LOCAL CONTROL IN FUSION SYSTEMS OF P-BLOCKS OF FINITE GROUPS

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August 2001

Abstract. If p is an odd prime, b a p-block of a finite group G such that $SL(2,p)$ is not involved in $N_G(Q,e)/C_G(Q)$ for any b-subpair (Q,e), then $N_G(Z(J(P)))$ controls b-fusion, where P is a defect group of b. This is a block theoretic analogue of Glauberman's ZJ-Theorem [6].

1 Introduction

Glauberman's ZJ-Theorem [6, Theorem B] states that if p is an odd prime and G is a finite group such that $Qd(p)$ is not involved in G, then $N_G(Z(J(P)))$ controls p-fusion in G, for P a Sylow p-subgroup of G. Here, $J(P)$ denotes the Thompson subgroup of P (that is, the subgroup generated by all abelian subgroups of P of maximal order) and $Qd(p)$ denotes the semi-direct product of $C_p \times C_p$ with $SL(2,p)$ (with the natural action). This has proved to be an extremely powerful tool in local group-theoretic analysis, as it gives a general condition which ensures that p-fusion is controlled by a single p-local subgroup.

In this paper, we establish block-theoretic analogues of this and other similar results. Along the way, we will obtain results which seem to be new even in the group-theoretic case. A key ingredient, allowing us to exploit the existing group-theoretic methods, is a result of Külshammer and Puig [11] on extensions of nilpotent blocks. We also show (both in a group-theoretic and in a block-theoretic context) that if a normal subgroup of a given group G has a single local subgroup which controls fusion, then G itself has a single local subgroup with the same property. We discuss some consequences of such control of fusion to other problems in block theory.

Throughout the paper, k will denote an algebraically closed field of prime characteristic p. A *block of a finite group* G is a primitive idempotent b in $Z(kG)$; following Alperin-Broué [1], a (G,b)-*subpair* is a pair (Q,e) consisting of a p-subgroup Q of G and a block e of $C_G(Q)$ such that $\mathrm{Br}_Q(b)e = e$, where $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$ is the *Brauer*

1991 *Mathematics Subject Classification.* 20C20.

homomorphism [5]. The set of (G, b)-subpairs is a partially ordered set on which G acts by conjugation, and the maximal (G, b)-subpairs with respect to this partial order are all G-conjugate. If (P, e) is a maximal (G, b)-subpair, then P is called a *defect group of the block* b (this notion is due to Brauer [2]); moreover, for any subgroup Q of P there is a unique block e_Q of $C_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$ (cf. [1]). A detailed account of subpairs and their properties may be found in [14] (where subpairs are referred to as Brauer pairs). The local structure of b is the G-set of (G, b)-subpairs viewed as category; the following definition makes this precise.

Definition 1.1. Let G be a finite group, let b be a block of G and let (P, e) be a maximal (G, b)-subpair. For any subgroup Q of P denote by e_Q the unique block of $C_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. We denote by $\mathcal{F}_{(P,e)}(G, b)$ the category whose objects are the subgroups of P and whose sets of morphisms $\mathrm{Hom}_{\mathcal{F}_{(P,e)}(G,b)}(Q, R)$ are the sets of group homomorphisms $\varphi : Q \to R$ for which there exists an element $x \in G$ satisfying ${}^x(Q, e_Q) \subseteq (R, e_R)$ and $\varphi(u) = xux^{-1}$ for all $u \in Q$, where Q, R run over the set of subgroups of P.

Since all maximal (G, b)-subpairs are G-conjugate, the category $\mathcal{F}_{(P,e)}(G, b)$ does not depend on the choice of (P, e) up to isomorphism of categories. If b is the principal block of G then P is a Sylow-p-subgroup of G and e_Q is the principal block of $C_G(Q)$ for any subgroup Q of P; in this case we write $\mathcal{F}_P(G) = \mathcal{F}_{(P,e)}(G, b)$. Glauberman's ZJ-Theorem reads then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(Z(J(P))))$, provided that p is odd and $Qd(p)$ is not involved in G.

We need a block-theoretic replacement for the hypothesis on $Qd(p)$. Recall that if G is a finite group and b is a block of G, then a (G, b)-subpair (Q, f) is called *centric* if $Z(Q)$ is a defect group of f and (Q, f) is called *radical* if $O_p(N_G(Q, f)/QC_G(Q)) = 1$. The notion of centric subpairs - frequently called self-centralising pairs in the literature - goes back to Brauer [3].

Definition 1.2. Let G be a finite group. A block b of G is called $SL(2, p)$*-free* if $SL(2, p)$ is not isomorphic to a subquotient of any of the groups $N_G(Q, f)/C_G(Q)$, where (Q, f) is a centric and radical (G, b)-subpair.

The definition of an $SL(2, p)$-free block is really a local condition on the block, in that it can be formulated purely in terms of the category $\mathcal{F}_{(P,e)}(G, b)$, where (P, e) is a maximal subpair of a block b of G. Indeed, b is $SL(2, p)$-free if and only if $SL(2, p)$ is not involved in the automorphism group in $\mathcal{F}_{(P,e)}(G, b)$ of any subgroup Q of P such that (Q, e_Q) is centric and radical for the unique e_Q such that $(Q, e_Q) \leq (P, e)$. It may well happen that a non principal block b of G is $SL(2, p)$-free even though $SL(2, p)$ is involved in G. If, however, the principal block of G is $SL(2, p)$-free, then $Qd(p)$ is not involved in G (cf. Proposition 5.1 and [7, Lemma 10.6]). In this case, our hypothesis "$SL(2, p)$-free" is in fact slightly more restrictive, since (in the principal block case) it effectively excludes faithful action of $SL(2, p)$ on any p-subgroup of G, not just the natural action of $SL(2, p)$ on $C_p \times C_p$.

Examples of $SL(2,p)$-free blocks include all blocks with abelian defect groups and, for $p \geq 5$, all blocks of finite p-solvable groups, or more generally, all blocks for which the groups $N_G(Q,f)/C_G(Q)$ occurring in 1.2 are p-solvable.

Since Glauberman's control of fusion theorems also apply to some characteristic subgroups of p-groups other than the center of the Thompson subgroup, we make the following definitions, the first of which is given in [9, §5].

Definition 1.3 A *positive characteristic p-functor* is a map W sending any finite p-group P to a subgroup $W(P)$ of P, with the property that $W(P) \neq 1$ if $P \neq 1$ and that any isomorphism of finite p-groups $P \cong Q$ maps $W(P)$ onto $W(Q)$. A *Glauberman functor* is a positive characteristic p-functor W with the following additional property: whenever P is a Sylow-p-subgroup of a finite group L which satisfies $C_L(O_p(L)) = Z(O_p(L))$ and which does not have a subquotient isomorphic to $Qd(p)$, then $W(P)$ is normal in L.

Of course, by Glauberman's ZJ-Theorem the map sending a finite p-group P to $Z(J(P))$ is a Glauberman functor; in fact showing that this map is a Glauberman functor is the essential ingredient of the ZJ-Theorem. By [7, Theorem 14.8] any of the maps sending a finite p-group P to $K_\infty(P)$ or $K^\infty(P)$ are Glauberman functors, where $K_\infty(P)$, $K^\infty(P)$ are defined in [7, Section 12].

If W is a positive characteristic p-functor, then $W(P)$ is characteristic in P, for any finite p-group P; in particular, if P is a p-subgroup of a finite group G, then $N_G(W(P))$ contains $N_G(P)$. If H is any subgroup of G containing $N_G(P)$, there is a unique block c of H such that $\mathrm{Br}_P(b) = \mathrm{Br}_P(c)$, the *Brauer correspondent of* b (cf. [1] or [14]). Then P is again a defect group of c, and since $C_G(P) \subseteq H$, every maximal (G,b)-subpair (P,e) is also a maximal (H,c)-subpair.

We are now ready to state our results. In what follows, refer to 2.1 and 2.3 for the exact definition of control of fusion that we are using.

Theorem 1.4. *Let G be a finite group, let b be a block of G and let (P,e) be a maximal (G,b)-subpair. Let W be a Glauberman functor, set $N = N_G(W(P))$ and denote by c the unique block of N such that $\mathrm{Br}_P(b) = \mathrm{Br}_P(c)$. If p is odd and b is $SL(2,p)$-free, then $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(N,c)$. In other words, the group N controls fusion in $\mathcal{F}_{(P,e)}(G,b)$.*

The proof of 1.4 is given in section 6. If we specialise Theorem 1.4 to the case of principal blocks and $W(P) = Z(J(P))$, we obtain the conclusion of Glauberman's ZJ-Theorem (but, as mentioned above, our hypothesis "$SL(2,p)$-free" is slightly more restrictive).

Our next result shows that the property of being locally controlled by the normaliser of a single non-trivial subgroup of a defect group carries through normal extensions of blocks.

Theorem 1.5. *Let G be a finite group, H a normal subgroup of G, c a G-stable block of H and b a block of G such that $bc = b$. Let (P, e) be a maximal (G, b)-subpair. There is a P-stable maximal (H, c)-subpair (Q, f) such that $Q = P \cap H$ and $fe_Q \neq 0$, where (Q, e_Q) is the unique (G, b)-subpair contained in (P, e).*

Furthermore, if there is a normal subgroup V of Q such that $N_H(V)$ controls fusion in $\mathcal{F}_{(Q,f)}(H, c)$, then $N_G(W)$ controls fusion in $\mathcal{F}_{(P,e)}(G, b)$ where W is the subgroup of P generated by the set of $N_G(Q, f)$-conjugates of V.

An interesting consequence of Theorem 1.5 is that it allows us to prove that any block b of a finite group G lying over an $SL(2, p)$-free block of a normal subgroup N of G with non-trivial defect groups has again a local structure which is controlled by the normaliser of a single non-trivial p-subgroup of G, even though b itself need not be $SL(2, p)$-free:

Corollary 1.6. *Let G be a finite group, let b be a block of G and let (P, e) be a maximal (G, b)-subpair. If there is a normal subgroup H of G such that $H \cap P \neq 1$ and such that b covers an $SL(2, p)$-free block c of H, then there is a non-trivial normal subgroup W in P such that $N_G(W)$ controls fusion in $\mathcal{F}_{(P,e)}(G, b)$.*

In [7, Section 12] Glauberman showed that for $W = K_\infty$ or $W = K^\infty$, the subgroup $W(P)$ of P is self-centralising; that is, $C_P(W(P)) = Z(W(P))$. Thus, in the situation of Theorem 1.4, the (G, b)-subpair $(W(P), e_{W(P)})$ is centric; in other words, the normaliser in G of some centric (G, b)-subpair controls b-fusion. The next Theorem shows that there is a canonical choice for such a centric subpair. By results of Külshammer and Puig in [11, Theorem 1.8], associated with any centric (G, b)-subpair (Q, f) and any choice of a maximal $(N_G(Q, f), f)$-subpair (R, g), there is a canonical group extension

$$1 \longrightarrow Q \longrightarrow L \longrightarrow N_G(Q, f)/QC_G(Q) \longrightarrow 1$$

having the property that R is a Sylow-p-subgroup of L and $\mathcal{F}_{(R,g)}(N_G(Q, f), f) = \mathcal{F}_R(L)$ (we explain this in some more detail in 2.4 below); moreover, $O_{p'}(L) = 1$ and $C_L(Q) = Z(Q)$. Thus, if b is $SL(2, p)$-free, then $Qd(p)$ is not involved in L, and hence $W(R)$ is normal in L for any Glauberman functor W.

Theorem 1.7. *Let G be a finite group, let b be a block of G and let (P, e) be a maximal (G, b)-subpair. Assume that p is odd and that b is $SL(2, p)$-free. There is a unique minimal subgroup Q of P such that (Q, f) is centric and radical, where f is the unique block of $C_G(Q)$ such that $(Q, f) \subseteq (P, e)$. Moreover, Q is normal in P and we have $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(L)$, where L is the middle term of the Külshammer-Puig extension associated with (Q, f).*

Remark 1.8. Theorems 1.5 and 1.7 seem to add some new information even in the principal block case. Theorem 1.5 implies that if N is a normal subgroup of a finite group G such that $N_N(V)$ controls strong p-fusion in $P \cap N$ with respect to N for some normal subgroup V of $P \cap N$ then the subgroup, W, of P generated by all $N_G(P \cap N)$-conjugates of V has the property that $N_G(W)$ controls strong p-fusion in P with respect to G. Theorem 1.7 translates to the following statement: given a finite group G with a Sylow-p-subgroup P such that $SL(2,p)$ is not involved in $N_G(Q)/C_G(Q)$ for any p-subgroup Q of G, there is a unique minimal subgroup Q of P such that $Z(Q)$ is a Sylow-p-subgroup of $C_G(Q)$ and such that $O_p(N_G(Q)/QC_G(Q)) = 1$; moreover, $N_G(Q)$ controls strong p-fusion in P with respect to G.

A *classifying space of* b is a p-complete space $B(G,b)$ having the homotopy type of the p-completion of an $\mathcal{L}$-system associated with $\mathcal{F}_{(P,e)}(G,b)$ in the sense of Broto, Levi and Oliver [4]. Note that in the situation of Theorem 1.7, the local structure of b is the same as the local structure of the principal block of L. Thus, if we take for $B(G,b)$ the p-completion $BL_p^\wedge$ of the classifying space BL of L we obtain the following immediate consequence.

Corollary 1.9. *If p is odd, any $SL(2,p)$-free block has a classifying space, which is unique up to homotopy.*

Theorems 1.4 and 1.5 provide many examples of blocks whose fusion pattern is determined by the normaliser of a single non-trivial p-subgroup. The existence of such controlling subgroups has ramifications for the Dade Projective Conjectures (DPC).

Theorem 1.10. *Let G be a finite group, let b be a block of G and let (P,e) be a maximal (G,b)-subpair. Assume that there is a normal subgroup R in P such that $N_G(P,e) \subseteq N_G(R)$ and such that $N_G(R)$ controls fusion in $\mathcal{F}_{(P,e)}(G,b)$. Let c be the block of $N_G(R)$ which satisfies $\mathrm{Br}_P(c)e = e$; that is, c is the Brauer correspondent in $N_G(Q)$ of b.*

(i) If every section of G satisfies DPC, then there is a defect preserving bijection between the sets of irreducible characters of b and irreducible characters of c.

(ii) If every proper section of G satisfies DPC, then DPC holds for b if and only if there is a defect preserving bijection between the sets of irreducible characters of b and irreducible characters of c.

2 On local categories of blocks

We collect in this Section some standard terminology and properties of local categories of blocks.We fix a finite group G, a block b of G and a maximal (G,b)-subpair (P,e). For any subgroup Q of P, denote by e_Q the unique block of $C_G(Q)$ such that $(Q,e_Q) \subseteq (P,e)$ (in particular, $e = e_P$).

By the uniqueness of the inclusion of subpairs (cf. [1]) we have $\mathcal{F}_P(P) \subseteq \mathcal{F}_{(P,e)}(G,b)$. If we choose a Sylow-$p$-subgroup S of G containing P, we have also $\mathcal{F}_{(P,e)}(G,b) \subseteq \mathcal{F}_S(G)$.

Two subgroups Q, R of P are isomorphic as objects in $\mathcal{F}_{(P,e)}(G,b)$ if there is $x \in G$ such that ${}^x(Q,e_Q) = (R,e_R)$. Any subgroup Q of P is isomorphic in $\mathcal{F}_{(P,e}(G,b)$ to a subgroup R of P such that $N_P(R)$ is a defect group of e_R viewed as block of $N_G(R,e_R)$ (cf. [1] or [14]). We say that (Q,e_Q) is an *Alperin-Goldschmidt- pair* (for $\mathcal{F}_{(P,e)}(G,b)$), if (Q,e_Q) is centric, radical and $N_P(Q)$ is a defect group of $kN_G(Q,e_Q)e_Q$. If Q is normal in P, then P is a defect group of e_Q as block of $N_G(Q,e_Q)$, and hence (P,e_P) is also a maximal $(N_G(Q,e_Q),e_Q)$-subpair. It has been shown by Puig, that (Q,e_Q) is centric if and only if $C_P(R) = Z(R)$ for any subgroup R of P which is isomorphic to Q in $\mathcal{F}_{(P,e)}(G,b)$. Thus the property of being centric can be read off the category $\mathcal{F}_{(P,e)}(G,b)$. Furthermore, the automorphism group of Q in $\mathcal{F}_{(P,e)}(G,b)$ is canonically isomorphic to $N_G(Q,e_Q)/C_G(Q)$.

A *conjugation family for* $\mathcal{F}_{(P,e)}(G,b)$ is a set $\mathcal{C}$ of subgroups of P with the following property: every isomorphism in $\mathcal{F}_{(P,e)}(G,b)$ is the composition of isomorphisms of the form $\varphi : Q \to R$, where Q, R are subgroups of P, such that there exists a subgroups S in $\mathcal{C}$ containing both Q, R and an element $x \in N_G(S,e_S)$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in Q$.

It is well-known and easy to check that if $\mathcal{C}$ is a conjugation family for $\mathcal{F}_{(P,e)}(G,b)$, then any subset $\mathcal{C}'$ of $\mathcal{C}$ such that any object in $\mathcal{C}$ is isomorphic to an object of $\mathcal{C}'$ in $\mathcal{F}_{(P,e)}(G,b)$ is again a conjugation family.

By Alperin's fusion theorem (in its refined version by Goldschmidt and adapted to blocks, cf. [1, §4]), the set of subgroups Q of P for which (Q,e_Q) is an Alperin-Goldschmidt pair is a conjugation family for $\mathcal{F}_{(P,e)}(G,b)$, called the *Alperin-Goldschmidt conjugation family for* $\mathcal{F}_{(P,e)}(G,b)$.

Definition 2.1 A subgroup H of G *controls fusion in* $\mathcal{F}_{(P,e)}(G,b)$ if H contains P and if $\mathcal{F}_{(P,e)}(G,b) \subseteq \mathcal{F}_S(H)$ for some Sylow-p-subgroup S of H which contains P.

By Alperin's fusion theorem, a subgroup H of G containing P controls fusion in $\mathcal{F}_{(P,e)}(G,b)$ if and only if $N_G(Q,e_Q) = N_H(Q,e_Q)C_G(Q)$ for any subgroup Q of P.

Lemma 2.2. *Let W be a normal subgroup in P, and let H be a subgroup of G such that $P \subseteq H \subseteq N_G(W)$. Assume that H controls fusion in $\mathcal{F}_{(P,e)}(G,b)$. Then W is contained in any subgroup Q of P such that (Q,e_Q) is centric and radical.*

Proof. Let Q be a subgroup of P such that (Q,e_Q) is centric and radical. Since $N_G(Q,e_Q) = N_H(Q,e_Q)C_G(Q)$ and W is normal in H, the image of $N_W(Q)$ is normal in $N_G(Q,e_Q)/QC_G(Q)$, hence $N_W(Q) \subseteq QC_G(Q)$ as (Q,e_Q) is radical. Thus $N_W(Q) \subseteq Q$ because (Q,e_Q) is centric, and therefore $W \subseteq Q$. □

The first statement of the following Proposition is a variation of [10, Statement 1]. The second statement makes precise what it means, in certain circumstances, for a subgroup to control fusion.

Proposition 2.3. *Let Q be a subgroup of P, let H be a subgroup of $N_G(Q)$ containing $QC_G(Q)$, and let c be the unique block of H such that $\mathrm{Br}_Q(c)e_Q = e_Q$. Assume that c has a defect group R contained in P. Then (R, e_R) is a maximal (H, c)-subpair, and we have $\mathcal{F}_{(R,e_R)}(H, c) \subseteq \mathcal{F}_{(P,e)}(G, b)$; moreover, this inclusion is an equality if and only if H controls fusion in $\mathcal{F}_{(P,e)}(G, b)$.*

Proof. Since Q is normal in H, Q is contained in any defect group of H. If R is a defect group of c contained in P, then $C_G(R) \subseteq C_G(Q) \subseteq H$, and thus (R, e_R) is a - necessarily maximal - (H, c)-subpair. Let (S, f) be a centric radical (H, c)-subpair contained in (R, e_R). Again, since Q is normal in H, we have $Q \subseteq S$ by 2.2. Then $C_G(S) = C_H(S)$, and so $f = e_S$. Thus $N_H(S, f) = N_H(S, e_S) \subseteq N_G(S, e_S)$. The inclusion $\mathcal{F}_{(R,e_R)}(H, c) \subseteq \mathcal{F}_{(P,e)}(G, b)$ follows, using Alperin's fusion theorem.

Assume that H controls fusion in $\mathcal{F}_{(P,e)}(G, b)$. Then in particular $R = P$ is a defect group of c, as Q is normal in H and P is contained in H. Thus (P, e) is also a maximal (H, c)-subpair. Let now S be a subgroup of P such that (S, e_S) is a radical centric (G, b)-subpair. Thus $Q \subseteq S$ by 2.2. But then $C_G(S) \subseteq H$, and so (S, e_S) is also a centric (H, c)-subpair. Thus the inclusion $N_G(S, e_S) \subseteq N_H(S, e_S)C_G(S)$ translates to $\mathcal{F}_{(P,e)}(G, b) \subseteq \mathcal{F}_{(P,e)}(H, c)$, hence equality by the first statement. The rest is clear. □

Proposition 2.3 applies in the following two situations. If H contains $N_G(P)$ and if c is the unique block of H such that $\mathrm{Br}_P(c) = \mathrm{Br}_P(b)$, then (P, e) is also a maximal (H, c)-subpair. Thus if P has a subgroup Q such that $C_G(Q) \subseteq H \subseteq N_G(Q)$, we have $\mathcal{F}_{(P,e)}(H, c) \subseteq \mathcal{F}_{(P,e)}(G, b)$. The second situation, in which we are going to apply 2.3 arises if $H = N_G(Q, e_Q)$ for some subgroup Q of P and if $c = e_Q$ such that $N_P(Q)$ is a defect group of c (viewed as block of H).

The next Proposition is a particular case of Külshammer-Puig [11, Theorem 1.8], translated to our terminology (see also [10, Statement 8]).

Proposition 2.4. *Assume that $G = N_G(Q, e_Q)$ for some subgroup Q of P such that (Q, e_Q) is centric. Then $b = e_Q$, and there is a short exact sequence of finite groups*

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/QC_G(Q) \longrightarrow 1$$

such that P is a Sylow-p-subgroup of L and such that $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(L)$. Moreover, we have $O_{p'}(L) = 1$ and $C_L(Q) = Z(Q)$; in particular, L is p-constrained.

Proof. As Q is normal in G, the block idempotent b is contained in $kC_G(Q)$, and as G stabilises e_Q, we have $b = e_Q$ (this is a standard argument; see [1]). To establish the link with the terminology in [11, 1.8], note first that P is also a defect group of $\{b\}$ viewed as point of G on $kC_G(Q)$, because P is maximal with respect to the property $\mathrm{Br}_P(b) \neq 0$. The existence of a canonical exact sequence as stated such that P is a Sylow-p-subgroup of L is a particular case of [11, 1.8]. This extension has the property, that for any $y \in L$, the outer automorphisms of Q induced by conjugation with y and by conjugation with some element $x \in G$ such that $xQC_G(Q)$ is the image of y in $G/QC_G(Q)$ coincide.

In particular, if $y \in C_L(Q)$ then $x \in QC_G(Q)$, and hence $y \in Q$. This shows that $C_L(Q) = Z(Q)$, and since Q is normal in L, we have $O_{p'}(L) = O_{p'}(C_L(Q)) = 1$. The equality $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(L)$ is essentially a reformulation of [11, 1.8.2]; we reproduce the argument from [10, Statement 8]. Since Q is normal in L and in G, it suffices to show that the images in $\mathrm{Aut}(R)$ of $N_G(R,e_R)$ and $N_L(R)$ are equal, where R is a subgroup of P containing Q. As (Q,e_Q) is centric, so is (R,e_R). Similarly, as $C_L(Q) = Z(Q)$, we have $C_L(R) = Z(R)$. Setting $\bar{G} = G/QC_G(Q)$, with the notation of [11, 1.8] (which is defined in [11, 2.8]) we have $E_{G,\bar{G}}(R,e_R) = E_{L,\bar{G}}(R)$. By [11, (2.8.1)], the canonical maps $E_{G,\bar{G}}(R,e_R) \to E_G(R,e_R)$ and $E_{L,\bar{G}}(R) \to E_L(R)$ are surjective. Thus $E_G(R,e_R) = E_L(R)$. This implies the equality $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(L)$. □

We need the following generalisation of [10, Statement 9].

Proposition 2.5. *Let Q be a normal subgroup of P, set $H = N_G(Q)$ and denote by c the unique block of H such that $e_Q c = e_Q$. Suppose there is a finite group L having P as Sylow-p-subgroup such that $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(L)$. Then (P,e) is a maximal (H,c)-subpair, P is a Sylow-p-subgroup of $N_L(Q)$, and we have $\mathcal{F}_{(P,e)}(H,c) = \mathcal{F}_P(N_L(Q))$.*

Proof. Since Q is normal in P, the pair (P,e_P) is also a maximal (H,c)-subpair, and clearly P is a Sylow-p-subgroup of $N_L(Q)$. By 2.3, we have $\mathcal{F}_{(P,e)}(H,c) \subseteq \mathcal{F}_{(P,e)}(G,b)$. In order to show the equality $\mathcal{F}_{(P,e)}(H,c) = \mathcal{F}_P(N_L(Q))$, it suffices to show that $N_H(S,f)$ and $N_L(S) \cap N_L(Q)$ have the same images in $\mathrm{Aut}(S)$, where (S,f) is an (H,c)-Brauer pair contained in (P,e). Since Q is normal in H and $N_L(Q)$, we may assume that $Q \subseteq S$, by 2.2. Then $C_G(S) \subseteq H$ and $f = e_S$. The assumption $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(L)$ implies that given any $x \in N_G(S,e_S)$, there is $y \in N_L(S)$ such that ${}^x u = {}^y u$ for all $u \in S$. Since $Q \subseteq S$, clearly $x \in N_H(S,e_S)$ if and only if $y \in N_L(S) \cap N_L(Q)$. The equality $\mathcal{F}_{(P,e)}(H,c) = \mathcal{F}_P(N_L(Q))$ follows. □

The following Lemma appears in a slightly more general version in Puig [12].

Lemma 2.6. *Let G be a finite group, let b be a block of G and let (Q,e), (R,f) be centric (G,b)-subpairs such that $(Q,e) \subseteq (R,f)$. We have*

$$N_G(R,f) \cap C_G(Q) = Z(Q)C_G(R) .$$

Proof. Clearly the right side is contained in the left side. For the converse, assume first that Q is normal in R. Let $x \in N_G(R,f) \cap C_G(Q)$. It is easy to check that $[R,x] \subseteq C_R(Q) = Z(Q)$. Thus $[R,x,x] = 1$. If x is a p'-element, this forces $x \in C_G(R)$ by standard properties of coprime group actions (cf. [8]). Note that the image of a defect group of f as block of $N_G(R,f)$ is a Sylow-p-subgroup of $N_G(R,f)/C_G(R)$. Thus if x is a p-element, we may assume that x belongs to a defect group of f as block of $N_G(R,f)$, which implies $x \in Z(Q)$, as (Q,e) is centric. The general case follows by induction. □

Proposition 2.7. *Assume that there is a unique minimal subgroup R of P such that (R, e_R) is centric and radical. Then R is normal in P, the pair (P, e) is a maximal $(N_G(R, e_R), e_R)$-subpair, and we have $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N_G(R, e_R), e_R)$.*

Proof. The uniqueness of R implies that R is normal in P, and hence (P, e) is also a maximal $(N_G(R, e_R), e_R)$-subpair. Let S be a subgroup of P such that (S, e_S) is centric and radical. Then $R \subseteq S$ by the uniqueness of (R, e_R). If $x \in N_G(S, e_S)$, then ${}^x(R, e_R) \subseteq (S, e_S)$, and again, by the uniqueness of (R, e_R), we deduce that ${}^x(R, e_R) = (R, e_R)$. In other words, $N_G(S, e_S) \subseteq N_G(R, e_R)$, which implies $\mathcal{F}_{(P,e)}(G, b) \subseteq \mathcal{F}_{(P,e)}(N_G(R, e_R), e_R)$, hence the equality by 2.3. □

We provide a criterion for when the Alperin-Goldschmidt conjugation family has a unique minimal element.

Proposition 2.8. *Assume that $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N_G(Q, e_Q), e_Q)$ for some normal subgroup Q of P such that (Q, e_Q) is centric. Then there is a unique subgroup R of P containing Q such that $O_p(N_G(Q, e_Q)/QC_G(Q)) = RC_G(Q)/QC_G(Q)$. The group R is then the unique minimal subgroup of P such that (R, e_R) is centric and radical. In particular, R is normal in P and $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N_G(R, e_R), e_R)$.*

Proof. We may assume that $G = N_G(Q, e_Q)$ and hence that $b = e_Q$. The image of P in $G/QC_G(Q)$ is a Sylow-p- subgroup; since (Q, e_Q) is centric, this image is isomorphic to P/Q. Therefore, there is a unique subgroup R of P containing Q such that the image of R in $G/QC_G(Q)$ is $O_p(G/QC_G(Q))$. The uniqueness of R implies that R is normal in P. Note that b is still a block of $RC_G(Q)$, and then (R, e_R) is a maximal $(RC_G(Q), b)$-subpair. By our choice of R, the group $RC_G(Q)$ is normal in G, and since $RC_G(Q)$ acts transitively on the set of maximal $(RC_G(Q), b)$-subpairs, the Frattini argument shows that $G = N_G(R, e_R)C_G(Q)$.

Let S be a subgroup of P such that (S, e_S) is centric and radical. By Lemma 2.6, we have $N_G(S, e_S) \cap QC_G(Q) = QC_G(S)$. Thus the inclusion $N_G(S, e_S) \subset G$ induces an injective group homomorphism $N_G(S, e_S)/QC_G(S) \to G/QC_G(Q)$. The image of R in $G/QC_G(Q)$ is $O_p(G/QC_G(Q))$; thus the image of $N_R(S)$ in $N_G(S, e_S)/QC_G(S))$ is contained in $O_p(N_G(S, e_S)/QC_G(S))$, and hence the image of $N_R(S)$ in $N_G(S, e_S)/SC_G(S)$ is contained in $O_p(N_G(S, e_S)/SC_G(S)) = 1$. This forces $N_R(S) \subseteq SC_G(S)$. As (S, e_S) is centric, we get $N_R(S) \subseteq S$, hence $R \subseteq S$.

By Lemma 2.6 again, we have $N_G(R, e_R) \cap RC_G(Q) = RC_G(R)$. As $G = N_G(R, e_R)C_G(Q)$, it follows that $N_G(R, e_R)/RC_G(R) \cong G/RC_G(Q)$, and hence $O_p(N_G(R, e_R)/RC_G(R)) = 1$ by our choice of R. This shows that R is indeed the unique minimal subgroup of P such that (R, e_R) is centric and radical. The rest is clear by 2.7. □

3 Local control of characteristic p-functors

Let G be a finite group, let b be a block of G, let (P,e) be a maximal (G,b)-subpair, and for any subgroup Q of P, denote by e_Q the unique block of $C_G(Q)$ such that $(Q,e_Q) \subseteq (P,e)$.

Given a positive characteristic p-functor W and a subgroup Q of P, we set $W_1(Q) = Q$ and $P_1(Q) = N_P(Q)$. For any positive integer i, we define inductively $W_{i+1}(Q) = W(P_i(Q))$ and $P_{i+1}(Q) = N_P(W_{i+1}(Q))$. For all positive integers i we have $W_i(Q) \subseteq P_i(Q)$, and if $P_i(Q)$ is a proper subgroup of P, in fact $P_i(Q)$ is a proper subgroup of $P_{i+1}(Q)$. In particular, $P_i(Q) = P$ for all large enough i. We will say that Q is *well-placed in* P (with respect to W and $\mathcal{F}_{(P,e)}(G,b)$) if $P_i(Q)$ is a defect group of the block $e_{W_i(Q)}$ as block of $N_G(W_i(Q), e_{W_i(Q)})$ for all positive integer i. Clearly P is always well-placed in P.

The next Lemma states essentially that every subgroup of P is isomorphic to a well-placed subgroup with respect to $\mathcal{F}_{(P,e)}(G,b)$ and a positive characteristic p-functor.

Lemma 3.1. *Let W be a positive characteristic p-functor. For any subgroup Q of P, there is an element $x \in G$ such that ${}^x(Q,e_Q) \subset (P,e)$, ${}^xN_P(Q) \subseteq P$ and such that ${}^xQ$ is well-placed in P.*

Proof. Define sequences of subgroups and blocks as follows. Let $V_1 := Q$, $v_1 := e_Q$. Let (R_1, r_1) be a b-subpair which is maximal with respect to normalising (V_1, v_1) and such that $(N_P(Q), e_{N_P(Q)}) \le (R_1, r_1)$. For $i \ge 1$ let $V_{i+1} = W(R_{i+1})$ and let (V_{i+1}, v_{i+1}) be the b-subpair contained in (R_{i+1}, r_{i+1}). Let (R_{i+1}, r_{i+1}) be a b-subpair which is maximal with respect to normalising (V_{i+1}, v_{i+1}) and such that $(R_i, r_i) \le (R_{i+1}, r_{i+1})$. Note that if (S,f) is a maximal b-subpair containing (R_{i+1}, r_{i+1}), then $R_{i+1} = N_S(V_{i+1})$. On the other hand, $N_S(R_i) \subset N_S(V_{i+1})$. Thus, either $R_i = S$ or R_{i+1} properly contains R_i. In other words, there exists an integer t such that for all $i \ge t$, $(R_i, r_i) = (R_t, r_t)$ is a maximal b-Brauer pair, $(V_i, v_i) = (W(R_t), v_t)$. Let $x \in G$ be such that ${}^x(R_t, r_t) = (P,e)$. Then ${}^x(Q,e_Q) \le (P,e)$, and since for every $i \ge 1$, ${}^xR_i \subset {}^gR_t = P$, it is clear that ${}^x(Q,e_Q)$ is well placed in (P,e). The second assertion is clear since $N_P(Q) \subset R_1 \subset {}^{x^{-1}}P$. □

The next results states roughly speaking, that "if a positive characteristic p-functor controls fusion locally, it controls fusion globally". This generalises a result by Alperin and Gorenstein (cf. [9, Ch. X, Theorem 9.3])

Proposition 3.2. *Let W be a positive characteristic p-functor. Assume that for any non-trivial subgroup Q of P and any maximal $(N_G(Q,e_Q), e_Q)$-subpair (R,f), the group $N_{N_G(Q,e_Q)}(W(R))$ controls fusion in $\mathcal{F}_{(R,f)}(N_G(Q,e_Q), e_Q)$. Then $N_G(W(P))$ controls fusion in $\mathcal{F}_{(P,e)}(G,b)$.*

Proof. Set $H = N_G(W(P))$. Suppose, if possible that the result is not true. Then by 3.1 above, there exists a non-trivial subgroup Q of P such that (Q, e_Q) is well placed in (P, e) such that $N_G(Q, e_Q)$ is not contained in $C_G(Q)N_H(Q, e_Q)$.

We introduce the following notation. For $i \geq 1$, let $W_i = W_i(Q)$, $P_i = P_i(Q)$, $e_i = e_{W_i}$, $N_i = N_G(W_i, e_i)$, $M_i = N_G(W_i)$ and $L_i = N_i \cap N_G(W_{i+1})$. Let f_i be the block of M_i satisfying $e_i f_i = e_i$. Let s_i be the block of L_i such that $\mathrm{Br}_{P_i}(s_i) = \mathrm{Br}_{P_i}(e_i)$.

Set $\mathcal{F}_i = \mathcal{F}_{(P_i, e_{P_i})}(N_i, e_i)$, set $\mathcal{G}_i = \mathcal{F}_{(P_i, e_{P_i})}(L_i, s_i)$, and set $\mathcal{H}_i = \mathcal{F}_{(P_i, e_{P_i})}(M_i, f_i)$.

It is clear from 2.3 that $\mathcal{G}_i \subset \mathcal{F}_i$. On the other hand, $P_i C_{M_{i+1}}(W_i) \subset L_i \subset N_{M_{i+1}}(W_i)$. Since $(W_{i+1}, e_{i+1}) \leq (P_i, e_{P_i})$, $\mathrm{Br}_{P_i}(f_i)e_{P_i} = e_{P_i}$ and hence by 2.3 it follows that $\mathcal{G}_i \subset \mathcal{H}_{i+1}$. Since, clearly $\mathcal{H}_{i+1} = \mathcal{F}_{i+1}$, we get that $\mathcal{G}_i \subset \mathcal{F}_{i+1}$.

By the hypothesis of proposition, we have that $\mathcal{G}_i = \mathcal{F}_i$, hence, we get that for all $i \geq 1$, $\mathcal{F}_1 \subset \mathcal{F}_i \subset \mathcal{F}_{i+1}$.

Let i be such that $P_i = P$, so that $\mathcal{F}_{i+1} = \mathcal{F}_{(P,e)}(H, c)$, where c is the Brauer correspondent of b. Let g be an element of $N_G(Q, e_Q)$. Then conjugation by g determines an element, say ϕ of of $\mathrm{End}_{\mathcal{F}_1}(Q)$. Then ϕ is induced by conjugation with an element $x \in H$, hence $g = zx$ for some $z \in C_G(Q)$. Thus, $N_G(Q, e_Q) \subset C_G(Q)(H \cap N_G(Q, e_Q))$, contradicting our choice of (Q, e_Q). □

4 On the local structure of central p-extensions

Let G be a finite group, let b be a block of G, and let (P, e) be a maximal (G, b)-subpair. We assume in this section that P contains a subgroup Z of $Z(G)$. We set $\bar{G} = G/Z$ and $\bar{P} = P/Z$; for any element or subset a of kG, we denote by $\bar{a}$ its canonical image in $k\bar{G}$. It is well-known that the image $\bar{b}$ of b in $k\bar{G}$ is a block of $\bar{G}$ having $\bar{P}$ as defect group. The following (equally well-known) Lemma relates the local structures of b and $\bar{b}$.

Lemma 4.1. *For every (G, b)-subpair (Q, f) there is a unique $(\bar{G}, \bar{b})$-subpair of the form $(\bar{Q}, g)$ such that $\bar{f}g = \bar{f}$, and then the canonical map $G \to \bar{G}$ induces a surjective group homomorphism $N_G(Q, f)/C_G(Q) \to N_{\bar{G}}(\bar{Q}, g)/C_{\bar{G}}(\bar{Q})$ whose kernel is an abelian p-group. In particular, if $O_p(N_G(Q, f)/QC_G(Q)) = 1$, this map induces an isomorphism $N_G(Q, f)/QC_G(Q) \cong N_{\bar{G}}(\bar{Q}, g)/\bar{Q}C_{\bar{G}}(\bar{Q})$.*

Proof. It is well-known (and easy to check) that the group $\overline{C_G(Q)}$ is a normal subgroup of $C_{\bar{G}}(\bar{Q})$ and that $C_{\bar{G}}(\bar{Q})/\overline{C_G(Q)}$ is an abelian p-group. Thus any block of $C_{\bar{G}}(\bar{Q})$ is contained in $k\overline{C_G(Q)}$. Hence the sum of the different $C_{\bar{G}}(\bar{Q})$- conjugates of $\bar{f}$ is the unique block g of $C_{\bar{G}}(\bar{Q})$ fulfilling $\bar{f}g = \bar{f}$, and we have $N_{\bar{G}}(\bar{Q}, g) = \overline{N_G(Q, f)}C_{\bar{G}}(\bar{Q})$. The Lemma follows. □

The above Lemma implies in particular, that the maximal (G, b)-subpair (P, e) determines a unique maximal $(\bar{G}, \bar{b})$-subpair $(\bar{P}, f)$ by the condition $\bar{e}f = \bar{e}$. With this choice of maximal subpairs, 4.1 translates to the following statement.

Proposition 4.2. *The canonical map $G \to \bar{G}$ induces a surjective functor $\mathcal{F}_{(P,e)}(G,b) \to \mathcal{F}_{(\bar{P},f)}(\bar{G},\bar{b})$. In particular, b is $SL(2,p)$-free, if and only if $\bar{b}$ is $SL(2,p)$-free.*

Proof. Clear by 4.1. □

Proposition 4.3. *Let H be a subgroup of G containing $N_G(P)$ and denote by c the unique block of H such that $\mathrm{Br}_P(c) = \mathrm{Br}_P(b)$. Assume that there is a subgroup Q of P containing Z such that Q is normal in H and such that $C_{\bar{G}}(\bar{Q}) \subseteq \bar{H}$. Then $\mathrm{Br}_{\bar{P}}(\bar{c}) = \mathrm{Br}_{\bar{P}}(\bar{b})$, and we have $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(H,c)$ if and only if $\mathcal{F}_{(\bar{P},f)}(\bar{G},\bar{b}) = \mathcal{F}_{(\bar{P},f)}(\bar{H},\bar{c})$.*

Proof. The equality $\mathrm{Br}_{\bar{P}}(\bar{c}) = \mathrm{Br}_{\bar{P}}(\bar{b})$ is clear by [10, Statement 5]. Suppose that $\mathcal{F}_{\bar{G},\bar{b}} = \mathcal{F}_{\bar{H},\bar{c}}$. Let (R,t) be a centric radical (G,b)-subpair. Let s be the unique block of $C_{\bar{G}}(\bar{R})$ such that $\bar{t}s = \bar{t}$. By Lemma 4.1, we have $N_G(R,t)/RC_G(R) \cong N_{\bar{G}}(\bar{R},s)/\bar{R}C_{\bar{G}}(\bar{R}) = N_{\bar{H}}(\bar{R},s)/\bar{R}C_{\bar{G}}(\bar{R}) \cap \bar{H}$. Now Q is normal in H, and thus the canonical image of $N_Q(R)$ is normal in $N_G(R,t)/RC_G(R)$. Therefore we have $N_Q(R) \subseteq RC_G(R)$. As the subpair (R,t) is centric, we have $N_Q(R) \subseteq R$, which forces $Q \subseteq R$. Thus $C_{\bar{G}}(\bar{R}) \subseteq \bar{H}$ by the assumptions, and so $(\bar{R},s)$ is also an $(\bar{H},\bar{c})$-subpair and (R,t) is an (H,c)-subpair. Therefore $N_H(R,t)/RC_G(R)$ is a subgroup of $N_G(R,t)/RC_G(R) \cong N_{\bar{G}}(\bar{R},s)/\bar{R}C_{\bar{G}}(\bar{R}) = N_{\bar{H}}(\bar{R},s)/\bar{R}C_{\bar{H}}(\bar{R})$. But then Lemma 4.1, applied to H and c instead of G and b, respectively, shows that $N_H(R,t) = N_G(R,t)$, which implies the equality $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(H,c)$. The converse is trivial. □

5 On $SL(2,p)$-free blocks

Proposition 5.1. *Let G be a finite group and let b be a block of G. Suppose that $SL(2,p)$ is involved in $N_G(Q,f)/C_G(Q)$ for some non-trivial (G,b)-subpair (Q,f). Then $SL(2,p)$ is involved in $N_G(Q,e)/C_G(Q)$ for some centric and radical (G,b)-subpair (Q,e).*

Proof. Fix a maximal (G,b)-subpair (P,e_P), and for any subgroup Q of P, denote by (Q,e_Q) the unique (G,b)-subpair contained in (P,e_P). Let Q be a subgroup of P with $|Q|$ maximum such that $SL(2,p)$ is involved in $N_G(Q,e_Q)/C_G(Q)$. Replacing (Q,e_Q) with a G-conjugate if necessary, we may assume that $N_P(Q)$ is a defect group of $kN_G(Q,e_Q)e_Q$, so that in particular, $R = QC_P(Q)$ is a defect group of $kQC_G(Q)e_Q$ and (R,e_R) is a maximal $(QC_G(Q),e_Q)$-pair. Since $QC_G(Q)$ is normal in $N_G(Q,e_Q)$, the Frattini argument gives $N_G(Q,e_Q) = C_G(Q)[N_G(R,e_R) \cap N_G(Q,e_Q)]$. But then, $N_G(Q,e_Q)/C_G(Q) \cong N_G(R,e_R) \cap N_G(Q,e_Q)/N_G(R,e_R) \cap C_G(Q)$. On the other hand, since $C_G(R) \subseteq C_G(Q)$, $N_G(R,e_R) \cap N_G(Q,e_Q)/N_G(R,e_R) \cap C_G(Q)$ is isomorphic to a subquotient of $N_G(R,e_R)/C_G(R)$. Hence $SL(2,p)$ is involved in $N_G(R,e_R)/C_G(R)$. The choice of Q now implies that $R = Q$ whence (Q,e_Q) is a centric (G,b)-pair.

Let M be the inverse image of $O_p(N_G(Q,e_Q)/QC_G(Q))$ in $N_G(Q,e_Q)$ and let $S = M \cap N_P(Q)$. Then S is a defect group of kMe_Q, (S,e_S) is a maximal (M,e_Q)-pair. Since $N_G(Q,e_Q)$ normalises M, the Frattini argument again gives that

$N_G(Q, e_Q) = M[N_G(S, e_S) \cap N_G(Q, e_Q)]$. But $M = (QC_G(Q))S$ whence $N_G(Q, e_Q) = C_G(Q)[N_G(S, e_S) \cap N_G(Q, e_Q)]$. Arguing as before, we conclude that $S = Q$ and hence that $M = QC_G(Q)$. This completes the proof. □

The main application of 5.1 is the following proposition which shows that the property of being $SL(2, p)$-free passes down to corresponding blocks of normalisers of subpairs.

Proposition 5.2. *Let G be a finite group and let b be an $SL(2, p)$-free block of G. For every (G, b)-subpair (Q, f) the block f of $N_G(Q, f)$ is $SL(2, p)$-free.*

Proof. Let (R, g) be a centric radical $(N_G(Q, f), f)$-subpair. Then $Q \subseteq R$ by 2.2, and hence $C_G(R) \subseteq N_Q(Q, f)$. Thus (R, g) is a (G, b)-Brauer pair, and hence $SL(2, p)$ is not a subquotient of $N_G(R, g)/C_G(R)$ by 5.1. But then $SL(2, p)$ is obviously not a subquotient of $N_{N_G(Q,f)}(R, g)/C_{N_G(Q,f)}(R)$. □

6 Proof of Theorem 1.4 and Theorem 1.7

Proof of Theorem 1.4. Let G be a finite group, let b be a block of G, let (P, e) be a maximal (G, b)-subpair, and for any subgroup Q of P, denote by e_Q the unique block of $C_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. Let W be a Glauberman functor, set $N = N_G(W(P))$ and denote by c the unique block of N such that $\mathrm{Br}_P(c) = \mathrm{Br}_P(b)$. Assume that p is odd.

Suppose that Theorem 1.4 fails for the blocks b and c of G and N, respectively, and assume that $|G|$ has minimal order with this property. We are going to derive a contradiction, proceeding in several steps.

6.1. *We have $O_p(G) \neq 1$.*

Proof. If $O_p(G) = 1$, then for any nontrivial (G, b)-Brauer pair (Q, f), the group $N_G(Q, f)$ is a proper subgroup of G. Since f is $SL(2, p)$-free by 5.2, the induction hypothesis implies that Theorem 1.4 holds for the block f of $N_G(Q, f)$. But then 3.2 implies, that Theorem 1.4 holds for the block b of G, contradicting our choice of b. □

From now on, we set $Q = O_p(G)$. Since Q is normal in G, the block b lies in $kC_G(Q)$ (cf. [1, (2.9)(1)]). Thus $b = \mathrm{Tr}^G_{N_G(Q,e_Q)}(e_Q)$. But then $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N_G(Q, e_Q), e_Q)$. If $N_G(Q, e_Q)$ is a proper subgroup of G, the induction hypothesis implies that Theorem 1.4 holds for the block e_Q of $N_G(Q, e_Q)$, and hence for the block b, contradicting again our choice of b. This proves the following.

6.2. *We have $G = N_G(Q, e_Q)$ and $b = e_Q$.*

Then b is a block for any subgroup of G containing $C_G(Q)$. In particular, b is a block of $QC_G(Q)$. Set $R = QC_P(Q)$. Then (R, e_R) is a maximal $(QC_G(Q), b)$-subpair (cf. [1, (2.9)(6)]). Note that $C_G(R) \subseteq C_G(Q)$ and that R is normal in P. Since the maximal $(QC_G(Q), b)$-subpairs are $QC_G(Q)$-conjugate, a Frattini argument shows that

6.3. *we have* $G = N_G(R, e_R)C_G(Q)$.

If $R = Q$, then $(R, e_R) = (Q, b)$ is (G, b)-centric. The group L occurring in the Külshammer-Puig-extension [11, Theorem 1.8]

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/QC_G(Q) \longrightarrow 1$$

is p-constrained and does not have $SL(2, p)$ as subquotient (cf. 2.4). Thus $W(P)$ is normal in L. Since $\mathcal{F}_P(L) = \mathcal{F}_{(P,e)}(G, b)$, this implies $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N, c)$, contradicting our choice of b.

Thus Q is a proper subgroup of R. Since $Q = O_p(G)$, it follows that $N_G(R, e_R)$ is a proper subgroup of G. So Theorem 1.4 applies to the block e_R of $N_G(R, e_R)$, which has still P as defect group as R is normal in P.

We assume now that $PC_G(Q)$ is a proper subgroup of G, and derive a contradiction; that is, we are going to show that then (G, b) cannot be a counterexample to Theorem 1.4. We do this by showing that $N_G(S, e_S) = N_N(S, e_S)C_G(S)$ for any subgroup S of P containing Q such that (S, e_S) is (G, b)-centric. We argue by induction over the order of S. Up to replacing (S, e_S) by some G-conjugate, we may assume that $N_P(S)$ is a defect group of e_S as block of $N_G(S, e_S)$. The subgroup $N_G(S, e_S) \cap SC_G(Q)$ is normal in $N_G(S, e_S)$ and contains $C_G(S)$. Thus e_S is a block of $N_G(S, e_S) \cap SC_G(Q)$ having as defect group the group $T = N_P(S) \cap SC_G(Q) = N_P(S) \cap SC_P(Q) = N_{SR}(Q)$, as $R = QC_P(Q)$. Therefore, (T, e_T) is a maximal $(N_G(S, e_S) \cap SC_G(Q), e_S)$-subpair. The Frattini argument yields

$$N_G(S, e_S) = (N_G(S, e_S) \cap N_G(T, e_T)) \cdot (N_G(S, e_S) \cap SC_G(Q)) \ .$$

Now (S, e_S) is also a $(PC_G(Q), b)$-subpair contained in (P, e). As $PC_G(Q)$ is assumed to be a proper subgroup of G, it follows that $N_G(S, e_S) \cap SC_G(Q) = (N_N(S, e_S) \cap SC_G(Q))C_G(S)$. If S does not contain R, then S is properly contained in SR, hence properly contained in $T = N_{SR}(S)$. By induction, we get $N_G(T, e_T) = N_N(T, e_T)C_G(T)$. Together we get $N_G(S, e_S) \subseteq NC_G(S)$, hence $N_G(S, e_S) = N_N(S, e_S)C_G(S)$.

Thus, we may assume that $R \subseteq S$. Then $C_G(S) \subseteq C_G(R) \subseteq N_G(R, e_R) \cap PC_G(Q)$. Therefore, e_R is a block of the group $N_G(R, e_R) \cap PC_G(Q)$, having still (P, e) as maximal subpair. Let $x \in N_G(S, e_S)$. Since $G = N_G(R, e_R)C_G(Q)$, we can write $x = nc$ for some $n \in N_G(R, e_R)$ and some $c \in C_G(Q)$. Then ${}^c(S, e_S) = {}^{n^{-1}}(S, e_S)$. This implies that ${}^cS \subseteq N_G(R, e_R) \cap PC_G(Q)$ and that $(R, e_R) \subseteq {}^c(S, e_S)$. Thus ${}^c(S, e_S)$ is a $(N_G(R, e_R) \cap PC_G(Q), e_R)$-subpair. Therefore, there is $y \in N_G(R, e_R) \cap PC_G(Q)$ such that ${}^{yc}(S, e_S) \subseteq (P, e)$. We have $x = nc = (ny^{-1})(yc)$. The element yc belongs to the group $PC_G(Q)$, and conjugation by yc is a morphism in the category $\mathcal{F}_{(P,e)}(PC_G(Q), b)$ from S to ${}^{yc}S$. As $PC_G(Q)$ is assumed to be a proper subgroup of G, this implies that $yc \in (N \cap PC_G(Q))C_G(S)$. The element ny^{-1} belongs to the group $N_G(R, e_R)$, and conjugation by ny^{-1} is a morphism from ${}^{yc}S$ to ${}^xS = S$ in the category $\mathcal{F}_{(P,e)}(N_G(R, e_R), e_R)$. Since $N_G(R, e_R)$ is a proper subgroup of G, it follows that $ny^{-1} \in N_N(R, e_R)C_G({}^{yc}S)$. Together, we get $x = (ny^{-1})(yc) \in NC_G(S)$, hence $N_G(S, e_S) = N_N(S, e_S)C_G(S)$. This contradicts the fact that (G, b) is a counterexample to the Theorem. Therefore,

6.4. *we have $G = PC_G(Q)$.*

Set $Z = Z(P) \cap Q$; since Q is normal in G, the group Z is non-trivial. Set $\bar{G} = G/Z$, and denote by $\bar{b}$ the image of b in $k\bar{G}$. Thus $\bar{b}$ is a block of $k\bar{G}$ with defect group $\bar{P} = P/Z$. By 4.2, the block $\bar{b}$ is $SL(2,p)$-free. Denote by H the inverse image in G of $N_{\bar{G}}(W(\bar{P}))$. Then H is the normaliser in G of a subgroup of P which contains Q properly, and so H is a proper subgroup of G fulfilling the hypotheses of 4.3. Denote by d the unique block of H such that $\mathrm{Br}_P(d) = \mathrm{Br}_P(b)$, and denote by $\bar{d}$ the image of d in $k\bar{H}$. By the induction hypothesis, we have $\mathcal{F}_{(\bar{P},f)}(\bar{G},\bar{b}) = \mathcal{F}_{(\bar{P},f)}(\bar{H},\bar{d})$, where f is the unique block of $C_{\bar{G}}(\bar{P})$ such that $\bar{e}f = f$. But then 4.3 implies that we have $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(H,d)$. Since H is a proper subgroup of G, by induction again, we have $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(N,c)$ by 2.3. This contradicts our choice of b and completes the proof of Theorem 1.4. □

Proof of Theorem 1.7. Since b is $SL(2,p)$-free, we apply Theorem 1.4 with the Glauberman functor W mapping P to $Q = K_\infty(P)$. Then the subpair (Q, e_Q) is centric, and its normaliser controls fusion in $\mathcal{F}_{(P,e)}(G,b)$. The Theorem follows immediately from 2.8 and 2.7. □

7 Proof of Theorem 1.5 and 1.6

Proof of Theorem 1.5. In this section, we keep the notation of Theorem 1.5; that is, we let G be a finite group, and let H be a normal subgroup of G. Let c be a G-stable block of H and let b be a block of G, which covers c; that is, b satisfies $bc = b$. Let (P,e) be a maximal (G,b)-subpair and set $Q = P \cap H$. Then clearly Q is a defect group of the block c of N. Let (Q, e_Q) be the unique (G,b)-subpair contained in (P,e) and let f be a block of $C_H(Q)$ covered by the block e_Q of $C_G(Q)$, i.e. such that $e_Q f \neq 0$. Then (Q,f) is a maximal (H,c)-subpair. If $x \in N_G(Q,e_Q)$, then ${}^x f$ is a block of $kC_H(Q)$ which is covered by e_Q, hence $x = yz$ for some $y \in C_G(Q)$ and some $z \in [N_G(Q,e_Q) \cap N_G(Q,f)]$. In other words, we have

$$N_G(Q,e_Q) = C_G(Q)[N_G(Q,e_Q) \cap N_G(Q,f)].$$

The group in square brackets has a block which induces up to the block $kN_G(Q,e_Q)e_Q$ and thus contains a defect group of $kN_G(Q,e_Q)e_Q$. For some $y \in C_G(Q)$, we thus have

$${}^y P \leq N_G(Q,e_Q) \cap N_G(Q,f),$$

hence

$$P \leq N_G({}^{y^{-1}}(Q,e_Q)) \cap N_G({}^{y^{-1}}(Q,f)),$$

Since ${}^{y^{-1}}(Q) = Q$ and since $e_Q {}^{y^{-1}} f = {}^{y^{-1}}(e_Q f) \neq 0$, on replacing (Q,f) by ${}^{y^{-1}}(Q,f)$, we may assume that P stabilises f, and this proves the first statement of the Theorem.

For any subgroup R of P, we let e_R be the unique block of $C_G(R)$ such that $(R, e_R) \leq (P, e)$, and for a subgroup S of Q, we let f_S be the unique block of $C_H(S)$ such that $(S, f_S) \leq (Q, f)$. Note that whenever R is a subgroup of P, the pair $(R \cap H, f_{R\cap H})$ is stabilised by $N_P(R \cap H)$ because this last group stabilzes $R \cap H$ and (Q, f).

Let $\mathcal{F}$ denote the Brauer category $\mathcal{F}_{(P,e)}(G, b)$ and let $\mathcal{H}$ denote the Brauer category $\mathcal{F}_{(Q,f)}(H, c)$. Let $\mathcal{C}$ denote the Alperin-Goldschmidt conjugation family for $\mathcal{F}$.

Let $\mathcal{D}$ denote the set of objects R of $\mathcal{F}$ such that

(i) $N_P(R)$ is a defect group of $kN_G(R, e_R)e_R$.

(ii)$N_P(R \cap H)$ is a defect group of $kN_G(R \cap H, e_{R\cap H})e_{R\cap H}$.

(iii) $N_P(R \cap H)$ stabilises $f_{R\cap H}$.

7.1. *Every object in $\mathcal{F}$ is isomorphic to an object in $\mathcal{D}$ and $\mathcal{C} \cap \mathcal{D}$ is a conjugation family for $\mathcal{F}$.*

Proof. Consider $(R, e_R) \leq (P, e)$. Let (S, u) be a (G, b)-Brauer pair such that S is maximal with respect to normalising (R, e_R). Since $N_G(R, e_R) \leq N_G(R \cap H, e_{R\cap H})$, we may find a (G, b)- subpair (T, v) such that T is maximal with respect to normalising $(R\cap H, e_{R\cap H})$ and such that $S \leq T$. Note that T is a defect group of $kN_G(R\cap H, e_{R\cap H})e_{R\cap H}$ and S is a defect group of $N_G(R, e_R)e_R$.

For some element x of G, we have ${}^x(T, v) \leq (P, e)$. Thus we have ${}^x(R \cap H, e_{R\cap H}) \leq {}^x(R, e_R) \leq {}^x(S, u) \leq {}^x(T, v) \leq (P, e)$.

Clearly, ${}^xT$ is a defect group of $kN_G({}^x(R\cap H, e_{R\cap H}))^x e_{R\cap H}$, and ${}^xS$ is a defect group of $kN_G({}^x(R, e_R))^x e_R$. Also, ${}^xS = N_P({}^xR)$ and ${}^xT = N_P({}^x(R \cap H))$.

Hence, on replacing (R, e_R) by ${}^x(R, e_R)$, we may assume that (R, e_R) satisfies (i) and (ii) above. Statement (iii) is immediate from (i) and (ii), since P stabilses (Q, f). This proves the first part of the proposition. Since the set of objects R of $\mathcal{F}_{(P,e)}(G, b)$ for which (R, e_R) is a centric and radical (G, b)-subpair is invariant under $\mathcal{F}$ isomorphism, this proves also the second part of Statement 7.1. □

Let $\mathcal{E}$ be the Alperin-Goldschmidt conjugation family for $\mathcal{F}_{(Q,f)}(H, c)$.

7.2. *If $R \in \mathcal{C} \cap \mathcal{D}$, then $R \cap H \in \mathcal{E}$.*

Proof. Let $R \in \mathcal{C}\cap\mathcal{D}$ and let $\tilde{e}_{R\cap H}$ and $\tilde{f}_{R\cap H}$ respectively denote the blocks of $N_G(R\cap H)$ and $N_H(R \cap H)$ induced from $e_{R\cap H}$ and $f_{R\cap H}$. Since $N_P(R \cap H)$ is a defect group of $kN_G(R \cap H, e_{R\cap H})e_{R\cap H}$, $N_P(R \cap H)$ is a defect group of $kN_G(R \cap H)\tilde{e}_{R\cap H}$. Since the block $\tilde{e}_{R\cap H}$ of $kN_G(R \cap H)$ covers the block $\tilde{f}_{R\cap H}$ of $kN_H(R \cap H)$, $N_Q(R \cap H)$ is a defect group of $kN_H(R \cap H)\tilde{f}_{R\cap H}$; hence the defect groups of $kN_H(R \cap H, f_{R\cap H})f_{R\cap H}$ have order $|N_Q(R \cap H)|$. On the other hand, since $(R, e_R) \in \mathcal{D}$, $N_Q(R \cap H) \subseteq N_H(R \cap H, f_{R\cap H})$, thus $N_Q(R \cap H)$ is a defect group of $kN_H(R \cap H, f_{R\cap H})f_{R\cap H}$.

Next we show that $(R \cap H, f_{R\cap H})$ is (H, c)-centric. For this, by the above remarks, it suffices to show that $C_Q(R \cap H) = Z(R \cap H)$. Choose p-regular $y \in C_H(R \cap H) \cap N_G(R, e_R)$. Then $[R, y] \subseteq R \cap H$, so that $[R, y, y] = 1$, and hence $[R, y] = 1$ as y is p-regular. Hence $[C_H(R \cap H) \cap N_G(R, e_R)]/C_H(R)$ is a p-group. On the other hand, $C_H(R\cap H)\cap N_G(R, e_R)$ is clearly a normal subgroup of $N_G(R, e_R)$, and $O_p(N_G(R, e_R)/RC_G(R)) = 1$. Hence, $C_H(R \cap H) \cap N_G(R, e_R) \subseteq RC_G(R)$. Since

$C_P(R) = Z(R)$, we get $C_Q(R \cap H) \cap N_Q(R) \subseteq R$. Since R normalises $C_Q(R \cap H)$, this means that R is its own normaliser in the p-group $C_Q(R \cap H)R$ whence $C_Q(R \cap H) \subseteq R$.

It remains to show that $O_p(N_H(R \cap H, f_{R \cap H})/(R \cap H)C_H(R \cap H)) = 1$. So, let M be the full inverse image of $O_p(N_H(R \cap H, f_{R \cap H})/(R \cap H)C_H(R \cap H))$ in $N_H(R \cap H, f_{R \cap H})$. Since $N_P(R \cap H)C_H(R \cap H)/(R \cap H)C_H(R \cap H)$ is a Sylow-p subgroup of $N_H(R \cap H, f_{R \cap H})/(R \cap H)C_H(R \cap H))$, we have $M = (M \cap P)C_H(R \cap H)$. We will show that $M \cap P \subset R \cap H$.

We have $N_G(R \cap H, e_{R \cap H}) = C_G(R \cap H)[N_G(R \cap H, f_{R \cap H}) \cap N_G(R \cap H, e_{R \cap H})]$, and $C_H(R \cap H)$ is normal in $N_G(R \cap H, e_{R \cap H})$; hence $C_H(R \cap H)[M \cap N_G(R \cap H, e_{R \cap H})]$ is normal in $N_G(R \cap H, e_{R \cap H})$. Since $C_H(R \cap H) \subset [M \cap N_G(R \cap H, e_{R \cap H})]$, this means that $[M \cap N_G(R \cap H, e_{R \cap H})]$ is normal in $N_G(R \cap H, e_{R \cap H})$ and hence is normal in $N_G(R, e_R)$. By the definition of M, it follows that $M \cap N_G(R \cap H, e_{R \cap H})/C_H(R \cap H)$ is a p-group. On the other hand, we have shown before that $C_H(R \cap H) \cap N_G(R, e_R)/C_H(R)$ is a p-group. Hence, $M \cap N_G(R, e_R)/C_H(R)$ is a normal p subgroup of $N_G(R, e_R)/C_H(R)$, and is therefore isomorphic to a normal p-subgroup of $N_G(R, e_R)/C_G(R)$. But then by choice of (R, e_R) it follows that $M \cap N_G(R, e_R) \subset RC_G(R)$ whence $M \cap N_P(R) \subset RC_P(R) \cap H \subset R \cap H$. Since R normalises $M \cap P$, we see that $M \cap P \subset R \cap H$. This completes the proof. □

Now let V be a normal subgroup of Q and suppose that $N_H(V)$ controls fusion in $\mathcal{H}$ and let W be as in the statement of the Theorem.

7.3. *$N_H(W)$ controls fusion in $\mathcal{H}$. Further, if S is a subgroup of Q containing W then $N_G(S, e_S) \subset N_G(W)$.*

Proof. Let $(S, f_S) \leq (Q, f)$ and let $x \in N_G(Q, f)$. Since ${}^{x^{-1}}(S, f_S) \leq (Q, f)$, we have that $N_H({}^{x^{-1}}(S, f_S)) \subset C_H({}^{x^{-1}}S)N_H(V)$ whence $N_H(S, f_S) \subset C_H(S)N_H({}^{x}V)$.Thus $N_H({}^{x}V)$ controls fusion in $\mathcal{F}_{H,c}$ for all $x \in N_G(Q, f)$. It follows by Lemma 2.1 that if $S \in \mathcal{E}$, then $N_H(S, e_S) \subseteq N_H({}^{x}V)$ for all $x \in N_G(Q, f)$, so that in particular, $N_H(S, e_S) \subseteq N_H(W)$. Hence $N_H(W)$ controls fusion in $\mathcal{F}_{H,c}$.

Let S be a subgroup of Q containing W and let $x \in N_G(S, e_S)$. By the Frattini argument, we may write $x = yz$, where $y \in N_G(Q, f)$ and $z \in H$. Then ${}^{z}(S, f_S) = {}^{y^{-1}x}(S, f_S) \leq (Q, f)$. Since $N_H(W)$ controls fusion in $\mathcal{F}_{H,c}$, we may write $z = ct$, where $c \in C_H(S) \subset N_H(W)$ and $t \in N_H(W)$. Since by definition of W, $y \in N_G(W)$, we have $x = yct \in N_G(W)$. □

Let $R \in \mathcal{C} \cap \mathcal{D}$. Then by 7.2, $R \cap H \in \mathcal{E}$. In particular, by Lemma 2.1, we have that $W \subset R \cap H$ and it follows by 7.3 that $N_G(R \cap H, f_{R \cap H}) \subset N_G(W)$. Hence, $N_G(R, e_R) \subset N_G(R \cap H, e_{R \cap H}) \subset C_G(R \cap H)[N_G(R \cap H, e_{R \cap H}) \cap N_G(R \cap H, f_{R \cap H})] \subset N_G(W)$. Theorem 1.5 now follows from 7.2 and the fact that $P \subseteq N_G(W)$. □

Proof of 1.6. By a standard argument we may assume that G stabilises the block c. Then 1.6 is an immediate consequence of 2.3 and Theorems 1.4 and 1.5. □

Remark 7.4 The advantage of Theorem 1.6 is that if we wish to produce a single local subgroup controlling fusion in $\mathcal{F}_{(P,e)}(G, b)$, it is not really necessary to assume that b is $SL(2,p)$ free. This could be useful in some instances; for example, suppose that $G = XwrS_n$ for some large integer n and some non-Abelian finite simple group X, while H is the base-group of the wreath product. It is quite possible for automizers of "diagonal-type" (G,b)-subpairs to involve $SL(2,p)$ because of the action of the S_n, while automizers (in H) of (H,c) might not involve $SL(2,p)$.

8 Proof of 1.10

Proof of 1.10. It is clear that the pair (P,e) is a maximal $(N_G(R),c)$-subpair. For a subgroup Q of R, we let (Q,f_Q) be the unique $(N_G(R),c)$ subpair contained in (P,e).

In [13], it is shown that if we are considering a group G such that DPC holds in every section of G, then in calculating the various quantities $k_d(B,\lambda)$, it is only necessary to consider chains of (G,b)-pairs whose initial objects are pairs (Q,e_Q) contained in (P,e) which are (G,b)-centric and radical. By Lemma 2.2, we have that for any such subpair (Q,e_Q), $R \leq Q$, and thus $N_G(Q,e_Q) \subset C_G(Q)N_G(R) \subset N_G(R)$. The fact that $R \leq Q$ also implies that $f_Q = e_Q$. It follows that in the subpair version of (W)DPC, the contribution in kGb from chains beginning with (Q,e_Q) is the same as the contribution in $kN_G(R)c$ from chains beginning with (Q,e_Q). Similarly, it follows that if DPC holds in every proper section of G, then checking DPC for G reduces to checking that there is a defect-preserving bijection between irreducible characters of B lying over λ and irreducible characters in c lying over λ. □

Acknowledgement. This work was completed while the third author was a Visiting Fellow at All Souls College, Oxford. The author is grateful to the College for its support and hospitality.

Note added in proof: Since this work was written, a related result of M. Lechuga (Theorem 7.11 in his thesis *Contribution à l'étude locale dans les groups finis*, Publ. Math. Univ. Paris 7, tome IV, 1994) has been brought to our attention. Lechuga's result concerns the particular Glauberman functor ZL (defined by L. Puig), is valid for $p \geq 5$, and makes use of J. G. Thompson's classification of quadratic pairs. While, as stated, it does not imply the involvement of $SL(2,p)$ in the relevant automizer, the $PSL(2,p^n)$ and $PSU(3,p^m)$ components he mentions arise because of quadratic action, so the presence of a genuine $SL(2,p)$ is implicit.

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