Koshitani, S. & Linckelmann, M. (2005). The indecomposability of a certain bimodule given by the Brauer construction. Journal of Algebra, 285(2), 726 - 729. doi: 10.1016/j.jalgebra.2004.08.031 <http://dx.doi.org/10.1016/j.jalgebra.2004.08.031>

The indecomposability of a certain bimodule given by the Brauer construction

Shigeo Koshitani and Markus Linckelmann

Abstract.

Broué's abelian defect conjecture [3, 6.2] predicts for a p-block of a finite group G with an abelian defect group P a derived equivalence between the block algebra and its Brauer correspondent. By a result of Rickard [11], such a derived equivalence would in particular imply a stable equivalence induced by tensoring with a suitable bimodule - and it appears that these stable equivalences in turn tend to be obtained by "gluing" together Morita equivalences at the local levels of the considered blocks; see e.g. [4, 6.3], [8, 3.1], [12, 4.1], and [13, 5.6, A.4.1]. This note provides a technical indecomposability result which is intended to verify in suitable circumstances the hypotheses that are necessary to apply gluing results as mentioned above. This is used in [7] to show that Broué's abelian defect group conjecture holds for nonprincipal blocks of the simple Held group and the sporadic Suzuki group.

Keywords: Broué's conjecture; Brauer construction; block; Brauer pair

Throughout this note, p is a prime and $\mathcal{O}$ is a complete discrete valuation ring having an algebraically closed residue field k of characteristic p. We allow the case $\mathcal{O} = k$. We state our result and explain the terminology below.

Theorem. *Let G be a finite group, let b be a block of $\mathcal{O}G$ and let (P, e) be a maximal b-Brauer pair. Set $H = N_G(P, e)$. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$ satisfying $(Q, e_Q) \subseteq (P, e)$ and $(Q, f_Q) \subseteq (P, e)$, respectively. Let f be a primitive idempotent in $(\mathcal{O}Gb)^{\Delta H}$ such that $\mathrm{Br}_{\Delta P}(f)e = e$ and set $X = \mathcal{O}Gf$. Then, as $\mathcal{O}(G \times H)$-module X is indecomposable with vertex ΔP, and for any subgroup Q of $Z(P)$ the $k(C_G(Q) \times C_H(Q))$-module $e_Q X(\Delta Q) f_Q$ is up to isomorphism the unique indecomposable direct summand of $e_Q kC_G(Q) f_Q$ with vertex ΔP.*

This Theorem is used in [7] to verify Broué's abelian defect group conjecture for nonprincipal blocks of the simple Held group and the sporadic Suzuki group.

Given a finite group G, a *block of* $\mathcal{O}G$ is a primitive idempotent in $Z(\mathcal{O}G)$. We denote by ΔG the diagonal subgroup $\Delta G = \{(g,g) \mid g \in G\}$ of $G \times G$. Unless stated otherwise, modules are left modules. If G and H are two finite groups, by an $(\mathcal{O}G, \mathcal{O}H)$-bimodule we mean a bimodule whose left and right $\mathcal{O}$-module structure coincide, so that we can view any such bimodule X as $\mathcal{O}(G \times H)$-module via $(g,h)x = gxh^{-1}$ for any $(g,h) \in G \times H$ and any $x \in X$. If furthermore Q is a common subgroup of G and H, we set $X^{\Delta Q} = \{x \in X \mid (u,u)x = x, \forall u \in Q\} = \{x \in X \mid uxu^{-1} = x, \forall u \in Q\}$. If Q is actually a p-group, the *Brauer construction* is defined to be the quotient $X(\Delta Q) = X^{\Delta Q}/(\sum_{Q'} \mathrm{Tr}_{Q'}^{Q}(X^{\Delta Q'}) + J(\mathcal{O})X^{\Delta Q})$, where in the sum Q' runs over the set of proper subgroups of Q, and where $\mathrm{Tr}_{Q'}^{Q}$ is the usual relative trace map. This construction is functorial in X. Moreover, since $C_{G\times H}(\Delta Q) = C_G(Q) \times C_H(Q) \subseteq N_{G\times H}(\Delta Q)$, we can regard $X(\Delta Q)$ as a $(kC_G(Q), kC_H(Q))$-bimodule. When applied to $X = \mathcal{O}G$, there is a canonical isomorphism $(\mathcal{O}G)(\Delta Q) \cong kC_G(Q)$, and the map $\mathrm{Br}_{\Delta Q} : (\mathcal{O}G)^{\Delta Q} \longrightarrow kC_G(Q)$ obtained from composing the canonical epimorphism $(\mathcal{O}G)^{\Delta Q} \to (\mathcal{O}G)(\Delta Q)$ with this isomorphism is in fact an algebra homomorphism, called the *Brauer homomorphism*. More explicitly, every element in $(\mathcal{O}G)^{\Delta Q}$ is an $\mathcal{O}$-linear combination of Q-conjugacy class sums of elements in G, and $\mathrm{Br}_{\Delta Q}$ maps the Q-conjugacy class sum of an element $x \in G$ to zero unless $x \in C_G(Q)$, in which case x is mapped to its canonical image in $kC_G(Q)$.

Given a finite group G and a block b of $\mathcal{O}G$, a *b-Brauer pair* is a pair (Q,f) consisting of a p-subgroup Q of G and a block f of $kC_G(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(b)f = f$. By results of Alperin and Broué [1], the set of b-Brauer pairs is a G-poset with a single G-conjugacy class of maximal b-Brauer pairs. If (P,e) is such a maximal b-Brauer pair then P is a defect group of b. A primitive idempotent $i \in (\mathcal{O}Gb)^{\Delta P}$ satisfying $\mathrm{Br}_{\Delta P}(i) \neq 0$ is then called a *source idempotent* of the block b. Since $\mathrm{Br}_{\Delta P}$ is a surjective algebra homomorphism, $\mathrm{Br}_{\Delta P}(i)$ is a primitive idempotent in $kC_G(P)$, and we may thus always choose i such that $\mathrm{Br}_{\Delta P}(i)e \neq 0$. By [5, 1.8], for any subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$, and then e_Q is the unique block of $kC_G(Q)$ such that $(Q,e_Q) \subseteq (P,e)$; in particular, $e_P = e$. See [9] and [14] for more details and background information. For the proof of the above Theorem we need the following Lemma, the first part of which is well-known.

Lemma. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let (P,e) be a maximal b-Brauer pair, and let $i \in (\mathcal{O}Gb)^{\Delta P}$ be a source idempotent of b such that* $\mathrm{Br}_{\Delta P}(i)e \neq 0$*. Let Q be a subgroup of $Z(P)$ and let e_Q be the unique block of $kC_G(Q)$ satisfying $(Q, e_Q) \subseteq (P,e)$. Then P is a defect group of e_Q and* $\mathrm{Br}_{\Delta Q}(i)$ *is a source idempotent of the block e_Q in $(kC_G(Q)e_Q)^{\Delta P}$.*

Proof. Since $Q \subseteq Z(P)$ we have $P \subseteq C_G(Q)$, and hence P is a defect group of e_Q by [8, 7.6]. Now $\mathrm{Br}_{\Delta Q}$ maps $(\mathcal{O}G)^{\Delta Q}$ onto $kC_G(Q)$; since P normalises $C_G(Q)$, any P-conjugacy class of elements in G is either contained in $C_G(Q)$ or in $G - C_G(Q)$. Hence $\mathrm{Br}_{\Delta Q}$ maps $(\mathcal{O}G)^{\Delta P}$ onto $(kC_G(Q))^{\Delta P}$. This implies that $\mathrm{Br}_{\Delta Q}(i)$ is a primitive idempotent in $(kC_G(Q))^{\Delta P}$. Moreover, by [5, 1.8] we have $\mathrm{Br}_{\Delta Q}(i) \in kC_G(Q)e_Q$ and clearly $\mathrm{Br}_{\Delta P}(\mathrm{Br}_{\Delta Q}(i)) = \mathrm{Br}_{\Delta P}(i) \neq 0$, which proves the second statement of the Lemma. □

Proof of the Theorem. Let $\hat{e}$ be the block of $\mathcal{O}C_G(P)$ which corresponds to the block e of $kC_G(P)$. Note first that $\hat{e}$ is still a block of $\mathcal{O}H$ with (P,e) as unique maximal Brauer pair. Let $j \in (\mathcal{O}H\hat{e})^{\Delta P}$ be a source idempotent of $\hat{e}$ as block of $\mathcal{O}H$. Then, by [6, 4.10] (or also [2, Theorem 5(ii) and p.265, line 3]) the idempotent $i = jf$ is a source idempotent of the block b in $(\mathcal{O}Gb)^{\Delta P}$, and since f was chosen such that $\mathrm{Br}_{\Delta P}(f)e = e$ we have $\mathrm{Br}_{\Delta P}(i)e \neq 0$. Let Q be a subgroup of $Z(P)$. By the above Lemma, $i_Q = \mathrm{Br}_{\Delta Q}(i)$ is a source idempotent of the block e_Q, and $j_Q = \mathrm{Br}_{\Delta Q}(j)$ is a source idempotent of the block f_Q. Since $i = jf = fj$ we have $i_Q = \mathrm{Br}_{\Delta Q}(f)j_Q$, and this is therefore in particular a primitive idempotent in $(kC_G(Q)e_Q)^{\Delta P}$.

Since $X = \mathcal{O}Gf$ we have $X(\Delta Q) = kC_G(Q)\,\mathrm{Br}_{\Delta Q}(f)$, and therefore

$$e_Q X(\Delta Q) j_Q = e_Q kC_G(Q)\,\mathrm{Br}_{\Delta Q}(f) j_Q = e_Q kC_G(Q) i_Q \ .$$

As $i_Q \in kC_G(Q)e_Q$ this implies in particular that $e_Q X(\Delta Q) j_Q$ is non zero. The point now is that since i_Q is primitive in $(kC_G(Q)e_Q)^{\Delta P}$, the $(kC_G(Q)e_Q, kP)$-bimodule $e_Q kC_G(Q) i_Q$ is indecomposable. Since kP is isomorphic to a subalgebra of the source algebra $j_Q kC_H(Q) j_Q$ via multiplication by j_Q, it follows that $e_Q X(\Delta Q) j_Q$ is indecomposable as $(kC_G(Q)e_Q, j_Q kC_H(Q) j_Q)$-bimodule. By [10, 3.4], the block algebra $kC_H(Q)f_Q$ and its source algebra $j_Q kC_H(Q) j_Q$ are Morita equivalent, which implies that indeed $e_Q X(\Delta Q) f_Q$ is indecomposable as $k(C_G(Q) \times C_H(Q))$-module.

Since X is a direct summand of $\mathcal{O}Gb$ as $\mathcal{O}(G \times H)$-module, $X(\Delta Q)$ is a direct summand of $kC_G(Q)\,\mathrm{Br}_{\Delta Q}(b)$ as $k(C_G(Q) \times C_H(Q)$-module, and hence $e_Q X(\Delta Q) f_Q$ is a direct summand of $e_Q kC_G(Q) f_Q$.

Since f is primitive in $(\mathcal{O}G)^{\Delta H}$, the $\mathcal{O}(G \times H)$-module X is indecomposable. As $\mathcal{O}(G \times G)$-module, $\mathcal{O}Gb$ has ΔP as vertex. Thus X has a vertex contained in a $(G \times G)$-conjugate of ΔP. Since $\mathrm{Br}_{\Delta P}(f)e = e \neq 0$, we have $X(\Delta P) \neq 0$ and thus X has ΔP as a vertex by [14, 27.7]. Similarly, we have $e = e\,\mathrm{Br}_{\Delta P}(e_Q) = e\,\mathrm{Br}_{\Delta P}(f_Q) = e\,\mathrm{Br}_{\Delta P}(f)$ by [5, 1.8(3)] and the assumption. Thus, if we denote by $\bar{f}$ the canonical image of f in $(kG)^{\Delta H}$, we get $e\,\mathrm{Br}_{\Delta P}(e_Q \bar{f} f_Q) = e \neq 0$, so that $\mathrm{Br}_{\Delta P}(e_Q \bar{f} f_Q) \neq 0$, hence $(e_Q X(\Delta Q) f_Q)(\Delta P) \neq 0$, and so ΔP is a vertex of $e_Q X(\Delta Q) f_Q$.

For the last part we observe that the $k(C_G(Q) \times C_H(Q))$-module $e_Q kC_G(Q) f_Q$ is a direct summand of $kC_G(Q) f_Q = \mathrm{Ind}_{C_H(Q) \times C_H(Q)}^{C_G(Q) \times C_H(Q)}(kC_H(Q) f_Q)$. Moreover, the $k(C_H(Q) \times C_H(Q))$-module $kC_H(Q) f_Q$ is indecomposable with ΔP as vertex, and the normaliser of ΔP in $C_G(Q) \times C_H(Q)$ is contained in $C_H(Q) \times C_H(Q)$. Thus the Green correspondence implies that the $k(C_G(Q) \times C_H(Q))$-module $kC_G(Q) f_Q$ has exactly one indecomposable direct summand with ΔP as vertex, up to isomorphism. The result follows. □

Remark. With the notation of the Theorem, if Q is a subgroup of $Z(P)$ then $f_Q = e$. Indeed, P is normal in H, hence in $C_H(Q)$, and thus every block of $kC_H(Q)$ is contained in $kC_H(P) = kC_G(P)$. The last argument in the proof of the Theorem shows the seemingly stronger statement that $e_Q X(\Delta Q) f_Q$ is the unique direct summand with vertex ΔP of the $k(C_G(Q) \times C_H(Q))$-module $kC_G(Q) f_Q$, but since $\mathrm{Br}_{\Delta P}(e_Q)e = e = f_Q$, every direct summand of $kC_G(Q) f_Q$ with vertex ΔP is already a direct summand of $e_Q kC_G(Q) f_Q$.

Acknowledgments

A part of this work was done while the first author was staying at the Department of Mathematics, the Ohio State University, May 2003. He thanks the department for its hospitality. The first author was in part supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for JSPS Fellows 01016, 2002–2003; and the JSPS (Japan Society for Promotion of Science), Grant-in-Aid for Scientific Research C(2) 14540009, 2002–2003. The authors would like to thank the referee for his comments.

References

[1] J. L. Alperin, M. Broué, Local methods in block theory, Ann. of Math. (2) **110** (1979), 143–157.

[2] J. L. Alperin, M. Linckelmann, R. Rouquier, Source algebras and source modules, J. Algebra **239** (2001), 262–271.

[3] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, Astérisque **181/182** (1990), 61–92.

[4] M. Broué, Equivalences of blocks of group algebras, in: "Finite Dimensional Algebras and Related Topics", V. Dlab and L. L. Scott (eds), Kluwer Acad. Pub. (1994), 1–26.

[5] M. Broué, L. Puig, Characters and local structure in G-algebras, J. Algebra **73** (1980), 306–317.

[6] Y. Fan, L. Puig, On blocks with nilpotent coefficient extensions, Algebras and Representation Theory **2** (1999).

[7] S. Koshitani, N. Kunugi, K. Waki, Broué's abelian defect group conjecture for the Held group and the sporadic Suzuki group, J. Algebra, in press (2004).

[8] M. Linckelmann, On splendid derived and stable equivalences between blocks of finite groups, J. Algebra **242** (2001), 819–843.

[9] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, New York, 1988.

[10] L. Puig, Pointed groups and construction of characters, Math. Z. **176** (1981), 265–292.

[11] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. **43** (1991), 37–48.

[12] J. Rickard, Splendid equivalences: derived categories and permutation modules, Proc. London Math. Soc. **72** (1996), 331–358.

[13] R. Rouquier, Block theory via stable and Rickard equivalences, in: "Modular Representation Theory of Finite Groups", M. J. Collins, B. J. Parshall and L. L. Scott (eds.), de Gruyter, Berlin, (2001), 101–146.

[14] J. Thévenaz, G-Algebras and Modular Representation Theory, Clarendon Press, Oxford, 1995.