a $\mathbb{Z}[\zeta_{\varphi(N)}]$-module. In particular the set of operators T_{p^2} for primes $p \le R$ forms a generating set as an algebra. Moreover, $f \in S_{k/2}(N, \chi)$ is an eigenform for all Hecke operators if and only if it is an eigenform for T_{p^2} for $p \le R$.

Proof. Note that $S_{k-1}(N', \chi^2) \subset S_{k-1}(\Gamma_1(N'))$. Now the corollary follows by applying Theorem 3.4.3 and Theorem 3.4.4 to the congruence subgroup $\Gamma_1(N')$ and using the formula for $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N')]$ (see Proposition 2.1.3). $\square$

Corollary 3.4.6. *With the same hypothesis as in the above corollary, further suppose that χ is a quadratic character. Then the same result holds as above with*

$$m = N' \prod_{p \mid N'} \left(1 + \frac{1}{p}\right), \qquad R = \frac{(k-1)m}{12} - \frac{m-1}{N'}.$$

Proof. Since χ is a quadratic character $S_{k-1}(N', \chi^2) = S_{k-1}(N')$. So we apply Theorem 3.4.4 to the group $\Gamma_0(N')$ and we now use the formula for $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N')]$. $\square$

Example 3.4.7. We now return to Example 3.4.2. We found that the space $S^{\perp}_{3/2}(72) = S_{3/2}(72)$ consists entirely of the new subspace, with basis f_1, f_2

given in (3.12). Moreover, f_1, f_2 are eigenfunctions with the same eigenvalue for T_{p^2} for primes $p < 50$. From Corollary 3.4.6 we find that T_{p^2} with $p = 2, 3, 5, 7$ generate the Hecke algebra. Therefore f_1, f_2 are eigenfunctions with the same eigenvalue for all Hecke operators. We note here the failure of 'multiplicity-one'.

3.5 Shimura's Decomposition

In this chapter we shall state and refine a theorem of Shimura that conveniently decomposes the space of cusp forms of half-integral weight.

Fix positive integer k, N with k odd and $4 \mid N$. Let χ be an even Dirichlet character of modulus N. Let $N' = N/2$. For $M \mid N'$ such that $\mathrm{Cond}(\chi^2) \mid M$ and a newform $\phi \in S_{k-1}^{\mathrm{new}}(M, \chi^2)$ define

$$S_{k/2}(N, \chi, \phi) = \{f \in S_{k/2}^{\perp}(N, \chi) : T_{p^2}(f) = \lambda_p(\phi) f \text{ for almost all } p \nmid N\};$$

here $T_p(\phi) = \lambda_p(\phi)\phi$.

Theorem 3.5.1. *(Shimura) We have* $S_{k/2}^{\perp}(N, \chi) = \bigoplus_{\phi} S_{k/2}(N, \chi, \phi)$ *where* ϕ *runs through all newforms* $\phi \in S_{k-1}^{\mathrm{new}}(M, \chi^2)$ *with* $M \mid N'$ *and* $\mathrm{Cond}(\chi^2) \mid M$.

This theorem is attributed to Shimura by Waldspurger [45, Proposition 1] although no reference is given. It is also stated without reference in [19, page 60]. For us this theorem is not suitable for computation since for any particular prime $p \nmid N$, we do not know if it is included or excluded in the 'almost all'. In fact we shall prove this theorem with a more precise definition for the spaces $S_{k/2}(N, \chi, \phi)$.

From now on and for the rest of the thesis we take the following as the definition of the space $S_{k/2}(N, \chi, \phi)$.

Definition 3.5.2. *With notation as above take*

$$S_{k/2}(N, \chi, \phi) = \{f \in S_{k/2}^{\perp}(N, \chi) : T_{p^2}(f) = \lambda_p(\phi) f \textit{ for all } p \nmid N\}.$$

We say that $f \in S_{k/2}^{\perp}(N, \chi)$ is *Shimura equivalent* to ϕ if f belongs to the space $S_{k/2}(N, \chi, \phi)$.

Theorem 3.5.3. *Shimura's decomposition in Theorem 3.5.1 holds with this new definition.*

Proof. Let $f_1, f_2, \ldots, f_n$ be an eigenbasis for $S_{k/2}^{\perp}(N, \chi)$ with respect to the operators T_{p^2} for $p \nmid N$. Let f be one of the f_i. Let $\psi = \mathrm{Sh}_t(f)$ (i.e. the image of f under Shimura's correspondence (Theorem 3.2.1)) with any square-free t. We know that $\psi \in S_{k-1}(N', \chi^2)$. Moreover, for all $p \nmid N$ we know that ψ is an eigenfunction for T_p, with eigenvalue the same as that of f under T_{p^2}; see Proposition 3.2.5. By the theory of newforms (see Proposition 2.2.13) we know that there exists uniquely a divisor M of N' with $\mathrm{Cond}(\chi^2) \mid M$ and a newform $\phi \in S_{k-1}^{\mathrm{new}}(M, \chi^2)$ such that ϕ has the same T_p-eigenvalues as ψ for all primes $p \nmid N'$. Thus $f \in S_{k/2}(N, \chi, \phi)$. We show that that the above decomposition is actually a direct sum. For this, we just need to show that if $h_1, h_2, \ldots, h_r$ are all the elements of the above eigenbasis that belong to $S_{k/2}(N, \chi, F_0)$ where F_0 is a fixed newform in $S_{k-1}^{\mathrm{new}}(M_0, \chi^2)$ with $M_0 \mid N'$ and $\mathrm{Cond}(\chi^2) \mid M_0$, then they actually form a basis for the space $S_{k/2}(N, \chi, F_0)$. We can reorder our basis elements such that $f_i = h_i$ for $1 \le i \le r$. Let $h \in S_{k/2}(N, \chi, F_0)$ and suppose $h = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$. We show that $\alpha_i = 0$ for $r+1 \le i \le n$. We will show that $\alpha_{r+1} = 0$ and the same argument follows for the others. We know that $f_{r+1} \in S_{k/2}(N, \chi, F)$ for some suitable newform F and $F_0 \neq F$. This implies there exists a prime p such that $\lambda_p^0 \neq \lambda_p$ where λ_p^0 and λ_p are corresponding T_p-eigenvalues of F_0 and F. Applying T_{p^2} to h we get $\alpha_{r+1} = 0$. The theorem follows. $\square$

In fact, as a corollary to the proof of Theorem 3.5.3 we can deduce the following precise relationship between the Shimura lift ψ and the newform ϕ.

Corollary 3.5.4. *Let ϕ be a newform belonging to $S_{k-1}^{\mathrm{new}}(M, \chi^2)$ where $M \mid N'$ and $\mathrm{Cond}(\chi^2) \mid M$. Let $f \in S_{k/2}(N, \chi, \phi)$ and let $\psi = \mathrm{Sh}_t(f)$ for any square-free t. Then we can write ψ as a linear combination*

$$\psi = \sum_{d \mid (N'/M)} \alpha_d V_d(\phi).$$

We need later the following fact.

Lemma 3.5.5. *Our definition of* $S_{k/2}(N,\chi,\phi)$ *agrees with Shimura's definition. In other words, write*

$$S^{\mathrm{Sh}}_{k/2}(N,\chi,\phi) = \{f \in S^{\perp}_{k/2}(N,\chi) : T_{p^2}(f) = \lambda_p(\phi) f \text{ for almost all } p \nmid N\};$$

then $S^{\mathrm{Sh}}_{k/2}(N,\chi,\phi) = S_{k/2}(N,\chi,\phi)$.

Proof. Clearly, the right-hand side is contained in the left-hand side. Suppose f is in left-hand side. We use the decomposition Theorem 3.5.3 with our definition of summands. Let θ run through the newforms of levels dividing $N/2$. Then we can write $f = \sum f_\theta$ where $f_\theta \in S_{k/2}(N,\chi,\theta)$. Here ϕ is one of the θs. We know that for almost all primes p,

$$T_{p^2} f = \lambda_p^\phi f = \sum \lambda_p^\phi f_\theta$$

where $T_p\phi = \lambda_p^\phi \phi$. But,

$$T_{p^2}(f) = \sum T_{p^2}(f_\theta) = \sum \lambda_p^\theta f_\theta$$

where $T_p\theta = \lambda_p^\theta \theta$. Thus

$$\sum (\lambda_p^\phi - \lambda_p^\theta) f_\theta = 0.$$

By the fact that the summands belong to a direct sum, we see that each summand must individually be zero. If $f_\theta \neq 0$ then $\lambda_p^\phi = \lambda_p^\theta$ for almost all p which forces $\theta = \phi$ by the multiplicity-one theorem [28, Theorem 4.6.19]. Thus $f = f_\phi \in S_{k/2}(N,\chi,\phi)$ as required. $\square$

Example 3.5.6. As we shall see in Chapter 5, we may obtain some cuspforms of weight 3/2 by taking differences of theta-series of positive-definite ternary quadratic forms belonging to the same genus. Let

$$Q_1 = x_1^2 + 11x_2^2 + 11x_3^2,$$
$$Q_2 = 3x_1^2 + 2x_1x_2 + 4x_2^2 + 11x_3^2.$$

Let θ_1 and θ_2 be the theta-series associated to these positive-definite ternary forms Q_1 and Q_2. It turns out that

$$\theta_1(z) = 1+2q+2q^4+2q^9+O(q^{10}), \qquad \theta_2(z) = 1+2q^3+2q^4+2q^5+2q^9+O(q^{10}).$$

Let $F = \theta_1 - \theta_2$. Then $F \in S_{3/2}(44)$ (see 5.1 for details). Note that

$$F = 2q - 2q^3 - 2q^5 + O(q^{10}).$$

In Basmaji's thesis [3, page 61] it is claimed that F is a simultaneous eigenform for all the Hecke operators. It is easy to check using the formula for Hecke operators (Theorem 2.3.6) that F is indeed an eigenform for T_{p^2} for $p = 3, 5, 7, 11$. However,

$$T_{2^2}(F)(z) = 4q^3 - 4q^4 + O(q^{10})$$

which is clearly not a multiple of F. The space spanned by theta-forms $S_0(44) = 0$. Thus $S_{3/2}(44) = S_0^{\perp}(44)$. By Shimura's Theorem 3.5.3,

$$S_{3/2}(44) = \bigoplus S_{3/2}(44, \phi)$$

where the sum is taken over all newforms ϕ of weight 2 and level dividing $44/2 = 22$. There is precisely one such newform which is at level 11, which we denote by ψ. Thus $S_{3/2}(44) = S_{3/2}(44, \psi)$. In particular, for all $p \nmid 44$, $T_{p^2}F = \lambda_p(\psi)F$. From the above computations, F is an eigenform for T_{p^2} for all odd primes p, but not for $p = 2$.

3.6 Algorithm for Computing Shimura's Decomposition

We recall Shimura's decomposition (Theorem 3.5.3). Fix positive integer k, N with k odd and $4 \mid N$. Let χ be an even Dirichlet character of modulus N. Let $N' = N/2$. For $M \mid N'$ such that $\mathrm{Cond}(\chi^2) \mid M$ and a newform $\phi \in S_{k-1}^{\mathrm{new}}(M, \chi^2)$ define

$$S_{k/2}(N, \chi, \phi) = \{f \in S_{k/2}^{\perp}(N, \chi) : T_{p^2}(f) = \lambda_p(\phi)f \text{ for all } p \nmid N\};$$

here $T_p(\phi) = \lambda_p(\phi)\phi$. Theorem 3.5.3 states that

$$S^{\perp}_{k/2}(N,\chi) = \bigoplus_{\phi} S_{k/2}(N,\chi,\phi) \tag{3.16}$$

where ϕ runs through all newforms $\phi \in S^{\text{new}}_{k-1}(M,\chi^2)$ with level $M \mid N'$ and $\text{Cond}(\chi^2) \mid M$. The following lemma is obvious.

Lemma 3.6.1. *Each $S_{k/2}(N,\chi,\phi)$ is contained in a single T_{p^2}-eigenspace for every prime $p \nmid N$.*

The following theorem gives our algorithm for computing the Shimura decomposition.

Theorem 3.6.2. *Let $\phi_1,\dots,\phi_m$ be the newforms of weight $k-1$, character χ^2 and level dividing N'. For prime p, and ϕ one of these newforms, write $T_p(\phi) = \lambda_p(\phi)\phi$. Let $p_1,\dots,p_n \nmid N$ be primes such that the m vectors of eigenvalues $(\lambda_{p_1}(\phi),\dots,\lambda_{p_n}(\phi))$, with $\phi = \phi_1,\dots,\phi_m$, are pairwise distinct. If $f \in S^{\perp}_{k/2}(N,\chi)$ is an eigenform for $T_{p_i^2}$ for $i = 1,\dots,n$ then f belongs to one of the summands $S_{k/2}(N,\chi,\phi)$.*

Proof. Suppose $f \in S^{\perp}_{k/2}(N,\chi)$ is an eigenform for $T_{p_i^2}$ for $i = 1,\dots,n$. Write $T_{p_i^2} f = \mu_i f$. By Shimura's decomposition, we can write

$$f = \sum_{\phi} f_{\phi}$$

for some unique $f_\phi \in S_{k/2}(N,\chi,\phi)$; here ϕ varies over ϕ_i, $1 \le i \le m$. Thus

$$\sum_{\phi} \lambda_{p_i}(\phi) f_\phi = T_{p_i^2} f = \mu_i \sum_{\phi} f_\phi.$$

As the decomposition is a direct sum, we find that

$$(\lambda_{p_i}(\phi) - \mu_i) f_\phi = 0, \qquad i = 1,\dots,n.$$

We will show that at most one f_ϕ is non-zero. This will force f to be in one of the components $S_{k/2}(N,\chi,\phi)$ which is what we want to prove. Suppose therefore that $f_{\phi_1} \neq 0$ and $f_{\phi_2} \neq 0$. Then

$$\lambda_{p_i}(\phi_1) = \mu_i = \lambda_{p_i}(\phi_2), \qquad i = 1, 2, \ldots, n.$$

This contradicts the assumption that the vectors of eigenvalues are distinct, and completes the proof. $\square$

An Alternative Proof of Theorem 3.6.2. This proof is inspired by a similar argument in [2, page 18] (however there is a certain step in that paper that we were unable to follow).

Let $\mathbb{T}'$ be the subalgebra of the Hecke algebra of $S^{\perp}_{k/2}(N, \chi)$ generated by T_{p^2} for $p \neq p_i$ such that $p \nmid N$. Let

$$V = \mathrm{Span}\{Tf : T \in \mathbb{T}'\}.$$

We note the following:

(i) We claim that V is fixed under the action of the Hecke operators T_{p^2} for $p \nmid N$. If $T_{p^2} \in \mathbb{T}'$ then this is clear. If $p = p_i$, then $T_{p_i^2}$ commutes with every $T \in \mathbb{T}'$. But f is an eigenform for $T_{p_i^2}$, which proves the claim. Hence, we can write an eigenbasis $g_1, \ldots, g_r$ for V with respect to the Hecke operators T_{p^2} for $p \nmid N$.

(ii) Every element of V is an eigenfunction for $T_{p_i^2}$ having the same eigenvalues as f. This again follows from the fact that each $T_{p_i^2}$ commutes with each $T \in \mathbb{T}'$. Thus for each i, the eigenfunctions $g_1, \ldots, g_r$ share the same $T_{p_i^2}$-eigenvalue.

Let g be one of the g_j. Consider $\mathrm{Sh}_t(g)$. This is an eigenfunction for all the Hecke operators T_p with $p \nmid N$ acting on $S_{k-1}(N', \chi^2)$. By Proposition 2.2.13, there is a unique ϕ_i such that $\mathrm{Sh}_t(g)$ and ϕ_i share the same T_p-eigenvalues for all $p \nmid N$. If g, $g' \in V$ are two elements of the eigenbasis then it follows from (ii) and the hypothesis about the vectors of eigenvalues that $\mathrm{Sh}_t(g)$, $\mathrm{Sh}_t(g')$ correspond to the same ϕ_i. By the properties of the Shimura lift, g, g' will have precisely the same T_{p^2}-eigenvalues for all $p \nmid N$. Because f is a linear combination of these eigenbasis elements, it is an eigenform for T_{p^2} for all $p \nmid N$. $\square$

Remark. Our first proof is not only simpler but it also gives a good idea of the strategy that we will use to compute the summands in (3.16).

We can reframe Theorem 3.6.2 as follows.

Corollary 3.6.3. *Let ϕ be a newform of weight $k-1$, level M dividing N', and character χ^2. Let $p_1, \ldots, p_n$ be primes not dividing N satisfying the following: for every newform $\phi' \neq \phi$ of weight $k-1$, level dividing N' and character χ^2, there is some p_i such that $\lambda_{p_i}(\phi') \neq \lambda_{p_i}(\phi)$, where $T_{p_i}(\phi) = \lambda_{p_i}(\phi) \cdot \phi$. Then*

$$S_{k/2}(N, \chi, \phi) = \left\{ f \in S_{k/2}^{\perp}(N, \chi) \ : \ T_{p_i^2}(f) = \lambda_{p_i}(\phi) f \quad \textit{for } i = 1, \ldots, n \right\}.$$

Recall that $S_{k/2}^{\perp}(N, \chi) = S_{k/2}(N, \chi)$ except possibly when $k = 3$. We have the following refinement of the above corollary which takes care of the case when $S_{k/2}^{\perp}(N, \chi) \subsetneq S_{k/2}(N, \chi)$, that is, $S_0(N, \chi) \neq 0$.

Corollary 3.6.4. *Assuming the notation in the above corollary, the following stronger statement holds:*

$$S_{k/2}(N, \chi, \phi) = \left\{ f \in S_{k/2}(N, \chi) \ : \ T_{p_i^2}(f) = \lambda_{p_i}(\phi) f \quad \textit{for } i = 1, \ldots, n \right\}.$$

Proof. Let $f_1, \ldots f_r$ be the basis of eigenforms for $S_0(N, \chi)$ as stated in Theorem 3.1.1. Recall that $f_i = V(t_i) h_{\psi_i}$ where ψ_i is primitive odd character of conductor r_{ψ_i} such that $4 r_{\psi_i}^2 t_i \mid N$ and $\chi = \left(\frac{-4t_i}{\cdot}\right) \psi_i$. Let $q = p_i$ for some fixed i. We claim that $T_{q^2}(f_i) \neq \lambda_q(\phi) f_i$ for any $1 \leq i \leq r$. Since ϕ is a newform of weight 2 we know by Deligne's work on Weil conjectures that $|\lambda_q(\phi)| \leq 2\sqrt{q}$. By Lemma 3.1.2, $T_{q^2}(f_i) = \psi_i(q)(1+q) f_i$ as $q \nmid N$. Clearly $|\psi_i(q)(1+q)| = |1+q| > 2\sqrt{q}$. Hence the claim follows.

Let $g \in S_{k/2}(N, \chi)$ such that $T_{p_i^2}(g) = \lambda_{p_i}(\phi) g$ for $1 \leq i \leq n$. We can write $g = g_1 + g_2$ where $g_1 \in S_0(N, \chi)$ and $g_2 \in S_{k/2}^{\perp}(N, \chi)$. Since g_1 and g_2 are linearly independent we get that $T_{p_i^2}(g_j) = \lambda_{p_i}(\phi) g_j$ for all $1 \leq i \leq n$ and $j = 1, 2$. Thus by the above corollary $g_2 \in S_{k/2}(N, \chi, \phi)$. We show that $g_1 = 0$. Let $g_1 = \sum_{i=1}^{r} a_i f_i$. In particular for the prime q we must have $a_i T_{q^2}(f_i) = a_i \lambda_q(\phi) f_i$. The above claim implies that $a_i = 0$ for all $1 \leq i \leq r$. Hence we are done. $\square$

3.7 An Example of Non-Injectivity of Shimura Lifts

In this section we take the notation as above. We study the following problem.

Suppose ϕ is a newform belonging to $S_{k-1}^{\text{new}}(M, \chi^2)$ where $M \mid N'$ and $\text{Cond}(\chi^2) \mid M$. Let $f \in S_{k/2}(N, \chi)$ such that $\text{Sh}_t(f) = \phi$. Then does f belong to $S_{k/2}(N, \chi, \phi)$?

We show by providing an example that the above statement is not true in general. However in the cases where the Shimura Correspondence is injective, the above is clearly true because the Hecke operators commutes with Shimura lifts (see Proposition 3.2.5) and we have

$$\text{Sh}_t(T_{p^2}(f)) = T_p(\text{Sh}_t(f)) = T_p(\phi) = \lambda_p \phi = \lambda_p(\text{Sh}_t(f)) = \text{Sh}_t(\lambda_p f),$$

where λ_p is the eigenvalue of ϕ under T_p.

We first provide an example where Shimura Correspondence is not injective. Consider $S_{3/2}(68, \chi_{\text{triv}})$ where χ_{triv} is the trivial character modulo 68. A basis for this space is given by

$$\begin{aligned}
f_1(z) &= q - q^2 + q^4 - q^8 - q^9 - 2q^{13} + q^{16} + q^{17} + 3q^{18} - 2q^{19} + O(q^{20}) \\
f_2(z) &= q^3 - q^7 - q^{11} + O(q^{20}) \\
f_3(z) &= q^5 - q^6 - q^7 + q^{10} + q^{12} - q^{17} + O(q^{20}).
\end{aligned}$$

We claim that $Sh_1(f_2) = 0$. Recall that $Sh_1(f_2) \in S_2(34)$. Let $f_2(z) = \sum a_n q^n$. Then by definition of Shimura lifts, $\text{Sh}_1(f_2)(z) = \sum_{n=1}^{\infty} A_1(n) q^n$ where $A_1(n) = \sum_{ij=n} \chi_{\text{triv}}(i) \left(\frac{-1}{i}\right) a_{j^2}$. Since $a_1 = 0$, we have $A_1(1) = 0$. Similarly $A_1(2) = 0$ and $A_1(3) = 0$.

Using `MAGMA` [5] we get the following basis for the space $S_2(34)$,

$$\begin{aligned}
g_1(z) &= q - 2q^4 - 2q^5 + 4q^7 + 2q^8 - 3q^9 + O(q^{12}) \\
g_2(z) &= q^2 - q^4 - q^8 - q^{10} + O(q^{12}) \\
g_3(z) &= q^3 - 2q^4 - q^5 + q^6 + 4q^7 - 2q^9 + q^{10} - 3q^{11} + O(q^{12}).
\end{aligned}$$

This clearly shows that $A_1(n) = 0$ for all n and hence we are done with the claim.

Let

$$\phi_1 = q - q^2 - q^4 - 2q^5 + 4q^7 + 3q^8 - 3q^9 + 2q^{10} + O(q^{12}) \in S_2^{\text{new}}(17)$$
$$\phi_2 = q + q^2 - 2q^3 + q^4 - 2q^6 - 4q^7 + q^8 + q^9 + 6q^{11} + O(q^{12}) \in S_2^{\text{new}}(34).$$

Following our algorithm for computing the Shimura decomposition (see section 3.6) we get

$$\begin{aligned} S_{3/2}(68, \chi_{\text{triv}}) &= S_{3/2}(68, \chi_{\text{triv}}, \phi_1) \bigoplus S_{3/2}(68, \chi_{\text{triv}}, \phi_2) \\ &= \langle f_2,\ f_3 \rangle \bigoplus \langle f_1 \rangle. \end{aligned}$$

From Corollary 3.5.4, it follows that $Sh_1(f_1) = \phi_2$. Let $f = f_1 + f_2$. Then $Sh_1(f) = Sh_1(f_1) = \phi_2$, however clearly f does not belong to $S_{3/2}(68, \chi_{\text{triv}}, \phi_2)$. Hence we have our example.

3.8 Modular Forms are Determined by Coefficients Modulo n

As usual N is a positive integer divisible by 4, χ a Dirichlet character modulo N. Let k be an odd integer. Let ϕ be a newform of weight $k-1$, level dividing $N/2$ and character χ^2. To apply Waldspurger's Theorem, we need to know (see page 85) for certain primes p, certain $\omega \in \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ and certain forms $f = \sum a_n q^n \in S_{k/2}(N, \chi, \phi)$, whether there is some n such that the image of n in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ is ω and $a_n \neq 0$. Given such p, f and ω we can write down the first few coefficients of f and test whether the image of n in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ is ω and $a_n \neq 0$. If there is such an n then we should be able to find it by writing down enough coefficients. However, sometimes it appears that $a_n = 0$ for all n that are equivalent in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ to ω. To be able to prove that, we have developed the results in this section.

Theorem 3.8.1. *Let N be a positive integer such that $4 \mid N$ and χ be a Dirichlet character modulo N. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N, \chi)$. Let a, M be integers such that $(a, M) = 1$. Let $R = \frac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(NM^2)]$. Suppose $a_n = 0$ whenever $n \not\equiv a \pmod{M}$ for all integers n up to $R+1$. Then $a_n = 0$ whenever $n \not\equiv a \pmod{M}$ for all n.*

We will be requiring the following analogue of Theorem 2.2.15 in the case of half-integral weight forms.

Lemma 3.8.2. *Let Γ' be a congruence subgroup such that $\Gamma' \subseteq \Gamma_0(4)$, and let k' be a positive odd integer. Then the statement of Theorem 2.2.15 is valid for $\Gamma = \Gamma'$ and $k = k'/2$.*

Proof. Let $h := f - g \in S_{k'/2}(\Gamma')$. By assumption, $\mathrm{ord}_\lambda(h) > \frac{k'}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma']$. Let $h' = h^4$. Then $h' \in M_{2k'}(\Gamma')$. This is because for any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma'$ and $z \in \mathbb{H}$,

$$\begin{aligned} h'(\gamma z) &= h^4(\gamma z) \\ &= j(\gamma, z)^{4k'} h^4(z) \\ &= (cz + d)^{2k'} h'(z). \end{aligned}$$

Also, $\mathrm{ord}_\lambda(h') = 4 \cdot \mathrm{ord}_\lambda(h) > \frac{2k'}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma']$. So we apply Theorem 2.2.15 to h' to get that $\mathrm{ord}_\lambda(h') = \infty$. Hence $\mathrm{ord}_\lambda(h) = \infty$. $\square$

We note that the above lemma still holds if $f,\ g \in M_{k'/2}(\Gamma_0(N), \chi)$; the above proof works by taking $h' = h^{4n}$ where n is the order of Dirichlet character χ.

We will need the following lemmas for the proof of Theorem 3.8.1.

Lemma 3.8.3. *Let M be a positive integer and $a \in \mathbb{Z}$ such that $(a, M) = 1$. Define*

$$\mathrm{I_a}(n) := \begin{cases} 1 & \text{if } n \equiv a \pmod M \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathrm{I_a}(n) = \sum_{\psi \in \mathrm{X}(M)} \frac{\psi(a)^{-1}}{\varphi(M)} \psi(n)$$

where $\mathrm{X}(M)$ denotes the group of Dirichlet characters of modulus M and φ is Euler's phi function.

Proof. For the proof see [34, Page 63, Chapter 6]. $\square$

Before starting our next lemma we will recall Proposition 2.2.10. It is to be noted that an analogue of this proposition in the case of half-integral weight forms is quoted as a well known result in Chapter III of Ono's book [30] and no proof is given. We will give a proof below not only for the sake of completeness but also because later we will see that changing the proof in some places leads us to another useful version of this proposition. The proof essentially follows the proof of Proposition 2.2.10 for the integral weight case with some changes.

Proposition 3.8.4. *Let k be a positive odd integer, χ be a Dirichlet character modulo N where $4 \mid N$ and $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_{k/2}(N, \chi)$. If ψ is a Dirichlet character of conductor m, then*

$$f_\psi(z) = \sum_{n=0}^{\infty} \psi(n) a_n q^n \in M_{k/2}(Nm^2, \chi\psi^2).$$

Moreover, if f is a cusp form then so is f_ψ.

Proof. Let $\zeta = e^{2\pi i/m}$ and let $g = \sum_{j=0}^{m-1} \psi(j)\zeta^j$ be the Gauss sum. Note that

$$\frac{1}{m}\sum_{\nu=0}^{m-1} \zeta^{(l-n)\nu} = \begin{cases} 0 & \text{if } l \not\equiv n \pmod{m} \\ 1 & \text{if } l \equiv n \pmod{m}. \end{cases}$$

Thus we have

$$\sum_{l=0}^{m-1} \psi(l) \left(\frac{1}{m} \sum_{\nu=0}^{m-1} \zeta^{(l-n)\nu} \right) = \psi(n).$$

Hence we can write f_ψ as follows,

$$\begin{aligned}
f_\psi(z) &= \sum_{l=0}^{m-1} \psi(l) \sum_{n=0}^{\infty} \left(\frac{1}{m} \sum_{\nu=0}^{m-1} \zeta^{(l-n)\nu} \right) a_n q^n \\
&= \frac{1}{m} \sum_{l,\nu=0}^{m-1} \psi(l)\zeta^{l\nu} \sum_{n=0}^{\infty} a_n e^{2\pi i n(z-\nu/m)} \\
&= \frac{1}{m} \sum_{l,\nu=0}^{m-1} \overline{\psi}(\nu)\psi(l\nu)\zeta^{l\nu} \sum_{n=0}^{\infty} a_n e^{2\pi i n(z-\nu/m)} \\
&= \frac{1}{m} \sum_{\nu=0}^{m-1} \overline{\psi}(\nu) \left(\sum_{l=0}^{m-1} \psi(l\nu)\zeta^{l\nu} \right) f(z-\nu/m) \\
&= \frac{g}{m} \sum_{\nu=0}^{m-1} \overline{\psi}(\nu) f(z-\nu/m) \\
&= \frac{g}{m} \sum_{\nu=0}^{m-1} \overline{\psi}(\nu) f(\gamma_\nu z),
\end{aligned}$$

where for each $0 \le \nu < m$, γ_ν is the matrix $\left[\begin{smallmatrix} 1 & -\nu/m \\ 0 & 1 \end{smallmatrix}\right]$.

Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be any matrix in $\Gamma_0(Nm^2)$. We want to show that f_ψ is invariant under $[\tilde{\gamma}]_{k/2}$. Recall from Section 2.3 that $\tilde{\gamma}$ stands for $(\gamma, j(\gamma, z)) \in \Delta_0(Nm^2)$.

For each $0 \le \nu, \nu' < m$,

$$\gamma_\nu \gamma \gamma_{\nu'}^{-1} = \begin{bmatrix} a - c\nu/m & b + (\nu' a - \nu d)/m - c\nu\nu'/m^2 \\ c & d + c\nu'/m \end{bmatrix}.$$

Since a and d are coprime to m one can choose ν' uniquely for each ν such that $\nu' a \equiv \nu d \pmod{m}$ and for each such pair (ν, ν') we have $\gamma_\nu \gamma \gamma_{\nu'}^{-1} \in \Gamma_0(N)$.

Thus,

$$\begin{aligned}
f_\psi(\gamma z) &= \frac{g}{m}\sum_{\nu=0}^{m-1}\overline{\psi}(\nu) f(\gamma_\nu \gamma \gamma_{\nu'}^{-1}\gamma_{\nu'} z) \\
&= \frac{g}{m}\sum_{\nu=0}^{m-1}\overline{\psi}(\nu)\chi(d+c\nu'/m) j(\gamma_\nu\gamma\gamma_{\nu'}^{-1}, \gamma_{\nu'} z)^k f(\gamma_{\nu'} z) \\
&= \frac{g}{m}\sum_{\nu=0}^{m-1}\overline{\psi}(\nu)\chi(d)\ \epsilon_{d+c\nu'/m}^{-k}\left(\frac{c}{d+c\nu'/m}\right)^k (c\gamma_{\nu'} z + d + c\nu'/m)^{k/2} f(\gamma_{\nu'} z) \\
&= \frac{g}{m}\sum_{\nu=0}^{m-1}\overline{\psi}(\nu)\chi(d)\epsilon_d^{-k}\left(\frac{c}{d}\right)^k (cz+d)^{k/2} f(\gamma_{\nu'} z).
\end{aligned}$$

The last two equalities follow since $4 \mid N \mid (c\nu'/m)$ and $\left(\frac{c}{d+c\nu'/m}\right) = \left(\frac{c}{d}\right)$, the proof of which follows by Lemma 3.8.5 below. It is clear that $\overline{\psi}(\nu) = (\psi(d))^2\overline{\psi}(\nu')$. Hence,

$$f_\psi(\gamma z) = \chi(d)(\psi(d))^2 j(\gamma, z)^k \frac{g}{m}\sum_{\nu=0}^{m-1}\overline{\psi}(\nu') f(\gamma_{\nu'} z) = \chi\psi^2(d) j(\gamma, z)^k f_\psi(z).$$

Now we will show f_ψ is holomorphic on $\mathbb{H}$ and at all cusps, and that if f is a cusp form then so is f_ψ. It is to be noted that when f is a cusp form, $a_n = O(n^{k/4})$ (see [36]) and so $a_n\psi(n) = O(n^{k/4})$, thus it follows from [28, Lemma 4.3.3] that f_ψ is holomorphic on $\mathbb{H}$. In fact in the integral weight case we have coefficient estimates for the modular forms and so holomorphicity on $\mathbb{H}$ follows (see [28, Theorem 4.5.17, Theorem 4.7.3] for details).

We will be proving holomorphicity of f on $\mathbb{H}$ without the coefficient estimates. First, we will be dealing with the cusps. Let s be any cusp of $\Gamma_0(Nm^2)$ and $s = \alpha\infty$ for some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Let $\xi = (\alpha, \phi(z))$ be an element of G corresponding to α. Then,

$$f_\psi(z)|_{[\xi]_{k/2}} = f_\psi(\alpha z)(\phi(z))^{-k} = \frac{g}{m}\sum_{\nu=0}^{m-1}\overline{\psi}(\nu) f(\gamma_\nu \alpha z)(\phi(z))^{-k}.$$

One can easily show that an inverse image of γ_ν in G is $\tilde{\gamma}_\nu = (\gamma_\nu, t_{\gamma_\nu})$ where

t_{γ_ν} is a fourth root of unity. Hence,

$$f(z)|_{[\tilde{\gamma}_\nu\xi]_{k/2}} = f(\gamma_\nu \alpha z)(\phi(z)t_{\gamma_\nu})^{-k}.$$

Thus $f_\psi(z)|_{[\xi]_{k/2}}$ is a linear combination of $f(z)|_{[\tilde{\gamma}_\nu\xi]_{k/2}}$. Since s is a cusp so is $s-\nu/m$ and we are done. By the similar working as above for any z in $\mathbb{H}$, $f_\psi(z)$ is a linear combination of $f(z)|_{[\tilde{\gamma}_\nu]_{k/2}}$. Since $f(z)|_{[\tilde{\gamma}_\nu]_{k/2}} = f(\gamma_\nu z).t_{\gamma_\nu}^{-k}$ and f is holomorphic at $\gamma_\nu z$ we are done. □

Lemma 3.8.5. *Let* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$ *and* $m^2 \mid N$. *Let* $0 \le \nu' < m$ *and* $\frac{c\nu'}{m} \equiv 0 \pmod 4$. *Then,* $\left(\frac{c}{d+c\nu'/m}\right) = \left(\frac{c}{d}\right)$.

The proof of the above lemma requires the following reciprocity law as stated in Cassels and Fröhlich [9, Page 350]:

Proposition 3.8.6. *Let P, Q be positive odd integers and a be any non-zero integer with $a = 2^\alpha a_0$, a_0 odd. Then,*

$$\left(\frac{a}{P}\right) = \left(\frac{a}{Q}\right) \text{ if } P \equiv Q \pmod{8a_0}.$$

Proof of Lemma 3.8.5. We write $c = m^2 2^{2r} c'$ where $r \ge 0$ such that $\mathrm{ord}_2(c') \le 1$. Thus we want to show that $\left(\frac{m^2 2^{2r} c'}{d+m2^{2r}c'\nu'}\right) = \left(\frac{c}{d}\right)$. Since c is coprime to both d and $d + c\nu'/m$, this is equivalent to showing that $\left(\frac{c'}{d+m2^{2r}c'\nu'}\right) = \left(\frac{c'}{d}\right)$. By the hypothesis, $m2^{2r}c'\nu' \equiv 0 \pmod 4$, hence $r \ge 1$. We have following cases:

(i) Suppose $m2^{2r}c'\nu' \equiv 0 \pmod 8$. Let $c' = 2^\gamma c_0$, c_0 odd. Then $m2^{2r}c'\nu' \equiv 0 \pmod{8c_0}$. Using Proposition 3.8.6 we are done.

(ii) Suppose $m2^{2r}c'\nu' \not\equiv 0 \pmod 8$. Then $r = 1$ and c', m are odd. Hence $\left(\frac{d+4mc'\nu'}{c'}\right) = \left(\frac{d}{c'}\right)$. Now using the Quadratic Reciprocity Law, we are done.

□

Proposition 3.8.7. *Assume the hypotheses of Proposition 3.8.4 hold. In addition assume that $m^2 \mid N$. Then,*

(i) If $\frac{N}{m} \equiv 0 \pmod 4$ *then* $f_\psi \in M_{k/2}(N, \chi\psi^2)$.

(ii) If $\frac{N}{m} \equiv 2 \pmod 4$ *then* $f_\psi \in M_{k/2}(2N, \chi\psi^2)$.

Proof. The condition that $\frac{N}{m} \equiv 0 \pmod 4$ and $\frac{N}{m} \equiv 2 \pmod 4$ is to ensure that hypothesis of Lemma 3.8.5 holds so that we can replace the level Nm^2 by N and $2N$ respectively in the proof of Proposition 3.8.4. $\square$

Lemma 3.8.8. *Let* k*,* N *be positive integers such that* $4 \mid N$ *and* k *odd. Suppose* $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N, \chi)$*. Let* a*,* M *be positive integers such that* $(a, M) = 1$*. Define*

$$g(z) := \sum_{n=1}^{\infty} \mathrm{I_a}(n) a_n q^n.$$

Then $g \in S_{k/2}(\Gamma_1(NM^2))$*.*

Proof. We have

$$\begin{aligned} g(z) &= \sum_{n=1}^{\infty} \mathrm{I_a}(n) a_n q^n \\ &= \sum_{n=1}^{\infty} \sum_{\psi \in \mathrm{X}(M)} \frac{\psi(a)^{-1}}{\varphi(M)} \psi(n)\ a_n q^n \\ &= \sum_{\psi \in \mathrm{X}(M)} \alpha_\psi \sum_{n=1}^{\infty} \psi(n)\ a_n q^n \\ &= \sum_{\psi \in \mathrm{X}(M)} \alpha_\psi f_\psi\ , \end{aligned}$$

where $\alpha_\psi = \frac{\psi(a)^{-1}}{\varphi(M)}$. Since $S_{k/2}(NM^2, \chi\psi^2) \subset S_{k/2}(\Gamma_1(NM^2))$, using Proposition 3.8.4, for all $\psi \in \mathrm{X}(M)$ we have $f_\psi \in S_{k/2}(\Gamma_1(NM^2))$. Hence $g \in S_{k/2}(\Gamma_1(NM^2))$. $\square$

Now we are ready to prove Theorem 3.8.1.

Proof of Theorem 3.8.1. Let $h = f - g$ where we take g as in the above lemma. It is easy to see that $f \in S_{k/2}(\Gamma_1(NM^2))$ and hence, so does h. It is clear that

$$\text{coefficient of } q^n \text{ in } h = \begin{cases} a_n & \text{if } n \not\equiv a \pmod M \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $h(z) = \sum_{n \not\equiv a \pmod M} a_n q^n \in S_{k/2}(\Gamma_1(NM^2))$. Since we have assumed $a_n = 0$ whenever $n \not\equiv a \pmod M$ for all integers n up to $R+1$, we get that nth coefficient of h is zero for all integers n up to $R+1$. Applying Lemma 3.8.2 to h we get that $h = 0$. Hence the theorem follows. $\square$

We have the following corollary to the Lemma 3.8.8 which can be stated on the similar lines as Theorem 3.8.1.

Corollary 3.8.9. *Let N be a positive integer such that $4 \mid N$ and χ be a Dirichlet character modulo N. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N, \chi)$. Let a, M be integers such that $(a, M) = 1$. Let $R = \frac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(NM^2)]$. Suppose $a_n = 0$ whenever $n \equiv a \pmod M$ for all integers n up to $R+1$. Then $a_n = 0$ whenever $n \equiv a \pmod M$ for all n.*

Proof. Take g as in the Lemma 3.8.8. It is clear from the hypothesis that the coefficients of q^n in g are zero for all integers n up to $R+1$. Applying Lemma 3.8.2 we get that $g = 0$. Thus the result follows. $\square$

Remark. It is to be noted that the bound R in Theorem 3.8.1 and Corollary 3.8.9 in general can be very large and hence it might be practically impossible to check the Fourier coefficients until such a large R. For example, when $N = 1984$, $k = 3$ and $M = 8$ we get that $R = 1509949440$. However in certain special cases we can indeed work with comparatively much smaller values of R.

Theorem 3.8.10. *Let N be a positive integer such that $4 \mid N$ and χ be a Dirichlet character modulo N. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N, \chi)$. Let a, M be integers such that $(a, M) = 1$ and $M^2 \mid N$. Let*

$$R = \begin{cases} \frac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)] & \text{if } \frac{N}{M} \equiv 0 \pmod 4 \\ \frac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(2N)] & \text{if } \frac{N}{M} \equiv 2 \pmod 4. \end{cases}$$

Now suppose $a_n = 0$ whenever $n \not\equiv a \pmod M$ for all integers n up to $R+1$. Then $a_n = 0$ whenever $n \not\equiv a \pmod M$ for all n.

Proof. The proof basically follows as in the case of Theorem 3.8.1. The modification is due to applying Proposition 3.8.7 to Lemma 3.8.8. $\square$

It is to be noted that applying this theorem to the example given in the remark above, and since all Dirichlet characters modulo 8 are quadratic we in fact get a new improved bound which is given by $R = \frac{3}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(1984)] = 384$.

Chapter 4

Waldspurger's Theorem and Applications

We finally come to Waldspurger's Theorem which relates the critical values of L-functions of twists of newforms of integral weight to coefficients of cusp forms of half-integral weight. Our objective is to apply Waldspurger's Theorem to elliptic curves. In this chapter we state and simplify Waldspurger's Theorem for our purposes.

Waldspurger's Theorem uses the language of Hecke characters and automorphic representations. In Section 4.1 we review the correspondence between Dirichlet characters and Hecke characters and we prove a result that allows us to evaluate the components of a given Dirichlet character. Next, in Section 4.2 we review the correspondence between modular forms of even integral weight and automorphic representations and prove a result needed for simplifying the hypotheses of Waldspurger's Theorem. In Section 4.3 we state Waldspurger's Theorem in simplified form. To apply Waldspurger's Theorem in conjunction with the Birch and Swinnerton-Dyer Conjectures it is convenient to express the period of the n-th twist of a given elliptic curve in terms of the period of the elliptic curve itself. We do this in Section 4.4. The last section in this chapter is devoted to extensive examples computed using Waldspurger's Theorem.

4.1 Correspondence between Dirichlet Characters and Hecke Characters on $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ of Finite Order

We shall need the correspondence between Dirichlet characters and Hecke characters on $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ of finite order. This material is in Tate's thesis [9, Chapter XV], but we found the presentation in [6, Section 3.1] more useful. We refer to our Section 2.5 for some background and definitions.

Proposition 4.1.1. *Let $\boldsymbol{\chi} = (\chi_p)$ be a character on $\mathbb{A}_{\mathbb{Q}}^{\times}$. Then there exists a finite set S of places, including all the Archimedean ones, such that if $p \notin S$, then χ_p is trivial on the unit group $\mathbb{Z}_p^{\times}$.*

Recall that if χ_p is trivial on the unit group $\mathbb{Z}_p^{\times}$, then χ_p is unramified. Thus by the above proposition, χ_p is unramified for all but finitely many p.

Theorem 4.1.2. *([6, Proposition 3.1.2]) Suppose $\boldsymbol{\chi} = (\chi_p)$ is a character of finite order on $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$. There exists an integer N whose prime divisors are precisely the non-Archimedean primes p such that χ_p is ramified, and a primitive Dirichlet character χ modulo N such that if $p \nmid N$ is non-Archimedean then $\chi(p) = \chi_p(p)$. This correspondence $\boldsymbol{\chi} \mapsto \chi$ is a bijection between characters of finite order of $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ and the primitive Dirichlet characters.*

In our work, we shall need to start with a Dirichlet character χ of modulus N and then do computations with the corresponding adelic character $\boldsymbol{\chi}$. We collect here some facts that will help us with these computations.

Lemma 4.1.3. *We keep the notation of Theorem 4.1.2.*

(i) For any $\alpha \in \mathbb{Q}^{\times}$, $\prod \chi_p(\alpha) = 1$.

(ii) Suppose $p = \infty$ and $\alpha \in \mathbb{Q}_{\infty}^{\times} = \mathbb{R}^{\times}$. Then $\chi_{\infty}(\alpha) = 1$ if $\alpha > 0$, or if χ has odd order.

(iii) Let p be a non-Archimedean prime such that $p \mid N$ and α, $\beta \in \mathbb{Z}_p$ be non-zero. Suppose that $\beta \equiv \alpha \pmod{\alpha N \mathbb{Z}_p}$. Then $\chi_p(\beta) = \chi_p(\alpha)$.

(iv) Let p be non-Archimedean such that $p \nmid N$ then, χ_p is unramified.

Proof. (i) follows from the fact that χ is a character on $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$.

Now suppose that $p = \infty$ and $\alpha \in \mathbb{R}^{\times}$. Let d be the order of χ. If d is odd or α is positive, then we can write $\alpha = \beta^d$ for some $\beta \in \mathbb{R}$. Thus

$$1 = \chi_{\infty}^{d}(\beta) = \chi_{\infty}(\beta^d) = \chi_{\infty}(\alpha),$$

proving (ii).

In [6, Proposition 3.1.2] it is shown that for a non-Archimedean prime p with $p \mid N$, the character χ_p is trivial on $\{x \in \mathbb{Z}_p : x \equiv 1 \pmod{N\mathbb{Z}_p}\}$. Suppose that $\beta \equiv \alpha \pmod{\alpha N\mathbb{Z}_p}$. It is clear that $\beta/\alpha \in \mathbb{Z}_p^{\times}$ and $\beta/\alpha \equiv 1 \pmod{N\mathbb{Z}_p}$. Thus $\chi_p(\beta/\alpha) = 1$ and (iii) follows.

We again refer to [6, Proposition 3.1.2] for a proof of the fact that χ_p is trivial on $\mathbb{Z}_p^{\times}$ whenever $p \nmid N$ and hence (iv) follows.

$\square$

4.1.1 How to Evaluate χ_p?

In Waldspurger's Theorem (see Theorem 4.3.4) we start with a Dirichlet character χ modulo N and we need to evaluate $\chi_p(a)$ for certain primes p and certain non-zero $a \in \mathbb{Z}$. We have failed to find a reference for how to do these computations, so we give below our own method.

Proposition 4.1.4. *Let χ be a Dirichlet character modulo N (not necessarily primitive) and let $\boldsymbol{\chi} = (\chi_p)$ be the corresponding character on $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$. Let $a \in \mathbb{Z}$ be non-zero.*

(a) If $q \nmid N$ then $\chi_q(a) = \chi(q)^r$ where $r = \operatorname{ord}_q(a)$.

(b) Suppose q divides N and let $q_1, \dots, q_r$ be the other primes dividing N. Let b be a **positive** *integer satisfying*

$$b \equiv \begin{cases} a \pmod{aN\mathbb{Z}_q} \\ 1 \pmod{N\mathbb{Z}_{q_i}} \quad i = 1, \dots, r; \end{cases}$$

such b can easily be constructed by the Chinese Remainder Theorem.

Write

$$b = q^{\mathrm{ord}_q(a)} \prod_{j=1}^{s} \ell_j^{\beta_j}$$

where the ℓ_j are distinct primes. Then

$$\chi_q(a) = \prod_{j=1}^{s} \chi(\ell_j)^{-\beta_j}.$$

Proof. Let N' be the conductor of χ and note that $N' \mid N$. Now if $q \nmid N$ then, χ_q is unramified. Write $a = q^r a'$ where $q \nmid a'$. Then $a' \in \mathbb{Z}_q^\times$. Thus by definition of unramified, $\chi_q(a') = 1$. Moreover, from Theorem 4.1.2, $\chi_q(q) = \chi(q)$. This proves (a).

Now suppose $q \mid N$ and let $q_1, \ldots, q_r$ be the other primes dividing N. Let b be as in the proposition. Since $N' \mid N$, we have

$$b \equiv \begin{cases} a \pmod{aN'\mathbb{Z}_q} \\ 1 \pmod{N'\mathbb{Z}_{q_i}} \quad i = 1, \ldots, r. \end{cases}$$

By Lemma 4.1.3, $\chi_q(b) = \chi_q(a)$, and $\chi_{q_i}(b) = 1$ for $i = 1, \ldots, r$. Now

$$\begin{aligned} \chi_q(a) &= \chi_q(b) \\ &= \prod_{p \neq q} \chi_p(b)^{-1} \qquad \text{by (i) of Lemma 4.1.3,} \\ &= \prod_{p \nmid N} \chi_p(b)^{-1} \qquad \text{since } \chi_{q_i}(b) = 1, \\ &= \prod_{j=1}^{s} \chi(\ell_j)^{-\beta_j} \qquad \text{using part (a).} \end{aligned}$$

This completes the proof. □

Example 4.1.5. Here is an example of an evaluation that will be needed later in Section 4.5. Let χ_{triv} be the trivial character modulo 496. Let χ be the Dirichlet character modulo 496 given by

$$\chi(n) = \left(\frac{-1}{n}\right) \chi_{\mathrm{triv}}(n).$$

Note that $496 = 2^4 \times 31$. Let us evaluate $\chi_{31}(31)$. We follow the recipe in Proposition 4.1.4. We want a positive integer b such that

$$b \equiv \begin{cases} 31 \pmod{31^2} \\ 1 \pmod{2^4}. \end{cases}$$

Using the Chinese Remainder we can take $b = 1953$. Now $1953 = 3^2 \times 7 \times 31$. Thus by part (b) of Proposition 4.1.4

$$\chi_{31}(31) = \chi(3)^{-2}\chi(7)^{-1} = \left(\frac{-1}{3}\right)^{-2}\left(\frac{-1}{7}\right)^{-1} = -1.$$

4.2 Correspondence between Modular Forms of Even Integer Weight and Automorphic Representations

For the background on automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ and how they correspond to Hecke eigenforms, please refer to Section 2.5.

Let k be a positive odd integer with $k \geq 3$. Let $\phi = \sum_{n=1}^{\infty} a_n q^n \in S_{k-1}^{\text{new}}(N, \chi)$ be a newform of weight $k-1$, level N and character χ.

Recall that we can associate to ϕ an automorphic representation ρ. Let ρ_p be the local component of ρ at a prime p. Recall that ρ_p is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence ρ_p is either a principal series or a supercuspidal representation or it is some twist of the Steinberg representation (sometimes also referred to as a special representation).

Recall that if $\phi = \sum_{n=1}^{\infty} a_n q^n$ is an eigenform, then we have defined its twist by a character μ to be the modular form $\phi_\mu = \sum_{n=1}^{\infty} a_n \mu(n) q^n$. Waldspurger works with a different definition of twist:

Definition 4.2.1. *Let ϕ be a newform of weight $k-1$ and character χ. Let μ a Dirichlet character. We denote by $\phi \otimes \mu$ the (unique) newform of weight $k-1$ with character $\chi\mu^2$ satisfying $\lambda_p(\phi \otimes \mu) = \mu(p)\lambda_p(\phi)$ for almost all primes p, where λ_p is the eigenvalue under T_p.*

Now fix a prime number p. Let ξ_p be the set of primitive Dirichlet characters with p-power conductor. The following holds (see [45, Section III]):

(i) ρ_p is supercuspidal if and only if for all $\mu \in \xi_p$, the level of $\phi \otimes \mu$ is divisible by p and $\lambda_p(\phi \otimes \mu) = 0$.

(ii) ρ_p is an irreducible principal series if and only if either

  (a) there exists a character μ in ξ_p such that p does not divide the level of $\phi \otimes \mu$; or,

  (b) there exist two distinct characters μ_1, μ_2 in ξ_p such that $\lambda_p(\phi\otimes\mu_1) \neq 0$, $\lambda_p(\phi \otimes \mu_2) \neq 0$.

(iii) ρ_p is a special representation if and only if the following conditions hold:

  (a) for all $\mu \in \xi_p$, the level of $\phi \otimes \mu$ is divisible by p;

  (b) there exists a unique μ in ξ_p such that $\lambda_p(\phi \otimes \mu) \neq 0$.

We shall need the following theorem which is extracted from the paper of Atkin and Li [1].

Theorem 4.2.2. *(Atkin and Li) Let $\phi = \sum_{n=1}^{\infty} a_n q^n$ be a newform of weight $k-1$, character χ and level N. Let μ be a primitive character of conductor m. Then*

(a) If $\gcd(m, N) = 1$ then $\phi \otimes \mu = \phi_\mu$, and it is a newform of weight $k-1$, character $\chi\mu^2$ and level Nm^2 ([1, Introduction]).

(b) Suppose μ is of q-power conductor where $q \mid N$ and write $N = q^s M$ where $q \nmid M$. Then $\phi \otimes \mu$ is a newform of weight $k-1$, character $\chi\mu^2$ and level $q^{s'}M$ for some $s' \geq 0$. Moreover, $\lambda_p(\phi \otimes \mu) = \mu(p)\lambda_p(\phi)$ for all primes $p \nmid N$ ([1, Theorem 3.2]). In particular if $s = 1$ and χ is trivial, then for μ with conductor q^r, $r \geq 1$, it turns out that $\phi \otimes \mu = \phi_\mu$ is a newform of level $q^{2r}M$ and character μ^2 ([1, Corollary 4.1]).

(c) Let $q \mid N$. Suppose ϕ is q-primitive and $a_q = 0$. Then for all characters μ of q-power conductor, $\phi \otimes \mu = \phi_\mu$ is a newform of level divisible by N (Recall that ϕ is q-primitive if ϕ is not a twist of any newform of level lower than N by a character of conductor equal to some power of q) ([1, Proposition 4.1]).

(d) Let $N = q^s M$ where $q \nmid M$; let $Q = q^s$. Let χ_Q be the Q-part [1] of the character χ. If s is odd and $\mathrm{cond}\,\chi_Q \leq \sqrt{Q}$ *then ϕ is q-primitive.*

Now suppose $q = 2$. Then, if $s = 2$ then ϕ is always 2-primitive; if s is odd then ϕ is 2-primitive if and only if $\mathrm{cond}\,\chi_Q < \sqrt{Q}$*; if s is even and $s \geq 4$ then ϕ is 2-primitive if and only if* $\mathrm{cond}\,\chi_Q = \sqrt{Q}$ *([1, Theorem 4.4]).*

We deduce the following corollaries which we will be using later.

Corollary 4.2.3. *Let $\phi = \sum_{n=1}^{\infty} a_n q^n \in S_{k-1}^{\mathrm{new}}(N)$ be a newform with trivial character. Let ρ_2 be the local component at 2 of the corresponding automorphic representation. Suppose either*

(i) N is odd; or

(ii) $\nu_2(N) = 1$ and $a_2 \neq 0$.

Then ρ_2 is not supercuspidal.

Further if $\nu_2(N) \geq 2$ and ϕ is 2-primitive then ρ_2 is supercuspidal. In particular, if either $\nu_2(N) = 2$ or $\nu_2(N)$ is odd then ρ_2 is supercuspidal.

Proof. If N is odd, take μ to be the identity character. Thus $\mu \in \xi_2$ and the level of $\phi \otimes \mu$ is odd and hence ρ_2 is not supercuspidal. If $N = 2M$ such that M is odd and $a_2 \neq 0$, again taking μ as identity character we get that $\lambda_2(\phi \otimes \mu) = a_2 \neq 0$ and thus ρ_2 is not supercuspidal.

Let $\nu_2(N) \geq 2$. Then $a_2 = 0$ (see Theorem 2.2.12). If ϕ is 2-primitive then it follows using part (c) of the Theorem 4.2.2 that for any $\mu \in \xi_2$, $\phi \otimes \mu = \phi_\mu$ is newform of level divisible by 2. Write $T_2(\phi_\mu) = \sum_{n=1}^{\infty} b_n q^n$. By Proposition 2.2.5, $b_n = a_{2n}\mu(2n) + \mu^2(2)2^{k-2}a_{n/2}\mu(n/2)$ for all n. Thus $T_2(\phi_\mu) = 0$. Therefore, $\lambda_2(\phi \otimes \mu) = \lambda_2(\phi_\mu) = 0$ and ρ_2 is supercuspidal. Note that we have not yet used the condition that ϕ has trivial character, but we need it to prove the final statement which is indeed a direct application of part (d) of the Theorem 4.2.2. □

Corollary 4.2.4. *Let ϕ be as in the above corollary.*

[1] Let χ be a Dirichlet character with modulus $p_1^{r_1} \cdots p_n^{r_n}$ where the p_i are distinct primes. Then χ can be written uniquely as a product $\prod \chi_{p_i^{r_i}}$ where $\chi_{p_i^{r_i}}$ has modulus $p_i^{r_i}$. See [1].

(i) If $N = pM$ with M coprime to p and $a_p \neq 0$, then ρ_p is a special representation.

(ii) If $p \nmid N$, then ρ_p is an irreducible principal series.

Proof. We first prove (i). By part (b) of the Theorem 4.2.2, for any $\mu \in \xi_p$, the level of $\phi \otimes \mu$ is divisible by p. Further if μ is the identity character then $\lambda_p(\phi \otimes \mu) = a_p \neq 0$; we claim that this is unique such character in ξ_p. Let $\mu \in \xi_p$ be such that μ is a character of conductor p^r, $r \geq 1$. Then $\phi \otimes \mu = \phi_\mu$ is a newform in $S_{k-1}(p^{2r}M, \mu^2)$ such that $\lambda_p(\phi_\mu) = a_p\mu(p) = 0$ (see Theorem 2.2.12) and hence $\lambda_p(\phi \otimes \mu) = 0$.

The proof of (ii) is obvious and does not require the condition that newform ϕ has trivial character. $\square$

4.3 Waldspurger's Theorem and Notation

In this section we will present Waldspurger's Theorem. We will introduce and simplify the notation used in the theorem. This is needed in the following section where we will discuss how to use the theorem for elliptic curves and compute critical values of L-functions in terms of coefficients of corresponding half-integral weight forms. An important application is the computation of orders of the Tate-Shafarevich groups assuming the Birch and Swinnerton-Dyer Conjecture.

Let k be positive integers with $k \geq 3$ odd. Let χ be an even Dirichlet character with modulus divisible by 4. Fix a newform ϕ of level M_ϕ in $S_{k-1}^{\text{new}}(M_\phi, \chi^2)$. Let p be a prime number. Let ν_p be the p-adic valuation on $\mathbb{Q}$ and $\mathbb{Q}_p^\times$. Let $m_p = \nu_p(M_\phi)$ and λ_p be the Hecke eigenvalue of ϕ corresponding to the Hecke operator T_p.

Let ρ be the automorphic representation associated to ϕ and ρ_p be the local component of ρ at p. Let S be the (finite) set of primes p such that ρ_p is not irreducible principal series. If $p \notin S$, ρ_p is equivalent to $\pi(\mu_{1,p}, \mu_{2,p})$ where $\mu_{1,p}$ and $\mu_{2,p}$ are two continuous characters on $\mathbb{Q}_p$ such that $\mu_{1,p}\mu_{2,p} \neq |\cdot|^{\pm 1}$. Let (H1) be the following hypothesis:

$$\text{(H1)} \qquad \text{For } p \notin S,\ \mu_{1,p}(-1) = \mu_{2,p}(-1) = 1.$$

Theorem 4.3.1. *(Flicker) There exists N such that $S_{k/2}(N, \chi, \phi) \neq \{0\}$ if and only if the hypothesis* (H1) *holds.*

It is to be noted that Flicker [20] made this statement with Shimura's definition of $S_{k/2}(N, \chi, \phi)$. However, we saw in Lemma 3.5.5 that this agrees with our definition. We shall also need the following theorem of Vigneras.

Theorem 4.3.2. *(Vigneras) Flicker's condition* (H1) *always holds whenever ϕ is a newform of even weight with trivial character.*

Proof. For the proof refer to [44]. □

From the theorems of Flicker and Vigneras we have the following easy corollary.

Corollary 4.3.3. *Let ϕ be a newform of weight $k-1$, level M_ϕ and trivial character χ_{triv}. Let χ be a Dirichlet character satisfying $\chi^2 = \chi_{\text{triv}}$. Then there exists some N such that $S_{k/2}(N, \chi, \phi) \neq \{0\}$.*

Henceforth, we will always assume that ϕ has trivial character and χ is quadratic, thus the conclusion of the corollary holds. We will now introduce several pieces of notation used by Waldspurger [45, Section VIII] before stating his main theorem.

Let χ_0 be the Dirichlet character associated to χ given by

$$\chi_0(n) := \chi(n) \left(\frac{-1}{n}\right)^{(k-1)/2}.$$

Note that χ_0 has modulus N and its conductor is equal to conductor of χ whenever $k \equiv 1 \pmod 4$. Let $\chi_{0,p}$ be the local component of χ_0 at prime p. For each prime p we will later define non-negative integer $\widetilde{n_p}$ that depends only on the local components ρ_p and $\chi_{0,p}$. Let $\widetilde{N_\phi}$ be given by

$$\widetilde{N_\phi} := \prod_p p^{\widetilde{n_p}}.$$

For prime p and natural number e, we will later define a set $\mathrm{U}_p(e, \phi)$ which consists of some finite number of complex-valued functions on $\mathbb{Q}_p^\times$ having support in $\mathbb{Z}_p \cap \mathbb{Q}_p^\times$.

Let $\mathbb{N}^{\text{sc}}$ be the set of positive square-free numbers and for $n \in \mathbb{N}$, let n^{sc} be the square-free part of n. Let A be a function on the set $\mathbb{N}^{\text{sc}}$ having values in $\mathbb{C}$ and E be an integer such that $\widetilde{N_\phi} \mid E$. Denote $e_p = \nu_p(E)$ for all prime numbers p and let $\underline{c} = (c_p)$ be any element of $\prod_p \mathrm{U}_p(e_p, \phi)$. Define

$$f(\underline{c}, A)(z) := \sum_{n=1}^{\infty} A(n^{\text{sc}}) n^{(k-2)/4} \prod_p c_p(n) q^n, \qquad z \in \mathbb{H}$$

and let $\overline{\mathrm{U}}(E, \phi, A)$ be the space generated by these functions $f(\underline{c}, A)$ on $\mathbb{H}$ where $\underline{c} \in \prod_p \mathrm{U}_p(e_p, \phi)$.

With the above notation, we are now ready to state the main theorem of Waldspurger [45, Page 481].

Theorem 4.3.4. *(Waldspurger) Let* (H2) *be the following hypothesis: One of the following holds:*

(a) the local component ρ_2 is not supercuspidal;

(b) the conductor of χ_0 is divisible by 16*;*

(c) $16 \mid M_\phi$.

Let χ be a Dirichlet character and ϕ be a newform of weight $k-1$ and character χ^2 such that (H1) *and* (H2) *hold. Then there exists a function A_ϕ on $\mathbb{N}^{\text{sc}}$ such that for $t \in \mathbb{N}^{\text{sc}}$:*

$$A_\phi(t)^2 := \mathrm{L}(\phi \otimes \chi_0^{-1}\chi_t, 1) \cdot \epsilon(\chi_0^{-1}\chi_t, 1/2).$$

Moreover, for $N \geq 1$,

$$S_{k/2}(N, \chi, \phi) = \bigoplus \overline{\mathrm{U}}(E, \phi, A_\phi)$$

where the sum is over all $E \geq 1$ such that $\widetilde{N_\phi} \mid E \mid N$.

Recall from Section 2.3 that $\chi_t = \left(\frac{t}{\cdot}\right)$ is a quadratic character with conductor $|t|$ if $t \equiv 1 \pmod 4$, otherwise with conductor $|4t|$ if $t \equiv 2, 3 \pmod 4$.

Remark. Note that the function A_ϕ depends only on χ and ϕ. However A_ϕ is not deterministic, so we cannot use this theorem for computing the basis for the space $S_{k/2}(N,\chi,\phi)$. However, if we know a basis for the space $S_{k/2}(N,\chi,\phi)$ and if $f(z)=\sum_{n=1}^{\infty} a_n q^n$ is one of the basis elements, then we can express the critical value of the L-function of twist of the newform ϕ with character $\chi_0^{-1}\chi_t$, in terms of the square of the Fourier coefficient a_t and the factor $\epsilon(\chi_0^{-1}\chi_t,1/2)$ which depends on the local components of ϕ and χ_0.

It is to be noted that $\epsilon(\chi,1/2)$ for any Hecke character χ can be computed as shown in Tate's article [41] (see also Tunnell [42]). In particular, when χ is quadratic, $\epsilon(\chi,1/2)$=1. Since we will be only dealing with the quadratic characters, we can ignore the ϵ-factor. Moreover, note that if χ is quadratic, then the conductor of χ_0 is at most divisible by 8, so we do not need to consider possibility (b) of the hypothesis (H2).

Further by Corollary 4.2.3, possibilities (a) and (c) of the hypothesis (H2) can be simply stated in terms of the level M_ϕ. Assuming χ to be quadratic, Waldspurger's Theorem is applicable whenever either M_ϕ is odd; or $\nu_2(M_\phi)=1$ and $\lambda_2\neq 0$; or $\nu_2(M_\phi)\geq 4$. The last condition is the same as possiblility (c) of (H2).

We also state the following corollaries of Waldspurger; the proofs can be found in [45, Page 483].

Corollary 4.3.5. *(Waldspurger) Let $N\geq 1$. If the conductor of χ is not divisible by* 16*, it is assumed that N is not divisible by* 8*. Then we have the following decomposition* :

$$S_{k/2}(N,\chi)=\bigoplus_{\phi,E}\overline{\mathrm{U}}(E,\phi,A_\phi)$$

where the sum is over all newforms $\phi\in S_{k-1}^{\mathrm{new}}(M_\phi,\chi^2)$ for M_ϕ dividing $N/2$ such that ϕ satisfies (H1) *and over the integers $E\geq 1$ such that $\widetilde{N_\phi}\mid E\mid N$.*

Corollary 4.3.6. *(Waldspurger) Let $\phi\in S_{k-1}^{\mathrm{new}}(M_\phi,\chi^2)$ be a newform such that ϕ satisfies* (H1)*. Suppose* [2] *$f(z)=\sum_{n=1}^{\infty} a_n q^n\in S_{k/2}(N,\chi,\phi)$ for some $N\geq 1$ such that M_ϕ divides $N/2$. Suppose that $n_1,n_2\in\mathbb{N}^{\mathrm{sc}}$ such that $n_1/n_2\in\mathbb{Q}_p^{\times 2}$ for all $p\mid N$. Then we have the following relation:*

$$a_{n_1}^2\mathrm{L}(\phi\chi_0^{-1}\chi_{n_2},1)\chi(n_2/n_1)n_2^{k/2-1}=a_{n_2}^2\mathrm{L}(\phi\chi_0^{-1}\chi_{n_1},1)n_1^{k/2-1}.$$

[2]In this corollary we do not require f to be of the form $f(\underline{c},A_\phi)$.

In what follows $(\cdot\,,\,\cdot)_p$ stands for the Hilbert symbol defined on $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. Recall that (see for example, [10]) if $p=2$ and a, b are odd then

$$(2^s a, 2^t b)_2 = \left(\frac{2}{|a|}\right)^t \left(\frac{2}{|b|}\right)^s (-1)^{\frac{(a-1)(b-1)}{4}}.$$

For an odd prime p and a, b coprime to p,

$$(p^s a, p^t b)_p = \left(\frac{-1}{p}\right)^{st} \left(\frac{a}{p}\right)^t \left(\frac{b}{p}\right)^s.$$

In particular, for an odd n, $(n,-1)_2 = (-1)^{\frac{n-1}{2}}$ and $(2,n)_2 = (-1)^{\frac{n^2-1}{8}}$. Also, if $\nu_p(n)=0$ then $(p,n)_p = \left(\frac{n}{p}\right)$ and, if $\nu_p(n)=1$ and $n=pn'$, then $(p,n)_p = \left(\frac{-n'}{p}\right)$.

We now write down explicitly the definitions of the integers $\widetilde{n_p}$ and the local factors $U(e,\phi)$ used in Waldspurger's Theorem, but only in the cases we need for the purposes of this thesis. Recall that $U(e,\phi)$ will be a finite set of complex-valued functions on $\mathbb{Q}_p^\times$ having support in $\mathbb{Z}_p \setminus \{0\}$. It is to be noted that for Waldspurger's Theorem, we would be only requiring the values of the functions in $U_p(e,\phi)$ at square-free positive integers. We will first define a certain set of functions.

Case 1. *p odd.*
Waldspurger considered the following set of functions which we will be denoting as Λ_p:

$$\Lambda_p = \{c_p^0[\delta],\ c_p^*[\delta],\ c_p'[\delta],\ {}'c_p[\delta],\ {}''c_p[\delta],\ c_p^s[\delta],\ {}^s c_p[\delta]\ : \delta \in \mathbb{C}\}.$$

We will simplify the notation of Waldspurger and for any $\delta \in \mathbb{C}$ we will denote $c_p^0[\delta]$ as $c_{p,\delta}^{(0)}$, $c_p^*[\delta]$ as $c_{p,\delta}^{(1)}$, $c_p'[\delta]$ as $c_{p,\delta}^{(2)}$, ${}'c_p[\delta]$ as $c_{p,\delta}^{(3)}$, ${}''c_p[\delta]$ as $c_{p,\delta}^{(4)}$, $c_p^s[\delta]$ as $c_{p,\delta}^{(5)}$ and ${}^s c_p[\delta]$ as $c_{p,\delta}^{(6)}$. Hence with our notation,

$$\Lambda_p = \{c_{p,\delta}^{(0)},\ c_{p,\delta}^{(1)},\ c_{p,\delta}^{(2)},\ c_{p,\delta}^{(3)},\ c_{p,\delta}^{(4)},\ c_{p,\delta}^{(5)},\ c_{p,\delta}^{(6)} : \delta \in \mathbb{C}\}.$$

We will be only interested in values of the functions in Λ_p at square-free numbers in $\mathbb{Z}_p \setminus \{0\}$. Let $n \in \mathbb{Z}_p \setminus \{0\}$ be square-free, hence we have $\nu_p(n)=0$

or $\nu_p(n) = 1$. We get the following after simplification:

$$c^{(0)}_{p,\delta}(n) = \begin{cases} 1 & \nu_p(n) = 0 \\ 1 & \nu_p(n) = 1, \end{cases}$$

$$c^{(1)}_{p,\delta}(n) = \begin{cases} 1 & \nu_p(n) = 0 \\ \delta & \nu_p(n) = 1, \end{cases}$$

$$c^{(2)}_{p,\delta}(n) = \begin{cases} 1 - (p,n)_p \chi_{0,p}(p) p^{-1/2}\delta^{-1} & \nu_p(n) = 0 \\ 1 & \nu_p(n) = 1, \end{cases}$$

$$c^{(3)}_{p,\delta}(n) = \begin{cases} 1 & \nu_p(n) = 0 \\ \delta - (p,n)_p \chi_{0,p}(p) p^{-1/2} & \nu_p(n) = 1, \end{cases}$$

$$c^{(4)}_{p,\delta}(n) = \begin{cases} 0 & \nu_p(n) = 0 \\ \delta(p-1)^{-1} & \nu_p(n) = 1, \end{cases}$$

$$c^{(5)}_{p,\delta}(n) = \begin{cases} 2^{1/2} & \nu_p(n) = 0,\ (p,n)_p = -p^{1/2}\chi_{0,p}(p^{-1})\delta \\ 0 & \nu_p(n) = 0,\ (p,n)_p = p^{1/2}\chi_{0,p}(p^{-1})\delta \\ 1 & \nu_p(n) = 1, \end{cases}$$

$$c^{(6)}_{p,\delta}(n) = \begin{cases} 1 & \nu_p(n) = 0 \\ 2^{1/2}\delta & \nu_p(n) = 1,\ (p,n)_p = -p^{1/2}\chi_{0,p}(p^{-1})\delta \\ 0 & \nu_p(n) = 1,\ (p,n)_p = p^{1/2}\chi_{0,p}(p^{-1})\delta. \end{cases}$$

Case 2. $p = 2$.
As in the above case, here again we will simplify the notation of Waldspurger and for any $\delta \in \mathbb{C}$ we will denote $c_2^*[\delta]$ as $c^{(0)}_{2,\delta}$, $c_2'[\delta]$ as $c^{(1)}_{2,\delta}$, $c_2''[\delta]$ as $c^{(2)}_{2,\delta}$ $'c_2[\delta]$

as $c^{(3)}_{p,\delta}$, $''c_2[\delta]$ as $c^{(4)}_{2,\delta}$, $c_2^s[\delta]$ as $c^{(5)}_{2,\delta}$ and ${}^s c_2[\delta]$ as $c^{(6)}_{2,\delta}$. Hence, we consider the following set of functions which we will be denoting as Λ_2:

$$\Lambda_2 = \{c^{(0)}_{2,\delta},\ c^{(1)}_{2,\delta},\ c^{(2)}_{2,\delta},\ c^{(3)}_{2,\delta},\ c^{(4)}_{2,\delta},\ c^{(5)}_{2,\delta},\ c^{(6)}_{2,\delta} : \delta \in \mathbb{C}\},$$

Let $n \in \mathbb{Z}_2 \setminus \{0\}$ be square-free so that either $\nu_2(n) = 0$ or $\nu_2(n) = 1$. We have:

$$c^{(0)}_{2,\delta}(n) = \begin{cases} 1 & \nu_2(n) = 0 \\ \delta & \nu_2(n) = 1, \end{cases}$$

$$c^{(1)}_{2,\delta}(n) = \begin{cases} \delta - (2,n)_2\chi_{0,2}(2)2^{-1/2} & \nu_2(n) = 0,\ (n,-1)_2 = \chi_{0,2}(-1) \\ 1 & \nu_2(n) = 0,\ (n,-1)_2 = -\chi_{0,2}(-1) \\ 1 & \nu_2(n) = 1, \end{cases}$$

$$c^{(2)}_{2,\delta}(n) = \begin{cases} \delta & \nu_2(n) = 0,\ (n,-1)_2 = \chi_{0,2}(-1) \\ 0 & \nu_2(n) = 0,\ (n,-1)_2 = -\chi_{0,2}(-1) \\ 0 & \nu_2(n) = 1, \end{cases}$$

$$c^{(3)}_{2,\delta}(n) = \begin{cases} \delta^{-1} & \nu_2(n) = 0 \\ \delta - (2,n)_2\chi_{0,2}(2)2^{-1/2} & \nu_2(n) = 1,\ (n,-1)_2 = \chi_{0,2}(-1) \\ 1 & \nu_2(n) = 1,\ (n,-1)_2 = -\chi_{0,2}(-1), \end{cases}$$

$$c^{(4)}_{2,\delta}(n) = \begin{cases} 0 & \nu_2(n) = 0 \\ 2\delta - (2,n)_2\chi_{0,2}(2)2^{-1/2} & \nu_2(n) = 1,\ (n,-1)_2 = \chi_{0,2}(-1) \\ 1 & \nu_2(n) = 1,\ (n,-1)_2 = -\chi_{0,2}(-1), \end{cases}$$

$$c_{2,\delta}^{(5)}(n) = \begin{cases} 0 & \nu_2(n)=0,\ (n,-1)_2=\chi_{0,2}(-1),\ (2,n)_2=2^{1/2}\chi_{0,2}(2^{-1})\delta \\ 2^{1/2}\delta & \nu_2(n)=0,\ (n,-1)_2=\chi_{0,2}(-1),\ (2,n)_2=-2^{1/2}\chi_{0,2}(2^{-1})\delta \\ 1 & \nu_2(n)=0,\ (n,-1)_2=-\chi_{0,2}(-1) \\ 1 & \nu_2(n)=1, \end{cases}$$

$$c_{2,\delta}^{(6)}(n) = \begin{cases} \delta^{-1} & \nu_2(n)=0 \\ 0 & \nu_2(n)=1,\ (n,-1)_2=\chi_{0,2}(-1),\ (2,n)_2=2^{1/2}\chi_{0,2}(2^{-1})\delta \\ 2^{1/2}\delta & \nu_2(n)=1,\ (n,-1)_2=\chi_{0,2}(-1),\ (2,n)_2=-2^{1/2}\chi_{0,2}(2^{-1})\delta \\ 1 & \nu_2(n)=1,\ (n,-1)_2=-\chi_{0,2}(-1). \end{cases}$$

We will be interested in the above functions for only particular values of δ. We will specify and further simplify them later.

Recall that λ_p is the Hecke eigenvalue of ϕ corresponding to the Hecke operator T_p for any prime p, and $m_p = \nu_p(M_\phi)$. Let $\lambda'_p = p^{1-k/2}\lambda_p$. For $p \nmid M_\phi$ let α_p and α'_p be such that

$$\alpha_p + \alpha'_p = \lambda'_p,$$

$$\alpha_p \cdot \alpha'_p = 1.$$

It is to be noted that if ϕ is rational newform of weight 2 then $\alpha_p \neq \alpha'_p$, since otherwise $\lambda_p^2 = 4p^{k-2}$, which is a contradiction as λ_p is rational (p-th Fourier coefficient of ϕ).

Next, we need to consider a subset of $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$, denoted by $\Omega_p(\phi)$, which is defined as

$$\Omega_p(\phi) = \{\omega \in \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} :\ \exists\ f \in S_{k/2}(N,\chi,\phi) \text{ for some } N \text{ and } \exists\ n \geq 1 \text{ such that } i) \text{ image of } n \text{ in } \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} \text{ is } \omega\ ;\ ii)\ n\text{th coefficient of } f \neq 0\}. \tag{4.1}$$

It is to be noted that the set $\Omega_p(\phi)$ depends on the newform ϕ and character χ that we started with. Computation of this set is important in our applications and we will see that we need this set only in the case when $m_p \geq 1$ and $\lambda_p = 0$. Since this set consists of at most eight elements when $p = 2$, and four when p is an odd prime, computation doesn't seem to be difficult. Indeed, we can use the results of Section 3.8 and our algorithm in Section 3.6 to compute most of the elements.

We define another set of local functions on $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ which takes values in $\mathbb{Z}/2\mathbb{Z}$ and denote this set by Γ_p,

$$\Gamma_p = \{\gamma_{e,\upsilon} : e \in \mathbb{Z},\ \upsilon \in \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} \text{such that } \nu_p(\upsilon) \equiv e \pmod 2\},$$

where

$$\gamma_{e,\upsilon}(u) = \begin{cases} 1 & u \in \upsilon\mathbb{Q}_p^{\times 2},\ \nu_p(u) = e \\ 0 & \text{else.} \end{cases}$$

If $p = 2$, we further define

$$\gamma'_{e,\upsilon} = \frac{1}{2}(\gamma_{e,\upsilon} + \gamma_{e,5\upsilon}),$$

$$\gamma''_e(u) = \begin{cases} 1 & \nu_2(u) = e \\ 0 & \text{else,} \end{cases}$$

and

$$\gamma^0_e(u) = \begin{cases} 1 & \nu_2(u) = e,\ (u,-1)_2 = -\chi_{0,2}(-1) \text{ or } \nu_2(u) = e+1 \\ 0 & \text{else.} \end{cases}$$

Now we are ready to define the local factors $\widetilde{n_p}$ and the set $U_p(e,\phi)$ for $e = \widetilde{n_p}$. We will be dealing with several cases and subcases and in each of them we will be simplifying Waldspurger's formulae and making them more explicit for our use.

Case 1. p *odd* and $m_p \geq 1$.

We consider the following subcases:

(a) $\lambda_p = 0$.
In this case we need to compute $\Omega_p(\phi)$. We know that $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} = \{1,\ p,\ u,\ pu\}$ where u is unit in $\mathbb{Z}_p$ which is a non-square mod p. If there exists a $\omega \in \Omega_p(\phi)$ such that $\nu_p(\omega) = 0$ then $\widetilde{n_p} = m_p$, and

$U_p(\widetilde{n_p}, \phi) = \{\gamma_{0,\omega} : \ \omega \in \Omega_p(\phi) \text{ and } \nu_p(\omega) = 0\}$. In this case, the set $U_p(\widetilde{n_p}, \phi)$ consists of at most the functions $\gamma_{0,1}$ and $\gamma_{0,u}$. Otherwise, for all $\omega \in \Omega_p(\phi)$, $\nu_p(\omega) = 1$. In this case $\widetilde{n_p} = m_p + 1$, and $U_p(\widetilde{n_p}, \phi) = \{\gamma_{1,\omega} : \ \omega \in \Omega_p(\phi) \text{ and } \nu_p(\omega) = 1\}$, hence $U_p(\widetilde{n_p}, \phi)$ consists of at most $\gamma_{1,p}$ and $\gamma_{1,pu}$. It is clear from the definition given above that $\gamma_{0,1}$, $\gamma_{0,u}$, $\gamma_{1,p}$, $\gamma_{1,pu}$ are characteristic functions of 1, u, p, pu modulo $\mathbb{Q}_p^{\times 2}$ respectively.

(b) $\lambda_p \neq 0$.

In this case we must have $m_p = 1$, since by the theory of newforms (see Section 2.2.12), $m_p \geq 2$ implies that $\lambda_p = 0$. Recall that S is the collection of primes p such that ρ_p is not irreducible principal series. We have further subcases:

(i) $p \notin S$.
By Waldspurger, in this case $\widetilde{n_p} = m_p = 1$. Let $\beta_p \in \mathbb{C}$ such that $\beta_p^2 = \lambda'_p$. Then $U_p(1, \phi) = \{c^{(1)}_{p,\beta_p}\}$.
However we note that we do not need to consider this subcase since by Corollary 4.2.4, ρ_p is a special representation and hence not a principal irreducible series. Thus in this case we always have $p \in S$.

(ii) $p \in S$.
Here we have the following subcases:

(i') $\chi_{0,p}$ is unramified.
Here again $\widetilde{n_p} = m_p = 1$ and $U_p(1, \phi) = \{c^{(5)}_{p,\lambda'_p}\}$. We use the theory of newforms (2.2.12) to simplify the function $c^{(5)}_{p,\lambda'_p}$. Since $m_p = 1$ we get that $\lambda_p = \pm p^{(k-3)/2}$ and $\lambda'_p = \pm p^{-1/2}$. Hence we have in this case,

$$c^{(5)}_{p,\lambda'_p}(n) = \begin{cases} 2^{1/2} & \nu_p(n) = 0, \ \left(\frac{n}{p}\right) = \mp\chi_{0,p}(p^{-1}) \\ 0 & \nu_p(n) = 0, \ \left(\frac{n}{p}\right) = \pm\chi_{0,p}(p^{-1}) \\ 1 & \nu_p(n) = 1. \end{cases}$$

(ii') $\chi_{0,p}$ is ramified.
We have $\widetilde{n_p} = m_p = 1$ and $U_p(1, \phi) = \{c^{(6)}_{p,\lambda'_p}\}$. As in the above

subcase, we get the following simplification:

$$c^{(6)}_{p,\lambda'_p}(n) = \begin{cases} 1 & \nu_p(n) = 0 \\ \pm 2^{1/2} p^{-1/2} & \nu_p(n) = 1,\ (p,n)_p = \mp \chi_{0,p}(p^{-1}) \\ 0 & \nu_p(n) = 1,\ (p,n)_p = \pm \chi_{0,p}(p^{-1}). \end{cases}$$

Case 2. p *odd* and $m_p = 0$.

We have the following subcases:

(a) $\chi_{0,p}$ is unramified.
Here, $\widetilde{n_p} = m_p = 0$ and $U_p(0, \phi) = \{c^{(0)}_{p,\lambda'_p}\}$. It is to be noted that $c^{(0)}_{p,\lambda'_p}$ takes the value 1 at any square-free n.

(b) $\chi_{0,p}$ is ramified.
We have $\widetilde{n_p} = 1$ and $U_p(1, \phi) = \{c^{(3)}_{p,\alpha_p},\ c^{(3)}_{p,\alpha'_p}\}$ if $\alpha_p \neq \alpha'_p$, else $U_p(1, \phi) = \{c^{(3)}_{p,\alpha_p},\ c^{(4)}_{p,\alpha_p}\}$.

We note that $M_\phi \mid (N/2)$, so if N has no factor of prime p, then we do not need to consider the part (b) because in this case $\chi_{0,p}$ is unramified by Lemma 4.1.3.

Case 3. $p = 2$ and $m_2 \geq 1$.

Consider the following subcases:

(a) $\lambda_2 = 0$.
We compute $\Omega_2(\phi)$. Note that $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2} = \{\pm 1,\ \pm 2,\ \pm 5,\ \pm 10\}$. If there exists a $\omega \in \Omega_2(\phi)$ such that $\nu_2(\omega) = 0$ then $\widetilde{n_2} = m_2 + 2$, and $U_2(\widetilde{n_2}, \phi) = \{\gamma_{0,\omega} :\ \omega \in \Omega_2(\phi) \text{ and } \nu_2(\omega) = 0\}$. In this case, the set $U_2(\widetilde{n_2}, \phi)$ consists of at most $\gamma_{0,1}$, $\gamma_{0,3}$, $\gamma_{0,5}$, and $\gamma_{0,7}$. Otherwise, for all $\omega \in \Omega_2(\phi)$, $\nu_2(\omega) = 1$ and then $\widetilde{n_2} = m_2 + 3$, and $U_2(\widetilde{n_2}, \phi) = \{\gamma_{1,\omega} :\ \omega \in \Omega_2(\phi) \text{ and } \nu_2(\omega) = 1\}$, hence $U_2(\widetilde{n_2}, \phi)$ consists of at most $\gamma_{1,2}$, $\gamma_{1,6}$, $\gamma_{1,10}$ and $\gamma_{1,14}$. As above, $\gamma_{0,i}$ for $i \in \{1,\ 3,\ 5,\ 7\}$ are the characteristic functions of an odd residue class modulo 8 and $\gamma_{1,j}$ for $j \in \{2,\ 6,\ 10,\ 14\}$ are the characteristic functions of even residue class modulo $\mathbb{Q}_2^{\times 2}$.

(b) $\lambda_2 \neq 0$.

By the similar argument as in Case 1 (b), we must have $m_2 = 1$. We have the following subcases:

(i) $2 \notin S$.

In this case $\widetilde{n_2} = m_2 + 1 = 2$. Let $\beta_2 \in \mathbb{C}$ such that $\beta_2^2 = \lambda_2'$. Then $U_2(2, \phi) = \{c^{(0)}_{2,\beta_2}\}$.

We point out that this subcase does not arise since as before by Corollary 4.2.4, ρ_2 is a special representation and hence $p \in S$.

(ii) $2 \in S$.

Then, we have the following subcases:

(*i'*) $\chi_{0,2}$ is trivial on $1 + 4\mathbb{Z}_2$.

Here $\widetilde{n_2} = 2$ and $U_2(2, \phi) = \{c^{(5)}_{2,\lambda_2'}\}$. Since $m_2 = 1$ we get that $\lambda_2 = \pm 2^{(k-3)/2}$ and $\lambda_2' = \pm 2^{-1/2}$. Hence we have,

$$c^{(5)}_{2,\lambda_2'}(n) = \begin{cases} 0 & \nu_2(n) = 0,\ (-1)^{\frac{n-1}{2}} = \chi_{0,2}(-1),\ (-1)^{\frac{n^2-1}{8}} = \pm\chi_{0,2}(2^{-1}) \\ \pm 1 & \nu_2(n) = 0,\ (-1)^{\frac{n-1}{2}} = \chi_{0,2}(-1),\ (-1)^{\frac{n^2-1}{8}} = \mp\chi_{0,2}(2^{-1}) \\ 1 & \nu_2(n) = 0,\ (-1)^{\frac{n-1}{2}} = -\chi_{0,2}(-1) \\ 1 & \nu_2(n) = 1. \end{cases}$$

(*ii'*) $\chi_{0,2}$ is nontrivial on $1 + 4\mathbb{Z}_2$.

Here $\widetilde{n_2} = 3$ and $U_2(3, \phi) = \{c^{(6)}_{p,\lambda_2'},\ \gamma_0''\}$ and we get the following simplification:

$$c^{(6)}_{2,\lambda_2'}(n) = \begin{cases} \pm 2^{1/2} & \nu_2(n) = 0 \\ 0 & \nu_2(n) = 1,\ (n,-1)_2 = \chi_{0,2}(-1),\ (2,n)_2 = \pm\chi_{0,2}(2^{-1}) \\ \pm 1 & \nu_2(n) = 1,\ (n,-1)_2 = \chi_{0,2}(-1),\ (2,n)_2 = \mp\chi_{0,2}(2^{-1}) \\ 1 & \nu_2(n) = 1,\ (n,-1)_2 = -\chi_{0,2}(-1). \end{cases}$$

Case 4. $p = 2$ and $m_2 = 0$.

We have the following subcases:

(a) $\chi_{0,2}$ is trivial on $1 + 4\mathbb{Z}_2$.

We have $\widetilde{n_2} = 2$ and $U_2(2, \phi) = \{c^{(1)}_{2,\alpha_2},\ c^{(1)}_{2,\alpha_2'}\}$ if $\alpha_2 \neq \alpha_2'$, else $U_2(2, \phi) = \{c^{(1)}_{2,\alpha_2},\ c^{(2)}_{2,\alpha_2}\}$.

(b) $\chi_{0,2}$ is nontrivial on $1+4\mathbb{Z}_2$.

Here $\widetilde{n_2}=3$ and $U_2(3,\phi)=\{c^{(3)}_{2,\alpha_2},\ c^{(3)}_{2,\alpha'_2},\ \gamma''_0\}$ if $\alpha_2\neq\alpha'_2$, else $U_2(3,\phi)=\{c^{(3)}_{2,\alpha_2},\ c^{(4)}_{2,\alpha_2},\ \gamma''_0\}$.

We would like to point out the following useful lemma:

Lemma 4.3.7. *Let χ be a quadratic character modulo N such that $\nu_2(N)$ is at most 2. Then, $\chi_{0,2}$ is trivial on $1+4\mathbb{Z}_2$.*

Proof. Since χ is a quadratic character, χ_0 is also quadratic with modulus $\mathrm{lcm}(4,N)=4N'$ where $2\nmid N'$. Now the lemma follows from part (iii) of Lemma 4.1.3. □

Remark. These simplifications along with our method to compute the basis for $S_{k/2}(N,\chi,\phi)$ for suitable N and χ lead to an algorithm for computing critical values of the L-functions of certain quadratic twists of ϕ. For example, if $M_\phi=p^\alpha$ for some odd prime p, then the possible choices for $\widetilde{N_\phi}$ are either $4p^\alpha$ or $4p^{\alpha+1}$, hence we compute bases for spaces $S_{k/2}(4p^\alpha,\chi_{\mathrm{triv}},\phi)$ and $S_{k/2}(4p^{\alpha+1},\chi_{\mathrm{triv}},\phi)$ and the sets $U_2(2,\phi)$, $U_p(\alpha,\phi)$, $U_p(\alpha+1,\phi)$ to apply Theorem 4.3.4 in order to get the desired results.

It is to be noted that in the above we have discussed computation of $U_p(e,\phi)$ only for $e=\widetilde{n_p}$. But in certain cases as we will see later, working with the level $\widetilde{N_\phi}$ is not sufficient to get the complete information and one might need to go to higher levels.

4.4 Period

Lemma 4.4.1. *Let E be an elliptic curve, given by a minimal Weierstrass model, and let E_n be the minimal model of its twist by square-free positive integer n. Then there is a computable non-zero rational number α_n such that*

$$\Omega(E_n)=\frac{\alpha_n\Omega(E)}{\sqrt{n}}.$$

The proof we give also explains how to compute α_n.

Proof. Let $\omega = dx/(2y + a_1x + a_3)$ be the invariant differential for the model

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

By definition, the period

$$\Omega(E) = \int_{E(\mathbb{R})} |\omega|.$$

Recall [37, page 49] that a change of variable

$$x = u^2x' + r, \qquad y = u^3y' + u^2sx' + t$$

leads to a model E' with invariant differential $\omega' = u\omega$; thus the periods are related by $\Omega(E') = |u|\Omega(E)$. Completing the square in y we obtain the model

$$E' : y'^2 = x'^3 + Ax'^2 + Bx' + C$$

where

$$A = \frac{b_2}{4}, \qquad B = \frac{b_4}{2}, \qquad C = \frac{b_6}{4}.$$

Since $u = 1$ in this change of variable, $\omega' = \omega$ and $\Omega(E') = \Omega(E)$. Now let the model E'' be the twist of E' by n:

$$E'' : y''^2 = x''^3 + Anx''^2 + Bn^2x'' + Cn^3.$$

Note that these are related by the change of variable

$$y'' = n^{3/2}y', \qquad x'' = nx'.$$

Thus the invariant differentials satisfy

$$\omega'' = \frac{dx''}{2y''} = \frac{\omega'}{\sqrt{n}}.$$

Thus

$$\Omega(E'') = \frac{\Omega(E')}{\sqrt{n}} = \frac{\Omega(E)}{\sqrt{n}}.$$

Now the model E'' is not necessarily minimal (nor even integral at 2), but by

Tate's algorithm there is a change of variables

$$x'' = u^2X + r, \qquad y'' = u^3Y + u^2sX + t$$

with rational u, s, t (and $u \neq 0$) such that the resulting model E_n is minimal. By the above

$$\Omega(E_n) = u\Omega(E'') = \frac{|u|\Omega(E)}{\sqrt{n}}.$$

□

Example 4.4.2. Let $E : Y^2 = X^3 - 5^3$ (which is already in minimal Weierstrass model). Then $E_5 : Y^2 = X^3 - 5^6$. This model is clearly non-minimal. A minimal model is given by $E_5 : Y^2 = X^3 - 1$. Following the above argument we see that $\alpha_5 = 5$. To check our computations we find using `MAGMA` that $\sqrt{5}\Omega(E_5)/\Omega(E)$ is equal to 5 to 29 decimal places.

Lemma 4.4.3. *Let $E : Y^2 = X^3 + AX^2 + BX + C$ be an elliptic curve with A, B, $C \in \mathbb{Z}$. Suppose that the discriminant of this model is sixth-power free. Let n be a square-free positive integer. Then a minimal model for the n-th twist is $E_n : Y^2 = X^3 + AnX^2 + Bn^2X + Cn^3$. Moreover, the periods are related by the formula*

$$\Omega(E_n) = \frac{\Omega(E_1)}{\sqrt{n}}.$$

Proof. Let Δ be the discriminant of the model $E : Y^2 = X^3 + AX^2 + BX + C$. We are assuming that Δ is sixth-power free. Thus it is 12-th power free, and so E is minimal. Now the model $E_n : Y^2 = X^3 + AnX^2 + Bn^2X + Cn^3$ has discriminant $\Delta_n = \Delta \cdot n^6$. Since n is square-free this is 12-th power free. Thus the model for E_n is minimal. The argument in the proof of Lemma 4.4.1 completes the proof. □

4.5 Applications of Waldspurger's Theorem

In this section we will present a few examples explaining how to use Waldspurger's Theorem. The idea of using Waldspurger's Theorem for an elliptic curve is motivated by Tunnell's famous work on the congruent number problem. We will see however that our case needs many more computations to get any desired result. In the examples that follow we will first use our algorithm (Section 3.6) to compute the space of cusp forms that are Shimura equivalent to the given elliptic curve and then use Waldspurger's Theorem to get some interesting results. We will follow the notation adopted in the previous section.

4.5.1 A First Example

Our first example will be the elliptic curve E over $\mathbb{Q}$ given by

$$E : Y^2 = X^3 + X + 1.$$

The conductor of E is $496 = 16 \times 31$ and E does not have complex multiplication. Let $\phi \in S_2^{\text{new}}(496, \chi_{\text{triv}})$ be the corresponding newform given by the Modularity Theorem; ϕ has the following q-expansion,

$$\phi(z) = q - 3q^5 + 3q^7 - 3q^9 - 2q^{11} - 4q^{13} - q^{19} + O(q^{20}).$$

It is to be noted that ϕ satisfies the hypothesis (H1)–this follows by Theorem 4.3.2, and since $16 \mid M_\phi$, ϕ satisfies (H2). Let χ be a Dirichlet character with $\chi^2 = \chi_{\text{triv}}$. Hence by Theorem 4.3.1 there exists N such that $S_{3/2}(N, \chi, \phi) \neq \{0\}$. Note that we must have $496 \mid (N/2)$.

In order to apply Waldspurger's Theorem we would like to compute an eigenbasis for the summand $S_{3/2}(N, \chi, \phi)$ for a suitable N and χ. We will assume χ to be the trivial character χ_{triv}. We use our algorithm on Shimura's decomposition, see Section 3.6 for details. Using Corollary 3.6.4 it turns out that $S_{3/2}(992, \chi, \phi) = \{0\}$. At level 1984 however one can compute using dimension formula 2.3.5 that the space $S_{3/2}(1984, \chi)$ is 119-dimensional and using Corollary 3.6.4 we get that the space $S_{3/2}(1984, \chi, \phi)$ has a basis $\{f_1, f_2, f_3\}$ where f_1, f_2 and f_3 have the following q-expansions:

$$f_1(z) = q^3 + q^{43} - 2q^{75} + 2q^{83} + q^{91} + 3q^{115} - 3q^{123} + O(q^{145}) := \sum_{n=1}^{\infty} a_n q^n$$
$$f_2(z) = q^{15} + q^{23} - q^{31} + 2q^{55} + q^{79} - 3q^{119} + O(q^{145}) := \sum_{n=1}^{\infty} b_n q^n$$
$$f_3(z) = q^{17} + q^{57} + q^{65} + 2q^{73} - q^{89} - q^{105} + q^{137} + O(q^{145}) := \sum_{n=1}^{\infty} c_n q^n.$$

We are now ready to apply Waldspurger's Theorem. We are interested in the level $N = 1984$. In this case $\chi_0 = \chi_{\text{triv}}(\cdot)\left(\frac{-1}{\cdot}\right)$ is a Dirichlet character modulo 1984. By Waldspurger's Theorem 4.3.4 there exists a function A_ϕ on square-free positive integers n such that

$$A_\phi(n)^2 = \mathrm{L}(E_{-n}, 1)$$

and

$$S_{3/2}(1984, \chi, \phi) = \bigoplus \overline{\mathrm{U}}(E, \phi, A_\phi),$$

where the sum is over all $E \geq 1$ such that $\widetilde{N_\phi} \mid E \mid 1984$. We already know the left-hand side of the above identity. Henceforth we will be interested in computing the right-hand side. We will first compute $\widetilde{N_\phi}$ and then $\overline{\mathrm{U}}(E, \phi, A_\phi)$ for $\widetilde{N_\phi} \mid E \mid 1984$.

Recall that $\widetilde{N_\phi} = \prod_p p^{\widetilde{n_p}}$ and so we need to compute local components $\widetilde{n_p}$ for each prime p. We consider the following cases. Please refer to the Section 4.3 for details.

Case 1. p odd and $p \neq 31$.

In this case $m_p = 0$ and since $p \nmid N$ the local character $\chi_{0,p}$ is unramified. Hence we get that $\widetilde{n_p} = 0$.

Case 2. $p = 31$.

Here $m_{31} = 1$. Since $\lambda_{31} \neq 0$ using Corollary 4.2.4 it follows that the local component ρ_{31} is a special representation of $\mathrm{GL}_2(\mathbb{Q}_{31})$ and so $31 \in S$. Also, note that $\mathbb{Z}_{31}^\times/{\mathbb{Z}_{31}^\times}^2$ is generated by 11 mod ${\mathbb{Z}_{31}^\times}^2$ and using Proposition 4.1.4 we can show that $\chi_{0,31}(11) = 1$. Thus $\chi_{0,31}$ is unramified and so, $\widetilde{n_{31}} = 1$.

Case 3. $p = 2$.

In this case $m_2 = 4$ and it is clear from the q-expansion of ϕ that $\lambda_2 = 0$. We need some information about the set $\Omega_2(\phi)$ (see Equation 4.1). In our case, looking at f_1, f_2 and f_3, we get that $\{1, 3, 7\} \subseteq \Omega_2(\phi)$. Since $\nu_2(1) = \nu_2(3) = \nu_2(7) = 0$, we get $\widetilde{n_2} = m_2 + 2 = 6$.

Hence

$$\widetilde{N_\phi} = 31 \times 2^6 = 1984.$$

Thus we have $E = \widetilde{N_\phi} = 1984$ and we would like to know how the space $\overline{\mathrm{U}}(1984, \phi, A_\phi)$ looks. For that the next immediate task will be to compute $\mathrm{U}_p(e_p, \phi)$ where $e_p = \nu_p(1984)$. We consider the following cases and again refer to the previous section for details:

Case 1. p odd and $p \neq 31$.

Here, $e_p = 0$ and $\mathrm{U}_p(0, \phi)$ consists of only one function $c^{(0)}_{p,\lambda'_p}$ defined on $\mathbb{Q}_p^\times$. Recall that $c^{(0)}_{p,\lambda'_p}(n) = 1$ for n square-free.

Case 2. $p = 31$.

In this case $e_{31} = 1$ and as already seen, $31 \in S$ and $\chi_{0,31}$ is unramified. So, $\mathrm{U}_{31}(1, \phi) = \{c^{(5)}_{31,\lambda'_{31}}\}$. Note that $\lambda_{31} = -1$ and hence $\lambda'_{31} = (31)^{-1/2}\lambda_{31} = -(31)^{-1/2}$. Again using Proposition 4.1.4 we can show that $\chi_{0,31}(31^{-1}) = -1$. Also note that $(31, n)_{31} = \left(\frac{n}{31}\right)$. So for n square-free we have,

$$c^{(5)}_{31,\lambda'_p}(n) = \begin{cases} 2^{1/2} & \nu_{31}(n) = 0,\ \left(\frac{n}{31}\right) = -1 \\ 0 & \nu_{31}(n) = 0,\ \left(\frac{n}{31}\right) = 1 \\ 1 & \nu_{31}(n) = 1. \end{cases}$$

Case 3. $p = 2$.

Here $e_2 = 6$. Since $\lambda_2 = 0$ and $\{1, 3, 7\} \subseteq \Omega_2(\phi)$, we see that $\mathrm{U}_2(6, \phi)$ consists of $\gamma_{0,1}, \gamma_{0,3}, \gamma_{0,7}$ which are the characteristic functions of residue classes of $1, 3, 7$ modulo 8 respectively. By our methods so far we do not know whether 5 belongs to $\Omega_2(\phi)$ or not.

Recall that $\overline{\mathrm{U}}(E,\phi,A_\phi)$ is the space generated by the functions $f(\underline{c},A_\phi)$ where $\underline{c}\in\prod_p \mathrm{U}_p(e_p,\phi)$. Thus in our case $\underline{c}=(c_p)_p$ where, for odd primes $p\neq 31$ we have $c_p=c^{(0)}_{p,\lambda'_p}$, $c_{31}=c^{(5)}_{31,\lambda'_{31}}$ and for c_2 the possible choices are $\gamma_{0,1}$, $\gamma_{0,3}$, $\gamma_{0,5}$ and $\gamma_{0,7}$. By using Waldspurger's Theorem 4.3.4

$$S_{3/2}(1984,\chi,\phi)=\overline{\mathrm{U}}(1984,\phi,A_\phi)$$

and so every cusp form in the space on the left-hand side can be written in terms of

$$f(\underline{c},A_\phi)(z):=\sum_{n=1}^{\infty}A_\phi(n^{\mathrm{sc}})n^{1/4}\prod_p c_p(n)q^n$$

for some $\underline{c}=(c_p)\in\prod U_p(e_p,\phi)$.

We use Theorem 3.8.10 to conclude that f_1 have non-zero n-th coefficients only for $n\equiv 3 \pmod 8$, f_2 have non-zero coefficients only for $n\equiv 7 \pmod 8$ and f_3 have non-zero coefficients only for $n\equiv 1 \pmod 8$.

Since f_1 have non-zero a_n only for $n\equiv 3 \pmod 8$, taking $\underline{c}$ as above with $c_2=\gamma_{0,3}$ we get that for n square-free,

$$a_n=\beta_1A_\phi(n)n^{1/4}c_2(n)c_{31}(n)=\begin{cases}2^{1/2}\beta_1A_\phi(n)n^{1/4} & \nu_{31}(n)=0,\left(\frac{n}{31}\right)=-1, n\equiv 3 \pmod 8\\ \beta_1A_\phi(n)n^{1/4} & \nu_{31}(n)=1, n\equiv 3 \pmod 8\\ 0 & \text{otherwise,}\end{cases} \tag{4.2}$$

for some complex constant β_1. Similarly, taking $c_2=\gamma_{0,7}$ for f_2 and $c_2=\gamma_{0,1}$ for f_3 respectively we get that

$$b_n=\beta_2A_\phi(n)n^{1/4}c_2(n)c_{31}(n)=\begin{cases}2^{1/2}\beta_2A_\phi(n)n^{1/4} & \nu_{31}(n)=0,\left(\frac{n}{31}\right)=-1, n\equiv 7 \pmod 8\\ \beta_2A_\phi(n)n^{1/4} & \nu_{31}(n)=1, n\equiv 7 \pmod 8\\ 0 & \text{otherwise,}\end{cases} \tag{4.3}$$

for some complex constant β_2 and

$$c_n = \beta_3 A_\phi(n) n^{1/4} c_2(n) c_{31}(n) = \begin{cases} 2^{1/2}\beta_3 A_\phi(n) n^{1/4} & \nu_{31}(n) = 0, \left(\frac{n}{31}\right) = -1, n \equiv 1 \pmod 8 \\ \beta_3 A_\phi(n) n^{1/4} & \nu_{31}(n) = 1, n \equiv 1 \pmod 8 \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

for some complex constant β_3.

We have the following theorem which allows us to calculate the critical values of the L-functions of E_{-n}, the $(-n)$-th quadratic twists of E.

Theorem 4.5.1. *Let E be as above and n be a positive square-free integer.*

(i) If $\nu_{31}(n) = 0$, $n \equiv 3 \pmod 8$ and $\left(\frac{n}{31}\right) = -1$ then,

$$\mathrm{L}(E_{-n}, 1) = \frac{a_n^2}{2{\beta_1}^2\sqrt{n}}.$$

(ii) If $\nu_{31}(n) = 1$, $n \equiv 3 \pmod 8$ then,

$$\mathrm{L}(E_{-n}, 1) = \frac{a_n^2}{{\beta_1}^2\sqrt{n}}.$$

(iii) If $\nu_{31}(n) = 0$, $n \equiv 7 \pmod 8$ and $\left(\frac{n}{31}\right) = -1$ then,

$$\mathrm{L}(E_{-n}, 1) = \frac{b_n^2}{2{\beta_2}^2\sqrt{n}}.$$

(iv) If $\nu_{31}(n) = 1$, $n \equiv 7 \pmod 8$ then,

$$\mathrm{L}(E_{-n}, 1) = \frac{b_n^2}{{\beta_2}^2\sqrt{n}}.$$

(v) If $\nu_{31}(n) = 0$, $n \equiv 1 \pmod 8$ and $\left(\frac{n}{31}\right) = -1$ then,

$$\mathrm{L}(E_{-n}, 1) = \frac{c_n^2}{2{\beta_3}^2\sqrt{n}}.$$

(vi) If $\nu_{31}(n) = 1$, $n \equiv 1 \pmod 8$ *then,*

$$\mathrm{L}(E_{-n}, 1) = \frac{c_n^2}{{\beta_3}^2\sqrt{n}}.$$

Proof. Using Waldspurger's Theorem 4.3.4 we know the existence of a function A_ϕ on square-free numbers such that $A_\phi(n)^2 = \mathrm{L}(E_{-n}, 1)$. The proof follows now using Equations (4.2), (4.3) and (4.4). □

We have the following lemma which gives a partial result when $n \equiv 5 \pmod 8$.

Lemma 4.5.2. *Let* E *be as above and* n *be a positive square-free integer such that* $n \equiv 5 \pmod 8$. *Then* $\mathrm{L}(E_{-n}, 1) = 0$ *if either (i)* $\nu_{31}(n) = 1$ *or (ii)* $\nu_{31}(n) = 0$ *and* $\left(\frac{n}{31}\right) = -1$.

Proof. Recall that the space $S_{3/2}(1984, \chi, \phi)$ is generated by functions of the form $\sum_{n=1}^{\infty} A_\phi(n^{\mathrm{sc}}) n^{1/4} \prod_p c_p(n) q^n$. Recall that for c_2 the choices are characteristic functions of an odd residue class modulo 8. Since f_1, f_2, f_3 spans $S_{3/2}(1984, \chi, \phi)$ and none of them have a non-zero coefficient for $n \equiv 5 \pmod 8$ we get that

$$A_\phi(n) c_{31}(n) = 0 \text{ whenever } n \equiv 5 \pmod 8.$$

Since $c_{31}(n) \neq 0$ if either $\nu_{31}(n) = 1$ or, $\nu_{31}(n) = 0$ and $\left(\frac{n}{31}\right) = -1$, the lemma follows. □

Later on, in Proposition 4.5.4, we will give another proof of this result using root number calculations.

We will show now how we use the above to calculate the order of the Tate-Shafarevich group $Ш(E_{-n}/\mathbb{Q})$. We will be assuming the Birch and Swinnerton-Dyer Conjecture for rank zero elliptic curves:

$$\mathrm{L}(E_{-n}, 1) = \frac{|Ш(E_{-n}/\mathbb{Q})| \cdot \Omega_{E_{-n}} \cdot \prod_p c_p}{|E_{-n,\mathrm{tor}}|^2} \tag{4.5}$$

where $\Omega_{E_{-n}}$ stands for the real period of E_{-n} (since $E_{-n}(\mathbb{R})$ is connected), c_p for the p-th Tamagawa number of E_{-n} and $E_{-n,\mathrm{tor}}$ stands for the torsion group of E_{-n}, all of which are easily computable.

We have the following lemma.

Lemma 4.5.3. *Let $E : Y^2 = X^3 + X + 1$. Then $E_{n,\mathrm{tor}} = 0$ for all square-free integers n.*

Proof. Let $K = \mathbb{Q}(\sqrt{n})$. It is well-known that the map

$$E_n(\mathbb{Q}) \to E(K)$$

given by

$$O \mapsto O, \qquad (X, Y) \mapsto \left(\frac{X}{n}, \frac{Y}{n\sqrt{n}}\right)$$

is an injective group homomorphism [3]. Thus it is sufficient to show that $E(K)$ has trivial torsion subgroup. Recall that the discriminant of E is $-496 = -16 \times 31$. Let $p \neq 2, 31$ be a rational prime and let $\mathfrak{P}$ be a prime ideal of K dividing p. Then E has good reduction at $\mathfrak{P}$. Moreover, if $e_{\mathfrak{P}} < p - 1$ then the reduction map $E(K)_{\mathrm{tor}} \to E(\mathbb{F}_{\mathfrak{P}})$ is injective [22, page 501], where $e_{\mathfrak{P}}$ is the ramification index for $\mathfrak{P}$ and $\mathbb{F}_{\mathfrak{P}}$ denotes the residue field of $\mathfrak{P}$. Thus if $p \geq 5$ and $p \neq 31$ then this map is injective. Now we take $p = 5, 7$, so $E(\mathbb{F}_{\mathfrak{P}})$ is a subgroup of $E(\mathbb{F}_{25})$ and $E(\mathbb{F}_{49})$ respectively. Using `MAGMA` we find

$$E(\mathbb{F}_{25}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}, \qquad E(\mathbb{F}_{49}) \cong \mathbb{Z}/55\mathbb{Z}.$$

Since these two groups have coprime orders, it follows that $E(K)_{\mathrm{tor}} = 0$ and so $E_{n,\mathrm{tor}} = 0$. $\square$

Further, since the discriminant of E_{-1} is $-496 = 2^4 \times 31$, by Lemma 4.4.3 we know that $\Omega(E_{-n}) = \Omega(E_{-1})/\sqrt{n}$.

From (4.5) it is clear that the quantity $\frac{\mathrm{L}(E_{-n},1)}{\Omega_{E_{-n}} R(E_{-n}/\mathbb{Q})}$ is an integer. Using `MAGMA` we compute this integer for $n \in \{3, 15, 17\}$ and using Lemma 4.4.3, one gets that

$$\Omega_{E_{-1}} = \frac{1}{4{\beta_1}^2} = \frac{1}{4{\beta_2}^2} = \frac{1}{8{\beta_3}^2}. \tag{4.6}$$

Now recall that $W(E_{-n}/\mathbb{Q})$ denotes the root number for elliptic curve E_{-n} over rational numbers. We have the following proposition.

[3] As the map simply scales the variables, it takes lines to lines and so must define a homomorphism of Mordell-Weil groups.

Proposition 4.5.4. *For E as above and n positive square-free the following holds.*

(i) If $\nu_{31}(n) = 0$ then,

$$W(E_{-n}/\mathbb{Q}) = \begin{cases} -1 & n \equiv 1,3,7 \pmod 8,\ \left(\frac{n}{31}\right) = 1 \text{ or} \\ & n \equiv 5 \pmod 8,\ \left(\frac{n}{31}\right) = -1 \text{ or} \\ & n \text{ even},\ \left(\frac{n}{31}\right) = -1; \\ 1 & n \equiv 1,3,7 \pmod 8,\ \left(\frac{n}{31}\right) = -1 \text{ or} \\ & n \equiv 5 \pmod 8,\ \left(\frac{n}{31}\right) = 1 \text{ or} \\ & n \text{ even},\ \left(\frac{n}{31}\right) = 1. \end{cases}$$

(ii) If $\nu_{31}(n) = 1$ then,

$$W(E_{-n}/\mathbb{Q}) = \begin{cases} -1 & n \equiv 5 \pmod 8 \text{ or} \\ & n \text{ even}; \\ 1 & n \equiv 1,3,7 \pmod 8. \end{cases}$$

Proof. The methods used here to compute the root numbers are well-known and we refer to [11]. We can express the global root number $W(E_{-n}/\mathbb{Q})$ as a product of local root numbers

$$W(E_{-n}/\mathbb{Q}) = \prod_p W(E_{-n}, p)$$

where the product is taken over all primes including ∞; here $W(E_{-n}, \infty) = -1$. The value of the local root number $W(E_{-n}, p)$ depends only on the isomorphism class of E_{-n} over $\mathbb{Q}_p$ and hence only on the value of n modulo $(\mathbb{Q}_p^*)^2$. For a fixed value of n and a fixed prime p we can use the computer algebra package `MAGMA` to compute $W(E_{-n}, p)$. By writing down all the possibilities for n modulo squares in $\mathbb{Q}_2$, $\mathbb{Q}_3$ and $\mathbb{Q}_{31}$ we find the following:

$$W(E_{-n}, 2) = \begin{cases} -1 & n \equiv 1 \pmod 8 \\ 1 & n \equiv 3, 5, 7 \pmod 8 \\ 1 & 2 \mid n, \ n/2 \equiv 1, 5 \pmod 8 \\ -1 & 2 \mid n, \ n/2 \equiv 3, 7 \pmod 8), \end{cases}$$

and

$$W(E_{-n}, 3) = \begin{cases} -1 & 3 \mid n \\ 1 & 3 \nmid n, \end{cases} \qquad W(E_{-n}, 31) = \begin{cases} -1 & 31 \mid n \\ -1 & \left(\frac{n}{31}\right) = 1 \\ 1 & \left(\frac{n}{31}\right) = -1. \end{cases}$$

It remains to calculate the local root numbers at primes $p \neq 2, 3, 31$. We consider the elliptic curve E_{-1},

$$E_{-1} : Y^2 = X^3 + X - 1.$$

The conductor of E_{-1} is 248 and the discriminant $\Delta_{E_{-1}}$ is $-496 = -2^4 \times 31$. Fix n positive and square-free, the n-th quadratic twist of E_{-1} is given by the Weierstrass model,

$$E_{-n} : Y^2 = X^3 + n^2 X - n^3.$$

The discriminant $\Delta_{E_{-n}}$ of E_{-n} is $-2^4 \times 31 \times n^6$. Since n is square-free, $\Delta_{E_{-n}}$ is 12-th power free and hence the model for E_{-n} is minimal at every prime p. For primes p such that p is odd and coprime to 31 and $p \nmid n$, $W(E_{-n}, p) = 1$.

Let $p \neq 2, 3, 31$ be a prime such that $p \mid n$. Then E has additive reduction modulo p. Since $\nu_p(\Delta_{E_{-n}}) = 6$, we get that [11, page 96]

$$W(E_{-n}, p) = \left(\frac{-1}{p}\right).$$

Thus we can summarize for all primes $p \neq 2, 31$ (note that we are now including

$p = 3$)

$$W(E_{-n}, p) = \begin{cases} 1 & p \nmid n \\ \left(\frac{-1}{p}\right) & p \mid n. \end{cases}$$

Write $n = 2^i 31^j n'$ where $2,\ 31 \nmid n'$. Then

$$W(E_{-n}/\mathbb{Q}) = -\left(\frac{-1}{n'}\right) W(E_{-n}, 2) W(E_{-n}, 31).$$

The proof now follows by combining all the possibilities. $\square$

Before computing the order of the Tate-Shafarevich group Ш $(E_{-n}/\mathbb{Q})$, we have the following refinement of Theorem 4.5.1.

Theorem 4.5.5. *Let* $E : Y^2 = X^3 + X + 1$ *and* $f = f_1 + f_2 + \sqrt{2} f_3 = \sum d_n q^n$. *Then, for positive square-free* $n \equiv 1,\ 3,\ 7 \pmod 8$

$$\mathrm{L}(E_{-n}, 1) = \frac{2^{(\nu_{31}(n)+1)} \Omega_{E_{-1}}}{\sqrt{n}} \cdot d_n^2.$$

Proof. Note that $d_n = a_n + b_n + \sqrt{2} c_n$. It is important for the proof to note that $a_n = 0$ for $n \not\equiv 3 \pmod 8$, and $b_n = 0$ for $n \not\equiv 7 \pmod 8$, and $c_n = 0$ for $n \not\equiv 1 \pmod 8$; we proved this by applying Theorem 3.8.10. It follows from equations (4.2), (4.3) and (4.4) that $d_n = 0$ whenever $n \equiv 1, 3, 7 \pmod 8$ and the Kronecker symbol $\left(\frac{n}{31}\right) = 1$. Further by Proposition 4.5.4 if $n \equiv 1, 3, 7 \pmod 8$ and $\left(\frac{n}{31}\right) = 1$ then $W(E_{-n}, \mathbb{Q}) = -1$ and so $\mathrm{L}(E_{-n}, 1) = 0$. Thus the theorem follows when $\left(\frac{n}{31}\right) = 1$.

In the case when $\left(\frac{n}{31}\right) = -1$, the refinement follows by using Equation (4.6) in Theorem 4.5.1. $\square$

We have now the following corollary which computes the order of the Tate-Shafarevich group Ш $(E_{-n}/\mathbb{Q})$.

Corollary 4.5.6. *Let* $E : Y^2 = X^3 + X + 1$ *and* $f = f_1 + f_2 + \sqrt{2} f_3 = \sum d_n q^n$. *Let* n *be positive square-free number such that* $n \equiv 1,\ 3,\ 7 \pmod 8$ *and* E_{-n} *has rank zero. Then, assuming the Birch and Swinnerton-Dyer conjecture,*

$$|\text{Ш}\,(E_{-n}/\mathbb{Q})| = \frac{2^{(\nu_{31}(n)+1)}}{\prod_p c_p} \cdot d_n^2$$

where the Tamagawa numbers c_p of E_{-n} are given by

$$c_2 = \begin{cases} 1 & n \equiv 3,7 \pmod 8 \\ 2 & n \equiv 1,5 \pmod 8, \end{cases} \qquad c_{31} = \begin{cases} 1 & 31 \nmid n, \\ 4 & 31 \mid n, \left(\frac{n/31}{31}\right) = 1 \\ 2 & 31 \mid n, \left(\frac{n/31}{31}\right) = -1, \end{cases}$$

and $c_p = \#E_{-1}(\mathbb{F}_p)[2]$ for $p \mid n$, $p \neq 31$, and $c_p = 1$ for all other primes p.

Proof. From Lemma 4.5.3 we have $E_{-n,\mathrm{tor}} = 0$ for all square-free integers n. Further since we are assuming that E_{-n} has rank zero, $R(E_{-n}/\mathbb{Q}) = 1$. Substituting these facts and $\Omega(E_{-n}) = \Omega(E_{-1})/\sqrt{n}$ in Equation (4.5) we get that

$$|\text{Ш}\,(E_{-n}/\mathbb{Q})| = \frac{\mathrm{L}(E_{-n},1)\cdot\sqrt{n}}{\Omega_{E_{-1}}\cdot\prod_p c_p} = \frac{2^{(\nu_{31}(n)+1)}}{\prod_p c_p}\cdot d_n^2 \ ;$$

the last equality follows by Theorem 4.5.5.

We will be using Tate's algorithm (see [38, Pages 364–368]) to compute the Tamagawa numbers c_p for E_{-n} for n odd and square-free. Recall that the Weierstrass model for E_{-n} is given by

$$E_{-n} : Y^2 = X^3 + n^2 X - n^3.$$

The discriminant $\Delta_{E_{-n}}$ is $2^6 \times 31 \times n^6$; since n is odd and square-free the above model is minimal at every prime p. We note that Weierstrass coefficients are $a_1 = a_2 = a_3 = 0$, $a_4 = n^2$, $a_6 = -n^3$ and $b_2 = 0$, $b_4 = 2n^2$, $b_6 = -4n^3$, $b_8 = -n^4$.

Now fix a prime p such that $p \neq 31$ and $p \mid n$. Hence $p^3 \mid b_6$ and we are in the step 6 of Tate's algorithm. We need to consider the polynomial $P(T) = T^3 + m^2 T - m^3$ where $m = n/p$. Note $\nu_p(m) = 0$ and $p \nmid \mathrm{Disc}(P) = -31 \times m^6$. Therefore,

$$c_p = 1 + \#\{\alpha \in \mathbb{F}_p : P(\alpha) = 0\} = 1 + \#\{\alpha \in \mathbb{F}_p : \alpha^3 + \alpha - 1 = 0\} = \#E_{-1}(\mathbb{F}_p)[2].$$

Let $p = 31$ and suppose $p \mid n$. Then the above polynomial $P(T)$ factorizes as $P(T) = (T + 3m)(T + 14m)^2$ over $\mathbb{F}_p$. We are now in the step 7 of Tate's algorithm. We translate X-coordinate in the Weierstrass equation so that double root of $P(T)$ is $T = 0$. This gives the following Weierstrass equation for E_{-n},

$$Y^2 = X^3 - 42nX^2 + 589n^2X - 2759n^3.$$

We must consider the factorization of the polynomial $Y^2 + 89m^3$ over $\mathbb{F}_p$ (note $2759n^3/p^4 = 89m^3$). By the recipe in step 7, if $\left(\frac{m}{31}\right) = 1$, then $c_{31} = 4$; else $c_{31} = 2$.

It is to be noted that for a fixed prime p, the value of c_p depends only on isomorphism classes of E_{-n} over $\mathbb{Q}_p$ and thus only on n modulo $(\mathbb{Q}_p^*)^2$. In particular for $p = 2$ using `MAGMA` we get that $c_2(E_{-1}) = c_2(E_{-5}) = 2$ and $c_2(E_{-3}) = c_2(E_{-7}) = 1$. Similarly for $p = 31$ such that $p \nmid n$, we have $c_{31}(E_{-1}) = c_{31}(E_{-3}) = 1$. Now the result follows combining all these possibilities. $\square$

The following is a small check that our computed order of Tate-Shafarevich group $Ш(E_{-n}/\mathbb{Q})$ is indeed a square. Note that

$$d_n^2 = \begin{cases} square & n \equiv 3, 7 \pmod 8 \\ 2 \times square & n \equiv 1 \pmod 8. \end{cases}$$

Let $f := x^3 + x - 1$; discriminant of f is $\Delta_f = -31$. By the above corollary for $p \neq 31$ and $p \mid n$,

$$c_p = \begin{cases} 1 & f \text{ has no roots over } \mathbb{F}_p \\ 2 & f \text{ has one root over } \mathbb{F}_p \\ 4 & f \text{ has three roots over } \mathbb{F}_p \end{cases}$$

It is easy to see that Galois group of f over $\mathbb{F}_p$ is either C_1 or C_3 if and only if $\left(\frac{\Delta_f}{p}\right) = 1$. Thus,

$$\prod_{\substack{p \mid n \\ p \neq 31}} c_p = \begin{cases} square & \left(\frac{n}{31}\right) = 1 \\ 2 \times square & \left(\frac{n}{31}\right) = -1. \end{cases}$$

We assume $\nu_{31}(n) = 0$ and $\left(\frac{n}{31}\right) = -1$. If $n \equiv 3, 7 \pmod 8$ then $c_2 = c_{31} = 1$ and so $\prod_p c_p = 2 \times$ *square*. If $n \equiv 1 \pmod 8$ we have $c_2 = 2$ and $c_{31} = 1$ and so $\prod_p c_p$ is a square. Thus in these cases, $|\text{Ш}(E_{-n}/\mathbb{Q})| = \frac{2}{\prod_p c_p} \cdot d_n^2$ is a square. The other cases follow similarly.

We have the following easy corollary to Theorem 4.5.5.

Corollary 4.5.7. *Suppose* $n \equiv 1,\ 3,\ 7 \pmod 8$ *and* $\left(\frac{n}{31}\right) = -1$. *Then assuming the Birch and Swinnerton-Dyer Conjecture,*

$$\text{Rank}(E_{-n}) \geq 2 \Leftrightarrow d_n = 0.$$

Proof. By Proposition 4.5.4, if $n \equiv 1,\ 3,\ 7 \pmod 8$ and $\left(\frac{n}{31}\right) = -1$ then $W(E_{-n}/\mathbb{Q}) = 1$. Thus the analytic rank is even, and so by BSD, the rank is even. The corollary now follows using Theorem 4.5.5. $\square$

In order to get a complete solution we need to know what happens when either n is even or $n \equiv 5 \pmod 8$. From Proposition 4.5.4 it follows that $\mathrm{L}(E_{-n}, 1) = 0$ whenever n is even or $n \equiv 5 \pmod 8$ and either $\left(\frac{n}{31}\right) = -1$ or $31 \mid n$. Thus we are unable to predict in these cases what happens [4] when $\nu_{31}(n) = 0$ and $\left(\frac{n}{31}\right) = 1$.

We will be able to get a complete answer if we are working with higher levels. So we are interested in similar computations as above for $S_{3/2}(N, \chi, \phi)$ where N varies so that $\widetilde{N_\phi} = 1984 \mid N$. We arrive at following conclusions:

(i) If $N = 1984 \times 2^\alpha$ then only interesting situation is when $\alpha = 1$; indeed if $\alpha > 1$ then choices for $c_2(n)$ are the functions such that $c_2(n) \neq 0$ only when $\nu_2(n) = \alpha$ and hence are zero on n square-free. Suppose $\alpha = 1$. Then, $\nu_2(N) = 7$, $\nu_{31}(N) = 1$ and so $c_{31}(n)$ remains the same and the possibilities for $c_2(n)$ are now the characteristic functions $\gamma_{1,2}$, $\gamma_{1,6}$, $\gamma_{1,10}$ or $\gamma_{1,14}$. Suppose $2,\ 6,\ 10,\ 14 \in \Omega_2(\phi)$. Then $c_2(n)c_{31}(n) \neq 0$ only when $n \equiv 2, 6, 10, 14 \pmod 8$ and, either $\nu_{31}(n) = 1$ or $\left(\frac{n}{31}\right) = -1$. From the root number argument above we have $\mathrm{L}(E_{-n}, 1) = 0$ in these cases. Using

[4] In fact doing computations using `MAGMA` we get for example, $\mathrm{L}(E_{-n}, 1) \neq 0$ for $n = 5$, 69, 101, 109, 133, 157, 165; these n satisfy the conditions $n \equiv 5 \pmod 8$ and $\left(\frac{n}{31}\right) = 1$. However for $n = 149$, 173, which also satisfy the same two conditions, we get that $\mathrm{L}(E_{-n}, 1) = 0$ (note thus using root number argument $\text{Rank}(E_{-n}) \geq 2$ for $n = 149$, 173). We do not detect a general pattern.

Waldspurger's Theorem we can conclude that $\overline{\mathrm{U}}(1984 \times 2, \phi, A_\phi) = \{0\}$ and hence $S_{3/2}(1984 \times 2, \chi_{\mathrm{triv}}, \phi) = S_{3/2}(1984, \chi_{\mathrm{triv}}, \phi)$. We do not get any new information.

(ii) Suppose now $N = 1984 \times 31^\alpha$. As before the only interesting case for us will be $\alpha = 1$ and we assume this. Hence $\nu_2(N) = 6$, $\nu_{31}(N) = 2$. Now we will have two choices for $c_{31}(n)$, namely $\gamma_{0,1}$ or $\gamma_{0,u}$ where $u \in \mathbb{Q}_{31}^\times/\mathbb{Q}_{31}^{\times\,2}$ such that $\left(\frac{u}{31}\right) = -1$ and, four choices for $c_2(n)$, namely $\gamma_{0,1}$, $\gamma_{0,3}$, $\gamma_{0,5}$ or $\gamma_{0,7}$. If $5 \in \Omega_2(\phi)$, choosing $c_2(n) = \gamma_{0,5}$ and $c_{31}(n) = \gamma_{0,1}$, we will be able to conclude what happens when $n \equiv 5 \pmod 8$ and $\left(\frac{n}{31}\right) = 1$ by computing bases for the space $S_{3/2}(1984 \times 31, \chi_{\mathrm{triv}}, \phi)$.

(iii) In fact from the above two cases one can easily see that we need to compute at least the bases for the space $S_{3/2}(1984 \times 31 \times 2, \chi_{\mathrm{triv}}, \phi)$ in order to hope to get the complete solution.

The computation for $S_{3/2}(1984 \times 31 \times 2, \chi_{\mathrm{triv}}, \phi)$ is still in progress. We note that the dimension of the space $S_{3/2}(1984 \times 31 \times 2, \chi_{\mathrm{triv}})$ is 7686.

4.5.2 Second Example

Our second example will be the rational elliptic curve E of conductor 144 given by

$$E : Y^2 = X^3 - 1.$$

The corresponding newform ϕ is given by

$$\phi(z) = q + 4q^7 + 2q^{13} - 8q^{19} - 5q^{25} + 4q^{31} - 10q^{37} - 8q^{43} + 9q^{49} + O(q^{50}).$$

Here $M_\phi = 144$. Since (H1) and (H2) are satisfied, there exists a N such that $S_{3/2}(N, \chi, \phi) \neq \{0\}$, where $144 \mid (N/2)$ and again $\chi^2 = \chi_{\mathrm{triv}}$. We assume that χ is the trivial character. Using Corollary 3.6.4 for computing Shimura's decomposition, we find that at the level 576, the space $S_{3/2}(576, \chi, \phi) \neq \{0\}$; and this space has a basis $\{f_1, f_2, f_3, f_4\}$ where f_1, f_2, f_3 and f_4 have the following q-expansion:

$$f_1(z) = q - q^{25} + 5q^{49} - 6q^{73} - 6q^{97} + O(q^{100}) := \sum_{n=1}^{\infty} a_n q^n$$

$$f_2(z) = q^5 + q^{29} - q^{53} - 2q^{77} + O(q^{100}) := \sum_{n=1}^{\infty} b_n q^n$$

$$f_3(z) = q^{13} - 2q^{61} + q^{85} + O(q^{100}) := \sum_{n=1}^{\infty} c_n q^n$$

$$f_4(z) = q^{17} - q^{41} - q^{89} + O(q^{100}) := \sum_{n=1}^{\infty} d_n q^n.$$

Doing similar calculations as in the previous example it turns out that $\widetilde{N_\phi} = 576$. Using Waldspurger's Theorem there exists a function A_ϕ on square-free numbers such that $S_{3/2}(576, \chi, \phi) = \overline{\mathrm{U}}(576, \phi, A_\phi)$. Following the computations we get that $\overline{\mathrm{U}}(576, \phi, A_\phi)$ is spanned by $\sum_{n=1}^{\infty} A_\phi(n^{\mathrm{sc}}) n^{1/4} \prod_p c_p(n) q^n$ where the choices for c_2 include the characteristic functions of 1, 5 modulo $\mathbb{Q}_2^{\times 2}$, while the choices for c_3 are characteristic functions of 1, 2 modulo $\mathbb{Q}_3^{\times 2}$.

The following lemma is a special case of a standard theorem on the torsion of Mordell elliptic curves (i.e. elliptic curves of the form $Y^2 = X^3 + B$). For the proof see [8, page 52].

Lemma 4.5.8. *Let E be as above and let n be a square-free integer. Then $E_{n,\mathrm{tor}} \cong \mathbb{Z}/2\mathbb{Z}$ unless $n = -1$ in which case $E_{-1,\mathrm{tor}} \cong \mathbb{Z}/6\mathbb{Z}$.*

The discriminant of the model $E_{-1} : Y^2 = X^3 + 1$ is $-432 = 2^4 \times 3^3$ which is sixth-power free. By Lemma 4.4.3, $\Omega(E_{-n}) = \Omega(E_{-1})/\sqrt{n}$.

We have the following lemma on root numbers which can be proved on similar lines as Proposition 4.5.4.

Lemma 4.5.9. *Let E be as above. For n positive square-free the following holds.*

(i) If $\nu_3(n) = 0$ then,

$$W(E_{-n}/\mathbb{Q}) = \begin{cases} 1 & n \equiv 1, 5 \pmod 8 \\ -1 & n \equiv 3, 7 \pmod 8 \\ -1 & n \text{ even.} \end{cases}$$

(ii) If $\nu_3(n) = 1$ then,

$$W(E_{-n}/\mathbb{Q}) = \begin{cases} 1 & n/3 \equiv 1,5 \pmod 8 \\ -1 & n/3 \equiv 3,7 \pmod 8 \\ 1 & n \text{ even.} \end{cases}$$

Finally, we have the following theorem.

Theorem 4.5.10. *Let $E : Y^2 = X^3 - 1$. Let*

$$f = f_1/2 + f_2 + \sqrt{2} f_3 + \sqrt{3} f_4 := \sum_{n=1}^{\infty} e_n q^n.$$

Let $n \neq 1$ [5] be positive square-free integer such that $n \equiv 1,2 \pmod 3$. Then,

$$\mathrm{L}(E_{-n}, 1) = \frac{\Omega_{E_{-1}}}{\sqrt{n}} \cdot e_n^2. \tag{4.7}$$

Further assuming BSD, if E_{-n} has rank zero then,

$$|\text{Ш}(E_{-n}/\mathbb{Q})| = \frac{4}{\prod_p c_p} \cdot e_n^2$$

where the Tamagawa numbers $c_2 = 3$ if $n \equiv 1 \pmod 8$, $c_2 = 1$ if $n \equiv 3,5,7 \pmod 8$; $c_3 = 2$; $c_p = \#E_{-1}(\mathbb{F}_p)[2]$ for $p \mid n$, $p \neq 3$; and $c_p = 1$ for all other primes p.

Proof. It is to be noted that using Theorem 3.8.10, we can prove that a_n is non-zero only for $n \equiv 1 \pmod{24}$, b_n is non-zero only for $n \equiv 5 \pmod{24}$, c_n is non-zero only for $n \equiv 13 \pmod{24}$ and d_n is non-zero only for $n \equiv 17 \pmod{24}$. Thus we can choose f as in the theorem (the choice for coefficients of f_i in f are done using similar calculations as in Theorem 4.5.5). Using Lemma 4.5.9, we see that both sides of equation (4.7) vanish if $n \equiv 1,2 \pmod 3$ and $n \equiv 3,7 \pmod 8$. So it is enough to consider the other cases. Recall that $A_\phi(n)^2 = \mathrm{L}(E_{-n}, 1)$ by Theorem 4.3.4. The proof of the first statement now follows.

[5] In the case $n = 1$ we still have $\mathrm{L}(E_{-n}, 1) = \frac{\Omega_{E_{-1}}}{\sqrt{n}} \cdot e_n^2$, but since $|E_{-1,\mathrm{tor}}| = 6$ we get that $|\text{Ш}(E_{-n}/\mathbb{Q})| = \frac{36}{\prod_p c_p} \cdot e_n^2$.

For the second statement, we use Lemma 4.5.8 and substitute $\Omega(E_{-n}) = \Omega(E_{-1})/\sqrt{n}$ in the equation (4.5). The calculation for Tamagawa numbers c_p are done as before (see Corollary 4.5.6). $\square$

In order to consider the case of E_{-n} when $3 \mid n$ we try to look at the space $S_{3/2}(1728, \chi_{\mathrm{triv}}, \phi)$ but it turns out that this space is equal to the space $S_{3/2}(576, \chi_{\mathrm{triv}}, \phi)$. Hence we do not get any new information.

Another possible way to deal with this situation is to work with the quadratic character $\chi_3 = \left(\frac{3}{\cdot}\right)$, instead of the trivial character. Our algorithm shows that $S_{3/2}(576, \chi_3, \phi) = \{0\}$ and $S_{3/2}(1728, \chi_3, \phi)$ has a basis consisting of g_1, g_2, g_3 and g_4 where g_i's are as follows:

$$g_1 = q^3 - q^{75} + 5q^{147} - 6q^{219} - 6q^{291} + O(q^{300}), \qquad g_2 = q^{39} - 2q^{183} + q^{255} + O(q^{300}),$$

$$g_3 = q^{15} + q^{87} - q^{159} - 2q^{231} + O(q^{300}), \qquad g_4 = q^{51} - q^{123} - q^{267} + O(q^{300}).$$

Waldspurger's Theorem now asserts the existence of a function A_ϕ (which now depends on χ_3 and ϕ) on $\mathbb{N}^{\mathrm{sc}}$ such that $A_\phi(n)^2 = \mathrm{L}(E_{-3n}, 1)$. Note that g_i's have non-zero n-th coefficient only for $n \equiv 3,\ 6 \pmod 9$. Further if $n = 3m$ then $\mathrm{L}(E_{-3n}, 1) = \mathrm{L}(E_{-m}, 1)$. This leads us to obtain exactly the same results as in Theorem 4.5.10.

Remark. It is to be noted that we cannot apply Waldspurger's Theorem to the elliptic curve E' given by

$$E' : Y^2 = X^3 + 1$$

since it is easy to check that the hypothesis (H1) is not satisfied. However, $E' = E_{-1}$, hence by Theorem 4.5.10 we get information about the positive n-th quadratic twists of E' for n with $3 \nmid n$. Further note that E_3 is isogenous to E_{-1}, hence $\mathrm{L}(E_n, 1) = \mathrm{L}(E_{-3n}, 1)$ for all n. Thus computation of $\mathrm{L}(E_{-3n}, 1)$ for n positive square-free will lead to a formula for $\mathrm{L}(E_n, 1)$ and hence for $\mathrm{L}(E'_n, 1)$ for all n square-free.

4.5.3 Example with a Non-Rational Newform

In this example we start with a non-rational newform ψ and we show that we can get similar formulae as before for the critical values of L-functions of $\psi \otimes \chi_{-n}$.

Let $\psi \in S_2^{\text{new}}(62, \chi_{\text{triv}})$ be a newform of weight 2, level 62 and trivial character given by the following q-expansion,

$$\psi(z) = q - q^2 + aq^3 + q^4 + (-2a+2)q^5 - aq^6 + 2q^7 - q^8 + (2a-1)q^9 + O(q^{10})$$

where a has minimal polynomial $x^2 - 2x - 2$.

As before using our algorithm (Corollary 3.6.4) we get that the space $S_{3/2}(124, \chi_{\text{triv}}, \psi) = \langle f \rangle$ where f has the following q-expansion,

$$f(z) = q + (a+1)q^2 - q^4 - 2aq^5 - aq^7 + (-a-1)q^8 + (a+1)q^9 - 2q^{10} + O(q^{12}).$$

Note that Waldspurger's theorem is applicable for the newform ψ as the local automorphic representation of ψ at 2 is not supercuspidal; this follows since $\nu_2(62) = 1$ and the second coefficient of ψ is non-zero (see Corollary 4.2.3).

We have the following proposition.

Proposition 4.5.11. *Let ψ and $f := \sum_{n=1}^{\infty} a_n q^n$ be as above. Let n be square-free such that $n \not\equiv 3 \pmod 8$ and $\left(\frac{n}{31}\right) \neq -1$. Then*

$$\mathrm{L}(\psi \otimes \chi_{-n}, 1) = \begin{cases} \frac{\beta}{\sqrt{n}} \cdot a_n^2 & \text{if } \nu_{31}(n) = 1 \\ \frac{\beta}{2\sqrt{n}} \cdot a_n^2 & \text{if } \nu_{31}(n) = 0 \end{cases}$$

where $\beta = 2 \cdot \mathrm{L}(\psi \otimes \chi_{-1}, 1)$.

Proof. The proof follows by the similar calculations as shown in the previous examples. □

Remark. Using `MAGMA`, we have numerically checked the above formula for the first ten values of n and we find that the two sides of the formula agree to 30 decimal places. It is to be noted that as we increase the values of n, the level of the newform $\psi \otimes \chi_{-n}$ becomes very large, for example the level of newform $\psi \otimes \chi_{-n}$ for $n = 1, 2, 3, 5, 7, 10$ are $496, 1984, 558, 12400, 3038, 49600$ respectively.

In the next chapter we will study the relation between modular forms of weight $3/2$ and positive-definite ternary quadratic forms. In fact given a quadratic character χ and a rational newform ϕ, we would like to compute the subspace of $S_{3/2}(N, \chi, \phi)$ (for a suitable N) that is coming from the ternary quadratic forms in a sense explained in the next chapter. This will lead us to give Tunnell-like formulae for critical values of n-th quadratic twists of ϕ in terms of ternary quadratic forms. We point out that given a newform it might not always be possible to find forms of weight $3/2$ that are Shimura equivalent to the newform and that come from ternary quadratic forms. In particular for the elliptic curve in our first example, $E : Y^2 = X^3 + X + 1$, the space $S_{3/2}(1984, \chi_{\text{triv}}, \phi_E)$ has trivial intersection with the subspace of $S_{3/2}(1984, \chi_{\text{triv}})$ coming from ternary quadratic forms. We also note that the space $S_{3/2}(1984, \chi_{\text{triv}}, \phi_E)$ does not consist of any forms that one gets by multiplying weight one and weight half forms as explained in Chapter 1. However for the elliptic curve in the second example, $E : Y^2 = X^3 - 1$, we will see (Example 5.3.3) that each of the basis elements f_i of $S_{3/2}(576, \chi_{\text{triv}}, \phi_E)$ comes from ternary quadratic forms.

Chapter 5

Ternary Quadratic Forms

The reader will recall that in Tunnell's Theorem, the critical value of the L-function of the n-th twist of the $E : Y^2 = X^3 - X$ is expressed in terms of ternary quadratic forms. In the previous chapter we saw several examples where such critical values are expressed in terms of coefficients of cusp forms of weight 3/2. It turns out that for a given level N and *quadratic* character χ, a subspace of $S_{3/2}(N, \chi)$ is spanned by theta-series coming from positive-definite ternary quadratic forms. To express our critical values in terms of quadratic forms we need to compute theses subspaces.

5.1 Positive-Definite Quadratic Forms and associated Theta-Series

Let $F \in \mathbb{Z}[x_1, \ldots, x_k]$ be a positive-definite quadratic form. Associated to F is a theta-series

$$\theta_F(z) := \sum_{\mathbf{m} \in \mathbb{Z}^k} q^{F(\mathbf{m})} = \sum_{n=0}^{\infty} \#\{\mathbf{m} \in \mathbb{Z}^k : F(\mathbf{m}) = n\} \cdot q^n; \qquad q = e^{2\pi i z}.$$

Theorem 5.1.1. *(Shimura [36]) With notation as above, let A_F be the $k \times k$ matrix*

$$A_F = \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right).$$

Define N_F to be the smallest positive integer so that $N_F {A_F}^{-1}$ is an even matrix, *that is, has integral entries, and even integers on the main diagonal. Then* $\theta_F \in M_{k/2}(N_F, \chi_{d_F})$, *where* $\chi_{d_F} = \left(\frac{d_F}{\cdot}\right)$ *and* $d_F = \det(A_F)$ *if* $k \equiv 0 \pmod 4$, $d_F = -\det(A_F)$ *if* $k \equiv 2 \pmod 4$ *and* $d_F = \det(A_F)/2$ *if* $k \equiv 1 \pmod 2$.

We shall call N_F as in Shimura's Theorem the *level of* F, the integer d_F the *discriminant of* F, χ_{d_F} the *character of* F and A_F the *matrix* of F.

Let R be either $\mathbb{Z}$ or $\mathbb{Z}_p$ (where we take $\mathbb{Z}_p = \mathbb{R}$ if $p = \infty$). Let F, G be homogeneous quadratic forms in $R[x_1, \ldots, x_k]$. We say that F *and* G *are* R*-equivalent* if there exists a unimodular matrix U with coefficients in R such that $F(\mathbf{x}) = G(\mathbf{x}U)$. Now suppose F, G are homogeneous quadratic forms in $\mathbb{Z}[x_1, \ldots, x_k]$ with the same level and discriminant. We say that F *and* G *are in the same genus* if F is $\mathbb{Z}_p$-equivalent to G for all p (including ∞).

It is clear that if F and G are $\mathbb{Z}$-equivalent, then $\theta_F = \theta_G$.

Theorem 5.1.2. *(Siegel [33]) Suppose* F *and* G *are in the same genus. Let* N *be their level and* χ_d *be their character. Then* $\theta_F - \theta_G \in S_{k/2}(N, \chi_d)$.

Now if F, G are homogeneous quadratic forms in $\mathbb{Z}[x_1, \ldots, x_k]$ and $F = rG$ for some integer r, then $\theta_F(q) = \theta_G(q^r)$. Hence $\theta_F = V(r)(\theta_G)$ where $V(r)$ is the V-operator. It is for this reason that we restrict to *primitive* quadratic forms. It is clear that if a form is primitive, then all other forms belonging to the same genus are primitive. We can therefore speak of *primitive genera*. As we are most interested in modular forms of weight $3/2$ we shall restrict ourselves to the case $k = 3$; i.e. to the case of ternary quadratic forms, and follow the exposition in Lehman's paper [27].

Let F be a positive-definite, primitive ternary quadratic form with integer coefficients given by

$$F = ax^2 + by^2 + cz^2 + ryz + sxz + txy.$$

Let A_{ij} be the ij-th cofactor of A_F and $M = \gcd(A_{11}, A_{22}, A_{33}, 2A_{23}, 2A_{13}, 2A_{12})$. Let $\alpha = A_{11}/M$, $\beta = A_{22}/M$, $\gamma = A_{33}/M$, $\rho = A_{23}/M$, $\sigma = A_{13}/M$, $\tau = A_{12}/M$. Let

$$\phi = \alpha x^2 + \beta y^2 + \gamma z^2 + \rho yz + \sigma xz + \tau xy.$$

Then ϕ is a primitive positive-definite form and is called *reciprocal* of F. It turns out that $N_F = N_\phi$ and $d_\phi = N_F^3/4d_F$. Moreover, the reciprocal of equivalent forms are equivalent and if F and G are in the same genus, their reciprocals are in the same genus; see [27, page 410].

Given N there are only finitely many choices for d such that we have ternary quadratic forms of level N and discriminant d. In particular,

Theorem 5.1.3. *([27, Theorem 2]) Let F be as above. Suppose that*

$$N_F = 2^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

is the prime factorization of N_F. Then $n_0 \geq 2$ and d_F is of the form

$$d_F = 2^{d_0} p_1^{d_1} p_2^{d_2} \cdots p_r^{d_r}$$

with following restrictions on d_is:

(i) either $d_0 = n_0 - 2$ or, $d_0 = 2n_0$ or, $n_0 \leq d_0 \leq 2n_0 - 2$, and

(ii) for $1 \leq i \leq r$ we must have $n_i \leq d_i \leq 2n_i$.

Further if n_i is even for $0 \leq i \leq r$, then either $n_0 \leq d_0 \leq 2n_0 - 2$ or, d_i is odd for some $1 \leq i \leq r$.

Fix a level N and discriminant d. There are finitely many primitive genera having level N and discriminant d. Each genus has finitely many forms up to $\mathbb{Z}$-equivalence.

Below we recall a standard algorithm, due to Dickson [17], for writing down the primitive genera of ternary quadratic forms of given level and discriminant, and for each genus writing down a representative of each $\mathbb{Z}$-equivalence class. We shall follow the exposition of Dickson's algorithm given in [27].

We say that F is *reduced* if the following are true:

- $a \leq b \leq c$;
- r, s and t are all positive or all non-positive;
- $a \geq |t|$; $a \geq |s|$; $b \geq |r|$;

- $a+b+r+s+t \geq 0$;
- if $a=t$ then $s \leq 2r$; if $a=s$ then $t \leq 2r$; if $b=r$ then $t \leq 2s$;
- if $a=-t$ then $s=0$; if $a=-s$ then $t=0$; if $b=-r$ then $t=0$;
- if $a+b+r+s+t=0$ then $2a+2s+t \leq 0$;
- if $a=b$ then $|r| \leq |s|$; if $b=c$ then $|s| \leq |t|$.

Theorem 5.1.4. *([27, Proposition 3]) Every primitive positive-definite ternary quadratic form is equivalent to one and only one reduced form. Also, if f is reduced and has discriminant d, then*

$$\frac{d}{4} \leq abc \leq \frac{d}{2}.$$

It follows from the above inequalities that if F is a reduced form of discriminant d then

$$1 \leq a \leq \sqrt[3]{\frac{d}{2}}, \qquad a \leq b \leq \sqrt{\frac{d}{2a}}, \qquad \max\left(b, \frac{d}{4ab}\right) \leq c \leq \frac{d}{2ab},$$

and either

$$-b \leq r \leq 0, \qquad -a \leq s \leq 0, \qquad -a \leq t \leq 0,$$

or

$$1 \leq r \leq b, \qquad 1 \leq s \leq a, \qquad 1 \leq t \leq a.$$

It is clear now, how in principle we can list all reduced forms of a given level N and discriminant d. In fact, Lehman [27] gives additional bounds on the coefficients. First $c \leq N/2$. Thus

$$1 \leq a \leq \min\left(\frac{N}{2}, \sqrt[3]{\frac{d}{2}}\right), \qquad a \leq b \leq \min\left(\frac{N}{2}, \sqrt{\frac{d}{2a}}\right). \tag{5.1}$$

Let $m = 4d/N$ and $\mu = N^2/d$. Then, moreover, either $a \equiv 0$ or $-\mu \pmod 4$. The same is true for b, c in place of a. To this we add our own improvement, given by the following lemma.

Lemma 5.1.5. *Let $\alpha = 4ab - t^2$. Then r is a root modulo α of the polynomial $aX^2 - stX + (d + bs^2)$. Moreover,*

$$c = \frac{ar^2 - str + d + bs^2}{\alpha}.$$

Proof. The discriminant $d = \det(A_f)/2$ and hence can be given by following expression,

$$d = 4abc + rst - ar^2 - bs^2 - ct^2.$$

The lemma now follows. $\square$

To enumerate all primitive reduced forms of level N and discriminant d, we run through the pairs a, b satisfying the inequalities (5.1) and the above congruences. We then enumerate the pairs s, t that satisfy

$$-a \leq s \leq 0, \qquad -a \leq t \leq 0, \qquad \text{or} \qquad 1 \leq s \leq a, \qquad 1 \leq t \leq a.$$

Next we use the lemma to determine the possibilities for r modulo α, and write down all r satisfying the above inequalities and these congruences. Finally, the lemma gives the value of c. Once we have all the coefficients, we can check that they indeed define a primitive reduced form of level N and discriminant d.

In order to write down the cusp forms of level N and quadratic character $\chi_D = \left(\frac{D}{\cdot}\right)$ that are coming from primitive ternary quadratic forms, we first need to consider the possible choices of discriminants d given by Theorem 5.1.3 with square-free part D. For each such choice of discriminant, we can use the above algorithm to write down the reduced representatives in primitive genera of ternary quadratic forms of level N. However since discriminants can be very large, we modify the algorithm by using reciprocals. In particular, if $d > N^3/4d$ we compute the reduced ternary forms of level N and discriminant $N^3/4d$ and take their reciprocals which are now primitive forms of level N and discriminant d. Note that taking reciprocal need not keep the forms reduced but as remarked earlier it preserves each genus. Now we can use Theorem 5.1.2 to compute the subspace of $S_{3/2}(N, \chi_d)$ which comes from primitive ternary quadratic forms. Here we can test for forms being in the same genus using an algorithm of Conway and Sloane [13, Chapter 15], which fortunately is implemented in `MAGMA`.

Notation. We will denote by $[a, b, c, r, s, t]$, the ternary quadratic form given by $ax^2 + by^2 + cz^2 + ryz + sxz + txy$.

5.2 Action of Hecke operators on Theta-Series

The following theorem is a reformulation by Bungert [7] of the results of Eichler [18] and Schulze-Pillot [32] which gives an explicit description of the action of Hecke operators on theta-series of ternary quadratic forms.

Theorem 5.2.1. *[7, Proposition 4] Let F be an integral positive-definite ternary quadratic form with matrix A_F. Let p be a prime not dividing the level N_F of the theta-series θ_F of F. Then the action of Hecke operator T_{p^2} is given by*

$$T_{p^2}(\theta_F)(z) = \sum_{S \in M/\operatorname{GL}_3(\mathbb{Z})} \theta_{\frac{S^T A_F S}{p^2}}(z),$$

where M denotes the set of 3×3 matrices S over $\mathbb{Z}$ such that S has elementary divisors 1, p, p^2 and $\frac{S^T A_F S}{p^2}$ has integral entries and $\theta_{\frac{S^T A_F S}{p^2}}$ stands for theta-series of the ternary quadratic form with matrix $\frac{S^T A_F S}{p^2}$.

Let F be as in the theorem, having matrix $A = A_F$, and let G be the quadratic form represented by the matrix $B = \frac{S^T A S}{p^2}$. The reader might be wondering why F and G have the same level. It is clear that the determinants of A and B, and therefore discriminants of F and G, are equal. We know by Theorem 5.1.3 that the two levels N_F and N_G have precisely the same prime divisors. Since $p \nmid N_F$, we know $p \nmid N_G$. Now for any prime $\ell \neq p$, the forms F and G are $\mathbb{Z}_\ell$-equivalent. Therefore, $\nu_\ell(N_F) = \nu_\ell(N_G)$. Hence $N_F = N_G$.

To be able to compute the action of Hecke operators on theta-series, we proved the following lemma.

Lemma 5.2.2. *Let p be a prime and*

$$M' = \{S \in \mathrm{M}_3(\mathbb{Z}) : S \text{ has elementary divisors } 1,\ p \text{ and } p^2\ \}.$$

Then the following are representatives of $M'/\operatorname{GL}_3(\mathbb{Z})$:

$$\begin{bmatrix} p & a & b \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}_{\substack{0<a<p \\ 0\le b<p}}, \qquad \begin{bmatrix} p & 0 & b \\ 0 & p & c \\ 0 & 0 & p \end{bmatrix}_{\substack{0<c<p \\ 0\le b<p}}, \qquad \begin{bmatrix} p & 0 & b \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}_{0<b<p},$$

$$\begin{bmatrix} p^2 & a & b \\ 0 & p & c \\ 0 & 0 & 1 \end{bmatrix}_{\substack{0\le a,b,c<p^2 \\ p\mid a}}, \qquad \begin{bmatrix} p^2 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix}_{\substack{0\le a,b<p^2 \\ p\mid b}}, \qquad \begin{bmatrix} p & 0 & b \\ 0 & p^2 & c \\ 0 & 0 & 1 \end{bmatrix}_{\substack{0\le c<p^2 \\ 0\le b<p}},$$

$$\begin{bmatrix} p & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^2 \end{bmatrix}_{0\le a<p}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & p^2 & c \\ 0 & 0 & p \end{bmatrix}_{\substack{0\le c<p^2 \\ p\mid c}}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p^2 \end{bmatrix}.$$

Proof. Recall that given any S in $\mathrm{M}_3(\mathbb{Z})$ there exists a unimodular matrix $U \in \operatorname{GL}_3(\mathbb{Z})$ such that S has unique Hermite normal form H and $H = SU$. So we list matrices in Hermite normal form with elementary divisors 1, p and p^2. $\square$

Given a newform $\phi \in S_2(M_\phi)$ we would like to compute the subspace of $S_{3/2}(N, \chi, \phi)$ for a suitable N with $2M_\phi \mid N$ and χ quadratic that comes from the theta-series of ternary quadratic forms. For a choice of N and character χ_d we use the algorithm in Section 5.1 to compute the subspace of $S_{3/2}(N, \chi_d)$ that comes from the theta-series of ternary quadratic forms. We now apply the above Lemma to compute the Hecke action on this subspace and use the algorithm in Section 3.6 to compute the subspace of $S_{3/2}(N, \chi, \phi)$ coming from ternary quadratic forms. In the upcoming section we will illustrate this algorithm by presenting several examples.

5.3 Examples

For this section, we need to recall the methods used in Section 4.3, in addition to the algorithm mentioned in the previous section.

Example 5.3.1. Let E be an elliptic curve of conductor 50 given by

$$E : Y^2 + XY + Y = X^3 + X^2 - 3X + 1.$$

Let ϕ be the newform corresponding to E,

$$\phi_E : q + q^2 - q^3 + q^4 - q^6 - 2q^7 + q^8 - 2q^9 - 3q^{11} + O(q^{12}).$$

Note that $\nu_2(50) = 1$ and second coefficient of ϕ_E is non-zero, hence ρ_2 is not supercuspidal and so we can apply Waldspurger's Theorem. Please refer to Section 4.3 for notation and details of the calculation.

We get that $\widetilde{N_\phi} = 100$ and $S_{3/2}(100, \chi_{\text{triv}}, \phi_E)$ has a basis consisting of f_1 and f_2 where

$$f_1 = q + q^4 - q^6 - q^{11} - 2q^{14} + O(q^{15}) := \sum_{n=1}^{\infty} a_n q^n$$

$$f_2 = q^2 - q^3 + q^8 - q^{12} + 2q^{13} + O(q^{15}) := \sum_{n=1}^{\infty} b_n q^n.$$

In fact it turns out that $f_1 = (\theta_{Q_1} - \theta_{Q_2})/2$ and $f_2 = (\theta_{Q_3} - \theta_{Q_4})/2$ where Q_i's are quadratic ternary forms of level 50 given by

$$Q_1 = [25, 25, 1, 0, 0, 0], \qquad Q_2 = [14, 9, 6, 4, 6, 2],$$

$$Q_3 = [25, 13, 2, 2, 0, 0], \qquad Q_4 = [17, 17, 3, -2, -2, 16].$$

We have the following proposition which can be now proved on the similar lines as Theorem 4.5.5.

Proposition 5.3.2. *Let E be as above. Let n be positive square-free number such that $5 \nmid n$. Then,*

$$\mathrm{L}(E_{-n}, 1) = \frac{\mathrm{L}(E_{-1}, 1)}{\sqrt{n}} \cdot c_n^2$$

where

$$c_n = \sum_{i=1}^{4} \frac{(-1)^{i-1}}{2} \cdot \#\{(x, y, z) : Q_i(x, y, z) = n\}.$$

Again we can compute the order of $Ш(E_{-n}/\mathbb{Q})$ assuming the BSD. For example, we get that

$$|Ш(E_{-9318}/\mathbb{Q})| = 33^2 = 1089.$$

We can further consider the real quadratic twists E_n. For this we work with the elliptic curve E_{-1} of conductor 400,

$$E_{-1} : Y^2 = X^3 + X^2 - 48X - 172.$$

We can show that if $5 \nmid n$ then,

$$\mathrm{L}(E_n, 1) = \begin{cases} \frac{\mathrm{L}(E_1,1)}{\sqrt{n}} \cdot c_n^2 & \left(\frac{n}{5}\right) = 1 \\ \mathrm{L}(E_{17}, 1) \cdot \sqrt{\frac{17}{n}} \cdot c_n^2 & \left(\frac{n}{5}\right) = -1, \end{cases}$$

where c_n is the n-th coefficient of the following linear combination of theta-series of weight 3/2 and level 1600 coming from the ternary quadratic forms:

$$\begin{aligned}
&-\frac{1}{5}\cdot\theta_{[5,5,17,-2,-4,0]} + \frac{1}{5}\cdot\theta_{[5,9,10,2,2,4]} + \frac{1}{10}\cdot\theta_{[1,4,400,0,0,0]} - \frac{1}{10}\cdot\theta_{[5,17,20,-8,0,-2]} \\
&-\frac{1}{10}\cdot\theta_{[5,17,20,4,4,2]} + \frac{1}{10}\cdot\theta_{[8,13,20,12,8,4]} - \frac{1}{5}\cdot\theta_{[1,32,52,-16,0,0]} + \frac{1}{5}\cdot\theta_{[8,13,17,6,4,4]} \\
&+\frac{1}{10}\cdot\theta_{[4,5,400,0,0,-4]} - \frac{1}{10}\cdot\theta_{[4,16,101,0,-4,0]} + \frac{1}{10}\cdot\theta_{[400,100,1,0,0,0]} \\
&-\frac{1}{10}\cdot\theta_{[125,100,4,0,0,100]} + \frac{1}{5}\cdot\theta_{[89,56,9,-4,-2,-44]} - \frac{1}{5}\cdot\theta_{[49,36,29,24,22,16]} \\
&-\frac{1}{2}\cdot\theta_{[400,13,8,4,0,0]} - \frac{1}{10}\cdot\theta_{[100,25,17,10,0,0]} + \frac{1}{10}\cdot\theta_{[52,32,25,0,0,16]} \\
&+\frac{1}{2}\cdot\theta_{[53,33,25,-10,-10,-14]} + \frac{1}{2}\cdot\theta_{[400,400,1,0,0,0]} + \frac{9}{10}\cdot\theta_{[400,25,16,0,0,0]} \\
&-\frac{1}{2}\cdot\theta_{[201,201,4,4,4,2]} + \frac{1}{10}\cdot\theta_{[224,89,9,-2,-8,-88]} - \frac{1}{10}\cdot\theta_{[209,36,25,20,10,36]} \\
&-\frac{9}{10}\cdot\theta_{[129,100,16,0,-16,-100]} - \frac{4}{5}\cdot\theta_{[84,81,25,10,20,4]} + \frac{4}{5}\cdot\theta_{[89,49,41,-6,-14,-38]} \\
&-\frac{1}{5}\cdot\theta_{[400,29,16,16,0,0]} + \frac{1}{5}\cdot\theta_{[125,100,16,0,0,100]} - \frac{2}{5}\cdot\theta_{[100,96,21,8,20,80]} \\
&+\frac{2}{5}\cdot\theta_{[84,69,29,2,12,28]} - \frac{2}{5}\cdot\theta_{[400,32,13,8,0,0]} + \frac{2}{5}\cdot\theta_{[117,52,32,-16,-24,-44]} \\
&+\frac{1}{5}\cdot\theta_{[400,25,17,10,0,0]} + \frac{1}{5}\cdot\theta_{[212,48,17,8,4,48]} + \frac{1}{10}\cdot\theta_{[208,32,25,0,0,32]}
\end{aligned}$$

$$- \frac{1}{5} \cdot \theta_{[212,33,25,-10,-20,-28]} - \frac{1}{10} \cdot \theta_{[208,33,32,32,32,16]} - \frac{1}{5} \cdot \theta_{[113,52,32,16,8,52]}.$$

Further using the root number arguments, we get that $\mathrm{L}(E_{-5n}, 1) = 0$ whenever $n \not\equiv 3 \pmod 8$ and $\mathrm{L}(E_{5n}, 1) = 0$ whenever $n \equiv 5 \pmod 8$. For the remaining cases, we look at the space $S_{3/2}(8000, \phi_E)$.

Example 5.3.3. This example formulates Theorem 4.5.10 in terms of ternary quadratic forms. Let $E : Y^2 = X^3 - 1$. Let n be positive square-free integer such that $n \equiv 1, 2 \pmod 3$. Then

$$\mathrm{L}(E_{-n}, 1) = \frac{\Omega_{E_{-1}}}{\sqrt{n}} \cdot a_n^2$$

where a_n is the n-th coefficient of the cusp form f of weight $3/2$ and level 576 that can be written as follows as a linear combination theta series:

$$\begin{aligned}
f = \sum_{n=1}^{\infty} a_n q^n = \\
&+ \frac{1}{6} \cdot \theta_{[1,4,144,0,0,0]} - \frac{1}{6} \cdot \theta_{[4,4,37,0,-4,0]} + \frac{1}{6} \cdot \theta_{[4,5,36,0,0,-4]} - \frac{1}{6} \cdot \theta_{[4,13,13,-10,0,0]} \\
&+ \frac{1}{3} \cdot \theta_{[1,20,32,-16,0,0]} + \frac{1}{6} \cdot \theta_{[4,5,29,-2,0,0]} - \frac{1}{2} \cdot \theta_{[4,9,17,-6,0,0]} + \frac{1}{2} \cdot \theta_{[1,36,45,-36,0,0]} \\
&- \frac{1}{2} \cdot \theta_{[4,9,37,0,-4,0]} + \frac{1}{6} \cdot \theta_{[144,16,1,0,0,0]} - \frac{1}{6} \cdot \theta_{[16,16,9,0,0,0]} - \frac{1}{3} \cdot \theta_{[144,5,4,4,0,0]} \\
&+ \frac{1}{6} \cdot \theta_{[37,16,4,0,4,0]} + \frac{1}{6} \cdot \theta_{[16,13,13,10,0,0]} + \frac{1}{6} \cdot \theta_{[32,21,4,-4,0,-16]} - \frac{1}{6} \cdot \theta_{[29,16,5,0,2,0]} \\
&- \frac{1}{2} \cdot \theta_{[144,36,1,0,0,0]} + 1 \cdot \theta_{[144,9,4,0,0,0]} - \frac{1}{2} \cdot \theta_{[45,36,4,0,0,36]} - \frac{1}{6} \cdot \theta_{[144,144,1,0,0,0]} \\
&- \frac{1}{2} \cdot \theta_{[144,16,9,0,0,0]} + \frac{2}{3} \cdot \theta_{[49,36,16,0,-16,-36]} + \frac{1}{4} \cdot \theta_{[144,13,13,10,0,0]} \\
&- \frac{1}{4} \cdot \theta_{[45,36,16,0,0,36]} + \frac{1}{2} \cdot \theta_{[144,29,5,2,0,0]} - \frac{1}{2} \cdot \theta_{[32,29,29,22,16,16]} \\
&- \frac{1}{6} \cdot \theta_{[80,32,9,0,0,32]} + \frac{1}{2} \cdot \theta_{[80,17,17,-2,-16,-16]} - \frac{1}{3} \cdot \theta_{[41,32,20,16,20,8]}.
\end{aligned}$$

Example 5.3.4. Let $E : Y^2 + Y = X^3 - 7$ be an elliptic curve of conductor 27 and let ϕ be the corresponding newform. Using Corollary 4.2.3, we get that ρ_2, the local component of ϕ at 2 is not supercuspidal and hence we can apply

Waldspurger's Theorem. We have the following proposition.

Proposition 5.3.5. *With E as above let n be a square-free integer.*

(i) Suppose $n \equiv 1 \pmod 3$. Let f be given by

$$f = \sum_{n=1}^{\infty} a_n q^n = -\frac{1}{2} \cdot \theta_{[1,6,15,-6,0,0]} + \frac{1}{2} \cdot \theta_{[4,4,7,4,4,2]} + \theta_{[27,27,1,0,0,0]} - \theta_{[28,27,4,0,4,0]} - \frac{1}{2} \cdot \theta_{[27,7,4,2,0,0]} - \frac{1}{2} \cdot \theta_{[16,9,7,-6,-4,-6]} + \theta_{[31,16,7,4,2,16]}.$$

If either $\nu_2(n) = 1$ or, $\nu_2(n) = 0$ and $n \equiv 1, 5 \pmod 8$ then

$$\mathrm{L}(E_{-n}, 1) = \frac{\mathrm{L}(E_{-1}, 1)}{\sqrt{n}} \cdot a_n^2.$$

Otherwise,

$$\mathrm{L}(E_{-n}, 1) = \frac{\kappa}{\sqrt{n}} \cdot a_n^2$$

where $\kappa = \sqrt{19} \cdot \mathrm{L}(E_{-19}, 1)$ if $n \equiv 3 \pmod 8$ and $\kappa = \sqrt{7} \cdot \mathrm{L}(E_{-7}, 1)$ if $n \equiv 7 \pmod 8$.

(ii) Suppose $n \equiv 0 \pmod 3$ and let $n = 3m$. Let $h \in S_{3/2}(324, \chi_{\mathrm{triv}}, \phi)$ be the cusp form having the following q-expansion

$$h = q^3 - q^{21} + 2q^{30} - q^{39} - 2q^{48} - q^{57} - 2q^{66} + q^{75} + O(q^{80}) := \sum_{n=1}^{\infty} b_n q^n.$$

Further suppose $\left(\frac{m}{3}\right) = 1$. If either $\nu_2(n) = 1$ or, $\nu_2(n) = 0$ and $n \equiv 1$, $5 \pmod 8$ then

$$\mathrm{L}(E_{-n}, 1) = \mathrm{L}(E_{-21}, 1) \cdot \sqrt{\frac{21}{n}} \cdot b_n^2.$$

If $n \equiv 3, 7 \pmod 8$ then

$$\mathrm{L}(E_{-n}, 1) = \frac{\kappa}{\sqrt{n}} \cdot b_n^2$$

where $\kappa = \sqrt{3} \cdot \mathrm{L}(E_{-3}, 1)$ if $n \equiv 3 \pmod 8$ and $\kappa = \sqrt{39} \cdot \mathrm{L}(E_{-39}, 1)$ if $n \equiv 7 \pmod 8$.

(iii) If $n = 3m$ *and* $\left(\frac{m}{3}\right) = -1$ *then* $\mathrm{L}(E_{-n}, 1) = 0$.

(iv) If $n \equiv 2 \pmod 3$ *then* $\mathrm{L}(E_{-n}, 1) = 0$.

The proof of (i) and (ii) follows as in the previous examples, while for (iii) and (iv) one can use root number arguments. We point out that the cusp form h which appears in (ii) does not come from ternary quadratic forms. Moreover since E is isogenous to E_{-3}, for n positive square-free $\mathrm{L}(E_n, 1) = \mathrm{L}(E_{-3n}, 1)$. Thus using above proposition we are able to compute the critical values $\mathrm{L}(E_n, 1)$ for all n square-free.

Given a rational elliptic curve E of level N odd and square-free, Böcherer and Schulze-Pillot [4] showed that an inverse Shimura lift of ϕ_E comes from ternary quadratic forms if and only if $\mathrm{L}(E, 1) \neq 0$.

In each of the above examples, the level is not odd and square-free but the result of Böcherer and Schulze-Pillot still holds.

Appendix A

Tables

A.1 Dimensions

In the following table we give the dimension of the space $S_{3/2}(N)$ of cusp forms of weight 3/2, level N and trivial character, for $1 \leq N \leq 2000$ with $4 \mid N$. We compare it with the dimensions of the subspaces $S_0(N)$ and $\Theta(N)$, the latter being the subspace spanned by theta-series of positive-definite ternary quadratic forms, and with the intersection

$$\Theta_0(N) := S_0(N) \cap \Theta(N).$$

Table A.1: Dimensions of Theta Subspace

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
4	0	0	0	0
8	0	0	0	0
12	0	0	0	0
16	0	0	0	0
20	0	0	0	0
24	0	0	0	0
28	1	0	1	0
32	0	0	0	0
36	0	0	0	0

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Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
40	1	0	1	0
44	2	0	2	0
48	0	0	0	0
52	2	0	2	0
56	2	0	2	0
60	3	0	3	0
64	1	1	1	1
68	3	0	3	0
72	2	0	2	0
76	4	0	4	0
80	2	0	2	0
84	5	0	4	0
88	4	0	3	0
92	5	0	5	0
96	2	0	2	0
100	2	0	2	0
104	5	0	5	0
108	5	1	5	1
112	4	0	4	0
116	6	0	5	0
120	7	0	5	0
124	7	0	7	0
128	3	1	3	1
132	9	0	7	0
136	7	0	6	0
140	9	0	8	0
144	4	0	4	0
148	8	0	6	0
152	8	0	7	0
156	11	0	9	0
160	6	0	6	0

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Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
164	9	0	8	0
168	11	0	8	0
172	10	0	8	0
176	8	0	8	0
180	10	0	9	0
184	10	0	7	0
188	11	0	11	0
192	7	1	5	0
196	6	0	5	0
200	8	0	8	0
204	15	0	11	0
208	10	0	9	0
212	12	0	9	0
216	11	1	8	0
220	15	0	13	0
224	10	0	10	0
228	17	0	11	0
232	13	0	9	0
236	14	0	13	0
240	14	0	12	0
244	14	0	11	0
248	14	0	11	0
252	18	0	16	0
256	8	2	7	2
260	17	0	12	0
264	19	0	13	0
268	16	0	12	0
272	14	0	13	0
276	21	0	15	0
280	19	0	13	0
284	17	0	15	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
288	12	0	12	0
292	17	0	13	0
296	17	0	12	0
300	20	0	16	0
304	16	0	15	0
308	21	0	15	0
312	23	0	14	0
316	19	0	15	0
320	15	1	15	1
324	15	1	10	1
328	19	0	14	0
332	20	0	17	0
336	22	0	17	0
340	23	0	15	0
344	20	0	14	0
348	27	0	20	0
352	18	0	14	0
356	21	0	17	0
360	26	0	20	0
364	25	0	18	0
368	20	0	18	0
372	29	0	18	0
376	22	0	16	0
380	27	0	21	0
384	19	1	15	0
388	23	0	17	0
392	18	0	16	0
396	30	0	24	0
400	16	0	14	0
404	24	0	19	0
408	31	0	18	0

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Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
412	25	0	21	0
416	22	0	22	0
420	41	0	24	0
424	25	0	16	0
428	26	0	19	0
432	22	2	19	2
436	26	0	17	0
440	31	0	21	0
444	35	0	25	0
448	23	1	18	0
452	27	0	18	0
456	35	0	20	0
460	33	0	24	0
464	26	0	21	0
468	34	0	28	0
472	28	0	20	0
476	33	0	27	0
480	34	0	24	0
484	20	0	10	0
488	29	0	20	0
492	39	0	24	0
496	28	0	25	0
500	28	0	21	0
504	38	0	27	0
508	31	0	23	0
512	21	3	19	3
516	41	0	25	0
520	37	0	22	0
524	32	0	26	0
528	38	0	28	0
532	37	0	24	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
536	32	0	21	0
540	44	1	35	0
544	30	0	26	0
548	33	0	22	0
552	43	0	25	0
556	34	0	25	0
560	38	0	31	0
564	45	0	27	0
568	34	0	21	0
572	39	0	31	0
576	30	2	25	2
580	41	0	27	0
584	35	0	24	0
588	42	0	27	0
592	34	0	24	0
596	36	0	25	0
600	44	0	28	0
604	37	0	27	0
608	34	0	30	0
612	46	0	36	0
616	43	0	26	0
620	45	0	31	0
624	46	0	33	0
628	38	0	27	0
632	38	0	23	0
636	51	0	34	0
640	35	1	30	1
644	45	0	31	0
648	39	1	28	0
652	40	0	26	0
656	38	0	32	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
660	65	0	35	0
664	40	0	27	0
668	41	0	33	0
672	50	0	36	0
676	30	0	13	0
680	49	0	28	0
684	54	0	43	0
688	40	0	30	0
692	42	0	29	0
696	55	0	31	0
700	50	0	38	0
704	39	1	31	0
708	57	0	32	0
712	43	0	29	0
716	44	0	32	0
720	52	0	42	0
724	44	0	30	0
728	51	0	31	0
732	59	0	36	0
736	42	0	30	0
740	53	0	34	0
744	59	0	31	0
748	51	0	35	0
752	44	0	39	0
756	62	1	46	1
760	55	0	31	0
764	47	0	37	0
768	44	2	30	0
772	47	0	29	0
776	47	0	31	0
780	77	0	44	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
784	36	0	28	0
788	48	0	30	0
792	62	0	41	0
796	49	0	37	0
800	40	0	38	0
804	65	0	37	0
808	49	0	32	0
812	57	0	38	0
816	62	0	42	0
820	59	0	35	0
824	50	0	34	0
828	66	0	51	0
832	47	1	40	1
836	57	0	39	0
840	85	0	41	0
844	52	0	33	0
848	50	0	36	0
852	69	0	41	0
856	52	0	30	0
860	63	0	45	0
864	52	2	42	0
868	61	0	39	0
872	53	0	30	0
876	71	0	40	0
880	62	0	48	0
884	59	0	38	0
888	71	0	39	0
892	55	0	40	0
896	51	1	42	0
900	64	0	36	0
904	55	0	31	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
908	56	0	40	0
912	70	0	43	0
916	56	0	38	0
920	67	0	40	0
924	89	0	51	0
928	54	0	38	0
932	57	0	35	0
936	74	0	51	0
940	69	0	42	0
944	56	0	48	0
948	77	0	44	0
952	67	0	40	0
956	59	0	45	0
960	75	1	49	0
964	59	0	39	0
968	50	0	32	0
972	66	2	51	2
976	58	0	43	0
980	66	0	43	0
984	79	0	40	0
988	67	0	45	0
992	58	0	46	0
996	81	0	44	0
1000	63	0	42	0
1004	62	0	47	0
1008	76	0	62	0
1012	69	0	42	0
1016	62	0	37	0
1020	101	0	56	0
1024	46	4	31	4
1028	63	0	42	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
1032	83	0	43	0
1036	73	0	45	0
1040	74	0	49	0
1044	82	0	59	0
1048	64	0	40	0
1052	65	0	47	0
1056	82	0	56	0
1060	77	0	45	0
1064	75	0	43	0
1068	87	0	47	0
1072	64	0	45	0
1076	66	0	45	0
1080	92	1	57	0
1084	67	0	49	0
1088	63	1	54	1
1092	105	0	54	0
1096	67	0	38	0
1100	80	0	57	0
1104	86	0	57	0
1108	68	0	41	0
1112	68	0	41	0
1116	90	0	70	0
1120	82	0	56	0
1124	69	0	45	0
1128	91	0	44	0
1132	70	0	47	0
1136	68	0	53	0
1140	113	0	59	0
1144	79	0	47	0
1148	81	0	56	0
1152	70	2	60	2

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
1156	56	0	20	0
1160	85	0	48	0
1164	95	0	52	0
1168	70	0	50	0
1172	72	0	48	0
1176	90	0	49	0
1180	87	0	54	0
1184	70	0	52	0
1188	98	1	68	0
1192	73	0	42	0
1196	81	0	57	0
1200	88	0	62	0
1204	85	0	49	0
1208	74	0	42	0
1212	99	0	56	0
1216	71	1	56	0
1220	89	0	51	0
1224	98	0	65	0
1228	76	0	51	0
1232	86	0	59	0
1236	101	0	54	0
1240	91	0	49	0
1244	77	0	59	0
1248	98	0	60	0
1252	77	0	50	0
1256	77	0	48	0
1260	130	0	84	0
1264	76	0	54	0
1268	78	0	46	0
1272	103	0	52	0
1276	87	0	56	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
1280	76	2	60	2
1284	105	0	58	0
1288	91	0	51	0
1292	87	0	54	0
1296	78	2	54	2
1300	92	0	61	0
1304	80	0	45	0
1308	107	0	62	0
1312	78	0	58	0
1316	93	0	59	0
1320	133	0	59	0
1324	82	0	51	0
1328	80	0	63	0
1332	106	0	74	0
1336	82	0	49	0
1340	99	0	65	0
1344	107	1	71	0
1348	83	0	50	0
1352	72	0	44	0
1356	111	0	62	0
1360	98	0	60	0
1364	93	0	62	0
1368	110	0	72	0
1372	86	1	60	1
1376	82	0	60	0
1380	137	0	65	0
1384	85	0	49	0
1388	86	0	54	0
1392	110	0	73	0
1396	86	0	55	0
1400	104	0	63	0

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Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
1404	116	1	85	1
1408	83	1	57	0
1412	87	0	56	0
1416	115	0	54	0
1420	105	0	65	0
1424	86	0	66	0
1428	137	0	70	0
1432	88	0	50	0
1436	89	0	64	0
1440	116	0	90	0
1444	72	0	25	0
1448	89	0	52	0
1452	110	0	59	0
1456	102	0	68	0
1460	107	0	61	0
1464	119	0	57	0
1468	91	0	63	0
1472	87	1	61	0
1476	118	0	80	0
1480	109	0	59	0
1484	105	0	66	0
1488	118	0	68	0
1492	92	0	56	0
1496	103	0	60	0
1500	133	0	81	0
1504	90	0	66	0
1508	101	0	61	0
1512	128	1	79	0
1516	94	0	57	0
1520	110	0	76	0
1524	125	0	66	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
1528	94	0	54	0
1532	95	0	69	0
1536	101	3	67	1
1540	137	0	73	0
1544	95	0	51	0
1548	126	0	86	0
1552	94	0	65	0
1556	96	0	59	0
1560	157	0	71	0
1564	105	0	71	0
1568	84	0	72	0
1572	129	0	67	0
1576	97	0	51	0
1580	117	0	73	0
1584	124	0	93	0
1588	98	0	60	0
1592	98	0	58	0
1596	153	0	80	0
1600	90	2	64	2
1600	90	2	64	2
1604	99	0	61	0
1608	131	0	64	0
1612	109	0	64	0
1616	98	0	74	0
1620	135	1	87	0
1624	115	0	62	0
1628	111	0	71	0
1632	130	0	76	0
1636	101	0	64	0
1640	121	0	61	0
1644	135	0	72	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
1648	100	0	76	0
1652	117	0	69	0
1656	134	0	82	0
1660	123	0	72	0
1664	99	1	85	1
1668	137	0	68	0
1672	115	0	65	0
1676	104	0	72	0
1680	170	0	93	0
1684	104	0	62	0
1688	104	0	54	0
1692	138	0	99	0
1696	102	0	68	0
1700	122	0	81	0
1704	139	0	67	0
1708	121	0	71	0
1712	104	0	71	0
1716	161	0	83	0
1720	127	0	68	0
1724	107	0	75	0
1728	115	5	88	3
1732	107	0	67	0
1736	123	0	67	0
1740	173	0	83	0
1744	106	0	66	0
1748	117	0	72	0
1752	143	0	66	0
1756	109	0	73	0
1760	130	0	88	0
1764	132	0	66	0
1768	121	0	65	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
1772	110	0	68	0
1776	142	0	90	0
1780	131	0	70	0
1784	110	0	65	0
1788	147	0	82	0
1792	108	2	75	0
1796	111	0	67	0
1800	148	0	96	0
1804	123	0	78	0
1808	110	0	70	0
1812	149	0	76	0
1816	112	0	64	0
1820	161	0	93	0
1824	146	0	86	0
1828	113	0	68	0
1832	113	0	65	0
1836	152	1	103	0
1840	134	0	89	0
1844	114	0	75	0
1848	181	0	80	0
1852	115	0	73	0
1856	111	1	84	1
1860	185	0	87	0
1864	115	0	59	0
1868	116	0	77	0
1872	148	0	112	0
1876	133	0	73	0
1880	139	0	69	0
1884	155	0	84	0
1888	114	0	82	0
1892	129	0	71	0

Continued on next page

Table A.1 – continued from previous page

Level N	**Dim** $S_{3/2}(N)$	**Dim** $S_0(N)$	**Dim** $\Theta(N)$	**Dim** $\Theta_0(N)$
1896	155	0	73	0
1900	140	0	97	0
1904	134	0	97	0
1908	154	0	105	0
1912	118	0	66	0
1916	119	0	87	0
1920	163	1	97	0
1924	129	0	76	0
1928	119	0	67	0
1932	185	0	90	0
1936	100	0	58	0
1940	143	0	77	0
1944	138	2	85	0
1948	121	0	75	0
1952	118	0	84	0
1956	161	0	81	0
1960	146	0	81	0
1964	122	0	80	0
1968	158	0	91	0
1972	131	0	80	0
1976	135	0	75	0
1980	202	0	130	0
1984	119	1	86	0
1988	141	0	83	0
1992	163	0	73	0
1996	124	0	73	0
2000	126	0	91	0

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