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THE GRADED CENTER OF THE STABLE CATEGORY OF A BRAUER TREE ALGEBRA

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Abstract. We calculate the graded center of the stable category of a Brauer tree algebra. The canonical map from the Tate analogue of Hochschild cohomology to the graded center of the stable category is shown to induce an isomorphism module taking quotients by suitable nilpotent ideals. More precisely, we show that this map is surjective with nilpotent kernel in even degrees, while this map need not be surjective in odd degrees in general.

1 Introduction

The graded center of a triangulated category $(\mathcal{C}, \Sigma)$ is the graded ring $Z^*(\mathcal{C})$, which in degree $n \in \mathbb{Z}$, consists of all natural transformations $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ satisfying $\Sigma\varphi = (-1)^n \varphi\Sigma$. This condition implies that $Z^*(\mathcal{C})$ is graded commutative; see e.g. [14] for more details. Examples of triangulated categories include derived module categories and stable module categories of self-injective algebras. The background motivation, from block theory, for calculating graded centers is that comparing the graded center of the stable module category of a block algebra to its block cohomology may shed some light on the rather mysterious connections between the stable module category of the block and its local structure. If A is a self-injective algebra over a field k there is a canonical map from the Tate analogue $\widehat{HH}^*(A)$ of Hochschild cohomology to the graded center $Z^*(\overline{\mathrm{mod}}(A))$ of the stable category $\overline{\mathrm{mod}}(A)$ of finitely generated A-modules. For Brauer tree algebras we get the following result:

Theorem 1.1. *Let A be a Brauer tree algebra over a field k. The canonical homomorphism of graded k-algebras $\widehat{HH}^*(A) \longrightarrow Z^*(\overline{\mathrm{mod}}(A))$ induces an isomorphism modulo nilpotent elements and the graded algebra $Z^*(\overline{\mathrm{mod}}(A))$ is commutative.*

If A is a block with cyclic defect groups then A is a Brauer tree algebra, and the above theorem combined with [20, Theorem 1] or [17, 3.5] or [9, 1.4] shows that the nonnegative part of $Z^*(\overline{\mathrm{mod}}(A))$ and the block cohomology of A are isomorphic modulo nilpotent ideals. In even degree, the graded center of the stable category of a Brauer tree algebra is in fact a quotient of the Tate analogue of Hochschild cohomology by a nilpotent ideal:

Theorem 1.2. *Let A be a Brauer tree algebra over a field k with exceptional multiplicity s and e isomorphism classes of simple A-modules. The canonical homomorphism of graded k-algebras $\widehat{HH}^{ev}(A) \longrightarrow Z^{ev}(\overline{\mathrm{mod}}(A))$ is surjective and its kernel is a nilpotent ideal. Moreover, for any integer d we have*

$$\dim_k(Z^{2d}(\overline{\mathrm{mod}}(A))) = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even} \\ \frac{s+1}{2} & \text{if } s \text{ is odd and } 0 \le r < \frac{e}{2} \\ \frac{s-1}{2} & \text{if } s \text{ is odd and } \frac{e}{2} \le r < e \end{cases}$$

where r is the unique integer such that $0 \le r < e$ and $r \equiv d \pmod{e}$.

In odd degree, the dimensions are as follows:

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Theorem 1.3. *Let A be a Brauer tree algebra over a field k with exceptional multiplicity s and e isomorphism classes of simple A-modules. For any integer d we have*

$$\dim_k(Z^{2d-1}(\overline{\mathrm{mod}}(A))) = \begin{cases} 0 & \text{if } e \text{ does not divide } d \\ \frac{es}{2} & \text{if } es \text{ is even and } e \text{ divides } d \\ \frac{es-1}{2} & \text{if } es \text{ is odd, } e \text{ divides } d \text{ and } \mathrm{char}(k) \neq 2 \\ \frac{es+1}{2} & \text{if } es \text{ is odd, } e \text{ divides } d \text{ and } \mathrm{char}(k) = 2 \end{cases}$$

When specialised to degree zero, Theorem 1.2 implies the following statement:

Corollary 1.4. *Let A be a Brauer tree algebra over a field k with e isomorphism classes of simple modules and exceptional multiplicity s. The canonical map $Z(A) \longrightarrow Z^0(\overline{\mathrm{mod}}(A))$ is surjective and $Z^0(\overline{\mathrm{mod}}(A))$ is a uniserial commutative k-algebra of dimension $\frac{s}{2}$ if s is even and dimension $\frac{s+1}{2}$ if s is odd.*

By a result of Rickard [18], a Brauer tree algebra is derived equivalent to a serial symmetric algebra, and hence it suffices to prove the above results in that case. It is possible to describe $Z^*(\mathrm{mod}(A))$ in terms of generators and relations; we do this for group algebras of cyclic p-groups for future reference:

Corollary 1.5. *Let k be a field of positive characteristic p and let P be a finite cyclic p-group.*
(i) If $|P| = 2$ then $Z^*(\overline{\mathrm{mod}}(kP)) \cong k[\zeta, \zeta^{-1}]$, where $\deg(\zeta) = 1$.
(ii) If $|P| > 2$ then $Z^*(\overline{\mathrm{mod}}(kP)) \cong k[\pi, \zeta, \zeta^{-1}, \tau_1, \tau_2, .., \tau_d]/J$ where $\deg(\pi) = 0$, $\deg(\zeta) = 2$, $\deg(\tau_i) = -1$ for $1 \leq i \leq d$, where d is the largest integer less or equal to $\frac{|P|}{2}$, and where J is the ideal generated by the set of elements π^d, $\tau_i\tau_j$, $\pi\tau_i$, with $1 \leq i, j \leq d$.

After recalling some background information on stable categories, graded centers, serial algebras and their Hochschild cohomology in §2, §3, §4 and §5, respectively, we prove Theorem 1.2 and its Corollary in §6, and we prove Theorem 1.3 in §7.

Remark. Here are two further reasons why graded centers might be worthwile considering in the context of stable categories of p-blocks of finite groups. For one, a non-principal block does not have, in general, a canonical module which would play the role of the trivial module in the principal block, and in particular, whose $\widehat{\mathrm{Ext}}$-algebra would be a good analogue for Tate cohomology. Considering the graded center is a way to avoid choosing a particular module. Second, even in the principal block case, the thick subcategory of the stable category generated by the trivial module will not always be the entire stable category, and so the graded center of the stable category of a principal block contains Tate cohomology as graded subalgebra but need not be equal to it.

2 Background material on stable categories

Let k be a field and let A be a finite-dimensional self-injective k-algebra; that is, the classes of finitely generated projective and injective A-modules coincide. This case occurs in particular if A is *symmetric*; that is, if A is isomorphic to its k-dual $A^* = \mathrm{Hom}_k(A, k)$ as A-A-bimodule. Denote by $\overline{\mathrm{mod}}(A)$ the stable category of the category $\mathrm{mod}(A)$ of finitely generated A-modules; that is, the objects of $\overline{\mathrm{mod}}(A)$ are the finitely generated A-modules, and for any two finitely generated A modules U, V, the k-space of morphisms in $\overline{\mathrm{mod}}(A)$ from U to V is the quotient

$$\overline{\mathrm{Hom}}_A(U, V) = \mathrm{Hom}_A(U, V)/\mathrm{Hom}_A^{pr}(U, V)$$

where $\mathrm{Hom}_A^{pr}(U, V)$ is the subspace of A-homomorphisms from U to V which factor through a finitely generated projective A-module. The composition of morphisms in $\overline{\mathrm{mod}}(A)$ is induced by that in $\mathrm{mod}(A)$. For any finitely generated A-module U choose a projective cover (P_U, π_U) and an injective envelope (I_U, ι_U); that is, P_U is a projective A-module and $\pi_U : P_U \to U$ a surjective A-homomorphism inducing an isomorphism $P_U/\mathrm{rad}(P_U) \cong U/\mathrm{rad}(U)$ and I_U is an injective A-module and $\iota_U : U \to I_U$ is an injective A-homomorphism inducing an isomorphism $\mathrm{soc}(U) \cong \mathrm{soc}(I_U)$. The operators defined by $\Omega_A(U) = \ker(\pi_U)$ and $\Sigma_A(U) = \mathrm{coker}(\iota_U)$ are unique up to *unique* isomorphisms in the stable category; more precisely, any other pair (P, π) consisting of a finitely generated projective A-module P and a surjective A-homomorphism $\pi : P \to U$, determines a *unique* isomorphism $\ker(\pi) \cong \Omega_A(U)$ in $\overline{\mathrm{mod}}(A)$ which is induced by a homomorphism $\beta : P \to P_U$ satisfying $\pi_U \circ \beta = \pi$. In particular, Ω_A and Σ_A induce equivalences on $\overline{\mathrm{mod}}(A)$ which are inverse to each other and up to isomorphism of functors independent of the choices of projective covers and injective envelopes. In what follows, whenever we use the notation Ω_A and Σ_A, we implicitly assume a choice of projective covers and injective envelopes. The category $\overline{\mathrm{mod}}(A)$ together with the equivalence Σ_A becomes a triangulated category, where exact triangles are induced by short exact sequences. See [7] for details. It is well-known that for any integer n the functor $\Sigma^n_{A \underset{k}{\otimes} A^0}(A) \underset{A}{\otimes} -$ on $\overline{\mathrm{mod}}(A)$ is canonically isomorphic to the functor Σ^n_A; this follows from the fact that if $\mathcal{P}$ is a projective resolution of A as $A \underset{k}{\otimes} A^0$-module then, for any A-module X, the complex $\mathcal{P} \underset{A}{\otimes} X$ is a projective resolution of X, in conjunction with the uniqueness observations above. Whenever the underlying algebra is clear from the context we write Ω and Σ instead of Ω_A and Σ_A. Given an A-homomorphism $\varphi : U \to V$ we denote by $\bar{\varphi}$ its image in $\overline{\mathrm{Hom}}_A(U, V)$. Conversely, if $\bar{\varphi} \in \overline{\mathrm{Hom}}_A(U, V)$ we implicitly assert that φ is a representative of $\bar{\varphi}$ in $\mathrm{Hom}_A(U, V)$. The following is a well-known tool to lift commutative diagrams in $\overline{\mathrm{mod}}(A)$ to commutative diagrams in $\mathrm{mod}(A)$.

Lemma 2.1. *Let A be a finite-dimensional self-injective algebra over a field k. Let*

$$\begin{tikzcd} U \arrow[r, "\bar{\varphi}"] \arrow[d, "\bar{\alpha}"'] & X \arrow[d, "\bar{\beta}"] \\ V \arrow[r, "\bar{\psi}"'] & Y \end{tikzcd}$$

be a commutative square in $\overline{\mathrm{mod}}(A)$. Suppose that $\bar{\alpha}$ has an injective representative α. Then for any choice of representatives φ of $\bar{\varphi}$ and β of $\bar{\beta}$ there is a representative ψ of $\bar{\psi}$ such that the square

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & X \arrow[d, "\beta"] \\ V \arrow[r, "\psi"'] & Y \end{tikzcd}$$

commutes in $\mathrm{mod}(A)$.

Proof. Let α, β, φ, ψ be representatives of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\varphi}$, $\bar{\psi}$, respectively. Suppose that α is injective. The commutativity of the square in $\overline{\mathrm{mod}}(A)$ means that $\beta\varphi - \psi\alpha : U \to Y$ factors through a projective (and hence also injective) A-module I; say $\beta\varphi - \psi\alpha = \sigma\rho$ for some $\rho \in \mathrm{Hom}_A(U, I)$ and $\sigma \in \mathrm{Hom}_A(I, Y)$. Since α is injective, there is a morphism $\gamma \in \mathrm{Hom}_A(V, I)$ such that $\rho = \gamma\alpha$. Set $\psi' = \psi + \sigma\gamma$. Then $\bar{\psi}' = \bar{\psi}$ and we have $\psi'\alpha = \psi\alpha + \sigma\gamma\alpha = \beta\varphi - \sigma\rho + \sigma\rho = \beta\varphi$. Hence replacing ψ by ψ' proves the result. □

§3 Background material on graded centers

3.1. Let k be a commutative ring, let $\mathcal{C}$ be a k-linear triangulated category with shift functor $\Sigma : \mathcal{C} \to \mathcal{C}$. For any integer n denote by $Z^n(\mathcal{C})$ the k-module of natural transformations $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ satisfying

$\Sigma\varphi = (-1)^n\varphi\Sigma$; that is, $\Sigma(\varphi(X)) = (-1)^n\varphi(\Sigma(X)) : \Sigma(X) \to \Sigma^{n+1}(X)$ for any object X in $\mathcal{C}$. The graded k-module $Z^*(\mathcal{C})$ becomes a graded commutative k-algebra with product induced by composing natural transformations; that is, for $\varphi \in Z^m(\mathcal{C})$ and $\psi \in Z^n(\mathcal{C})$ the product $\psi\varphi \in Z^{m+n}(\mathcal{C})$ is defined by $(\psi\varphi)(X) = \psi(\Sigma^m(X)) \circ \varphi(X)$ for all objects X in $\mathcal{C}$. We call $Z^*(\mathcal{C})$ the *graded center of the triangulated category* $\mathcal{C}$; see e.g. [14] for more details. Given a k-algebra A which is finitely generated projective as k-module, it is well-known that there is a canonical graded k-algebra homomorphism $HH^*(A) \longrightarrow Z^*(D^b(A))$ which need not be surjective or injective; cf. [4], [11].

3.2. Let k be a field and let A be a self-injective k-algebra. One checks that there is a canonical graded k-algebra homomorphism $\widehat{HH}^*(A) \longrightarrow Z^*(\overline{\mathrm{mod}}(A))$ where $\widehat{HH}^*(A)$ is the Tate analogue of Hochschild cohomology. More explicitly, $\widehat{HH}^n(A) = \overline{\mathrm{Hom}}_{A\underset{k}{\otimes}A^0}(A, \Sigma^n_{A\underset{k}{\otimes}A^0}(A))$ for any integer n. In particular, $\widehat{HH}^0(A) \cong Z(A)/Z^{pr}(A)$, where $Z^{pr}(A)$ is the ideal of all elements $z \in Z(A)$ with the property that the $A \underset{k}{\otimes} A^0$-endomorphism of A given by multiplication with z factors through a projective $A \underset{k}{\otimes} A^0$-module. The map $\widehat{HH}^*(A) \to Z^*(\overline{\mathrm{mod}}(A))$ sends $\varphi \in \widehat{HH}^n(A)$ to the family of morphisms in the stable module category

$$X \xrightarrow{\cong} A \underset{A}{\otimes} X \xrightarrow{\varphi\otimes \mathrm{Id}_X} \Sigma^n_{A\underset{k}{\otimes}A^0}(A) \underset{A}{\otimes} X \xrightarrow{\cong} \Sigma^n_A(X)$$

for any finitely generated A-module X. Note that by the remarks in §2 the isomorphisms $\Sigma^n_{A\underset{k}{\otimes}A^0}(A) \underset{A}{\otimes} X \cong \Sigma^n_A(X)$ are uniquely determined in $\overline{\mathrm{mod}}(A)$. Evaluation at X induces a graded algebra homomorphism $Z^*(\overline{\mathrm{mod}}(A)) \longrightarrow \widehat{\mathrm{Ext}}^*(X,X)$ The composition of the two canonical homomorphisms $\widehat{HH}^*(A) \to Z^*(\overline{\mathrm{mod}}(A)) \to \widehat{\mathrm{Ext}}^*(X,X)$ is the graded algebra homomorphism induced by the functor $- \underset{A}{\otimes} X$ modulo the canonical identification of the functors $\Sigma^n_{A\underset{k}{\otimes}A^0}(A) \underset{A}{\otimes} -$ and Σ^n_A for $n \in \mathbb{Z}$.

3.3. Let A be a symmetric k-algebra. Then A is in particular self-injective, hence $\widehat{HH}^0(A) \cong \bar{Z}(A) = Z(A)/Z^{pr}(A)$, where the notation is as in 3.2. By [10, (4.E)] we have $Z^{pr}(A) \subseteq Z(A) \cap \mathrm{soc}(A)$. The canonical map $Z(A) \to Z^0(\overline{\mathrm{mod}}(A))$ sending $z \in Z(A)$ to the familiy of endomorphisms in $\overline{\mathrm{mod}}(A)$ given by multiplication with z on each A-module has certainly $Z^{pr}(A)$ in its kernel. In fact, its kernel contains $Z(A) \cap \mathrm{soc}(A)$ because $\mathrm{soc}(A)$ annihilates $J(A)$ and hence annihilates any indecomposable non projective A-module as A is symmetric.

3.4. Let A, B be symmetric k-algebras. Following [3] an A-B-bimodule M is said to induce a *stable equivalence of Morita type* if M is finitely generated projective as left A-module and as right B module, and if we have isomorphisms $M \underset{B}{\otimes} M^* \cong A$ in $\overline{\mathrm{mod}}(A \underset{k}{\otimes} A^0)$ and $M^* \underset{A}{\otimes} M \cong B$ in $\overline{\mathrm{mod}}(B \underset{k}{\otimes} B^0)$. In that case the map sending $\varphi \in Z^n(\overline{\mathrm{mod}}(A))$ to $\mathrm{Id}_{M^*} \otimes \varphi$ induces an isomorphism of graded k-algebras $Z^*(\overline{\mathrm{mod}}(A)) \cong Z^*(\overline{\mathrm{mod}}(B))$; similarly, the map sending $\zeta \in \widehat{HH}^n(A)$ to $\mathrm{Id}_{M^*} \otimes \zeta \otimes \mathrm{Id}_M$ induces an isomorphism of graded k-algebras $\widehat{HH}^*(A) \cong \widehat{HH}^*(B)$. There is an obvious commutative diagram of graded algebra homomorphisms

$$\begin{array}{ccccc}
\widehat{HH}^*(A) & \longrightarrow & Z^*(\overline{\mathrm{mod}}(A)) & \longrightarrow & \widehat{\mathrm{Ext}}^*_A(X,X) \\
\cong\downarrow & & \downarrow\cong & & \downarrow\cong \\
\widehat{HH}^*(B) & \longrightarrow & Z^*(\overline{\mathrm{mod}}(B)) & \longrightarrow & \widehat{\mathrm{Ext}}^*_B(M^* \underset{A}{\otimes} X, M^* \underset{A}{\otimes} X)
\end{array}$$

where X is a finitely generated A-module.

§4 On symmetric serial algebras

By a result of Gabriel and Riedtmann [6], a Brauer tree algebra is stably equivalent to a symmetric serial algebra with the same number e of isomorphism classes of simple modules and the same exceptional multiplicity s. Rickard proved in [18] that one can lift some stable equivalence to a derived equivalence, and showed in [19] that this implies the existence of a stable equivalence of Morita type. As mentioned in 3.4 above, a stable equivalence of Morita type between two symmetric algebras induces compatible isomorphisms between their Tate-Hochschild cohomology algebras and between the graded centers of their stable categories. Therefore, in order to prove any of the results stated in §1 it suffices to give proofs in the case of symmetric serial algebras. The structure theory of symmetric serial algebras is very well understood, going back to work of Nakayama [16] and Morita [15]; see e.g. [2] or [12] for more details and references. We review the main aspects, without proofs, as needed for the explicit calculations of the graded center of the stable category.

Throughout this section we denote by A a split basic indecomposable symmetric serial k-algebra. In order to avoid trivialities we assume that A is not simple. Let I be a primitive idempotent decomposition of 1_A in A. Since A is basic, the set $\{Ai\}_{i\in I}$ is a set of representatives of the isomorphism classes of projective indecomposable A-modules and we have $A = \bigoplus_{i\in I} Ai$ as left A-modules. Set $S_i = Ai/J(A)i$ for $i \in I$. Then $\{S_i\}_{i\in I}$ is a set of representatives of the isomorphism classes of simple A-modules.

4.1. *All projective indecomposable A-modules have the same composition length m.*

4.2. *There is a cyclic transitive permutation π of the set I such that $J(A)i/J(A)^2 i \cong S_{\pi(i)}$ for all $i \in I$.*

4.3. *For any $i \in I$ the composition series (from top to bottom) of Ai is $S_i, S_{\pi(i)}, S_{\pi^2(i)}, .., S_{\pi^{m-1}(i)} \cong S_i$; in particular, the number $|I|$ of isomorphism classes of simple A-modules divides $m-1$.*

4.4. *There is an element $c \in J(A)$ such that $J(A) = cA = Ac$ and such that $ci = \pi(i)c$ for all $i \in I$.*

4.5. *For any $i \in I$ the set $\{i, ci, c^2 i, .., c^{m-1} i\}$ is a k-basis of Ai.*

Set $e = |I|$. The integer $s = \frac{m-1}{e}$ is called the *exceptional multiplicity of A*. This is the unique integer such that $s+1$ is the largest Cartan invariant of A. This characterisation of s makes sense for an arbitrary Brauer tree algebra and is invariant under stable equivalences.

4.6. *Set $t = c^e$. The set $\{1, t, t^2, .., t^{s-1}\} \cup \{t^s i \mid i \in I\}$ is a k-basis of $Z(A)$ and the set $\{t^s i \mid i \in I\}$ is a k-basis of $Z(A) \cap \mathrm{soc}(A)$.*

Using the explicit description of $Z(A)$ one easily verifies the next statement:

4.7. *For any $i \in I$ the set $\{i, ti, t^2 i, .., t^s i\}$ is a k-basis of iAi; in particular we have $iAi = Z(A)i$.*

Every indecomposable A-module U is uniserial, hence isomorphic to a submodule of a projective indecomposable A-module Ai for some $i \in I$. Thus every endomorphism of U extends to an endomorphism of Ai and therefore 4.7 implies the following statement:

4.8. *For any indecomposable A-module and any $\varphi \in \mathrm{End}_A(U)$ there is $z \in Z(A)$ such that $\varphi(u) = zu$ for all $u \in U$.*

Right multiplication by c sends Ai to $A\pi^{-1}(i)$, where $i \in I$. Thus right multiplication by c^r sends Ai to Aj, where $j = \pi^{-r}(i)$. Combined with 4.7 and 4.8 this yields a description of homomorphisms between all projective indecomposable A-modules:

4.9. *Let $i, j \in I$. Let r be the smallest non-negative integer such that $\pi^r(j) = i$. For any $\varphi \in \mathrm{Hom}_A(Ai, Aj)$ there is an element $z \in Z(A)$ such that $\varphi(ai) = aic^r z$ for all $a \in A$.*

The fact that every indecomposable A-module is uniserial yields the following classification of indecomposable A-modules:

4.10. *For any $i \in I$ and any integer A such that $1 \leq a \leq m$ there is, up to isomorphism, a unique indecomposable A-module $U_{(i,a)}$ such that $\mathrm{soc}(U_{(i,a)}) \cong S_i$ and $\ell(U_{(i,a)}) = a$. The set $\{U_{(i,a)} \mid i \in I,\ 1 \leq a \leq m\}$ is a complete set of representatives of the isomorphism classes of indecomposable A-modules.*

The effect of the equivalences Ω and Σ on $\overline{\mathrm{mod}}(A)$ as described in §2 can be made explicit:

4.11. *For any $i \in I$ and any integer a such that $1 \leq a \leq m-1$ we have $\Omega(U_{(i,a)}) \cong U_{(\pi^{1-a}(i),m-a)}$ and $\Omega^2(U_{(i,a)}) \cong U_{(\pi(i),a)}$.*

4.12. *For any $i \in I$ and any integer a such that $1 \leq a \leq m-1$ we have $\Sigma(U_{(i,a)}) \cong U_{(\pi^{-a}(i),m-a)}$ and $\Sigma^2(U_{(i,a)}) \cong U_{(\pi^{-1}(i),a)}$.*

Thus Σ and Ω have period $2e$ unless $es = 1$, in which case they have period 1. Combining the above statements implies that for any $i \in I$ we have chains of monomorphisms

$$S_i \cong U_{i,1} \hookrightarrow U_{i,2} \hookrightarrow \cdots \hookrightarrow U_{i,m-1} \hookrightarrow U_{i,m} \cong Ai$$

and chains of epimorphisms

$$Ai \cong U_{(i,m)} \twoheadrightarrow U_{\pi^{-1}(i),m-1)} \twoheadrightarrow U_{(\pi^{-2}(i),m-2)} \twoheadrightarrow \cdots \twoheadrightarrow U_{(\pi^{1-m}(i),1)} = U_{(i,1)} \cong S_i\ .$$

In what follows we assume that we have made a fixed choice of such monomorphisms and epimorphisms in such a way that applying Ω to the morphism in $\overline{\mathrm{mod}}(A)$ represented by the monomorphism $U_{(i,a)} \hookrightarrow U_{(i,a+1)}$ is the morphism in $\overline{\mathrm{mod}}(A)$ represented by the epimorphism $U_{(\pi^{1-a}(i),m-a)} \twoheadrightarrow U_{(\pi^{-a}(i),m-a-1)}$.

4.13. *(Alperin [1, 21.3]) Let U, V be indecomposable A-modules and let $\varphi : U \to V$ be a non zero A-homomorphism. Then φ factors through a projective A-module if and only if $\ell(\mathrm{Im}(\varphi)) \leq \ell(U) + \ell(V) - m$.*

The following two observations are special cases of 4.13:

4.14. *Let U, V be indecomposable A-modules such that $\ell(U) + \ell(V) \leq m$. Then $\overline{\mathrm{Hom}}_A(U,V) \cong \mathrm{Hom}_A(U,V)$, or equivalently, no nonzero A-homomorphism from U to V factors through a projective A-module.*

4.15. *Let Ube an indecomposable A-module and let n be an odd integer. Then $\overline{\mathrm{Hom}}_A(U,\Sigma^n(U)) \cong \mathrm{Hom}_A(U,\Sigma^n(U))$, or equivalently, no nonzero A-homomorphism from U to $\Sigma^n(U)$ factors through a projective A-module.*

4.16. *Every homomorphism between two indecomposable A-modules is a composition of finitely many irreducible homomorphisms, and a homomorphism is irreducible if it is either injective with simple cokernel or surjective with simple kernel.*

The relevance of 4.16 for graded centers is that in order to verify whether a given family of morphisms $\{\varphi(U) : U \to \Sigma^n(U)\}_U$, with U running over a set of representatives of indecomposable A-modules and for some integer n, is actually an element in $Z^n(\overline{\mathrm{mod}}(A))$ it suffices to check the compatibility with irreducible homomorphisms.

5 On the Hochschild cohomology of A

Let A be a split basic indecomposable non simple symmetric serial k-algebra. In order to relate the graded center of $\overline{\mathrm{mod}}(A)$ to $\widehat{HH}^*(A)$ we adapt material from Holm [8] and Erdmann-Holm [5] to the Tate analogue of Hochschild cohomology. It is shown in [8] that for any even integer n there is an automorphism β of A such that $\Sigma^n_{A\otimes A^0}(A) \cong {}_\beta A$ in the stable category of A-A-bimodules. Here we denote by ${}_\beta A$ the A-A-bimodule which is equal to A as right A-module and whose left A-module structure is given by left multiplication with $\beta(a)$ on A for any $a \in A$. Hence any element in $\widehat{HH}^n(A)$ is represented by a bimodule homomorphism $A \to {}_\beta A$. It is further shown that any such bimodule homomorphism is induced by left multiplication on A with an element in the k-vector space

$$Z_\beta(A) = \{z \in A \mid za = \beta(a)z \text{ for all } a \in A\}$$

Using the material from the previous section one can describe the automorphism group of A explicitly. We use the notation from the previous section.

5.1. *There is a unique algebra automorphism α of A satisfying $\alpha(i) = \pi(i)$ for all $i \in I$ and $\alpha(c) = c$. Moreover, we have $ca = \alpha(a)c$ for all $a \in A$.*

Since π is a transitive cycle on I the automorphism α has order $e = |I|$. A straightforward verification shows:

5.2. *Let r be an integer such that $0 \le r \le e-1$ and set $\beta = \alpha^r$. Then $Z_\beta(A) = c^r Z(A)$. If r is positive, the set $\{c^r, c^r t, .., c^r t^{s-1}\}$ is a k-basis of $Z_\beta(A)$.*

For U an A-module and β an automorphism of A we denote by ${}_\beta U$ the A-module which is equal to U as k-vector space, with $a \in A$ acting as multiplication by $\beta(a)$ on U. In this way, restriction along β becomes an equivalence on $\mathrm{mod}(A)$ which is isomorphic to the functor ${}_\beta A \underset{A}{\otimes} -$.

5.3. *Let r be an integer such that $0 \le r \le e-1$ and set $\beta = \alpha^r$. For any $i \in I$ and $a \in A$, the map sending ai to $\beta(a)\pi^r(i)$ induces an isomorphism $Ai \cong {}_\beta A\pi^r(i)$. In particular, for any indecomposable A-module U we have ${}_\beta U \cong \Sigma^{2r}(U)$.*

The last statement in 5.3 follows from the first combined with 4.11. For even n we have $\Sigma^n_{A \otimes A^0}(A) \cong {}_\beta A$ in $\overline{\mathrm{mod}}(A \underset{k}{\otimes} A^0)$ for some automorphism β of A, by [8]. By [5, 4.2] (or by 5.3 and some explicit verifications) we can identify β in terms of n:

5.4. *Let n be an even integer and let r be the unique integer such that $0 \le r \le e-1$ and such that $r \equiv \frac{n}{2} (\mathrm{mod}\ e)$. Set $\beta = \alpha^r$. We have ${}_\beta A \cong \Sigma^n_{A \underset{k}{\otimes} A^0}(A)$ in the stable category of A-A-bimodules; in particular, $\Sigma^{2e}_{A \underset{k}{\otimes} A^0}(A) \cong A$.*

Thus, as $A \underset{k}{\otimes} A^0$-module, A has period dividing $2e$. In fact, by [5, 4.2], A has period exactly $2e$ unless $es = 1$ and $\mathrm{char}(k) = 2$, in which case A has period 1. The following observation will be needed in order to determine the image of an element $\zeta \in \widehat{HH}^n(A)$ in $Z^n(\overline{\mathrm{mod}}(A))$ for n even. It says, roughly speaking, that an element ζ given by left multiplication with $c^r z$ as in 5.2 induces an element in the graded center induced by right multiplication with $c^r z$ on projective indecomposable modules.

5.5. *Let $i, j \in I$ and let r be the unique integer such that $0 \le r \le e-1$ and such that $\pi^r(j) = i$. Set $\beta = \alpha^r$. Let $z \in Z(A)$. Define $\varphi \in \mathrm{Hom}_A(Ai, Aj)$ by $\varphi(ai) = aic^r z$ and define $\zeta \in \mathrm{Hom}_{A \otimes A^0}(A, {}_\beta A)$ by $\zeta(a) = c^r za$ for all $a \in A$. We have a commutative diagram of A-modules*

$$\begin{array}{ccc}
A \underset{A}{\otimes} Ai & \xrightarrow{\zeta \otimes \mathrm{Id}_{Ai}} & {}_\beta A \underset{A}{\otimes} Ai \\
\cong \big\downarrow & & \big\downarrow \cong \\
Ai & \xrightarrow[\varphi]{} & Aj
\end{array}$$

where the right vertical isomorphism is the composition of the isomorphisms ${}_\beta A \underset{A}{\otimes} Ai \cong {}_\beta Ai \cong Aj$ *from 5.3.*

Proof. Let $a \in A$. The map $\zeta \otimes \mathrm{Id}_{Ai}$ sends $a \otimes i$ to $c^r za \otimes i$. The isomorphism ${}_\beta A \underset{A}{\otimes} Ai \cong {}_\beta Ai$ sends this to $c^r zai$, and the isomorphism ${}_\beta Ai \cong Aj$ sends this to $\beta^{-1}(c^r za)j = c^r z\beta^{-1}(a)j = ac^r zj = aic^r z = \varphi(ai)$ as claimed. □

§6 The even part of the graded center of $\overline{\mathrm{mod}}(A)$

Proof of Theorem 1.2. Let A be a split basic indecomposable serial symmetric k-algebra. We use the notation of section 4. Let d be an integer and set $n = 2d$. Let $i \in I$. Since n is even we have $\Sigma^n(S_i) \cong S_j$ for a unique $j \in I$. Then $\Sigma^n(U_{(i,a)}) \cong U_{(j,a)}$ for $1 \le a \le m-1$. Let $\bar\varphi \in Z^n(\overline{\mathrm{mod}}(A))$ and set $\bar\varphi_{(a,i)} = \bar\varphi(U_{(i,a)})$ for any $i \in I$ and any integer a such that $1 \le a \le m-1$. By 2.1 we may choose representatives $\varphi_{(i,a)}$ of $\bar\varphi_{(i,a)}$ such that the diagram

6.1.

$$\begin{array}{ccc}
U_{(i,1)} & \xrightarrow{\varphi_{(i,1)}} & U_{(j,1)} \\
\downarrow & & \downarrow \\
U_{(i,2)} & \xrightarrow{\varphi_{(i,2)}} & U_{(j,2)} \\
\downarrow & & \downarrow \\
U_{(i,3)} & \xrightarrow{\varphi_{(i,3)}} & U_{(j,3)} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
U_{(i,m-1)} & \xrightarrow{\varphi_{(i,m-1)}} & U_{(j,m-1)}
\end{array}$$

commutes in $\mathrm{mod}(A)$, where the vertical arrows column are the chosen inclusions. The above diagram determines $\bar\varphi$ uniquely because every indecomposable non projective A-module is a Heller translate of one of the $U_{(i,a)}$ with $1 \le a \le \frac{m}{2}$, by 4.11 and 4.12. Note also that then as a consequence of 4.14,

6.2. *we have* $\bar\varphi = 0$ *if and only if* $\varphi_{(i,a)} = 0$ *for* $1 \le a \le \frac{m}{2}$.

Conversely, we show that any commutative diagram as in 6.1 determines an element in $Z^n(\overline{\mathrm{mod}}(A))$ and that this element is in the image of the canonical map $\widehat{HH}^n(A) \to Z^n(\overline{\mathrm{mod}}(A))$. Since A is symmetric, the projective A-module Aj is also injective. Thus we can extend $\varphi_{(i,m-1)}$ to a morphism $\varphi_i : Ai \to Aj$. Then φ_i is given by right multiplication with an element of the form $c^r z$ for some non negative integer r and some $z \in Z(A)$. Here r is the smallest non negative integer such that $\pi^{-r}(i) = j$, or equivalently, such that $ic^r = c^r j$. This is also the smallest non negative integer such that $r \equiv d \pmod e$ because Σ has period $2e$ (if $\dim_k(A) > 2$) or 1 (if $\dim_k(A) = 2$). Set $\beta = \alpha^r$. Then ${}_\beta A \cong \Sigma^n_{A \underset{k}{\otimes} A^0}(A)$ in $\overline{\mathrm{mod}}(A \underset{k}{\otimes} A^0)$. Note also that $\beta(j) = i$. The map $\zeta : A \to {}_\beta A$ defined by $\zeta(a) = c^r za$ is an A-A-bimodule homomorphism, hence determines an element in $\widehat{HH}^n(A)$. Let $\bar\varphi$ be the image of ζ in $Z^n(\overline{\mathrm{mod}}(A))$. We will show that $\bar\varphi$ coincides with $\bar\varphi_{(a,i)}$ on $U_{(i,a)}$ for $1 \le a \le m-1$. It suffices to show that the map $\zeta \otimes \mathrm{Id}_{Ai} : A \underset{A}{\otimes} Ai \to {}_\beta A \underset{A}{\otimes} Ai$ becomes φ_i upon composition with the isomorphisms $Ai \cong A \underset{A}{\otimes} Ai$ and ${}_\beta A \underset{A}{\otimes} Ai \cong {}_\beta Ai \cong Aj$, where the last isomorphism sends ai

to $\beta^{-1}(ai) = \beta^{-1}(a)j$. This, however, is clear by 5.5. This shows the surjectivity of the map $\widehat{HH}^{even}(A) \to Z^{even}(\overline{\mathrm{mod}}(A))$. The kernel of this map is contained in $J(Z(A)) \cdot \widehat{HH}^{even}(A)$ and hence nilpotent.

Using 6.2 one can be more precise regarding the dimension of $Z^n(\overline{\mathrm{mod}}(A))$. This dimension will be equal to the smallest non negative integer b with the property that right multiplication by $c^r t^b$ annihilates all modules of length at most $\frac{m}{2}$, by 6.2. Now $c^r t^b \in J(A)^{r+eb}$ but $c^r t^b \notin J(A)^{r+eb+1}$. Thus $c^r t^b$ annihilates all modules of length at most $r+eb$. Therefore b is the smallest integer satisfying

$$r + eb \geq \begin{cases} \frac{m}{2} & \text{if } m \text{ is even} \\ \frac{m-1}{2} & \text{if } m \text{ is odd} \end{cases}$$

Note that $m = es + 1$. Thus the previous inequality is equivalent to

$$b \geq \begin{cases} \frac{s}{2} - \frac{2r-1}{2e} & \text{if } es \text{ is odd} \\ \frac{s}{2} - \frac{r}{e} & \text{if } es \text{ is even} \end{cases}$$

If s is even, so is es, and hence in this case we get $b \geq \frac{s}{2} - \frac{r}{e}$. Since $0 \leq \frac{r}{e} < 1$ and since $\frac{s}{2}$ is already an integer, this forces $b \geq \frac{s}{2}$, and hence $\dim_k(Z^{2d}(\overline{\mathrm{mod}}(A))) = \frac{s}{2}$. For s odd, similar arguments show that regardless whether e is even or odd, we get $b \geq \frac{s+1}{2}$ if $0 \leq r < \frac{e}{2}$ and $b \geq \frac{s-1}{2}$ if $r \geq \frac{e}{2}$. Hence, in that case we get the formulae

$$\dim_k(Z^{2d}(\overline{\mathrm{mod}}(A))) = \begin{cases} \frac{s+1}{2} & \text{if } 0 \leq r < \frac{e}{2} \\ \frac{s-1}{2} & \text{if } \frac{e}{2} \leq r < e \end{cases}$$

This completes the proof of Theorem 1.2. □

Proof of Corollary 1.4. The kernel I of the canonical map $Z(A) \to Z^0(\overline{\mathrm{mod}}(A))$ contains $Z(A) \cap \mathrm{soc}(A)$, and hence $Z(A)/I$ is uniserial. This map factors through the canonical map $\widehat{HH}^0(A) \to Z^0(\overline{\mathrm{mod}}(A))$, hence is surjective. The dimension of $Z^0(\overline{\mathrm{mod}}(A))$ follows from 1.2 applied to $d = 0$. □

§7 The odd part of the graded center of $\overline{\mathrm{mod}}(A)$

Proof of Theorem 1.3. Let A be a split basic indecomposable serial symmetric k-algebra. Keep the notation from section 4. Let n be an odd integer and let $\bar{\varphi} \in Z^n(\overline{\mathrm{mod}}(A))$. Let $i \in I$. Then $\Sigma^n(U_{(i,1)}) \cong U_{(j,m-1)}$ for some $j \in I$. Using the effect of Σ on indecomposable non projective A-modules one gets

$$\Sigma^n(U_{(i,a)}) \cong U_{(\pi^{1-a}(j),m-a)}$$

for $1 \leq a \leq m-1$. By 2.1 we may choose representatives $\varphi_{(i,a)}$ of $\bar{\varphi}(U_{(i,a)})$ such that the diagram

7.1.

$$\begin{array}{ccc}
U_{(i,1)} & \xrightarrow{\varphi_{(i,1)}} & U_{(j,m-1)} \\
\downarrow & & \downarrow \\
U_{(i,2)} & \xrightarrow{\varphi_{(i,2)}} & U_{(\pi^{-1}(j),m-2)} \\
\downarrow & & \downarrow \\
U_{(i,3)} & \xrightarrow{\varphi_{(i,3)}} & U_{(\pi^{-2}(j),m-3)} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
U_{(i,m-1)} & \xrightarrow{\varphi_{(i,m-1)}} & U_{(\pi^{2-m}(j),1)}
\end{array}$$

commutes in $\mathrm{mod}(A)$. Here the vertical arrows in the left column are inclusions while they are surjections in the right column. As before, φ is uniquely determined by such a diagram. The commutativity of the diagram 7.1 implies that

7.2. *for* $1 \leq a \leq m-1$ *we have* $\mathrm{Im}(\varphi_{(i,a)}) \subseteq \mathrm{soc}(U_{(\pi^{1-a}(j),m-a)})$.

Indeed, this is true for $a = 1$ because $U_{(i,1)}$ is simple. Since the vertical maps in the left column of the diagram 7.1 are injective with simple cokernel and the vertical maps in the right column of the diagram 7.1 are surjective with simple kernel, an easy induction shows that 7.2 holds for all a such that $1 \leq a \leq m-1$.

The top composition factor of $U_{(i,a)}$ is isomorphic to $S_{\pi^{1-a}(i)}$. The bottom composition factor of $U_{(\pi^{1-a}(j),m-a)}$ is isomorphic to $S_{\pi^{1-a}(j)}$. It follows that

7.3. *if* $i \neq j$ *then* $\bar{\varphi} = 0$.

Together with 4.11 and 4.12 this implies that

7.4. *we have* $i = j$ *if and only if* $n = 2d-1$ *for some integer* d *which is divisible by* e.

This proves that $Z^{2d-1}(\overline{\mathrm{mod}}(A))$ is zero unless possibly if e divides d. We need to consider the case where e divides d. In that case we have $i = j$ and any choice of a family of morphisms $\varphi_{(i,a)} : U_{(i,a)} \to U_{(\pi^{1-a}(i),a)}$ whose images are in the socle of $U_{(\pi^{1-a}(i),a)}$ make the diagram 7.1 commutative, and hence defines a natural transformation $\mathrm{Id} \to \Sigma^n$. Not any such choice will define an element in $Z^n(\overline{\mathrm{mod}}(A))$, because we have in addition to make sure that the compatibility $\Sigma\bar{\varphi}(U) = -\bar{\varphi}(\Sigma(U))$ is satisfied for all indecomposable non-projective A-modules U. In particular, $\bar{\varphi}$ is already determined by the family $\varphi_{(i,a)}$ with $1 \leq a \leq \frac{m}{2}$. Thus

$$\dim_k(Z^n(\overline{\mathrm{mod}}(A))) \leq \frac{m}{2}$$

if n is an odd integer of the form $n = 2d-1$ for some integer d.

If m is odd then any family $\varphi_{(i,a)}$ with $1 \leq a \leq \frac{m-1}{2}$ which makes the upper half of 7.1 commutative gives rise to an element in $Z^n(\overline{\mathrm{mod}}(A))$ by applying powers of Σ to the morphisms in this familiy and modifying by signs as appropriate. Thus, if m is odd then

$$\dim_k(Z^n(\overline{\mathrm{mod}}(A))) = \frac{m-1}{2} = \frac{es}{2}$$

Consider the case where m is even. In that case, any indecomposable A-module U of length $\frac{m}{2}$ has an odd period and hence the compatibility condition with Σ implies that $\bar{\varphi}(U) = -\bar{\varphi}(U)$. If $\mathrm{char}(k) \neq 2$ this forces $\bar{\varphi}(U) = 0$, and so $\bar{\varphi}$ is determined by the familiy $\varphi_{(i,a)}$ with $1 \leq a < \frac{m}{2}$, which is equivalent to $1 \leq a \leq \frac{es-1}{2}$. Thus $\dim_k(Z^n(\overline{\mathrm{mod}}(A))) = \frac{es-1}{2}$ in that case. If $\mathrm{char}(k) = 2$ then the condition $\bar{\varphi}(U) = -\bar{\varphi}(U)$ holds trivially, and so we get an extra dimension, leading to the formula $\dim_k(Z^n(\overline{\mathrm{mod}}(A))) = \frac{es+1}{2}$ as stated. This completes the proof of Theorem 1.3. $\square$

Proof of 1.5. If $|P| = 2$ then the trivial kP-module is, up to isomorphism, the unique indecomposable non projective kP-module, and hence there is an isomorphism of functors $\zeta : \mathrm{Id} \cong \Sigma$ on $\overline{\mathrm{mod}}(A)$. Thus $Z^*(\overline{\mathrm{mod}}(kP)) \cong \hat{H}^*(P;k) \cong k[\zeta,\zeta^{-1}]$ as claimed. If $|P| > 2$, or equivalently, $es \geq 2$, it follows from 7.2 is that any odd degree element in $Z^*(\overline{\mathrm{mod}}(A))$ composed with any other odd degree element or any even degree element in the radical of $Z^*(\overline{\mathrm{mod}}(A))$ yields zero. More explicitly, not only is the odd degree part of $Z^*(\overline{\mathrm{mod}}(A))$ an ideal which squares to zero, but it is in fact annihilated by $Z^{2n}(\overline{\mathrm{mod}}(A))$ for any integer n not divisible by e and by the radical of $Z^0(\overline{\mathrm{mod}}(A))$. The description of the graded center in terms of generators and relations is obtained by taking for π the image of a generator of the radical of $Z(A)/(Z(A) \cap \mathrm{soc}(A))$, for ζ the isomorphism $\mathrm{Id} \cong \Sigma^2$ and for the τ_i any k-basis of $Z^{-1}(\overline{\mathrm{mod}}(kP))$. $\square$

Proof of 1.1. If $\mathrm{char}(k) = 2$ and if $es = 1$ the result is a trivial verification (this is the case $|P| = 2$ in 1.5). If $es \geq 2$ or if $\mathrm{char}(k) \neq 2$, any two odd degree elements in $Z^*(\overline{\mathrm{mod}}(A))$ multiply to zero. Since the even degree part of $Z^*(\overline{\mathrm{mod}}(A))$ is a quotient of the even part of the Tate analogue of Hochschild cohomology by 1.2, the result follows. $\square$

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