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Bounds for Hochschild cohomology of block algebras

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Abstract

We show that for any block algebra B of a finite group over an algebraically closed field of prime characteristic p the dimension of $HH^n(B)$ is bounded by a function depending only on the nonnegative integer n and the defect of B. The proof uses in particular a theorem of Brauer and Feit which implies the result for $n = 0$.

Let p be a prime and k an algebraically closed field of characteristic p. Let G be a finite group and B a block algebra of kG; that is, B is an indecomposable direct factor of kG as a k-algebra. A *defect group of* B is a minimal subgroup P of G such that B is isomorphic to a direct summand of $B \otimes_{kP} B$ as a B-B-bimodule. The defect groups of B form a G-conjugacy class of p-subgroups of G, and the *defect of* B is the integer $d(B)$ such that $p^{d(B)}$ is the order of the defect groups of B. The *weak Donovan conjecture* states that the Cartan invariants of B are bounded by a function depending only on the defect $d(B)$ of B. As a consequence of a theorem of Brauer and Feit [3], the number of isomorphism classes of simple B-modules is bounded by a function depending only on $d(B)$. Thus the weak Donovan conjecture would imply that the dimension of a basic algebra of B is bounded by a function depending on $d(B)$. This in turn would imply that the dimension of the term in any fixed degree n of the Hochschild complex of a basic algebra of B is bounded by a function depending on n and $d(B)$; since Hochschild cohomology is invariant under Morita equivalences, we would thus get that the dimension of $HH^n(B)$ is bounded by a function depending on n and $d(B)$. The purpose of this note is to show that this consequence of the weak Donovan conjecture does indeed hold.

Theorem 1. *There is a function $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ such that for any integer $n \geq 0$, any finite group G and any block algebra B of kG with defect d we have*

$$\dim_k(HH^n(B)) \leq f(n, d)$$

For $n = 0$ this follows from the aforementioned theorem of Brauer and Feit [3], since $HH^0(B) \cong Z(B)$. Using Tate duality, the theorem above extends to Tate cohomology for negative n. A result of Külshammer and Robinson [7, Theorem 1] implies that it suffices to show theorem 1 for finite groups with a non-trivial normal p-subgroup. We follow a slightly different strategy in the proof below, reducing the problem directly to finite groups with a non-trivial central p-subgroup.

Remark 2. We make no effort to construct a best possible bound; we define the function f in theorem 1 inductively as follows: we set $f(0, 0) = 1$, $f(n, 0) = 0$ for $n > 0$; for all $d > 0$, $f(0, d)$ is the largest integer less or equal to the bound $\frac{1}{4}p^{2d} + 1$ given in the Brauer-Feit theorem (one could, of course, also take $f(0,1) = p$ and $f(0,d) = p^{2d-2}$ for $d \geq 2$; cf. [6, Ch. VII, 10.14]), and for $n > 0$, $d > 0$ we set

$$f(n,d) = p \cdot c(d) \cdot \sum_{i=0}^{n} f(i, d-1)$$

where $c(d)$ is the maximum of the numbers of subgroups in any finite group of order p^d.

Let G be a finite group and U a kG-module. We denote as usual by U^G the subspace of G-fixed points in U. If H is a subgroup of G then $U^G \subseteq U^H$, and there is a *trace map* $\mathrm{tr}_H^G : U^H \to U^G$ sending $u \in U^H$ to $\sum_{x \in [G/H]} xu$, where $[G/H]$ is a set of representatives of the H-cosets in G; one checks that this map is independent of the choice of $[G/H]$ and that its image, denoted U_H^G, is contained in U^G. For Q a p-subgroup of G, we denote the Brauer construction of U with respect to Q by $U(Q) = U^Q / \sum_{R;R<Q} U_R^Q$ and by $\mathrm{Br}_Q^U : U^Q \to U(Q)$ the canonical surjection, called *Brauer homomorphism.* A block algebra B of kG can be viewed as an indecomposable $k(G \times G)$-module, with $(x,y) \in G \times G$ acting by left multiplication with x and right multiplication with y^{-1}. For H a subgroup of G, we denote by ΔH the 'diagonal' subgroup $\Delta H = \{(h,h) \mid h \in H\}$ in $G \times G$. In particular, the action of ΔG on B can be identified with the conjugation action of G on B. The Brauer construction applied to B with respect to ΔQ is canonically isomorphic to $kC_G(Q)c$, where Q is a p-subgroup of G and $c = \mathrm{Br}_{\Delta Q}(1_B)$. A *$B$-Brauer pair* is a pair (Q,e) consisting of a p-subgroup Q of G and of a block idempotent e of $kC_G(Q)$ satisfying $e\mathrm{Br}_Q(1_B) = e$. The set of B-Brauer pairs is a G-poset in which the maximal pairs are all conjugate. The maximal B-Brauer pairs are exactly the B-Brauer pairs (Q,e) for which Q is a defect group of B. See [2] and [9, §11, §40] for details. In what follows we use without further comment the canonical graded isomorphism $HH^*(B) \cong H^*(\Delta G; B)$; see [8, (3.2)]. The following result is certainly well-known but not always stated in exactly the form we need it; we therefore give a proof for the convenience of the reader.

Proposition 3. *Let G be a finite group, B be a block algebra of kG and Q a p-subgroup of G. Set $b = 1_B$ and $c = \mathrm{Br}_Q(b)$. Suppose that $c \neq 0$ and set $B_Q = kC_G(Q)cb$. Then we have a direct sum decomposition of $kN_{G\times G}(\Delta Q)$-modules*

$$\mathrm{Res}^{G\times G}_{N_{G\times G}(\Delta Q)}(B) = B_Q \oplus C_Q$$

such that multiplication by b is an isomorphism of $kN_{G\times G}(\Delta Q)$-modules $kC_G(Q)c \cong B_Q$ and such that $C_Q(\Delta Q) = \{0\}$.

The proof we present here uses the following well-known lemma, which is a special case of expressing relative projectivity in terms of the splitting of adjunction maps (the general theme behind this is developed in [4], [5], for instance).

Lemma 4. *Let $\alpha : B \to A$ be a homomorphism of k-algebras. Suppose that B is isomorphic to a direct summand of A as a B-B-bimodule. Then α is injective and $\mathrm{Im}(\alpha)$ is a direct summand of A as a B-B-bimodule.*

Proof. The left or right action of an element $b \in B$ on A is given by left or right multiplication with $\alpha(b)$. Let $\iota : B \to A$ and $\pi : A \to B$ be B-B-bimodule homomorphisms satisfying $\pi \circ \iota = \mathrm{Id}_B$. Then $\iota(1_B)$ commutes with $\mathrm{Im}(\alpha)$, the map β sending $a \in A$ to $a\iota(1_B)$ is an A-B-bimodule endomorphism of A, and we have $\beta(\alpha(b)) = \alpha(b)\iota(1_A) = \iota(b)$, hence $\beta \circ \alpha = \iota$. Thus $\pi \circ \beta \circ \alpha = \mathrm{Id}_B$, which shows that as a B-B-bimodule homomorphism, α is split injective with $\pi \circ \beta$ as a retraction. □

Proof of Proposition 3. For any block of $kN_G(Q)$ which appears in $kN_G(Q)c$, the block B of kG is the corresponding 'induced' block. By [1, §14, Lemma 1], $kN_G(Q)c$ is isomorphic to a direct summand of B as a $kN_G(Q)$-$kN_G(Q)$-bimodule, and thus of cBc, as a $kN_G(Q)c$-$kN_G(Q)c$-bimodule. By lemma 4, multiplication by b induces an algebra homomorphism $kN_G(Q)c \to cBc$ which is split injective as a homomorphism of $kN_G(Q)c$-$kN_G(Q)c$-bimodules. Since $kC_G(Q)c$ is a direct summand of $kN_G(Q)c$ as an $N_{G\times G}(\Delta Q)$-module we get that $kC_G(Q)c \cong B_Q$ and that B_Q is a direct summand of B as an $N_{G\times G}(\Delta Q)$-module. Moreover, $B(\Delta Q) \cong B_Q$, and hence any complement C_Q of B_Q in B, as an $N_{G\times G}(\Delta Q)$-module, satisfies $C_Q(\Delta Q) = \{0\}$. □

We will make use of the following well-known fact on transfer in cohomology (we include a short proof for the convenience of the reader).

Lemma 5. *Let G be a finite group, H a subgroup of G and V a kH-module. Let U be a direct summand of $\mathrm{Ind}_H^G(V)$. Then $H^*(G;U) = \mathrm{tr}_H^G(H^*(H;\mathrm{Res}_H^G(U)))$.*

Proof. By Higman's criterion there is a kH-endomorphism φ of U such that $\mathrm{Id}_U = \mathrm{tr}_H^G(\varphi)$. Let $n \geq 0$ and let $\zeta : \Omega^n(k) \to U$ be a kG-homomorphism, representing an element in $H^n(G;U)$. Then $\zeta = \mathrm{Id}_U \circ \zeta = \mathrm{tr}_H^G(\varphi \circ \zeta)$, whence the result. □

This is applied in the following situation:

Lemma 6. *Let G be a finite group, B a block algebra of kG and P a defect group of B. We have $H^*(\Delta G;B) = \mathrm{tr}_{\Delta P}^{\Delta G}(H^*(\Delta P;B))$.*

Proof. As a $k(G\times G)$-module, B has vertex ΔP and trivial source, thus is isomorphic to a direct summand of $\mathrm{Ind}_{\Delta P}^{G\times G}(k)$. Mackey's formula shows that $\mathrm{Res}_{\Delta G}^{G\times G}(B)$ is still relatively ΔP-projective, hence lemma 5 implies the result. Alternatively, this follows from the fact that $b = 1_B$ can be written as a relative trace of the form $b = \mathrm{Tr}_{\Delta P}^{\Delta G}(y)$ for some $y \in B^{\Delta P}$. □

Proposition 7. *Let G be a finite group and B be a block algebra of kG. Set $b = 1_B$ and for every B-Brauer pair (Q,e) set $B_{(Q,e)} = kC_G(Q)eb$. Then $B_{(Q,e)}$ is a direct summand of B as a $k(C_G(Q)\times C_G(Q))\Delta Q$-module, isomorphic to $kC_G(Q)e$. In particular, $H^*(\Delta Q;B_{(Q,e)})$ is a direct summand, as a graded vector space, of $H^*(\Delta Q;B)$, and we have*

$$H^*(\Delta G;B) = \sum_{(Q,e)} \mathrm{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q;B_{(Q,e)}))$$

where in the sum (Q,e) runs over a set of representatives of the G-conjugacy classes of B-Brauer pairs.

Proof. The proof adapts techniques that have been used in the proof of a result of Watanabe [10, Lemma 1]. Clearly $H^*(\Delta G;B)$ contains the right side in the displayed equation. We need to show that $H^*(\Delta G;B)$ is contained in the right side. Since any summand of the right side of the form $\mathrm{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q; B_{(Q,e)})$ depends only on the G-conjugacy class of (Q, e) it suffices to prove the inclusion

$$H^*(\Delta G; B) \subseteq \sum_Q \mathrm{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q; B_Q))$$

where Q runs over the p-subgroups of G for which $\mathrm{Br}_Q(b) \neq 0$. Note that this makes sense since B_Q is a direct summand of B as a $k\Delta Q$-module, hence $H^*(\Delta Q; B_Q)$ is a subspace of $H^*(\Delta Q; B)$, to which we then apply the transfer map $\mathrm{tr}_{\Delta Q}^{\Delta G}$. Since $H^*(\Delta G; B) = \mathrm{tr}_{\Delta P}^{\Delta G}(H^*(\Delta P; B))$ by lemma 6 it suffices to show that the right side contains $\mathrm{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B))$ for any p-subgroup R of G. This will be shown by induction. For $R = \{1\}$ this holds trivially because $B_{\{1\}} = B$ and $C_{\{1\}} = \{0\}$. For $R \neq \{1\}$ we have a direct sum decomposition $B = B_R \oplus C_R$ of $kN_{G\times G}(\Delta R)$-modules as in proposition 3, and hence

$$H^*(\Delta R; B) = H^*(\Delta R; B_R) + H^*(\Delta R; C_R)$$

Since $C_R(\Delta R) = \{0\}$ we have

$$H^*(\Delta R; C_R) \subseteq \sum_{S; S<R} \mathrm{tr}_{\Delta S}^{\Delta R}(H^*(\Delta S; B))$$

by lemma 5. Applying the transfer map $\mathrm{tr}_{\Delta R}^{\Delta G}$ yields

$$\mathrm{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; C_R)) \subseteq \sum_{S; S<R} \mathrm{tr}_{\Delta S}^{\Delta G}(H^*(\Delta S; B))$$

hence

$$\mathrm{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B)) \subseteq \mathrm{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B_R)) + \sum_{S; S<R} \mathrm{tr}_{\Delta S}^{\Delta G}(H^*(\Delta S; B))$$

The result follows by induction. □

Lemma 8. *Let G be a finite group and B be a block algebra of kG. Set $b = 1_B$ and for every B-Brauer pair (Q, e) set $B_{(Q,e)} = kC_G(Q)eb$. For any integer $n \geq 0$ we have*

$$\dim_k(\mathrm{tr}_{\Delta Q}^{\Delta G}(H^n(\Delta Q; B_{(Q,e)}))) \leq \dim_k(H^n(\Delta QC_G(Q); kQC_G(Q)e))$$

Proof. Clearly $\dim_k(\mathrm{tr}_{\Delta Q}^{\Delta G}(H^n(\Delta Q; B_{(Q,e)}))) \leq \dim_k(\mathrm{tr}_{\Delta Q}^{\Delta QC_G(Q)}(H^n(\Delta QC_G(Q); B_{(Q,e)})))$. Moreover, since $B_{(Q,e)} \cong kC_Q(Q)e$ is isomorphic to a direct summand of $kQC_G(Q)e$, the lemma follows. □

Lemma 9. *Let G be a finite group, B a block of kG and Z a subgroup of order p of $Z(G)$. Set $\bar{G} = G/Z$ and denote by $\bar{B}$ the image of B in $k\bar{G}$ under the canonical algebra homomorphism $kG \to k\bar{G}$. For any integer $n \geq 0$ we have*

$$\dim_k(H^n(\Delta G; B)) \leq p \cdot \sum_{i=0}^{n} \dim_k(H^i(\Delta\bar{G}; \bar{B}))$$

Proof. The Lyndon-Hochschild-Serre spectral sequence associated with G, Z, $\bar{G}$ and B endowed with the conjugation action of G reads

$$H^i(\Delta\bar{G}; H^j(\Delta Z; B)) \Rightarrow H^{i+j}(\Delta G; B)$$

Since ΔZ acts trivially on kG, hence on B, we have $H^j(\Delta Z; B) \cong H^j(\Delta Z; k) \otimes_k B \cong B$, where the last isomorphism uses that we have $H^j(\Delta Z; k) \cong k$ because Z is cyclic. Thus $H^n(\Delta G; B)$ is filtered by subquotients of $H^i(\Delta\bar{G}; B))$, with $0 \le i \le n$; in particular, $\dim_k(H^n(\Delta G; B)) \le \sum_{i=0}^n \dim_k(H^i(\Delta\bar{G}; B))$. Let z be a generator of Z. As a $k\Delta\bar{G}$-module, B has a filtration of the form

$$B \supseteq B(1-z) \supseteq B(1-z)^2 \supseteq \cdots \supseteq B(1-z)^{p-1} \supseteq \{0\}$$

and since B is projective as a right kZ-module, the quotient of any two consecutive terms in this filtration is isomorphic to $\bar{B}$. Thus the appropriate long exact sequences in cohomology imply that $\dim_k(H^i(\Delta\bar{G}; B)) \le p \cdot \dim_k(H^i(\Delta\bar{G}; \bar{B}))$, whence the result. □

Proof of Theorem 1. Let f be the function defined in remark 2. Note that $f(n, d) \ge f(n, d-1)$ for all $n \ge 0$ and all $d > 0$. Denote by $c(d)$ the maximum of the numbers of subgroups in finite groups of order p^d. As mentioned before, theorem 1 holds for $n = 0$. Clearly theorem 1 holds for $d = 0$ because a defect zero block is a matrix algebra. Let n and d be a positive integers. Then $\mathrm{tr}_{\Delta 1}^{\Delta G}(H^n(1; B)) = \{0\}$. Thus, by proposition 7 and lemma 8 we have $\dim_k(HH^n(B)) \le \sum_{(Q,e)} \dim_k(HH^n(kQC_G(Q)e))$ where in the sum (Q, e) runs over a set of representatives of the G-conjugacy classes of non-trivial B-Brauer pairs. Any such pair (Q, e) has a conjugate with Q contained in a fixed defect group P, and hence the number of summands in this sum is at most $c(d)$. Moreover, $Z(QC_G(Q))$ contains $Z(Q)$, and hence $QC_G(Q)$ has a non-trivial central subgroup Z_Q of order p. After replacing (Q, e) by a suitable G-conjugate, we may assume that $QC_P(Q)$ is a defect group of e viewed as a block of $kQC_G(Q)$; in particular the defect groups of e have order at most $|P| = p^d$. Thus the defect groups of the image $\bar{e}$ of e in $kQC_G(Q)/Z_Q$ have order at most $|P|/p = p^{d-1}$, hence $\dim_k(HH^n(kQC_G(Q)/Z_Q\bar{e})) \le f(n, d-1)$. It follows from lemma 9 that $\dim_k(HH^n(kQC_G(Q)e)) \le p \cdot \sum_{i=0}^n f(i, d-1)$. Together with the above remarks we get the inequality $\dim_k(HH^n(B) \le p \cdot c(d) \cdot \sum_{i=0}^n f(i, d-1) = f(n, d)$, as required. □

Remark 10. The strong version of Donovan's conjecture states that for a fixed integer $d \ge 0$ there should be only finitely many Morita equivalence classes of blocks with defect at most d. If true, this would imply that there are only finitely many isomorphism classes of Hochschild cohomology algebras of blocks with defect at most d; this remains an open problem.

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