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ON PERFECT ISOMETRIES FOR TAME BLOCKS

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January 2000

Abstract. Any 2-block of a finite group G with a quaternion defect group Q_8 is Morita equivalent to the corresponding block of the centraliser H of the unique involution of Q_8 in G ; this answers positively an earlier question raised by M. Broué.

Mathematics subject classification: 20C20.

1 INTRODUCTION

It seems to be a general intuition that there should be some blocktheoretic analogue of Glauberman's Z^* -theorem [15]. For the particular case of the Brauer-Suzuki theorem [3], M. Broué raises in [4] explicitly the question, whether any block having a quaternion defect group Q_8 over an algebraically closed field k of characteristic 2 is Morita equivalent to the corresponding block of the centraliser of the unique involution in Q_8 . Combining K. Erdmann's classification of blocks with a quaternion defect group up to Morita equivalence [13] with Theorem 2 from Cabanes-Picarony [8] we show that the answer is positive:

Theorem 1. *Let G be a finite group and let b be a block of kG having a quaternion defect group Q_8 . Denote by Z the unique subgroup of order 2 of Q_8 and set $H = C_G(Z)$. Let c be the Brauer correspondent of b ; that is, $c = Br_Z(b)$. Then the block algebras kGb and kHc are Morita equivalent.*

If we replace Q_8 by a generalised quaternion group of order 2^n for some $n > 3$, the theorem remains true under the assumption that the considered block b has either one or three isomorphism classes of simple modules; in the case of two isomorphism classes arises the problem that in the classification up to Morita equivalence, some scalars remain undetermined (see [13, p. 294 sqq.]). Of course, Theorem 1 follows from the Brauer-Suzuki theorem [3], if the block b is the principal block of kG , because in this case we actually have an isomorphism $kGb \cong kHc$.

The idea of the proof of Theorem 1 is simple: Theorem 2 of Cabanes-Picarony [8] implies that there is a perfect isometry between the blocks b and c all of whose signs are positive. Using the list of possible decomposition matrices as determined by Erdmann [13], an easy combinatorial argument will show that the decomposition matrices of b and c have to be equal. Again, thanks to Erdmann's work [13], the decomposition

matrix of a block with a quaternion defect group determines its Morita equivalence class over k (and vice versa), which proves Theorem 1.

The last remark in [8] gives evidence that the hypothesis of the existence of a sign preserving perfect isometry holds in cases other than blocks with quaternion defect groups. Together with the above arguments, this suggests that on the way towards a block theoretic analogue of the Z^* -theorem, one useful piece in the puzzle might be a statement which gives information on the ordinary decomposition matrices under the assumption that there is a sign preserving perfect isometry. This is conclusive for tame blocks:

Theorem 2. *If there is a perfect isometry all of whose signs are positive between two blocks having isomorphic defect groups and tame representation type, then the ordinary decomposition matrices of the two blocks coincide.*

A block has tame representation type if and only if its defect groups are generalised quaternion, dihedral or semidihedral 2-groups (cf. [2], [10]). Thus Theorem 2 follows from the Theorems 3, 4, 5 below. Contrary to the case of blocks with quaternion defect groups of order 8, we cannot conclude at this point that the decomposition matrix determines the Morita equivalence class. On one hand, there is the problem mentioned above of certain undetermined scalars in the relations of the quivers of tame blocks. On the other hand, the cases $\mathcal{SD}(3\mathcal{B})_1$ and $\mathcal{SD}(3\mathcal{D})$ in [13, pp. 300-301] have equal decomposition matrices, but they cannot be Morita equivalent since their quivers are different (and both cases do occur as blocks).

The unsatisfactory aspect of Theorem 1 is that it does not say anything about the Morita equivalence classes over a complete discrete valuation ring, or about source algebras (cf. [17]). In fact, a good generalisation of the Brauer-Suzuki theorem should determine the source algebras of b in terms of the source algebras of c .

The paper is organised as follows: in section 2, we introduce our notation and terminology, section 3 is devoted to proving Theorem 1, and the sections 4 and 5 deal with the remaining cases of tame blocks.

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2 NOTATION, TERMINOLOGY AND QUOTED RESULTS

Let \mathcal{O} be a complete local Noetherian commutative ring with a residue field k of prime characteristic p . We allow, for the time being, the case $\mathcal{O} = k$ (whereas from section 3 onwards we will assume that \mathcal{O} has characteristic zero and that $p = 2$).

2.1. Let G be a finite group. A *block of $\mathcal{O}G$* is a primitive idempotent b in the center $Z(\mathcal{O}G)$; its corresponding *block algebra* is the algebra $\mathcal{O}Gb$. A *defect group of the block b* is a minimal subgroup P of G with the property that the map $\mathcal{O}Gb \otimes_{\mathcal{O}P} \mathcal{O}Gb \rightarrow \mathcal{O}Gb$ induced by multiplication in $\mathcal{O}Gb$ splits as homomorphism of $\mathcal{O}Gb - \mathcal{O}Gb$ -bimodules. For any subgroup N of G containing $N_G(P)$ there is a unique block e of $\mathcal{O}N$ having P as defect group, such that $\mathcal{O}Ne$ is isomorphic to a

direct summand of $\mathcal{O}Gb$ as $\mathcal{O}N - \mathcal{O}N$ -bimodule; the block e is called the *Brauer correspondent* of b . The image \bar{b} of b in kG is then again a block of kG with P as defect group.

There are various reformulations of the Brauer correspondence - see e.g. [19]. The approach of Alperin-Broué [1] characterises e in terms of the Brauer homomorphism $Br_P : (\mathcal{O}G)^P \rightarrow kC_G(P)$; that is, e is the unique block of $\mathcal{O}N$ with P as defect group such that $Br_P(e) = Br_P(b)$. The description of the Brauer correspondent in Theorem 1 as $c = Br_Z(b)$ is specific for a defect group P with a unique minimal subgroup Z and $H = C_G(Z)$, since this is a “trivial intersection” situation (that is, we have ${}^xP \cap P = \{1\}$ for $x \in G - H$).

For what follows, we assume now that \mathcal{O} is a complete discrete valuation ring with a quotient field K of characteristic zero (and still with a residue field of prime characteristic p).

2.2 Let G be a finite group. Assume that K is “large enough” for G ; that is, KG is isomorphic to a direct products of matrix algebras over K . Let b be a block of $\mathcal{O}G$.

2.2.1 We denote by $\mathbb{Z}Irr_K(G, b)$ the free \mathbb{Z} -module over the set $Irr_K(G, b)$ of ordinary irreducible characters χ of G associated with b (i.e. satisfying $\chi(b) = \chi(1)$), endowed with the usual scalar product $\langle -, - \rangle_G$.

2.2.2 We denote by $Pr_{\mathcal{O}}(G, b)$ the \mathbb{Z} -submodule of $\mathbb{Z}Irr_K(G, b)$ consisting of all virtual characters of G associated with b vanishing outside the set G_{reg} of p' -elements in G .

2.2.3 We denote by $L^0(G, b)$ the \mathbb{Z} -submodule of $\mathbb{Z}Irr_K(G, b)$ of all virtual characters of G associated with b which are orthogonal to all elements of $Pr_{\mathcal{O}}(G, b)$ (with respect to $\langle -, - \rangle_G$); that is, $L^0(G, b)$ consists of all virtual characters of G associated with b which vanish on G_{reg} . In order to check whether a virtual character of G associated with b belongs to $L^0(G, b)$, it suffices to verify, that it is orthogonal to the characters of the projective indecomposable $\mathcal{O}Gb$ -modules.

Note that all nonzero elements in $L^0(G, b)$ are virtual characters, because their value at one has to be zero. This observation plays a role when it comes to determining elements of “small” norms in $L^0(G, b)$. Any element of norm 2 in $L^0(G, b)$ is of the form $\chi - \chi'$ for two different $\chi, \chi' \in Irr_K(G, b)$; similarly, every element of norm 3 in $L^0(G, b)$ is of the form plus or minus $\chi + \chi' - \chi''$ for three pairwise different $\chi, \chi', \chi'' \in Irr_K(G, b)$. Thus every norm 3 element in $L^0(G, b)$ determines a unique irreducible character which occurs with a sign different from the signs of the two other irreducible characters in that norm 3 element.

2.3 Let G, H be finite groups and let b, c be blocks of $\mathcal{O}G, \mathcal{O}H$, respectively, having a common defect group P .

2.3.1 A *perfect isometry between the blocks b and c* is a linear isomorphism $\Phi : \mathbb{Z}Irr_K(G, b) \cong \mathbb{Z}Irr_K(H, c)$ mapping every $\chi \in Irr_K(G, b)$ to $\epsilon_{\chi}\eta_{\chi}$ for some sign $\epsilon_{\chi} \in \{+1, -1\}$ and some $\eta_{\chi} \in Irr_K(H, c)$ satisfying in addition the following arithmetical condition: for any $g \in G$ and any $h \in H$, the value of $\sum_{\chi \in Irr_K(G, b)} \epsilon_{\chi}\chi(g)\eta_{\chi}(h^{-1})$ is

zero, if exactly one of g, h is a p' -element, and divisible in \mathcal{O} by the orders of $C_G(u), C_H(u)$, if the p -parts of g, h are conjugate (in G, H , respectively) to a common element $u \in P - \{1\}$. This notion is due to M. Broué [5]. The following statement is an immediate consequence of the definitions in [5]:

2.3.2 Any perfect isometry $\Phi : \mathbb{Z}Irr_K(G, b) \cong \mathbb{Z}Irr_K(H, c)$ induces isomorphisms $Pr_{\mathcal{O}}(G, b) \cong Pr_{\mathcal{O}}(H, c)$ and $L^0(G, b) \cong L^0(H, c)$.

2.3.3 If the blocks have actually the same *Brauer categories* (in the sense of [8,I.5]), Broué has refined perfect isometries to the concept of an *isotypy*, and the blocks b, c are said to have the same type if there is such an isotypy between them (see [5]). Loosely speaking, an isotypy is to a perfect isometry what is a splendid derived equivalence to a derived equivalence, namely compatible with the Brauer construction.

In order to recall the main results of Cabanes-Picaronny [8,9], we assume now that $p = 2$.

2.4 Let G, H be finite groups such that K is large enough for both G and H , let b, c be blocks of $\mathcal{O}G, \mathcal{O}H$, respectively, having a common defect group P , and assume that P has a cyclic subgroup of index 2.

2.4.1 ([8,Theorem 1]) If b and c have the same Brauer categories, they have the same type.

2.4.2 ([8,Theorem 2]) If P is a generalised quaternion 2–group, H the centraliser in G of the unique involution on P and c the Brauer correspondent of b , there is an isotypy $\Phi : \mathbb{Z}Irr_K(G, b) \cong \mathbb{Z}Irr_K(H, c)$ mapping $Irr_K(G, b)$ onto $Irr_K(H, c)$.

The condition that Φ maps $Irr_K(G, b)$ onto $Irr_K(H, c)$ is obviously equivalent to requiring that all the signs ϵ_χ in 2.3.1 are +1.

Remark 2.5 There is a gap in the proof of [8,Lemma 5], which is used for the proof of [8,Theorem 2], but M. Cabanes has shown in [9], that even without [8, Lemma 5], the proof of [8, Theorem 2] can be completed. It is worthwhile to point out that in the case where b, c are the principal blocks, [8, Lemma 5] is correct as it stands, and can be used to deduce an alternative proof of the Brauer-Suzuki Theorem [3] - see the Remark at the end of Section 3 below.

2.6 We refer to K. Erdmann’s book [13] for the classification of tame blocks. We are going to use, without further comment, standard properties of tame blocks - such as the fact that the number of isomorphism classes of simple modules of a tame block is at most three; if it is one, the considered block is *nilpotent* in the sense of Broué-Puig [7], and in this case its block algebra is Morita equivalent to the group algebra of its defect group by Puig [18]. Standard material on decomposition numbers can, for instance, be found in W. Feit’s book [14].

3 THE QUATERNION DEFECT CASE

Throughout the rest of the paper, we denote by \mathcal{O} a complete discrete valuation ring having a quotient field K of characteristic zero and an algebraically closed residue field k of characteristic 2. We assume that K is large enough for all the finite groups occurring in the statements of our results.

Theorem 3. *Let G, H be finite groups and b, c be blocks of $\mathcal{O}G, \mathcal{O}H$, respectively, having a common generalised quaternion defect group Q of order 2^n for some integer $n \geq 3$.*

If there is a perfect isometry $\Phi : \mathbb{Z}Irr_K(G, b) \cong \mathbb{Z}Irr_K(H, c)$ mapping $Irr_K(G, b)$ onto $Irr_K(H, c)$, then b and c have the same ordinary decomposition matrices.

Proof. If b (and whence c) has one isomorphism class of simple modules, the block algebras $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are Morita equivalent to $\mathcal{O}Q$, because both blocks are nilpotent (cf. 2.6 above as well as [7], [18], [19, Ch. 7]).

If b and c have two isomorphism classes of simple modules, we have $n \geq 4$, and by Erdmann [13] and Holm [16, 4.1], their decomposition matrices are of the following forms (corresponding to the cases $\mathcal{Q}(2\mathcal{A})$ and $\mathcal{Q}(2\mathcal{B})_1$ in Erdmann's list [13, pages 303-304]; the case $\mathcal{Q}(2\mathcal{B})_2$ does not occur as block by [16, 4.1]):

$$\text{Case I : } \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}^* \quad \text{Case II : } \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}^*$$

where the $*$ means, that the last row has to be repeated $2^{n-2} - 1$ times. If Λ is any set of cardinality $2^{n-2} - 1$, we can label the irreducible characters of b as follows:

$$Irr_K(G, b) = \{\chi_i\}_{1 \leq i \leq 5} \cup \{\chi_\lambda\}_{\lambda \in \Lambda}.$$

According to the two matrices, we can write out bases of $Pr_{\mathcal{O}}(G, b)$; either one of the two cases occurs as set of characters of the finitely generated projective indecomposable $\mathcal{O}Gb$ -modules:

$$\text{Case I: } \{\chi_1 + \chi_2 + \chi_3 + \chi_4 + 2 \sum_{\lambda \in \Lambda} \chi_\lambda, \chi_3 + \chi_4 + \chi_5 + \sum_{\lambda \in \Lambda} \chi_\lambda\}$$

$$\text{Case II: } \{\chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5, \chi_3 + \chi_4 + \chi_5 + \sum_{\lambda \in \Lambda} \chi_\lambda\}$$

The corresponding statements apply to c as well as to b . We have to show, that under the given assumption, it cannot happen that the two different cases occur for b and c ; that is, we have to show, that there is no perfect isometry between the two different cases mapping any irreducible character again to an irreducible character. We argue by contradiction.

Up to multiplication by a sign, the complete list of elements of norm 2 in $L^0(G, b)$ in both of the two above cases can be read off directly from the given bases of $Pr_{\mathcal{O}}(G, b)$:

$$\chi_1 - \chi_2, \chi_3 - \chi_4, \chi_\lambda - \chi_{\lambda'}, \text{ with } \lambda, \lambda' \in \Lambda, \lambda \neq \lambda'.$$

Since $n \geq 4$, we have $|\Lambda| \geq 3$. Therefore, in both cases, the characters χ_λ are the only characters which appear in more than one element of norm 2 with sign +1. Thus the characters labelled by the set Λ have to be preserved by any perfect isometry. An easy verification shows, that any norm 3 element in $L^0(G, b)$ involving one of the characters χ_λ is of the following form (again, up to multiplication with a sign):

$$\text{Case I: } \chi + \chi' - \chi_\lambda, \text{ for some } \chi, \chi' \in Irr_K(G, b);$$

$$\text{Case II: } \chi - \chi' + \chi_\lambda, \text{ for some } \chi, \chi' \in Irr_K(G, b).$$

That is, in case I, χ_λ appears systematically with a sign opposite to the sign of χ , χ' , while in case II, χ_λ has a sign equal to that of one of the two other characters. This shows, that there is no perfect isometry between the two cases all of whose signs are $+1$. This proves Theorem 3 in the case where b and c have two isomorphism classes of simple modules.

If b, c have three isomorphism classes of simple modules, their decomposition matrices are of the following forms (corresponding to the cases $\mathcal{Q}(3\mathcal{A})_2$, $\mathcal{Q}(3\mathcal{K})$ and $\mathcal{Q}(3\mathcal{B})$ in [9, pages 305-306]):

$$\begin{array}{l} \text{Case I :} \\ \text{Case II :} \\ \text{Case III :} \end{array} \begin{array}{l} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} *$$

where the subscript $*$ means again, that the last line occurs $2^{n-2} - 1$ times. Case III does not occur for $n = 3$, and it is not known, whether it occurs for $n > 4$ (but it does occur for $n = 4$ as principal block of the double cover of A_7).

As before, we can label the irreducible characters by

$$\text{Irr}_K(G, b) = \{\chi_i\}_{1 \leq i \leq 6} \cup \{\chi_\lambda\}_{\lambda \in \Lambda}$$

for some indexing set Λ having $2^{n-2} - 1$ elements. Just as before, one reads off the decomposition matrices the elements of small norm in $L^0(G, b)$: in all three cases, the only norm 2 elements in $L^0(G, b)$ are of the form $\chi_\lambda - \chi_{\lambda'}$ for some distinct elements λ, λ' in Λ . Thus if $n \geq 4$, any perfect isometry between these cases has to preserve the sets of characters labelled by the elements of Λ .

Up to multiplication with a sign, the complete lists of norm 3 elements in $L^0(G, b)$ are as follows:

Case I: $\chi_1 - \chi_2 + \chi_5, \chi_1 - \chi_3 + \chi_6, \chi_2 - \chi_4 + \chi_6, \chi_3 - \chi_4 + \chi_5, \chi_1 + \chi_4 - \chi_\lambda, \chi_2 + \chi_3 - \chi_\lambda$, where $\lambda \in \Lambda$;

Case II: $\chi_1 + \chi_2 - \chi_5, \chi_1 + \chi_3 - \chi_6, \chi_2 - \chi_4 + \chi_6, \chi_3 - \chi_4 + \chi_5, \chi_1 - \chi_4 + \chi_\lambda, \chi_2 + \chi_3 - \chi_\lambda$, where $\lambda \in \Lambda$;

Case III: $\chi_1 - \chi_3 + \chi_6, \chi_1 + \chi_4 - \chi_5, \chi_2 + \chi_3 - \chi_5, \chi_2 - \chi_4 + \chi_6, \chi_1 - \chi_2 + \chi_\lambda, \chi_3 - \chi_4 + \chi_\lambda$, where $\lambda \in \Lambda$.

If $n \geq 4$, the characters labelled by Λ have to be preserved under any perfect isometry between any two of the above cases. But this is not possible with a perfect isometry all of whose signs are $+1$: in Case I, χ_λ occurs in norm 3 elements only with a sign different from the signs of the two other characters in that norm 3 element, in Case III, χ_λ occurs systematically with the same sign as one other character in any norm 3 element, while in Case II, it occurs both ways.

Thus we have only to consider the situation $n = 3$. This rules out Case III. In the Cases I and II, there is a unique character occurring in norm 3 elements only with a sign opposite to that of the two other characters, namely χ_λ in Case I and χ_4 in Case II. But these two characters cannot correspond to each other under a perfect

isometry, because χ_λ occurs in Case I in exactly 2 elements of norm 3 (with sign +1) while χ_4 occurs in Case II in exactly 3 elements of norm 3 (with sign +1). This completes the proof of Theorem 3. \square

Proof of Theorem 1. Under the assumptions of Theorem 1, there is a perfect isometry between b and c all of whose signs are +1 by Cabanes-Picaronny [8, Theorem 2] (see 2.4.2 above). Therefore, by Theorem 3, the blocks b and c have the same decomposition matrices. It follows from Erdmann's classification [13, pages 305-306] that the decomposition matrix determines the Morita equivalence class of the block algebras over the residue field k . \square

Remark. If one applies the results of Cabanes-Picaronny [8] to principal blocks, one can get a proof of the Brauer-Suzuki theorem as follows: let G be a finite group with a quaternion Sylow-2-subgroup Q_8 , let $H = C_G(Z(Q_8))$ and denote by b_0 and c_0 the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$, respectively. Then there is, by [8, V. Step 2], a perfect isometry between b and c which coincides with the restriction from G to H on $L^0(G, b_0)$. If we denote by η_χ the irreducible character of H and by ϵ_χ the sign such that $\epsilon_\chi \eta_\chi$ is the image of $\chi \in \text{Irr}_K(G, b_0)$ under this perfect isometry, then [8, Lemma 10] implies in the last part of [8, Section V], that the quotient $\chi(1)/\epsilon_\chi \eta_\chi(1)$ is independent of $\chi \in \text{Irr}_K(G, b_0)$. Applying this to the trivial character shows that this quotient is equal to 1 for any $\chi \in \text{Irr}_K(G, b_0)$. This however is only possible if actually $\eta_\chi = \text{Res}_H^G(\chi)$ for any $\chi \in \text{Irr}_K(G, b_0)$. By a theorem of Broué [6, 0.1], the block algebras $\mathcal{O}Gb_0$ and $\mathcal{O}Hc_0$ are equal. Since $O_{2'}(G)$ is the kernel of b_0 , this is equivalent to saying, that the image of $Z(Q_8)$ in $G/O_{2'}(G)$ is in the center of that group.

4 THE DIHEDRAL DEFECT CASE

In order to conclude the proof of Theorem 2, it remains to generalise Theorem 3 to blocks with dihedral or semidihedral defect groups. In both cases, the proofs are similar to that of Theorem 3, so we only sketch the main steps. This section deals with the dihedral case (including blocks with a Klein four defect group):

Theorem 4. *Let G, H be finite groups and b, c be blocks of $\mathcal{O}G, \mathcal{O}H$, respectively, having a common dihedral defect group D of order 2^n for some integer $n \geq 2$.*

If there is a perfect isometry $\Phi : \mathbb{Z}\text{Irr}_K(G, b) \cong \mathbb{Z}\text{Irr}_K(H, c)$ mapping $\text{Irr}_K(G, b)$ onto $\text{Irr}_K(H, c)$, then b and c have the same ordinary decomposition matrices.

Proof. If b and c have only one isomorphism class of simple modules, they are both Morita equivalent to $\mathcal{O}D$, and we are done.

If b has two isomorphism classes, there are two possibilities for the decomposition matrices, namely $\mathcal{D}(2\mathcal{A})$ and $\mathcal{D}(2\mathcal{B})$ in Erdmann's list [13, pages 294-295]. If we label the irreducible characters analogously to what we did in the proof of Theorem 3, we can write out complete lists of norm 3 elements in $L^0(G, b)$:

Case $\mathcal{D}(2\mathcal{A})$: $\chi_1 + \chi_3 - \chi_\lambda, \chi_1 + \chi_4 - \chi_\lambda, \chi_2 + \chi_3 - \chi_\lambda, \chi_2 + \chi_4 - \chi_\lambda$, where $\lambda \in \Lambda$;

Case $\mathcal{D}(2\mathcal{B})$: $\chi_1 - \chi_3 + \chi_\lambda, \chi_1 - \chi_4 + \chi_\lambda, \chi_2 - \chi_3 + \chi_\lambda, \chi_2 - \chi_4 + \chi_\lambda$, where $\lambda \in \Lambda$.

In both cases, the χ_λ appear twice as often in norm 3 elements than any of the characters χ_i , $1 \leq i \leq 4$, and therefore, they have to be preserved by a perfect isometry between the two cases. But in the first case, the χ_λ appear with a sign opposite to both other characters in any norm 3 element, while in the second case, there is one other character with the same sign. Thus there is no perfect isometry between these two cases all of whose signs are +1.

If b has three isomorphism classes of simple modules, there are three possibilities for the decomposition matrices, namely $\mathcal{D}(3\mathcal{A})_1$, $\mathcal{D}(3\mathcal{B})_1$ and $\mathcal{D}(3\mathcal{K})$ in Erdmann's list [13, pages 295-296]. The second case occurs only for $n \geq 3$ and it is not known, whether it actually occurs for $n \geq 4$ (but it does occur for $n = 3$ as principal block of A_7). One easily checks that the third case $\mathcal{D}(3\mathcal{K})$ is the only case having a norm 4 element in $L^0(G, b)$ in which three characters have the same sign, namely $\chi_1 + \chi_2 + \chi_3 - \chi_4$ in the appropriate labelling of the irreducible characters. Thus we only have to compare the first two cases; in particular, $n \geq 3$. The complete lists of norm 3 elements in $L^0(G, b)$ are as follows:

Case $\mathcal{D}(3\mathcal{A})_1$: $\chi_1 + \chi_4 - \chi_\lambda$, $\chi_2 + \chi_3 - \chi_\lambda$, where $\lambda \in \Lambda$;

Case $\mathcal{D}(3\mathcal{B})_1$: $\chi_1 - \chi_2 + \chi_\lambda$, $\chi_3 - \chi_4 + \chi_\lambda$, where $\lambda \in \Lambda$.

In the first case, precisely the characters χ_λ appear in all norm 3 elements with a sign opposite to the two other characters in that norm 3 element, while in the second case, precisely the two characters χ_2, χ_4 have this property. Since Λ has odd cardinality, there is no perfect isometry between the two cases all of whose signs are +1. This proves Theorem 4. \square

5 THE SEMIDIHEDRAL DEFECT CASE

We use freely the notation of [13, pages 294-306] for the remaining case of blocks with semidihedral defect groups.

Theorem 5. *Let G, H be finite groups and b, c be blocks of $\mathcal{O}G, \mathcal{O}H$, respectively, having a common semidihedral defect group D of order 2^n for some integer $n \geq 4$.*

If there is a perfect isometry $\Phi : \mathbb{Z}\text{Irr}_K(G, b) \cong \mathbb{Z}\text{Irr}_K(H, c)$ mapping $\text{Irr}_K(G, b)$ onto $\text{Irr}_K(H, c)$, then b and c have the same ordinary decomposition matrices.

Proof. As before, if b and c have one isomorphism class of simple modules, they are Morita equivalent to $\mathcal{O}D$.

Assume that b has two isomorphism classes of simple modules. The case $\mathcal{SD}(2\mathcal{B})_3$ does not occur as block algebra (cf. [11, 8.16]).

If b is of type $\mathcal{SD}(2\mathcal{A})_1$ or $\mathcal{SD}(2\mathcal{B})_1$, the decomposition matrices are obtained by inserting a line for a character χ_5 in the matrices for $\mathcal{D}(2\mathcal{A})$ and $\mathcal{D}(2\mathcal{B})$, respectively. Moreover, χ_5 is the only character which occurs in no norm 2 element of $L^0(G, b)$ in these cases, and therefore has to be preserved by any perfect isometry. But then one concludes exactly as for $\mathcal{D}(2\mathcal{A}), \mathcal{D}(2\mathcal{B})$ in the proof of Theorem 4, that there is no perfect isometry between these two cases all of whose signs are +1. If b is of type $\mathcal{SD}(2\mathcal{A})_2$ or $\mathcal{SD}(2\mathcal{B})_2$, there is no perfect isometry with any of the two previous cases, because the numbers of irreducible characters are different. Moreover, their decomposition matrices are as for $\mathcal{D}(2\mathcal{A})$ and $\mathcal{D}(2\mathcal{B})$, and hence one concludes as in

the proof of Theorem 4, that there is no perfect isometry between these two cases all of whose signs are positive.

Assume that b has three isomorphism classes of simple modules. The decomposition matrix of the case $\mathcal{SD}(3\mathcal{D})$ is identical to that of $\mathcal{SD}(3\mathcal{B})_1$. (This case appears as principal block of M_0 if $n = 4$, but it is not known, whether it occurs for $n > 4$.) The case $\mathcal{SD}(3\mathcal{C})_2$ does not occur as block algebra by [12, 11.12].

Thus, according to Erdmann's list, we have to consider the four decomposition matrices coming from the cases $\mathcal{SD}(3\mathcal{A})_1$, $\mathcal{SD}(3\mathcal{B})_1$, $\mathcal{SD}(3\mathcal{B})_2$ and $\mathcal{SD}(3\mathcal{H})$. We label the irreducible characters following the pattern in the proof of Theorem 3.

In all cases, the norm 2 elements in $L^0(G, b)$ are exactly the elements of the form $\chi_\lambda - \chi_{\lambda'}$; thus the characters labelled by the λ are preserved under any perfect isometry.

By writing out the norm 3 elements involving some χ_λ , one finds that the case $\mathcal{SD}(3\mathcal{A})_1$ is the only case, where χ_λ occurs in all norm 3 elements of $L^0(G, b)$ with a sign opposite to the signs of the two other characters in that norm 3 element. Similarly, the case $\mathcal{SD}(3\mathcal{H})$ is the only case, in which for a given χ_λ , one can find a norm 3 element in $L^0(G, b)$, such that χ_λ appears with a sign opposite to both other characters and another norm 3 element in $L^0(G, b)$ in which one other character has the same sign as χ_λ .

Thus we may assume, that b and c are of type $\mathcal{SD}(3\mathcal{B})_1$ or $\mathcal{SD}(3\mathcal{B})_2$. In both cases, χ_5 is the only character which appears in no norm 3 element involving χ_λ . But $\mathcal{SD}(3\mathcal{B})_1$ is the only of these two cases, where χ_5 appears with the same sign as some other character in a norm 3 element not involving χ_λ . \square

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