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A REDUCTION THEOREM FOR FUSION SYSTEMS OF BLOCKS

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Abstract. Let k be an algebraically closed field of characteristic p and G a finite group. An interesting question for fusion systems is whether they can be obtained from the local structure of a block of the group algebra kG. In this paper we develop some methods to reduce this question to the case when G is a central p'-extension of a simple group. As an application of our result, we obtain that the 'exotic' examples of fusion systems discovered by Ruiz and Viruel [RV] do not occur as fuison systems of p-blocks of finite groups.

Dedicated to the memory of Walter Feit

1. Introduction

Let p be a prime number. Fusion systems (referred to as full Frobenius systems in [Pu], and as saturated fusion systems in [BLO]) on finite p-groups were introduced by L. Puig and provide an axiomatic framework for studying p-fusion in finite groups. This axiomatic point of view has been very useful in determining many properties of finite groups and of the p-completion of their classifying spaces as well as in modular representation theory. It also underlies the theory of p-local finite groups developed by C. Broto, R. Levi and R. Oliver in [BLO].

To each pair (G, P), where G is a finite group and P is a Sylow p-subgroup of G, is associated a fusion system $\mathcal{F}_P(G)$ on P called a p-fusion system of G. However, there exist fusion systems which do not arise in this way [RV],[BLO]; such systems are called *exotic*. These exotic examples are interesting from two different, albeit related points of view. First, the exoticity of a given fusion has ramifications for classification problems in finite groups. For instance, R. Solomon's theorem characterising the sporadic group .3 by the isomorphism type of its Sylow 2-subgroup may be restated, somewhat ahistorically, as asserting that certain fusion systems are exotic. Secondly, each known exotic fusion system has associated to it a unique p-local finite group. In general, [BLO] provides an obstruction theory for the existence and uniqueness of a p-local group associated to a given fusion system, and it is not known if the obstructions always vanish.

Fusion systems arise as well in block theory. To each quadruple (H, b, Q, e) where H is a finite group, b is a p-block of kH, and (Q, e) is a maximal b-Brauer pair (that is Q is a defect group of H and e is a p-block of $C_H(Q)$ in correspondence with b) is associated a fusion system $\mathcal{F}_{(Q,e)}(H, b)$ on Q, called a fusion system of b. If b is the principal block of H, then Q is a Sylow p-subgroup of H and by Brauer's third main theorem, it follows that $\mathcal{F}_{(Q,e)}(H, b) = \mathcal{F}_Q(H)$. However, if b is not the

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principal block, then Q may be a proper subgroup of a Sylow p-subgroup of H of arbitrarily large index. On the other hand, there are many examples of quadruples (H, b, Q, e) such that for some group L associated to H, Q is a Sylow p-subgroup of L and $\mathcal{F}_{(Q,e)}(H, b) = \mathcal{F}_Q(L)$. This is always the case, for instance, if H is a p-solvable group or a symmetric group.

The aim of this paper is to shed some light on the relationship between fusion systems of blocks and fusion systems of finite groups. We use as our starting point *simple* fusion systems as introduced by Linckelmann [Li, Definition 2.9]. If a simple fusion system occurs as $\mathcal{F}_P(G)$ for some finite group G, then it occurs as $\mathcal{F}_P(L)$ for some simple finite group L. It would be desirable to obtain an analogous result for fusion systems of blocks with L being possibly a quasi-simple group. There is however one complication which arises when one tries "descent to a normal subgroup" in the context of block theory: Let H be a finite group and L a normal subgroup of H. If P is a Sylow p-subgroup of H, then $P \cap L$ is a Sylow p-subgroup of L and $\mathcal{F}_{P\cap L}(L)$ is a normal subsystem of $\mathcal{F}_P(H)$ (see Definition 2.9). Now suppose that (H, b, Q, e) is a quadruple as above and c is a block of L covered by b. Then, $Q \cap L$ is a defect group of c, however it is not the case in general that for some p-block f of $C_L(Q \cap L)$ in correspondence with c, the system $\mathcal{F}_{(Q\cap L,f)}(L, c)$ is a subsytem of $\mathcal{F}_{(Q,e)}(H, b)$. In other words, there may be fusion in the covered block c, which is not seen in b. Our main result, Theorem 4.2 shows that under certain extra hypotheses this difficulty may be circumvented. In order to prove this theorem, we were led to consider categories that arise through the conjugation action of a finite group G on the block algebra kNd of a G-stable block d of a normal subgroup N of G. We show that these categories, which we call generalised Brauer categories are fusion systems (Theorem 3.4).

As an application of Theorem 4.2, we show in Theorem 6.4 that the examples of exotic fusion systems discovered by Ruiz and Viruel [RV] do not occur in block algebras.

The paper has 6 sections and an appendix. In section 2, we recall the relevant definitions and facts on fusion systems and block theory. In section 3 we study the generalized Brauer category and show that it is a fusion system. Section 4 contains the main reduction theorem. In section 5, we recall the properties of exotic fusion systems on extra-special p-groups of order 7^3 and exponent 7. Section 6 contains the proof of the fact that these exotic systems do not occur as fusion systems of blocks. The defining properties of fusion systems as stated by us are slightly different from those in [BLO]-in the Appendix we show the equivalence of our approach with that in [BLO]. We should reiterate that for us, "fusion systems" are what are called "saturated fusion systems" in [BLO].

2. Fusion systems. Definitions and basic properties

Definition 2.1. *A category $\mathcal{F}$ on a finite p-group P is a category whose objects are the subgroups of P and whose set of morphisms between the subgroups Q and R of P, is the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ of injective group homomorphisms from Q to R, with the following properties:*

(a) if $Q \leq R$ then the inclusion of Q in R is a morphism in $\mathrm{Hom}_{\mathcal{F}}(Q, R)$.

(b) for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ the induced isomorphism $Q \simeq \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$.

(c) composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Note that the above definition of a category on P differs from what Puig calls *divisible Frobenius system* and what, equivalently, Broto, Levi and Oliver call *fusion system* by the fact that we do not ask for the inner automorphisms of P to be in the category.

If there exists an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ we say that Q and R are *$\mathcal{F}$-conjugate.*

Definition 2.2. *Let $\mathcal{F}$ be a category on P. A subgroup Q of P is fully $\mathcal{F}$-centralized, respectively fully $\mathcal{F}$-normalized if $|C_P(Q)| \geq |C_P(Q')|$, respectively $|N_P(Q)| \geq |N_P(Q')|$, for all $Q' \leq P$ which are $\mathcal{F}$-conjugate to Q.*

For Q, R, $T \leq P$ we denote $\mathrm{Hom}_T(Q,R) := \{u \in T \,|\, {}^uQ \leq R\}/C_T(Q)$ and $\mathrm{Aut}_T(Q) := \mathrm{Hom}_T(Q,Q)$. Other useful notation is $\mathrm{Aut}_{\mathcal{F}}(Q) := \mathrm{Hom}_{\mathcal{F}}(Q,Q)$ and $\mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$.

Definition 2.3. *We say that a subgroup Q of P is $\mathcal{F}$-centric if $Z(Q') = C_P(Q')$ for any Q' in the $\mathcal{F}$-isomorphism class of Q. We say that Q is $\mathcal{F}$-radical if $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = \mathrm{Out}_P(Q)$.*

Definition 2.4. *A fusion system $\mathcal{F}$ on a finite p-group P is a category on P satisfying the following properties:*

(1) $\mathrm{Hom}_P(Q,R) \subset \mathrm{Hom}_{\mathcal{F}}(Q,R)$ for all $Q, R \leq P$.

(2) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(3) Every $\phi : Q \to P$ such that $\phi(Q)$ is fully $\mathcal{F}$-normalized extends to a morphism $\bar{\phi} : N_\phi \to P$ where $N_\phi = \{x \in N_P(Q) \,|\, \exists\, y \in N_P(\phi(Q)),\, \phi({}^xu) = {}^y\phi(u) \forall u \in Q\}$.

In the rest of the section we give some properties of fusion systems. Let us start with a characterization of being fully $\mathcal{F}$-normalized.

Proposition 2.5 ([Pu, Propositon 2.7])**.** *Let $\mathcal{F}$ be a fusion system on P and let Q be a subgroup of P. Then Q is fully $\mathcal{F}$-normalized if and only if Q is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.*

Definition 2.6. *Let $\mathcal{F}$ be a fusion system on P and let Q be a subgroup of P. The normalizer $N_{\mathcal{F}}(Q)$ is the category on $N_P(Q)$ having as morphisms, those morphisms $\psi \in \mathrm{Hom}_{\mathcal{F}}(R,T)$ such that there exists a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(QR,QT)$ with $\phi|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\phi|_R = \psi$. The centralizer $C_{\mathcal{F}}(Q)$ is the category on $C_P(Q)$ having as morphisms those morphisms $\psi \in \mathrm{Hom}_{\mathcal{F}}(R,T)$ such that there exists a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(QR,QT)$ with $\phi|_Q = \mathrm{id}_Q$ and $\phi|_R = \psi$.*

$N_{\mathcal{F}}(Q)$ is not, in general, a fusion system on $N_P(Q)$ (for instance, the property (2) in Definition 2.4 may fail to hold) but it is one if Q is fully $\mathcal{F}$-normalized. It is the same for $C_{\mathcal{F}}(Q)$ when Q is fully $\mathcal{F}$-centralized.

Proposition 2.7 ([Pu, Proposition 2.8])**.** *Let $\mathcal{F}$ be a fusion system on P. If $Q \leq P$ is fully $\mathcal{F}$-normalized then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$.*

A special role in our study is played by *strongly $\mathcal{F}$-closed* subgroups.

Definition 2.8. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and Q a subgroup of P. We say that Q is strongly $\mathcal{F}$-closed if for any subgroup R of Q and any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R,P)$ we have $\phi(R) \leq Q$.*

Linckelmann [Li] has introduced the notion of normal fusion subsystem.

Definition 2.9. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and $\mathcal{F}'$ a fusion subsystem of $\mathcal{F}$ on a subgroup P' of P. We say that $\mathcal{F}'$ is normal in $\mathcal{F}$ if P' is strongly $\mathcal{F}$-closed and if for every isomorphism $\phi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q \cap P'$ we have*

$$\phi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \phi^{-1} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\phi(R), \phi(R'))\,.$$

If $\mathcal{F}$ is a fusion system on a finite p-group P, a subsystem of $\mathcal{F}$ is a category on a subgroup of P that is contained in $\mathcal{F}$ and is itself a fusion system. We say that a fusion system is *simple* if it has no non-trivial normal fusion subsystem.

Finally, we record how fusion systems arise in finite groups.

Definition 2.10. *Let G be a finite group, and P a Sylow p-subgroup of G. We denote by $\mathcal{F}_P(G)$ the category on P with morphisms $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) := \mathrm{Hom}_G(Q, R)$.*

It is easy to check that $\mathcal{F}_P(G)$ is a fusion system. Also, $\mathcal{F}_P(G)$ and $\mathcal{F}_{P'}(G)$ are isomorphic for different Sylow p-subgroups P and P' of G.

3. The generalised Brauer category

Let k be an algebraically closed field of characteristic p, G a finite group, N a normal subgroup of G and c a G-stable block of kN, that is c is a primitive idempotent of $Z(kN)$, fixed by the conjugation action of G. Thus kNc is a primitive G-algebra. For any p-subgroup Q of G the canonical projection from kN to $kC_N(Q)$ induces an algebra morphism Br_Q^N from the subalgebra of fixed points of Q, $(kN)^Q$ onto $kC_N(Q)$ (see [AB]). This morphism is known in the literature as the Brauer morphism. We adopt the approach of Broué and Puig [BP] for generalized Brauer pairs.

Definition 3.1. *A (c, G)-Brauer pair is a pair (Q, e_Q) where Q is a p-subgroup of G such that $\mathrm{Br}_Q^N(c) \neq 0$ and e_Q is a block of $kC_N(Q)$ such that $\mathrm{Br}_Q^N(c)e_Q \neq 0$. When $G = N$, a (c, G)-Brauer pair is also known as a c-Brauer pair.*

Let (Q, e_Q) and (R, e_R) be two (c, G)-Brauer pairs; we say that (Q, e_Q) is contained in (R, e_R), and we write $(Q, e_Q) \leq (R, e_R)$, if $Q \leq R$ and for any primitive idempotent $i \in (kN)^R$ such that $\mathrm{Br}_R^N(i)e_R \neq 0$, we have $\mathrm{Br}_Q^N(i)e_Q \neq 0$. This defines an order relation on the set of (c, G)-Brauer pairs compatible with the conjugation action of G. We also have that given a (c, G)-Brauer pair (R, e_R) and $Q \leq R$ there exists an unique (c, G)-Brauer pair (Q, e_Q) contained in (R, e_R) [BP, Theorem 1.8 (i)]. By [BP, Theorem 1.14 (2)] all maximal (c, G)-Brauer pairs are G-conjugate. If (P, e_P) is a maximal (c, G)-Brauer pair then the group P is called a (c, G)-defect group. In the case that $G = N$, the group P is a defect group of c in the usual sense.

Before proceeding, we record a property characterising the inclusion of generalised Brauer pairs in the case of normal p-subgroups, which is just a reformulation of [BP, Theorem 1.8]. For a (c, G)-Brauer pair (Q, e), denote by $N_G(Q, e_Q)$ the stablilizer in $N_G(Q)$ of e_Q.

Proposition 3.2. *Let G be a finite group, N a normal subgroup of G, c a G-stable block of N and (Q, e) a (c, G)-Brauer pair. Let H be a group such that $QC_N(Q) \leq H \leq N_G(Q, e)$. Let S be a p-group such that $Q \leq S \leq H$ and let f be a block of $kC_N(S)$. The following are equivalent.*

(i) (S,f) is a (c,G)-Brauer pair such that $(Q,e) \leq (S,f)$.
(ii) (S,f) is a (e,H)-Brauer pair.

Proof. First of all, note that the statement makes sense, since $C_N(Q)$ is a normal subgroup of H and e is an H-stable block of $kC_N(Q)$. Also, since $Q \leq S$, $C_{C_N(Q)}(S) = C_N(S)$. Suppose first that (S,f) is a (c,G)-Brauer pair such that $(Q,e) \leq (S,f)$. Since S normalizes Q, it follows from [BP, Theorem 1.8 (iii)] $\mathrm{Br}_S^{C_N(Q)}(e)f = f$, hence (S,f) is an (e,H)-Brauer pair. Conversely, suppose that (S,f) is a (e,H)- Brauer pair. Then, $\mathrm{Br}_S^{C_N(Q)}(e)f = f$, which means that

$$\begin{aligned}
\mathrm{Br}_S^N(c)f &= \mathrm{Br}_S^N(c)\mathrm{Br}_S^{C_N(Q)}(e)f \\
&= \mathrm{Br}_S^{C_N(Q)}(\mathrm{Br}_Q^N(c))\mathrm{Br}_S^{C_N(Q)}(e)f \\
&= \mathrm{Br}_S^{C_N(Q)}(\mathrm{Br}_Q^N(c)e)f \\
&= \mathrm{Br}_S^{C_N(Q)}(e)f \\
&= f
\end{aligned}$$

This shows that (S,f) is a (c,G)-Brauer pair. Thus, by [BP, Theorem 1.8 (i)], there is a unique (c,G)-Brauer pair (Q,e') with $(Q,e') \leq (S,f)$ and by (iii) of the same theorem, this e' is S-stable and $\mathrm{Br}_S^{C_N(Q)}(e')f = f$. But by definition $\mathrm{Br}_S^{C_N(Q)}(e)f = f$. We claim that $e' = e$. Indeed, suppose not. Then, since e and e' are blocks of $kC_N(Q)$, $ee' = 0$. Since $\mathrm{Br}_S^{C_N(Q)}$ is an algebra homomophism from $(kC_N(Q))^S$ onto $kC_N(S)$, this would imply that $\mathrm{Br}_S^{C_N(Q)}(e)\mathrm{Br}_S^{C_N(Q)}(e') = 0$. Hence, f being a central idempotent of $kC_N(S)$,

$$f = f\mathrm{Br}_S^{C_N(Q)}(e)f\mathrm{Br}_S^{C_N(Q)}(e') = 0,$$

a contradiction. Thus, $e = e'$, showing that $(Q,e) \leq (S,f)$ as required.

Definition 3.3. *Let G be a finite group, N a normal subgroup of G, c a G-stable block of kN and (P,e_P) a maximal (c,G)-Brauer pair. For a subgroup Q of P, we let e_Q be the unique block of $kC_N(Q)$ such that $(Q,e_Q) \leq (P,e_P)$. Denote by $\mathcal{F}_{(P,e_P)}(G,N,c)$ the category on P with morphisms:*

$$\mathrm{Hom}_{\mathcal{F}_{(P,e_P)}(G,N,c)}(Q,R) := \{\mathrm{conj}_g : Q \to R | g \in G,\ {}^g(Q,e_Q) \leq (R,e_R)\}$$

If $G = N$, then $\mathcal{F}_{(P,e_P)}(G,N,c)$ is the usual fusion system of the block c, and we denote it by $\mathcal{F}_{(P,e_P)}(G,c)$

We now show that the category $\mathcal{F}_{(P,e_P)}(G,N,c)$ is a fusion system on P. The details of the proof for the case $G = N$ are given in [Li].

Theorem 3.4. *Let N be a normal subgroup of G, let c be a G-stable block of kN and let (P,e_P) be a maximal (c,G)-Brauer pair.*
(i) The category $\mathcal{F}_{(P,e_P)}(G,N,c)$ is a fusion system on P.
(ii) If $(P',e_{P'})$ is another maximal (c,G)-Brauer pair, then $\mathcal{F}_{(P',e_{P'})}(G,N,c)$ is isomorphic to $\mathcal{F}_{(P,e_P)}(G,N,c)$.

Proof. Denote $\mathcal{F} := \mathcal{F}_{(P,e_P)}(G,N,c)$. Let $u \in P$ and let $(Q,e_Q) \leq (P,e_P)$. Then, $({}^uQ, {}^ue_Q) \leq ({}^uP, {}^ue_P) = (P,e_P)$ which implies that ${}^ue_Q = e_{{}^uQ}$. This shows that property (1) of Definition 2.4 holds.

For the second property we check that $\mathrm{Aut}_P(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Denoting by $N_G(P,e_P)$ the normalizer of (P,e_P) and by $C_G(P,e_P)$ the intersection of $C_G(P)$

with $N_G(P, e_P)$, we have $\mathrm{Aut}_{\mathcal{F}}(P) \simeq N_G(P, e_P)/C_G(P, e_P)$ hence we must show that the index $[N_G(P, e_P) : PC_G(P, e_P)]$ is not divisible by p.

By [BP, Theorem 1.14(b)], (P, e_P) being a maximal (c, G)-Brauer pair means that $c \in \mathrm{Tr}_P^G((kN)^P)$ and hence that $\mathrm{Br}_P^N(c) = \mathrm{Tr}_1^{N_G(P)/P}(a')$ for some $a' \in kC_N(P)$.

Since the maximal (c, G)-Brauer pairs are all conjugate [BP, Theorem 1.14 (2)], the map $a \to \mathrm{Tr}_{N_G(P,e_P)}^{N_G(P)} a$ is an algebra isomorphism from $(kC_N(P)e_P)^{N_G(P,e_P)}$ to $(kC_N(P)\mathrm{Br}_P^N(c))^{N_G(P)}$. The reverse map is given by $a \to ae_P$.

Let $g \in N_G(P)$, and let $j \in kC_N(P)$. Then

$$\begin{aligned}\mathrm{Tr}_P^{N_G(P)}(j\ {}^g e_P) &= \mathrm{Tr}_{N_G(P,\ {}^g e_P)}^{N_G(P)}(\mathrm{Tr}_P^{N_G(P,\ {}^g e_P)}(j\ {}^g e_P)) \\ &= \mathrm{Tr}_{N_G(P,\ {}^g e_P)}^{N_G(P)}(\mathrm{Tr}_P^{N_G(P,\ {}^g e_P)}(j)\ {}^g e_P)\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Tr}_P^{N_G(P)}(j\ {}^g e_P)e_P &= {}^{g^{-1}}(\mathrm{Tr}_P^{N_G(P,\ {}^g e_P)}(j\ {}^g e_P)) \\ &= \mathrm{Tr}_P^{N_G(P,e_P)}((\ {}^{g^{-1}}j)e_P)\end{aligned}$$

The above calculation shows that the image of the ideal $\mathrm{Tr}_P^{N_G(P)}(kC_N(P)\mathrm{Br}_P(b))$ under the map $a \to ae_P$ is $\mathrm{Tr}_P^{N_G(P,e_P)}(kC_N(P)e_P)$. In particular, since $\mathrm{Br}_P^N(c) = \mathrm{Tr}_P^{N_G(P)}(a')$ for some $a' \in kC_N(P)$, it follows that $e_P = \mathrm{Tr}_P^{N_G(P,e_P)}(a)$ for some $a \in kC_N(P)e_P$.

As $N_G(P, e_P) \geq PC_G(P, e_P) \geq P$ we have furthermore

$$e_P = \mathrm{Tr}_{P(N_G(P,e_P)\cap C_G(P))}^{N_G(P,e_P)} \mathrm{Tr}_P^{PC_G(P,e_P)}(a).$$

Now $\mathrm{Tr}_P^{PC_G(P,e_P)}(a) \in Z(kC_N(P)e_P)$ and $Z(kC_N(P)e_P)$ is a local ring, hence

$$\mathrm{Tr}_P^{PC_G(P,e_P)}(a) = \alpha e_P + x,\ \ x \in J(kPC_N(P)), \alpha \in k.$$

If $[N_G(P, e_P) : PC_G(P, e_P)]$ is not prime to p then

$$e_P = \mathrm{tr}_{P(C_G(P,e_P))}^{N_G(P,e_P)}(x) \in J(kPC_N(P)),$$

which is impossible since e_P is an idempotent.

For the third property, let $(Q, e_Q) \leq (P, e_P)$ be fully $\mathcal{F}$-normalized. This means that $R := N_P(Q)$ is of maximal order among the normalizers in P of subgroups $\mathcal{F}$-isomorphic to Q. Let $H = RC_G(Q, e_Q)$. We claim that (R, e_R) is a maximal (e_Q, H)-Brauer pair. Indeed, it follows from Proposition 3.2 that (R, e_R) is an (e_Q, H)-Brauer pair. Now, suppose that (S, f) is a (e_Q, H)-Brauer pair with $(R, e_R) \leq (S, f)$. Then by Proposition 3.2, (S, f) is a (c, G)-Brauer pair with $(Q, e_Q) \leq (S, f)$. Since all maximal (c, G)-Brauer pairs are G-conjugate, there exists $g \in G$ such that ${}^g(S, f) \leq (P, e_P)$. Now, $(Q, e_Q) \leq (S, f)$ implies ${}^g(Q, e_Q) \leq {}^g(S, f) \leq (P, e_P)$, so that in particular, ${}^gQ$ is $\mathcal{F}$-isomorphic to Q. On the other hand, ${}^gS \leq N_P({}^gQ)$. The maximality of R forces $S = R$, proving the claim.

Now let $h \in G$ such that $({}^hQ, {}^he_Q) \leq (P, e_P)$ and denote by $\phi := conj_{h^{-1}} : {}^hQ \to Q$. We have to prove that ϕ extends to $\tilde{\phi} : N_\phi \to N_P(Q)$. Now N_ϕ consists of those elements x of $N_P(()Q)$ such that $\mathrm{conj}(h^{-1}xh) : Q \to P$ is equal to $\mathrm{conj}(y) : Q \to P$ for some $y \in N_P(()Q)$, that is

$$N_\phi = \{x \in N_P({}^hQ) | h^{-1}xh \in H\}\,.$$

So we have ${}^{h^{-1}}N_\phi \le H$. Thus it suffices to find a $z \in C_G(Q, e_Q)$ such that ${}^{zh^{-1}}N_\phi \le N_P(Q)$. Since ${}^{h}(Q, e_Q) \le (P, e_P)$ and $N_\phi \le N_P({}^{h}Q)$, there is a containment of (c, G)-Brauer pairs, ${}^{h}(Q, e_Q) \le (N_\phi, e_{N_\phi})$, giving $(Q, e_Q) \le {}^{h^{-1}}(N_\phi, e_{N_\phi})$. By Proposition 3.2 applied with $S = {}^{h^{-1}}N_\phi$ and $f = {}^{h^{-1}}e_{N_\phi}$, it follows that $({}^{h^{-1}}N_\phi, {}^{h^{-1}}e_{N_\phi})$ is a (e_Q, H)-Brauer pair. But we have shown above that (R, e_R) is a maximal (e_Q, H)-Brauer pair, hence there exists a $y \in H$ such that ${}^{yh^{-1}}N_\phi \le N_P(Q)$, proving the existence of z as desired.

This proves (i) of the theorem. Part (ii) is immediate since all maximal (c, G)-Brauer pairs are G-conjugate.

Theorem 3.5. *Let G be a finite group and let N be a normal subgroup of G. Let c be a G-stable block of N and b a block of kG such that $bc = b$. Let (P, e_P) be a maximal b-Brauer pair. Then there exists a maximal (c, G)-Brauer pair (S, e'_S) such that $P \le S$ and such that $\mathcal{F}_{(P,e_P)}(G, b) \le \mathcal{F}_{(S,e'_S)}(G, N, c)$. Furthermore, $P \cap N = S \cap N$, $(S \cap N, e'_{S\cap N})$ is a maximal (c, N)-Brauer pair and $\mathcal{F}_{(S\cap N,e'_{S\cap N})}(N, N, c) = \mathcal{F}_{(S\cap N,e'_{S\cap N})}(N, c)$ is a normal subsystem of $\mathcal{F}_{(S,e'_S)}(G, N, c)$.*

Proof. Let (P, e_P) be a maximal (b, G)-Brauer pair. As $bc = c$ and $\mathrm{Br}^G_P(b)e_P = e_P$ there exists a central primitive idempotent $e'_P \in kC_N(P)$ such that $\mathrm{Br}^N_P(c)e'_P = e'_P$ and e_P covers e'_P, i.e. $e_Pe'_P \ne 0$. Let (S, e'_S) be a maximal (c, G)-Brauer pair containing (P, e'_P). Let (Q, e_Q) be a (b, G)-Brauer pair contained in (P, e_P) and (Q, e'_Q) be a (c, G)-Brauer pair contained in (S, e'_S). We prove that $e_Qe'_Q \ne 0$.

Consider a primitive idempotent decomposition of 1 in $(kN)^P$:

$$1 = j_1 + j_2 + \cdots + j_n$$

We have

$$1 = \mathrm{Br}^N_P(1) = \mathrm{Br}^N_P(j_1) + \mathrm{Br}^N_P(j_2) + \cdots + \mathrm{Br}^N_P(j_n) \subset kC_G(P)$$

and by mulpilying by $e_Pe'_P$ we obtain

$$e_Pe'_P = \mathrm{Br}^N_P(j_1)e_Pe'_P + \mathrm{Br}^N_P(j_2)e_Pe'_P + \cdots + \mathrm{Br}^N_P(j_n)e_Pe'_P\,.$$

Thus, given that $e_Pe'_P \ne 0$, there exists a primitive idempotent j in $(kN)^P$ such that $\mathrm{Br}^N_P(j)e_Pe'_P \ne 0$. Moreover, as Br^N_P is surjective we have that $\mathrm{Br}^N_P(j)$ is also primitive in $kC_N(P)$ so $\mathrm{Br}^N_P(j)e'_P = \mathrm{Br}^N_P(j)$. Consider now a primitive idempotent decomposition of j in $(kG)^P$:

$$j = i_1 + i_2 + \cdots + i_m\,.$$

As before we have

$$\mathrm{Br}^N_P(j) = \mathrm{Br}^N_P(i_1) + \mathrm{Br}^N_P(i_2) + \cdots + \mathrm{Br}^N_P(i_m)$$

giving that

$$0 \ne \mathrm{Br}^N_P(j)e_Pe'_P = \mathrm{Br}^G_P(j)e_Pe'_P = \mathrm{Br}^G_P(i_1)e_Pe'_P + \mathrm{Br}^G_P(i_2)e_Pe'_P + \cdots + \mathrm{Br}^G_P(i_m)e_Pe'_P\,.$$

Thus there exists a primitive idempotent i in $(kG)^P$ such that $ij = i$ and $\mathrm{Br}^G_P(i)e_Pe'_P \ne 0$. Again $\mathrm{Br}^G_P(i)$ is primitive in $kC_G(P)$ so $\mathrm{Br}^G_P(i)e_P = \mathrm{Br}^G_P(i)$.

By definition, every primitive idempotent $i \in (kG)^P$ such that $\mathrm{Br}^G_P(i)e_P \ne 0$ satisfies $\mathrm{Br}^G_Q(i)e_Q \ne 0$ and every primitive idempotent $j \in (kN)^P$ such that $\mathrm{Br}^N_P(j)e'_P \ne 0$ satisfies $\mathrm{Br}^N_Q(j)e'_Q \ne 0$. More precisely, we have

$$\mathrm{Br}^G_Q(i)\mathrm{Br}^N_Q(j)e_Qe'_Q = \mathrm{Br}^G_Q(i)e_Q\mathrm{Br}^N_Q(j)e'_Q = \mathrm{Br}^G_Q(i)\mathrm{Br}^N_Q(j) = \mathrm{Br}^G_Q(i)\mathrm{Br}^G_Q(j) = \mathrm{Br}^G_Q(i)$$

This proves the claim.

Consider the orbit $\mathcal{O} = \{{}^g e'_Q | g \in N_G(Q, e_Q)\}$ of e'_Q by conjugation with elements of $N_G(Q, e_Q)$. As $e_Q e'_Q \neq 0$ for any $g \in N_G(Q, e_Q)$ we have $e_Q\, {}^g e'_Q = {}^g(e_Q e'_Q) \neq 0$. Since $C_G(Q)$ acts transitively on the set of blocks f of $kC_N(Q)$ satisfying $e_Q f \neq 0$ we have that $C_G(Q)$ acts transitively on $\mathcal{O}$. We apply the Frattini argument to the transitive actions of $N_G(Q, e_Q)$ and $C_G(Q)$ on $\mathcal{O}$ and we have $N_G(Q, e_Q) = C_G(Q)(N_G(Q, e_Q) \cap N_G(Q, e'_Q))$. So $N_G(Q, e_Q)/C_G(Q) \simeq (N_G(Q, e_Q) \cap N_G(Q, e'_Q))/(C_G(Q) \cap N_G(Q, e'_Q))$ giving the inclusion of $\mathrm{Aut}_{\mathcal{F}_{(P,e_P)}(G,b)}(Q)$ into $\mathrm{Aut}_{\mathcal{F}_{(S,e'_S)}(G,N,c)}(Q)$. By Alperin's fusion theorem $\mathcal{F}_{(P,e_P)}(G, b)$ is isomorphic to a subsystem of $\mathcal{F}_{(S,e'_S)}(G, N, c)$. This proves the first assertion.

Now, since P is a defect group of the block b of kG, $P \cap N$ is a defect group of the block c of kN [NT, Chapter 5, Theorem 5.16 (iii)]. On the other hand, clearly $\mathrm{Br}^N_{S \cap N}(c) \neq 0$, hence $N \cap S$ is contained in a defect group of the block c of kN. Thus $S \cap N = P \cap N$. It follows that $(S \cap N, e'_{S \cap N})$ is a maximal (c, N)-Brauer pair and that $\mathcal{F}_{(S \cap N, e'_{S \cap N})}(N, c)$ is the subcategory of $\mathcal{F}_{(S,e'_S)}(G, N, c)$ on $S \cap N$ with morphisms induced from conjugation by elements of N (note that for a subgroup R of $S \cap N$ a (c, G)-Brauer pair with first component R is a (c, N)-Brauer pair with first component R and that for (c, G)-Brauer pairs (R', f') and (R, f) such that $R \leq S \cap N$, $(R', f') \leq (R, f)$ as (c, G)-Brauer pairs if and only if $(R', f') \leq (R, f)$ as (c, N)-Brauer pairs). Since N is a normal subgroup of G, it is easy to check that $\mathcal{F}_{(S \cap N, e'_{S \cap N})}(N, c)$ is a normal subsystem of $\mathcal{F}_{(S,e'_S)}(G, N, c)$.

□

4. Main Result

Definition 4.1. *Let G be a finite group, k an algebraically closed field of characteristic p, b a block of kG and $\mathcal{F}$ a fusion system on a finite p-group. We say that b is a $\mathcal{F}$-block if $\mathcal{F}_{P,e_P}(G, b)$ is isomorphic to $\mathcal{F}$ for some (and hence any) maximal b-Brauer pair (P, e_P).*

We say that a finite group is a p'-group if its order is not divisible by p.

Theorem 4.2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fusion systems on P, $\mathcal{F}_1$ containing $\mathcal{F}_2$. Suppose that:*
a) P has no non-trivial proper strongly $\mathcal{F}_2$-closed subgroup (and a fortiori no non-trivial strongly $\mathcal{F}_1$-closed subgroup),
b) if $\mathcal{F}$ is a fusion system on P containing $\mathcal{F}_2$, then $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$.
c) if $\mathcal{F}$ is a non-trivial fusion system normal in $\mathcal{F}_1$ or $\mathcal{F}_2$ then $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$.
If there exists a finite group G having an $\mathcal{F}_1$ or an $\mathcal{F}_2$-block then there exists a quasi-simple group L with $Z(L)$ a p'-group having an $\mathcal{F}_1$ or an $\mathcal{F}_2$-block.

Proof. Let G be a minimal order group having an $\mathcal{F}_1$ or an $\mathcal{F}_2$-block b. By a standard reduction (see for example [Ke, Proposition 2.11]), if N is a normal subgroup of G and c is a block of N with $bc \neq 0$, then c is G-stable.

By abuse of notation, P is a (b, G)-defect group. Let $H :=< {}^g P \,|\, g \in G >$ be the normal subgroup of G generated by all G-conjugates of P. Let d be the unique block of kH covered by b. Given that d is G-stable, G acts by conjugation on kHd. Let N be the kernel of the homomorphism $G \to \mathrm{Out}(kHd) = \mathrm{Aut}(kHd)/\mathrm{Inn}(kHd)$. Then by ([Kü3]) G/N is a p'-group. We prove, using the minimality of G and the hypothesis on $\mathcal{F}_1$ and $\mathcal{F}_2$, that $G = N$.

Let c be the block of N covered by b, i.e. $bc \neq 0$; (in this case in fact we have $c = b$). Let (P, e_P) be a maximal b- Brauer pair and let (S, e'_S) be a maximal (G, c) Brauer pair as in Theorem 3.5. Since G/N is a p'-group, it follows that $S = P$. Hence, we have $\mathcal{F}_{(P,e_P)}(G, b)$ is a subsystem of $\mathcal{F}_{(P,e'_P)}(G, N, c)$ and that $\mathcal{F}_{(P,e'_P)}(N, c)$ is a normal subsystem of $\mathcal{F}_{(P,e'_P)}(G, N, c)$.

Given that b is a $\mathcal{F}_1$- or $\mathcal{F}_2$-block and that $\mathcal{F}_1$ and $\mathcal{F}_2$ are the only fusion systems on P that contain $\mathcal{F}_2$ we obtain that $\mathcal{F}_{(P,e_P)}(G, N, c)$ is either $\mathcal{F}_1$ or $\mathcal{F}_2$. Again, since the only normal proper fusion subsystem on P contained in $\mathcal{F}_1$ is $\mathcal{F}_2$ and $\mathcal{F}_2$ has no normal fusion subsystem it follows that $\mathcal{F}_{(P,e'_P)}(N, c)$ is either $\mathcal{F}_1$ or $\mathcal{F}_2$. By the minimality of G we deduce that $G = N$.

As b and d have the same defect group P and G acts on kHd by inner automorphisms, using another result of Külshammer ([Kü2, Theorem 7]), we have that kGb and kHd have isomorphic source algebras, so c is also a $\mathcal{F}_1$ or $\mathcal{F}_2$ block. Thus, once again by the minimality of G we have $G = H$.

Let M be a proper normal subgroup of G. Then $P \cap M$ is a strongly $\mathcal{F}_1$ (or $\mathcal{F}_2$)-closed subgroup of P, hence $P \cap M = 1$ or $P \cap M = P$. Suppose first that $P \cap M = P$. Then P and all its G-conjugates lie in M. Thus $G = M$, which is a contradiction. Thus we are in the case $P \cap M = 1$. A variation of Fong reduction allows us to deduce that there is a central p'-extension G' of G/M having an $\mathcal{F}_1$ or $\mathcal{F}_2$-block (see for example [Ke, Section 3;3.3 and below]).

□

5. The Ruiz-Viruel Exotic Fusion Systems

In their paper [RV], Ruiz and Viruel classified all possible fusion systems on extra-special p-groups of order p^3. They showed that there are three exotic fusion systems on the extraspecial 7-group of order 7^3 and exponent 7. Let P be such a 7-group. A fusion system $\mathcal{F}$ on P is completely determined by $\mathrm{Out}_{\mathcal{F}}(P)$ and the set of $\mathcal{F}$-automorphisms of $\mathcal{F}$-centric, $\mathcal{F}$-radical proper subgroups of P. The three exotic systems of Ruiz and Viruel correspond to the following data. As in Ruiz and Viruel's tables we denote by $\#\mathcal{F}^{ec}$ the number of $\mathcal{F}$-centric, $\mathcal{F}$-radical proper subgroups of P. An entry of the form $a + b$ in the $\#\mathcal{F}^{ec}$ column indicates that there are two $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups of cardinality a and b respectively.

$\mathrm{Out}_{\mathcal{F}}(P)$	$\#\mathcal{F}^{ec}$	$\mathrm{Aut}_{\mathcal{F}}(V)$
$D_{16} \times 3$	$4 + 4$	$SL_2(7) : 2$, $SL_2(7) : 2$
$6^2 : 2$	$6 + 2$	$SL_2(7) : 2$, $GL_2(7)$
$SD_{32} \times 3$	8	$SL_2(7) : 2$

The categories on P generated by the above sets of morphisms satisfy the properties of fusion systems [RV2]. For the convenience of the reader, we give here a proof of this fact. Let $\mathcal{F}$ be one of the categories on P described above. The properties (1) and (2) are trivially satisfied. For the property (3), we have to study two types of $\mathcal{F}$-morphisms: those between the elementary abelian subgroups of rank 2 and those between the cyclic subgroups of order 7.

1) Take $\phi : Q \to R$ be an isomorphism in $\mathcal{F}$ where Q and R are elementary abelian subgroups of rank 2. Remark that R and Q are both fully $\mathcal{F}$-normalized. Suppose that $N_\phi = P$. By the construction of homomorphism in $\mathcal{F}$, the morphism ϕ decomposes into $\alpha\psi\beta$ where $\alpha \in N_{\mathcal{F}}(R)$, $\beta \in N_{\mathcal{F}}(Q)$ and ψ is the restriction of a

morphism in $\mathrm{Aut}_{\mathcal{F}}(P)$. In fact we can suppose that $\alpha = id$ as $\alpha\psi\beta = \psi\psi^{-1}\alpha\psi\beta$ and $\psi^{-1}\alpha\psi\beta \in N_{\mathcal{F}}(Q)$. So without loss of generality we suppose that $\phi = \psi\beta$. Now as $N_\phi = P$ we have that $N_\beta = P$. Indeed for any $x \in N_\phi$, by the definition there exists a $y \in N_P(R)$ such that $\phi({}^x u) ={}^y \phi(u)$. Take $z = \tilde{\psi}^{-1}(y)$ where $\tilde{\psi}$ is the extension of ψ to P. Then $\psi\beta({}^x u) ={}^y \psi\beta(u)$ implies that $\beta({}^x u) ={}^z \beta(u)$. By construction, all the morphisms in $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q))$ can be extended to $\mathrm{Aut}_{\mathcal{F}}(P)$. So there exists $\tilde{\beta} \in \mathrm{Aut}_{\mathcal{F}}(P)$ extending β. Now $\tilde{\psi}\tilde{\beta} \in \mathrm{Aut}_{\mathcal{F}}(P)$ extends $\psi\beta$ and we are done.

2) Take $\phi : Q \to R$ be a isomorphism in $\mathcal{F}$ where Q and R are cyclic subgroups of order 7 with R fully $\mathcal{F}$-normalized. As the cyclic subgroups of order p are all $\mathcal{F}$-conjugated we have that necessary $R = Z(P)$ as $Z(P)$ is the only cyclic subgroup of order 7 having its normalizer equal to P. Now if $Q = Z(P)$ we are done as any $\mathcal{F}$-automorphism of $Z(P)$ lifts to P. If $Q \neq Z(P)$, by construction ϕ lifts to $\tilde{\phi} : T \to U$ where T and U are elementary abelian subgroups of rank 2 containing Q, respectively R. But then $T = N_P(Q)$ so ϕ lifts to $N_P(Q)$ and we are done.

Proposition 5.1. *Let $P = 7^{1+2}_2$ be the extra-special group of order 7^3 and exponent 7 and let $\mathcal{F}$ be an exotic fusion system on P. If $\mathrm{Out}_{\mathcal{F}}(P) = D_{16} \times 3$, let $\mathcal{F}_2 = \mathcal{F}$ and let $\mathcal{F}_1$ be the exotic fusion system on P with $\mathrm{Out}_{\mathcal{F}}(P) = SD_{32} \times 3$. If $\mathrm{Out}_{\mathcal{F}}(P) = SD_{32} \times 3$, let $\mathcal{F}_1 = \mathcal{F}$ and let $\mathcal{F}_2$ be the exotic fusion system on P with $\mathrm{Out}_{\mathcal{F}}(P) = D_{16} \times 3$. If $\mathrm{Out}_{\mathcal{F}}(P) = 6^2 : 2$, set $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$. Then $\mathcal{F}_1$ contains $\mathcal{F}_2$. Furthermore, $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy conditions (a), (b) and (c) of Theorem 4.2.*

Proof. Let $\mathcal{F}$ be one of the above three fusion systems on P. The proper nontrivial sugroups of P are either elementary abelian of rank 2 or cyclic of order p. There are eight elementary abelian subgroups of rank 2 of P and they are all $\mathcal{F}$-centric, $\mathcal{F}$-radical. Moreover they are not unique in their $\mathcal{F}$-conjugacy class so they are not strongly $\mathcal{F}$-closed. Another fact is that each of the automorphism groups of the elementary abelian subgroups of rank 2 of P contains $SL_2(7)$ so the cyclic subgroups in any $\mathcal{F}$-centric, $\mathcal{F}$-radical subroup of P are transitively permuted by these automorphisms. Thus none of the cyclic subgroups of P are strongly $\mathcal{F}$-closed. This proves that the condition (a) of Theorem 4.2 is satisfied.

If $\mathcal{F}_1$ is the exotic fusion system on P with $\mathrm{Out}_{\mathcal{F}_1}(P) = SD_{32} \times 3$ and $\mathcal{F}_2$ is the exotic fusion system on P with $\mathrm{Out}_{\mathcal{F}_2}(P) = D_{16} \times 3$ then $\mathcal{F}_1$ contains $\mathcal{F}_2$ by construction as $\mathrm{Out}_{\mathcal{F}_2}(P)$ is a subgroup of $\mathrm{Out}_{\mathcal{F}_1}(P)$ and the $\mathcal{F}$-automorphisms of $\mathcal{F}$-centric, $\mathcal{F}$-radical proper subgroups of P are the same for $\mathcal{F} = \mathcal{F}_1$ and $\mathcal{F} = \mathcal{F}_2$. From the classification of Ruiz and Viruel there is no fusion system on P containing the exotic fusion system on P with $\mathrm{Out}_{\mathcal{F}}(P) = SD_{32} \times 3$ or the exotic fusion system on P with $\mathrm{Out}_{\mathcal{F}}(P) = 6^2 : 2$. So the condition (b) is satisfied.

Again let $\mathcal{F}$ be one of the above three exotic fusion systems on P. Suppose that $\mathcal{F}$ has a normal non-trivial subsystem $\mathcal{N}$ on a subgroup R of P. We have that R is strongly $\mathcal{F}$-closed, thus, given that $\mathcal{F}$ satisfies property (a), we have that $R = P$ so $\mathcal{N}$ is a fusion system on P. Our aim is to prove that $\mathcal{N}$ is one of the three exotic fusion systems on P. For this it is sufficient to show that $\mathcal{N}$ has also eight $\mathcal{N}$-centric $\mathcal{N}$-radical proper subgroups of P since this characterizes the exotic fusion systems by the classification of Ruiz and Viruel. Take Q to be an elementary abelian subgroups of rank 2 of P. As $C_p \simeq \mathrm{Aut}_P(Q) \leq \mathrm{Aut}_{\mathcal{N}}(Q) \lhd \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$ contains $\mathrm{SL}_2(7)$, we have that $\mathrm{Aut}_{\mathcal{N}}(Q)$ also contains $\mathrm{SL}_2(7)$ so Q is an $\mathcal{N}$-centric $\mathcal{N}$-radical subgroup. Now if we take $\mathcal{F}_1$ and $\mathcal{F}_2$ as in the proposition we see that $\mathcal{F}_2 \lhd \mathcal{F}_1$ and no other exotic fusion system on P is contained in $\mathcal{F}_1$ or in $\mathcal{F}_2$. Thus $\mathcal{F}_1$ and $\mathcal{F}_2$ also satisfy the condition (c) of Theorem 4.2

□

6. An application

As in sections 3 and 4, in this section k will denote an algebraically closed field of characteristic p. We will be using the following two well known results.

Lemma 6.1. *Let G be a finite group and N a normal subgroup of G. Let b be a block of kG and let D be a defect group of b. Then there exists a block c of kN such that c is D-stable, $cb \neq 0$, $D \cap N$ is a defect group of c and $\mathrm{Br}_D^H(c) \neq 0$ for any subgroup H of G containing ND.*

Proof. Let d be a block of kN such that $bd \neq 0$ and let I be the stabilizer in G of d. Then there is a block b_1 of kI such that $db_1 \neq 0$ and such that any defect group of b_1 is a defect group of b [NT, Chapter 5, Theorem 5.10]. Let $D_1 \leq I$ be a defect group of b_1 and of b. Then $\mathrm{Br}_{D_1}^I b_1 \neq 0$. Since I stabilizes d, d is the unique block of kN such that $db_1 \neq 0$, hence $db_1 = b_1$. It follows that $\mathrm{Br}_{D_1}^I(d) \neq 0$ and hence that $\mathrm{Br}_{D_1}^H(d) \neq 0$ for any subgroup H of G containing ND. Also, $D_1 \cap N$ is a defect group of d [NT, Chapter 5, Theorem 5.16 (ii)]. Now, $D = {}^gD_1$ for some $g \in G$. Set $c = {}^gd$. It is easy to check that c has all the desired properties.

□

Lemma 6.2. *Let $H = LD$ be a finite group such that L is normal in H and D is a p-group. Let c be a block of kL , stabilized by D. Suppose that $D \cap L$ is a defect group of c as a block of kL and that $\mathrm{Br}_D^H(c) \neq 0$. Let D' be a subgroup of D containing $D \cap L$. Then,*

(i) The idempotent c is a block of LD' and D' is a defect group of c as a block of LD'.

(ii) If the elements of D' induce inner automorphisms of L, then $D' = (D' \cap L)C_{D'}(L)$.

Proof. The fact that c is a block of LD' is immediate since D' is a p-group. By hypothesis, $\mathrm{Br}_D^H(c) \neq 0$, hence $\mathrm{Br}_{D'}(c) \neq 0$. Hence there is a p-subgroup, say D'', of LD' containing D' such that D'' is a defect group of c as a block of LD'. Now $D'' \cap L$ is a defect group of c as block of L [NT, Chapter 5, Theorem 5.16 (ii)], hence $|D'' \cap L| = |D \cap L| = |D' \cap L|$. On the other hand, $D''L/L$ is a a subgroup of $D'L/L$, proving (i).

Now suppose that the elements of D' induce inner automorphisms of L. Let $x \in D'$, and let $w_x \in L$ be a p-element such that ${}^{w_x}u = {}^xu$ for all $u \in L$. Then $w_x^{-1}x$ is a central p-element of $L < x >$. In particular, $w_x^{-1}x$ is contained in any defect group of any block of $L < x >$. On the other hand, by (i), c is a block of $L < x >$ with defect group $(D \cap L) < x >$. Since $(D \cap L) < x > \leq D'$, it follows that $w_x \in D' \cap L$ and $w_x^{-1}x \in C_{D'}(L)$. The result follows. □

Proposition 6.3. *Let $p \geq 7$ be prime. and let D be an extra-special p-group. Let G be a quasisimple finite group, and let $\bar{G}$ be the simple group $G/Z(G)$. Suppose that $\bar{G} = G(q)$ is a finite group of Lie type with $p \nmid q$. If D is a defect group of a block of G, then there exists an integer n, a power q' of q and a subgroup H of the finite general linear group $GL_n(q')$ (or $GU_n(q')$) with $H \geq SL_n(q')$ (or $SU_n(q')$), a block c of H and a defect group $\tilde{D}$ of c such that $\tilde{D}/ < \zeta >$ is extra-special of* *order $|D|$ for some cyclic subgroup $< \zeta >$ of $\tilde{D} \cap Z(H)$. Consequently, G has no blocks with defect groups extra-special of order p^3.*

Proof. Suppose that G has a block with defect group isomorphic to D. Then $\bar{G}$ has non-abelian Sylow p-subgroups which means in particular that the order of the Weyl group of the algebraic group corresponding to $\bar{G}$ is divisible by p ([GLS, Theorem 4.10.2]). Since $p \geq 7$, this means that exceptional part of the Schur multiplier of $\bar{G}$ is trivial ([GLS, Table 6.1.3]). Thus there is a simple simply connected algebraic group $\bar{K}$ over the algebraic closure of the field of q elements and a Frobenius morphism $F : \bar{K} \to \bar{K}$ such that $\bar{K}^F$ is a central extension of G. If $\bar{K}$ is of type A, then set $H := \bar{K}^F$ and let c be the unique block of H whose image under the algebra homomorphism $kH \to kG$ induced by the canonical surjection of H onto G. Then c clearly has the required properties and the first assertion holds.

Thus we may assume that $\bar{K}$ is not of type A. Since $p \geq 7$, the kernel of the surjection K^F is an p'-group. In particular, $\bar{K}^F$ has a block with defect group isomorphic to D. Thus, we may assume that $G = \bar{K}^F$.

Let $Z(D) =< z >$. By Brauer's first main theorem, the group $C_G(z)$ has a block, say b with defect group D. Since $p \geq 7$, p is good for $\bar{K}$. Thus, since $\bar{K}$ is simply connected, $C_{\bar{K}}(z)$ is a Levi subgroup of $\bar{K}$.

Let $\bar{Z}$ denote the connected center of $C_{\bar{K}}(z)$. Then

$$C_{\bar{K}}(z) = [C_{\bar{K}}(z), C_{\bar{K}}(z)]\bar{Z}.$$

Furthermore, $[C_{\bar{K}}(z), C_{\bar{K}}(z)]$ being simply connected ([GLS, Theorem 1.13.2]) is a direct product of its components, each of which is also simply connected and which are permuted by F. That is, we may write

$$[C_{\bar{K}}(z), C_{\bar{K}}(z)] = \prod_{i=1}^{t} \prod_{j=1}^{r_i} \bar{L}_{ij},$$

where each $\bar{L}_{ij}$ is a simply connected simple group, such that for each i, $1 \leq j \leq r_i$, the groups $\bar{L}_{ij}$, $1 \leq j \leq r_i$ are in a single orbit under the action of F.

Set $L_i := (\prod_{j=1}^{r_i} \bar{L}_{ij})^F$. Then L_i is the diagonal subgroup consisting of elements $\prod_{j=0}^{r_i-1} F^j(u)$ where $u \in \bar{L}_{1i}^{F^{r_i}}$. In particular, $L_i \cong \bar{L}_{i1}^{F^{r_i}}$. Furthermore,

$$C_G(z) \cong (L_1 \times \cdots \times L_t)T,$$

where T is an abelian group of order prime to q, inducing inner-diagonal automorphisms ([GLS, Definition 2.5.13]) on each L_i (T is the subgroup of F-fixed points of a F-stable maximal torus of $C_{\bar{K}}(z)$).

Since T is abelian, $D \cap (L_1 \times \cdots \times L_t) \neq 1$. On the other hand, $D \cap (L_1 \times \cdots \times L_t)$ is a defect group of a block of $L_1 \times \cdots \times L_t$ (see Lemma 6.1). But a defect group of a block of a direct product of groups is the direct product of defect groups of blocks of each factor. Thus, since $Z(D)$ is cyclic of prime order, we may assume that $Z(D) \leq L_1$ and that $D \cap (L_2 \times \cdots \times L_t) = 1$. Since $Z(D)$ is central in $C_G(z)$, it follows that each $\bar{L}_{1j}$ is of type A and of Lie rank at least p, hence that L_1 is isomorphic to $SL_n(q')$ or $SU_n(q')$ for some power q' of q.

Let x be a non-central element of D. We claim that x does not centralize L_1. Indeed, first note that if $\bar{L} = SL_n(\bar{\mathbb{F}}_q)$ and $\sigma : \bar{L} \to \bar{L}$ is a Frobenius endomorphism, then $\bar{L}^\sigma \cong SL_n(q')$ or $\bar{L}^\sigma \cong SU_n(q')$ for some power q' of q and $C_{\bar{L}}(\bar{L}^\sigma) \leq Z(\bar{L})$.

Write

$$x = (\prod_{i=1}^{t}\prod_{j=1}^{r_i} x_{ij})t_1,\ x_{ij} \in \bar{L}_{ij},\ t_1 \in \bar{Z}$$

Let y in D be such that $[x, y] = z$. Write

$$y = (\prod_{i=1}^{t}\prod_{j=1}^{r_i} y_{ij})t_2,\ y_{ij} \in \bar{L}_{ij},\ t_2 \in \bar{Z}.$$

Since $[x, y] = z \in L_1$, $[x_{11}, y_{11}] \neq 1$. In particular, x_{11} is not in the center of $\bar{L}_{11}$ and by the remark above, x_{11} does not centralize $L_{11}^{F^{r_i}}$. It follows that x does not centralize L_1.

Let c be a block of kL_1 such that $bc \neq 0$, c is D-stable, $\mathrm{Br}_D^{L_1 D}(c) \neq 0$ and such that $D \cap L_1$ is a defect group of c (see Lemma 6.1). Let D_0 be the kernel of the map $D \to \mathrm{Out}(L_1)$. Then, $< z >= Z(D) \leq (D \cap L_1) \leq D_0$. Thus, by Lemma 6.2 applied to the group $L_1 D$, the block c of kL_1 and the subgroup D_0 of D, we have that $D_0 = (D_0 \cap L_1)C_{D_0}(L_1)$. But it was shown above that $C_D(L_1) = Z(D) \leq D \cap L_1$. Hence $D_0 \leq L_1$.

If $D_0 = D$, then the first assertion of the proposition holds with $H = L_1$, $\tilde{D} = D$ and $< \zeta >= 1$ for the block c. We assume from now on that $D \neq D_0$. The elements of T and hence of $C_G(z)$ induce inner diagonal automorphisms of L_1. Since L_1 is isomorphic to a special linear or special unitary group, $Inndiag(L_1)/Inn(L_1)$ is cyclic ([GLS, Section 2.7]). In particular, D/D_0 is cyclic. But since $Z(D) \leq D_0$ and D is extra-special, in fact $|D/D_0| = p$. Let $y \in D$ be such that $D/D_0 =< yD_0 >$ and let η be a p-element in $GL_n(q')$ (or $GU_n(q')$) such that ${}^{\eta}u = {}^{y}u$ for all $u \in L_1$. In particular, c is stabilized by $< \eta >$. Let $H = L_1 < \eta >$. Then H is a subgroup of $GL_n(q')$ (or $GU_n(q')$) containing $SL_n(q')$ (or $SU_n(q')$). Let $\tilde{D}$ be the subgroup of H generated by D_0 and η. Then $H = L_1\tilde{D}$ and c is an H-stable block of L_1. Also, since $C_D(L_1) = C_{\tilde{D}}(L_1)$, we have that $\mathrm{Br}_{\tilde{D}}^{H}(c) \neq 0$. Finally, $D_0 = \tilde{D} \cap L_1$ is a defect group of the block c of kH. Thus, by Lemma 6.2, applied with $H = L_1\tilde{D}$, the block c is of kL_1 and the subgroup $\tilde{D}$ of $\tilde{D}$, we have that c is a block of kH with $\tilde{D}$ as defect group.

Now $y^p \in Z(D) \leq Z(L_1)$, hence η^p centralizes L_1. Thus, η^p is a central element of $GL_n(q')$ (or $GU_n(q')$). It follows that $< \eta^p > \cap D_0 \leq Z(D) =< z >$. If $z \in< \eta >$, then η has order at least p^2 and we set $< \zeta >$ to be the subgroup of $< \eta >$ of index p^2. If $z \notin< \eta >$, then we set $< \zeta >$ to be the subgroup of $< \eta >$ of index p. Then it is easy to check that $\tilde{D}/ < \zeta >$ is extra-special of order $|D|$. Since $< \zeta >\leq< \eta^p >$, $< \zeta >$ is a central subgroup of H. This proves the first part of the proposition.

Now suppose that G has a block with a defect group D which is extra-special of order p^3. Let H, c and $\tilde{D}$ be as in the the first assertion of the proposition. Suppose first that $H \leq GL_n(q')$ and let p^a be the exact power of p dividing $q' - 1$. Then since $SL_n(q') \leq H$, it follows that $|\tilde{D}| \leq p^{3+a}$ and that there is a block of $kGL_n(q')$ covering c, with non-abelian defect groups of order at most p^{2a+3}. The structure of defect groups of finite general linear and unitary groups groups is well known. In particular, non-abelian defect groups of $GL_n(q')$ have order at least p^{pa+1} ([FS, Theorem 3C]). So, $p^{pa+1} \leq p^{2a+3}$, which is impossible since $p > 3$. A similar argument, taking p^a to be the exact power of p dividing $q' + 1$ handles the case $H \leq GU_n(q')$.

□

We now state and prove the main theorem of this section.

Theorem 6.4. *Let $\mathcal{F}$ be an exotic fusion system on the extra-special group P of order 7^3. Then $\mathcal{F}$ is not a fusion system of a 7 block of any finite group.*

Proof. Let G be a finite group with an $\mathcal{F}$-block, say b. If $\mathrm{Out}_{\mathcal{F}}(P) = D_{16} \times 3$, let $\mathcal{F}_2 = \mathcal{F}$ and let $\mathcal{F}_1$ be the exotic fusion system on P with $\mathrm{Out}_{\mathcal{F}}(P) = SD_{32} \times 3$. If $\mathrm{Out}_{\mathcal{F}}(P) = SD_{32} \times 3$, let $\mathcal{F}_1 = \mathcal{F}$ and let $\mathcal{F}_2$ be the exotic fusion system on P with $\mathrm{Out}_{\mathcal{F}}(P) = D_{16} \times 3$. If $\mathrm{Out}_{\mathcal{F}}(P) = 6^2 : 2$, set $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$. Then, by Proposition 5.1 and Theorem 4.2, we may assume that G is quasisimple, $7 \nmid |Z(G)|$ and that b is an $\mathcal{F}_1-$ or $\mathcal{F}_2-$ block. Also, by Proposition 5.1, we may assume that P is not Sylow in G since neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ is contained in a non-exotic fusion system on P.

Let $\bar{G}$ be the simple quotient of G. By the previous proposition, $\bar{G}$ is not a finite field of Lie type in characteristic different from 7. Suppose that $\bar{G}$ is a finite field of Lie type in characteristic 7. Then the exceptional part of the Schur multiplier of $\bar{G}$ is trivial, ([GLS, Table 6.1.3.]). Thus there is a central $7'$ extension $\tilde{G}$ of G such that $\tilde{G} = \bar{K}^F$ where $\bar{K}$ is a simply connected simple algebraic group and F is a Frobenius endomorphism of $\bar{K}$. Then it follows from the theory of finite groups with strongly split BN pair [CE, Theorem 6.18], (see also [Ke, Lemma 5.1]) that the defect groups of a 7 block of $\tilde{G}$ are either trivial or Sylow 7-subgroups of $\tilde{G}$. Hence, P is a Sylow 7-subgroup of G, a contradiction.

For odd p, a defect group of a p-block of a finite alternating group or a double cover of a finite alternating group is isomorphic to the Sylow p-subgroups of a finite symmetric group, hence $\bar{G}$ is not an alternating group.

Now, if $\bar{G}$ is a sporadic group then $\bar{G}$ must be one of He, $O'N$, Fi'_{24} and the monster F_1 as these are the only sporadic groups whose order is divisible by 7^3. Furthermore, if $\bar{G}$ is one of He, $O'N$, Fi'_{24}, then 7^3 is the exact power of 7 dividing $|G|$. Hence, P is a Sylow 7-subgroup of G, a contradiction.

Finally, suppose that $G = \bar{G} = F_1$. Thus G has two conjugacy classes of elements of order 7 denoted by $7A$ and $7B$(ATLAS notation). As in the ATLAS we denote by $7A^2$ and $7B^2$ the abelian elementary 7-groups of rank 2 generated by elements in $7A$, respectively $7B$. Then the maximal 7-local subgroups of $\bar{G}$ are of the type $T_1 = (7 : 3 \times He) : 2$, normalizer of an element in $7A$, $S_1 = (7^2 : (3 \times 2S_7) \times L_2(7)).2$ normalizer of a group of type $7A^2$, $T_2 = 7_+^{1+4} : (3 \times 2S_7)$, normalizer of an element in $7B$ and $S_2 = 7^2.7.7^2 : GL_2(7)$ normalizer of a group of type $7B^2$.

The cyclic subgroups of order 7 of P are all conjugate in $\mathcal{F}$ as they are in the conjugacy class of the centre of P given by the automorphisms of the elementary abelian subgroups of rank 2 of P. Thus the elements of order 7 of P are in the same $\mathcal{F}$-conjugacy class. Also $\mathrm{Aut}_{\mathcal{F}}(P)$ normalizes the centre of P. So we have that $\mathrm{Aut}_{\mathcal{F}}(P)$ is a section of T_i, for $i = 1$ or 2. Moreover, $\mathrm{Aut}_{\mathcal{F}}(V)$ is a section of S_i for the same index i as above (where V is an $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup of P). But this is not possible as S_1 has no section containing $SL_2(7) : 2$ and T_2 has no section containing $D_{16} \times 3$ or $6^2 : 2$.

□

7. Appendix

We prove in the appendix that the definition we give in this paper for fusion systems is equivalent to the definition of Broto, Levi and Oliver [BLO] for saturated fusion systems. Here is their approach.

Definition 7.1. *A fusion system on a finite p-group P is a category whose objects are the subgroups in P and whose morphism set $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following*

(1) $\mathrm{Hom}_P(Q,R) \subset \mathrm{Hom}_{\mathcal{F}}(Q,R) \subset \mathrm{Inj}(Q,R)$ *for all* $Q, R \leq P$.

(2) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

The definition Broto, Levi and Oliver give for a fully $\mathcal{F}$-centralized subgroup is the same as ours. But the definition for a fully $\mathcal{F}$-normalized subgroup is different.

Definition 7.2. *A subgroup Q of P is fully $\mathcal{F}$-centralized if $|C_P(Q)| \geq |C_P(Q')|$ for all $Q' \leq P$ which is $\mathcal{F}$-conjugated to Q.*

Definition 7.3. *A subgroup Q of P is fully $\mathcal{F}$-normalized if Q is fully $\mathcal{F}$-centralized and if* $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.

Now the definition of saturated fusion systems.

Definition 7.4. *$\mathcal{F}$ is a saturated fusion system if the two following conditions hold:*

(i) Each subgroup $Q \leq P$ is $\mathcal{F}$-conjugated to at least one fully $\mathcal{F}$-normalized subgroup.

(ii) If $Q \leq P$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ are such that $\phi(Q)$ is fully $\mathcal{F}$-centralized and one set $N_\phi := \{g \in N_P(Q) | \phi c_g \phi^{-1} \in \mathrm{Aut}_P(\phi(Q))\}$, then there is $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, P)$ such that $\bar{\phi}|_P = \phi$.

It is obvious that a saturated fusion system in Broto, Levi and Oliver approach satisfies the three properties in Definition 2.4 we give for fusion systems. We prove now that our definition implies the one in [BLO].

We start by proving that in a fusion system (Definition 2.4) the definition for a fully $\mathcal{F}$-normalized subgroup (Definition 7.3) as in [BLO] is obtained as a property from our setting.

Proposition 7.5. *Let $\mathcal{F}$ be a fusion system (Definition 2.4) on a finite p-group P and let Q be a subgroup of P. Then Q is fully $\mathcal{F}$-normalized (Definition 2.2) if and only if Q is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.*

Proof. First, a fully $\mathcal{F}$-normalized subgroup Q of P is also fully $\mathcal{F}$-centralized as for any $\mathcal{F}$-isomorphic subgroup Q' we have that the morphism $\phi : Q' \to Q$ extends to a morphism $\bar{\phi} : N_\phi \to N_P(Q)$. But $C_P(Q') \subset N_\phi$ and $\bar{\phi}(C_P(Q')) \subset C_P(Q)$ giving that $|C_P(Q')| \leq |C_P(Q)|$. So Q is fully $\mathcal{F}$-centralized.

Second, if Q is fully $\mathcal{F}$-normalized then $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. To prove this, we will follow the proof of Proposition 1.5 in [Li]. Let Q be a subgroup of maximal order such that $\mathrm{Aut}_P(Q)$ is not a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Then Q is a proper subgroup of P by property (2). Choose a p-subgroup S of $\mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\mathrm{Aut}_P(Q)$ is a proper normal subgroup of S. Let $\phi \in S \setminus \mathrm{Aut}_P(Q)$. Since ϕ normalizes $\mathrm{Aut}_P(Q)$, for every $y \in N_P(Q)$ there is $z \in N_P(Q)$ such that $\phi({}^y u) = {}^z\phi(u)$ for all $u \in Q$. Thus we have $N_\phi = N_P(Q)$. Since Q is fully $\mathcal{F}$-normalized, by property (3) ϕ extends to $\bar{\phi} : N_\phi \to N_P(Q)$ so $\bar{\phi} \in \mathrm{Aut}_{\mathcal{F}}(N_P(Q))$. Since ϕ has p-power order, by decomposing $\bar{\phi}$ into its p-part and its p'-part we may assume that $\bar{\phi}$ has also p-power order. Let $\psi : N_P(Q) \to P$ be a morphism such that $N' := \psi(N_P(Q))$ is fully $\mathcal{F}$-normalized. As the order on N' is greater then the order of Q, we have that $\mathrm{Aut}_P(N')$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(N')$. Now $\psi\bar{\phi}\psi^{-1}$ is a p-element of $\mathrm{Aut}_{\mathcal{F}}(N')$, thus conjugated to an element in $\mathrm{Aut}_P(N')$. Therefore we may choose ψ in such a way that there is $y \in N_P(N')$ satisfying $\psi\bar{\phi}\psi^{-1}(v) = {}^{y}v$ for all $v \in N'$. Since $\bar{\phi}|_Q = \phi$, the automorphism $\psi\bar{\phi}\psi^{-1}$ of N' stabilizes $\psi(Q)$. Thus $y \in N_P(\psi(Q))$. Since Q is fully $\mathcal{F}$-normalized and $\psi(N_P(Q)) \subset N_P(\psi(Q))$ we have that $\psi(N_P(Q)) = N_P(\psi(Q))$, hence $\bar{\phi}(u) = {}^{\tau^{-1}(y)}u$, for all $u \in N_P(Q)$. And, in particular, $\phi \in \mathrm{Aut}_P(Q)$, contradicting our first choice of ϕ.

The converse is straight forward as $|N_P(Q)| = |\mathrm{Aut}_P(Q)| \cdot |C_P(Q)|$.

□

The following proposition gives the last ingredient for the equivalence of the two approaches. In our approach, property (3) guarantees the extension to N_ϕ for the $\mathcal{F}$-isomorphisms ϕ ending in fully $\mathcal{F}$-normalized subgroups. But this is sufficient in order to have the extension to N_ϕ for all the $\mathcal{F}$-isomorphisms ending in fully $\mathcal{F}$-centralized subgroups.

Proposition 7.6. *Every $\phi : Q \to P$ such that $\phi(Q)$ is fully $\mathcal{F}$-centralized extends to a morphism $\bar{\phi} : N_\phi \to P$*

Proof. We note $Q' := \phi(Q)$. Choose $\theta : Q' \to P$ such that $\theta(Q')$ is fully $\mathcal{F}$-normalized and, as $\mathrm{Aut}_P(\theta(Q'))$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\theta(Q'))$ we can modify θ by a morphism in $\mathrm{Aut}_{\mathcal{F}}(\theta(Q'))$ and suppose that $N_\theta = N_P(Q')$.

By the property (3) we have that θ extends to $\bar{\theta} : N_\theta \to P$. Note $\psi := \theta\phi$. By the same property (3) ψ extends to $\bar{\psi} : N_\psi \to P$.

Our aim in what follows is to prove that $N_\phi \subset N_\psi$ and $\bar{\psi}(N_\phi) \subset \bar{\theta}(N_\theta)$ so that $(\bar{\theta})^{-1}\bar{\psi}|_{N_\phi}$ would be the extension of ϕ to N_ϕ.

Both are simple verifications. Take $y \in N_\phi$ then by definition, there exists $z \in N_P(Q')$ such that $\phi({}^{y}u) = {}^{z}\phi(u)$ for all $u \in Q$. By composing with θ we obtain $\theta\phi({}^{y}u) = \theta({}^{z}\phi(u))$. But as $N_\theta = N_P(Q')$ we have that there exists $x \in N_P(\theta(Q'))$ such that $\theta({}^{z}\phi(u)) = {}^{x}\theta(\phi(u)) = {}^{x}\psi(u)$. By resuming, we have $\psi({}^{y}u) = {}^{x}\psi(u)$ which means that $y \in N_\psi$. As this is true for all $y \in N_\phi$ we obtain that $N_\phi \subset N_\psi$.

Take now $x \in \bar{\psi}(N_\phi)$. Suppose that $x = \bar{\psi}(y)$, $y \in N_\phi$. By definition, there exists $z \in N_P(Q')$ such that $\phi({}^{y}u) = {}^{z}\phi(u)$ for all $u \in Q$. We obtain $\psi({}^{y}u) = {}^{x}\psi(u)$, so $\theta({}^{z}\phi(u)) = {}^{x}\theta(\phi(u))$, which is equivalent to ${}^{\bar{\theta}(z)}\psi(u) = {}^{x}\theta(\phi(u))$ for all $u \in Q$. This gives that $x = \bar{\theta}(z)c$ with $c \in C_P(\theta(Q))$. But as $C_P(Q') \subset N_\theta$ and $\bar{\theta}(C_P(Q')) \subset C_P(\theta(Q'))$ and using the fact that Q' is fully $\mathcal{F}$-centralized we have that $\bar{\theta}(C_P(Q')) = C_P(\theta(Q'))$. This means that $c \in \bar{\theta}(N_\theta)$, so $x \in \bar{\theta}(N_\theta)$. Now this is true for all $x \in \bar{\psi}(N_\phi)$ so $\bar{\psi}(N_\phi) \subset \bar{\theta}(N_\theta)$.

Thus we showed that $(\bar{\theta})^{-1} \circ \bar{\psi}|_{N_\phi}$ extends ϕ to N_ϕ.

□

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