Linckelmann, M. (2009). On dimensions of block algebras. Mathematical Research Letters, 16(6), 1011 - 1014.

Original citation: Linckelmann, M. (2009). On dimensions of block algebras. Mathematical Research Letters, 16(6), 1011 - 1014.

On dimensions of block algebras

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February 19, 2010

Abstract

Following a question by B. Külshammer, we show that an inequality, due to Brauer, involving the dimension of a block algebra, has an analogue for source algebras, and use this to show that a certain case where this inequality is an equality can be characterised in terms of the structure of the source algebra, generalising a similar result on blocks of minimal dimensions.

Let p be a prime and k an algebraically closed field of characteristic p. Let G be a finite group and B a block algebra of kG; that is, B is an indecomposable direct factor of kG as k-algebra. By a result of Brauer in [2], the dimension of B satisfies the inequality

$$\dim_k(B) \geq p^{2a-d} \cdot \ell(B) \cdot u_B^2$$

where p^a is the order of a Sylow-p-subgroup of G, p^d is the order of a defect group of B, $\ell(B)$ is the number of isomorphism classes of simple B-modules and u_B is the unique positive integer such that $p^{a-d} \cdot u_B$ is the greatest common divisor of the dimensions of the simple B-modules. It is well-known that u_B is prime to p. Külshammer raised the question whether an equality could be expressed in terms of the structure of a source algebra of B, generalising the result in [3] on blocks of minimal dimension. We show that this is the case. The first observation is an analogue for source algebras of Brauer's inequality. We keep the notation above and refer to [5] for block theoretic background material.

Theorem 1. *Let A be a source algebra of B. Then $\dim_k(A) \geq p^d \cdot \ell(B) \cdot u_A^2$, where u_A is the greatest common divisor of the dimensions of the simple A-modules.*

Proof. One can prove this by adapting Brauer's proof of [2, Theorem 1]. Alternatively, one can use the theorem of Wedderburn-Malcev, according to which A contains a unitary subalgebra isomorphic to $A/J(A)$, where $J(A)$ denotes the Jacobson radical of A. Every indecomposable factor of $A/J(A)$ is a matrix algebra of dimension n^2 for some integer n which is divisible by u_A, hence of the form $T \otimes_k S$ for some matrix algebra T of dimension u_A^2 and some matrix algebra S of dimension $\frac{n^2}{u_A^2}$. Thus A contains T as unitary subalgebra, hence $A \cong T \otimes_k A'$, where A' is the centraliser of T in A. Note that B, A, A' are Morita equivalent; in particular $\ell(B) = \ell(A) = \ell(A')$. Since A is projective as module over a defect group, every projective indecomposable A-module has dimension divisible by p^d. By [5, Corollary (44.9)], the integer u_A is prime to p. Thus every projective A'-module has still dimension divisible by p^d. Since A' is a direct sum, as left A'-module, of at least $\ell(A')$ indecomposable direct summands, it follows that $\dim_k(A') \geq p^d \cdot \ell(B)$, hence $\dim_k(A) = \dim_k(A') \cdot u_A^2 \geq p^d \cdot \ell(B) \cdot u_A^2$ as claimed. $\square$

An equality $\dim_k(A) = p^d \cdot \ell(B) \cdot u_A^2$ does not seem to imply strong structural conditions on A; for instance, if $\ell(B) = 1$ then $\dim_k(A) = p^d \cdot u_A^2$. Indeed, if $\ell(B) = 1$ then u_A is the dimension of the unique (up to isomorphism) simple A-module, and hence the algebra A' arising in the proof of Theorem 1 is a basic algebra, hence local, and thus its dimension is the unique Cartan invariant p^d of B. If one requires $u_A = 1$, one gets the following structural characterisation:

Theorem 2. *Let A be a source algebra of B with defect group P. The following are equivalent.*
(i) $\dim_k(A) = p^d \cdot \ell(B)$.
(ii) $A \cong k(P \rtimes E)$ for some abelian p'-subgroup E of $\mathrm{Aut}(P)$.

Proof. Suppose that (i) holds. Let J be a primitive decomposition of 1_A in A; that is, J is a set of primitive pairwise orthogonal idempotents in A such that $1_A = \sum_{j\in J} j$. Then $A = \oplus_{j\in J} Aj$ as left A-module, and Aj is projective as kP-module for each $j \in J$. Thus $\dim_k(A) = p^d \sum_{j\in J} \frac{\dim_k(Aj)}{p^d}$. The equality in (i) is therefore equivalent to $|J| = \ell(B)$ and $\dim_k(Aj) = p^d$ for all $j \in J$. Thus each point (cf. [5, §4]) of A has multiplicity 1, or equivalently, A is a basic k-algebra, hence each simple A-module has dimension 1, and each Aj restricted to kP is isomorphic to the regular module kP. Thus the radical of Aj as A-module is equal to the radical of Aj as kP-module, hence $J(kP)A = J(A)$. The same argument yields $J(A) = AJ(kP)$. By a result of Puig (cf. [4, 14.6] or [5, Theorem (44.3)] or also [1] for another proof), as kP-kP-bimodule, A is isomorphic to a direct sum of $k(P \rtimes E)$ for some p'-subgroup E of $\mathrm{Aut}(P)$ and indecomposable direct summands of the form $kP \otimes_{kQ}\, {}_{\varphi}kP$, where Q is a proper subgroup of P and $\varphi : Q \to P$ is an injective group homomorphism. The equality $J(kP)A = AJ(kP)$ forces that there is no summand of that form. To see this, note first that $J(kP)A$ is a kP-kP-submodule of A, thus so is the quotient $A/J(kP)A$. The elements of P act as identity on the left of $A/J(kP)A$, hence also on the right, but the elements in P outside of $\varphi(Q)$ do not act as identity on the right side of the kP-kP-bimodule $(kP \otimes_{kQ}\, {}_{\varphi}kP)/J(kP)(kP \otimes_{kQ}\, {}_{\varphi}kP)) \cong k \otimes_{kQ}\, {}_{\varphi}kP$. This implies that $A = k(P \rtimes E)$ as kP-kP-bimodule. But then, by [4, 14.6] again (or the alternative references [1, Theorem 1], [5, Theorem (45.11)]), A is isomorphic, as interior P-algebra (cf. [5, §10]), to a twisted group algebra $k_\alpha(P \rtimes E)$, for some $\alpha \in H^2(E; k^\times)$, inflated trivially to $P \rtimes E$. The fact that A has a simple module of dimension 1 implies that there is an algebra homomorphism $\epsilon : k_\alpha E \to k$. If x, $y \in E$ their product in $k_\alpha E$ is $x \cdot y = \alpha(x, y)xy$, where xy is the product of x, y in E. Thus $x \cdot y$ is mapped under ϵ to $\epsilon(x)\epsilon(y) = \alpha(x, y)\epsilon(xy)$, proving that α is a 2-coboundary, or equivalently, that its class is zero. But then, since every simple A-module has dimension 1 this also forces E to be abelian. Thus (i) implies (ii). The converse is easy. □

A block B satisfying the equivalent conditions of Theorem 2 is splendidly Morita equivalent to its Brauer correspondent, hence satisfies in particular all relevant conjectures by Alperin, Broué (for P abelian), Dade and Robinson. For S a subgroup of G we denote as usual by $(kG)^S$ the subalgebra of kG consisting of all elements in kG which are invariant under conjugation by elements in S.

Theorem 3. *Let A be a source algebra of B. We have* $\dim_k(B) \geq p^{2a-2d}\dim_k(A)$*, with equality if and only if the block idempotent 1_B remains primitive in $(kG)^S$ for any Sylow-p-subgroup S of G.*

Proof. Choose a Sylow-p-subgroup S of G and a defect group P of B such that $P \leq S$. Denote by $\mathrm{Br}_P : (kG)^P \to kC_G(P)$ the Brauer homomorphism (cf. [5, §11]). Since $\mathrm{Br}_P(1_B) \neq 0$ there is a primitive idempotent $e \in B^S$ such that $\mathrm{Br}_P(e) \neq 0$. Thus there is a primitive idempotent $i \in (eBe)^P$ such that $\mathrm{Br}_P(i) \neq 0$. Then e belongs to a point σ of S on B and i belongs to a local point γ of P on B such that $P_\gamma \leq S_\sigma$ (cf. [5, §13, 14] for background material on pointed groups, inclusion between pointed groups, and local pointed groups). Since P is maximal with the property $\mathrm{Br}_P(1_B) \neq 0$ it follows that P_γ is a defect pointed group of S_σ, hence $\sigma \subseteq \mathrm{Tr}_P^S(A^P\gamma A^P)$ (cf. [5, Proposition (18.5)]). Thus, by Higman's criterion (cf. [5, Corollary (17.3)]), as $k(G \times S)$-module, kGe is relatively $G \times P$-projective, and Green's indecomposablity theorem [5, Corollary (23.5)] implies therefore that $kGe \cong kGi \otimes_{kP} kS$ as $k(G \times S)$-modules. The k-dual of $kGi \otimes_{kP} kS$ is isomorphic to the $k(S \times G)$-module $kS \otimes_{kP} ikG$. Tensoring these two modules over kG yields an isomorphism of kS-kS-bimodules $eBe \cong kS \otimes_{kP} ikGi \otimes_{kP} kS$. Since $\mathrm{Br}_P(i) \neq 0$, the algebra $ikGi = iBi$ is a source algebra of B; in particular, $\dim_k(ikGi) = \dim_k(A)$. Clearly $\dim_k(kS \otimes_{kP} ikGi \otimes_{kP} kS) = p^{2a-2d}\dim_k(A)$ and $\dim_k(B) \geq \dim_k(eBe)$. This shows the inequality, and also shows that the equality holds if and only if $1_B = e$ is primitive in $(kG)^S$. Since 1_B is a central idempotent, 1_B is then primitive in $(kG)^{S'}$ for any Sylow-p-subgroup S' of G, whence the second statement. $\square$

The result in [3] on blocks of minimal dimension follows easily from the above:

Corollary 4 ([3, Theorem]). *If* $\dim_k(B) = p^{2a-d}$ *then* B *is a nilpotent block with source algebra* kP*, where* P *is a defect group of* B*.*

Proof. Combining the above Theorem 3 and Theorem 1 yields in particular

$$\dim_k(B) \geq p^{2a-2d}\dim_k(A) \geq p^{2a-d}\ell(B)$$

Thus the equality $\dim_k(B) = p^{2a-d}$ forces $\ell(B) = 1$ and $\dim_k(A) = p^d$. Since by Theorem 2 also $A \cong k(P \rtimes E)$ for some abelian p'-group E, this implies $E = 1$, hence kP is a source algebra of B, and in particular, B is nilpotent (cf. [5, Remark (50.10)]). $\square$

As pointed out by the referee, combining Theorem 1 and Theorem 3 yields the inequality $\dim_k(B) \geq p^{2a-d} \cdot \ell(B) \cdot u_A^2$; that is, Brauer's inequality with u_B replaced by u_A. Since u_A can be smaller than u_B this does not imply Brauer's original result. This is due to the fact that the inequality in Theorem 3 does not take into account the multiplicity of a local point of P on B. For this reason there is no obvious connection between u_A and u_B. Examples where $u_A = 1$ and u_B is an arbitrary p'-integer arise from blocks of $H \times P$, where H is a finite p'-group having an ordinary irreducible character of degree u_B; in such a situation, u_B is precisely the multiplicity of the unique point of P on the block algebra under consideration.

Acknowledgement. The author would like to thank the referee for many valuable comments.

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