Gorenstein rings and Kustin–Miller unprojection

Stavros Papadakis

A thesis submitted to the University of Warwick
for the degree of Doctor of Philosophy.

Mathematics Institute
University of Warwick

August 2001

Contents

Acknowledgements

It is a pleasure to express my deep gratitude to my supervisor Miles Reid, whose generosity in supplying guidance and ideas contributed immensely in the present work. His continued patient advice, encouragement and friendliness has been greatly appreciated during the past four years of my studies at Warwick.

Of the many other people who helped me, I am especially grateful to David Mond for useful discussions and encouragement.

I am very grateful to Greek State Scholarships Foundation (IKY) for providing the financial support during my research at Warwick, and to the Mathematics Institute for giving me the opportunity to study in a stimulating environment.

Part of this work was carried out at RIMS, Kyoto University during September-December 2000. I thank RIMS for providing excellent hospitality and financial support.

I am also thankful to my family for their support.

This thesis is dedicated to Bob Dylan; thank you for the poetry and the songs Bob!

Declaration

Section 2.3 and most results of Section 2.1 are joint research [PR] with M. Reid. Unless otherwise stated, the rest of the thesis is, to the best of my knowledge, original personal research.

Summary

Chapter 1 briefly describes the motivation for the thesis and presents some background material.

Chapter 2 develops the foundations of the theory of unprojection in the local and projective settings.

Chapter 3 develops methods that calculate the unprojection ring for two important families of unprojection, Tom & Jerry.

Finally, Chapter 4 proves some algebraic results concerning Catanese's rank condition for symmetric matrices of small size.

Chapter 1

Introduction

1.1 Introduction

Gorenstein rings appear often in algebraic geometry. For example, the anti-canonical ring

$$R = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(-mK_X))$$

of a (smooth, just for simplicity) Fano n-fold and the canonical ring

$$R = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

of a (smooth) regular surface of general type are Gorenstein. Another example is the ring

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$$

associated to an ample divisor D on a (smooth) K3 surface.

If $R = k[x_1, \ldots, x_n]/I$ is a Gorenstein graded ring, quotient of a polynomial ring, of codimension at most three, then there are good structure theorems. Serre proved that if the codimension is at most two then R is a complete intersection, while in 1977 Buchsbaum and Eisenbud [BE] proved a more general version of the following theorem. For generalities about Pfaffians see Section 1.3; for a proof see e.g. [BE] or [BH] Section 3.4.

Theorem 1.1.1 (Buchsbaum and Eisenbud [BE]) *Let R be a polynomial ring over a field, and $I \subset R$ a homogeneous ideal of codimension three* *such that R/I is a Gorenstein ring. Then I is generated by the $2n \times 2n$ Pfaffians of a skewsymmetric $(2n+1) \times (2n+1)$ matrix M with entries in R. Conversely, assume $I \subset R$ is a (not necessarily homogeneous) ideal of codimension 3 generated by the $2n \times 2n$ Pfaffians of a skewsymmetric $(2n+1) \times (2n+1)$ matrix M. Then the ring R/I is Gorenstein.*

In the 1980s, Kustin and Miller attacked the problem of finding a structure theorem for Gorenstein codimension four with a series of papers [KM1]-[KM6], [JKM]. Unfortunately they were not successful, although they managed to classify their Tor algebras [KM6], and get information about their Poincaré series [JKM]. Moreover, in [KM4] they introduced a procedure which constructs more complicated Gorenstein rings from simpler ones by increasing the codimension.

Some years later Altınok [Al] and Reid [R0] rediscovered what was essentially the same procedure while working with Gorenstein rings arising from K3s and 3-folds. The important observation is that under extra conditions which are reasonable in birational geometry this procedure corresponds to a contraction of a (Weil) divisor, possibly after a factorialization which would allow the divisor to be contractible (compare Sections 2.3, 2.4 and [R1] for examples). Therefore it is a modern and explicit version of Castelnuovo contractibility. It has found many applications in algebraic geometry, for example in the birational geometry of Fanos [CPR] and [CM], in the construction of weighted complete intersection K3s and Fanos [Al], and in the study of Mori flips [BrR]. [R1] contains more details about these and other applications.

The main topic of the present work is the study of some of the algebraic aspects of this procedure that we call *Kustin–Miller unprojection*, or for simplicity just *unprojection*.

1.2 Structure of thesis

The structure of the present work is as follows.

Chapter 2 is about the foundation of unprojection. In Section 2.1 we define it in the local setting (Definiton 2.1.3), and prove the fundamental result that it is Gorenstein (Theorem 2.1.10). In Sections 2.2 to 2.4 we give the formulation of unprojection in the setting of projective and birational geometry and some examples. In Section 2.5 we present a method, originally developed by Kustin and Miller [KM4], prove that it calculates the equations of the unprojection (Theorem 2.5.2), and give an application that generalises a calculation of [CFHR]. Finally, in Section 2.6 we discuss possible generalisations.

In Chapter 3 we study Tom and Jerry. These are two families of Gorenstein codimension four rings arising as unprojections, originally defined and named by Reid. The main results are Theorems 3.5.2 and 3.10.2 where we calculate their equations using multilinear and homological algebra. In addition, in Sections 3.6 and 3.11 we give relative Maple algorithms, and in Section 3.7 we present a combinatorial procedure which, conjecturally, also calculates Tom.

Finally, in Chapter 4 we study the algebra of Catanese's Rank Condition for 'generic' symmetric matrices of small size (Lemma 4.1.2, Theorem 4.1.5) and relate it with the unprojection (Example 4.1.7, Remark 4.1.8).

1.3 Notation

Unless otherwise mentioned all rings are commutative and with unit. By abuse of notation, when s is an element of a commutative ring R we sometimes write R/s for the quotient of R by the principal ideal (s) generated by s.

Radical If I is an ideal of a ring R we define the radical of I to be the ideal

$$\operatorname{Rad} I = \big\{a \in R : a^n \in I \text{ for all sufficiently large } n\big\}.$$

It is equal to the intersection of all prime ideals of R containing I.

Codimension Assume $I \subset R$ is an ideal with $I \neq R$, and set

$$V(I) = \big\{p \in \operatorname{Spec} R : I \subseteq p\big\}.$$

Following Eisenbud [Ei], we define the *codimension* of I in R to be the minimum of $\dim R_p$ for $p \in V(I)$. Many authors use the term height of I for the same notion.

Grade Assume R is a Noetherian ring and $I \subset R$ an ideal with $I \neq R$. The common length of all maximal R-sequences contained in I will be called the *grade* of I. The basic inequality is that the grade of I is less than or equal the codimension of I, see e.g. [BH] Section 1.2.

Depth Assume R is a Noetherian local ring with maximal ideal m, and N is a finite R-module. The common length of all maximal N-sequences contained in m will be called the *depth* of N.

Cohen–Macaulay rings A local Noetherian ring R is called Cohen–Macaulay if the depth of R as R-module is equal to the dimension of R. More generally, a Noetherian ring R is called Cohen–Macaulay, if for every maximal ideal m of R the localisation R_m is Cohen–Macaulay.

Gorenstein rings A local Noetherian ring R is called Gorenstein if it is Cohen–Macaulay, the dualising module ω_R exists, and ω_R is isomorphic to R as R-modules. More generally, a Noetherian ring R is called Gorenstein, if for every maximal ideal m of R the localisation R_m is Gorenstein. There are many equivalent characterizations of Gorenstein rings, see for example [M].

Pfaffians Assume $A = [a_{ij}]$ is a $k \times k$ skewsymmetric (i.e., $a_{ji} = -a_{ij}$ and $a_{ii} = 0$) matrix with entries in a Noetherian ring R.

For k even we define a polynomial $\mathrm{Pf}(A)$ in a_{ij} called the *Pfaffian* of A by induction on k. If $k = 2$ we set

$$\mathrm{Pf}(\begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}) = a_{12}.$$

For even $k \geq 4$ we define

$$\mathrm{Pf}(A) = \sum_{j=2}^{k} (-1)^j a_{1j} \, \mathrm{Pf}(A_{1j}),$$

where A_{1j} is the skewsymmetric submatrix of A obtained by deleting the first and the jth rows and the first and the jth columns of A. An interesting property is that

$$(\mathrm{Pf}(A))^2 = \det A.$$

Now assume that $k = 2l + 1$ is odd. In the present work, by *Pfaffians* of A we mean the set

$$\{\, \mathrm{Pf}(A_1), \mathrm{Pf}(A_2), \ldots, \mathrm{Pf}(A_k) \},$$

where for $1 \leq i \leq k$ we denote by A_i the skewsymmetric submatrix of A obtained by deleting the ith row and and the ith column of A. Moreover, there is a complex **L**:

$$0 \to R \xrightarrow{B_3} R^k \xrightarrow{B_2} R^k \xrightarrow{B_1} R \to 0 \tag{1.1}$$

associated to A, with $B_2 = A$, B_1 the $1 \times k$ matrix with ith entry equal to $(-1)^{i+1}\,\mathrm{Pf}(A_i)$ and B_3 the transpose matrix of B_1. We have the following theorem due to Eisenbud and Buchsbaum [BE].

Theorem 1.3.1 *Let R be a Noetherian ring, $k = 2l + 1$ an odd integer and A a skewsymmetric $k \times k$ matrix with entries in R. Denote by I the ideal generated by the Pfaffians of A. Assume that $I \neq R$ and the grade of I is three, the maximal possible. Then the complex* **L** *defined in (1.1) is acyclic, in the sense that the complex*

$$0 \to R \xrightarrow{B_3} R^k \xrightarrow{B_2} R^k \xrightarrow{B_1} R \to R/I \to 0$$

is exact. Moreover, if R is Gorenstein then the same is true for R/I.

For more details about Pfaffians and a proof of the theorem see e.g. [BE] or [BH] Section 3.4.

Chapter 2

Theory of unprojection

2.1 Local unprojection

Let $X = \operatorname{Spec} \mathcal{O}_X$ be a Gorenstein local scheme and $I \subset \mathcal{O}_X$ an ideal defining a subscheme $D = V(I) \subset X$ that is also Gorenstein and has codimension one in X. We assume in this section that all schemes are Noetherian. We do not assume anything else about the singularities of X and D, although an important case in applications is when X is normal and D a Weil divisor.

Since X is Cohen–Macaulay, the adjunction formula (compare [R2], p. 708 or [AK], p. 6) gives

$$\omega_D = \operatorname{Ext}^1(\mathcal{O}_D, \omega_X).$$

To calculate the Ext, we Hom the exact sequence $0 \to I \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ into ω_X, giving the usual adjunction exact sequence

$$0 \to \omega_X \to \operatorname{Hom}(I_D, \omega_X) \xrightarrow{\operatorname{res}_D} \omega_D \to 0, \tag{2.1}$$

where res_D is the residue map. For example, in the case that X is normal and D a divisor, the second map is the standard Poincaré residue map $\mathcal{O}_X(K_X + D) \to \mathcal{O}_D(K_D)$.

Lemma 2.1.1 *The $\mathcal{O}_X$-module* $\operatorname{Hom}(I, \omega_X)$ *is generated by two elements i and s, where i is a basis of ω_X and $s \in \operatorname{Hom}(I, \omega_X)$ satisfies*

(i) $s \colon I \to \omega_X$ is injective;

(ii) $\overline{s} = \operatorname{res}_D(s)$ is a basis of ω_D.

Proof Choose bases $i \in \omega_X$, $\overline{s} \in \omega_D$ and any lift $s \mapsto \overline{s}$. Using i we identify $\omega_X \cong \mathcal{O}_X$. Then everything holds except (i). I contains a regular element w (in fact $\operatorname{grade} I = \operatorname{codim} D = 1$, by [M] Theorem 17.4). We claim that there exists $f \in \mathcal{O}_X$ such that $s + fj \colon I \to \mathcal{O}_X$ is injective. It is enough to find f such that $s(w) + fw$ is a regular element of $\mathcal{O}_X$. Let $P_1, \dots, P_l$ be the associated primes of $\mathcal{O}_X$ ordered so that $s(w) \in P_i$ for $i < d$ and $s(w) \notin P_i$ for $i \geq d$. Because $\mathcal{O}_X$ is Cohen–Macaulay (unmixed) they are all minimal. Prime avoidance ([Ei] Lemma 3.3) gives an element f with $f \in P_i$ for $i \geq d$ and $f \notin P_i$ for $i < d$. Then $s(w) + fw$ is regular. QED

We view s as defining an isomorphism $I \to J$, where $J \subset \omega_X = \mathcal{O}_X$ is another ideal. Choose a set of generators $f_1, \dots, f_k$ of I and write $s(f_i) = g_i$ for the corresponding generators of J. We view $s = g_i/f_i$ as a rational function having I as ideal of denominators and J as ideal of numerators. Unprojection is simply the graph of s.

Remark 2.1.2 The total ring of fractions $K(X)$ is defined as $S^{-1}\mathcal{O}_X$ where S is the set of non–zerodivisors, that is, the complement of the union of the associated primes $P_i \in \operatorname{Ass} \mathcal{O}_X$. Then $s \colon I \to J$ is multiplication by an invertible rational function in $K(X)$. For I contains a regular element w (see the proof of Lemma 2.1.1), and

$$t = s(w)/w \in K(X)$$

is independent of the choice of w, because

$$0 = s(w_1 w_2 - w_2 w_1) = w_1 s(w_2) - w_2 s(w_1) \quad \text{for } w_1, w_2 \in I.$$

Moreover,

$$I = \{a \in \mathcal{O}_X : at \in \mathcal{O}_X\}. \tag{2.2}$$

Indeed, assume $at \in \mathcal{O}_X$ for some $a \in \mathcal{O}_X$. Then

$$0 = \operatorname{res}_D(as) = a \operatorname{res}_D(s) \in \omega_D,$$

so $a \in I$.

Definition 2.1.3 Let S be an indeterminate. The *unprojection ring* of D in X is the ring $\mathcal{O}_X[s] = \mathcal{O}_X[S]/(Sf_i - g_i)$; the *unprojection* of D in X is its Spec, that is,

$$Y = \operatorname{Spec} \mathcal{O}_X[s].$$

Clearly, Y is simply the subscheme of $\operatorname{Spec} \mathcal{O}_X[S] = \mathbb{A}^1_X$ defined by the ideal $(Sf_i - g_i)$. Usually Y is no longer local, see Example 2.1.8.

Remark 2.1.4 Clearly $J = \mathcal{O}_X$ if and only if I is principal. We exclude this case in what follows.

Remark 2.1.5 We only choose generators for ease of notation here. The ideal defining Y could be written $\{Sf - s(f) : f \in I\}$. The construction is independent of s: the only choice in Lemma 2.1.1 is $s \mapsto us + hi$ with $u, h \in \mathcal{O}_X$ and u a unit (here we use that $\mathcal{O}_X$ is local), which just gives the affine linear coordinate change $S \mapsto uS + h$ in $\mathbb{A}^1_X$.

Remark 2.1.6 We recall a number of standard facts about valuations of a normal domain. For details see e.g. [Bour] Section VI. Suppose $\mathcal{O}_X$ is a normal domain with field of fractions $K(X)$. For each prime ideal p of $\mathcal{O}_X$ of codimension one, the local ring $\mathcal{O}_{X,p}$ is a discrete valuation domain. Therefore, there exists natural valuation map

$$v_p \colon K(X)^* \to \mathbb{Z}$$

satisfying

$$\mathcal{O}_{X,p} = \{a \in K(X) : v_p(a) \geq 0\} \tag{2.3}$$

and

$$\mathcal{O}_X = \bigcap_p \mathcal{O}_{X,p} \tag{2.4}$$

the intersection over all prime ideals p of codimension one.

The following lemma is also contained in [N3], p. 39.

Lemma 2.1.7 *Define $\mathcal{O}_X[I^{-1}]$ to be the subring of $k(X)$ generated by $\mathcal{O}_X$ and $t = s(w)/w$. If X is normal and integral then the natural map*

$$\mathcal{O}_X[s] \to \mathcal{O}_X[I^{-1}]$$

with $s \mapsto t$ is an isomorphism of $\mathcal{O}_X$-algebras.

Proof Let S be an indeterminate and consider the surjective map

$$\phi\colon \mathcal{O}_X[S] \to \mathcal{O}_X[I^{-1}]$$

with $S \mapsto t$. To prove the lemma it is enough to show

$$\ker(\phi) \subseteq (Sf_i - g_i).$$

A first remark is that if

$$(a_n t^n + \cdots + a_1 t + a_0)t \in \mathcal{O}_X$$

with $a_i \in \mathcal{O}_X$, then $a_n t^n + \cdots + a_1 t + a_0 \in \mathcal{O}_X$. Indeed, if that is not true by (2.4) there exists codimension one prime p with $v_p(a_n t^n + \cdots + a_1 t + a_0) < 0$. It follows $v_p(t) < 0$, so $v_p((a_n t^n + \cdots + a_1 t + a_0)t) < 0$, a contradiction.

Assume now that $f(S) = a_n S^n + \cdots + a_0 \in \ker(\phi)$. Using induction on the degree of $f(S)$ we prove $f(S) \in (Sf_i - g_i)$.

If $f(S)$ is linear in S this follows from (2.2). Assume the result is true for all degrees less than $n+1$, and suppose

$$a_{n+1} S^{n+1} + \cdots + a_1 S + a_0 \in \ker \phi.$$

Then

$$tr = -a_0 \in \mathcal{O}_X,$$

where $r = a_{n+1} t^n + a_n t^{n-1} \cdots + a_1$. By what we said above $r \in \mathcal{O}_X$. Using the case $n = 1$ there exist $q_i \in \mathcal{O}_X[S]$ with

$$r = \sum q_i f_i,$$

so

$$-a_0 = tr = \sum q_i g_i.$$

Using the inductive hypothesis we can find $p_i \in \mathcal{O}_X[S]$ with

$$a_{n+1} S^n + \cdots + (a_1 - r) = \sum p_i (Sf_i - g_i).$$

Then

$$\begin{aligned} a_{n+1} S^{n+1} + \ldots \quad + \quad & a_1 S + a_0 = S(a_{n+1} S^n + \cdots + a_1) + a_0 \\ = \quad & S(r + \sum p_i (Sf_i - g_i)) - \sum q_i g_i \\ = \quad & \sum (Sp_i + q_i)(Sf_i - g_i), \end{aligned}$$

which finishes the proof of the lemma. QED

Example 2.1.8 The lemma is not true without the normality assumption, as the example $X =$ nodal curve, $D =$ reduced origin implies. Set $X : (x^2 - y^2 = 0)$ and $D : (x = y = 0)$. Then $t = x/y$ is an automorphism of $I = J = m$, and $Y \to X$ is an affine blowup, with an exceptional $\mathbb{A}^1$ over the node. Also $t^2 = 1$, so $S^2 - 1 \in \ker(\phi)$, but clearly $S^2 - 1$ is not in the ideal $(Sx - y, Sy - x)$. Y is not local, even if we assume that X is the Spec of the local ring of the nodal curve at the origin.

Lemma 2.1.9 *Write $N = V(J) \subset X$ for the subscheme with $\mathcal{O}_N = \mathcal{O}_X/J$.*

(a) No component of X is contained in N.

(b) Every associated prime of $\mathcal{O}_N$ has codimension 1.

If X is normal then D and N are both divisors, with $\operatorname{div} s = N - D$. More generally, set $n = \dim X$; then (a) says that $\dim N \leq n - 1$, and (b) says that $\dim N = n - 1$ (and has no embedded primes).

Proof I contains a regular element $w \in \mathcal{O}_X$. Then $v = s(w) \in J$ is again regular (obvious), and (a) follows.

Note first that $vI = wJ$. We prove that every element of $\operatorname{Ass}(\mathcal{O}_X/vI) = \operatorname{Ass}(\mathcal{O}_X/wJ)$ is a codimension 1 prime; the lemma follows, since $\operatorname{Ass}(\mathcal{O}_X/J) = \operatorname{Ass}(w\mathcal{O}_X/wJ) \subset \operatorname{Ass}(\mathcal{O}_X/wJ)$. Clearly,

$$\operatorname{Ass}(\mathcal{O}_X/vI) \ \subset \ \operatorname{Ass}(\mathcal{O}_X/I) \cup \operatorname{Ass}(I/vI).$$

For any $P \in \operatorname{Ass}(I/vI)$, choose $x \in I$ with $P = (vI : x) = \operatorname{Ann}(\overline{x} \in I/vI)$. One sees that

$$\begin{cases} x \in \mathcal{O}_X v \implies P \in \operatorname{Ass}(\mathcal{O}_X/I), \\ x \notin \mathcal{O}_X v \implies P \subset Q \text{ for some } Q \in \operatorname{Ass}(\mathcal{O}_X/v\mathcal{O}_X). \end{cases}$$

Since every associated prime of $\mathcal{O}_X/v\mathcal{O}_X$ has codimension 1, this gives

$$\operatorname{Ass}(\mathcal{O}_X/vI) \ \subset \ \operatorname{Ass}(\mathcal{O}_X/I) \cup \operatorname{Ass}(\mathcal{O}_X/v\mathcal{O}_X).$$

QED

Theorem 2.1.10 (Kustin and Miller [KM4]) *The element $s \in \mathcal{O}_X[s]$ is regular, and the ring $\mathcal{O}_X[s]$ is Gorenstein.*

Proof

Step 1 We first prove that

$$S\mathcal{O}_X[S] \cap (Sf_i - g_i) = S(Sf_i - g_i), \tag{2.5}$$

under the assumption that $s\colon I \to J$ is an isomorphism.

For suppose $b_i \in \mathcal{O}_X[S]$ are such that $\sum b_i(Sf_i - g_i)$ has no constant term. Write b_{i0} for the constant term in b_i, so that $b_i - b_{i0} = Sb'_i$. Then $\sum b_{i0}g_i = 0$. Since $s\colon f_i \mapsto g_i$ is injective, also $\sum b_{i0}f_i = 0$. Thus the constant terms in the b_i do not contribute to the sum $\sum b_i(Sf_i - g_i)$, which proves (2.5).

The natural projection $\mathcal{O}_X[S] \twoheadrightarrow \mathcal{O}_X$ takes $(Sf_i - g_i) \twoheadrightarrow J = (g_i)$, and (2.5) calculates the kernel. This gives the following exact diagram:

$$\begin{array}{ccccccccc}
0 & \to & (Sf_i - g_i) & \xrightarrow{S} & (Sf_i - g_i) & \to & J & \to & 0 \\
 & & \bigcap & & \bigcap & & \bigcap & & \\
0 & \to & \mathcal{O}_X[S] & \xrightarrow{S} & \mathcal{O}_X[S] & \to & \mathcal{O}_X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_X[s] & \xrightarrow{s} & \mathcal{O}_X[s] & \to & \mathcal{O}_X/J & \to & 0
\end{array}$$

The first part of the theorem follows by the Snake Lemma.

Step 2 To prove that N is Cohen–Macaulay, recall that

$$\operatorname{depth} M = \inf\left\{i \geq 0 \;\middle|\; \operatorname{Ext}^i_{\mathcal{O}_X}(k, M) \neq 0\right\}$$

for M a finite $\mathcal{O}_X$-module over a local ring $\mathcal{O}_X$ with residue field $k = \mathcal{O}_X/m$ (see [M], Theorem 16.7). We have two exact sequences

$$\begin{gathered}
0 \to I \to \mathcal{O}_X \to \mathcal{O}_X/I \to 0 \\
0 \to J \to \mathcal{O}_X \to \mathcal{O}_X/J \to 0.
\end{gathered} \tag{2.6}$$

By assumption, $\mathcal{O}_X$ and $\mathcal{O}_X/I$ are Cohen–Macaulay, therefore

$$\begin{aligned}
&\operatorname{Ext}^i_{\mathcal{O}_X}(k, \mathcal{O}_X) = 0 \quad \text{for } 0 \leq i < n \\
\text{and} \quad &\operatorname{Ext}^i_{\mathcal{O}_X}(k, \mathcal{O}_X/I) = 0 \quad \text{for } 0 \leq i < n-1,
\end{aligned}$$

where $n = \dim X$. Thus

$$\operatorname{Ext}^i_{\mathcal{O}_X}(k, I) = 0 \quad \text{for } 0 \le i < n, \tag{2.7}$$

and the Ext long exact sequence of (2.6) gives also

$$\operatorname{Ext}^i_{\mathcal{O}_X}(k, \mathcal{O}_X/J) = 0 \quad \text{for } 0 \le i < n-1.$$

Therefore $\mathcal{O}_N = \mathcal{O}_X/J$ is Cohen–Macaulay.

Step 3 We prove that $\omega_N \cong \mathcal{O}_N$ by running the argument of Lemma 2.1.1 in reverse. Recall that $\operatorname{Hom}(I, \omega_X)$ is generated by two elements i, s, where i is a given basis element of ω_X viewed as a submodule $\omega_X \subset \operatorname{Hom}(I, \omega_X)$, and s is our isomorphism $I \to J \subset \omega_X$.

We write j for the same basis element of ω_X viewed as a submodule of $\operatorname{Hom}(J, \omega_X)$, and $t = s^{-1}\colon J \to I \subset \omega_X$ for the inverse isomorphism. Now $s\colon I \to J$ induces a dual isomorphism

$$s^*\colon \operatorname{Hom}(J, \omega_X) \to \operatorname{Hom}(I, \omega_X),$$

which is defined by $s^*(\varphi)(v) = \varphi(s(v))$ for $\varphi\colon J \to \omega_X$. By our definitions, clearly $s^*(j) = s$ and $s^*(t) = i$. Since s^* is an isomorphism, it follows that $\operatorname{Hom}(J, \omega_X)$ is generated by t and j. Therefore the adjunction exact sequence

$$0 \to \omega_X \to \operatorname{Hom}(J, \omega_X) \to \omega_N \to 0$$

gives $\omega_N = \mathcal{O}_N \bar{t}$. This completes the proof that $\mathcal{O}_N$ is Gorenstein.

Step 4 The proof that $\mathcal{O}_X[s]$ is Gorenstein will be given in Subsection 2.1.1.

2.1.1 Step 4 of the proof of Theorem 2.1.10

In this subsection we will prove Step 4 of Theorem 2.1.10. First of all we prove some general lemmas that are needed.

Lemma 2.1.11 *Let R be a Noetherian ring and $s \in R$ a regular element. The following are equivalent:*

a) R/s is Gorenstein.

b) The localization R_n is Gorenstein, for every maximal ideal $n \subset R$ containing s.

Proof Assume R/s is Gorenstein. For every maximal ideal n containing s, the local ring $(R/s)_n = R_n/s$ is Gorenstein. Since R_n is local and s is a regular element, R_n is Gorenstein. Conversely, assume that the localization R_n is Gorenstein, for every maximal ideal $n \subset R$ containing s. Since the maximal ideals of R/s correspond to the maximal ideals of R containing s, and localization commutes with taking quotient, it follows that R/s localised at each maximal ideal is Gorenstein. Therefore, R/s is Gorenstein. QED

Lemma 2.1.12 *Assume that $f\colon A \to B$ is a faithfully flat ring homomorphism between two Noetherian rings (for example B is a free A-module). If B is Gorenstein then the same is true for A.*

Proof It follows from [BH] pp. 64 and 120. QED

Lemma 2.1.13 *Assume that $f\colon A \to B$ is a flat ring homomorphism between two Noetherian rings, $n \subset B$ a prime ideal and $m \subset A$ the inverse image of n under f. Then the ring homomorphism*

$$f\colon A_m \to B_n$$

is faithfully flat.

Proof By [M] Theorem 7.1 B_n is a flat A_m module. Since $mB_n \neq B_n$, the claim follows from [AM] p. 45.

Lemma 2.1.14 *Suppose R is a local Gorenstein ring of dimension d with maximal ideal m, and $I \subset R$ a codimension one ideal with R/I Gorenstein. Assume $f(X) \in R[X]$ is a monic, irreducible over R/m polynomial, and $s\colon I \to R$ an injective homomorphism. Let T be an indeterminate and set $R' = R[T]/f(T), I' = IR'$ and $s'\colon I' \to R'$ for the induced homomorphism. Then R' is local and Gorenstein, I' has codimension one in R', R'/I' is Gorenstein and s' is injective.*

Proof The ideal $m' = mR'$ is a maximal ideal of R'. Indeed,

$$R'/m' = (R/m)[T]/f(T)$$

is a field since $f(X)$ is irreducible over R/m. Every maximal ideal of R' contains m', because by [AM] Corollary 5.8 it contains m. Hence, R' is local.

Since $R[T]$ is Gorenstein and $f(T)$ is a regular element of $R[T]$, R' is also Gorenstein. The ring $R'/I' = (R/I)[T]/f(T)$ is Gorenstein by a similar argument. Also $\dim R' = d$, since R' is finite over R. For similar reasons $\dim R'/I' = d - 1$. It is clear that s' is injective. QED

We now continue the proof of Step 4 of Theorem 2.1.10. We denote by m the maximal ideal of $\mathcal{O}_X$, and for simplicity set $\mathcal{O}_Y = \mathcal{O}_X[s]$. For a triple $(\mathcal{O}_X, \mathcal{O}_Y, n)$, where $\mathcal{O}_Y = \mathcal{O}_X[s]$ is an unprojection of $\mathcal{O}_X$ and n is a maximal ideal of $\mathcal{O}_Y$, we will prove that $\mathcal{O}_{Y,n}$ is Gorenstein by induction on the degree

$$\deg(\mathcal{O}_X, \mathcal{O}_Y, n) = \min\big\{r > 0 : \text{ there exists } a_0, \dots, a_{r-1} \in \mathcal{O}_Y \text{ with } a_0 + \cdots + a_{r-1}s^{r-1} + s^r \in n\big\}.$$

deg = 1 Assume $s + a_0 \in n$. Using the proof of Lemma 2.1.1 there exists $u \in m \subset n$ with $s + a_0 + u \colon I \to \mathcal{O}_X$ injective. Then using Steps 1–3 of Theorem 2.1.10 we have that $s + a_0 + u$ is a regular element of $\mathcal{O}_Y$ and the quotient $\mathcal{O}_Y/(s + a_0 + u)$ is Gorenstein. Therefore, $\mathcal{O}_{Y,n}$ is Gorenstein.

deg = r Assume the result is true for all triples with degree at most $r - 1$, and that $\deg(\mathcal{O}_X, \mathcal{O}_Y, n) = r$. Choose $f(s) = a_0 + \cdots + a_{r-1}s^{r-1} + s^r \in n$.

Then $f(X) = a_0 + \cdots + a_{r-1}X^{r-1} + X^r \in \mathcal{O}_X[X]$ is irreducible when considered modulo $\mathcal{O}_X/m$. Indeed, assume

$$f(X) = p_1(X)p_2(X) + q(X)$$

where $p_1, p_2 \in \mathcal{O}_X[X]$ are monic of smaller degree than f, and $q \in m[X]$. Then $p_1(s)p_2(s) = f(s) - q(s) \in n$. Since n is prime this implies $p_1(s) \in n$ or $p_2(s) \in n$, contradicting the degree of the triple.

Therefore, by Lemma 2.1.14 we have an unprojection

$$\mathcal{O}_X[T]/f(T) \subset \mathcal{O}_Y[T]/f(T).$$

Choose a maximal ideal n' of $\mathcal{O}_Y[T]/f(T)$ containing n. Since it contains $f(s)$, the degree of the triple $(\mathcal{O}_X[T]/f(T), \mathcal{O}_Y[T]/f(T), n')$ is less than r, so $(\mathcal{O}_Y[T]/f(T))_{n'}$ is Gorenstein by the inductive hypothesis. The extension

$$\mathcal{O}_Y \subset \mathcal{O}_Y[T]/f(T)$$

is faithfully flat, since the second ring a free module over the first. Hence, by Lemma 2.1.13 the extension

$$\mathcal{O}_{Y,n} \subset (\mathcal{O}_Y[T]/f(T))_{n'}$$

is also faithfully flat, and by Lemma 2.1.12 $\mathcal{O}_{Y,n}$ is Gorenstein, which finishes the induction.

As a consequence, it follows immediately that $\mathcal{O}_Y$ is Gorenstein, which finishes the proof of Theorem 2.1.10.

The original argument

We worked out the above slick proof of Step 3 by untangling the following essentially equivalent argument, which may be more to the taste of some readers.

We set up the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & I & \xrightarrow{s} & \mathcal{O}_X & \to & \mathcal{O}_N & \to & 0 \\
 & & \cap & & \cap & & & & \\
0 & \to & \mathcal{O}_X & \xrightarrow{s_2} & \operatorname{Hom}(I, \omega_X) & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_D & \xrightarrow{s_3} & \omega_D & & & &
\end{array}$$

The first column is just the definition of $\mathcal{O}_D$. The second column is the identification of $\mathcal{O}_X$ with ω_X composed with the adjunction formula for ω_D.

The first row is the multiplication $s\colon I \to J$ composed with the definition of $\mathcal{O}_N$. To make the first square commute, the map s_2 must be defined by

$$s_2(a)(b) = s(ab) \quad \text{for } a \in \mathcal{O}_X \text{ and } b \in I. \tag{2.8}$$

We identify its cokernel L below. The first two rows induce the map s_3. Since s_2 takes $1 \in \mathcal{O}_X$ to $s \in \operatorname{Hom}(I, \omega_X)$, it follows that s_3 takes $1 \in \mathcal{O}_D$ to $\overline{s} \in \omega_D$ as in Lemma 2.1.1, and therefore s_3 is an isomorphism.

Now the second row is naturally identified with the adjunction sequence

$$0 \to \omega_X \to \operatorname{Hom}(J, \omega_X) \to \omega_N \to 0.$$

The point is just that $s\colon I \cong J$, and s_2 is the composite

$$0 \to \omega_X \hookrightarrow \operatorname{Hom}(I, \omega_X) \xrightarrow{s^*} \operatorname{Hom}(J, \omega_X),$$

by its definition in (2.8). The Snake Lemma now gives $\mathcal{O}_N \cong L = \omega_N$. Therefore, as before, N is Gorenstein.

2.2 Projective unprojection

Assume $Q = (q_0, \dots, q_r)$ is a set of positive weights. Denote by

$$R = k[x_0, \dots, x_r]$$

the polynomial ring with $\deg x_i = q_i$, and $\mathbb{P} = \operatorname{Proj} R$ the corresponding weighted projective space. Since our methods are algebraic, we do not need to assume anything else about the weights.

A closed subscheme $X \subseteq \mathbb{P}$ defines the satured homogeneous ideal $I_X \subset R$ and the homogeneous coordinate ring $S(X) = R/I_X$.

Notation If M is a graded R-module, we denote by M_d the degree d component of M, and by $M(n)$ the graded R-module with $M(n)_d = M_{n+d}$. If N is a finitely generated graded R-module, the R-module $\operatorname{Hom}_R(N, M)$ has a natural grading, with $\operatorname{Hom}_R(N, M)_d$ consisting of the degree d homomorphisms from N to M.

Definition 2.2.1 A subscheme $X \subseteq \mathbb{P}$ is called *projectively Gorenstein* if the homogeneous coordinate ring $S(X)$ is Gorenstein. Following Zariski, some authors also use the term arithmetically Gorenstein.

By [BH], Chapter 3.6, if X is projectively Gorenstein there exists (unique) $k_X \in \mathbb{Z}$ with $\omega_{S(X)} = S(X)(k_X)$.

Definition 2.2.2 $D \subset X \subseteq \mathbb{P}$ is an *unprojection pair* if X and D are projectively Gorenstein, $\dim X = \dim D + 1$ and $k_X > k_D$, where $k_X, k_D \in \mathbb{Z}$ with $\omega_{S(X)} = S(X)(k_X)$, $\omega_{S(D)} = S(D)(k_D)$.

Assume $D \subset X \subseteq \mathbb{P}$ is an unprojection pair, and set $l = k_X - k_D$, $I = I_X,\ J = I_D$. As in (2.1) there is an exact sequence of graded R-modules

$$0 \to \omega_{S(X)} \to \operatorname{Hom}_R(J/I, \omega_{S(X)}) \xrightarrow{\text{res}} \omega_{S(D)} \to 0, \tag{2.9}$$

which, using the assumptions, induces an exact sequence

$$0 \to S(X)(l) \to \operatorname{Hom}_R(J/I, S(X)(l)) \xrightarrow{\text{res}} S(D) \to 0. \tag{2.10}$$

Taking the homogeneous parts of degree 0 we have an exact sequence

$$0 \to S(X)_l \to \operatorname{Hom}_R(J/I, S(X))_l \xrightarrow{\text{res}} k \to 0. \tag{2.11}$$

Definition 2.2.3 An unprojection for the pair $D \subset X$ is a degree l homomorphism

$$s \colon J/I \to S(X) \tag{2.12}$$

such that $\operatorname{res}(s) \neq 0$.

Set $I = (f_1, \ldots, f_n)$ with each f_i homogeneous, and $g_i = s(f_i) \in S(X)$. Since $\deg g_i - \deg f_i = l$ for all i, the ring

$$A = S(X)[T]/(Tf_i - g_i)$$

has a natural grading extending the grading of $S(X)$, such that $\deg T = l$. Theorem 2.1.10 implies the following

Theorem 2.2.4 *The graded ring A is Gorenstein.*

2.2.1 Ordinary projective space

Under the assumption that $Q = (1, \ldots, 1)$, so $\mathbb{P} = \mathbb{P}^r$ is the usual projective space, we can reformulate the previous section in terms of coherent sheaves using the Serre correspondence between graded R-modules and coherent sheaves on $\mathbb{P}$. For simplicity, we also assume $\dim X \geq 2$. Similar geometric interpretation should exist also in the case of general Q or $\dim X = 1$.

The following result is well known (see e.g. [Ei], p. 468 and [Mi], p. 79).

Theorem 2.2.5 *Assume $X \subseteq \mathbb{P}^r$ with $\dim X \geq 1$. Then X is projectively Gorenstein if and only if the natural restriction map*

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}}(t)) \to H^0(X, \mathcal{O}_X(t))$$

is surjective for all $t \in \mathbb{Z}$,

$$H^i(X, \mathcal{O}_X(t)) = 0, \qquad \textit{for } 0 < i < \dim X \textit{ and } t \in \mathbb{Z},$$

and there exists $k_X \in \mathbb{Z}$ *with*

$$\omega_X = \mathcal{O}_X(k_X).$$

Remark 2.2.6 For $\dim X = 0$ see e.g. [Mi], p. 60.

Remark 2.2.7 Assume that $X \subseteq \mathbb{P}^r$ is projectively Gorenstein. It is well known that X is of pure dimension, locally Gorenstein and, provided $\dim X \geq 1$, connected. In addition, X is normal if and only if it is projectively normal if and only if it is nonsingular in codimension one.

Assume now that $D \subset X \subseteq \mathbb{P}^r$ is an unprojection pair with $\dim X \geq 2$ and $\omega_{S(X)} = S(X)(k_X)$, $\omega_{S(D)} = S(D)(k_D)$, $l = k_X - k_D > 0$. Since X is locally Cohen–Macaulay, we have as in (2.1) an exact sequence of coherent sheaves

$$0 \to \omega_X \to \mathcal{H}om(\mathcal{I}_D, \omega_X) \xrightarrow{\operatorname{res}_D} \omega_D \to 0, \tag{2.13}$$

which, by twisting, induces an exact sequence

$$0 \to \mathcal{O}_X(l) \to \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_X(l)) \xrightarrow{\operatorname{res}_D} \mathcal{O}_D \to 0. \tag{2.14}$$

Taking global sections, Theorem 2.2.5 implies the fundamental exact sequence

$$0 \to H^0(X, \mathcal{O}_X(l)) \to \operatorname{Hom}(\mathcal{I}_D, \mathcal{O}_X(l)) \xrightarrow{\operatorname{res}_D} H^0(\mathcal{O}_D) \to 0. \tag{2.15}$$

By the same theorem $H^0(\mathcal{O}_D) = k$. Hence, an unprojection s is just an element of $\operatorname{Hom}(\mathcal{I}_D, \mathcal{O}_X(l))$ with $\operatorname{res}_D(s) \neq 0$.

2.2.2 A reformulation of projective unprojection

Under the strong assumption that D is a Cartier (effective) divisor of X we give another, simpler and in more classical terms, reformulation of projective unprojection.

Assume that $D \subset X \subseteq \mathbb{P}^r$ is an unprojection pair with $\dim X \geq 2$ and $\omega_X = \mathcal{O}_X(k_X H)$, $\omega_D = \mathcal{O}_D(k_D H)$, $l = k_X - k_D > 0$, where H is a hyperplane divisor of $\mathbb{P}^r$. Moreover, we suppose that D is a Cartier divisor of X. By [KoM] Proposition 5.73 we have the adjunction formula

$$\omega_D = \omega_X \otimes \mathcal{O}_X(D) \otimes \mathcal{O}_D.$$

Hence,

$$\mathcal{O}_X(D) \otimes \mathcal{O}_D = \mathcal{O}_D(-lH). \tag{2.16}$$

Lemma 2.2.8 *a) For all* $0 \leq i \leq l-1$ *the natural injection*

$$H^0(X, \mathcal{O}_X(iH)) \hookrightarrow H^0(X, \mathcal{O}_X(iH + D))$$

is an isomorphism.

b) There is an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(lH)) \to H^0(X, \mathcal{O}_X(lH + D)) \to k \to 0.$$

Proof Taking into account Theorem 2.2.5, for every $i \in \mathbb{Z}$ the natural exact sequence

$$0 \to \mathcal{O}_X(iH) \to \mathcal{O}_X(iH + D) \to \mathcal{O}_D(iH + D) \to 0$$

induces an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(iH)) \to H^0(X, \mathcal{O}_X(iH + D)) \to H^0(D, \mathcal{O}_D(iH + D)) \to 0.$$

In light of (2.16), $H^0(D, \mathcal{O}_D(iH + D)) = 0$ for $i \leq l-1$ and
$H^0(D, \mathcal{O}_D(lH + D)) = k$. QED

Using the lemma, we can say that an unprojection s is any element in $H^0(X, \mathcal{O}_X(lH+D)) \setminus H^0(x, \mathcal{O}_X(lH))$ i.e., a rational function of homogeneous degree l with single pole on D. In addition, the unprojection rational map from X to the weighted projective space $\mathbb{P}(1^{r+1}, l)$ is defined by a basis of $H^0(X, \mathcal{O}_X(H))$ together with s.

2.3 Simplest example

This section is taken from [PR]. We discuss a case that has many consequences in birational geometry, even though the algebra itself is very simple. Consider the generic equations

$$X : (Bx - Ay = 0) \quad \text{and} \quad D : (x = y = 0) \tag{2.17}$$

defining a hypersurface X containing a codimension 2 complete intersection D in some as yet unspecified ambient space. The unprojection variable is

$$s = \frac{A}{x} = \frac{B}{y} \,. \tag{2.18}$$

We can view s as a rational function on X, or as an isomorphism from (x, y) to (A, B) in $\mathcal{O}_X$. The unprojection is the codimension two complete intersection $Y : (sx = A, sy = B)$.

For example, take $\mathbb{P}^3$ as ambient space, with x, y linear forms defining a line D, and A, B general quadratic forms. Then s has degree 1 , and the equations describe the contraction of a line on a nonsingular cubic surface to the point $P_s = (0 : 0 : 0 : 0 : 1) \in \mathbb{P}^4$ on a del Pezzo surface of degree 4. It is the inverse of the linear projection $Y \dashrightarrow X$ from P_s, eliminating s. But the equations are of course much more general. The only assumptions are that x, y and $Bx - Ay$ are regular sequences in the ambient space. For example, if A, B vanish along D, so that X is singular there, then Y contains the plane $x = y = 0$ as an exceptional component lying over D. Note that, in any case, Y has codimension 2 and is nonsingular at P.

The same rather trivial algebra lies behind the quadratic involutions of Fano 3-folds constructed in [CPR], 4.4–4.9. For example, consider the general weighted hypersurface of degree 5

$$X_5 : (x_0 y^2 + a_3 y + b_5 = 0) \subset \mathbb{P}(1, 1, 1, 1, 2),$$

with coordinates $x_0, \ldots, x_3, y$. The coordinate point $P_y = (0 : \cdots : 1)$ is a Veronese cone singularity $\frac{1}{2}(1, 1, 1)$. The anticanonical model of the blowup of P_y is obtained by eliminating y and adjoining $z = x_0 y$ instead, thus passing to the hypersurface

$$Z_6 : (z^2 + a_3 z + x_0 b_5 = 0) \subset \mathbb{P}(1, 1, 1, 1, 3).$$

The 3-fold Z_6 contains the plane $x_0 = z = 0$, the exceptional $\mathbb{P}^2$ of the blowup. Writing its equation as $z(z+a_3)+x_0b_5$ gives $y = \frac{z}{x_0} = -\frac{b_5}{z+a_3}$, and puts the birational relation between X_5 and Z_6 into the generic form (2.17–2.18). In fact Z_6 is the "midpoint" of the construction of the birational involution of X_5. The construction continues by setting $y' = \frac{z+a_3}{x_0} = -\frac{b_5}{z}$, thus unprojecting a different plane $x_0 = z + a_3 = 0$. For details, consult [CPR], 4.4–4.9. See [CM] for a related use of the same algebra, to somewhat surprising effect.

2.4 Birational geometry of unprojection, an example

We present a simple example that shows that even when we fix the algebra of an unprojection, the geometric picture can vary.

Suppose we have an irreducible cubic hypersurface $X \subset \mathbb{P}^3$, with equation $Ax - By = 0$, containing the codimension one subscheme D with $I_D = (x, y)$. According to Section 2.3, the unprojection variety is $Y \subset \mathbb{P}^4$ with ideal $I_Y = (sx - B, sy - A)$. There is a natural projection

$$\phi\colon Y \dashrightarrow X, \quad \text{with } [x, y, z, w, s] \mapsto [x, y, z, w],$$

with birational inverse the rational map (graph of s)

$$\phi^{-1}\colon X \dashrightarrow Y, \quad \text{with } [x, y, z, w] \mapsto [x, y, z, w, s = \frac{A}{y} = \frac{B}{x}].$$

Denote by $N \subset X$ the closed subscheme with $I_N = (A, B)$. Under ϕ^{-1}, D is (possibly after a factorialisation) contracted to the point $[0, 0, 0, 0, 1]$ while $\phi^{-1}(N)$ is the hyperplane section $s = 0$ of Y.

Generic case The generic case is when X is smooth. As a consequence $N \cap D = \emptyset$. It is easy to see that ϕ^{-1} is a regular map, the usual blowdown of the -1 line D.

Special case When $N \cap D \neq \emptyset$ the birational geometry is more complicated. Assume, for example, that the equation of X is $xz(w+x) - y(z+y)w$.

Then $N \cap D = \{p_1 = [0,0,1,0], p_2 = [0,0,0,1]\}$. Both points are A_1 singularities of X. We have the following factorization of $\phi^{-1}\colon X \dashrightarrow Y$

$$\begin{array}{ccc} & & Z \\ & \swarrow & \downarrow \\ X & \dashrightarrow & Y \end{array}$$

In the diagram $Z \to X$ is the blowup of X at the two points p_1, p_2, and $Z \to Y$ is the blowdown of the strict transform of D. Of course, Z is nothing but the graph of the projection $\phi\colon Y \dashrightarrow X$.

2.5 The link between Kustin–Miller theorem and our unprojection

Resolutions can be used to calculate unprojection and we will prove a more general version of the fact that the construction of Kustin–Miller [KM4] gives an unprojection in our sense, as described in Section 2.1.

We change to more convenient for our purposes 'algebraic' notation. Let R be a Gorenstein local ring and $I \subset J$ perfect (therefore Cohen–Macaulay) ideals of codimensions r and $r+1$ respectively. Unless otherwise indicated, all Hom and Ext modules and maps are over R.

Recall that we have the fundamental adjunction exact sequence (2.1)

$$0 \to \omega_{R/I} \xrightarrow{a} \operatorname{Hom}_R(J/I, \omega_{R/I}) \xrightarrow{\text{res}} \omega_{R/J} \to 0, \tag{2.19}$$

with a the natural map

$$a(x)(l) = lx, \quad \text{for all } l \in J \text{ and } x \in \omega_{R/I}.$$

In the following we identify $\omega_{R/I}$ with its image under a.

Let

$$\mathbf{L} \to R/I, \quad \mathbf{M} \to R/J \tag{2.20}$$

be minimal resolutions as R-modules. According to [FOV] Proposition A.2.12, the dual complexes

$$\mathbf{L}^* \to \omega_{R/I}, \quad \mathbf{M}^* \to \omega_{R/J} \tag{2.21}$$

are also minimal resolutions, where $*$ means $\operatorname{Hom}(-,R)$. More precisely we have an exact sequence

$$M_r^* \xrightarrow{b} M_{r+1}^* \xrightarrow{c} \omega_{R/J} \to 0, \tag{2.22}$$

and we set

$$T = \ker c. \tag{2.23}$$

For simplicity of notation, since M^*_{r+1} is free of rank (say) l, we identify it with R^l. There is an an exact sequence

$$0 \to T \to R^l \xrightarrow{c} \omega_{R/J} \to 0. \tag{2.24}$$

For the canonical base $e_i = (0, \dots, 1, \dots 0) \in R^l$, we fix liftings

$$q_i \in \operatorname{Hom}(J, \omega_{R/I}) \tag{2.25}$$

of the basis $\overline{e}_i = c(e_i) \in \omega_{R/J}$, under the map res in (2.19). Notice that $\operatorname{Hom}(J, \omega_{R/I})$ is generated by $\omega_{R/I}$ together with $q_1, \dots, q_l$. Moreover, clearly

$$T = \big\{(b_1, \dots, b_l) \in R^l : \sum_i b_i q_i \in \omega_{R/I}\big\}. \tag{2.26}$$

We denote by $s_i \colon J \to T$ the map

$$s_i(t) = (0, \dots, 0, t, 0, \dots, 0),$$

where t is in the ith coordinate. Define

$$\Phi \colon \operatorname{Hom}(T, \omega_{R/I}) \to \operatorname{Hom}(J, \omega_{R/I})^l$$

with

$$\Phi(e) = (e \circ s_1, \dots, e \circ s_l).$$

Lemma 2.5.1 *The map Φ is injective with image equal to*

$$L = \big\{(k_1, \dots, k_l) : \sum_i b_i k_i \in \omega_{R/I} \quad \textit{whenever } (b_1, \dots, b_l) \in T\big\}.$$

Proof The ring R/I is Cohen–Macaulay, so there exists $t \in J$ that is R/I-regular. Since $\omega_{R/I}$ is a maximal Cohen–Macaulay R/I-module, t is also $\omega_{R/I}$-regular (compare e.g. [Ei], p. 529). Assume $e \circ s_i = e' \circ s_i$ for all i, and let $b' = (b_1, \dots, b_l) \in T$. Then

$$te(b') = e(tb') = \sum_i b_i e \circ s_i(t) = \sum_i b_i e' \circ s_i(t) = te'(b').$$

Since t is $\omega_{R/I}$-regular we have $e = e'$. Moreover, this also shows that the image of ϕ is contained in L.

Now consider $(k_1, \dots, k_l) \in L$. Define $e\colon T \to \omega_{R/I}$ with $e(b_1, \dots, b_l) = \sum b_i k_i$. Then $e \circ s_i(t) = tk_i = k_i(t)$, so $\Phi(e) = (k_1, \dots, k_l)$. QED

Now we present a method, originally developed in [KM4], and prove that it calculates a set of generators for

$$\operatorname{Hom}(J, \omega_{R/I})/\omega_{R/I},$$

which was conjectured by Reid.

The natural map $R/I \to R/J$ induces a map of complexes $\psi\colon \mathbf{L} \to \mathbf{M}$ and the dual map $\psi^*\colon \mathbf{M}^* \to \mathbf{L}^*$. Using (2.21), we get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
M^*_{r-1} & \longrightarrow & M^*_r & \xrightarrow{b} & M^*_{r+1} & \xrightarrow{c} & \omega_{R/J} \\
\downarrow & & \downarrow & & & & \\
L^*_{r-1} & \longrightarrow & L^*_r & \longrightarrow & \omega_{R/I} & \longrightarrow & 0
\end{array}$$

By the definition of T, there is an induced map $\psi^*\colon T \to \omega_{R/I}$. Notice that this map is not canonical, but depends on the choice of ψ; we fix one such choice. Set

$$\Phi(\psi^*) = (k_1, \dots, k_l).$$

Theorem 2.5.2 *The R-module $\operatorname{Hom}_R(J, \omega_{R/I})$ is generated by $\omega_{R/I}$ together with $k_1, \dots, k_l$.*

Proof Since $\omega_{R/I}$ together with the q_i generate $\operatorname{Hom}(J, \omega_{R/I})$, we have equations

$$k_i = \sum_j a_{ij} q_j + \theta_i, \qquad \text{with } \ a_{ij} \in R, \ \theta_i \in \omega_{R/I}. \tag{2.27}$$

Clearly (q_i) and (θ_i) are in the image of Φ, which by Lemma 2.5.1 is equal to L, set $(q_i) = \Phi(Q)$, $(\theta_i) = \Phi(\Theta)$. We have an induced map $f_0\colon R^l \to R^l$ with

$$f_0(b_1, \dots b_l) = (b_1, \dots, b_l)[a_{ij}].$$

Using (2.26), T is invariant under f_0, so there is an induced map $f_1\colon T \to T$ and, using (2.24), a second induced map $f_2 \in \operatorname{Hom}(\omega_{R/J}, \omega_{R/J})$, with $f_2(\overline{e}_i) =$

$\sum_j a_{ij}\overline{e}_j$. We will show that f_2 is an automorphism of $\omega_{R/J}$, which will prove the theorem.

Using (2.20), (2.21) and the definition (2.23) of T, we get

$$\begin{aligned} \operatorname{Ext}^{r+1}(\omega_{R/J}, R) &= \operatorname{Ext}^r(T, R) = R/J, \\ \operatorname{Ext}^r(\omega_{R/I}, R) &= R/I. \end{aligned}$$

Moreover, by [BH] Theorem 3.3.11 the natural map $R/J \to \operatorname{Hom}(\omega_{R/J}, \omega_{R/J})$ is an isomorphism. This, together with the formal properties of the Ext functor, imply that the natural map

$$\operatorname{Ext}^{r+1}(-, \operatorname{id}_R)\colon \operatorname{Hom}(\omega_{R/J}, \omega_{R/J}) \to R/J = \operatorname{Hom}(R/J, R/J)$$

is the identity $R/J \to R/J$. Therefore it is enough to show that $\operatorname{Ext}^{r+1}(f_2)$ is a unit in R/J, and by (2.24) $\operatorname{Ext}^{r+1}(f_2) = \operatorname{Ext}^r(f_1)$, where for simplicity of notation we denote $\operatorname{Ext}^*(-, \operatorname{id}_R)$ by $\operatorname{Ext}^*(-)$.

By (2.27) and the injectivity of Φ (Lemma 2.5.1)

$$\psi^* = Q \circ f_1 + \Theta; \tag{2.28}$$

therefore

$$\operatorname{Ext}^r(\psi^*) = \operatorname{Ext}^r(f_1)\operatorname{Ext}^r(Q) + \operatorname{Ext}^r(\Theta)$$

as maps $R/I \to R/J$. Since Θ can be extended to a map $R^l \to \omega_{R/I}$, $\operatorname{Ext}^r(\Theta) = 0$. In addition, by the construction of ψ^*, $\operatorname{Ext}^r(\psi^*) = 1 \in R/J$. This implies that $\operatorname{Ext}^r(f_1)$ is a unit in R/J, which finishes the proof. QED

The arguments in the proof of Theorem 2.5.2 also prove the more general

Theorem 2.5.3 *Let*

$$f\colon T \to \omega_{R/I}$$

be an R-homomorphism, and set $f_i = f \circ s_i$ *for* $1 \le i \le l$. *Then* $\omega_{R/I}$ *together with* $f_1, \dots, f_l$ *generate* $\operatorname{Hom}(J, \omega_{R/I})$ *if and only if*

$$\operatorname{Ext}^r(f) : R/I \to R/J$$

is surjective.

Theorems 2.5.2 and 2.5.3 can be used to justify the part of the calculations of [R1] Section 9 related to the definition of the unprojection rings.

2.5.1 Unprojection of a complete intersection inside a complete intersection

Let R be a Gorenstein local ring and $I \subset J$ ideals of R, of codimensions r and $r+1$ respectively. We assume that each is generated by a regular sequence, say

$$I = (v_1, \dots, v_r), \quad J = (w_1, \dots, w_{r+1}). \tag{2.29}$$

Since $I \subset J$, there exists an $r \times (r+1)$ matrix A with

$$\begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = A \begin{pmatrix} w_1 \\ \vdots \\ w_{r+1} \end{pmatrix} \tag{2.30}$$

Definition 2.5.4 $\bigwedge^r A$ is the $1 \times (r+1)$ matrix whose ith entry $(\bigwedge^r A)_i$ is $(-1)^{i+1}$ times the determinant of the submatrix of A obtained by removing the ith column.

Lemma 2.5.5 (Cramer's rule) *For all i, j the element*

$$(\bigwedge^r A)_i w_j - (\bigwedge^r A)_j w_i$$

is in the ideal $(v_1, \dots, v_r)$.

Proof Simple linear algebra (Cramer's rule). QED

We define $g_i \in R$ by

$$\bigwedge^r A = (g_1, \dots, g_{r+1}).$$

The special case $r = 2$ of the following theorem was proven by direct methods in [CFHR] Lemma 6.11, compare also Section 2.3 for applications of the case $r = 1$.

Theorem 2.5.6 $\mathrm{Hom}_{R/I}(J/I, R/I)$ *is generated as R/I-module by two elements* id *and s, where*

$$s(w_i) = g_i, \quad \textit{for } 1 \le i \le r+1.$$

Proof Since R/I is Gorenstein, we have $\omega_{R/I} = R/I$. By Lemma 2.5.5 s is well defined. Consider the minimal Koszul complexes corresponding to the generators given in (2.29) that resolve $R/I, R/J$ as R-modules,

$$\mathbf{M} \to R/I$$
$$\mathbf{N} \to R/J.$$

The matrix A can be considered as a map $M_1 \to L_1$ making the following square commutative

$$\begin{array}{ccc} M_1 & \xrightarrow{(v_1,\dots,v_r)} & M_0 = R \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ L_1 & \xrightarrow{(w_1,\dots,w_{r+1})} & L_0 = R \end{array}$$

There are induced maps $\bigwedge^n A \colon M_n \to L_n$ (compare e.g. [BH], Proposition 1.6.8), giving a commutative diagram

$$\begin{array}{ccc} \mathbf{M} & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ \mathbf{N} & \longrightarrow & R/J \end{array}$$

Since the last nonzero map is given by $\bigwedge^r A$ the result follows from Theorem 2.5.2. QED

2.6 More general unprojections

In Section 2.1 we defined the unprojection ring S for a pair $I \subset R$, where R is a local Gorenstein ring and I a codimension one ideal with R/I Gorenstein. By Theorem 2.1.10 S is Gorenstein.

An important question, raised by Reid in [R1] Section 9, is whether we can define a Gorenstein 'unprojection ring' S for more general pairs $I \subset R$. Corti and Reid have indeed found such examples, see loc. cit., but a general definition of S is still lacking.

Assume, for example, that R is a Gorenstein local normal domain and $I \subset R$ is an ideal of pure codimension one (i.e., all associated primes of I have codimension one). Taking into account Lemma 2.1.7, a natural candidate for

S is the ring $R[I^{-1}]$, that is the R-subalgebra of the field of fractions $K(R)$ of R generated by the set

$$I^{-1} = \{a \in K(R) : aI \subseteq R\}.$$

An important question which the present author has been unable to decide is whether $R[I^{-1}]$ is Gorenstein under the above assumptions.

More generally we can ask whether $R[I^{-1}]$ is Gorenstein, assuming just that I is a pure codimension one ideal of a Gorenstein ring R (compare also Example 2.1.8).

Chapter 3

Tom & Jerry

3.1 Here come the heroes

In the following k will be the field of complex numbers $\mathbb{C}$. This is for simplicity, most arguments work in much greater generality.

3.1.1 Tom

We work over the polynomial ring $S = k[x_k, z_k, a_{ij}^k]$. More precisely we have indeterminates x_k, z_k, a_{ij}^k, for $1 \leq k \leq 4,\ 2 \leq i < j \leq 5$.

The generic *Tom* ideal is the ideal I of S generated by the five Pfaffians of the skewsymmetric matrix

$$A = \begin{pmatrix} . & x_1 & x_2 & x_3 & x_4 \\ & . & a_{23} & a_{24} & a_{25} \\ & & . & a_{34} & a_{35} \\ & -\mathrm{sym} & & . & a_{45} \\ & & & & . \end{pmatrix} \tag{3.1}$$

where

$$a_{ij} = \sum_{k=1}^{4} a_{ij}^k z_k.$$

The proof of the following theorem will be given in Section 3.2.

Theorem 3.1.1 *The ideal I is prime of codimension three and S/I is Gorenstein.*

3.1.2 Jerry

Here we work over the polynomial ring $S = k[x_i, z_k, c^k, a_i^k, b_i^k]$. More precisely we have indeterminates $x_1, x_2, x_3, z_k, a_i^k, b_i^k, c^k$, for $1 \le k \le 4$, $1 \le i \le 3$. The generic *Jerry* ideal is the ideal I of S generated by the five Pfaffians of the skewsymmetric Jerry matrix

$$B = \begin{pmatrix} . & c & a_1 & a_2 & a_3 \\ & . & b_1 & b_2 & b_3 \\ & & . & x_1 & x_2 \\ -\mathrm{sym} & & & . & x_3 \\ & & & & . \end{pmatrix} \tag{3.2}$$

where

$$a_i = \sum_{k=1}^{4} a_i^k z_k, \quad b_i = \sum_{k=1}^{4} b_i^k z_k, \quad c = \sum_{k=1}^{4} c^k z_k.$$

The methods of Section 3.2 also prove the following

Theorem 3.1.2 *The ideal I is prime of codimension three and S/I is Gorenstein.*

3.2 Tom ideal is prime

We give two proofs of Theorem 3.1.1. The first occupies Subsections 3.2.1 to 3.2.7, while the second is in Subsection 3.2.8.

3.2.1 Definition of X_1

Consider the affine space (over k) $\mathbb{A}_1 \cong \mathbb{A}^{10}$ with coordinates $x_1, \dots, x_4, w_{ij}$, for $2 \le i < j \le 5$, and the skewsymmetric matrix

$$M_1 = \begin{pmatrix} . & x_1 & x_2 & x_3 & x_4 \\ & . & w_{23} & w_{24} & w_{25} \\ & & . & w_{34} & w_{35} \\ & & & . & w_{45} \\ & & & & . \end{pmatrix}$$

We define $X_1 \subset \mathbb{A}_1$ (the affine cone over the Grassmanian $\mathrm{Gr}(2,5)$) by

$$X_1 = \left\{ (x_k, w_{ij}) \in \mathbb{A}_1 \text{ that satisfy the Pfaffians of } M_1 \right\}.$$

It is a well known fact (see e.g. [KL]) that X_1 is irreducible and $\dim X_1 = 7$.

3.2.2 Definition of X_2

Consider the affine space $\mathbb{A}_2 \cong \mathbb{A}^{17}$ with coordinates

$$x_1, \dots, x_4, z_1, \dots, z_4, a_{23}^1, \dots, a_{23}^4, w_{24}, w_{25}, w_{34}, w_{35}, w_{45}$$

and the skewsymmetric matrix

$$M_2 = \begin{pmatrix} \cdot & x_1 & x_2 & x_3 & x_4 \\ & \cdot & a_{23} & w_{24} & w_{25} \\ & & \cdot & w_{34} & w_{35} \\ & & & \cdot & w_{45} \\ & & & & \cdot \end{pmatrix}$$

where

$$a_{23} = a_{23}^1 z_1 + \cdots + a_{23}^4 z_4.$$

We define $X_2 \subset \mathbb{A}_2$ by

$$X_2 = \left\{ (x_k, z_k, a_{23}^k, w_{ij}) \in \mathbb{A}_2 \text{ that satisfy the Pfaffians of } M_2 \right\}.$$

3.2.3 Definition of X_3

Consider the affine space $\mathbb{A}_3 \cong \mathbb{A}^{32}$ with coordinates x_k, z_k, a_{ij}^k, for $1 \le k \le 4,\ 2 \le i < j \le 5$ and the skewsymmetric matrix

$$M_3 = \begin{pmatrix} \cdot & x_1 & x_2 & x_3 & x_4 \\ & \cdot & a_{23} & a_{24} & a_{25} \\ & & \cdot & a_{34} & a_{35} \\ & & & \cdot & a_{45} \\ & & & & \cdot \end{pmatrix}$$

where

$$a_{ij} = a_{ij}^1 z_1 + \cdots + a_{ij}^4 z_4.$$

We define $X_3 \subset \mathbb{A}_3$ by

$$X_3 = \left\{ (x_k, z_k, a_{ij}^k) \in \mathbb{A}_3 \text{ that satisfy the Pfaffians of } M_3 \right\}.$$

3.2.4 Proving irreducibility

We will use the irreducibility of X_1 and pass through X_2 to prove the irreducibility of X_3. We need a variant of the following general theorem which can be found for example in [Ha], p. 139 or [Sha], p. 77.

Theorem 3.2.1 *Let $\phi\colon X \to Y$ be a surjective morphism of reduced projective schemes. Suppose all fibers of ϕ are irreducible of the same dimension and Y is irreducible. Then X is irreducible.*

The theorem is not correct in general without the projectiveness assumption. (A counterexample in the affine case is the projection to the the x-axis of X, with $X \subset \mathbb{A}^2$ the union of the point $(0,0)$ with the hyperbola $xy = 1$.) However, if we assume that $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ and the map $\phi\colon X \to Y$ are homogeneous with respect to gradings of the coordinates in $\mathbb{A}^n$ and $\mathbb{A}^m$, in other words X and Y are cones over varieties in weighted projective spaces and ϕ respects this structure, the result still holds. In the following this condition will be satisfied by setting $\deg w_{ij} = 2,\ \deg x_k = \deg a^k_{ij} = \deg z_k = 1$.

3.2.5 Definition of the map $\phi_2\colon X_2 \to X_1$

Define $\phi_2\colon X_2 \to X_1$ with

$$\phi_2(x_k, z_k, a^k_{23}, w_{ij}) = (x_k, \sum_{k=1}^{4} a^k_{23} z_k, w_{ij}).$$

Lemma 3.2.2 *ϕ_2 is surjective with every fiber irreducible of dimension 7.*

Proof Indeed,

$$\phi_2^{-1}(x_k, w_{ij})$$

is given in X_2 by the single equation

$$\sum_{k=1}^{4} a^k_{23} z_k = w_{23},$$

so it is isomorphic with an irreducible hypersurface in $\mathbb{A}^8$. QED

Using Subsection 3.2.4 we have the following corollary.

Corollary 3.2.3 *X_2 is irreducible with* $\dim X_2 = 14$.

3.2.6 Definition of the map $\phi_3\colon X_3 \to X_2$

Define $\phi_3\colon X_3 \to X_2$ with

$$\phi_3(x_k, z_k, a_{ij}^k) = (x_k, z_k, a_{23}^1, \dots, a_{23}^4, \sum_{k=1}^{4} a_{24}^k z_k, \dots, \sum_{k=1}^{4} a_{45}^k z_k)$$

Let $U_2 \subset X_2$ be defined by

$$U_2 = \big\{(z_1, \dots, z_4) \neq (0, \dots, 0)\big\}$$

and set

$$U_3 = \phi_3^{-1}(U_2) \subset X_3.$$

Since X_2 is irreducible by Corollary 3.2.3, U_2 is also irreducible. Denote by $\psi\colon U_3 \to U_2$ the restriction of ϕ_3 to U_3.

Lemma 3.2.4 *ψ is surjective with every fiber isomorphic to $\mathbb{A}^{15}$.*

Proof Obvious. QED

Using Subsection 3.2.4 U_3 is irreducible with $\dim U_3 = 29$. To show that X_3 is irreducible it is enough to prove that U_3 is dense in it. Notice that since $\dim X_3 \setminus U_3 = 28 = \dim U_3 - 1$, we have $\dim X_3 = 29$.

By Hilbert's Nullstellensatz, the ideal of $X_3 \subset \mathbb{A}_3$ is $L = \operatorname{Rad} I$. As a consequence I has also codimension three, and since a polynomial ring over a field is Cohen–Macaulay it has grade equal to three. Since it is generated by Pfaffians, Theorem 1.3.1 implies that S/I is Gorenstein. By unmixedness each component of X_3 has dimension 29. The following general topological lemma completes the proof that X_3 is irreducible of codimension three in $\mathbb{A}_3$.

Lemma 3.2.5 *Let X be a topological space of finite dimension and $U \subseteq X$ an irreducible open subset such that $\dim X \setminus U < \dim U$ and each component of X has dimension at least equal to $\dim U$. Then U is dense in X.*

Corollary 3.2.6 *$X_3 \subset \mathbb{A}_3$ is irreducible of codimension three.*

3.2.7 Proof of Theorem 3.1.1

In Subsection 3.2.6 we proved that the radical $L = \operatorname{Rad} I$ of the ideal I of the generic Tom is prime of codimension three and that S/I is Gorenstein.

We claim that S/I is reduced, which will finish the proof of Theorem 3.1.1. Assume that it is not reduced. Consider the Jacobean matrix M of the five Pfaffian generators of I and and the ideal $J \subseteq S$ generated by the determinants of the 3×3 submatrices of M. By [Ei] Theorem 18.15, the ideal $(I + J)/I$ has codimension 0 in S/I. Since L is prime, L/I is the unique minimal ideal of S/I, hence $J \subseteq L$. This implies $V(L) \subseteq V(J)$. But an easy calculation shows that the point P with all coordinates a_{ij}^k, x_k, z_k equal to zero except $x_4 = a_{34}^1 = a_{24}^2 = a_{23}^3 = 1$ is in $V(L)$ but not in $V(J)$, a contradiction which finishes the proof of Theorem 3.1.1.

3.2.8 Second proof of Theorem 3.1.1

We give a second proof of Theorem 3.1.1 based on the ideas of [BV] Chapter 2. Set $R = S/I$ and $X = \operatorname{Spec} R$. We will prove that R is a domain.

Lemma 3.2.7 *For all $i = 1, \dots, 4$ the element $z_i \in R$ is not nilpotent. Moreover, $R[z_i^{-1}]$ is a domain and* $\dim R[z_i^{-1}] = 29$.

Proof Due to symmetry, it is enough to prove it for z_1. By the form of the generators of I it follows immediately that $z_1 \in R$ is not nilpotent. Consider the ring

$$T = k[z_1][z_1^{-1}][x_1, \dots, x_4, z_2, z_3, z_4][a_{ij}^k],$$

and the two skewsymmetric matrices N_1 and N_2 with

$$N_1 = \begin{pmatrix} . & x_1 & x_2 & x_3 & x_4 \\ & . & a_{23}^1 & a_{24}^1 & a_{25}^1 \\ & & . & a_{34}^1 & a_{35}^1 \\ & -\text{sym} & & . & a_{45}^1 \\ & & & & . \end{pmatrix}$$

and $N_2 = A$, the generic Tom matrix defined in (3.1). Denote by I_i the ideal of T generated by the Pfaffians of N_i for $i = 1, 2$. Consider the automorphism $f \colon T \to T$ that is the identity on $k[z_1][z_1^{-1}][x_1, \dots, x_4, z_2, z_3, z_4]$, $f(a_{ij}^t) = a_{ij}^t$ if $t \neq 1$ and $f(a_{ij}^1) = \sum_{k=1}^4 a_{ij}^k z_k$ (f is automorphism since z_1 is invertible in T). Because N_1 is the generic skewsymmetric matrix, I_1 is prime of codimension 3 (see e.g. [KL]). Hence $I_2 = f(I_1)$ is also prime of codimension 3, which proves the lemma. QED

Set $U_i = \operatorname{Spec} R[z_i^{-1}] \subset X$, by Lemma 3.2.7 U_i is irreducible. Since the prime ideal $(x_k, a_{ij}^k) \subset R$ is in the intersection of all U_i, we have that $V = \cup U_i$ is also irreducible of dimension 29. Moreover, $X \setminus V = \operatorname{Spec} S/J$ has dimension 28, where $J = (z_1, \dots, z_4) \subset S$. Therefore, X has dimension 29, so I has codimension three. Since I is generated by Pfaffians, Theorem 1.3.1 implies that $R = S/I$ is Gorenstein. It follows that I is unmixed, so J is not contained in any associated prime of I.

Since $R[z_1^{-1}]$ is a domain, there is exactly one associated prime ideal P of R such that $z_1 \notin P$. If P is the single associated prime ideal of R, then z_1 is a regular element of R and R is also a domain. Suppose there is a second associated ideal $Q \neq P$. By what we have stated above and since $z_1 \in Q$, there is some $z_i \notin Q$. Since $R[z_i^{-1}]$ is a domain, it follows as before that $z_i \in P$. Now $PR[z_1^{-1}] = 0$, but the image of z_i in $PR[z_1^{-1}]$ is different from 0 (otherwise $z_i z_1^t \in I$ which clearly doesn't happen), a contradiction which finishes the second proof of Theorem 3.1.1.

3.3 Fundamental calculation for Tom

We work with the generic Tom over $S = k[x_k, z_k, a_{ij}^k]$ (see Section 3.1.1) and prove useful identities. Define I to be the ideal generated by the Pfaffians of the generic Tom matrix

$$A = \begin{pmatrix} . & x_1 & x_2 & x_3 & x_4 \\ & . & a_{23} & a_{24} & a_{25} \\ & & . & a_{34} & a_{35} \\ -\text{sym} & & & . & a_{45} \\ & & & & . \end{pmatrix} \tag{3.3}$$

where

$$a_{ij} = \sum_{k=1}^{4} a_{ij}^k z_k.$$

Explicitly, $I = (P_0, \dots, P_4)$ with

$$\begin{aligned} P_0 &= a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34} \\ P_1 &= x_2a_{45} - x_3a_{35} + x_4a_{34} \\ P_2 &= x_1a_{45} - x_3a_{25} + x_4a_{24} \\ P_3 &= x_1a_{35} - x_2a_{25} + x_4a_{23} \\ P_4 &= x_1a_{34} - x_2a_{24} + x_3a_{23} \end{aligned} \tag{3.4}$$

Clearly $I \subset J$, where $J = (z_1, z_2, z_3, z_4)$.

Since each P_i, for $1 \le i \le 4$, is linear in z_j, there exists (unique) 4×4 matrix Q independent of the z_j such that

$$\begin{pmatrix} P_1 \\ \vdots \\ P_4 \end{pmatrix} = Q \begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix} \tag{3.5}$$

We denote by Q_i the ith row of Q, and by $\widehat{Q}_i$ the submatrix of Q obtained by deleting the ith row. Since (compare (1.1))

$$x_4P_4 = x_1P_1 - x_2P_2 + x_3P_3,$$

and, as we noticed above, Q is independent of the z_j, it follows that

$$x_4Q_4 = x_1Q_1 - x_2Q_2 + x_3Q_3. \tag{3.6}$$

For $i = 1, \dots, 4$ we define a 1×4 matrix H_i by

$$H_i = \bigwedge^3 \widehat{Q}_i,$$

where $\bigwedge$ as in Definition 2.5.4.

Lemma 3.3.1 *For all i, j*

$$x_iH_j = x_jH_i.$$

Proof Equation (3.6) implies, for example, that

$$\bigwedge^3 \widehat{Q}_3 = \bigwedge^3 \begin{pmatrix} Q_1 \\ Q_2 \\ Q_4 \end{pmatrix} = \bigwedge^3 \begin{pmatrix} Q_1 \\ Q_2 \\ \frac{x_3}{x_4}Q_3 \end{pmatrix} = \frac{x_3}{x_4} \bigwedge^3 \widehat{Q}_4.$$

QED

Using the previous lemma we can define four polynomials g_i by

$$(g_1, g_2, g_3, g_4) = \frac{H_j}{x_j} \tag{3.7}$$

and this definition is independent of the choice of j.

Lemma 3.3.2 *For all i, j*

$$g_i z_j - g_j z_i \in I.$$

Proof The definition (3.5) of Q implies

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix}$$

By Cramer's rule (Lemma 2.5.5)

$$(H_4)_i z_j - (H_4)_j z_i \in I,$$

so

$$x_4(g_i z_j - g_j z_i) \in I.$$

I is prime by Theorem 3.1.1, so the result follows. QED

Remark 3.3.3 Of course, we can also express directly $g_i z_j - g_j z_i$ as a combination of the P_k. For example, Magma [Mag] gives:

```
g3*z4-g4*z3 =
    (x1*a241*a352  - x1*a251*a342 - x2*a241*a252 +
     x2*a242*a251  + x3*a231*a252 - x3*a232*a251 -
     x4*a231*a242  + x4*a232*a241)* P2+
    (-x1*a341*a352 + x1*a342*a351 - x2*a242*a351 +
     x2*a252*a341  - x3*a231*a352 + x3*a232*a351 +
     x4*a231*a342  - x4*a232*a341)* P3+
    (x1*a341*a452  - x1*a342*a451 - x2*a241*a452 +
     x2*a242*a451  + x3*a231*a452 - x3*a232*a451 +
```

```
 x3*a241*a352  - x3*a252*a341 - x4*a241*a342 +
 x4*a242*a341) * P4 +
(-x1*a351*a452 + x1*a352*a451 + x2*a251*a452 -
 x2*a252*a451  - x3*a251*a352 + x3*a252*a351 -
 x4*a231*a452  + x4*a232*a451 - x4*a242*a351 +
 x4*a251*a342) * P5
```

where we denote a_{ij}^k by $aijk$ etc.

Lemma 3.3.4 *There is no homogeneous polynomial $F \in S$ with $g_1 - Fz_1 \in I$.*

Proof Assume that such F exists. Then after specializing to the original Tom (see Subsection 3.3.1) we have a contradiction with Lemma 3.3.5. QED

3.3.1 The original Tom

The subsection is due to Reid. Write $Y \subset \mathbb{P}^8$ for the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. It is easy to see that Y is projectively Gorenstein of codimension four. The defining equations are $\operatorname{rank} L \leq 1$, where L is the generic 3×3 matrix

$$L = \begin{pmatrix} a & x_3 & x_4 \\ x_1 & z_1 & z_2 \\ x_2 & z_3 & z_4 \end{pmatrix}$$

and $a, x_1, \ldots, x_4, z_1, \ldots z_4$ are indeterminates. Let $X \subset \mathbb{P}^7$ be the image of the projection of Y from the point $((1,0,0),(1,0,0)) \in \mathbb{P}^2 \times \mathbb{P}^2$. Clearly, the ideal of X is generated by the five polynomials (the five minors of L *not* involving a)

$$I(X) = (x_3z_2 - x_4z_1, x_3z_4 - x_4z_3, x_1z_3 - x_2z_1, x_1z_4 - x_2z_2).$$

These are the Pfaffians of the skewsymmetric 5×5 matrix

$$M = \begin{pmatrix} . & x_1 & x_2 & x_3 & x_4 \\ & . & 0 & z_1 & z_2 \\ & & . & z_3 & z_4 \\ & & & . & 0 \\ & & & & . \end{pmatrix} \tag{3.8}$$

therefore we have a special Tom. X contains the complete intersection D, with $I(D) = (z_1, \dots, z_4)$.

Equations (3.7) specialize to

$$g_1 = x_1x_3, \quad g_2 = x_1x_4, \quad g_3 = x_2x_3, \quad g_4 = x_2x_4.$$

Lemma 3.3.5 *There is no homogeneous polynomial f with $g_1 - fz_1 \in I(X)$.*

Proof Clear, since each monomial appearing in an element of $I(X)$ is divisible by at least one of the z_j. QED

3.4 Generic projective Tom

In this section we calculate the unprojection of the generic projective Tom variety. The ambient space is $\mathbb{P} = \mathbb{P}^{31}$ with homogeneous coordinates x_k, z_k, a^k_{ij} for $1 \le k \le 4$, $2 \le i < j \le 5$. D is the complete intersection with ideal $I(D) = (z_1, \dots, z_4)$, and X the codimension three projectively Gorenstein subscheme with ideal $I(X) = (P_0, \dots, P_4)$ generated by the five Pfaffians (3.4) of the skewsymmetric matrix A defined in (3.3).

Since D is a complete intersection, $\omega_D = \mathcal{O}_D(-28)$. The minimal resolution for $I(X)$ has the form

$$0 \to \mathcal{O}(-8) \to \mathcal{O}(-4) \oplus \mathcal{O}(-5)^4 \to \mathcal{O}(-4) \oplus \mathcal{O}(-3)^4 \to \mathcal{O},$$

therefore $\omega_X = \mathcal{O}_X(-24)$.

The exact sequence (2.13) for the pair $D \subset X$ becomes

$$0 \to \mathcal{O}_X(4) \to \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_X(4)) \to \mathcal{O}_D \to 0.$$

Taking global sections we have the exact sequence

$$0 \to H^0(\mathcal{O}_X(4)) \to \operatorname{Hom}(\mathcal{I}_D, \mathcal{O}_X(4)) \xrightarrow{\operatorname{res}_D} H^0(\mathcal{O}_D) \to 0.$$

Each g_i defined in (3.7) is homogeneous of degree 5, therefore using Lemma 3.3.2 and the Serre correspondence there is a well defined map of sheaves

$$g \colon \mathcal{I}_D \to \mathcal{O}_X(4),$$

with $z_i \mapsto g_i$. Since $\operatorname{res}_D(g) = 0$ contradicts Lemma 3.3.4, we have proved the following theorem.

Theorem 3.4.1 *The map g is an unprojection, in the sense that $\operatorname{res}_D(g) \neq 0$ as an element of $H^0(\mathcal{O}_D) = k$.*

3.5 Local Tom

3.5.1 The commutative diagram

In this subsection we work over the polynomial ring $S = \mathbb{Z}[x_k, z_k, a_{ij}^k]$ with indices as in Subsection 3.1.1. Let A be the skewsymmetric matrix defined in (3.3), $I \subset S$ the ideal generated by the Pfaffians of A (see (3.4)) and $J = (z_1, \ldots, z_4)$. The methods of Subsection 3.2.8 prove that I is a prime ideal of codimension three.

Consider the Koszul complex $\mathbf{M}$ that gives a resolution of the ring S/J

$$0 \to S \xrightarrow{B_4} S^4 \xrightarrow{B_3} S^6 \xrightarrow{B_2} S^4 \xrightarrow{B_1} S \to 0,$$

with

$$B_1 = (z_1, z_2, z_3, z_4),$$
$$B_2 = \begin{pmatrix} -z_2 & -z_3 & 0 & -z_4 & 0 & 0 \\ z_1 & 0 & -z_3 & 0 & -z_4 & 0 \\ 0 & z_1 & z_2 & 0 & 0 & -z_4 \\ 0 & 0 & 0 & z_1 & z_2 & z_3 \end{pmatrix}$$

and

$$B_3 = \begin{pmatrix} z_3 & z_4 & 0 & 0 \\ -z_2 & 0 & z_4 & 0 \\ z_1 & 0 & 0 & z_4 \\ 0 & -z_2 & -z_3 & 0 \\ 0 & z_1 & 0 & -z_3 \\ 0 & 0 & z_1 & z_2 \end{pmatrix}, \quad B_4 = (-z_4, z_3, -z_2, z_1)^t.$$

Moreover, the skewsymetric matrix A defines as in (1.3.1) a complex $\mathbf{L}$:

$$0 \to S \xrightarrow{C_3} S^5 \xrightarrow{C_2} S^5 \xrightarrow{C_1} S \to 0 \qquad (3.9)$$

resolving the ring S/I. Here $C_2 = A, C_1 = (P_0, -P_1, P_2, -P_3, P_4)$, and C_3 is the transpose matrix of C_1. Define the 4×1 matrix D_3 with

$$D_3 = (-g_4, g_3, -g_2, g_1)^t,$$

where the g_i are as in (3.7).

Theorem 3.5.1 *There exist matrices D_2, D_1, D_0 (of suitable sizes) making the following diagram commutative.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \xrightarrow{C_3} & S^5 & \xrightarrow{C_2} & S^5 & \xrightarrow{C_1} & S \\
 & & \downarrow{\scriptstyle D_3} & & \downarrow{\scriptstyle D_2} & & \downarrow{\scriptstyle D_1} & & \downarrow{\scriptstyle D_0} \\
S & \xrightarrow{B_4} & S^4 & \xrightarrow{B_3} & S^6 & \xrightarrow{B_2} & S^4 & \xrightarrow{B_1} & S
\end{array}$$

In addition we can assume that $D_0 = 1 \in \mathbb{Z}$.

Proof

Step 1 As in (2.21), the dual complexes

$$S^* \to S^{5*} \to S^{5*} \to S^*$$

and

$$S^* \to S^{4*} \to S^{6*} \to S^{4*}$$

are exact. Using Lemma 3.3.2, there exists D_2^* making the (dual) square commutative. Then, the existence of D_1^* and D_0^* follows by simple homological algebra.

We get $D_0 \in \mathbb{Z}$ by checking the degrees in the commutative diagram.

Step 2 We prove that we can take $D_0 = 1$ by a specialization argument to the original Tom (compare Subsection 3.3.1).

For the original Tom, an easy calculation using (the specialization of) the complexes **L** and **M** gives that we can take in (the specialization of) the diagram of the theorem

$$D_2' = \begin{pmatrix} 0 & -x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_4 & 0 \\ 0 & 0 & x_2 & x_3 & 0 \\ 0 & x_1 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & x_1 & 0 & 0 \end{pmatrix},$$

$$D_1' = \begin{pmatrix} -z_4 & 0 & x_4 & 0 & -x_2 \\ 0 & 0 & -x_3 & x_2 & 0 \\ z_2 & -x_4 & 0 & 0 & x_1 \\ 0 & x_3 & 0 & -x_1 & 0 \end{pmatrix}$$

and $D'_0 = 1$.

Using the uniqueness up to homotopy of a map between resolutions of modules induced by a fixed map between the modules, the last part of the theorem follows from $D'_0 = 1$.

QED

3.5.2 Local Tom

Let R be a Gorenstein local ring, $a^k_{ij} \in R$ and $x_k, z_k \in m$, the maximal ideal of R, with indices as in Subsection 3.1.1. Let A be the skewsymmetric matrix (with entries in R) defined in (3.3), I the ideal generated by the Pfaffians of A (see (3.4)) and $J = (z_1, \ldots, z_4)$.

We assume that $z_1, \ldots, z_4$ is a regular sequence and that I has codimension three, the maximal possible. Since R is Cohen–Macaulay, the grade of I is also three. By Theorem 1.3.1, the complex $\mathbf{L}$ defined in (1.1) is the minimal resolution of R/I and R/I is Gorenstein. According to Section 2.1, we can unproject the pair $I \subset J$.

Recall that in (3.7) we defined elements g_i which are polynomials of a^k_{ij} and x_k. Define a map $\psi\colon J/I \to R/I$ with $z_i \mapsto g_i$. By res we denote the residue map defined in (2.1).

Theorem 3.5.2 *The element* $\operatorname{res}(\psi) \in S/J$ *is a unit, and the ideal*

$$(P_0, \ldots, P_4, Tz_1 - g_1, \ldots, Tz_4 - g_4) \tag{3.10}$$

of the polynomial ring $S[T]$ *is Gorenstein of codimension four.*

Proof The theorem follows immediately from Theorem 3.5.1 (since the diagram is defined over $\mathbb{Z}$), Theorem 2.5.2, and Theorem 2.1.10. QED

3.6 A Maple routine that calculates Tom

The following is a Maple [Map] routine that calculates Tom unprojection. The input is a Tom matrix, and returns the unprojection vector $(g_1, \ldots, g_4)$ defined in (3.7).

```
pfaf := proc (a,b,c,d,e,f);
  pfaf := a*f-b*e+c*d;
end:

tomunproj := proc (data) local N,P1,P2,P3,
             P4,P5,L,o4,o3,o2,o1;
N := data:
P1 := expand(pfaf ( N[2,3], N[2,4], N[2,5],
                 N[3,4], N[3,5], N[4,5] )):
P2 := pfaf ( N[1,3], N[1,4], N[1,5], N[3,4],
                 N[3,5], N[4,5] ):
P3 := expand(pfaf ( N[1,2], N[1,4], N[1,5],
                 N[2,4], N[2,5], N[4,5] )):
P4 := expand(pfaf ( N[1,2], N[1,3], N[1,5],
                 N[2,3], N[2,5], N[3,5] )):
P5 := expand(pfaf ( N[1,2], N[1,3], N[1,4],
                 N[2,3], N[2,4], N[3,4] )):
L := matrix ( 3,4, [coeff(P2,z1),coeff(P2,z2),
       coeff(P2,z3),coeff(P2,z4), coeff(P3,z1),
       coeff(P3,z2),coeff(P3,z3),coeff(P3,z4),
       coeff(P4,z1),coeff(P4,z2), coeff(P4,z3),
       coeff(P4,z4)]);
divide( det(submatrix(L, [1,2,3], [1,2,3])),
        x4, 'temp'):   o4 := -temp;
divide( det(submatrix(L, [1,2,3], [1,2,4])),
        x4, 'temp'):o3 := temp;
divide( det(submatrix(L, [1,2,3], [1,3,4])),
        x4, 'temp'):o2 := -temp;
divide( det(submatrix(L, [1,2,3], [2,3,4])),
        x4, 'temp'):o1:=temp;
 matrix (1,4, [o1,o2,o3,o4]):
end:
```

Example 3.6.1 For

$$A = \begin{pmatrix} . & x_1 & x_2 & x_3 & x_4 \\ & . & 0 & z_1 & z_2 \\ & & . & z_3 & z_4 \\ & & & & z_3 \\ & & & & . \end{pmatrix}$$

it returns

$$(x_1x_3, x_4x_1 + x_1x_2, x_2x_3, x_2x_4 + x_2^2).$$

3.7 Triadic decomposition for Tom

In the following we give a combinatorial procedure which we conjecture (Conjecture 3.7.7) it calculates the Tom unprojection, reducing it to a sum of elementary cases. We do not have at present applications of the conjecture, our main motivation is the analogy with combinatorial results in representation theory and Schubert calculus.

We work over the polynomial ring $S = k[x_k, z_k, f_{ij}^k]$ for $1 \leq k \leq 4,\ 2 \leq i < j \leq 5$.

For this section a Tom matrix is one of the form

$$A = A(a_{ij}^k) = \begin{pmatrix} . & x_1 & x_2 & x_3 & x_4 \\ & . & a_{23} & a_{24} & a_{25} \\ & & . & a_{34} & a_{35} \\ & & & . & a_{45} \\ & & & & . \end{pmatrix}$$

where

$$a_{ij} = \sum_{k=1}^{4} a_{ij}^k z_k,$$

with $a_{ij}^k \in k[f_{ij}^k]$, the polynomial ring in a single indeterminate f_{ij}^k.

We denote by $\mathcal{T}$ the set of all such A, and by $\mathcal{T}_1 \subset \mathcal{T}$ the set of matrices $A \in \mathcal{T}$ with at least three a_{ij} nonzero.

We define projection maps $\eta_{ij}^k \colon \mathcal{T} \to k[f_{ij}^k]$ with

$$\eta_{ij}^k(A) = a_{ij}^k, \tag{3.11}$$

and the map

$$\mathrm{red}\colon \mathcal{T} \to \mathcal{T} \tag{3.12}$$

specified by the property $\eta^k_{ij}(\mathrm{red}(A)) = 1$ if $\eta^k_{ij}(A) \neq 0$ and 0 otherwise, i.e., that changes all nonzero coefficients a^k_{ij} to one. For $A \in \mathcal{T}$ we call content of A and denote by $\mathrm{cont}(A)$ the product of all nonzero $\eta^k_{ij}(A)$

$$\mathrm{cont}(A) = \prod_{\eta^k_{ij}(A)\neq 0} \eta^k_{ij}(A). \tag{3.13}$$

Definition 3.7.1 An *almost elementary* Tom matrix A is one with $\eta^k_{ij}(A) \neq 0$ for exactly three indices (i_l, j_l, k_l), $1 \leq l \leq 3$ and moreover, the three such (i_l, j_l) are distinct. It is called *elementary* if in addition $\eta^{k_l}_{i_l j_l}(A) = 1$ for $1 \leq l \leq 3$.

Notation Since all matrices $A \in \mathcal{T}$ are skewsymmetric and have the same first row, we will not write down the first and the last row and the first two columns of A.

Example 3.7.2 The matrix $C_1 \in \mathcal{T}_1$ with

$$C_1 = \begin{pmatrix} 0 & z_1 & (8 + f^3_{25})z_3 \\ & 2z_3 & 0 \\ & & 0 \end{pmatrix}$$

is almost elementary but not elementary with $\mathrm{cont}(C_1) = 16 + 2f^3_{25}$, while

$$\mathrm{red}(C_1) = \begin{pmatrix} 0 & z_1 & z_3 \\ & z_3 & 0 \\ & & 0 \end{pmatrix}$$

is elementary.

Definition 3.7.3 Assume $A \in \mathcal{T}_1$. An almost elementary Tom matrix B is a component of A if $\eta^k_{ij}(B) = \eta^k_{ij}(A)$ whenever $\eta^k_{ij}(B) \neq 0$. Clearly, A has a finite set of components $\{B_i\}$, we denote this set by $\mathrm{comp}(A)$.

Example 3.7.4 The set of components of

$$A = \begin{pmatrix} 0 & z_1 & z_2 + (8 + f^3_{25})z_3 \\ & z_3 & 2z_4 \\ & & 0 \end{pmatrix}$$

is

$$\mathrm{comp}(A) = \Big\{ \begin{pmatrix} 0 & z_1 & z_2 \\ & z_3 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & z_1 & z_2 \\ & 0 & 2z_4 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & z_1 & (8+f_{25}^3)z_3 \\ & z_3 & 0 \\ & & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & z_1 & (8+f_{25}^3)z_3 \\ & 0 & 2z_4 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & z_2 \\ & z_3 & 2z_4 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & (8+f_{25}^3)z_3 \\ & z_3 & 2z_4 \\ & & 0 \end{pmatrix} \Big\}.$$

Denote by V the free $k[f_{ij}^k, x_k]$-module

$$V = k[f_{ij}^k, x_k]^4. \tag{3.14}$$

We will define a map

$$s\colon \mathcal{T} \to V \tag{3.15}$$

such that whenever unprojection for a Tom matrix A makes sense, it will be given by $z_i \mapsto s_i$, where $s(A) = (s_1, s_2, s_3, s_4)$.

The main point is to define it for elementary Toms. The general definition will follow using the decomposition to components and the coefficient function cont.

3.7.1 Definition of s for elementary Tom

To keep track of an elementary Tom we need the following set of indices

$$\begin{aligned} I_1 &= \big\{(i,j) : 1 \le i < j \le 4\big\} \\ I_2 &= \big\{1,2,3,4\big\}. \end{aligned}$$

We use the lexicographic ordering $<$ for I_1 (e.g. $(1,2) < (1,4) < (2,3)$), and we define the sets

$$\begin{aligned} I_3 &= \big\{(t_k, r_k) \in (I_1 \times I_2)^3 : t_1 < t_2 < t_3\big\} \\ I_4 &= \big\{(t_k, r_k) \in I_3 : r_1, r_2, r_3 \text{ are distinct } \big\} \subset I_3 \\ I_5 &= \big\{(t_k, k) \in I_4\big\} \subset I_4. \end{aligned}$$

An element $u = (t_k = (i_k, j_k), r_k) \in I_3$ specifies uniquely the elementary Tom matrix $A = A(u)$ with the property

$$A_{i_k+1, j_k+1} = z_{r_k}, \quad \text{for } 1 \le k \le 3. \tag{3.16}$$

For example, the corresponding vector u to the matrix $\mathrm{red}(C_1)$ in Example 3.7.2 is $u = \{(131, 143, 233)\}$. In this way we identify the set of elementary Tom matrices with I_3. Moreover, using the natural projection $(t_k, k) \mapsto (t_k)$ we identify the set I_5 with I_6, where

$$I_6 = \big\{(t_k) \in (I_1)^3 : t_1 < t_2 < t_3\big\}.$$

We define an auxiliary map

$$q\colon I_4 \to k[x_1, x_2, x_3, x_4].$$

First of all, assume $u = (i_1j_1, i_2j_2, i_3j_3) \in I_6$. Consider the sequence

$$e = (i_1, i_2, i_3, j_1, j_2, j_3) \in (I_2)^6. \tag{3.17}$$

If an index $a \in I_2$ appears more than two times in e we set $q(u) = 0$. Otherwise, either there are exactly two distinct indices $a, b \in I_2$ appearing once in e, or a single index a doesn't appear in e. In the first case we set $q(u) = w(u)x_ax_b$, in the second $q(u) = w(u)x_a^2$. Now we define signs $w(u) \in \{1, -1\}$. If $u \notin \{(14, 23, 24), (13, 23, 24)\}$ we set $w(u) = (-1)^p$ with $p = i_1+i_2+i_3+j_1+j_2+j_3$, but we set $w(14, 23, 24) = -1, w(13, 23, 24) = 1$.

Remark 3.7.5 The choice of the sign $(-1)^p$, $p = i_1+i_2+i_3+j_1+j_2+j_3$ has a straightforward combinatorial meaning. Consider the basic configuration $\{(i, j) : 1 \le i < j \le 4\}$ with signs as in

$$\begin{pmatrix} - & + & - \\ & - & + \\ & & - \end{pmatrix}$$

and three positions (i_l, j_l) of the six. Then $(-1)^p$ is the product of the signs of the three positions.

So, using the identification of I_5 with I_6 we have defined $q(u)$ for $u \in I_5$. Now assume $u = (t_k, r_k) \in I_4$. Consider the unique permutation $\sigma \in \mathfrak{S}_4$, the symmetric group in four elements, with $\sigma(r_k) = k$ for $k = 1, 2, 3$. Define $u_1 \in I_5$ with

$$u_1 = (t_k, k)$$

and set

$$q(u) = \mathrm{sign}(\sigma)q(u_1),$$

where sign: $\mathfrak{S}_4 \to \{1,-1\}$ is the standard group homomorphism with kernel the alternating group.

Using the function q we define the function s for elementary Toms. Assume $u = (t_k, r_k) \in I_3$ and $A = A(u) \in \mathcal{T}$ the corresponding elementary Tom matrix. If $u \in I_3 \setminus I_4$, i.e., some z_i is repeated in A we set $s(A) = 0$. Assume now that $u \in I_4$. Let r_4 be such that $\{r_1, r_2, r_3, r_4\} = \{1,2,3,4\}$. Then we define $s(A)$ to have 0 in the r_1, r_2, r_3 coordinates and $(-1)^{r_4} q(u)$ in the r_4th. Therefore, we have defined the function s for elementary Tom matrices. In the next subsection we will extend the definition to general Tom matrices.

3.7.2 Definition of s for general Tom

Suppose that A is almost elementary. Then $\mathrm{red}(A)$ is elementary and we set

$$s(A) = \mathrm{cont}(A) s(\mathrm{red}(A)).$$

For $A \in \mathcal{T}_1$ with component set $\mathrm{comp}(A) = \{B_i\}$, with B_i almost elementary Tom, we define

$$s(A) = \sum_i s(B_i),$$

the addition being pointwise in V.

Finally, if $A \in \mathcal{T} \setminus \mathcal{T}_1$, i.e., A has less than three nonzero coefficients a_{ij} we set

$$s(A) = (0,0,0,0).$$

We have completed the definition of the function $s\colon \mathcal{T} \to V$.

Example 3.7.6 We calculate $s(A)$ for

$$A = \begin{pmatrix} 0 & z_1 & z_2 \\ & z_3 & z_4 \\ & & z_3 \end{pmatrix}$$

the matrix of Example 3.6.1. Clearly

$$\mathrm{comp}(A) = \Big\{ \begin{pmatrix} 0 & z_1 & z_2 \\ & z_3 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & z_1 & z_2 \\ & 0 & z_4 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & z_1 & z_2 \\ & 0 & 0 \\ & & z_3 \end{pmatrix}, \begin{pmatrix} 0 & z_1 & 0 \\ & z_3 & z_4 \\ & & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & 0 \\ & z_3 & z_4 \\ & & z_3 \end{pmatrix} \Big\}$$

Then

$$s\left(\begin{pmatrix}0 & z_1 & z_2\\ & z_3 & 0\\ & & 0\end{pmatrix}\right) = (0,0,0,x_2x_4),$$

since the occupied positions are $13, 14, 23$ (compare (3.16)), and the missing z_i is z_4. Similarly

$$s\left(\begin{pmatrix}0 & z_1 & z_2\\ & 0 & z_4\\ & & 0\end{pmatrix}\right) = (0,0,x_2x_3,0),\ s\left(\begin{pmatrix}0 & z_1 & z_2\\ & 0 & 0\\ & & z_3\end{pmatrix}\right) = (0,0,0,x_2^2),$$

$$s\left(\begin{pmatrix}0 & z_1 & 0\\ & z_3 & z_4\\ & & 0\end{pmatrix}\right) = (0,x_1x_4,0,0),\ s\left(\begin{pmatrix}0 & z_1 & 0\\ & z_3 & 0\\ & & z_3\end{pmatrix}\right) = (0,0,0,0),$$

$$s\left(\begin{pmatrix}0 & z_1 & 0\\ & 0 & z_4\\ & & z_3\end{pmatrix}\right) = (0,x_1x_2,0,0),\ s\left(\begin{pmatrix}0 & 0 & z_2\\ & z_3 & z_4\\ & & 0\end{pmatrix}\right) = (x_1x_3,0,0,0),$$

$$s\left(\begin{pmatrix}0 & 0 & z_2\\ & z_3 & 0\\ & & z_3\end{pmatrix}\right) = (0,0,0,0),\ s\left(\begin{pmatrix}0 & 0 & 0\\ & z_3 & z_4\\ & & z_3\end{pmatrix}\right) = (0,0,0,0).$$

Therefore,

$$s(A) = (x_1x_3, x_4x_1 + x_1x_2, x_2x_3, x_2x_4 + x_2^2),$$

as predicted by the Maple routine in Example 3.6.1

3.7.3 Justification of triadic decomposition

Denote by A the 'generic' Tom with $\eta_{ij}^k(A) = f_{ij}^k$. According to Section 3.3, x_4 divides the four elements h_t of $\bigwedge^3 M = (h_1, \dots, h_4)$, where

$$M = \begin{pmatrix} x_2f^1_{45}-x_3f^1_{35}+x_4f^1_{34} & x_2f^2_{45}-x_3f^2_{35}+x_4f^2_{34} & x_2f^3_{45}-x_3f^3_{35}+x_4f^3_{34} & x_2f^4_{45}-x_3f^4_{35}+x_4f^4_{34}\\ x_1f^1_{45}-x_3f^1_{25}+x_4f^1_{24} & x_1f^2_{45}-x_3f^2_{25}+x_4f^2_{24} & x_1f^3_{45}-x_3f^3_{25}+x_4f^3_{24} & x_1f^4_{45}-x_3f^4_{25}+x_4f^4_{24}\\ x_1f^1_{35}-x_2f^1_{25}+x_4f^1_{23} & x_1f^2_{35}-x_2f^2_{25}+x_4f^2_{23} & x_1f^3_{35}-x_2f^3_{25}+x_4f^3_{23} & x_1f^4_{35}-x_2f^4_{25}+x_4f^4_{23} \end{pmatrix}$$

and the unprojection is given by

$$g_t = \frac{h_t}{x_4} \qquad \text{for } 1 \le t \le 4.$$

In the previous subsections we defined a map

$$s\colon \mathcal{T} \to V.$$

Conjecture 3.7.7 We have

$$s(A) = (g_1, g_2, g_3, g_4). \tag{3.18}$$

In the following we study in more detail the two parts of (3.18) in an effort to justify the conjecture.

For simplicity we argue without taking care of the signs. Set $M = [m_{ij}]$. Then

$$\begin{aligned} m_{1k} &= x_2 f^k_{45} - x_3 f^k_{35} + x_4 f^k_{34} \\ m_{2k} &= x_1 f^k_{45} - x_3 f^k_{25} + x_4 f^k_{24} \\ m_{3k} &= x_1 f^k_{35} - x_2 f^k_{25} + x_4 f^k_{23} \end{aligned}$$

We notice that if we change in h_1 all higher indices from t to 1 we get h_t (up to sign!), for $2 \le t \le 4$. Moreover, by the definition of $s(A) = (s_1, s_2, s_3, s_4)$ the same property is true for the s_i. Therefore, it is enough to compare h_4 with s_4.

Now

$$h_4 = \sum m_{1i_1} m_{2i_2} m_{3i_3},$$

the sum for $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ (as usual we forget the signs!). But $m_{1i_1} m_{2i_2} m_{3i_3}$ is $m_{11} m_{22} m_{33}$ after changing higher indices t to i_t. So it enough to concentrate on

$$m = m_{11} m_{22} m_{33}.$$

Consider the matrix

$$L = \begin{pmatrix} x_2 f^1_{45} & x_3 f^1_{35} & x_4 f^1_{34} \\ x_1 f^2_{45} & x_3 f^2_{25} & x_4 f^2_{24} \\ x_1 f^3_{35} & x_2 f^3_{25} & x_4 f^3_{23} \end{pmatrix}$$

We define $L(ijk) = L_{1i} L_{2j} L_{3k}$, then

$$m = \sum L(ijk),$$

the sum for $1 \le i, j, k \le 3$.

Since x_4 divides m, terms $L(ijk)$ do not contribute in m whenever $\{i, j, k\} \subseteq \{1, 2\}$. So we are left with terms $L(3, j, k), L(i, 3, k), L(i, j, 3)$.

Consider for example the term $L(3, 1, 1) = f^1_{34} f^2_{45} f^3_{35} x_1^2 x_4$. We have

$$s(\begin{pmatrix} 0 & 0 & 0 \\ & f^1_{34} z_1 & f^3_{35} z_3 \\ & & f^2_{45} z_2 \end{pmatrix}) = (0, 0, 0, f^1_{34} f^2_{45} f^3_{35} x_1^2) = L(3, 1, 1). \tag{3.19}$$

Now, calculating each term $L(ijk)$ contributing to m, we get the following matrix of equations

$$O = \begin{pmatrix} L(3,1,1)/x_4 = f^1_{34} f^2_{45} f^3_{35} x_1^2 \\ L(3,1,2)/x_4 = f^1_{34} f^2_{45} f^3_{25} x_1 x_2 \\ L(3,1,3)/x_4 = f^1_{34} f^2_{45} f^3_{23} x_1 x_4 \\ L(3,2,1)/x_4 = f^1_{34} f^2_{25} f^3_{35} x_1 x_3 \\ L(3,2,2)/x_4 = 0 \\ L(3,2,3)/x_4 = f^1_{34} f^2_{25} f^3_{23} x_3 x_4 \\ L(3,3,1)/x_4 = f^1_{34} f^2_{24} f^3_{35} x_1 x_4 \\ L(3,3,2)/x_4 = f^1_{34} f^2_{24} f^3_{25} x_2 x_4 \\ L(3,3,3)/x_4 = f^1_{34} f^2_{24} f^3_{23} x_4^2 \\ L(1,3,1)/x_4 = f^1_{45} f^2_{24} f^3_{35} x_1 x_2 \\ L(1,3,2)/x_4 = f^1_{45} f^2_{24} f^3_{25} x_2^2 \\ L(1,3,3)/x_4 = f^1_{45} f^2_{24} f^3_{23} x_2 x_4 \\ L(2,3,1)/x_4 = 0 \\ L(2,3,2)/x_4 = f^1_{35} f^2_{24} f^3_{25} x_2 x_3 \\ L(2,3,3)/x_4 = f^1_{35} f^2_{24} f^3_{23} x_3 x_4 \\ L(1,1,3)/x_4 = 0 \\ L(1,2,3)/x_4 = f^1_{45} f^2_{25} f^3_{23} x_2 x_3 \\ L(2,1,3)/x_4 = f^1_{35} f^2_{45} f^3_{23} x_1 x_3 \\ L(2,2,3)/x_4 = f^1_{35} f^2_{25} f^3_{23} x_3^2 \end{pmatrix}$$

As one can check, the elements of the matrix O are 'compatible' (in the sense of (3.19)) with the monomials in definition of $s(A)$ (at least up to sign). We believe that the previous arguments justify, but certainly not prove, Conjecture 3.7.7.

3.8 Fundamental calculation for Jerry

We work with the generic Jerry over $S = k[x_i, z_k, a_i^k, b_i^k, c^k]$ (see Section 3.1.2).

Define I to be the ideal generated by the Pfaffians of the generic Jerry matrix

$$B = \begin{pmatrix} . & c & a_1 & a_2 & a_3 \\ & . & b_1 & b_2 & b_3 \\ & & . & x_1 & x_2 \\ -\text{sym} & & & . & x_3 \\ & & & & . \end{pmatrix} \tag{3.20}$$

where

$$a_i = \sum_{k=1}^{4} a_i^k z_k, \quad b_i = \sum_{k=1}^{4} b_i^k z_k, \quad c = \sum_{k=1}^{4} c^k z_k.$$

Explicitly, $I = (P_1, \dots, P_5)$ with

$$\begin{aligned} P_1 &= b_1x_3 - b_2x_2 + b_3x_1 \\ P_2 &= a_1x_3 - a_2x_2 + a_3x_1 \\ P_3 &= cx_3 - a_2b_3 + a_3b_2 \\ P_4 &= cx_2 - a_1b_3 + a_3b_1 \\ P_5 &= cx_1 - a_1b_2 + a_2b_1 \end{aligned} \tag{3.21}$$

Clearly $I \subset J = (z_1, z_2, z_3, z_4)$.

Unlike the Tom case, we only have two Pfaffians, P_1 and P_2, linear in z_k. P_3 is quadratic in z_k but after choosing to consider a_2, a_3 as indeterminates it can be considered linear. Using this convention we write

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = Q \begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix} \tag{3.22}$$

Q is a 3×4 matrix, with

$$\begin{aligned} Q_{1k} &= b_1^k x_3 - b_2^k x_2 + b_3^k x_1 \\ Q_{2k} &= a_1^k x_3 - a_2^k x_2 + a_3^k x_1 \\ Q_{3k} &= c^k x_3 - a_2 b_3^k + a_3 b_2^k \end{aligned}$$

We define h_i by

$$\bigwedge^3 Q = (h_1, \dots, h_4).$$

($\bigwedge$ as in Definition 2.5.4.)

Lemma 3.8.1 *For $i = 1, \dots, 4$ there exist polynomials K_i, L_i with*

$$h_i = x_3 K_i + (a_2x_2 - a_3x_1)L_i.$$

Therefore, we can write

$$h_i = x_3(K_i + a_1L_i) - L_iP_2.$$

Proof Let M be the matrix obtained from Q by substituting $x_3 = 0$. Since

$$\begin{aligned} M_{1k} &= x_1 b_3^k - x_2 b_2^k \\ M_{3k} &= -a_2 b_3^k + a_3 b_2^k \end{aligned}$$

we get

$$M = \begin{pmatrix} x_1 & 0 & -x_2 \\ 0 & 1 & 0 \\ -a_2 & 0 & a_3 \end{pmatrix} \begin{pmatrix} b_3^1 & b_3^2 & b_3^3 & b_3^4 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ b_2^1 & b_2^2 & b_2^3 & b_2^4 \end{pmatrix}$$

The lemma follows from elementary properties of determinants. QED

We fix the polynomials K_i, L_i defined (implicitly) in the proof of Lemma 3.8.1. For $i = 1, \ldots, 4$ we define polynomials g_i by

$$g_i = K_i + a_1 L_i. \tag{3.23}$$

Lemma 3.8.2 *For all i, j*

$$g_i z_j - g_j z_i \in I.$$

Proof Using Cramer's rule (Lemma 2.5.5) (3.22) implies

$$h_i z_j - h_j z_i \in I.$$

Therefore

$$x_3(g_i z_j - g_j z_i) \in I.$$

I is prime by Theorem 3.1.2, so the result follows. QED

Lemma 3.8.3 *There is no homogeneous polynomial $F \in S$ with $g_3 - F z_3 \in I$.*

Proof Assume that such F exists. Then after specializing to the original Jerry (see Subsection 3.8.1) we have a contradiction with Lemma 3.8.4. QED

3.8.1 The original Jerry

The treatment here is due to Reid, for more details see [R1], Example 6.10.

Write $Y \subset \mathbb{P}^7$ for the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Y is projectively Gorenstein of codimension four. Let $X \subset \mathbb{P}^6$ be the image of the projection of Y from the point $((1,0),(1,0),(1,0)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. If $z_1, \dots, z_4, x_1, \dots, x_3$ are homogeneous coordinates for $\mathbb{P}^6$, the homogeneous ideal of X is given by the Pfaffians of the skewsymmetric matrix

$$M = \begin{pmatrix} . & z_1 & z_2 & z_3 & 0 \\ & . & 0 & z_3 & z_4 \\ & & . & x_1 & x_2 \\ & & & & x_3 \\ & & & & . \end{pmatrix}$$

Namely,

$$I(X) = (-z_3x_2 + z_4x_1, z_2x_3 - z_3x_2, z_1x_3 - z_3z_4, z_1x_2 - z_2z_4, z_1x_1 - z_2z_3).$$

Equations (3.23) specialize to

$$g_1 = z_2x_3, \quad g_2 = x_1x_2, \quad g_3 = x_1x_3, \quad g_4 = x_2x_3.$$

Lemma 3.8.4 *There is no homogeneous polynomial f with $g_3 - fz_3 \in I(X)$.*

Proof Clear, since each monomial appearing in an element of $I(X)$ is divisible by at least one of the z_j. QED

3.9 Generic projective Jerry

In this section we calculate the unprojection of the generic projective Jerry variety. The arguments are similar with those in Section 3.4, but in order to avoid working in a weighted projective space we change our base field to the algebraic closure F of the field of rational functions $k(c^k, a_i^k, b_i^k)$ for $1 \le i \le 3,\ 1 \le k \le 4$. The ambient space is $\mathbb{P}_F = \mathbb{P}^6_F$ with homogeneous coordinates $x_1, x_2, x_3, z_1, \dots, z_4$. D is the complete intersection with ideal $I(D) = (z_1, \dots, z_4)$ and X is the codimension three projectively Gorenstein subscheme with ideal $I(X) = (P_1, \dots, P_5)$ generated by the five Pfaffians written in (3.21) of the skewsymmetric matrix B defined in (3.20).

Since D is a complete intersection, $\omega_D = \mathcal{O}_D(-3)$. The minimal resolution for $I(X)$ has the form

$$0 \to \mathcal{O}(-5) \to \mathcal{O}(-3)^5 \to \mathcal{O}(-2)^5 \to \mathcal{O},$$

therefore $\omega_X = \mathcal{O}_X(-2)$.

The exact sequence for the pair $D \subset X$ defined in Section 2.13 becomes

$$0 \to \mathcal{O}_X(1) \to \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_X(1)) \to \mathcal{O}_D \to 0.$$

Taking global sections there is an exact sequence

$$0 \to H^0(\mathcal{O}_X(1)) \to \operatorname{Hom}(\mathcal{I}_D, \mathcal{O}_X(1)) \xrightarrow{\operatorname{res}_D} H^0(\mathcal{O}_D) \to 0.$$

Each g_i defined in (3.23) is homogeneous (in the x_i, z_k) of degree 2, therefore using Lemma 3.8.2 and Serre correspondence we have a well defined map of sheaves

$$g \colon \mathcal{I}_D \to \mathcal{O}_X(1),$$

with $z_i \mapsto g_i$. Since $\operatorname{res}_D(g) = 0$ contradicts Lemma 3.8.3, we have proved the following theorem.

Theorem 3.9.1 *The map g is an unprojection, in the sense that* $\operatorname{res}_D(g) \neq 0$ *as an element of $H^0(\mathcal{O}_D) = k$.*

3.10 Local Jerry

This section is the Jerry counterpart of Section 3.5, and the arguments are very similar.

3.10.1 The commutative diagram

In this subsection we work over the polynomial ring

$$S = \mathbb{Z}[x_k, z_k, a_i^k, b_i^k, c^k]$$

with indices as in Subsection 3.1.2. Let B be the skewsymmetric matrix defined in (3.2), $I \subset S$ the ideal generated by the Pfaffians of B (see (3.21)), and $J = (z_1, \dots, z_4)$.

Consider as in Subsection 3.5.1 the Koszul complex $\mathbf{M}$ resolving S/J, the complex $\mathbf{L}$ resolving S/I, and define the 4×1 matrix D_3 with

$$D_3 = (-g_4, g_3, -g_2, g_1)^t,$$

where the g_i are as in (3.23).

Except from the part $D_0 = 1 \in \mathbb{Z}$, the following theorem follows immediately using the arguments in the proof of Theorem 3.5.1.

Theorem 3.10.1 *There exist matrices D_2, D_1, D_0 (of suitable sizes) making the following diagram commutative.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \xrightarrow{C_3} & S^5 & \xrightarrow{C_2} & S^5 & \xrightarrow{C_1} & S \\ & & {\scriptstyle D_3}\downarrow & & {\scriptstyle D_2}\downarrow & & {\scriptstyle D_1}\downarrow & & {\scriptstyle D_0}\downarrow \\ S & \xrightarrow{B_4} & S^4 & \xrightarrow{B_3} & S^6 & \xrightarrow{B_2} & S^4 & \xrightarrow{B_1} & S \end{array}$$

Moreover, we can assume that $D_0 = 1 \in \mathbb{Z}$.

The part $D_0 = 1$ follows, as in the Tom case, by a specialization argument using the original Jerry defined in Subsection 3.8.1.

3.10.2 Local Jerry

Let S be a Gorenstein local ring, $a_i^k, b_i^k, c^k \in S$ and $x_i, z_k \in m$, the maximal ideal of S, with indices as above. Let B be the skewsymmetric matrix defined in (3.2), I the ideal generated by the Pfaffians of B (see (3.21)) and $J = (z_1, \ldots, z_4)$.

We assume that $z_1, \ldots, z_4$ is a regular sequence and that I has codimension three, the maximal possible. Since S is Cohen–Macaulay, the grade of I is also three. By Theorem 1.3.1, the complex $\mathbf{L}$ defined in (1.1) is the minimal resolution of S/I and S/I is Gorenstein.

Recall that in (3.23) we defined elements g_i which are polynomials in a_i^k, b_i^k, c_i^k, z_k and x_i. Define a map $\psi\colon J/I \to S/I$ with $z_i \mapsto g_i$. By res we denote the residue map defined in (2.19).

The proof of the following theorem is very similar to the proof of Theorem 3.5.2.

Theorem 3.10.2 *The element* $\operatorname{res}(\psi) \in S/J$ *is a unit, and the ideal*

$$(P_1, \ldots, P_5, Tz_1 - g_1, \ldots, Tz_4 - g_4)$$

of the polynomial ring $S[T]$ is Gorenstein of codimension four.

3.11 A Maple routine that calculates Jerry

The following is a Maple [Map] routine that calculates Jerry unprojection. The input is a Jerry matrix, it returns the unprojection vector $(g_1, \ldots, g_4)$ defined in (3.23).

```
 pfaf := proc (a,b,c,d,e,f);
    pfaf := a*f-b*e+c*d;
 end:

 jerunproj := proc (data) local d3, d4,N,
       P1,P2,P3,L,o4,o3,o2, o1,oo4,oo3,oo2,
       oo1,det1,det2,det3,det4;
 N := data:
 P1 := pfaf (N[2,3] , N[2,4], N[2,5], N[3,4],
             N[3,5], N[4,5] ):
 P2 := pfaf ( N[1,3], N[1,4], N[1,5], N[3,4],
             N[3,5], N[4,5] ):
 P3 := pfaf ( N[1,2], d3, d4, N[2,4], N[2,5],
              N[4,5] ):
 L := matrix ( 3,4, [coeff(P1,z1),coeff(P1,z2),
               coeff(P1,z3),coeff(P1,z4),
               coeff(P2,z1),coeff(P2,z2),
               coeff(P2,z3),coeff(P2,z4),
               coeff(P3,z1),coeff(P3,z2),
               coeff(P3,z3),coeff(P3,z4)]);
det4 := -det(submatrix(L, [1,2,3], [1,2,3])):
det3 := det(submatrix(L, [1,2,3], [1,2,4]));
det2 := -det(submatrix(L, [1,2,3], [1,3,4])):
det1 := det(submatrix(L, [1,2,3], [2,3,4]));
divide(subs(x3=0,det4),d3*x2-x1*d4,'temp4'):
divide(subs(x3=0,det3),d3*x2-x1*d4,'temp3'):
divide (subs(x3=0,det2),d3*x2-x1*d4,'temp2'):
divide(subs(x3=0,det1),d3*x2-x1*d4,'temp1'):
divide( det4-subs(x3=0,det4)+N[1,3]*x3*temp4,x3,
  'o4');   oo4 := subs(d3=N[1,4],d4=N[1,5],o4);
```

```
divide( det3-subs(x3=0,det3)+N[1,3]*x3*temp3,x3,
  'o3');    oo3:= subs(d3=N[1,4],d4=N[1,5],o3);
divide( det2-subs(x3=0,det2)+N[1,3]*x3*temp2,x3,
  'o2');    oo2 := subs(d3=N[1,4],d4=N[1,5],o2);
divide( det1-subs(x3=0,det1)+N[1,3*x3*temp1,x3,
  'o1');    oo1 := subs(d3=N[1,4],d4=N[1,5],o1);
jerunproj := matrix (1,4, [oo1,oo2,oo3,oo4]):
 end:
```

3.12 Special Jerry

A Jerry matrix M that often appears in applications (compare [BrR]) is of the special form

$$M = \begin{pmatrix} \cdot & x & \sum_k a_1^k z_k & \sum_k a_2^k z_k & \sum_k a_3^k z_k \\ & \cdot & \sum_k b_1^k z_k & \sum_k b_2^k z_k & \sum_k b_3^k z_k \\ & & \cdot & y_3 & -y_2 \\ & & & \cdot & y_1 \\ & & & & \cdot \end{pmatrix} \tag{3.24}$$

with all sums for $k = 1, 2, 3$. Here the indeterminates are $x, y_k, z_k, a_i^k, b_i^k$, for $1 \leq i, k \leq 3$, I is generated by the Pfaffians of M and $J = (x, z_1, z_2, z_3)$. Denote by s the unprojection variable, and consider the 'dual' skewsymmetric matrix

$$N = \begin{pmatrix} \cdot & s & \sum_k a_k^1 y_k & \sum_k a_k^2 y_k & \sum_k a_k^3 y_k \\ & \cdot & \sum_k b_k^1 y_k & \sum_k b_k^2 y_k & \sum_k b_k^3 y_k \\ & & \cdot & z_3 & -z_2 \\ & & & \cdot & z_1 \\ & & & & \cdot \end{pmatrix} \tag{3.25}$$

with all sums for $k = 1, 2, 3$. It easy to check that the three Pfaffians of N involving s calculate sz_j. Computer calculations suggest that the remaining relation is

$$sx = -\sum y_{i_1} z_{j_1} D_{i_2,j_2} E_{i_3,j_3},$$

where the summation is for $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\} = \{1, 2, 3\}$, $D = \mathrm{w}^3([a_i^k])$, $E = \mathrm{w}^3([b_i^k])$, and by $\mathrm{w}^3(A)$ of a 3×3 matrix A we mean the 3×3 matrix with ij entry equal to the determinant of the submatrix of A obtained by deleting the ith row and the jth column.

Chapter 4

Catanese's rank condition

4.1 Some calculations on Catanese's rank condition

Let A be an $n \times n$ symmetric matrix (over a commutative ring R) and B the submatrix of A obtained by deleting the last row of A. Write I_A for the ideal generated by the determinants of the $(n-1) \times (n-1)$ submatrices of A, and I_B for the ideal generated by the determinants of the $(n-1) \times (n-1)$ submatrices of B. Clearly $I_B \subseteq I_A$.

Definition 4.1.1 The symmetric matrix A satisfies the *Rank Condition* if

$$I_A = I_B.$$

The Rank Condition was defined by Catanese in [C1], where he used it to study the canonical ring of regular surfaces of general type. More precisely, he gave a general procedure that constructs from a (sufficiently general) symmetric matrix satisfying the Rank Condition a Gorenstein ring. The aim of this chapter is to study the algebra of the Rank Condition for 'generic' symmetric matrices of small size (Lemma 4.1.2, Theorem 4.1.5) and to relate it with the unprojection (Example 4.1.7, Remark 4.1.8).

In the following k is an arbitrary field.

4.1.1 3×3 case

We work over the polynomial ring $R = k[x_0, x_1, x_2, z_i]$. Assume

$$A = \begin{pmatrix} x_0 & x_1 & C_0 \\ x_1 & x_2 & C_1 \\ C_0 & C_1 & D \end{pmatrix}$$

where $C_1, D \in R$ and $C_0 \in k[x_2, z_i]$ (i.e., x_0 and x_1 do not appear on C_0, we can always achieve this by subtracting columns and rows), and let B be the submatrix of A obtained by deleting the last row of A.

Lemma 4.1.2 *a) If A satisfies the Rank Condition then x_2 divides C_0 and $C_1 \in (x_0, x_1, x_2)$.*

b) Conversely, if x_2 divides C_0 and $C_1 \in (x_0, x_1, x_2)$, then there exists $D \in R$ such that $A = A(D)$ satisfies the Rank Condition.

Proof a) Since

$$\begin{vmatrix} x_0 & C_0 \\ C_0 & D \end{vmatrix} \in I_A = I_B \subseteq (x_0, x_1, x_2),$$

it follows that $C_0^2 \in (x_0, x_1, x_2)$. The elements x_0 and x_1 do not appear on C_0, therefore x_2 divides C_0. Similarly,

$$\begin{vmatrix} x_2 & C_1 \\ C_1 & D \end{vmatrix} \in (x_0, x_1, x_2)$$

implies $C_1 \in (x_0, x_1, x_2)$.

b) A solution according to Catanese ([C1], p. 101) is the symmetric matrix

$$A = \begin{pmatrix} x_0 & x_1 & gx_2 \\ & x_2 & l_0x_0 + l_1x_1 + l_2x_2 \\ & & l_0l_2x_0 + (l_1l_2 + gl_0)x_1 + (l_1g + l_2^2)x_2 \end{pmatrix} \qquad (4.1)$$

where l_i and g are arbitrary elements of R. QED

Example 4.1.3 In [C1], Section 4 Catanese describes a general procedure that in particular constructs a Gorenstein codimension three ring R from the matrix A defined in (4.1). First of all, consider $E = [e_{ij}]$, the adjoint matrix of A (i.e., the ij entry of E is $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting the ith column and the jth row of A). An easy calculation gives

$$\begin{aligned} e_{11} &= -l_1 e_{31} + l_0 e_{32} \\ e_{12} &= -l_2 e_{31} + g l_0 e_{33} \\ e_{22} &= g e_{31} - l_2 e_{32} + g l_1 e_{33} \end{aligned}$$

By loc. sit. the quadratic relations for R are

$$\begin{aligned} y_2^2 &= -l_1 y_2 + l_0 y_3 \\ y_2 y_3 &= -l_2 y_2 + g l_0 \\ y_3^2 &= g y_2 - l_2 y_3 + g l_1 \end{aligned}$$

It is easy to see that R can be described as the codimension three Gorenstein ring with ideal generated by the Pfaffians of the 5×5 skewsymmetric matrix

$$\begin{pmatrix} y_2 + l_1 & y_3 + l_2 & -l_0 & 0 \\ & x_0 & x_2 & -y_3 \\ & & -x_1 & -g \\ & & & y_2 \end{pmatrix}$$

Remark 4.1.4 The symmetric matrix

$$\begin{pmatrix} x_0^2 & 0 & x_0 x_1 \\ 0 & x_1^2 & 0 \\ x_0 x_1 & 0 & 0 \end{pmatrix}$$

satisfies the Rank Condition, but $x_0 x_1$ is not an element of (x_0^2, x_1^2).

4.1.2 4×4 case

We work over the polynomial ring $R = k[x_0, x_1, x_2, y_0, y_1, y_2, z_i]$. Let

$$A = \begin{pmatrix} x_0 & y_2 & y_1 & C_0 \\ y_2 & x_1 & y_0 & C_1 \\ y_1 & y_0 & x_2 & C_2 \\ C_0 & C_1 & C_2 & D \end{pmatrix} \qquad (4.2)$$

where $C_1, C_2, D \in R$ and $C_0 \in k[x_1, x_2, y_0, z_i]$ (i.e., x_0, y_2, y_1 do not appear on C_0, we can always achieve this by subtracting columns and rows), and write B for the submatrix of A obtained by deleting the last row of A.

The ideal J generated by the determinants of 2×2 submatrices of the matrix

$$T = \begin{pmatrix} x_0 & y_2 & y_1 \\ y_2 & x_1 & y_0 \\ y_1 & y_0 & x_2 \end{pmatrix}$$

is prime, since it is the ideal of the Veronese surface $S \subset \mathbb{P}^5$ (see e.g. [Ha], p. 24). Denote the generators of J by

$$\begin{aligned} q_1 &= x_1x_2 - y_0^2, \quad q_2 = y_2x_2 - y_0y_1, \quad q_3 = y_2y_0 - x_1y_1, \\ q_4 &= x_0x_2 - y_1^2, \quad q_5 = x_0y_0 - y_1y_2, \quad q_6 = x_0x_1 - y_2^2. \end{aligned}$$

Clearly $I_B \subseteq J$.

Theorem 4.1.5 *a) Assume that A satisfies the Rank Condition. Then q_1 divides C_0 and $C_1, C_2 \in J$.*

b) Conversely, if q_1 divides C_0 and $C_1, C_2 \in J$ then there exists $D \in R$ such that $A = A(D)$ satisfies the Rank Condition.

Proof a) Assume that A satisfies the Rank Condition. We have

$$\begin{vmatrix} x_0 & y_1 & C_0 \\ y_2 & y_0 & C_1 \\ C_0 & C_2 & D \end{vmatrix} \in I_B \subseteq J,$$

therefore

$$\begin{vmatrix} x_0 & y_1 & C_0 \\ y_2 & y_0 & C_1 \\ C_0 & C_2 & D \end{vmatrix} = \sum l_i q_i. \tag{4.3}$$

Substitute $x_0 = y_1 = y_2 = 0$ in (4.3) to get

$$y_0 C_0^2 = \bar{l}_1 q_1,$$

hence q_1 divides C_0. Now

$$\begin{vmatrix} x_0 & y_2 & C_0 \\ y_2 & x_1 & C_1 \\ C_0 & C_1 & D \end{vmatrix} \in I_B \subseteq J$$

implies that $x_0C_1^2 \in J$, therefore $C_1 \in J$ since J is prime. By a similar argument $C_2 \in J$.

b) It will be proved in Subsection 4.1.3. QED

Example 4.1.6 The symmetric matrix

$$A = \begin{pmatrix} x_0 & y_2 & y_1 & c_0(x_1x_2 - y_0^2) \\ & x_1 & y_0 & c_1(x_0x_2 - y_1^2) \\ & & x_2 & c_2(x_0x_1 - y_2^2) \\ & & & D \end{pmatrix}$$

with

$$D = c_1c_2x_0(x_0y_0 - y_1y_2) + c_0c_2x_1(x_1y_1 - y_0y_2) + c_0c_1x_2(x_2y_2 - y_0y_1)$$

satisfies the Rank Condition. Actually, it is a specialization of the matrix constructed in the proof of part b) of Theorem 4.1.5.

Example 4.1.7 The symmetric matrix

$$F = \begin{pmatrix} (s+1)x_0 & y_2 & y_1 & -sx_0y_0 - y_1y_2 \\ & (s+1)x_1 & y_0 & -sx_1y_1 - y_0y_2 \\ & & (s+1)x_2 & -sx_2y_2 - y_0y_1 \\ & & & D \end{pmatrix} \quad (4.4)$$

with

$$D = -y_0y_1y_2 + s(s+1)x_0x_1x_2 - s(x_0y_0^2 + x_1y_1^2 + x_2y_2^2)$$

satisfies the Rank Condition, but is not of the form (4.2).

We describe how we arrived at the matrix F. Consider the ideal I generated by the Pfaffians of the 5×5 skewsymmetric matrix

$$N = \begin{pmatrix} . & x_0 & y_1 & x_1 & y_0 \\ & . & 0 & y_2 & z_1 \\ & & . & z_0 & -sx_2 \\ & & & & 0 \\ & & & & . \end{pmatrix} \quad (4.5)$$

N is a Tom matrix, a deformation with parameter s of the matrix M defined in (3.8). Unproject I with respect to the ideal $J = (y_2, z_1, z_0, x_2)$ to get a codimension four Gorenstein ideal T. Explicitly, $T = (R_i, S_i, T_i)$ for $0 \le i \le 2$, with

$$\begin{array}{lll} R_0 = x_0z_0 - y_1y_2, & R_1 = x_1z_1 - y_0y_2, & R_2 = x_2z_2 - y_0y_1, \\ S_0 = y_0z_0 + sx_1x_2, & S_1 = y_1z_1 + sx_0x_2, & S_2 = y_2z_2 + sx_0x_1, \\ T_0 = z_1z_2 + sx_0y_0, & T_1 = z_0z_2 + sx_1y_1, & T_2 = z_1z_0 + sx_2y_2, \end{array}$$

where z_2 is the new unprojection variable.

After the linear change of coordinates $y_i = y_i + z_i$, the addition $T_i = T_i + R_i$ and the new change of coordinates $y_i = -y_i$ (all three transformations for $0 \le i \le 2$), we get equations

$$\begin{array}{c} -z_1z_2 + x_0z_0 - y_1y_2 + y_1z_2 + y_2z_1 \\ -z_0z_2 + x_1z_1 - y_0y_2 + y_0z_2 + y_2z_0 \\ -z_0z_1 + x_2z_2 - y_0y_1 + y_0z_1 + y_1z_0 \\ z_0^2 - y_0z_0 + sx_1x_2 \\ z_1^2 - y_1z_1 + sx_0x_2 \\ z_2^2 - y_2z_2 + sx_0x_1 \\ (s+1)x_0z_0 + y_2z_1 + y_1z_2 - sx_0y_0 - y_1y_2 \\ y_2z_0 + (s+1)x_1z_1 + y_0z_2 - sx_1y_1 - y_0y_2 \\ y_1z_0 + y_0z_1 + (s+1)x_2z_2 - sx_2y_2 - y_0y_1 \end{array}$$

Using the procedure described in [C1] Section 4 the symmetric matrix F follows. Indeed, the last three equations, which are the ones that z_i appear only linearly, give the first three rows of F, while the fourth row of F multiplied by $[z_1, z_2, z_3, 1]^t$ is a combination of the above polynomials.

Remark 4.1.8 The procedure in [C1] Section 4 produces from a (sufficiently general) symmetric 4×4 matrix A satisfying the Rank Condition a Gorenstein codimension four ring R (compare also Example 4.1.3). In Example 4.1.7 we did the opposite. We started from a codimension four Gorenstein ring belonging to the Tom family and calculated the corresponding symmetric matrix. It will be interesting to find out if all codimension four skewsymmetric Gorenstein rings produced by Catanese's method are related to the Tom and Jerry unprojection families.

4.1.3 Proof of existence

We now prove part b) of Theorem 4.1.5. The proof is based on computer calculations.

By the assumptions

$$\begin{aligned} C_0 &= kq_1, \\ C_1 &= \sum_{i=1}^{6} l_i q_i, \\ C_2 &= \sum_{i=1}^{6} m_i q_i, \end{aligned} \tag{4.6}$$

for some polynomials $k, l_i, m_i \in R$.

Notation We denote by H_{ij} the determinant of the submatrix of A obtained by deleting the ith column and the jth row.

Lemma 4.1.9 *Assume that I_B is prime, C_0, C_1, C_2 as in (4.6) and $A = A(D)$ satisfies*

$$H_{11} \in I_B.$$

Then A satisfies the Rank Condition.

Proof The vanishing

$$\begin{vmatrix} y_2 & x_1 & y_0 & C_1 \\ y_2 & x_1 & y_0 & C_1 \\ y_1 & y_0 & x_2 & C_2 \\ C_0 & C_1 & C_2 & D \end{vmatrix} = 0$$

implies that

$$y_2 H_{11} - x_1 H_{12} + y_0 H_{13} \in I_B,$$

and similarly

$$y_1 H_{11} - y_0 H_{12} + x_2 H_{13} \in I_B.$$

Eliminate H_{13} to get

$$(x_2 y_2 - y_0 y_1) H_{11} - (x_1 x_2 - y_0^2) H_{12} \in I_B.$$

Using the assumptions $H_{12} \in I_B$. Similar arguments prove that all H_{ij} are in I_B. QED

Since proving that I_B is prime doesn't seem to be very easy we will not be able to use Lemma 4.1.9. Nevertheless, it gives an idea for the proof of the existence of D. It says find D such that $H_{11} \in I_B$, and then it it highly probable that the Rank Condition holds for $A = A(D)$.

Hence, we are looking for D such that $H_{11} \in I_B$. Since

$$H_{11} = q_1 D + n,$$

it is enough to find $n_1 \in I_B$ such that q_1 divides $n+n_1$, say $n+n_1 = q_1 n_2$ and then take $D = -n_2$. A long Maple [Map] aided hand calculation establishes that such n_1 exists. More precisely, define the 1×11 matrix R_1 with

$$R_1 = (k, l_1, l_2, \ldots, l_6, m_1, m_2, \ldots, m_6),$$

the 6×1 matrix R_3 with

$$R_3 = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_6 \end{pmatrix}$$

the 11×6 matrix R_2 with

```
R_2 =
[[ -x_1m_3-l_3y_0+m_2y_0+2l_6y_2+l_2x_2, x_2l_4+2m_4y_0
       -x_1m_5+x_1l_6,   x_2m_4-x_1m_6+x_2l_5, 0, 0, 0],
 [ l_3y_1+l_6x_0+2m_1y_0-m_3y_2+x_2l_1, l_2x_2+m_2y_0 ,
        l_3x_2+m_3y_0-m_6y_2, x_2l_4+x_1m_5, x_2l_5-x_2m_4+
        2m_5y_0, l_6x_2+m_6y_0],
 [ m_1y_1+m_3x_0,  l_3y_1+l_6x_0+m_1y_0+m_2y_1+m_5x_0,
        m_6x_0, 0, 0, 0],
 [ 0, 0, l_6x_0+m_1y_0-m_3y_2+l_3y_1,  4y_1, l_5y_1-m_4y_1
        -m_5y_2, l_6y_1-m_6y_2],
 [ 0, m_3x_0, 0, l_6x_0+2m_1y_0+m_2y_1+m_5x_0,
         m_1x_2+m_6x_0, 0],
 [ m_1x_0, 0, m_3x_0, -m_1x_1, l_6x_0+m_2y_1+m_5x_0, m_6x_0],
 [ 0, 0, 0, 0,  -m_1x_1+m_2y_2-m_4x_0, l_6x_0],
 [-m_2y_2+m_4x_0+m_1x_1, m_2x_1, x_1m_3, m_4x_1,
         x_1m_5, x_1m_6],
 [0, m_4x_0-m_2y_2, -m_3y_2, -m_4y_2, -m_5y_2, -m_6y_2],
 [ 0, 0, m_4x_0, 0, 0, 0],
 [ 0, 0, 0, m_4x_0, m_5x_0, m_6x_0 ]]
```

and finally set

$$D = R_1 R_2 R_3. \tag{4.7}$$

Calculations using the computer program Magma [Mag] proved that for D as in (4.7) we have $H_{ij} \in I_B$ for all i, j. Therefore, $A = A(D)$ satisfies the Rank Condition, which finishes the proof of part b) of Theorem 4.1.5.

We hope in the future to give a more conceptual definition of D and proof of Theorem 4.1.5.

Bibliography

[Al] Altınok S., *Graded rings corresponding to polarised K3 surfaces and $\mathbb{Q}$-Fano 3-folds*. Univ. of Warwick Ph.D. thesis, Sep. 1998, 93 + vii pp.

[AK] Altman A. and Kleiman S., *Introduction to Grothendieck duality theory*. Lecture Notes in Math. 146, Springer–Verlag 1970

[AM] Atiyah M. and Macdonald I., *Introduction to commutative algebra*. Addison–Wesley Publishing Co. 1969

[BE] Buchsbaum D. and Eisenbud D., *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3. Amer. J. Math. **99** (1977), 447–485

[Bour] Bourbaki N., *Commutative algebra*. English translation, Springer–Verlag 1998

[BrR] Brown G. and Reid M., *Mory flips of Type A* (provisional title), in preparation

[BH] Bruns W. and Herzog J., *Cohen–Macaulay rings*. Revised edition, Cambridge Studies in Advanced Mathematics 39, CUP 1998

[BV] Bruns, W. and Vetter, U., *Determinantal rings*. Lecture Notes in Math. 1327. Springer 1988

[C1] Catanese F., *Commutative algebra methods and equations of regular surfaces*. Algebraic geometry, Bucharest 1982, Lecture Notes in Math. 1056, Springer 1984, 28–111

[C2] Catanese F., *Homological algebra and algebraic surfaces*. Algebraic geometry, Santa Cruz 1995, 3–56, Proc. Sympos. Pure Math. 62, Part 1, Amer. Math. Soc. 1997

[CFHR] Catanese F., Franciosi M., Hulek K. and Reid M., *Embeddings of curves and surfaces*. Nagoya Math. J. **154** (1999), 185–220

[CM] Corti A. and Mella M., *Birational geometry of terminal quartic 3-folds I*, in preparation

[CPR] Corti A., Pukhlikov A. and Reid M., *Birationally rigid Fano hypersurfaces*, in Explicit birational geometry of 3-folds, A. Corti and M. Reid (eds.), CUP 2000, 175–258

[Do] Dolgachev I., *Weighted projective varieties*, in Group actions and vector fields (Vancouver 1981), Lecture Notes in Math. 956, Springer 1982, 34–71

[Ei] Eisenbud D., *Commutative algebra, with a view toward algebraic geometry*. Graduate Texts in Mathematics 150, Springer–Verlag 1995

[FOV] Flenner H., O'Carrol L. and Vogel W. *Joins and intersections*. Springer Monographs in Mathematics. Springer–Verlag 1999

[GW] Goto S. and Watanabe K., *On graded rings I*. J. Math. Soc. Japan **30** (1978), 179–213

[Ha] Harris J., *Algebraic geometry*. Graduate Texts in Mathematics 133, Springer–Verlag 1995

[Har] Hartshorne R., *Algebraic geometry*. Graduate Texts in Mathematics 52, Springer–Verlag 1977

[HK] Herzog J. and Kunz E., (editors) *Der kanonische Modul eines Cohen–Macaulay–Rings*. Lecture Notes in Mathematics 238, Springer–Verlag 1971

[JKM] Jacobsson C., Kustin A. and Miller M., *The Poincaré series of a codimension four Gorenstein ring is rational*. J. Pure Appl. Algebra **38** (1985), 255–275

[KL] Kleppe H. and Laksov D., *The algebraic structure and deformation of Pfaffian schemes*. J. Algebra **64** (1980), 167–189

[KoM] Kollár J. and Mori S., *Birational geometry of algebraic varieties*. Cambridge Tracts in Mathematics 134, CUP 1998

[KM1] Kustin A. and Miller M., *Algebra structures on minimal resolutions of Gorenstein rings of embedding codimension four*. Math. Z. **173** (1980), 171–184

[KM2] Kustin A. and Miller M., *A general resolution for grade four Gorenstein ideals*. Manuscripta Math. **35** (1981), 221–269

[KM3] Kustin A. and Miller M., *Structure theory for a class of grade four Gorenstein ideals*. Trans. Amer. Math. Soc. **270** (1982), 287–307

[KM4] Kustin A. and Miller M., *Constructing big Gorenstein ideals from small ones*. J. Algebra **85** (1983), 303–322

[KM5] Kustin A. and Miller M., *Deformation and linkage of Gorenstein algebras*. Trans. Amer. Math. Soc. **284** (1984), 501–534

[KM6] Kustin A. and Miller M., *Classification of the Tor-algebras of codimension four Gorenstein local rings*. Math. Z. **190** (1985), 341–355

[M] Matsumura H., *Commutative ring theory*. Second edition, CUP 1989

[Mag] Magma (Computer algebra system): Bosma, W., Cannon, J. and Playoust, C., *The Magma algebra system. I. The user language*. J. Symbolic Comput. **24** (1997), 235–265. See also www.maths.usyd.edu.au:8000/u/magma

[Map] Maple V (Computer algebra system, Copyright by Waterloo Maple Inc.): *Mathematical computation with Maple V: ideas and applications*. Proceedings of the Maple Summer Workshop and Symposium (MSWS '93) held at the University of Michigan, Ann Arbor. Edited by Thomas Lee. Birkhäuser 1993

[Mi] Migliore J., *Introduction to liaison theory and deficiency modules*. Progress in Mathematics 165, Birkhäuser 1998

[N3] Nagata M., *Local rings*. Interscience Tracts in Pure and Applied Mathematics 13, Interscience Publishers 1962

[PR] Papadakis S. and Reid M., *Kustin–Miller unprojection without complexes*, to appear in J. Algebraic Geometry, math.AG/0011094, 15 pp.

[R0] Reid M., *Unprojection and Gorenstein rings in small codimension.* Personal notes

[R1] Reid M., *Graded Rings and Birational Geometry*, in Proc. of algebraic symposium (Kinosaki, Oct 2000), K. Ohno (Ed.) 1–72

[R2] Reid M., *Nonnormal del Pezzo surfaces.* Publ. Res. Inst. Math. Sci. **30** (1994), 695–727

[Sha] Shafarevich I., *Basic algebraic geometry 1.* Second edition, Springer–Verlag 1994