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FUSION CATEGORY ALGEBRAS

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Abstract. The fusion system $\mathcal{F}$ on a defect group P of a block b of a finite group G over a suitable p-adic ring $\mathcal{O}$ does not in general determine the number $l(b)$ of isomorphism classes of simple modules of the block. We show that conjecturally the missing information should be encoded in a single second cohomology class α of the constant functor with value $k^\times$ on the orbit category $\bar{\mathcal{F}}^c$ of $\mathcal{F}$-centric subgroups Q of P of b which "glues together" the second cohomology classes $\alpha(Q)$ of $\mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$ with values in $k^\times$ in Külshammer-Puig [13, 1.8]. We show that if α exists, there is a canonical quasi-hereditary k-algebra $\bar{\mathcal{F}}(b)$ such that Alperin's weight conjecture becomes equivalent to the equality $l(b) = l(\bar{\mathcal{F}}(b))$. By work of Broto, Levi, Oliver [3], the existence of a classifying space of the block b is equivalent to the existence of a certain extension category $\mathcal{L}$ of $\mathcal{F}^c$ by the center functor $\mathcal{Z}$. If both invariants α, $\mathcal{L}$ exist we show that there is an $\mathcal{O}$-algebra $\mathcal{L}(b)$ associated with b having $\bar{\mathcal{F}}(b)$ as quotient such that Alperin's weight conjecture becomes again equivalent to the equality $l(b) = l(\mathcal{L}(b))$; furthermore, if b has an abelian defect group, $\mathcal{L}(b)$ is isomorphic to a source algebra of the Brauer correspondent of b.

1 Twisted category algebras

We introduce in this section twisted category algebras as a straightforward generalistation of twisted group algebras. In order to settle some notation, we briefly describe the "untwisted" case first.

1.1. Let $\mathcal{C}$ be a finite category; that is, the object class $\mathrm{Ob}\,\mathcal{C}$ of $\mathcal{C}$ is a finite set and for any two objects Q, R in $\mathcal{C}$, the morphism set $\mathrm{Hom}_{\mathcal{C}}(Q,R)$ is finite. Let $\mathcal{O}$ be a commutative ring.

Denote by $\mathbf{F}(\mathcal{C};\mathcal{O})$ the category of contravariant functors from $\mathcal{C}$ to the category of left $\mathcal{O}$-modules $\mathrm{Mod}(\mathcal{O})$; the morphisms in $\mathbf{F}(\mathcal{C};\mathcal{O})$ are the natural transformations of functors. This is an abelian category with enough projectives. We denote by $\underline{\mathcal{O}}$ the constant contravariant functor mapping each object in $\mathcal{C}$ to $\mathcal{O}$ and each morphism in $\mathcal{C}$ to the identity map on $\mathcal{O}$.

For any non negative integer n, the degree n cohomology of $\mathcal{C}$ with coefficients in a functor $\mathcal{F} \in \mathbf{F}(\mathcal{C};\mathcal{O})$ is the $\mathcal{O}$-module defined by means of the usual general abstract nonsense

$$H^n(\mathcal{C};\mathcal{F}) = \mathrm{Ext}^n_{\mathbf{F}(\mathcal{C};\mathcal{O})}(\underline{\mathcal{O}},\mathcal{F}) = H^n(\mathrm{Hom}_{\mathbf{F}(\mathcal{C};\mathcal{O})}(\mathcal{P},\mathcal{F}))\ ,$$

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where $\mathcal{P}$ is a projective resolution of the constant functor $\underline{\mathcal{O}}$ in $\mathbf{F}(\mathcal{C};\mathcal{O})$, and where $\mathrm{Hom}_{\mathbf{F}(\mathcal{C};\mathcal{O})}(\mathcal{P},\mathcal{F})$ is the cochain complex of $\mathcal{O}$-modules obtained from applying the contravariant functor $\mathrm{Hom}_{\mathbf{F}(\mathcal{C};\mathcal{O})}(-,\mathcal{F})$ to the chain complex $\mathcal{P}$. See [6], [7] for more background information on functor cohomology.

It is well-known that there is a natural isomorphism $\varprojlim \mathcal{F} \cong \mathrm{Hom}_{\mathbf{F}(\mathcal{C};\mathcal{O})}(\underline{\mathcal{O}},\mathcal{F})$; that is, the inverse limit functor $\varprojlim$ from $\mathbf{F}(\mathcal{C};\mathcal{O})$ to $\mathrm{Mod}(\mathcal{O})$ is isomorphic to the covariant functor $\mathrm{Hom}_{\mathbf{F}(\mathcal{C};\mathcal{O})}(\underline{\mathcal{O}},-)$, and hence we have isomorphisms between their higher derived functors $H^n(\mathcal{C};\mathcal{F}) \cong \varprojlim^n \mathcal{F}$.

It is equally well-known that $\mathbf{F}(\mathcal{C};\mathcal{O})$ is equivalent to the module category of the *category algebra* $\mathcal{O}\mathcal{C}$ of $\mathcal{C}$ over $\mathcal{O}$ (terminology by P. Webb); that is, as $\mathcal{O}$-module, $\mathcal{O}\mathcal{C}$ is free having as basis the set of all morphisms in $\mathcal{C}$, with the unitary associative $\mathcal{O}$-bilinear multiplication induced by composition of morphisms in $\mathcal{C}$. More precisely, for any two morphisms φ, ψ in $\mathcal{C}$ we define the multiplication by

$$\varphi\psi = \psi \circ \varphi$$

provided that the composition $Q \xrightarrow{\varphi} R \xrightarrow{\psi} S$ is defined, and $\varphi\psi = 0$ otherwise. The unit element of $\mathcal{O}\mathcal{C}$ is

$$1_{\mathcal{O}\mathcal{C}} = \sum_{Q} \mathrm{Id}_Q \ ,$$

where in the sum Q runs over the set of objects in $\mathcal{C}$. This is in fact a decomposition of $1_{\mathcal{O}\mathcal{C}}$ as a sum of pairwise orthogonal (not necessarily primitive) idempotents in $\mathcal{O}\mathcal{C}$. With this notation, we have an obvious equivalence of categories

$$\begin{cases} \mathbf{F}(\mathcal{C};\mathcal{O}) & \cong \mathrm{Mod}(\mathcal{O}\mathcal{C}) \\ \mathcal{F} & \mapsto \bigoplus_{Q\in \mathrm{Ob}\,\mathcal{C}} \mathcal{F}(Q) \ , \end{cases}$$

where the module structure on the right side is given, for any morphism $\varphi : Q \to R$ in $\mathcal{C}$ by $\varphi.m = \mathcal{F}(\varphi)(m)$ if $m \in \mathcal{F}(R)$, and $\varphi.m = 0$ if $m \in \mathcal{F}(R')$ for some object $R' \neq R$ in $\mathcal{C}$. The inverse of this equivalence maps a left $\mathcal{O}\mathcal{C}$-module M to the functor sending any object Q of $\mathcal{C}$ to $\mathrm{Id}_Q.M$ and any morphism $\varphi : Q \to R$ to the map $\mathrm{Id}_R.M \to \mathrm{Id}_Q.M$ induced by left multiplication with φ on M; this makes sense as $\varphi\mathrm{Id}_R = \varphi = \mathrm{Id}_Q\varphi$ in $\mathcal{O}\mathcal{C}$.

Since $\mathcal{C}$ is finite, this equivalence maps the abelian subcategory $\mathbf{f}(\mathcal{C};\mathcal{O})$ of $\mathbf{F}(\mathcal{C};\mathcal{O})$ of contravariant functors from $\mathcal{C}$ to the category of finitely generated $\mathcal{O}$-modules $\mathrm{mod}(\mathcal{O})$ onto $\mathrm{mod}(\mathcal{O})$.

Example 1.2. Let G be a finite group. Denote by $\underline{G}$ the category having one object $*$ such that the set of endomorphisms of $*$ in $\underline{G}$ corresponds bijectively to the set of elements of G, with composition induced by multiplication in G. Then the category algebra $\mathcal{O}\underline{G}$ is isomorphic to the group algebra $\mathcal{O}G$.

1.3. Denote by $\underline{\mathcal{O}}^\times$ the constant contravariant functor from $\mathcal{C}$ to the category of abelian groups $\mathrm{Mod}(\mathbb{Z})$, mapping each object in $\mathcal{C}$ to the multiplicative group of invertible elements $\mathcal{O}^\times$ of $\mathcal{O}$ and each morphism in $\mathcal{C}$ to the identity on $\mathcal{O}^\times$. Let $\alpha \in H^2(\mathcal{C}; \underline{\mathcal{O}}^\times)$. Similarly to what happens in group theory, α can be represented by a 2-cocycle, abusively still denoted by α; that is, α is a function mapping any sequence of two composable morphisms $Q \xrightarrow{\varphi} R \xrightarrow{\psi} S$ in $\mathcal{C}$ to an element $\alpha(\psi, \varphi) \in \mathcal{O}^\times$ with the property that for any sequence of three composable morphisms

$$Q \xrightarrow{\varphi} R \xrightarrow{\psi} S \xrightarrow{\tau} T$$

in $\mathcal{C}$ we have the equality

$$\alpha(\tau, \psi \circ \varphi)\alpha(\psi, \varphi) = \alpha(\tau \circ \psi, \varphi)\alpha(\tau, \psi) \quad .$$

We define the *twisted category algebra* $\mathcal{O}_\alpha\mathcal{C}$ *of* $\mathcal{C}$ *by* α as follows: as $\mathcal{O}$-module, $\mathcal{O}_\alpha\mathcal{C}$ is equal to $\mathcal{O}\mathcal{C}$ - that is, $\mathcal{O}$-free with the morphism set of $\mathcal{C}$ as basis - , and the multiplication in $\mathcal{O}_\alpha\mathcal{C}$ is defined by

$$\varphi\psi = \alpha(\psi, \varphi) \cdot (\psi \circ \varphi)$$

for any sequence of two composable morphisms $Q \xrightarrow{\varphi} R \xrightarrow{\psi} S$ in $\mathcal{C}$ and $\varphi\psi = 0$ otherwise. As before, this is a unitary associative $\mathcal{O}$-bilinear product; the associativity is in fact equivalent to the above condition which defines α as 2-cocycle.

1.4. If α, α' are two cocycles on $\mathcal{C}$ with values in $\mathcal{O}^\times$, then α and α' represent the same class in $H^2(\mathcal{C}, \underline{\mathcal{O}}^\times)$ if and only if there is a 1-cochain β (that is, a map sending any morphism in $\mathcal{C}$ to an element in $\mathcal{O}^\times$) satisfying $\alpha'(\psi, \varphi) = \alpha(\psi, \varphi)\beta(\psi \circ \varphi)\beta(\psi)^{-1}\beta(\varphi)^{-1}$ for any two composable morphisms φ, ψ in $\mathcal{C}$. The map sending a morphism φ in $\mathcal{C}$ to $\beta(\varphi)\varphi$ induces then an algebra isomorphism $\mathcal{O}_\alpha\mathcal{C} \cong \mathcal{O}_{\alpha'}\mathcal{C}$. In particular, if we set $\beta(\varphi) = \alpha(\varphi, \mathrm{Id}_Q)$ then $\alpha'(\mathrm{Id}_Q, \mathrm{Id}_Q) = 1$ for any object Q in $\mathcal{C}$. Thus the isomorphism class of $\mathcal{O}_\alpha\mathcal{C}$ depends only on the class of α in $H^2(\mathcal{C}; \underline{\mathcal{O}}^\times)$, which justifies our notational abuse of denoting a 2-cocycle and its cohomology class by the same letter. Moreover, one may (and we do) choose α to be *normalised*; that is, $\alpha(\psi, \varphi) = 1$ whenever one of φ, ψ is the identity automorphism of some object. Then Id_Q is an idempotent in $\mathcal{O}_\alpha\mathcal{C}$ and the unit element of $\mathcal{O}_\alpha\mathcal{C}$ is $1_{\mathcal{O}_\alpha\mathcal{C}} = \sum_{Q \in \mathrm{Ob}\,\mathcal{C}} \mathrm{Id}_Q$.

Example 1.5. The previous construction applied to the example $\mathcal{C} = \underline{G}$ yields the usual definition of twisted group algebras.

Remark 1.6. If $\varphi : Q \cong Q'$ is an isomorphism in $\mathcal{C}$ then the idempotents Id_Q, $\mathrm{Id}_{Q'}$ are conjugate. To see this we may assume that $Q \neq Q'$. Set $t = \varphi + \varphi^{-1} + \sum_T \mathrm{Id}_T$, with T running over $\mathrm{Ob}\,\mathcal{C} - \{Q, Q'\}$. Then t is invertible and $t\mathrm{Id}_{Q'} = \mathrm{Id}_Q t = \varphi$ in $\mathcal{O}_\alpha\mathcal{C}$.

2 Algebras over EI-categories

Let $\mathcal{C}$ be a finite EI-category; that is, $\mathcal{C}$ is finite and every endomorphism of an object in $\mathcal{C}$ is an automorphism. The set of isomorphism classes of objects in $\mathcal{C}$ is then partially ordered by $[Q] \leq [R]$ whenever $\mathrm{Hom}_{\mathcal{C}}(Q, R) \neq \emptyset$, where $[Q]$, $[R]$ are the isomorphism classes of objects Q, R in $\mathcal{C}$. Let $\mathcal{O}$ be a commutative ring and let α be a normalised 2-cocycle on $\mathcal{C}$ with values in $\mathcal{O}^\times$. The aim of this section is to describe some basic properties of the twisted category algebra of $\mathcal{C}$ by α.

Recall that an $\mathcal{O}$-algebra A is called *symmetric* if A is finitely generated projective as $\mathcal{O}$-module and A is isomorphic to its $\mathcal{O}$-dual $\mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ as A-A-bimodule. Twisted group algebras of finite groups and their block algebras are symmetric (cf. [20, (10.4)]).

Set $A = \mathcal{O}_\alpha\mathcal{C}$. Denote by B the free $\mathcal{O}$-submodule of A having as basis the set of all isomorphisms in $\mathcal{C}$. Let N be the free $\mathcal{O}$-submodule of A having as basis the set of all non-isomorphisms in $\mathcal{C}$. We keep this notation throughout this section.

Proposition 2.1. *(i) B is a unitary subalgebra of A.*

(ii) N is a nilpotent 2-sided ideal in A.

(iii) $A = B \oplus N$ and $J(A) = J(B) \oplus N$.

Proof. The composition of any two isomorphisms is obviously again an isomorphism in $\mathcal{C}$, whence (i). The composition of a non-isomorphism with any morphism is a non-isomorphism in $\mathcal{C}$ thanks to the fact that $\mathcal{C}$ is an EI-category. Thus N is a 2-sided ideal. Moreover, since $\mathcal{C}$ is finite, there is an upper bound for the length of a sequence of composable non-isomorphisms in $\mathcal{C}$. Thus N is nilpotent, from which (ii) and (iii) follow. □

We can be more precise about the structure of B.

Proposition 2.2. *The algebra B is Morita equivalent to the direct product of twisted group algebras $\prod_Q \mathcal{O}_{\alpha(Q)}\mathrm{Aut}_{\mathcal{C}}(Q)^{op}$, where Q runs over a set of representatives of the isomorphism classes of objects in $\mathcal{C}$, and where $\alpha(Q) \in H^2(\mathrm{Aut}_{\mathcal{C}}(Q), \mathcal{O}^\times)$ is the restriction of α to $\mathrm{Aut}_{\mathcal{C}}(Q)$. In particular, the algebra B is symmetric.*

Proof. For any object Q in $\mathcal{C}$ denote by i_Q the idempotent in A which is the sum of all idempotents $\mathrm{Id}_{Q'}$, with Q' running over the set of all objects in $\mathcal{C}$ which are isomorphic to Q in $\mathcal{C}$. We have $B = \prod_Q i_Q \cdot A \cdot i_Q$, where Q runs over a set of representatives of the set of isomorphism classes in $\mathcal{C}$. For any two isomorphic objects Q, Q' in $\mathcal{C}$, the idempotents Id_Q, $\mathrm{Id}_{Q'}$ are conjugate in A by 1.6, and hence the algebra $i_Q \cdot A \cdot i_Q$ is Morita equivalent to the twisted group algebra $\mathrm{Id}_Q A \mathrm{Id}_Q = \mathcal{O}_{\alpha(Q)}\mathrm{Aut}_{\mathcal{C}}(Q)^{op}$. The proposition follows. □

Since N is a 2-sided ideal in A, the projection of A onto B with kernel N is a canonically split surjective algebra homomorphism. In general, A need not be projective as left or right B-module. If every morphism in $\mathcal{C}$ is in addition a monomorphism, then $\mathrm{Aut}_{\mathcal{C}}(Q)$ acts freely on the set $\mathrm{Hom}_{\mathcal{C}}(Q, R)$, whenever the latter set is non-empty. Similarly, if every morphism in $\mathcal{C}$ is an epimorphism, then $\mathrm{Aut}_{\mathcal{C}}(R)$ acts freely on $\mathrm{Hom}_{\mathcal{C}}(Q, R)$ whenever the latter set is non-empty. Together with the observation that $\mathrm{Hom}_{\mathcal{C}}(Q, R)$ is an $\mathcal{O}$-basis of $\mathrm{Id}_Q \cdot A \cdot \mathrm{Id}_R$, this translates to our algebra theoretic language as follows:

Proposition 2.3. *With the notation above, let Q be an object in $\mathcal{C}$.*

(i) If every morphism in $\mathcal{C}$ is a monomorphism then $\mathrm{Id}_Q \cdot A$ is free as left $\mathrm{Id}_Q \cdot A \cdot \mathrm{Id}_Q$-module; in particular, A is projective as left B-module.

(ii) If every morphism in $\mathcal{C}$ is an epimorphism then $A \cdot \mathrm{Id}_Q$ is free as right $\mathrm{Id}_Q \cdot A \cdot \mathrm{Id}_Q$-module; in particular, A is projective as right B-module.

The following theorem is the main result of this section. Quasi-hereditary algebras and highest weight categories were introduced by Cline, Parshall, Scott [4]; we refer to [5, Appendix] for an account and further references on quasi-hereditary algebras.

Theorem 2.4. *Suppose that $\mathcal{O} = k$ is a field, and let e be an idempotent in B such that $J(eBe) = \{0\}$. Then the k-algebra eAe is quasi-hereditary.*

We break up the proof of 2.4 in a series of easy lemmas, for which we introduce the following notation. For any positive integer n let $\mathcal{C}(n)$ be the full subcategory of $\mathcal{C}$ consisting of all objects Q in $\mathcal{C}$ for which there exists a sequence of n composable non-isomorphisms of the form

$$Q_0 \overset{\psi_0}{\to} Q_1 \overset{\psi_1}{\to} \cdots \overset{\psi_{n-1}}{\to} Q_n = Q \ .$$

We set $\mathcal{C}(0) = \mathcal{C}$. Clearly $\mathcal{C}(n+1) \subseteq \mathcal{C}(n)$ for any $n \geq 0$ and $\mathcal{C}(n) = \emptyset$ for n large enough, as $\mathcal{C}$ is a finite EI-category. For any $n \geq 0$ we define an idempotent e_n in A by

$$e_n = \sum_{Q \in \mathrm{Ob}(\mathcal{C}(n))} \mathrm{Id}_Q \ .$$

Then $e_0 = 1_A$ and $e_n e_{n+1} = e_{n+1} = e_{n+1} e_n$ for any $n \geq 0$.

Lemma 2.5. *For every integer $n \geq 0$, the projective A-module Ae_n is a 2-sided ideal in A and we have $Ae_{n+1} \subseteq Ae_n$.*

Proof. Let $\varphi : Q \to R$ be a morphism in $\mathcal{C}$. If $e_n\varphi = 0$ (the product taken in A) there is nothing to prove. If $e_n\varphi \neq 0$ then $Q \in \mathrm{Ob}(\mathcal{C}(n))$. But then clearly also $R \in \mathrm{Ob}(\mathcal{C}(n))$, hence $e_n\varphi = \varphi e_n$, which shows that Ae_n is a 2-sided ideal in A. The inclusion $Ae_{n+1} \subseteq Ae_n$ is a trivial consequence of the equality $e_{n+1} = e_{n+1}e_n$. □

Lemma 2.6. *For every ring E and any idempotent $f \in E$ such that Ef is a 2-sided ideal in E we have $\mathrm{Hom}_E(Ef, E/Ef) = \{0\}$.*

Proof. Any E-homomorphism from Ef to E/Ef lifts to a homomorphism from Ef to E, hence is induced by right multiplication with an element in E. Since Ef is a 2-sided ideal, right multiplication with an element in E maps Ef to itself, hence induces the zero map from Ef to E/Ef. □

Lemma 2.7. *For every integer $n \geq 0$ we have $e_n N \subseteq Ae_{n+1}$.*

Proof. If $Q \in \mathrm{Ob}(\mathcal{C}(n))$ and $\varphi : Q \to R$ is a non-isomorphism in $\mathcal{C}$ then obviously $R \in \mathrm{Ob}(\mathcal{C}(n+1))$. □

Proof of Theorem 2.4. Since any two primitive decompositions of 1_A are conjugate by an element in $A^\times$, up to replacing e by a suitable conjugate, we may assume that e commutes with all idempotents Id_Q, where $Q \in \mathrm{Ob}(\mathcal{C})$. Then e commutes with all idempotents e_n, $n \geq 0$. Hence, by 2.5, the k-subspace $H_n = eAe_ne$ of eAe is a projective left eAe-module and a 2-sided ideal in eAe. In order to prove 2.4, we need to observe the following three facts:

(1) H_n/H_{n+1} is a projective left eAe/H_{n+1}-module;
(2) $\mathrm{Hom}_{eAe/H_{n+1}}(H_n/H_{n+1}, eAe/H_n) = \{0\}$;
(3) $H_n/H_{n+1} \cdot J(eAe/H_{n+1}) \cdot H_n/H_{n+1} = \{0\}$.

As for (1), there is nothing to prove since in fact H_n is projective as left eAe-module. Statement (2) is an immediate consequence of 2.6 applied to $E = eAe$ and $f = e_ne$. Finally, (3) is equivalent to $e_nJ(eAe) \subseteq Ae_{n+1}$. Now, by 2.7, we have $e_nN \subseteq Ae_{n+1}$, and since $J(eAe) = J(eBe) \oplus eNe$ by 2.1(iii), statement (3) follows from the hypothesis $J(eBe) = \{0\}$. □

3 Local structure of blocks and Alperin's weight conjecture

In this section we provide some background material and notation that we are going to use in the next section for the definition of fusion category algebras.

3.1. Let $\mathcal{O}$ be a complete local commutative Noetherian ring whose residue field $k = \mathcal{O}/J(\mathcal{O})$ is perfect of prime characteristic $p > 0$. Let G be a finite group and let b be a block of $\mathcal{O}G$; that is, b is a primitive idempotent in $Z(\mathcal{O}G)$. Denote by $\bar{b}$ the canonical image of b in $Z(kG)$. A *Brauer pair* is defined to be a pair (Q, e) consisting of a p-subgroup Q of G and a block e of $kC_G(Q)$. The set of Brauer pairs is obviously a G-set with respect to the action of G by conjugation. It has been shown by Alperin and Broué [1] that this set can be endowed in a canonical way with a partial order "$\subseteq$" which is compatible with the G-action; in other words, the set of Brauer pairs is a G-poset. Note that in particular $(1, \bar{b})$ is a Brauer pair. A Brauer pair (Q, e) is called a *b-Brauer pair* if $(1, \bar{b}) \subseteq (Q, e)$. The following two fundamental properties of this partial order have been proved in [1]:

3.1.1. *for any Brauer pair (Q, e) and any subgroup R of Q there is a unique block f of $kC_G(R)$ such that $(R, f) \subseteq (Q, e)$;*

3.1.2. *any two maximal b-Brauer pairs are G-conjugate.*

In other words, the set of b-Brauer pairs is a G-subposet of the set of all Brauer pairs in which the maximal pairs form a single G-conjugacy class. Furthermore, if (P, e) is a maximal b-Brauer pair, then P is called a *defect group of b*, notion due to R. Brauer, and the group $E = N_G(P, e)/PC_G(P)$ is a p'-group, called *inertial quotient of b*.

3.2. Fix a maximal b-Brauer pair (P, e_P). Thus P is a defect group of the block b. For every subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e_P)$ and (Q, e_Q) is still a b-Brauer pair. By standard properties (cf. [1]), the idempotent e_Q remains a block of $kN_G(Q, e_Q)$.

The *fusion system of the block* b is the category $\mathcal{F}$ defined as follows.
• the objects of $\mathcal{F}$ are the subgroups of P;
• for any two subgroups Q, R of P, a morphism from Q to R in $\mathcal{F}$ is an injective group homomorphism $\varphi : Q \to R$ for which there exists an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in Q$ and ${}^x(Q, e_Q) \subseteq (R, e_R)$ (or equivalently, $xe_Qx^{-1} = e_{xQx^{-1}}$).

Loosely speaking, a morphism in $\mathcal{F}$ is a group homomorphism induced by conjugation with an element in G which is compatible with Brauer pairs. We have a canonical isomorphism $\mathrm{Aut}_{\mathcal{F}}(Q) \cong N_G(Q, e_Q)/C_G(Q)$. If b is the principal block of $\mathcal{O}G$ then e_Q is the principal block of $kC_G(Q)$, and we have $N_G(Q, e_Q) = N_G(Q)$ in that case. In general, since any two maximal b-Brauer pairs are G-conjugate, the equivalence class of $\mathcal{F}$ does not depend on the choice of (P, e_P).

The *orbit category of* $\mathcal{F}$ is the quotient category $\bar{\mathcal{F}}$ defined as follows:
• the objects of $\bar{\mathcal{F}}$ are again the subgroups of P;
• the morphism set $\mathrm{Hom}_{\bar{\mathcal{F}}}(Q, R)$ is the set of orbits $\mathrm{Inn}(R)\backslash\mathrm{Hom}_{\mathcal{F}}(Q, R)$ of the group of inner automorphisms $\mathrm{Inn}(R)$ of R, acting on the left of $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ by composition of homomorphisms, for any two subgroups Q, R of P.

We have an obvious canonical functor $\mathcal{F} \longrightarrow \bar{\mathcal{F}}$ which is the identity on objects and surjective on morphisms. For any subgroup Q of P we have a canonical isomorphism $\mathrm{Aut}_{\bar{\mathcal{F}}}(Q) \cong N_G(Q, e_Q)/QC_G(Q)$.

A subgroup Q of P is called $\mathcal{F}$-*centric* if $Z(Q)$ is a defect group of $kC_G(Q)e_Q$, and Q is called $\mathcal{F}$-*radical* if $\mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$ has no non trivial normal p-subgroup. By Alperin's fusion theorem (adapted to blocks in [1]) the fusion category $\mathcal{F}$ of the block b is completely determined by P and the automorphism groups $\mathrm{Aut}_{\mathcal{F}}(Q)$ of the $\mathcal{F}$-centric radical subgroups Q of P.

3.3. Alperin's weight conjecture states that there should be an equality

$$l(b) = \sum_{Q} z(Q, e_Q) ,$$

where the notation is as follows: as before, $l(b)$ is the number of isomorphism classes of simple $kG\bar{b}$-modules, $z(Q, e_Q)$ is the number blocks of defect 0 of $k(N_G(Q, e_Q)/Q)\bar{e}_Q$ (where $\bar{e}_Q$ is the canonical image of e_Q in $kN_G(Q, e_Q)/Q$), and where the sum is taken over a set of representatives of the $\mathcal{F}$-isomorphism classes of subgroups of P.

A key observation - which is implicitly in [11] and more explicitly in [19] - is that if $z(Q, e_Q) \neq 0$ then Q is $\mathcal{F}$-centric, and one easily sees that Q is also $\mathcal{F}$-radical (since normal p-subgroups are contained in all defect groups). In other words, only the $\mathcal{F}$-centric radical subgroups of P are relevant for the right hand expression of Alperin's weight conjecture.

We denote by $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$ consisting of $\mathcal{F}$-centric subgroups of P; similarly, we denote by $\bar{\mathcal{F}}^c$ the full subcategory of $\bar{\mathcal{F}}$ consisting of all $\mathcal{F}$-centric subgroups of P. It is easy to see that a subgroup Q of P is $\mathcal{F}$-centric if and only if $C_P(Q') = Z(Q')$ for any subgroup Q' of P which is isomorphic to Q in the category $\mathcal{F}$. In particular, if $Q \subseteq R \subseteq P$ and Q is $\mathcal{F}$-centric then so is R, and we have $Z(R) \subseteq Z(Q)$. From this follws that the map sending an $\mathcal{F}$-centric subgroup Q of P to its center $Z(Q)$ extends canonically to a contravariant functor

$$\mathcal{Z} : \mathcal{F}^c \longrightarrow \mathrm{mod}(\mathbb{Z}_{(p)}) \ .$$

Since inner automorphisms of Q act trivially on $Z(Q)$, this functor factors uniquely through $\bar{\mathcal{F}}^c$.

4 Fusion category algebras

We define in this section certain suitably twisted algebras over fusion systems and related categories of blocks. At this point this is still somewhat speculative as the existence of what one would like to call a "suitable twist" is not settled in general. Nonetheless, we are certain that such twists exist - see Conjecture 4.2 below - and that the resulting algebras are invariants of blocks of finite groups carrying a large amount of information which makes them worthwhile to be considered.

4.1. We keep the notation of the preceding section. The starting point of what follows is a theorem of Külshammer and Puig in [13], which associates with every $\mathcal{F}$-centric subgroup Q of P two pieces of information which are quite different in nature in that the first is an invariant of the fusion system $\mathcal{F}$, while the second is not:

4.1.1. *there is a canonical class $\zeta(Q) \in H^2(\mathrm{Aut}_{\mathcal{F}}(Q), Z(Q))$, or equivalently, a canonical group extension*

$$1 \longrightarrow Z(Q) \longrightarrow L_Q \longrightarrow \mathrm{Aut}_{\mathcal{F}}(Q) \longrightarrow 1$$

with the property that if $N_P(Q)$ is a defect group of $kN_G(Q, e_Q)e_Q$ then $N_P(Q)$ is a Sylow-p-subgroup of L_Q and the above exact sequence restricts to the obvious exact sequence $1 \to Z(Q) \to N_P(Q) \to N_P(Q)/Z(Q) \to 1$;

4.1.2. *there is a canonical class $\alpha(Q) \in H^2(\mathrm{Aut}_{\bar{\mathcal{F}}}(Q), k^\times)$ such that the twisted group algebra $k_{\alpha(Q)}\mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$ is Morita equivalent to $k(N_G(Q, e_Q)/Q)\bar{e}_Q$.*

The work of Broto, Levi and Oliver [3] shows what it means for a block b to have a classifying space; in particular, the existence of such a classifying space is shown to be equivalent to the existence of a certain extension category $\mathcal{L}$ of $\mathcal{F}^c$ by the center functor $\mathcal{Z}$, called *centric linking system* in [3]. More precisely, the objects of $\mathcal{L}$ are again the $\mathcal{F}$-centric subgroups in P, for every $\mathcal{F}$-centric subgroup Q in P we have $\mathrm{Aut}_{\mathcal{L}}(Q) = L_Q$ and there is a functor $\mathcal{L} \to \mathcal{F}^c$ which is the identity on objects, surjective on morphisms and which induces for each $\mathcal{F}$-centric Q the surjective map $L_Q \to \mathrm{Aut}_{\mathcal{F}}(Q)$ from the

Külshammer-Puig exact sequence in 4.1.1. The classifying space of b is then obtained as the p-completion $|\mathcal{L}|_p^\wedge$ of the nerve of $\mathcal{L}$. By the time of this writing, neither the existence nor the uniqueness of $\mathcal{L}$ seem to be known in general.

If $\mathcal{L}$ exists it determines a cohomology class $\zeta \in H^2(\mathcal{F}^c, \mathcal{Z})$ whose restriction to $\mathrm{Aut}_{\mathcal{F}}(Q)$ is the class $\zeta(Q)$ in 4.1.1 above, for every $\mathcal{F}$-centric subgroup subgroup Q of P. In other words, ζ "glues together" the classes $\zeta(Q)$.

The idea of gluing together the cohomology classes $\zeta(Q)$ may well be applied to the classes $\alpha(Q)$ in 4.1.2:

Conjecture 4.2. There is a second cohomology class $\alpha \in H^2(\bar{\mathcal{F}}^c, \underline{k}^\times)$ whose restriction to $\mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$ is the class $\alpha(Q)$ from 4.1.2 for any $\mathcal{F}$-centric subgroup Q of P.

Neither the existence nor the uniqueness of α is established at this point in general; see the next section for a (non exhaustive) list of examples. See also Remark 5.4 for a brief explanation regarding the difficulties that arise in attempting to construct α directly from block theoretic information. Since k is perfect we have a canonical group isomorphism $\mathcal{O}^\times \cong k^\times \times (1 + J(\mathcal{O}))$ from which we get in particular an inclusion $H^2(\bar{\mathcal{F}}^c, \underline{k}^\times) \subseteq H^2(\bar{\mathcal{F}}^c, \underline{\mathcal{O}}^\times)$.

If α exists, Alperin's weight conjecture admits the following reformulation:

Theorem 4.3. *Assume that α exists as in 4.2. Then Alperin's weight conjecture is equivalent to the following equality: the number $l(b)$ is equal to the number of isomorphism classes of simple $k_\alpha \bar{\mathcal{F}}^c$-modules whose diagonal entry in the Cartan matrix of $k_\alpha \bar{\mathcal{F}}^c$ is* 1.

Proof. Let S be a simple $k_\alpha \bar{\mathcal{F}}^c$-module. Let j be a primitive idempotent in $k_\alpha \bar{\mathcal{F}}^c$ such that $k_\alpha \bar{\mathcal{F}}^c j$ is a projective cover of S. Since the unit element of $k_\alpha \bar{\mathcal{F}}^c$ is the sum of the idempotents Id_Q we may actually choose j such that $j = \mathrm{Id}_Q j = j \mathrm{Id}_Q$ for some $\mathcal{F}$-centric subgroup Q of P. The diagonal entry of S in the Cartan matrix is the k-dimension of $j k_\alpha \bar{\mathcal{F}}^c j$. Since α glues together the classes $\alpha(Q)$ we have

$$\mathrm{Id}_Q \cdot k_\alpha \bar{\mathcal{F}}^c \cdot \mathrm{Id}_Q = k_{\alpha(Q)} \mathrm{Aut}_{\bar{\mathcal{F}}}(Q) ,$$

and this algebra is Morita equivalent to $k(N_G(Q, e_Q)/Q)\bar{e}_Q$. Thus the dimension of $j k_\alpha \bar{\mathcal{F}}^c j$ is also the diagonal entry of the Cartan matrix of $k_{\alpha(Q)} \mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$ corresponding to the simple $k_{\alpha(Q)} \mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$-module $\mathrm{Id}_Q S$. Since the twisted group algebra $k_{\alpha(Q)} \mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$ is symmetric, this Cartan number is 1 if and only if $\mathrm{Id}_Q S$ is projective (cf. [20, (6.8)]), which in turn holds if and only if $\mathrm{Id}_Q S$ belongs to a block of defect 0 of $k_{\alpha(Q)} \mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$. The Theorem follows. □

The argument proving the above Theorem shows a little more: the projective cover $k_\alpha \bar{\mathcal{F}}^c j$ of S looks like a highest weight module: its top composition factor is S, and all other composition factors are associated with $\mathcal{F}$-centric subgroups R of P such that $|R| < |Q|$. The slogan to restate the above Theorem is "Alperin's weight conjecture is a highest weight conjecture".

4.4. Assume that α exists as in 4.2; that is, such that the restriction of α to $\mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$ is $\alpha(Q)$ for every $\mathcal{F}$-centric subgroup Q of P. Denote by $\bar{e}$ an idempotent in $k_\alpha\bar{\mathcal{F}}^c$ such that $\bar{e}S = S$ for every simple $k_\alpha\bar{\mathcal{F}}^c$-module S whose diagonal entry in the Cartan matrix is 1 and such that $\bar{e}S' = \{0\}$ for any other simple $k_\alpha\bar{\mathcal{F}}^c$-module S'. This determines $\bar{e}$ up to conjugation in $k_\alpha\bar{\mathcal{F}}^c$, but there is in fact a canonical choice for $\bar{e}$: it follows from the proof of 4.3 that we can take for $\bar{e}$ the sum of all defect zero blocks of all twisted group algebras $k_{\alpha(Q)}\mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$, with Q running over the set of $\mathcal{F}$-centric subgroups of P. Set

$$\bar{\mathcal{F}}(b) = \bar{e}(k_\alpha\bar{\mathcal{F}}^c)\bar{e}\ .$$

The following combines 4.3 and 2.4:

Theorem 4.5. *Assume that α exists as in 4.2. The k-algebra $\bar{\mathcal{F}}(b)$ is quasi-hereditary, and Alperin's weight conjecture is equivalent to the equality $l(b) = l(\bar{\mathcal{F}}(b))$.*

Proof. The idempotent $\bar{e}$ collects precisely the defect zero blocks of the twisted group algebras $k_{\alpha(Q)}\mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$, hence 2.4 applies, showing that $\bar{\mathcal{F}}(b)$ is quasi-hereditary. The statement on Alperin's weight conjecture is obvious from 4.3. □

4.6. Suppose now that both α and ζ exist as above. Then, as explained in 4.1, the extension category $\mathcal{L}$ of $\mathcal{F}^c$ by $\mathcal{Z}$ determined by ζ comes along with a canonical functor $\mathcal{L} \longrightarrow \mathcal{F}^c$. Composing this with the canonical functor $\mathcal{F} \to \bar{\mathcal{F}}$ yields a functor

$$\mathcal{L} \longrightarrow \bar{\mathcal{F}}^c$$

through which we can restrict α to a class $H^2(\mathcal{L}, \underline{\mathcal{O}}^\times)$, still called α (we can replace $k^\times$ by $\mathcal{O}^\times$ by the remark preceding 4.3). The above functor induces thus a surjective algebra homomorphism

$$\mathcal{O}_\alpha\mathcal{L} \longrightarrow k_\alpha\bar{\mathcal{F}}^c\ ,$$

and the kernel of this algebra homomorphism is contained in the radical $J(\mathcal{O}_\alpha\mathcal{L})$, because at each object we are just dividing by a normal p-subgroup.

Thus there is an idempotent e in $\mathcal{O}_\alpha\mathcal{L}$ which lifts $\bar{e}$ with respect to the canonical algebra homomorphism $\mathcal{O}_\alpha\mathcal{L} \to k_\alpha\bar{\mathcal{F}}^c$; then e is unique up to conjugation. We define the $\mathcal{O}$-algebra $\mathcal{L}(b)$ by setting

$$\mathcal{L}(b) = e\mathcal{O}_\alpha\mathcal{L}e\ .$$

The surjective algebra homomorphism above induces a surjective algebra homomorphism $\mathcal{L}(b) \longrightarrow \bar{\mathcal{F}}(b)$ whose kernel is contained in the radical of $\mathcal{L}(b)$. In particular, $l(\mathcal{L}(b)) = l(\bar{\mathcal{F}}(b))$. As an immediate consequence of Theorem 4.3 and the construction of $\mathcal{L}(b)$, Alperin's weight conjecture reads then as follows:

Theorem 4.7. *If α, ζ exist, then, with the above notation, Alperin's weight conjecture is equivalent to the equality $l(b) = l(\mathcal{L}(b))$. Moreover, if P is abelian then $\mathcal{L}(b)$ is a source algebra of the Brauer correspondent of b.*

Proof. The first statement is a trivial consequence of the remarks in 4.6 above, and the second statement follows from the fact that if P is abelian, then P is the only $\mathcal{F}$-centric subgroup of P and $\mathrm{Aut}_{\mathcal{F}}(P) = N_G(P, e_P)/C_G(P) = E$ is a p'-group, namely the inertial quotient of b. Thus $\mathcal{L}(b) = \mathcal{O}_\alpha(P \rtimes E)$, which is a source algebra of the Brauer correspondent of b by results in [12] or [17]. □

As pointed out by G. R. Robinson, there should be a refinement of 4.7 which formulates Dade's conjectures in terms of a suitable variation of $\mathcal{L}(b)$. In fact, Robinson observed that Dade's projective conjecture admits a formulation in terms of chains of $\mathcal{F}$-centric subgroups of P and that in order to proceed further, one needs to find a way to "amalgamate" the local subgroups $N_G(Q, e_Q)$ (cf. [18]). The definition of the 2-cocycle α is an attempt in that direction.

One might want to speculate to what extent the algebra $\mathcal{L}(b)$ could be used for a generalisation of Broué's abelian defect conjecture. One certainly cannot expect a derived or even stable equivalence between the algebras $\mathcal{O}Gb$ and $\mathcal{L}(b)$, since in general, $\mathcal{L}(b)$ need not be symmetric and $K \underset{\mathcal{O}}{\otimes} \mathcal{L}(b)$ need not be semi-simple (where in the last statement we assume that $\mathcal{O}$ has characteristic zero and K is the quotient field of $\mathcal{O}$).

5 Examples

5.1 If b is the principal block, both ζ and α exist and are canonical. The category $\mathcal{L}$ is constructed explicitly in [2] and for α we can take the constant 2-cocycle mapping every pair of composable morphisms to 1 (that is, α represents the zero cohomology class).

5.2. As pointed out by G. R. Robinson, if b is tame (that is, $p = 2$ and P is either generalised quaternion or dihedral or semidihedral) both ζ and α exist and are canonical. For ζ this is due to the fact that the fusion system of a tame block is that of a suitable principal block, and for α we can again take the zero class.

5.3 Whenever there is a single $\mathcal{F}$-centric subgroup Q of P such that $N_G(Q, e_Q)$ controls $\mathcal{F}$-fusion - or equivalently, such that $\mathcal{F} = N_\mathcal{F}(Q)$ - then both ζ and α exist, since they are canonically determined by $\zeta(Q)$ and $\alpha(Q)$, respectively. This is the case if P is abelian (since here $N_G(P, e_P)$ controls fusion) or if G is p-solvable (by results of Puig [15]) or if b is $SL(2,p)$-free (by results of Kessar, Linckelmann, Robinson [9]).

Any element of $H^2(\mathrm{Aut}_{\bar{\mathcal{F}}}(Q), k^\times)$ shows up as an $\alpha(Q)$ in this way for some block of a finite group: if we start with a given $\alpha(Q) \in H^2(\mathrm{Aut}_{\bar{\mathcal{F}}}(Q), k^\times)$, we can view $\alpha(Q)$ as an element of $H^2(L_Q, k^\times)$, where L_Q is as in 4.1.1. This gives rise to a central group extension

$$1 \longrightarrow Z \longrightarrow H \longrightarrow L_Q \longrightarrow 0$$

of L_Q by a finite cyclic p'-group Z, which represents an element of $H^2(L_Q, Z)$ whose image in $H^2(L_Q, k^\times)$ via a suitable group homomorphism $\epsilon : Z \to k^\times$ is $\alpha(Q)$. Then $e = \sum_{z\in Z} \epsilon(z^{-1})z$ is a block of kH such that the block algebra kHe is in fact isomorphic to the twisted group algebra $k_{\alpha(Q)}L_Q$, the fusion system of e is precisely $\mathcal{F}$ and the 2-cocycle α on $\mathcal{F}$ for this block is the one determined by $\alpha(Q)$.

Remark 5.4. The 2-cocycles $\alpha(Q)$ in 4.1.2 can be characterised in terms of block theoretic information (cf. [17]) and one should expect that there is a similar way to construct a 2-cocycle α satisfying 4.2. We outline some of the difficulties that arise on the way.

Let Q, R be $\mathcal{F}$-centric subgroups of P. Let $i \in (kGb)^Q$ and $j \in (kGb)^R$ be primitive idempotents such that $\mathrm{Br}_Q(i)e_Q = \mathrm{Br}_Q(i) \neq 0$ and $\mathrm{Br}_R(j)e_R = \mathrm{Br}_R(j) \neq 0$. In other words, i and j belong to the unique local points of Q and R on kGb associated with e_Q and e_R, respectively.

Let $\varphi : Q \to R$ be a group homomorphism such that there exists an invertible element $a \in (kGb)^\times$ satisfying ${}^a(uj) = \varphi(u)({}^aj)$ for all $u \in Q$ and $({}^ai)j = {}^ai = j({}^ai)$. Then, by the main result in [16], the group homomorphism φ belongs to $\mathrm{Hom}_{\mathcal{F}}(Q, R)$, and moreover every element of $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ arises this way. In other words, if we denote by $T_{jkGi}(Q, R)$ the set of all elements of $jkGi$ of the form jai for some $a \in (kGb)^\times$ for which there is a group homomorphism $\varphi : Q \to R$ satisfying ${}^a(uj) = \varphi(u)({}^aj)$ for all $u \in Q$ and $({}^ai)j = {}^ai = j({}^ai)$, then the correspondence sending such an element a to φ induces a surjective map

$$T_{jkGi}(Q, R) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(Q, R) \ .$$

The group $((ikGi)^Q)^\times$ acts by right multiplication on the set $T_{jkGi}(Q, R)$, and thus the previous map induces a surjective map

$$T_{jkGi}(Q, R)/((ikGi)^Q)^\times \longrightarrow \mathrm{Hom}_{\mathcal{F}}(Q, R) \ .$$

For $Q = R$ and $i = j$ this is obviously a group isomorphism

$$T_{ikGi}(Q, Q)/((ikGi)^Q)^\times \cong \mathrm{Aut}_{\mathcal{F}}(Q) \ .$$

The left side in this isomorphism has a canonical central $k^\times$-extension, namely the group

$$T_{ikGi}(Q, Q)/(i + J((ikGi)^Q)) \ ;$$

indeed, since i is primitive in $(kGb)^Q$, the algebra $(ikGi)^Q$ is local, and hence $((ikGi)^Q)^\times \cong k^\times \times (i + J((ikGi)^Q))$. It is shown in [17] that this central $k^\times$-extension determines precisely $\alpha(Q)$. It is tempting to think that this might lead to a construction of the 2-cocycle α on $\mathcal{F}$ - and this idea had been our starting point to conjecture the existence of α as in 4.2. As it stands, this construction doesn't work, though.

The problem here is that in general the surjective map $T_{jkGi}(Q, R)/((ikGi)^Q)^\times \longrightarrow \mathrm{Hom}_{\mathcal{F}}(Q, R)$ considered above is not a bijection. Counterexamples occur systematically when the relative multiplicity of the involved local points of Q and R is greater than 1. To see this, consider the case where $Q \subset R$ and where $i \in (jkGj)^Q$ such that there exists another primitive idempotent i' in $(jkGj)^Q$ which is both orthogonal and conjugate to i by an element $a \in ((kGb)^Q)^\times$. Then both $ji = i$ and $jai = i'a$ belong to $T_{jkGi}(Q, R)$; their images in $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ are both equal to the inclusion homomorphism $Q \subset R$. However, i and $i'a$ cannot be in the same $((ikGi)^Q)^\times$-orbit because left multiplication by i annihilates the entire orbit $i'a((ikGi)^Q)^\times$ as i and i' are orthogonal.

This problem suggests what could be a remedy to it: if in addition Q is normal in R, then R acts on the simple algebra $kC_G(Q)/Z(Q)\bar{e}_Q$-algebra, and hence this algebra is isomorphic to $\mathrm{End}_k(W)$ for some endo-permutation kR/Q-module W. Then W determines the relative multiplicities of the local points of Q and R. Thus one would need to "glue together" the endo-permutation modules W obtained in this way to an endo-permutation kP-module V in such a way that "modifying" a source algebra of b by $\mathrm{End}_k(V)$ yields a "reduced source algebra" - that is, an algebra which is still Morita equivalent to the source algebra and which still contains the information on $\mathcal{F}$ but which now has the property that all relative multiplicities between local points of centric subgroups are 1. If that could be done, the above idea might still stand a chance to lead to an explicit description of α.

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