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THE STRUCTURE OF BLOCKS WITH A KLEIN FOUR DEFECT GROUP

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Abstract. We prove Erdmann's conjecture [16] stating that every block with a Klein four defect group has a simple module with trivial source, and deduce from this that Puig's finiteness conjecture holds for source algebras of blocks with a Klein four defect group. The proof uses the classification of finite simple groups.

1. Introduction

A conjecture of Puig states that for a given prime p and a finite p-group P there are only finitely many isomorphism classes of interior P-algebras arising as source algebras of p-blocks of finite groups with defect groups isomorphic to P. While Puig's conjecture has been verified for many classes of finite groups (cf. [25], [26], [29], [30], [31], [32], [41]), the case where P is cyclic has been to date the only case where this conjecture has been shown to hold in full generality (cf. [38]). Using the classification of the finite simple groups we show that Puig's conjecture holds for all blocks with a Klein four defect group.

Theorem 1.1. *Let $\mathcal{O}$ be a complete local Noetherian commutative ring with an algebraically closed residue field k of characteristic 2. Let G be a finite group and A a source algebra of a block of $\mathcal{O}G$ with a Klein four defect group P. Then A is isomorphic to one of $\mathcal{O}P$, $\mathcal{O}A_4$ or the principal block algebra of $\mathcal{O}A_5$. In particular, there are exactly three isomorphism classes of source algebras of blocks with a Klein four defect group.*

Here $\mathcal{O}A_4$ and the principal block algebra of $\mathcal{O}A_5$ are viewed as interior P-algebras via some identification of P with the Sylow 2-subgroups of A_4 and A_5. The interior P-algebra structure is independent of the choice of an identification because any automorphism of a Sylow 2-subgroup of A_4 or A_5 extends to an automorphism of A_4 or A_5, respectively. In view of the description of the source algebras of blocks with a Klein four defect group in [37], Theorem 1.1 is equivalent to the following theorem, first conjectured by Erdmann [16].

Theorem 1.2. *Every 2-block of a finite group over a field of characteristic 2 with a Klein four defect group P has at least one simple module with trivial source.*

Erdmann's results on sources of simple modules in [16] tell us that a block b of kG with a Klein four defect group P satisfies one of the following: b has one simple module, with source $\Omega_P^n(k)$ for some $n \in \mathbb{Z}$, in which case kGb is Morita equivalent to kP; or b has three simple modules, all having isomorphic source $\Omega_P^n(k)$ for some $n \in \mathbb{Z}$, in which case kGb is Morita equivalent to kA_4; or b has three simple modules and there is $n \in \mathbb{Z}$ such that one simple module has source $\Omega_P^n(k)$, whilst the other two have sources $\Omega_P^n(U)$, $\Omega_P^n(V)$ for indecomposable kP-modules U, V of dimension two, in which case kGb is Morita equivalent to the principal block algebra of kA_5. Recall that the *Dade group* $D_k(P)$ is the abelian group consisting of the isomorphism classes of indecomposable endo-permutation kP-modules with vertex P, with product induced by the tensor product of modules and inverse induced by duality; the unit element of $D_k(P)$ is the isomorphism class of the trivial kP-module k. Any Heller translate of an endo-permutation module is again an endo-permutation module. By the classification of endo-permutation modules, if P is a Klein four group, the only torsion element of the Dade group $D_k(P)$ is trivial. It then follows that Theorem 1.2 is equivalent to any of the following statements:

(1) any source of a simple module of a block with a Klein four defect group has dimension at most 2;
(2) no simple module of a block with a Klein four defect group P can have a source of the form $\Omega^n(k)$ for any non-zero integer n;
(3) no simple module of a block with a Klein four defect group P can have a source which is an endo-permutation module whose image in the Dade group $D_k(P)$ has infinite order.

It is this last version that we will reduce, in Theorem 4.1, to the situation where the underlying group G is of the form $G = NP$ for some quasi-simple finite group N and then run through the classification of the finite simple groups to exclude this scenario. In the first of the above equivalent statements one can be more precise about which modules actually do occur as sources.

Corollary 1.3. *There are, up to isomorphism, exactly three indecomposable kP-modules which occur as sources of simple modules of blocks with a Klein four defect group P, namely the trivial kP-module and two 2-dimensional kP-modules which are self-dual and Galois conjugate, and which are characterized by the fact that their isomorphism classes are stable under the automorphisms of order 3 of P.*

Given our reduction Theorem 4.1, our proof of Theorem 1.1 relies on the following result about dualities in Klein four defect blocks of almost simple groups which may be of independent interest. For the definition and relevance of automorphically dual and Galois dual blocks, we refer the reader to Section 5, in particular 5.4.

Theorem 1.4. *Let k be an algebraically closed field of characteristic 2, G a finite group and b a block of kG with a Klein four defect group $P \cong C_2 \times C_2$. Suppose that $G = N \rtimes R$ where N is a quasi-simple group with centre of odd order, R is a complement of $P \cap N$ in P, and $O_2(G) = 1$. Then either b is real, or automorphically dual, or Galois dual, or there exists a finite group L containing G as a subgroup of odd index and a block f of kL such that kGb and kLf are source-algebra equivalent, and such that f is automorphically dual or f is Galois dual.*

The paper is divided into twelve sections. Section 2 contains some background results from block theory. In Section 3, we develop some Clifford-theoretic methods to be used in later sections. The main result of Section 4 is the Theorem 4.1, which describes the structure of a minimal counter-example to Theorem 1.1 as a quasi-simple group extended by a 2-group of automorphisms. Section 5 discusses various dualities in block theory that are of relevance to us. Sections 6 and 7 contain some general results on finite groups of Lie type. The remainder, Sections 8–12, run through the various classes of finite simple group. We prove Theorem 1.1, and Theorem 1.4 at the end of Section 12.

2. Quoted results

We collect in this section, mostly without proofs, background material which will be needed at a later stage. Let $\mathcal{O}$ be a complete local Noetherian commutative ring having an algebraically closed residue field k of positive characteristic p.

Proposition 2.1 (Külshammer [35, Proposition 5.5])**.** *Let G be a finite group, b be a block of $\mathcal{O}G$, P be a defect group of b, N be a normal subgroup of G, and c be a G-stable block of $\mathcal{O}N$ such that $cb = b$. Set $C = \mathcal{O}Nc$. Suppose that every element $x \in G$ acts as an inner automorphism on C. Choose for any element $x \in G$ an element $a_x \in C^\times$ such that ${}^x a = a_x a a_x^{-1}$ for all $a \in C$ and such that $a_x n = a_{xn}$ for all $x \in G$ and $n \in N$. For any two $x, y \in G$ denote by $\alpha(x, y)$ the unique element in $Z(C)^\times$ satisfying $a_x a_y = \alpha(x, y)^{-1} a_{xy}$. Then the value $\alpha(x, y)$ depends only on the cosets xN, yN, hence α can be viewed as element in $Z^2(G/N; Z(C)^\times)$ and we have algebra isomorphisms*

$$\begin{cases} \mathcal{O}Gc & \cong & C \otimes_{Z(C)} Z(C)_\alpha(G/N) \\ xa & \to & a_x \otimes \bar{x} \\ xa_x^{-1}a & \leftarrow & a \otimes \bar{x} \end{cases}$$

where $x \in G$, $a \in C$ and where $\bar{x}$ is the canonical image of x in the twisted group algebra $Z(C)_\alpha(G/N)$ of G/N over $Z(C)$ with respect to α.

By results of Berger [1], [2] and Feit [17], simple modules of finite p-soluble groups are algebraic in the sense of Alperin. By a result of Puig (cf. [44, (30.5)]), their sources are endo-permutation modules, and hence their sources are torsion in the Dade group (a fact also proved by Puig [41] and Boltje–Külshammer [3, Prop. 4.4]). For odd p this requires the fact that the outer automorphism group of a simple p'-group has p-rank at most one, known only thanks to the classification of the finite simple groups. For $p = 2$, this follows from the Odd Order Theorem. We need the following particular case:

Proposition 2.2. *Let N be a finite p'-group, P a p-subgroup of $\mathrm{Aut}(N)$ and let c be a P-stable block of kN. Denote by V an endo-permutation kP-module such that $kNc \cong \mathrm{End}_k(V)$ as P-algebras. The image of V in the Dade group $D_k(P)$ is a torsion element.*

Proposition 2.3 (Cabanes [8])**.** *Let N be a normal subgroup of a finite group G such that G/N is a p-group and let b be a G-stable nilpotent block of kN. Then b is a nilpotent block of kG.*

Proposition 2.4. *Let N be a normal subgroup of a finite group G such that G/N is an abelian p-group and $O_p(G) = 1$, and let P be a finite p-subgroup of G such that $G = NP$. Then the canonical map $P/P \cap N \to \mathrm{Out}(N)$ is injective.*

Proof. Let $u \in P$ such that u acts as an inner automorphism of N. Then there is a p-element $y \in N$ such that $uy^{-1} \in C_G(N)$. Thus uy^{-1} and y commute, hence u and y commute, and therefore uy^{-1} is a p-element. Write $H = \langle uy^{-1}, N\rangle = \langle u, N\rangle$; then $uy^{-1} \in O_p(H)$. But $H \lhd G$ since G/N is abelian, so $O_p(H) \leq O_p(G) = 1$. Hence $u = y \in N$ and we are done. □

3. Some Clifford-theoretic results

Proposition 3.1. *Let G be a finite group, b be a block of $\mathcal{O}G$, P be a defect group of b, N be a normal subgroup of G, and c be a G-stable block of $\mathcal{O}N$ such that $cb = b$. Set $Q = P \cap N$. Suppose that $Q \subseteq Z(P)$. Assume that c is a nilpotent block. Then the elements of P act as inner automorphisms on $\mathcal{O}Nc$.*

Proof. The hypotheses imply that Q is a defect group of c and that Q is abelian. Since c is nilpotent, the block algebra $\mathcal{O}Nc$ is isomorphic to a matrix algebra over $\mathcal{O}Q$. In particular, $\mathrm{Out}(\mathcal{O}Nc) \cong \mathrm{Aut}(\mathcal{O}Q)$. We have $\mathrm{Br}_P(b) \neq 0$, hence $\mathrm{Br}_P(c) \neq 0$. The idempotent $\mathrm{Br}_Q(c)$ is a sum of blocks of $kC_N(Q)$ that are permuted by P. At least one of them is fixed by P because if not we would get the contradiction $0 = \mathrm{Br}_P(\mathrm{Br}_Q(c)) = \mathrm{Br}_P(c)$. Let e be a block of $kC_N(Q)$ that is fixed by P. Then, for any subgroup R of Q, the unique block e_R of $kC_N(R)$ satisfying $(R, e_R) \subseteq (Q, e)$ is fixed by P. Since c is nilpotent, so are the blocks e_R for R a subgroup of Q. In particular, e_R has a unique Brauer character, or equivalently, there is a unique local point $\epsilon(R)$ of R on $\mathcal{O}Nc$ associated with e_R. Thus, in particular, the action of P fixes all local pointed elements of the form $(u, \epsilon(u))$, where u is an element in Q and $\epsilon(u) = \epsilon(\langle u \rangle)$. If we choose $j_u \in \epsilon(u)$, then the matrix consisting of the numbers $\chi(uj_u)$ with $\chi \in \mathrm{Irr}_K(N, c)$ and $u \in Q$ is the matrix of generalized decomposition numbers of the block c. Thus all generalized decomposition numbers are fixed by P, hence all irreducible characters belonging to c are fixed, by Brauer's permutation lemma. This means that P acts as the identity in $Z(\mathcal{O}Nc)$. But then P acts as the identity on $\mathcal{O}Q$ via the Morita equivalence between $\mathcal{O}Nc$ and $\mathcal{O}Q$. Again, via this Morita equivalence, the identity automorphism on $\mathcal{O}Q$ corresponds to an inner automorphism of $\mathcal{O}Nc$. □

Remark 3.2. The previous result remains true if Q is cyclic of order p, even if c is not nilpotent, because the outer automorphism group of a block over $\mathcal{O}$ with defect group of order p is a p'-group. This could be useful for potential extensions of the next section to odd p.

The following trivial lemma is on identifying actions of p-groups in tensor products; we use the well-known fact that if a finite-dimensional k-algebra A has a unitary matrix subalgebra C then multiplication in A induces an isomorphism $C \otimes_k C_A(C) \cong A$ (cf. [44, (7.5)]).

Lemma 3.3. *Let P be a finite p-group and A a finite-dimensional interior P-algebra over k with structural homomorphism $\rho : P \to A^\times$. Suppose that C is a P-stable unitary matrix subalgebra of A, and set $D = C_A(C)$. Identify A with $C \otimes_k D$. Then there are unique group homomorphisms $\sigma : P \to C^\times$ and $\tau : P \to D^\times$ lifting the actions of P on C and D, respectively, such that $\rho(u) = \sigma(u) \otimes \tau(u)$ for all $u \in P$.*

Proof. Since any automorphism of C is inner and since $H^2(P; k^\times) = H^1(P; k^\times) = 0$, the action of P on C lifts uniquely to a group homomorphism $\sigma : P \to C^\times$. Let $u \in P$. Then $\rho(u)(\sigma(u)^{-1} \otimes 1)$ acts trivially on C, hence belongs to D. Setting $\tau(u) = \rho(u)(\sigma(u)^{-1} \otimes 1)$ yields the result. □

The next observation is a sufficient criterion for when a normal p'-extension of a nilpotent block remains nilpotent.

Lemma 3.4. *Let N be a normal subgroup of p'-index of a finite group G and c a G-stable nilpotent block of kN having a defect group Q such that $N_G(Q) = QC_G(Q)$. Then every block b of kG satisfying $bc \neq 0$ is nilpotent, with Q as a defect group.*

Proof. Let b be a block of kG such that $bc \neq 0$. Then $bc = b$ since c is G-stable, and Q is a defect group of b since $|G : N|$ is prime to p. Let d_Q be a block of $C_N(Q)$ such that $\mathrm{Br}_Q(c)d_Q \neq 0$; that is, (Q, d_Q) is a maximal (N, c)-Brauer pair. Since N acts transitively on the set of maximal (N, c)-Brauer pairs, a Frattini argument yields $G = N_G(Q, d_Q)N$. The hypothesis $N_G(Q) = QC_G(Q)$ implies that $G = C_G(Q, d_Q)N$. We show first that any (N, c)-Brauer pair (R, d) is $C_G(R)$-stable. After possibly replacing (R, d) by a suitable N-conjugate, we may assume that $(R, d) \leq (Q, d_Q)$. Since $G = C_G(Q, d_Q)N$ we have $C_G(R) = C_G(Q, d_Q)C_N(R)$. The uniqueness of the inclusion of Brauer pairs implies that $C_G(R)$ stabilizes (R, d). Let (R, e) be a (G, b)-Brauer pair, such that $R \leq Q$ and d a block of $C_N(R)$ such that $de \neq 0$. Then (R, d) is an (N, c)-Brauer pair, hence d is $C_G(R)$-stable, and thus $de = e$. It follows that $N_G(R, e) \leq N_G(R, d) = C_G(Q)N_N(R, d)$, and hence $N_G(R, e)/RC_G(R)$ is isomorphic to a subgroup of $N_N(R, d)/RC_N(R)$, which is a p-group since c is nilpotent. Thus b is nilpotent as well. □

The following two results are slight generalizations of [34, Lemmas 4.1 and 4.2].

Lemma 3.5. *Let N and J be normal subgroups of a finite group H such that $N \leq J$. Suppose that H/J is a p-group and that J/N is a cyclic p'-group. Let b be a nilpotent H-stable block of kN. Then there is an H-stable block f of kJ such that $bf = f$. Moreover, if $J/N \leq Z(H/N)$ then every block f of kJ satisfying $bf = f$ is H-stable.*

Proof. Let W be a simple kNb-module. Since b is nilpotent, W is unique up to isomorphism. Since b is also H-stable it follows that the isomorphism class of W is H-stable. As J/N is a cyclic p'-group, W extends, up to isomorphism, in exactly $|J : N|$ different ways to a simple kJ-module, and these are exactly the simple kJb-modules, up to isomorphism. The group H acts on the set S of isomorphism classes of the simple kJb-modules extending W. Clearly J acts trivially on S, so this induces an action of H/J on S. Since H/J is a p-group while $|J : N|$ is prime to p, this action has at least one fixed point. Thus there is an H-stable simple kJb-module V that extends W. The block f of kJ containing V is then also H-stable and satisfies $bf = f$. Moreover, any other simple kJb-module U extending W is then isomorphic to $V \otimes_k \mathrm{Inf}^J_{J/N}(T)$ for some 1-dimensional kJ/N-module T. If $J/N \leq Z(H/N)$ then the isomorphism class of $\mathrm{Inf}^J_{J/N}(T)$ is H-stable: hence U is H-stable, and the result follows. □

Lemma 3.6. *Let N and J be normal subgroups of a finite group L, and let G be a subgroup of L such that $L = JG$ and such that $J \cap G = N$. Suppose that G/N is a p-group and that J/N is a cyclic p'-group. Let b be an L-stable nilpotent block of kN and denote by (Q, d) a maximal (N, b)-Brauer pair. Let f be an L-stable block of kJ such that $bf = f$. Suppose that $N_J(Q) = QC_J(Q)$.*

(i) *The blocks kGb and kLf are nilpotent.*
(ii) *Any defect group P of kGb is a defect group of kLf.*
(iii) *If U is a simple kGb-module and V a simple kLf-module then $\mathrm{Res}^L_G(V) \cong U$.*
(iv) *The map $kGb \to kLf$ sending $a \in kGb$ to af is an interior G-algebra isomorphism; in particular, it induces an isomorphism of source algebras.*
(v) *Any vertex-source pair of a simple kGb-module U is a vertex-source pair of a simple kLf-module V.*

Proof. (i) Since G/N is a p-group and b is a G-stable nilpotent block of kN it follows from Proposition 2.3 that kGb is a nilpotent block. The block kJf is nilpotent by Lemma 3.4, whence kLf is nilpotent by Proposition 2.3.

(ii) Let P be a defect group of kGb. Since $L/J \cong G/N \cong P/P \cap N$ are p-groups and since $P \cap N$ is a defect group of kNb it follows that kLf has a defect group P_1 containing $Q_1 := P \cap N$ such that $P_1/Q_1 \cong P/Q_1$. Note that $|L : G| = |J/N|$ is prime to p. Let $x \in (kG)^P$ such that

$b = \mathrm{Tr}_P^G(x)$. Then

$$f = bf = \mathrm{Tr}_P^G(xf) = \mathrm{Tr}_P^L\left(\frac{1}{|L:G|}xf\right),$$

where we use that L stabilizes f and that $|L:G|$ is prime to p. Thus P contains a defect group of kLf, and so P is a defect group of kLf.

(iii) Since kGb is nilpotent, it has, up to isomorphism, a unique simple module U. Since kNb is also nilpotent and $|G:N|$ is a power of p it follows that $\mathrm{Res}_N^G(U)$ is, up to isomorphism, the unique simple kNb-module. The fact that b is L-stable implies in particular that $\mathrm{Res}_N^G(U)$ is J-stable. Since J/N is a cyclic p'-group, $\mathrm{Res}_N^G(U)$ extends in precisely $|J:N|$ non-isomorphic ways to a simple kJ-module, and these are exactly the simple kJ-modules whose restriction to N contains $\mathrm{Res}_N^G(U)$ as a direct summand; equivalently, these are exactly the simple kJb-modules. If V is a simple kLf-module then $\mathrm{Res}_J^L(V)$ is a simple kJf-module, and as $bf = f$ this implies that $\mathrm{Res}_J^L(V)$ is a simple kJb-module. Thus

$$\mathrm{Res}_N^G(\mathrm{Res}_G^L(V)) = \mathrm{Res}_N^J(\mathrm{Res}_J^L(V)) \cong \mathrm{Res}_N^G(U).$$

But U is, up to isomorphism, the unique extension of $\mathrm{Res}_N^G(U)$ to a kGb-module, and hence $\mathrm{Res}_G^L(V) \cong U$ as claimed.

(iv) Since f is a central idempotent in kL, the map sending $a \in kGb$ to af is an algebra homomorphism. By the preceding statements, kGb and kLf are nilpotent blocks with isomorphic defect groups and whose simple modules have the same dimension. Thus kGb and kLf have the same dimension. Moreover, kLf becomes a projective kGb-module through the map $a \mapsto af$. Since kGb has only one isomorphism class of projective indecomposable modules, it follows that the map $a \mapsto af$ is injective, and hence an isomorphism. This is obviously an isomorphism of interior G-algebras and thus, for any choice of a defect group P of kGb and a source idempotent $i \in (kGb)^P$, the image if is a source idempotent for kLf and we have an isomorphism of source algebras $ikGi \cong ifkLif$.

(v) This is an immediate consequence of the previous statement. □

Lemma 3.7. *Let $\tilde{G}$ be a finite group, G a subgroup of $\tilde{G}$ and P a p-subgroup of G. Let N be a normal subgroup of G such that $G = NP$ and $\tilde{N}$ a normal subgroup of $\tilde{G}$ such that $\tilde{G} = \tilde{N}P$. Suppose further that N is normal in $\tilde{N}$ and that $\tilde{N}/N$ is cyclic of order prime to p. Let b be a P-stable block of defect 0 of kN. Then there is a P-stable block $\tilde{b}$ of $k\tilde{N}$ such that $b\tilde{b} \neq 0$ and such that the blocks kGb and $k\tilde{G}\tilde{b}$ are source-algebra equivalent.*

Proof. Denote by H the stabilizer in $\tilde{N}$ of b. Then N is normal in H and H/N is cyclic of order prime to p. Thus the unique irreducible character χ associated with kNb extends in exactly $|H:N|$ different ways to an irreducible character of H, and each such character belongs to a block of defect 0 of kH. Thus, there are $|H:N|$ different blocks c of kH such that $bc = c$, the set of such blocks is permuted by P, and hence at least one such block c is P-stable. Moreover, multiplication by c is an isomorphism $kNb \cong kHc$, and if c is chosen P-stable, this extends to an isomorphism $kGb \cong kHPc$. In particular, kGb and $kHPc$ are source-algebra equivalent. Thus, repeating this step as necessary, we may assume that $H = N$. In that case, using that the different $\tilde{N}$-conjugates of b are pairwise orthogonal, we get that $\tilde{b} = \mathrm{Tr}_N^{\tilde{N}}(b)$ is a P-stable block of $k\tilde{N}$ satisfying $k\tilde{G}\tilde{b} \cong \mathrm{Ind}_{HP}^{\tilde{G}}(kGb)$. This shows that the block algebras kGb and $k\tilde{G}\tilde{b}$ are source-algebra equivalent. □

4. Reduction to quasi-simple groups

Let k be an algebraically closed field of characteristic 2.

Theorem 4.1. *Let G be a finite group and let b be a block of kG having a Klein four defect group $P \cong C_2 \times C_2$. Suppose that $|G/Z(G)|$ is minimal with the property that kGb has a simple module S with vertex P and an endo-permutation kP-module U as source such that the class of U in the Dade group has infinite order. Then $G = N \rtimes R$ where N is a normal subgroup of G isomorphic to an odd central extension of a non-abelian simple group and where R is a complement of $P \cap N$ in P. Moreover, the canonical map $R \to \mathrm{Out}(N)$ is injective and we have $O_2(G) = 1$. By also choosing $|Z(G)|$ to be minimal we may take N to be quasi-simple.*

Proof. By a standard Clifford theoretic argument, the minimality of $|G/Z(G)|$ implies that for any normal subgroup N of G containing $Z(G)$ there is a unique block b_N of kN such that $bb_N \neq 0$; thus b_N is G-stable and $bb_N = b$. Indeed, if b_N is not G-stable we can replace G by the stabilizer of b_N in G (which contains N, hence $Z(G)$ and which also contains a conjugate of P because $\mathrm{Br}_P(b) \neq 0$), contradicting the minimality of $|G/Z(G)|$. Let H be the normal subgroup of G generated by $Z(G)$ and all G-conjugates of P. By Clifford's theorem, $\mathrm{Res}^G_H(S)$ is semi-simple, and thus b_H has a simple module with vertex P and U as source. The minimality of $|G/Z(G)|$ implies that $G = H$. In particular, for any proper normal subgroup N of G containing $Z(G)$ the intersection $P \cap N$ is a proper subgroup of P, hence either trivial or cyclic of order 2. We have $O_2(G) = 1$ because $O_2(G) \leq P$ acts trivially on the source U, which would imply that if $O_2(G) \neq 1$ then $P/O_2(G)$ is cyclic and hence U has finite order in the Dade group $D_k(P)$. We show that $O_{2'}(G) = Z(G)$. Indeed, since $O_2(G) = 1$ we have $Z(G) \leq O_{2'}(G)$. If $N = O_{2'}(G)$ properly contains $Z(G)$, consider the isomorphism

$$kGb_N \cong C \otimes_k k_\alpha(G/N)$$

from Proposition 2.1, where $C = kNb_N$, which is a matrix algebra since N has odd order, hence $Z(C) \cong k$, for a suitable $\alpha \in Z^2(G/N; k^\times)$. Write

$$S = V \otimes_k W,$$

where $\mathrm{End}_k(V) \cong C$ and W is a simple $k_\alpha(G/N)$-module. Note that the restriction of V to P is an endo-permutation kP-module as P acts by permuting the elements of N. Let $V^\vee = \mathrm{Hom}_k(V, k)$ denote the contragradient dual of the kP-module V. Since U is a direct summand of $V \otimes_k W$ as a kP-module, $V^\vee \otimes_k U$ is a summand of $V^\vee \otimes_k V \otimes_k W$ as a kP-module, which in turn is isomorphic to a direct sum of copies of W and summands of smaller vertex. Thus, a summand of $V^\vee \otimes_k U$ is a source of W; in particular, the sources of W are endo-permutation. Therefore, at least one of V and W has a direct summand whose order in the Dade group is infinite. The order of the image of any summand of V of vertex P is finite because N has odd order (cf. Proposition 2.2). Thus we may replace G by a central extension of G/N, but that reduces the order of $G/Z(G)$, and hence is impossible. This proves that $O_{2'}(G) = Z(G)$. Let N be a proper normal subgroup of G that properly contains $Z(G)$. We first deal with the case where $P \cap N = 1$. Indeed, if $P \cap N = 1$ then b_N is a defect 0 block, and the isomorphism

$$kGb_N \cong kNb_N \otimes_k k_\alpha(G/N),$$

Writing, as above, $S = V \otimes_k W$ shows that W has an endo-permutation source which is torsion in the Dade group because the order of G/N is smaller than that of $G/Z(G)$. Thus V yields a non-torsion element in the Dade group for the block b_N viewed as block of $N \rtimes P$. The minimality of G implies that $G = N \rtimes P$, and then by Salminen's reduction [43], it follows that N is an odd central extension of a non-abelian simple group, as asserted. We may assume from now on that $Q = P \cap N$ is cyclic of order 2 for any proper normal subgroup N of G that properly contains $Z(G)$. We show next that $G = NP$ for any such N. By Proposition 3.1, the normal subgroup of G consisting of all elements acting as inner automorphisms on kNb_N contains P, and thus is equal to G by the above; thus we may apply Proposition 2.1. Setting $C = kNb_N$, we get a k-algebra isomorphism

$$kGb_N \;\cong\; C \otimes_{Z(C)} Z(C)_\alpha(G/N)$$

for some $\alpha \in Z^2(G/N; Z(C)^\times)$. The ideal in kGb_N generated by $J(Z(C))$ is nilpotent because N is normal in G, and hence is contained in $J(kGb_N)$. In order to detect sources of simple modules we lose no information if we divide out by this ideal. Since $Z(C)$ is local, we have that $Z(C)/J(Z(C)) \cong k$ and hence we get an algebra isomorphism

$$kGb_N/J(Z(C))kGb_N \cong \bar{C} \otimes_k k_\alpha(G/N),$$

where we now abusively use α to denote the image of α in $Z^2(G/N; k^\times)$ under the canonical map $Z(C)^\times \to k^\times$ induced by the canonical surjection $Z(C) \to Z(C)/J(Z(C)) \cong k$, and where $\bar{C} = C/J(Z(C))C$. Note that since C is a nilpotent block with an abelian defect group (namely Q) we have $J(C) = J(Z(C))C$ and hence $\bar{C}$ is a matrix algebra. Thus we can write as before $S \cong X \otimes_k Y$ for some simple C-module X and some simple $k_\alpha(G/N)$-module Y. We need to identify the action of P. The matrix algebra $\bar{C}$ is not only the semi-simple quotient of kNb_N but also of the nilpotent block $kNPb_N$ thanks to the structure theory of these blocks. Thus X is the restriction to kNb_N of the unique simple $kNPb_N$-module – we again abuse notation and denote this by X – and as such, has an endo-permutation kP-module V as source. The module Y belongs to a block of an odd central extension of G/N with defect group P/Q of order 2, and hence has a trivial source W. By Lemma 3.3, U is a summand of $V \otimes_k W$ for some choice of the source V of X. This means that V cannot be a torsion element in $D(P)$. In other words, the simple $kNPb_N$-module X has a source which is not torsion in $D(P)$ and hence $G = NP$ by the minimality of $|G/Z(G)|$. Since $|N \cap P| = 2$ for any proper normal subgroup N of G containing $Z(G)$, it follows that G has a unique minimal normal subgroup N strictly containing $Z(G)$, and $N/Z(G)$ must be a direct product of isomorphic non-abelian simple groups since $O_{2'}(G)O_2(G) \leq Z(G)$. But N must have a G-stable block of defect 1, so in fact $N/Z(N)$ is simple. Since P is a Klein four group, the subgroup $P \cap N$ has a complement R in P, and then $G = N \rtimes R$. This proves the first assertion of the theorem. By Proposition 2.4 the canonical map $R \to \mathrm{Out}(N)$ is injective (since $O_2(G) = 1$). Since $N/Z(G)$ is non-abelian simple we have $N = Z(G)[N, N]$, and hence $[N, N]P$ is normal in $G = NP$ of odd index. Then, by Clifford's theorem, $\mathrm{Res}^G_{[N,N]P}(S)$ is semi-simple, and some simple summand has P as vertex and U as source; hence we may replace G by $[N, N]P$. Note that $Z(G) \subseteq [N, N]$ if and only if $Z(G) \subseteq [N, N]P$ because $Z(G)$ is a $2'$-group. Repeating this as necessary implies that if we also choose $|Z(G)|$ minimal, then $Z(G) \subseteq [N, N] = N$, hence N is quasi-simple. □

Remark 4.2. Everything in the above proof works for p odd, except the argument showing that V and W are endo-permutation: the problem is that while any block with a defect group of order 2 is nilpotent, there are non-nilpotent blocks with defect group of order p for odd primes p. If we assume in addition that b is nilpotent, then the above seems to work for p odd as well.

After some preliminary work, from Section 8 onwards we will prove the main theorem for N each of the finite quasi-simple groups. Section 8 deals with the sporadic and alternating groups, together with the groups of Lie type in characteristic 2, Section 9 deals with the special linear groups in odd characteristic, Section 10 deals with the special unitary groups in odd characteristic, Section 11 will prove the result for the remaining classical groups in odd characteristic, and the final section, Section 12, will prove the result for the exceptional groups of Lie type in odd characteristic. In each of these sections, the following notation will be used: G will denote a finite group with $O_2(G) = 1$, and $N = G'$ will be quasi-simple of the appropriate type. A block b of G will have Klein four defect group P, and we will denote $P \cap N$ by D. A block of N that is covered by b will be denoted by c. Denote by R a complement to N in G, so that $G = N \rtimes R$, and $P = D \times R$. We have that the natural map from R to $\mathrm{Out}(N)$ is injective. It is important to note that $Z(G) \leq N$, so that R centralizes $Z(N)$.

5. On dualities in blocks

In this section, let p be a prime number and let $\mathcal{O}$ be a complete local Noetherian commutative ring with residue field k an algebraic closure of the field of p elements and K the quotient field of $\mathcal{O}$. Let $\sigma : k \to k$ be a field automorphism. For V a k-vector space, we let ${}^{\sigma}V$ be the twisted k-vector space, i.e., ${}^{\sigma}V = V$ as an abelian group, and the scalar multiplication in ${}^{\sigma}V$ is given by $\lambda \cdot v := \sigma(\lambda)v$ for $\lambda \in k$ and $v \in V$. Similarly, for A a finitely generated k-algebra, we let ${}^{\sigma}A$ be the corresponding twisted algebra, i.e., ${}^{\sigma}A = A$ as a ring, and the scalar multiplication is given by $\lambda \cdot a := \sigma(\lambda)a$ for $\lambda \in k$ and $a \in A$. If V is an A-module, then ${}^{\sigma}V$ is a ${}^{\sigma}A$-module via $a \cdot v = av$. For a finite group G, the map $\hat{\sigma} : kG \to {}^{\sigma}kG$ given by $\sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} \sigma(\alpha_g) g$ is an isomorphism of k-algebras. Hence, if V is a kG-module, then ${}^{\sigma}V$ is also a kG-module via restriction through $\hat{\sigma}$, i.e., ${}^{\sigma}V$ is a kG-module via $a \cdot v = \hat{\sigma}(a)v$ for $a \in kG$, and $v \in {}^{\sigma}V$. In what follows, we will use this kG-module structure on ${}^{\sigma}V$ without comment. Note that $\hat{\sigma}$ is a ring automorphism of kG, and as such permutes the blocks of kG. Furthermore, if V belongs to the block b of kG, then ${}^{\sigma}V$ belongs to the block $\hat{\sigma}^{-1}(b)$ of kG. Also, if $\phi : H \to G$ is a group isomorphism, and V a kG-module, we denote by ${}^{\phi}V$ the kH-module with action given by $hv := \phi(h)v$. Write $\hat{\phi} : kH \to kG$ for the isomorphism of k-algebras given by $\sum_{h \in H} \alpha_h h \to \sum_{h \in H} \alpha_h \phi(h)$. If $G = H$, then $\hat{\phi}$ is a ring automorphism of kG, and so permutes the blocks of kG. In this case, if V belongs to the block b of kG, then ${}^{\phi}V$ belongs to the block $\hat{\phi}^{-1}(b)$ of kG. If V is finite-dimensional over k, we denote by $V^{\vee}$ the contragradient dual of V, considered as a kG-module. If $b = \sum_{g \in G} \alpha_g g$ is a block of kG, we denote by $b^{\vee}$ the block $\sum_{g \in G} \alpha_g g^{-1}$. Note that if V belongs to b, then $V^{\vee}$ belongs to $b^{\vee}$. The notion of automorphically dual modules was introduced in [34]. We recall their definition, and introduce also the notion of Galois dual modules.

Definition 5.1. Let G be a finite group, V a finitely generated kG-module, and b a block of kG.

(i) V is automorphically dual if there exists a group automorphism $\phi : G \to G$ such that ${}^{\phi}V \cong V^{\vee}$ as kG-modules. Similarly, b is automorphically dual if $\hat{\phi}(b) = b^{\vee}$ for some automorphism $\phi : G \to G$.

(ii) V is Galois dual if there exists a field automorphism $\sigma : k \to k$ such that ${}^{\sigma}V \cong V^{\vee}$ as kG-modules. Similarly, b is Galois dual if $\hat{\sigma}(b) = b^{\vee}$ for some field automorphism $\sigma : k \to k$.

Before proceeding we establish some further notation. If P is a p-group, denote as before by $D_k(P)$ the Dade group of P and by $D_k^{\mathrm{tor}}(P)$ the torsion subgroup of $D_k(P)$ (for definitions and basic results about the Dade group, see for example [45]). For an indecomposable kP-module M, write $[M]$ for the class in $D_k(P)$ containing M.

Lemma 5.2. *Let P be an elementary abelian p-group of order p^2, and let W be an indecomposable endo-permutation kP-module with vertex P. If there is a field automorphism $\sigma : k \to k$ and $\phi \in \mathrm{Aut}(P)$ such that ${}^{\phi\sigma}W \cong W^\vee$, then W is self-dual, or equivalently, the order of the image of W in $D_k(P)$ is at most 2.*

Proof. If W is automorphically dual, then this is [34, Lemma 2.2], and the proof of this result is in the same spirit. Since P is elementary abelian, $D_k(P)/D_k^{\mathrm{tor}}(P)$ is an infinite cyclic group generated by the class of $\Omega_P(k)$. Hence $[W] = [\Omega_P^n(k) \otimes_k V]$ for some $n \in \mathbb{Z}$ and some self-dual kP-module V (which we may take to be trivial in the case $p = 2$), where n and $[V]$ are uniquely determined. Hence $[{}^{\phi\sigma}W] = [\Omega_P^n(k) \otimes_k {}^{\phi\sigma}V]$ and $[W^\vee] = [\Omega_P^{-n}(k) \otimes_k V]$. But ${}^{\phi\sigma}W \cong W^\vee$, so $n = 0$. □

Proposition 5.3. *Let G be a finite group and let V be an indecomposable kG-module, with vertex-source pair (P, W) such that P is elementary abelian of order p^2 and W is an endo-permutation module. If there is $\phi \in \mathrm{Aut}(G)$ or a field automorphism $\sigma : k \to k$ such that ${}^\phi V$ or ${}^\sigma V$ shares a vertex-source pair with $V^\vee$ (up to isomorphism), then W is self-dual. In particular, the class of W is a torsion element of the Dade group of P. Furthermore, if $p = 2$, then $W = k$. This condition is satisfied if V is automorphically dual or Galois dual.*

Proof. Let τ be either a group automorphism $\phi \in \mathrm{Aut}(G)$ or a field automorphism $\sigma : k \to k$ such that ${}^\tau V$ shares a vertex-source pair with $V^\vee$. By [34, Lemma 2.3] $(P, W^\vee)$ is a vertex-source pair for $V^\vee$, and by transport of structure $(P, {}^\tau W)$ or $({}^\tau P, {}^\tau W)$ is a vertex-source pair for ${}^\tau V$, according as τ is a field or a group automorphism respectively (hence it is clear that if V is automorphically dual or Galois dual, then it satisfies the hypotheses). Since ${}^\tau V$ shares a vertex-source pair with $V^\vee$, it follows that there is $g \in G$ such that ${}^{g\tau}W \cong W^\vee$ and ${}^{g\tau}P \cong P$ or ${}^gP = P$ according as τ is a group or a field automorphism respectively. Since conjugation by g induces an automorphism of P, it follows that W is self-dual by Lemma 5.2, and hence W has order at most 2 in the Dade group. If $p = 2$ then $D_k^{\mathrm{tor}}(P)$ is trivial, whence the result. □

Corollary 5.4. *Let L and G be finite groups and let f and b be source algebra equivalent blocks of L and G respectively with Klein four defect group P. If f is Galois or automorphically dual, then every endo-permutation module which is a source of a simple kGb-module lies in a class of finite order in the Dade group.*

Proof. We use freely the fact that every simple module in a block with defect group P has vertex P and that b (and so f) is Morita equivalent to either kP or kA_4 or the principal block algebra of kA_5. Suppose first that b (and so f) is Morita equivalent to kP or the principal block of kA_5, then kGb has a unique simple module with endo-permutation source, and the corresponding simple kLf-module V has isomorphic endo-permutation source W. By assumption there is a ring isomorphism $\hat{\tau}$ of kL given by τ, where τ is either a field automorphism $k \to k$ or an automorphism of L, such that $\tau(f) = f^\vee$. In either case ${}^\tau V$ belongs to $\tau(f)$, since $\tau^2(f) = f$. The result follows in this case from Proposition 5.3 since $V^\vee$ (resp. ${}^\tau V$) is also the unique simple $kLf^\vee$-module (resp. $kL\tau(f)$-module) with endo-permutation source. Now suppose that b (and so f) is Morita equivalent to kA_4. Then all simple modules have endo-permutation sources. By

[16, Theorem 3], it follows that all have isomorphic sources. Let V be a simple kLf-module, with endo-permutation source W. Then ${}^{\tau}V$ and $V^{\vee}$ both belong to $\tau(b)$, which is also Morita equivalent to kA_4, and the result follows by Proposition 5.3. □

Proposition 5.5. *Let p be a prime number, $(K, \mathcal{O}, k)$ a p-modular system such that k is an algebraic closure of $\mathbb{F}_p$ and K contains a primitive $|G|$th root of unity. Let G be a finite group and b a block of kG. Let $\tilde{b} \in Z(\mathcal{O}G)$ be the block of $\mathcal{O}G$ lifting b and let $\chi : G \to K$ be an irreducible character in $\tilde{b}$. Set $\mathbb{Q}[\chi] := \mathbb{Q}[\chi(g) : g \in G]$. Suppose that $\mathbb{Q}[\chi]$ is contained in a field of the form $\mathbb{Q}[\zeta]$, where ζ is an nth root of unity in K such that, writing $n = n_+n_-$, where n_+ is the p-power part of n and n_- is the p'-part of n, we have that $p^t \equiv -1 \bmod n_-$ for some integer t. Then b is Galois dual.*

Proof. Let $\mathcal{O}_0$ be the ring of Witt vectors in $\mathcal{O}$ and $\alpha : k \to k$ be the automorphism $x \mapsto x^{p^t}$. Then there is a unique isomorphism $\tilde{\alpha} : \mathcal{O}_0 \to \mathcal{O}_0$ such that $\tilde{\alpha}$ induces α on k. Denote by $\tilde{\alpha}$ also any extension of $\tilde{\alpha}$ to a field automorphism of K. By an easy argument using decomposition numbers, we see that it suffices to prove that $\chi^{\vee}|_{G_{p'}} = \tilde{\alpha} \circ \chi|_{G_{p'}}$, where $\chi^{\vee}$ is the character of G defined by $\chi^{\vee}(g) = \chi(g^{-1})$. (Note that the values of χ on p'-elements are in $\mathcal{O}_0$.) Let η be a primitive $\mathrm{lcm}(n, |G|)$-th root of unity in K and let $\phi : \mathbb{Q}[\eta] \to \mathbb{Q}[\eta]$ be the field automorphism such that $\phi(\eta) = \eta^{-1}$. Then, clearly $\chi^{\vee} = \phi \circ \chi$. Thus, it suffices to show that $\phi \circ \chi|_{G_{p'}} = \tilde{\alpha} \circ \chi|_{G_{p'}}$. For this, we note that for any $g \in G_{p'}$, we have that $\chi(g) \in \mathbb{Q}(\zeta) \cap \mathbb{Q}(\eta_-)$, where $\eta = \eta_+\eta_-$, with η_+ having order $(\mathrm{lcm}(n, |G|))_+$ and η_- having order $(n|G|)_-$. The result follows since on the one hand, $\mathbb{Q}(\zeta) \cap \mathbb{Q}(\eta_-) \subseteq \mathbb{Q}(\zeta_-)$ (see for instance [39, Theorem 4.10(v)]), and on the other hand, by hypothesis, $\tilde{\alpha}|_{\mathbb{Q}(\zeta_-)} = \phi|_{\mathbb{Q}(\zeta_-)}$. (Note that $\tilde{\alpha}$ raises every p'th root of unity to its p^tth power.) □

6. Background results on finite groups of Lie type

The quoted results in this section are culled from different sources, and this can occasionally lead to confusion as different authors may employ different terminology for the same concept and vice versa. In case of ambiguity, we specify the source of our notation. The following notation will be in effect in this section: let $\mathbb{F}$ denote an algebraic closure of a field $\mathbb{F}_q$ of q elements, $\mathbf{G}$ a connected reductive group over $\mathbb{F}$ and let $F : \mathbf{G} \to \mathbf{G}$ be an endomorphism such that $F^{\delta} : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism of $\mathbf{G}$ (for some $\delta \geq 1$) with respect to an $\mathbb{F}_q$-structure on $\mathbf{G}$ in the sense of [15, Definition 3.1]. We point out that the term 'Frobenius endomorphism' as employed here may have a different meaning from that of Definition 2.1.9 of [22], where the map F as above would also count as a Frobenius endomorphism. But in any case, the map F is a Steinberg endomorphism in the sense of [22, Definition 1.15.1]. Let $(\mathbf{G}^*, F^*)$ be dual to $(\mathbf{G}, F)$ through an isomorphism $X(\mathbf{T}) \xrightarrow{\sim} Y(\mathbf{T}^*)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$. If $s \in {\mathbf{G}^*}^{F^*}$ is semi-simple, we denote by $[s]$ and by (s), the rational conjugacy class and the geometric conjugacy class respectively of s. Let $\mathcal{E}(\mathbf{G}^F, [s])$ and $\mathcal{E}(\mathbf{G}^F, (s))$ denote the rational and geometric Lusztig series respectively; they are subsets of the set of ordinary ($\bar{\mathbb{Q}}_l$-valued) characters of $\mathbf{G}^F$, where l is a prime not dividing q. Let $\mathbf{G_{ad}} = \mathbf{G}/Z(\mathbf{G})$ be the adjoint type of $\mathbf{G}$. Then $\mathbf{G_{ad}}$ is a direct product of simple adjoint algebraic groups. Let Δ be the set of connected components of the Dynkin diagram of $\mathbf{G}$. For each orbit ω of F on Δ, let $\mathbf{G}_{\omega}$ be the direct product of the subgroups of $\mathbf{G}$ corresponding to elements of Δ in ω, and let $G_{\omega} = \mathbf{G}_{\omega}^F$; then $\mathbf{G}_{\mathbf{ad}}^F = \prod_{\omega} \mathbf{G}_{\omega}^F$. Each component G_{ω} is characterized by its simple type, twisted or not, $\mathbf{X}_{\omega} \in \{\, \mathbf{A}_n,\ {}^2\mathbf{A}_n,\ \mathbf{B}_n,\ \mathbf{C}_n$ $(n \geq$

1), $\mathbf{D}_m$, ${}^2\mathbf{D}_m$ $(m \geq 2)$, ${}^3\mathbf{D}_4$, $\mathbf{E}_6$, ${}^2\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_2\}$, and the extension field $\mathbb{F}_{q^{a(\omega)}}$ of $\mathbb{F}_q$ of degree the length $a(\omega)$ of the orbit ω. Then $(\mathbf{X}_\omega, q^{a(\omega)})_\omega$ is called the adjoint rational type of $\mathbf{G}$.

In the following proposition, we will be in the situation where the centre of $\mathbf{G}$ is connected, whence the centralizers of semi-simple elements of $\mathbf{G}^*$ are connected, and so $[s] = (s)$.

Proposition 6.1. *Suppose that q is odd and that $\mathbf{G}$ has connected centre. Let χ be an ordinary irreducible character of $\mathbf{G}^F$, and let $s \in \mathbf{G}^{*F^*}$ be a semi-simple element such that $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$. Let $|Z^\circ(C_{\mathbf{G}^*}(s))^{F^*}|_2 = 2^\alpha$. Then the 2-defect of χ is at least α. Further,*

(i) *If the 2-defect of χ is exactly α, then the adjoint rational type of $C_{\mathbf{G}^*}(s)^{F^*}$ contains no component of any of the classical types $\mathbf{A}_n$, ${}^2\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$, ${}^2\mathbf{D}_n$ or ${}^3\mathbf{D}_4$.*

(ii) *If the 2-defect of χ is $\alpha + 1$, then the adjoint rational type of $C_{\mathbf{G}^*}(s)$ contains at most one component of classical type and no component of the form $\mathbf{A}_n$ or ${}^2\mathbf{A}_n$.*

In particular, if the 2-defect of χ is 1, then either the adjoint rational type of $C_{\mathbf{G}^*}(s)$ contains no components of classical type or $Z^\circ(C_{\mathbf{G}^*}(s))^{F^*}$ has odd order.

Proof. Set $\mathbf{C} := C_{\mathbf{G}^*}(s)$ and let $\pi : \mathbf{C} \to \mathbf{C_{ad}}$ be the natural epimorphism. By Lusztig's parametrization of the irreducible characters of $\mathbf{G}^F$, (see for instance [15, Theorem 13.23, Remark 13.24]) there is a bijection $\theta \mapsto \psi_s(\theta)$ between $\mathcal{E}(\mathbf{C}^{F^*}, [1])$ and $\mathcal{E}(\mathbf{G}^F, [s])$ satisfying

$$\psi_s(\theta)(1) = \frac{|\mathbf{G}^F|_{q'}}{|\mathbf{C}^{F^*}|_{q'}}\theta(1).$$

On the other hand, by [15, Proposition 13.20], the map $\phi \mapsto \phi \circ \pi$ is a degree-preserving bijection between $\mathcal{E}(\mathbf{C}_{\mathbf{ad}}^{F^*}, [1])$ and $\mathcal{E}(\mathbf{C}^{F^*}, [1])$. Also, $|\mathbf{C}^{F^*}| = |Z^\circ(\mathbf{C})^{F^*}| \cdot |[\mathbf{C}, \mathbf{C}]^{F^*}|$ (see for instance [11, Section 2.9]) and $|[\mathbf{C}, \mathbf{C}]^{F^*}| = |\mathbf{C}_{\mathbf{ad}}^{F^*}|$ since $[\mathbf{C}, \mathbf{C}]^{F^*}$ and $\mathbf{C}_{\mathbf{ad}}^{F^*}$ are isogenous groups (see [5, 16.8]). Thus, $\phi \mapsto \psi_s(\phi \circ \pi)$ gives a bijection between $\mathcal{E}(\mathbf{C}_{\mathbf{ad}}^{F^*}, [1])$ and $\mathcal{E}(\mathbf{G}^F, [s])$ such that the 2-defect of $\psi_s(\phi \circ \pi)$ is the 2-defect of ϕ plus α.

Let $\Delta(s)$ be the set of components of the Dynkin diagram of $\mathbf{C}$ and let $\mathrm{Orb}_F(\Delta(s))$ be the set of orbits of F on $\Delta(s)$. For each $\omega \in \mathrm{Orb}_F(\Delta(s))$, let $\mathbf{C}_\omega$ be the product of all subgroups of $\mathbf{C_{ad}}$ corresponding to the elements of ω and set $C_\omega = \mathbf{C}_\omega^F$. Thus, $\mathbf{C}_{\mathbf{ad}}^{F^*} = \prod_{\omega \in \mathrm{Orb}_F(\Delta(s))} C_\omega$ and the irreducible unipotent characters of $\mathbf{C}_{\mathbf{ad}}^{F^*}$ are products of the irreducible unipotent characters of the C_ω. Let $\chi = \psi_s(\prod_\omega \phi_\omega \circ \pi)$. Hence,

$$(\text{2-defect of } \chi) = \alpha + \sum_\omega (\text{2-defect of } \phi_\omega).$$

Now suppose that the 2-defect of χ is α. Then, each ϕ_ω is a unipotent character of C_ω of defect 0. On the other hand, if C_ω is of classical type, then by [9, Proposition 11], the principal block of C_ω is the unique unipotent 2-block of C_ω so, in particular, C_ω does not have a unipotent block of defect 0; this proves (i). Similarly, if the 2-defect of χ is $\alpha + 1$, then one of the ϕ_ω is of defect 1 and the rest are of defect 0, which means in particular that there is at most one component of classical type. Therefore, in order to prove (ii) it suffices to prove that no finite projective (or equivalently special) linear or projective (or equivalently special) unitary group has a unipotent character of 2-defect 1. When the degree n is 1, the only unipotent characters of a general linear or unitary group are the trivial and the Steinberg character, both of full defect. On the other hand, when $n \geq 2$, any partition of $n + 1$ contains at least three hooks, hence the result follows from the degree formulae for the unipotent characters of finite general linear groups and unitary groups (see for instance [19, 1.15]). The last statement of (ii) is an immediate consequence of the above displayed equation and the fact that a component C_ω of classical type does not have a unipotent character of 2-defect 0. □

We keep the notation introduced in the beginning of this section. In addition, we will use terminology from Sections 1.5 and 2.5 of [22]. In particular, we will refer to as graph automorphisms of $\mathbf{G}^F$ the images of graph automorphisms of $\mathbf{G}$, even when $\mathbf{G}^F$ is a twisted group. Also, note that when $\mathbf{G}$ is simple and simply connected, then by [22, Theorem 2.2.6], $\mathbf{G}^F \in \mathcal{L}ie$ in the sense of [22]. We will use this fact without comment. We recall a result of Feit and Zuckerman from [18]. However, we need a slightly stronger version which follows easily from their proof. Recall that an element h of a group H is real in H if h is H-conjugate to h^{-1}. Given an isometry γ of the underlying Dynkin diagram of $\mathbf{G}$, and an automorphism τ of the algebraic group $\mathbf{G}$, we will say that 'τ induces γ' if τ stabilizes $\mathbf{T}$, as well as a basis of positive roots in the character group $X(\mathbf{T})$ and induces γ on this basis.

Proposition 6.2. *Let τ be an automorphism of order 2 of $\mathbf{G}$ which commutes with F and induces an isometry of order 2 on the Dynkin diagram of $\mathbf{G}$, if such an isometry exists. If no such isometry exists, let τ be the identity map. Suppose that $\mathbf{G}$ is simple and simply connected and let $x \in \mathbf{G}^F$ be semi-simple. Then x is real in $\mathbf{G}^F\langle\tau\rangle$. If $\mathbf{G}$ is not of type $\mathbf{D}_n$ for n even, and if α is a graph automorphism of $\mathbf{G}^F$ (as in [22, 2.15.13]) such that the* $\mathrm{Inndiag}(\mathbf{G}^F)$*-coset of α has order 2 in* $\mathrm{Aut}(\mathbf{G}^F)/\mathrm{Inndiag}(\mathbf{G}^F)$*, then x^{-1} is $\mathbf{G}^F$-conjugate to $\alpha(x)$. If $\mathbf{G}^F$ does not have such a graph automorphism, then x and x^{-1} are $\mathbf{G}^F$-conjugate.*

Proof. The first assertion is just Theorem A of [18]. If $\mathbf{G}^F$ does not have have a graph automorphism satisfying the order condition of the statement, then $\tau = 1$ and it follows from the first assertion that x and x^{-1} are $\mathbf{G}^F$-conjugate.

Now suppose that α is a graph automorphism of $\mathbf{G}^F$ satisfying the order condition of the statement, and that $\mathbf{G}$ is not of type $\mathbf{D}_n$, n even. Then, by [22, Definition 2.5.13b(2), 2.5.13c(2)] and the remark following [22, Definition 2.5.13], $\tau \neq 1$ and α is the restriction to $\mathbf{G}^F$ of $c_g \circ \tau$, for some element $g \in \mathbf{G}$ such that $g^{-1}F(g) \in Z(\mathbf{G})$ and for some $1 \neq \tau$ as above. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ containing x. The proof of Theorem A of [18] proceeds by observing that the group $N_{\mathbf{G}\langle\tau\rangle}(\mathbf{T})/\mathbf{T}$ contains an element -1 (negating every root, and which may or may not lie in the Weyl group $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$). In particular -1 inverts every element $z \in \mathbf{T}$ (for details of this, see VI.1 and VI.4 of [6]). Since $\mathbf{G}$ is not of type $\mathbf{D}_n$, n even, $\mathbf{G}$ must be one of the types $\mathbf{A}_n$ (for $n \neq 1$), $\mathbf{D}_n$ (for n odd) or $\mathbf{E}_6$. Following [6, VI.4], this implies that $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ contains no central involution, whence no lift of -1 in $N_{\mathbf{G}\langle\tau\rangle}(\mathbf{T})$ is contained in $N_{\mathbf{G}}(\mathbf{T})$. It follows that for every $z \in \mathbf{T}$, the elements z^{-1} and $\tau(z)$ are $\mathbf{G}$-conjugate and hence $\alpha(z)$ is $\mathbf{G}$-conjugate to z^{-1}. Since F commutes with α and since $\mathbf{G}$ is simply connected, it follows that $\alpha(x)$ is $\mathbf{G}^F$-conjugate to x^{-1}, as required. □

Proposition 6.3. *Suppose that $Z(\mathbf{G})$ is connected and that q is odd. Let $\mathbf{N}$ be a connected reductive F-stable subgroup of $\mathbf{G}$ such that $\mathbf{N}$ contains the derived subgroup of $\mathbf{G}$, $\mathbf{G} = \mathbf{N}Z(\mathbf{G})$, and $Z(\mathbf{N}) = Z(\mathbf{G}) \cap \mathbf{N}$. Let c be a block of $k\mathbf{N}^F$, and let $I(c)$ be the inertial subgroup of c in $\mathbf{G}^F$. Then the index of $I(c)$ in $\mathbf{G}^F$ is odd.*

Proof. Write $G = \mathbf{G}^F$, and let b be a block of kG covering c. There exists an odd semi-simple element $s \in \mathbf{G}^{*F^*}$ such that the block b of kG contains an ordinary irreducible character, say ϕ, in the rational series indexed by the conjugacy class of s (see [10, Theorem 9.12]). Let χ be an ordinary character of $\mathbf{N}^F$ in the block c of $k\mathbf{N}^F$ such that χ is covered by ϕ. Then the index in G of the inertial subgroup, $I(\chi)$, of χ in G divides the order of s (see Corollary 11.3 of [4]). The result follows as $I(\chi) \leq I(c)$. □

We restate the above proposition in the notation of [22, Section 2.5].

Corollary 6.4. *Let r be an odd prime and let $H \in \mathcal{L}ie(r)$, not of Ree or Suzuki type. Let c be a 2-block of H and let τ be a 2-element of $\mathrm{Inndiag}(H)$ such that the image of τ in $\mathrm{Outdiag}(H) = \mathrm{Inndiag}(H)/\mathrm{Inn}(H)$ is a 2-element. Then $\tau(c) = c$.*

Proof. By [22, Section 2.5, Theorem 2.5.14], there exists a simple, simply connected algebraic group $\mathbf{N}$, an isogeny $F : \mathbf{N} \to \mathbf{N}$ such that F^δ is a Frobenius morphism relative to some $\mathbb{F}_{r^a}$-structure on $\mathbf{N}$ (for some natural numbers δ and a) and a short exact sequence of groups

$$1 \to Z \to \mathbf{N}^F \to H \to 1,$$

where Z is a cyclic subgroup of $Z(\mathbf{N}) \cap \mathbf{N}^F$ such that every automorphism ϕ of H lifts to a unique automorphism $\tilde{\phi}$ of $\mathbf{N}^F$ and the resulting mappings $\mathrm{Aut}(H) \to \mathrm{Aut}(\mathbf{N}^F)$ and $\mathrm{Aut}(H)/\mathrm{Inn}(H) \to \mathrm{Aut}(\mathbf{N}^F)/\mathrm{Inn}(\mathbf{N}^F)$ are injective group homomorphisms. Furthermore, if $\phi \in \mathrm{Inndiag}(H)$, then there exists $x \in \mathbf{N}$ with $x^{-1}F(x) \in Z(\mathbf{N})$ such that $\tilde{\phi}$ is conjugation by x. By Lusztig, (see for instance [4, Section 2.B]), there is a connected reductive group $\mathbf{G}$ of the same type as $\mathbf{N}$ such that F extends to an isogeny of $\mathbf{G}$, (which we denote again by F), F^δ is a Frobenius morphism on $\mathbf{G}$ relative to $\mathbb{F}_{r^a}$, and such that $Z(\mathbf{G})$ is connected, $\mathbf{N}$ contains the derived subgroup of $\mathbf{G}$, $\mathbf{G} = \mathbf{N}Z(\mathbf{G})$, and $Z(\mathbf{N}) = Z(\mathbf{G}) \cap \mathbf{N}$. Now, let τ be as in the hypothesis of the theorem and let $\tilde{\tau}$ be induced by conjugation by y, where $y \in \mathbf{N}$ with $y^{-1}F(y) \in Z(\mathbf{N})$. Let $\tilde{c}$ be the unique block of $k\mathbf{N}^F$ which lifts the block c of kH. It suffices to prove that $\tilde{c}$ is $\tilde{\tau}$-stable. For this, we first claim that $\tilde{\tau}$ is induced by conjugation by some element of $\mathbf{G}^F$. Indeed, since $y^{-1}F(y) \in Z(\mathbf{N}) \leq Z(\mathbf{G})$, the connectivity of $Z(\mathbf{G})$ (along with the Lang–Steinberg theorem) implies that $y = y_1 z_1$ for some $z_1 \in Z(\mathbf{G})$ and some $y_1 \in \mathbf{G}^F$. Replacing y by y_1 proves the claim. Moreover, since by hypothesis the image of τ in $\mathrm{Out}(H)$ is a 2-element, the same is true of the image of $\tilde{\tau}$ in $\mathrm{Out}(\mathbf{N}^F)$. The result is now immediate from Proposition 6.3. □

In the final part of this section, we note some miscellaneous facts for some particular classes of groups.

Proposition 6.5. *Suppose that $\mathbf{G}$ is simple of type $\mathbf{E}_6$ with $Z(\mathbf{G}) = 1$. Let τ be an automorphism of order 2 of $\mathbf{G}$ as in Proposition 6.2. Let q be an odd prime power and suppose that $\mathbf{G}^F$ is of type $E_6(q)$. Let b be a block of $k\mathbf{G}^F$ of defect 0. If b is τ-stable, then b is Galois dual.*

Proof. Let $\tau^* : \mathbf{G}^* \to \mathbf{G}^*$ be an automorphism of order 2, commuting with F^*, stabilizing $\mathbf{T}^*$, some choice of positive roots and inducing the isometry of order 2 of the underlying Dynkin diagram (with respect to some τ^*-stable basis of the roots $X(\mathbf{T}^*)$). Let χ be the unique ordinary irreducible character of $\mathbf{G}^F$ in the block b and let s be a semi-simple element of $\mathbf{G}^{*F^*}$ such that $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$. We claim that $[s] = [s^{-1}]$. Indeed, since b is τ-stable, so is χ. By the compatibility of Lusztig series with outer automorphisms [40, Corollary 2.4] it follows that $[\tau^*(s)] = [s]$. The claim follows from Proposition 6.2 applied to the simply connected group $\mathbf{G}^*$, and the morphisms τ^* and F^*.

Let us first suppose that s is a central element of $\mathbf{G}^{*F^*}$. Then, $\chi = \hat{s} \otimes \chi'$, where χ' is a unipotent character of $\mathbf{G}^F$ of defect 0 and $\hat{s}$ is a linear character of $\mathbf{G}^F$ of order dividing 3.

The degrees of the unipotent characters of $\mathbf{G}^F$ are given in [11, p. 480] in terms of cyclotomic polynomials $\Phi_t = \Phi_t(q)$. Noting that $\Phi_t(q)$ is odd except when t is a power of 2, inspection of the degrees shows that the characters with minimal defect are those labelled $E_6[\theta]$ and $E_6[\theta^2]$. By [11, p. 461] these are precisely the cuspidal unipotent characters. Hence we see that χ' is in fact a cuspidal unipotent character. By [21], the character field of χ' is $\mathbb{Q}[\omega]$ where ω is a primitive third root of unity. Since $\hat{s}$ has order dividing 3, the same is true of χ. The result follows by Proposition 5.5.

Let us suppose now that s is non-central. Then every component of the adjoint rational type of $C_{\mathbf{G}^*}(s)$ is of classical type. It follows from Proposition 6.1(i) that $C_{\mathbf{G}^*}(s)$ is a maximal torus (necessarily F^*-stable) of $\mathbf{G}^*$; that is, that s is a regular semi-simple element. It follows that χ is the unique character in $\mathcal{E}(\mathbf{G}^F, [s])$. But we showed above that $[s] = [s^{-1}]$. The result follows from the fact that the dual character $\chi^\vee$ belongs to $\mathcal{E}(\mathbf{G}^F, [s^{-1}])$. □

Lemma 6.6. *Let $Z(\mathbf{G}) = 1$ and let $\mathbf{G}$ be of type $\mathbf{E}_7$ such that $\mathbf{G}^F$ is of type $E_7(q)$. Let s be a semi-simple element of $\mathbf{G}^*$ such that the Dynkin diagram of $C_{\mathbf{G}}(s)$ contains a component of type $\mathbf{E}_6$. Suppose further that $s \in \mathbf{G}^{*F^*}$. Then the adjoint rational type of $C_{\mathbf{G}^*}(s)$ is either $(\mathbf{E}_6, q)$ or $({}^2\mathbf{E}_6, q)$. In the former case, s has order dividing $(q-1)\cdot \gcd(3, q-1)$ and in the latter, s has order dividing $(q+1)\cdot\gcd(3, q+1)$.*

Proof. This is immediate from the description of centralizers of semi-simple elements given in [14, Table 1] once we observe that $s \in Z(C_{\mathbf{G}^*}(s)^F)$. Note that in the twisted case, we use the notation ${}^2E_6(q)$ whereas [14] uses ${}^2E_6(q^2)$. □

Lemma 6.7. *Suppose that $\mathbf{G}$ is of type $\mathbf{E}_7$, $Z(\mathbf{G}) = 1$, $q = q'^2$ and that $F = F_1^2$, where $F_1 : \mathbf{G} \to \mathbf{G}$ is a Frobenius morphism with respect to a $\mathbb{F}_{q'}$-structure on $\mathbf{G}$, such that $\mathbf{G}^{F_1}$ is a group of type $E_7(q')$ and $\mathbf{G}^F$ is a group of type $E_7(q)$. Let σ denote the restriction of F_1 to $\mathbf{G}^F$. Let χ be an ordinary irreducible character of $\mathbf{G}^F$ of 2-defect 1, such that $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$ for some semi-simple element s of odd order. If χ is σ-stable, then the 2-block of G containing χ is Galois dual.*

Proof. Let $F_1^* : \mathbf{G}^* \to \mathbf{G}^*$ be a Frobenius morphism with respect to an $\mathbb{F}_{q'}$-structure on $\mathbf{G}^*$, such that $(\mathbf{G}, F_1)$ is dual to $(\mathbf{G}^*, F_1^*)$ and $F^* = F_1^{*2}$. Denote by σ^* the restriction of F_1^* to $\mathbf{G}^{*F^*}$. Since χ is σ-stable, $[s]$ is σ^*-stable by [40, Corollary 2.4]. Thus, $F_1^*(s) = xsx^{-1}$ for some $x \in \mathbf{G}^*$. By the Lang theorem, there exists $y \in \mathbf{G}^*$ such that $x = F(y)y^{-1}$; set $t := y^{-1}sy$. Then t is F_1^*-fixed and is $\mathbf{G}^*$-conjugate to s. Since $F^* = F_1^{*2}$, t is also F^*-fixed. Furthermore, $\mathbf{G}^*$ is simply connected, and s and t are semi-simple elements of $\mathbf{G}^{F^*}$ which are conjugate in $\mathbf{G}^*$, and hence s and t are also conjugate in $\mathbf{G}^{*F^*}$. So, by replacing s by t we may assume that $F_1^*(s) = s$. The 2-defect of χ being 1, by Proposition 6.1(ii) the adjoint rational type of $C_{\mathbf{G}^*}(s)$ has no component of type $\mathbf{A}$ or ${}^2\mathbf{A}$, has at most one component of classical type, and if the adjoint rational type of $C_{\mathbf{G}^*}(s)$ has a component of classical type, then $|Z^\circ(C_{\mathbf{G}^*}(s))^{F^*}|$ has odd order. Also, note by the first assertion of Proposition 6.1 that in any case $|Z^\circ(C_{\mathbf{G}^*}(s))^{F^*}|$ is not divisible by 4. By the list of possible centralizer structures given in [14, Table 1], we see that there are only three possibilities: either $C_{\mathbf{G}^*}(s)$ is a maximal torus, or the adjoint rational type of $C_{\mathbf{G}^*}(s)$ consists of a single component, either $({}^2\mathbf{E}_6, q)$ or $(\mathbf{E}_7, q)$. (Note that we use the fact that q is odd and a square.) By Proposition 6.2, we see that $[s] = [s^{-1}]$. Hence, if $C_{\mathbf{G}^*}(s)$ is a torus, then we may argue as in Proposition 6.5 to conclude that χ is real. Next, we show that the case where the adjoint rational type of $C_{\mathbf{G}^*}(s)$ is $({}^2\mathbf{E}_6, q)$ does not occur. Suppose if possible that the adjoint rational type of $C_{\mathbf{G}^*}(s)$ is $({}^2\mathbf{E}_6, q)$. By Lemma 6.6, the order of s is a divisor of $(q+1)\cdot\gcd(q+1,3)$. Since q is a square, the order of s is a divisor of $(q+1)$. As s is F_1^*-fixed, by Lemma 6.6, applied with q' instead of q and F_1^* instead of F^*, it follows that the order of s is either a divisor of $(q'-1)\cdot\gcd(3,q'-1)$ or of $(q'+1)\cdot\gcd(3,q'+1)$. Thus, the order of s is either a divisor of $\gcd(q+1,q'+1)=2$ or of $\gcd(q+1,q'-1)=2$, a contradiction. Finally, suppose that the adjoint rational type of $C_{\mathbf{G}^*}(s)$ is $(\mathbf{E}_7,q)$; in this case, $s=1$ and χ is unipotent. By [11, Section 13.9], $\mathbf{G}^F$ has seventy-six unipotent characters, sixty in the principal series, ten arising from cuspidal unipotent characters of a Levi subgroup of type $D_4(q)$, four arising from cuspidal characters of a Levi subgroup of type $E_6(q)$, and two cuspidal unipotent characters. By [21, Table 1], the character field of the two cuspidal unipotent characters is $\mathbb{Q}[\sqrt{-q}]=\mathbb{Q}[i]$ as q is a square. By [21, Proposition 5.6], all except two of the non-cuspidal unipotent characters have character fields the same as the character fields of the cuspidal unipotent characters they arise from, and hence by [21, Table 1] the character field of these is either $\mathbb{Q}$ or $\mathbb{Q}[\omega]$, where ω is a primitive third root of unity. Finally, the two exceptions of [21, Proposition 5.6] have character field $\mathbb{Q}[\sqrt{q}]=\mathbb{Q}$. Thus, the result follows by Proposition 5.5. □

Next, we record a result concerning actions of graph automorphisms on centralizers of fixed automorphisms in groups of type $\mathbf{E}_6$. We will use the notation $H.n$ to denote a finite group having H as normal subgroup of index n such that the image of $H.n$ in $\mathrm{Out}(H)$ is cyclic of order n.

Proposition 6.8. *Let $\mathbf{G}$ be simple and simply connected of type $\mathbf{E}_6$ and let α be a graph automorphism of $G=\mathbf{G}^F$ in the sense of* [22].

If $G=E_6(q)$ and if in the notation of [22], *u is an involution of type t_j in G, then $C_{G\langle\alpha\rangle}(u)\cong(2.\mathrm{P}\Omega_{10}(q).2\times\mathbb{Z}_{(q-1)/2})\langle\gamma i\rangle$ or $(2.(\mathrm{PSL}_2(q)\times Z(N).\mathrm{PSL}_6(q)).2)\langle\gamma\rangle$, for $j=1,2$ respectively, where γ is a graph automorphism and i inverts elements in $\mathbb{Z}_{(q-1)/2}$.*

If $G={}^2E_6(q)$ and if u is an involution of type t_j in G, then $C_{G\langle\alpha\rangle}(u)\cong(2.\mathrm{P}\Omega^-_{10}(q).2\times\mathbb{Z}_{(q+1)/2})\langle\gamma i\rangle$ or $(2.(\mathrm{PSL}_2(q)\times Z(N).\mathrm{PSU}_6(q)).2)\langle\gamma\rangle$, for $j=1,2$ respectively, where γ is a graph automorphism and i inverts elements in $\mathbb{Z}_{(q+1)/2}$)

Proof. Suppose first that $G=E_6(q)$. By replacing by a conjugate if necessary, we may choose u so that it is fixed by α. Then the result follows from [22, 4.3.4, 4.3.4A] and [22, 4.5.2]. The other case is similar. □

We also require an improvement of a result of Wonenburger in [49], which is implicit in [49].

Lemma 6.9. *Any element t of $\mathrm{SO}^{\pm}_{4n+2}(q)$, q odd, may be written as a product $t=rs$, where $r,s\in\mathrm{GO}^{\pm}_{4n+2}(q)\backslash\mathrm{SO}^{\pm}_{4n+2}(q)$, $r^2=s^2=1$. In particular t is inverted by the graph automorphism induced by r.*

Proof. This may be seen by close inspection of the proofs of the main results in [49]. □

7. On 2-blocks of special linear groups

For this section, we introduce the following notation: let q be a power of an odd prime such that $q=q'^2$ for some q'. Let $G=\mathrm{GL}_n(q)$, $N=\mathrm{SL}_n(q)$. Let σ be the automorphism of $\mathrm{GL}_n(q)$ which raises every matrix entry to its q'th power and let τ be the inverse-transpose automorphism. Let $\phi:G\to G$ denote either σ or $\sigma\tau$. The aim of this section is to prove the following analogue to [34, Theorem 1.4] for the prime 2. Note that the defect groups of any 2-block of kN contain the Sylow 2-subgroup of $Z(N)$.

Theorem 7.1. *Suppose that q and n are as above and that n is even. Suppose that c is a block of kN such that the index of the Sylow 2-subgroup of $Z(N)$ in a defect group of kNc is at most 2, and set $I(c) = Stab_G(c)$. If c is stable under ϕ, then $I(c) \rtimes \langle\phi\rangle$ is a normal subgroup of $G \rtimes \langle\phi\rangle$.*

In order to prove the above, we gather together a few standard facts on the block theory of G and N (for pointers to the proof, see [34]). For an element s of G, we denote by $[s]$ the G-conjugacy class of s. We set $Z(s) = \{z \in Z(G) : [sz] = [s]\}$ and we denote by $d_{[s]}$ the number of distinct conjugacy classes of the form $[sz]$, where $z \in Z(G)$. Note that $d_{[s]} = (q-1)/|Z(s)|$.

Proposition 7.2. *Suppose that q is odd. There is a one-to-one correspondence $[s] \leftrightarrow b_{[s]}$ between the G-conjugacy classes of odd semi-simple elements of G and the blocks of kG; this correspondence has the following properties.*

Let s be an odd semi-simple element of G, let c be a block of kN covered by $b_{[s]}$ and let $I(c)$ denote the stabilizer of c in G. Then

(i) ${}^{\phi}b_{[s]} = b_{[\phi^{-1}(s)]}$,
(ii) *any Sylow 2-subgroup of $C_G(s)$ is a defect group of $kGb_{[s]}$,*
(iii) *any block of kG covering c is of the form $b_{[zs]}$ for some $z \in Z(G)$,*
(iv) *$I(c)/N$ has order divisible by $(q-1)/|Z(s)|$, and*
(v) *$|Z(s)|$ is a divisor of $\gcd(n, q-1, o(s))$, where $o(s)$ denotes the order of s. In particular, $|G : I(c)|$ is odd.*

In what follows we will identify $\mathrm{GL}_n(q)$ with the group of invertible linear transformations of n-dimensional vector space over a field of q elements. Given a positive integer d we denote by d_+ the largest power of 2 which divides d.

Lemma 7.3. *Suppose that q is square and that n is even. Let $d = \gcd(q-1, n)$. Let c be a block of $k\,\mathrm{SL}_n(q)$ whose defect groups have order at most $2d_+$. let $b = b_{[t]}$ be a block of $G = \mathrm{GL}_n(q)$ covering c, and let $f(x)$ be the characteristic polynomial of t (over $\mathbb{F}_q$). Then, either $f(x)$ is irreducible, or $f(x)$ is a product $p_1(x)p_2(x)$ of two irreducible polynomials such that $\deg(p_1(x)) \neq \deg(p_2(x))$.*

Proof. Let $f(x) = \prod_{1\le i\le u} p_i(x)^{m_i}$ be a prime factorization of $f(x)$ in $\mathbb{F}_q[x]$. Let R be a defect group of the block b of G. Then by Proposition 7.2, R is conjugate to a Sylow p-subgroup of $C_G(t)$. Moreover, $R \cap \mathrm{SL}_n(q)$ is a defect group of $k\,\mathrm{SL}_n(q)c$, and the restriction of the determinant map from $\mathrm{GL}_n(q)$ to R has image the Sylow 2-subgroup of $\mathbb{F}_q^\times$. Since $Z(\mathrm{SL}_n(q))$ has order d, $R \cap \mathrm{SL}_n(q)$ has order at least d_+. The hypothesis therefore implies that either $R\cap\mathrm{SL}_n(q)$ has order d_+ or $R\cap\mathrm{SL}_n(q)$ has order $2d_+$. In the first case, we get $|R| = d_+(q-1)_+$ and in the second, $|R| = 2d_+(q-1)_+$.

By the prime decomposition of $f(x)$ above, if n_i is the degree of $p_i(x)$ for $1 \le i \le u$, we have

$$(*) \qquad C_{\mathrm{GL}_n(q)}(t) \cong \prod \mathrm{GL}_{m_i}(q^{n_i}).$$

Suppose first that $R\cap\mathrm{SL}_n(q)$ has order $(q-1)_+d_+$, which means that $R\cap\mathrm{SL}_n(q) \subseteq Z(\mathrm{SL}_n(q))$. In particular $R \cap \mathrm{SL}_n(q)$ is cyclic. Since $R/R \cap \mathrm{SL}_n(q)$ is also cyclic this means that R is metacyclic. Since R is a Sylow p-subgroup of $C_{\mathrm{GL}_n(q)}(t)$, from $(*)$ we get $\sum_{i=1}^u m_i \le 2$. If $u = 2$, then $m_1 = m_2 = 1$, and $f(x)$ is a product $p_1(x)p_2(x)$ of two irreducible polynomials, and $(*)$ gives

$$(**) \qquad (q-1)_+d_+ = |R| = (q^{n_1}-1)_+(q^{n_2}-1)_+.$$

If $n_1 = n_2$ are both odd, then $n = n_1 + n_2$ would be twice an odd integer, $d_+ = 2$, and $(**)$ along with the fact that q is a square and hence congruent to 1 modulo 4, would give

$$2(q-1)_+ = |R| = (q-1)_+^2 \geq 4(q-1)_+,$$

a contradiction. If $n_1 = n_2$ are both even, then the right-hand side of $(**)$ is greater than $4(q-1)_+^2 \geq 4(q-1)_+d_+$, again an impossibility. Hence, if $u = 2$, then (ii) holds. If $u = 1$ and $m_1 = 2$, then R is a Sylow 2-subgroup of $\mathrm{GL}_2(q^{n_1})$ and in particular has order at least $2(q-1)_+^2$, which is strictly greater than $(q-1)_+d_+$. Thus this case cannot occur. If $u = 1$ and $m_1 = 1$, then $f(x)$ is irreducible and (i) holds.

Now suppose that $R \cap \mathrm{SL}_n(q)$ has order $2(q-1)_+d_+$, which means that $R \cap \mathrm{SL}_n(q)$ is metacyclic and R has 2-rank at most 3. By $(*)$, we have that $\sum_{1 \leq i \leq u} m_i \leq 3$.

We first show that the case $\sum_{1 \leq i \leq u} m_i = 3$ does not hold. Thus, suppose if possible that $\sum_{1 \leq i \leq u} m_i = 3$. Then, from $(*)$,

$$2(q-1)_+d_+ = |R| \geq (q-1)_+^3,$$

which implies that $2 = (q-1)_+ \leq n_+$. But this is impossible as q is a square. Now, suppose that $\sum_{1 \leq i \leq u} m_i = 2$ and $u = 1$. Then $m_1 = 2$, $n = 2n_1$ and by $(*)$ and the fact that q is a square,

$$2(q-1)_+d_+ = |R| = (q^n - 1)_+(q^{n/2} - 1)_+ = 2(q^{n/2} - 1)_+^2.$$

The above holds only if $n/2$ is odd and $d_+ = (q-1)_+$. But $n/2$ being odd means that n_+ and hence d_+ is at most 2, whereupon $d_+ = (q-1)_+$ forces $(q-1)_+ = 2$, again contradicting the fact that q is a square. Next, we consider $\sum_{1 \leq i \leq u} m_i = 2$ and $u = 2$. Then, $m_1 = m_2 = 1$, $f(x)$ is a product of two distinct irreducible polynomials, and $(*)$ gives

$$2(q-1)_+d_+ = |R| = (q^{n_1} - 1)_+(q^{n_2} - 1)_+.$$

If $n_1 = n_2$ are both odd, then $n = 2n_1$ is twice an odd integer, $d_+ = 2$ and the above forces $(q-1)_+ = 4$, contradicting the fact that q is a square. If $n_1 = n_2$ are both even, then the right-hand side of the above is at least $4(q-1)_+^2$, whereas the left is at most $2(q-1)_+^2$, a contradiction. Thus, if $\sum_{1 \leq i \leq u} m_i = 2$ and $u = 2$, then (ii) holds. Finally, if $\sum_{1 \leq i \leq u} m_i = 1$, then (i) holds. $\square$

Definition 7.4. (See [34]) Let $\lambda \in \bar{\mathbb{F}}_q^\times$. We define $\mathcal{M}(\lambda, q)$ to be the set of all non-negative integers u such that $\lambda^{q^u - 1} \in \mathbb{F}_q$, and we let $\mathfrak{M}(\lambda, q)$ be the subset of $\mathbb{F}_q$ consisting of all elements of the form λ^{q^u-1}, with $u \in \mathbb{Z}_{\geq 0}$.

Note the following: for any positive integers u and v, we have $\gcd(q^u - 1, q^v - 1) = q^{\gcd(u,v)} - 1$, and hence if $u_0(\lambda, q)$ denotes the least positive integer in $\mathcal{M}(\lambda, q)$, then $\mathcal{M}(\lambda, q) = u_0(\lambda, q)\mathbb{Z}$. Furthermore, if $\lambda^{q^u - 1} = \alpha \in \mathbb{F}_q$, and $\lambda^{q^v - 1} = \beta \in \mathbb{F}_q$, then

$$\lambda^{q^{u+v}} = (\lambda^{q^u})^{q^v} = \alpha^{q^v} \lambda^{q^v} = \alpha\beta\lambda,$$

from which it follows that $\mathfrak{M}(\lambda, q)$ is a subgroup of $\mathbb{F}_q^\times$. The identity

$$q^{uv} - 1 = (q^u - 1)(1 + q^u + \cdots + q^{u(v-1)})$$

for all positive u and v, gives that $\mathfrak{M}(\lambda, q)$ is generated by $\lambda^{u(\lambda,q)-1}$.

We now prove the result stated at the beginning of this section.

Proof of Theorem 7.1. Let c be as in the statement and let $b = b_t$ be a block of $k\,\mathrm{GL}_n(q)$ covering c. Let λ be an eigenvalue of t and let $y = \mathrm{diag}(\zeta, \zeta, \dots, \zeta) \in Z(t)$. We claim that $\zeta \in \mathfrak{M}(\lambda, q)$. Indeed, the eigenvalues of yt are of the form $\zeta\alpha$, where α is an eigenvalue of t. Hence $[t] = [yt]$ implies $\lambda\zeta$ is also an eigenvalue of t. Furthermore, since $\zeta \in \mathbb{F}_q$, clearly the minimal polynomial of $\lambda\zeta$ over $\mathbb{F}_q$ has the same degree as the minimal polynomial of λ over $\mathbb{F}_q$. Now, by Lemma 7.3, the characteristic polynomial of t is either irreducible or is a product of two irreducible factors of unequal degree over $\mathbb{F}_q$. Thus, λ and $\lambda\zeta$ have the same minimal polynomial over $\mathbb{F}_q$. In other words, $\lambda\zeta = \lambda^{q^u}$ for some u, proving the claim. Thus $|Z(t)|$ divides $|\mathfrak{M}(\lambda, q)|$.

Suppose first that $\phi = \sigma$. Since c is ϕ-stable, both b and ${}^{\phi}b$ cover c. By Proposition 7.2, ${}^{\phi}b$ has label $[\phi^{-1}(t)]$. Replacing t with $\phi^{-1}(t)$, we get by Proposition 7.2 that $[\phi(t)] = [tz]$ for some $z = \mathrm{diag}(\eta, \eta, \dots, \eta) \in Z(\mathrm{GL}_n(q))$. In particular, $\lambda^{q'}$ is an eigenvalue of tz; i.e., $\lambda^{q'} = \lambda'\eta$ for some eigenvalue λ' of t. The minimal polynomials of λ, $\lambda^{q'}$, $\lambda'\eta$ and λ' over $\mathbb{F}_q$ all having equal degree, it follows from Lemma 7.3, that $\lambda^{q'} = \lambda^{q^u}\eta$ for some positive integer u. This gives $\lambda^{q'(q'^{2u-1}-1)} = \eta^{-1} \in \mathbb{F}_q$ so, in particular, $\lambda^{(q'^v-1)} \in \mathbb{F}_q$ for some odd positive integer v. Thus,

$$\lambda^{(q^v-1)} = \lambda^{(q'^{2v}-1)} = \lambda^{(q'^v-1)(q'^v+1)} = \left(\lambda^{q'^v-1}\right)^{q'^v+1} \in \mathbb{F}_q,$$

which means that $u_0(\lambda, q) \mid v$. In particular, $u_0(\lambda, q)$ is odd.

Let v_0 be the least odd positive integer such that $\lambda^{(q'^{v_0}-1)} \in \mathbb{F}_q$ and let $u_0 = u_0(\lambda, q)$. We claim that $u_0 = v_0$. Arguing as above, we see that $u_0 \mid v_0$. Denote by $\bar{\lambda}$ the class of λ in the quotient group $\bar{\mathbb{F}}_q^\times/\mathbb{F}_q^\times$. Then by the definition of v_0, the order $o(\bar{\lambda})$ of $\bar{\lambda}$ is a divisor of $q'^{v_0} - 1$. On the other hand, by the definition of u_0, $o(\bar{\lambda})$ is a divisor of $q^{u_0} - 1 = q'^{2u_0} - 1$. Hence, $o(\bar{\lambda})$ divides

$$\gcd(q'^{v_0} - 1, q'^{2u_0} - 1) = q'^{\gcd(2u_0, v_0)} - 1 = q'^{u_0} - 1,$$

where the last equality holds because v_0 is odd and because $u_0 \mid v_0$. Since v_0 is the least odd positive integer such that $o(\bar{\lambda})$ divides $q'^{v_0} - 1$, it follows that $u_0 \geq v_0$, proving the claim.

Now, since $u_0 = v_0$ is odd, $q' + 1$ divides $q'^{v_0} + 1$, and hence

$$\left(\lambda^{q^{u_0}-1}\right)^{q'-1} = \left(\lambda^{q'^{v_0}-1}\right)^{(q'^{v_0}+1)(q'-1)} = \left(\lambda^{q'^{v_0}-1}\right)^{r(q-1)} = 1,$$

where $r = (q'^{v_0}+1)(q'+1)$. This along with the fact that $\mathfrak{M}(\lambda, q)$ is generated by $\lambda^{q^{u_0}-1}$ proves that $\mathfrak{M}(\lambda, q)$ has order dividing $q' - 1$.

But we have shown above that $|Z(t)|$ divides $|\mathfrak{M}(\lambda, q)|$. Thus, we get that $|Z(t)|$ divides $q' - 1$. On the other hand, by Proposition 7.2, $\mathrm{GL}_n(q)/I$ has order dividing $|Z(t)|$, and hence $\mathrm{GL}_n(q)/I$ has order dividing $q' - 1$. This means that if $h \in \mathrm{GL}_n(q)$ is such that $\det(h) = \alpha^{q'-1}$ for some $\alpha \in \mathbb{F}_q$, then $h \in I$. Now let $g \in \mathrm{GL}_n(q)$ and set $h = {}^{\sigma}gg^{-1}$. Then $\det(h) = \det(g)^{q'-1}$, hence $h \in I$. But this means exactly that $I \rtimes \langle\sigma\rangle$ is normal in $\mathrm{GL}_n(q) \rtimes \langle\sigma\rangle$.

Now let us consider the case that $\phi = \sigma\tau$. In this case, it suffices to show that if $\alpha \in \mathfrak{M}(\lambda, q)$, then $\alpha^{q'+1} = 1$. By the same arguments as before, the block b being $\sigma\tau$-stable means that there exists an integer u such that $\lambda^{-q'} = \lambda^{q^u}\eta$ for some $\eta \in \mathbb{F}_q$. In particular, $\lambda^{q'^v+1} \in \mathbb{F}_q$ for some odd v.

Let w_0 be the least such odd positive integer. We claim that $u_0 = w_0$. Indeed, on the one hand,

$$\lambda^{q^{w_0}-1} = \lambda^{q'^{2w_0}-1} = \lambda^{(q'^{w_0}+1)(q'^{w_0}-1)} \in \mathbb{F}_q,$$

which implies that u_0 is a divisor of w_0. Also, as above we have that $o(\bar{\lambda})$ is a divisor of

$$\gcd(q'^{w_0}+1, q'^{2u_0}-1) = \gcd(q'^{w_0}+1, (q'^{u_0}-1)(q'^{u_0}+1)).$$

Since $u_0 \mid w_0$, and w_0 is odd, $(q'^{u_0}+1) \mid (q'^{w_0}+1)$. On the other hand, $\gcd(q'^{w_0}-1, q'^{w_0}+1) = 2$. Hence, $\gcd(q'^{u_0}-1, q'^{w_0}+1) = 2$. Since t and hence λ have odd order, it follows from the above displayed equation that $o(\bar{\lambda})$ is a divisor of $q'^{u_0}+1$. The definition of w_0 now forces $u_0 \geq w_0$, proving our claim. Now

$$(\lambda^{q^{u_0}-1})^{q'+1} = (\lambda^{q'^{w_0}+1})^{(q'^{w_0}-1)(q'+1)} = (\lambda^{q'^{w_0}+1})^{s(q-1)} = 1,$$

where $s = (q'^{w_0}-1)/(q'-1)$. Thus $\mathfrak{M}(\lambda, q)$ has order dividing $q'+1$, as required. $\square$

8. Sporadic, alternating, and defining-characteristic Lie type groups

By Theorem 4.1 (and using the notation described at the end of that section) it suffices to consider blocks b with Klein four defect group P of finite groups $G = N \rtimes R$, where N is quasi-simple with centre of odd order and R is a complement of $D = P \cap N$ in P, and further $O_2(G) = 1$ and we may identify R with a subgroup of $\mathrm{Out}(N)$. In this section we consider the case that $N/Z(N)$ is a sporadic or alternating simple group. We fix this notation throughout this section and the subsequent ones. While the 2-blocks of the sporadic simple groups are classified in [36], this list is not sufficient for our purposes since we must consider quasi-simple groups, not just simple ones. Also, there are some gaps and errors in the list given in [36], most notably for us the existence of a block of Fi_{24}' with Klein four defect group. Our approach will be to determine the existence of blocks with Klein four defect groups principally by making use of the decomposition matrices, where available, given in [7], and to obtain the remaining information by consideration of centralizers of involutions given in [22, Table 5.3]. We also make use of the following special case of the main result of [13], related to Proposition 8.9.

Lemma 8.1. *Let H be a perfect finite group with a (B, N)-pair in characteristic 2. Then H has a unique 2-block of defect 0.*

Lemma 8.2. *If $N/Z(N) \cong M_{11}$, M_{23}, M_{24}, Co_3, Co_2, Co_1, Fi_{23}, Th, BM, M, J_1, Ly, Ru, or J_4, then $G = N$ is simple, and G possesses no block with defect group P.*

Proof. That $G = N$ follows immediately from [12]. The non-existence of blocks with Klein four defect group follows from [7] in all cases except Co_1, Th, M, Ly, and J_4. Let $Q \leq P$ with $|Q| = 2$. If there is a block b of G with defect group P, then there is a block b_P of $C_G(P)$ with defect group P and $(b_P)^G = b$. Hence $(b_P)^{C_G(Q)}$ has defect group P. In the following we make frequent use of [22, Table 5.3]. If $G \cong Th$ or J_4, then $|O_2(C_G(Q))| > 4$, so such a block cannot exist. If $G \cong Ly$, then $C_G(Q) \cong 2.A_{11}$. However, by [7], $2.A_{11}$ has no block with defect group P. If $G \cong Co_1$, then Q must possess an element in the conjugacy class $2A$, as otherwise $|O_2(C_G(Q))| > 4$. In this case $C_G(Q) \cong (P \times G_2(4)).2$. But by Lemma 8.1, $G_2(4)$ possesses a unique (and hence $C_G(Q)$-stable) 2-block of defect 0, and $C_G(Q)$ cannot possess a block with defect group P. If $G \cong BM$ (where BM is the 'baby monster'), then Q must possess non-trivial elements only in the conjugacy classes $2A$ or $2C$, otherwise $|O_2(C_G(Q))| > 4$. If Q possesses an element of the class $2C$, then $C_G(Q) \cong (P \times F_4(2)).2$. But by Lemma 8.1, this cannot possess a block with defect group P, a contradiction. Hence the non-trivial elements of P must belong to the class $2A$. But from [12], no irreducible character of G vanishes on all three of the conjugacy classes $2B$, $2C$, and $2D$, and so G cannot possess a block with defect group P. Finally suppose that $G \cong M$. Then Q must possess an element in the conjugacy class

$2A$, otherwise $|O_2(C_G(Q))| > 4$. In this case $C_G(Q) \cong 2.BM$ is the central extension of the baby monster. It suffices to show that BM cannot possess a 2-block of defect 1. But if it did, then the stabilizer of a defect group would be $2.{}^2E_6(2).2$. However, by Lemma 8.1, ${}^2E_6(2)$ has a unique (and so stable) 2-block of defect 0, so every block of $2.{}^2E_6(2).2$ has 2-defect at least 2, and we are done. □

Lemma 8.3. *Suppose that $N/Z(N) \cong M_{12}$, M_{22}, J_2, J_3, HS, Suz, $O'N$, He, HN, or Fi'_{24}. Then $|G : N| \leq 2$, and N possesses no 2-block of defect 1. If $N/Z(N) \cong M_{22}$, He, HN, Suz, $O'N$, or J_3, then N possesses no 2-block with defect group P, and otherwise N possesses a unique 2-block with defect group P. Hence every block of G with defect group P is real.*

Proof. In all cases except Fi'_{24} this follows using [7], noting that if $|G : N| = 2$, then a 2-block of G of defect 2 covers only 2-blocks of N of defect 1 or 2. Suppose that $N/Z(N) \cong Fi'_{24}$. Then we may take $N \cong 3.Fi'_{24}$. Note that $|\operatorname{Out}(N)| = 2$. Also, outer automorphisms invert the non-trivial elements of the centre, so no faithful 2-block of defect 1 of N can be covered by a block of defect 2 of G. Suppose that there is a 2-block of defect 1 of N, with defect group Q say. Write u for the non-trivial element of Q. Then $uZ(N)$ must lie in the conjugacy class labelled $2A$, otherwise $|O_2(C_{N/Z(N)}(R))| > 4$, and so $|O_2(C_N(Q))| > 4$. (Here we are using the fact that $C_{N/Z(N)}(uZ(N)) = C_N(u)/Z(N)$ when the orders of x and $Z(N)$ are coprime, as we shall do often.) Hence $C_N(Q) \cong Z(N) \times (2.Fi_{22}.2)$. But by [7] this group possesses no 2-block of defect 1. Hence G does not possess a block with defect group P covering a block of defect 1 of N. Now suppose that $G = N$, and let $Q \leq P$ with $|Q| = 2$. As before there must be a block b_Q of $C_G(Q)$ with defect group P and $b_Q^G = b$ (if b exists). If $Q = \langle x \rangle$ with $xZ(N) \in 2B$, then $|O_2(C_{G/Z(N)}(QZ(N)/Z(N)))| > 4$. Hence also $|O_2(C_G(Q))| > 4$, a contradiction. Since Q was taken to be an arbitrary subgroup of order 2 of P, it follows that all non-trivial elements of $\overline{P}$ belong to $2A$. Hence all non-trivial elements of P belong to the corresponding conjugacy class $2A$ of G. Examination of the character table for G in [12] gives precisely five faithful irreducible characters vanishing on the conjugacy class $2B$ (recall that we have taken $G \cong 3.Fi'_{24}$). By consideration of their degrees we cannot have a faithful block with defect group P. Hence it suffices to consider non-faithful blocks, and we assume now that $G \cong Fi'_{24}$. By the above arguments we have $Q \leq P$ of order 2, and a block b_Q of $C_G(Q)$ with defect group P and $b_Q^G = b$. We have $C_G(Q) \cong 2.Fi_{22}.2$. But by [7] there is a unique 2-block of $C_G(Q)$ with defect 2, which must then be real. Hence $b = b_Q^G$ must be the unique 2-block of G with defect group P, and hence is real. (In fact there does exist a block of Fi'_{24} with Klein four defect group.) □

Lemma 8.4. *If $N/Z(N) \cong McL$ or Fi_{22}, then N has no 2-blocks of defect 2, $|G : N| \leq 2$, and if $G \neq N$, then G has a unique, and so real, 2-block of defect 2.*

Proof. This follows from [7]. □

Lemma 8.5. *If $N/Z(N) \cong A_n$ for some $n \geq 5$, then b is real.*

Proof. Suppose first that $n \neq 6, 7$; then $Z(N) = 1$ and $|G : N| \leq 2$. Thus $P \cap N \neq 1$. If $|P \cap N| = 2$, then $G \cong S_n$, and we are done since every irreducible character of S_n is real. Hence we assume that $P \leq N$, so $G = N$. Write $N \leq E \cong S_n$. Suppose that b is not E-stable. Then b is covered by a block of E with defect group P. Let $u \in P \setminus \{1\}$ be a product of t disjoint transpositions. Then $C_E(u) \cong (\mathbb{Z}_2 \wr S_t) \times S_{n-2t}$. We have $t \leq 2$, since if $t > 2$, then $|O_2(C_E(u))| \geq 2^t > 4$, a contradiction since there is a block of $C_E(u)$ with defect group P. But there must be some $u \in P$ which is a product of two transpositions, in which case

$C_E(u) \cong D_8 \times S_{n-4}$, and $|O_2(C_E(u))| = 8$, a contradiction. Hence b is E-stable, and so is real, since a block of E covering b is real. If $n = 6$ or $n = 7$, then we refer to [7], which includes tables for the exceptional Schur multipliers $3.A_6$ and $3.A_7$. (In $3.A_7$ there are two non-real blocks of defect 0, but the outer automorphism group has order 2, so this is not a problem.) □

Collecting the last four lemmas, we have:

Proposition 8.6. *Let b be a block with Klein four defect group P of a finite group G with $N \lhd G$, where N is a quasi-simple group with odd centre and $N/Z(N)$ a sporadic or alternating simple group, and suppose that $C_G(N) = Z(N)$. Then b is real.*

To end this section, we will prove our result when $N/Z(N)$ is a simple group of Lie type in characteristic 2. We begin with the following result.

Proposition 8.7. *Let N be a quasi-simple group, such that $N/Z(N)$ is a finite simple group of Lie type in characteristic 2, and suppose that $N/Z(N)$ is not isomorphic with the Tits group ${}^2F_4(2)'$, $\mathrm{PSp}_4(2)' \cong A_6$, or $G_2(2)' \cong \mathrm{PSU}_3(3)$. Suppose that $|Z(N)|$ is odd. Let b be a 2-block of N. Then the defect group of b is either a Sylow 2-subgroup of N or the trivial group.*

The proof of this is Lemma 5.1 and the subsequent paragraph from [33], derived from [10, Theorem 6.18]. Together with Lemma 8.1 above, this implies that if N is a quasi-simple group of Lie type in characteristic 2, with the three exceptions given in the above proposition, then there is a unique 2-block of defect 0, and all other blocks have defect group a Sylow 2-subgroup of N. In particular, the block of defect 0 is real. Let b be a block of the finite group G with Klein four defect group P, and suppose that $N/Z(N)$ is a simple group of Lie type in characteristic 2. Then either b has defect group containing the Sylow 2-subgroup of N, or $D = 1$ and the defect group of b is R, a complement of N in G, and hence b is real. Since N must have a non-cyclic Sylow 2-subgroup, the only possibility is that P is a Sylow 2-subgroup of N, and $G = N$. However, it is well-known that the only groups of Lie type in characteristic 2 with abelian Sylow 2-subgroups are $\mathrm{PSL}_2(2^n)$, and hence $N/Z(N) = \mathrm{PSL}_2(4) \cong A_5$, which was dealt with in Lemma 8.5. We are left with the three simple groups ${}^2F_4(2)'$, $\mathrm{PSp}_4(2)'$, and $G_2(2)'$. The group $\mathrm{PSp}_4(2)' \cong A_6$ was considered in Lemma 8.5, and the group $\mathrm{PSU}_3(3)$ will be considered in Proposition 10.4. The Tits group has trivial Schur multiplier and outer automorphism group of order 2 (see, for example, [12]), and so we are only interested in 2-blocks of defects 2 and 1.

Lemma 8.8. *Let $N \cong F_4(2)'$. Then N has no 2-block of defect 1 or 2.*

Proof. Let D be a defect group of order 2 or 4 for a block of N. By [12] N has two conjugacy classes of involutions, and these have centralizer sizes $2^{11} \cdot 5$ and $2^9 \cdot 3$. Let $x \in Z(D)$ be an involution, and write $H = C_N(x)$. Since D is abelian, H possesses a block with defect group D, and so $|O_2(H)| \leq 4$. Suppose $O_{2'}(H) \neq 1$. Then $[H : C_H(O_{2'}(H))] \mid 4$, and so $|O_2(C_H(O_{2'}(H)))| \geq 2^7$. But $C_H(O_{2'}(H)) \lhd H$, contradicting $|O_2(H)| \leq 4$. Hence $O_{2'}(H) = 1$, and so $C_H(O_2(H)) \leq O_2(H)$ (since H is soluble), contradicting $[H : C_H(O_2(H))] \mid 6$. □

Actually, calculations using GAP [20] show that the Tits group has three 2-blocks: the principal block, and two 2-blocks of defect 0. We arrive at the following result.

Proposition 8.9. *Let b be a block with Klein four defect group P of a finite group G with $N \lhd G$, where N is a quasi-simple group with odd centre and $N/Z(N)$ a simple group of Lie type in characteristic 2, and suppose that $C_G(N) = Z(N)$. Then b is real.*

9. Linear Groups in odd characteristic

We will continue with the notation as outlined at the end of Section 4; in particular, $G = N \rtimes R$, where R is a complement of $D = N \cap P$ in the Klein four group P. Note that R necessarily stabilizes c. In this section, we consider the case that $N/Z(N)$ is a projective special linear group $\mathrm{PSL}_n(q)$, with q odd. If $N/Z(N) = \mathrm{PSL}_2(9) \cong A_6$, then by Proposition 8.6, the block b is real. By [22, Table 6.1.3], unless $N/Z(N) = \mathrm{PSL}_2(9)$, the exceptional part of the Schur multiplier of $N/Z(N)$ is trivial, and hence N is isomorphic to a quotient of $\mathrm{SL}_n(q)$. By the above, it suffices to consider the case where $N = \mathrm{SL}_n(q)/Z_0$ for some central subgroup Z_0 of $\mathrm{SL}_n(q)$ containing the Sylow 2-subgroup of $Z(\mathrm{SL}_n(q))$. We let Z_+ denote the Sylow 2-subgroup of $Z(\mathrm{GL}_n(q))$. Then N is naturally isomorphic to the subgroup $\mathrm{SL}_n(q)Z_+/Z_+Z_0$ of $\mathrm{GL}_n(q)/Z_+Z_0$. Henceforth, we will identify N with $\mathrm{SL}_n(q)Z_+/Z_+Z_0$. Thus, N is a normal subgroup of $\mathrm{GL}_n(q)/Z_+Z_0$, and we have that $\mathrm{GL}_n(q)/Z_+Z_0N$ is cyclic and has even order if and only if $\gcd(q-1,n)$ is even. Let $\pi : \mathrm{Aut}(N) \to \mathrm{Out}(N)$ be the natural homomorphism. Let $\mathcal{D} := \mathrm{Outdiag}(N) \le \mathrm{Out}(N)$ in the sense of [22, Definition 2.5.10]. Let $\mathcal{F}$ be the subgroup of $\mathrm{Out}(N)$ generated by the image of the automorphism induced by the automorphism of $\mathrm{GL}_n(q)$ which raises every matrix entry to its lth power, where $q = r^a$, and r is a prime. Let τ be the automorphism of N induced by the inverse-transpose map on $\mathrm{GL}_n(q)$ and let $\mathcal{T}$ be the image in $\mathrm{Out}(N)$ of the subgroup generated by τ. Then $\mathrm{Out}(N) = \mathcal{D}\mathcal{F}\mathcal{T}$. Here $\mathcal{D}$ is normal in $\mathrm{Out}(N)$ and $\mathcal{F}$ and $\mathcal{T}$ centralize each other; $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{T}$ are all cyclic, $|\mathcal{D}| := d = \gcd(q-1,n)$, $|\mathcal{F}| = a$ and $|\mathcal{T}| = 2$. If q is a square, say $q = q'^2$, for $q' > 0$, let σ be the automorphism of $\mathrm{GL}_n(q)$ which raises every matrix entry to its q'th power and denote by σ also the induced automorphism of N. If $n > 2$, then $\mathrm{Out}(N)$ is a semidirect product of $\mathcal{D}$ with $\mathcal{F} \times \mathcal{T}$. If $n = 2$, then $\mathcal{T} \le \mathcal{D}$ and $\mathrm{Out}(N)$ is a semi-direct product of $\mathcal{D}$ with $\mathcal{F}$.

Lemma 9.1. *Let ϕ be an involution in $\mathrm{Aut}(N)$. If $\pi(\phi) \in \mathcal{D}\mathcal{F} \setminus \mathcal{D}$, then ϕ is $\mathrm{Aut}(N)$-conjugate to σ. If $\pi(\phi) \in \mathrm{Out}(N) \setminus (\mathcal{D}\mathcal{F} \cup \mathcal{D}\langle\pi(\tau)\rangle)$, then ϕ is $\mathrm{Aut}(N)$-conjugate to $\sigma\tau$.*

Proof. If $\pi(\phi) \in \mathcal{D}\mathcal{F} \setminus \mathcal{D}$, then since ϕ has order 2, $\phi = c_g \circ \sigma$ for some $g \in \mathrm{GL}_n(q)$, where c_g denotes conjugation by g. Thus, by [22, Proposition 2.5.17], ϕ is a field automorphism. Now it follows by Proposition 4.9.1(d) of [22] that ϕ is conjugate to σ. If $\pi(\phi) \in \mathrm{Out}(N) \setminus (\mathcal{D}\mathcal{F} \cup \mathcal{D}\langle\pi(\tau)\rangle)$, then arguing similarly as above $\phi = c_g \circ \sigma\tau$. By [22, Proposition 2.5.17], ϕ is a graph-field automorphism. The result follows as before from Proposition 4.9.1(d) of [22]. $\square$

In the sequel, we will identify R with its image in $\mathrm{Aut}(N)$ and in $\mathrm{Out}(N)$.

Lemma 9.2. *If $R \le \mathcal{D}$, then b is automorphically dual.*

Proof. By hypothesis, $|R| \le 2$. If $R = 1$, then $G = N$. If $|R| = 2$, then d and hence the index of N in $\mathrm{GL}_n(q)/Z_+Z_0$ is even. By Lemma 4.4 (and the subsequent remarks) of [34], it follows that G is the subgroup of $\mathrm{GL}_n(q)/Z_+Z_0$ containing N with index 2. In both cases, by [34, Lemma 3.2], every simple kG-module is automorphically dual, and hence every block of kG is automorphically dual. $\square$

Lemma 9.3. *If $R \cap \mathcal{D}\pi(\tau) \ne \emptyset$, then b is real.*

Proof. By hypothesis, there is a $g \in \mathrm{GL}_n(q)$ such that ${}^{g\tau}b = b$. Let e be the unique block of $k\,\mathrm{SL}_n(q)Z_+$ which is the pull-back of b. Then, since Z_+ centralizes $\mathrm{SL}_n(q)Z_+$, e is also a block of $k\,\mathrm{SL}_n(q)$, and clearly ${}^{g\tau}e = e$. Let $\tilde{e}$ be a block of $k\,\mathrm{GL}_n(q)$ covering the block e of $k\,\mathrm{SL}_n(q)$. Then ${}^{g\tau}\tilde{e}$ also covers e. Since τ inverts conjugacy classes of $\mathrm{GL}_n(q)$, we have $\tilde{e}^\vee = {}^{\tau}\tilde{e}$, and since

$g \in \mathrm{GL}_n(q)$, we have ${}^{g\tau}\tilde{e} = {}^{\tau}\tilde{e}$. Hence, both $\tilde{e}$ and $\tilde{e}^\vee$ cover e, which means that e and $e^\vee$ are covered by the same blocks of $k\,\mathrm{GL}_n(q)$. Hence the $\mathrm{GL}_n(q)$-orbit of e is closed under taking duals. But by Proposition 7.2, the number of blocks in the $\mathrm{GL}_n(q)$-orbit of e is odd. This means that $f = f^\vee$ for some f in the $\mathrm{GL}_n(q)$-orbit of e; that is, for any element x of $\mathrm{SL}_n(q)$, the coefficient of x in f is the same as the coefficient of x^{-1}, but then the same is true for e. Hence e and therefore b are self-dual. $\square$

The above two lemmas can be combined to give:

Lemma 9.4. *If $R \leq \mathcal{DT}$, or if $R \cap \mathcal{D}\pi(\tau) \neq \emptyset$, then b is automorphically dual.*

We cannot show in general that b is automorphically dual (as a block of kG). Our strategy is to show that kGb is source-algebra equivalent to a block of a certain overgroup of G which is automorphically dual.

Lemma 9.5. *If d is even, let M be the subgroup of $\mathrm{GL}_n(q)/Z_+Z_0$ containing N with index 2. Suppose that R is not contained in $\mathcal{DT}$ and that $R \cap \mathcal{D}\pi(\tau) = \emptyset$. Then q is a square. Furthermore,*

(i) *if $|R| = 2$, then $G \cong N \rtimes \langle\phi\rangle$, through an isomorphism preserving N, and where ϕ is either σ or $\tau\sigma$, and*
(ii) *if $|R| = 4$, then n is even and $G \cong M \rtimes \langle\phi\rangle$, through an isomorphism preserving N and where ϕ is either σ or $\tau\sigma$.*

Proof. Suppose if possible that q is not a square. Then $\mathcal{F}$ has odd order, which means that all involutions of $\mathrm{Out}(N)$ are contained in $\mathcal{DT}$. In particular, $R \leq \mathcal{DT}$, a contradiction. Suppose now that $|R| = 2$. Then, since $R \not\leq \mathcal{D}$, the involution in R is either in the coset $\mathcal{D}\pi(\sigma)$ or in the coset $\mathcal{D}\pi(\sigma\tau)$. Hence, by Lemma 9.1, ϕ is $\mathrm{Aut}(N)$-conjugate to either σ or $\sigma\tau$. Thus, (i) follows from Lemma 4.4 and the subsequent remark of [34]. Suppose finally that $|R| = 4$. Then $R \cap \mathcal{D} \neq \emptyset$, as otherwise $R \cap \mathcal{DT} \neq \emptyset$, which is not the case. In particular, $\mathcal{D}$ is cyclic of even order, which means that d and hence n are even. Let δ be the unique involution in $\mathcal{D}$. Then $R = \langle\delta\rangle \times \langle x\rangle$ for some involution $x \notin \mathcal{D}$. But $x \notin \mathcal{D}\langle\pi(\tau)\rangle$, and so either $x \in \mathcal{DF} \setminus \mathcal{D}$ or $x \in \mathrm{Out}(N) \setminus (\mathcal{DF} \cup \mathcal{D}\langle\pi(\tau)\rangle)$. Since x is the image of an involution in $\mathrm{Aut}(N)$, by Lemma 9.1, x is $\mathrm{Out}(N)$-conjugate to one of $\pi(\sigma)$ or $\pi(\sigma\tau)$. The result follows from [34, Lemma 4.4] since δ is central in $\mathrm{Out}(N)$. $\square$

We now introduce some notation: for the rest of this section, we let ϕ denote either σ or $\sigma\tau$ and we let M denote either N or the subgroup of $\mathrm{GL}_n(q)/Z_+Z_0$ containing N with index 2. We let I denote the stabilizer of the block b of kN in $\mathrm{GL}_n(q)/Z_+Z_0$ and let e be the block of $k\,\mathrm{SL}_n(q)Z_+$ lifting the block of kN. Since Z_+ is a central 2-subgroup of $\mathrm{SL}_n(q)Z_+$, e is a block of $k\,\mathrm{SL}_n(q)$, and the stabilizer in $\mathrm{GL}_n(q)$ of e is the full inverse image of I in $\mathrm{GL}_n(q)$. By Corollary 6.4, the stabilizer in $\mathrm{GL}_n(q)$ of e has odd index in $\mathrm{GL}_n(q)$, and hence I has odd index in $\mathrm{GL}_n(q)/Z_+Z_0$. In particular, $M \leq I$. Let I_- be the subgroup of I such that $|I_- : N|$ is odd and $|I : I_-|$ is a power of 2. Set $L = MI_- \rtimes \langle\phi\rangle$, and note that since L/MI_- is a 2-group, any L-stable block of kMI_- is a block of kL.

Lemma 9.6. *Suppose that $G = M\langle\phi\rangle$. There is a block f of kL which is $(I \rtimes \langle\phi\rangle)$-stable and such that kGb and kLf are source-algebra equivalent. In particular, any vertex-source pair of a simple kGb-module U is a vertex-source pair of a simple kLf-module V.*

Proof. Set $J := MI_-$, $H := I \rtimes \langle\phi\rangle$, and let (Q, u) be a maximal (M, b)-Brauer pair. Then $M \leq J$ are normal subgroups of H, H/J is a 2-group and J/M is a cyclic group of odd order. Furthermore, since as a block of kG, b has Klein four defect groups, and since b is a block of kN, Q is cyclic of order 1 or 2, so in particular kMb is nilpotent and by the definition of I, b is H-stable. So by Lemma 3.5, (applied with the group N of the lemma being replaced by M), there is a block f of kJ which is H-stable and such that $bf = f$. Since $L \leq H$, f is also L-stable. Furthermore, since Q is cyclic of order 1 or 2, $N_J(Q) = C_J(Q)$. By construction, $L = JG$ and $G \cap J = M$. Hence, the result follows by Lemma 3.6(iv) and (v), again applied with the group N of the lemma being replaced by M. □

Lemma 9.7. *Let f be an $(I \rtimes \langle\phi\rangle)$-stable block of kJ. Then f is automorphically dual as a block of kL.*

Proof. As before, let e be the block of $k\,\mathrm{SL}_n(q)Z_+$ which is the pull-back of b, and let $\hat{I}$ be the stabilizer in $\mathrm{GL}_n(q)$ of e, so $\hat{I}$ is the full inverse image of I in $\mathrm{GL}_n(q)$. As a block of kN, b is ϕ-stable, hence as a block of $k\,\mathrm{SL}_n(q)$, e is ϕ-stable. Also, as a block of kN, b is of defect 0 or of defect 1, and hence the defect groups of e as a block of $k\,\mathrm{SL}_n(q)$ contain the Sylow 2-subgroup of $Z(\mathrm{SL}_n(q))$ with index at most 2. By Theorem 7.1, therefore, $\hat{I} \rtimes \langle\phi\rangle$ is a normal subgroup of $\mathrm{GL}_n(q) \rtimes \langle\phi\rangle$, from which it follows that $I \rtimes \langle\phi\rangle$ is a normal subgroup of $\mathrm{GL}_n(q)/Z_0Z_+ \rtimes \langle\phi\rangle$. Let v be a block of $k\,\mathrm{GL}_n(q)/Z_0Z_+$ covering the block f of kJ. Then ${}^{\tau}v = v^{\vee}$ covers both ${}^{\tau}f$ and $f^{\vee}$. Thus, there is $g \in \mathrm{GL}_n(q)$ such that $f^{\vee} = {}^{g\tau}f$. Since I has odd index in $\mathrm{GL}_n(q)/Z_0Z_+$, we may assume that g has odd order. We claim that g normalizes $L = J \rtimes \langle\phi\rangle$. Indeed, denoting by $\bar{\phi}$ the automorphism of $\mathrm{GL}_n(q)/Z_0Z_+N$ induced by ϕ and by $\bar{g}$ the coset of g in $\mathrm{GL}_n(q)/Z_0Z_+N$, it suffices to prove that $\bar{g}^{-1}\bar{\phi}(\bar{g}) \in J/N$. Since $I \rtimes \langle\phi\rangle$ is a normal subgroup of $\mathrm{GL}_n(q)/Z_0Z_+ \rtimes \langle\phi\rangle$, we have that $\bar{g}^{-1}\bar{\phi}(\bar{g}) \in I/N$. Since $\mathrm{GL}_n(q)/Z_0Z_+N$ is cyclic, the fact that g has odd order implies that $\bar{g}^{-1}\bar{\phi}(\bar{g})$ also has odd order. On the other hand, I/J is a 2-group. The claim follows. Since τ extends to an automorphism of L, it follows from the above claim that the map $x \mapsto {}^{g\tau}x$ is an automorphism of L sending f to $f^{\vee}$. This proves the lemma. □

Combining the above results give the following.

Proposition 9.8. *Let b be a block of kG with Klein four defect group P, where $G = N \rtimes R$, N is a quasi-simple group with centre of odd order such that $N/Z(N)$ is a projective special linear group $\mathrm{PSL}_n(q)$, q odd and such that $O_2(G) = 1$ and R injects into the outer automorphism group of N. There is a finite group L containing G as a subgroup of odd index and a block f of kL, such that kGb is source-algebra equivalent to kLf and f is automorphically dual.*

10. Unitary Groups over odd primes

Let b be a block with Klein four defect group P of a finite group $G = N \rtimes R$, where N is a quasi-simple group with centre of odd order, R is a complement of $P \cap N$ in P, $O_2(G) = 1$, and the natural map from R into $\mathrm{Out}(N)$ is injective. In this section, we consider the case that $N/Z(N)$ is a projective special unitary group $\mathrm{PSU}_n(q)$, for q odd. We first consider the case $N/Z(N) = \mathrm{PSU}_4(3)$, since this possesses an exceptional automorphism.

Lemma 10.1. *Suppose $N/Z(N) = \mathrm{PSU}_4(3)$. Then b is real.*

Proof. By [7], N possesses no 2-blocks of defect 1 or 2, and in the case $Z(N) \neq 1$ there is a unique block of defect 0 of N covering each faithful linear character of $Z(N)$. Hence, since $Z(G) = Z(N)$, we may assume that $Z(N) = 1$, and consider only the case $D = 1$, i.e., $R = P$. Now N has three blocks of defect 0, which we label b_1, b_2, and b_3, with b_1 and b_2 dual and b_3 self-dual (and having irreducible character degrees 640, 640 and 896 respectively). Since any block of G covering b_3 is also self-dual (as G/N is a 2-group), we consider b_1 and b_2. We have $\mathrm{Out}(N) \cong D_8$, so $\mathrm{Out}(N)$ has three conjugacy classes of involutions, whose actions are given in [12]. Only one class of involutions preserves b_1, so it follows that G cannot have a block with defect group P covering b_1 or b_2 after all. □

By [22, Table 6.1.3], unless $N/Z(N) = \mathrm{PSU}_4(3)$, the exceptional part of the Schur multiplier of $N/Z(N)$ is trivial, and hence N is isomorphic to a quotient of $\mathrm{SU}_n(q)$. By the above, it suffices to consider the case that $N = \mathrm{SU}_n(q)/Z_0$ for some central subgroup Z_0 of $\mathrm{SU}_n(q)$ containing the Sylow 2-subgroup of $Z(\mathrm{SU}_n(q))$. We let Z_+ denote the Sylow 2-subgroup of $Z(\mathrm{GU}_n(q))$. Then N is naturally isomorphic to the subgroup $\mathrm{SU}_n(q)Z_+/Z_+Z_0$ of $\mathrm{GU}_n(q)/Z_+Z_0$. Henceforth, we will identify N with $\mathrm{SU}_n(q)Z_+/Z_+Z_0$. So, N is a normal subgroup of $\mathrm{GU}_n(q)Z_+/Z_+Z_0$, the group $\mathrm{GU}_n(q)Z_+/Z_+Z_0N$ is cyclic and has even order if and only if $\gcd(q+1, n)$ is even. We view $\mathrm{GU}_n(q)$ through its standard embedding in $\mathrm{GL}_n(q^2)$ as the subgroup of matrices whose transpose equals their inverse. Let τ be the automorphism of $\mathrm{GU}_n(q)$ which sends every matrix entry to its qth power. Note that τ inverts conjugacy classes of $\mathrm{GU}_n(q)$. Denote also by τ the automorphism induced by τ on $\mathrm{SU}_n(q)$ and on N. Let $\pi : \mathrm{Aut}(N) \to \mathrm{Out}(N)$ be the natural homomorphism. Let $\mathcal{D} = \mathrm{Outdiag}(N)$ in the sense of [22, Definition 2.5.13] and let $\mathcal{F}$ be the subgroup of $\mathrm{Out}(N)$ generated by the image of the automorphism induced by the automorphism of $\mathrm{GU}_n(q)$ which raises every matrix entry to its rth power, where $q = r^a$, and r is a prime. Then $\mathrm{Out}(N) = \mathcal{D}\mathcal{F}$, $\mathcal{D}$ is normal in $\mathrm{Out}(N)$ and cyclic of order $d := \gcd(q+1, n)$, $\mathcal{F}$ is cyclic of order $2a$. Note that $\pi(\tau)$ is the unique involution in $\mathcal{F}$. We identify R with its image in $\mathrm{Aut}(N)$ and in $\mathrm{Out}(N)$.

Lemma 10.2. *If $R \leq \mathcal{D}$, then b is automorphically dual.*

Proof. By hypothesis, $|R| \leq 2$. If $R = 1$, then $G = N$. If $|R| = 2$, then d and hence the index of N in $\mathrm{GU}_n(q)/Z_+Z_0$ is even. By Lemma 4.4 (and subsequent remarks) of [34], it follows that G is the subgroup of $\mathrm{GU}_n(q)/Z_+Z_0$ containing N with index 2. In both cases, by [34, Lemma 3.2], every simple kG-module is automorphically dual, hence every block of kG is automorphically dual. □

Lemma 10.3. *If $R \cap \mathcal{D}\pi(\tau) \neq \emptyset$, then b is real.*

Proof. By hypothesis, there is a $g \in \mathrm{GU}_n(q)$ such that ${}^{g\tau}b = b$. Let e be the unique block of $k\,\mathrm{SU}_n(q)Z_+$ which is the pull-back of c. Then, since Z_+ centralizes $\mathrm{SU}_n(q)Z_+$, e is also a block of $k\,\mathrm{SU}_n(q)$, and clearly ${}^{g}e = e$. Let $\tilde{e}$ be a block of $k\,\mathrm{GU}_n(q)$ covering the block e of $k\,\mathrm{SU}_n(q)$. Then ${}^{g\tau}\tilde{e}$ also covers e. Since τ inverts conjugacy classes of $\mathrm{GU}_n(q)$, we have $\tilde{e}^\vee = {}^{\tau}\tilde{e}$, and since $g \in \mathrm{GU}_n(q)$, we have ${}^{g\tau}\tilde{e} = {}^{\tau}\tilde{e}$. Hence, both $\tilde{e}$ and $\tilde{e}^\vee$ cover e, which means that e and $e^\vee$ are covered by the same blocks of $k\,\mathrm{GU}_n(q)$. But by Corollary 6.4, the number of blocks in the $\mathrm{GU}_n(q)$-orbit of e is odd. In particular, $f = f^\vee$ for some f in the $\mathrm{GU}_n(q)$-orbit of e; that is for any element x of $\mathrm{SU}_n(q)$, the coefficient of x in f is the same as the coefficient of x^{-1}, but then the same is true for e. Hence e and therefore b are self-dual. □

The above two lemmas can be combined to give:

Proposition 10.4. *Let b be a block of kG with Klein four defect group P, where $G = N \rtimes R$, N is a quasi-simple group with centre of odd order such that $N/Z(N)$ is a projective special unitary group $\mathrm{PSU}_n(q)$, q odd and such that $O_2(G) = 1$, and R injects into the outer automorphism group of N. Then b is automorphically dual as a block of kG.*

11. Classical groups in odd characteristic

The remaining classical groups are the groups $\mathrm{PSp}_{2n}(q)$ for $n \geq 2$, $\mathrm{P\Omega}_{2n+1}$ for $n \geq 3$, $\mathrm{P\Omega}^+_{2n}(q)$ and $\mathrm{P\Omega}^-_{2n}(q)$ for $n \geq 4$, and ${}^3D_4(q)$. In each of these cases, all 2-blocks are real or automorphically dual, as we will prove now.

Proposition 11.1. *Let N be a quasi-simple group such that $N/Z(N)$ is either $\mathrm{PSp}_{2n}(q)$, $\mathrm{P\Omega}^+_{2n}(q)$, $\mathrm{P\Omega}^-_{2n}(q)$, $\mathrm{P\Omega}_{2n+1}(q)$, or ${}^3D_4(q)$. Then all 2-blocks of N are real except when (a) $N/Z(N) \cong \mathrm{P\Omega}^\pm_{4m+2}(q)$ for some m, in which case each block is real or automorphically dual via an automorphism of graph type, or (b) N is the exceptional 3-fold cover of $\mathrm{P\Omega}_7(3)$, in which case every block is real or automorphically dual via a diagonal automorphism. In case (a), real blocks are fixed by graph automorphisms.*

Proof. The Schur multipliers of the simple groups $N/Z(N)$ given in the statement are either 1 or 2, with the exception of $\mathrm{P\Omega}_7(3)$, for which there is a 3-fold central extension. Suppose that $N/Z(N) \cong \mathrm{P\Omega}_7(3)$. By the character table in [12] and [7] every 2-block of $\mathrm{P\Omega}_7(3)$ is real, so it suffices to consider the faithful blocks of the (exceptional) 3-fold extension $3.\mathrm{P\Omega}_7(3)$. By [7] there is a dual pair of faithful 2-blocks of defect nine and another dual pair of defect three. However, by [12] non-inner automorphisms do not centralize $Z(3.\mathrm{P\Omega}_7(3))$, and it follows that the faithful blocks are automorphically dual (in this case via a diagonal automorphism, which we note does not contradict Corollary 6.4 since, being an exceptional extension, $3.\mathrm{P\Omega}_7(3)$ is not a group of Lie type). Since factoring by a central 2-subgroup induces a bijection between the sets of 2-blocks, and since every automorphism of $N/Z(N)$ lifts to an automorphism of N, when $Z(N)$ is a 2-group, we may therefore assume that $Z(N) = 1$. Note that, for each of our N, the group of outer diagonal automorphisms is a 2-group. Consider an extension H of N by a subgroup of the group of diagonal outer automorphisms. By Corollary 6.4, all 2-blocks of N are H-stable, since H/N is a 2-group. In particular, if a 2-block c of N is covered by a real block of H, then c is itself real. Note also that if a block of N is real, then the unique block of H covering it is also real. The groups $\mathrm{SO}_{2n+1}(q)$ and $\mathrm{SO}^\pm_{4n}(q)$ have the property that all irreducible characters, and so all blocks, are real, as proved by Gow [23, Theorem 2], and the result follows from the above observation in this case. If $q \equiv 1 \bmod 4$, then all characters, and so all blocks, of $\mathrm{PSp}_{2n}(q)$ are real by [46], whereas if $q \equiv 3 \bmod 4$, then this is no longer true. However, the group H, obtained by extending the group $\mathrm{PSp}_{2n}(q)$ by the outer diagonal automorphism, only has real-valued characters by [47, Theorem 4]. Hence the result again follows in this case by the above observation. If N is the Steinberg triality group ${}^3D_4(q)$, then all characters of N are real by [46], and so certainly all 2-blocks of N are real. It remains to consider the orthogonal groups $\Omega^\pm_{4n+2}(q)$. Here the full orthogonal group $\mathrm{GO}^\pm_{4n+2}(q)$ has the property that all irreducible characters are real (see [49], [23]), but the same is not true for $\mathrm{SO}^\pm_{4n+2}(q)$. In this case, $\mathrm{PGO}^\pm_{4n+2}(q)/\operatorname{Inndiag}(\mathrm{P\Omega}^\pm_{4n+2}(q))$ has order 2 and is generated by the image of a graph automorphism (see [22, 2.7]). Consider a non-real 2-block c of N. It follows from Corollary 6.4 that c and its dual are stable in $\operatorname{Inndiag}(\mathrm{P\Omega}^\pm_{4n+2}(q))$ and covered by distinct dual blocks c_1 and $c_1^\vee$ of $\operatorname{Inndiag}(\mathrm{P\Omega}^\pm_{4n+2}(q))$ (since $\operatorname{Outdiag}(\mathrm{P\Omega}^\pm_{4n+2}(q))$ is a 2-group and each 2-block of N is covered by a unique block of $\mathrm{Inndiag}(\mathrm{P}\Omega^{\pm}_{4n+2}(q))$). Since every block of $\mathrm{GO}^{\pm}_{4n+2}(q)$ is real and $Z(\mathrm{GO}^{\pm}_{4n+2}(q))$ is a 2-group, it follows that c_1 and $c_1^{\vee}$ are conjugate in $\mathrm{PGO}^{\pm}_{4n+2}(q)$, and so c and $c^{\vee}$ are conjugate via a graph automorphism. It follows from Lemma 6.9 that real blocks of $\mathrm{Inndiag}(\mathrm{P}\Omega^{\pm}_{4n+2}(q))$ are stable in $\mathrm{PGO}^{\pm}_{4n+2}(q)$. □

Corollary 11.2. *Let G be a finite group with a normal subgroup N of index a power of 2, such that N is quasi-simple with odd centre, and $N/Z(N)$ is a symplectic or orthogonal group, or ${}^3D_4(q)$. Suppose further that $C_G(N) = Z(N)$. Then all 2-blocks of G are real or automorphically dual via a graph automorphism, except in the case N is the exceptional 3-fold cover of $\mathrm{P}\Omega_7(3)$, where non-real blocks are automorphically dual via a diagonal automorphism.*

Proof. Let b be a 2-block of G, and c a block of N covered by b. If c is real, then b is real since b is the unique block of G covering c. Hence assume that c is not real. If $N/Z(N) \cong \mathrm{P}\Omega_7(3)$, then $|G : N|$ is 1 or 2, and the result follows immediately from Proposition 11.1. It remains to consider the case $N/Z(N) \cong \mathrm{P}\Omega^{\pm}_{4n+2}(q)$. The result follows from Proposition 11.1 in this case since the subgroup of $\mathrm{Out}(N/Z(N))$ generated by diagonal and field automorphisms is stable under graph automorphisms. □

12. Exceptional groups of Lie type in odd characteristic

In this section, we will make frequent use of the following result.

Lemma 12.1. *Every 2-block of $\mathrm{PSL}_2(q)$ is real.*

We continue the notation of the previous sections, as described at the end of Section 4; in particular, $G = N \rtimes R$, where R is a complement of $D = N \cap P$ in the Klein four group P. Suppose that $O_2(G) = 1$, and identify R with a subgroup of $\mathrm{Out}(N)$. In this section we consider the case where $N/Z(N)$ is a simple exceptional group. For convenience, we will use the following additional notation throughout this section, for which we largely follow Section 4.5 of [22]. Let $q = r^m$, $r \neq 2$ a prime, and let N be a quotient of a (quasi-simple) universal finite group of Lie type by its (central) maximal normal 2-subgroup. Let u be an involution in D (if such an involution exists). Writing $L = O^{r'}(C_N(u))$, we have that $C_N(u) = LT$, where T is an abelian r'-group acting on L via (possibly trivial) inner-diagonal automorphisms. Note that u induces an involution in the automorphism group of the universal version of the group, so we may make use of Tables 4.5.1 and 4.5.2 of [22]. Write $Q = \langle u \rangle$, $H = C_N(Q)$ and $\overline{H} = H/Q$. Let c_Q be a block of H with $c_Q^N = c$. Then c_Q may be chosen to have defect group D, and corresponds to a unique block c_Q' of $\overline{H}$ with defect group D/Q under the natural map. We make frequent use of [22, 4.5.1,4.5.2], and only make a citation when we wish to draw the reader's attention to a particular table. Note that Table 4.5.1 of [22] gives the centralizer of an involution in $\mathrm{Inndiag}(N)$, so it will sometimes be necessary to consider $E = \mathrm{Inndiag}(N)$ even when $G = N$. For ease of notation, we use the convention that $a.X$ could be any central extension of X by a subgroup of order a, including the case of a direct product.

Lemma 12.2. *If $N/Z(N)$ is of type $G_2(q)$, then $P \leq N$ and b is real.*

Proof. Note that if $r \neq 3$, then $|G : N| \leq 2$ and $Z(N) = 1$, since we are assuming $r \neq 2$. If $r = 3$, then $|G : N| \mid 4$ and $|Z(N)| = 3$ if $q = 3$ and $Z(N) = 1$ otherwise. In each case $\mathrm{Out}(N)$ is cyclic, and so we need not consider the case where $D = 1$. Now $N/Z(N)$ (and so N) has a unique class of involutions, and $C_{N/Z(N)}(uZ(N)) \cong 2.(\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)).2$, where the outer automorphisms of $\mathrm{PSL}_2(q)$ are inner-diagonal; thus $C_N(u) \cong Z(N) \times (2.(\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)).2)$. If $q = 3$, then $|O_2(C_N(u))| = 32$, and so we have no block with defect group P in this case. If $q \neq 3$, then $Z(N) = 1$, and by Lemma 12.1 the block of $C_N(u)'$ covered by c_Q is real. Hence by Corollary 6.4 the block c_Q, and so b, is real. □

We note that blocks with Klein four defect group do indeed exist for $G_2(q)$ for certain q by [27].

Lemma 12.3. *If $N/Z(N)$ is of type ${}^2G_2(q)$, then b is real.*

Proof. Here $r = 3$ and m is odd. We have that $Z(N) = 1$ and that $|\operatorname{Out}(N)|$ is odd; hence $D = P$. In [48, II.7], Ward notes that the ordinary characters in 2-blocks of defect 2 are real. Therefore all blocks with defect group P in $N = G$ are real. □

Lemma 12.4. *If $N/Z(N)$ is of type $F_4(q)$, then b is real.*

Proof. We have $Z(N) = 1$ and $|G : N| \leq 2$, since q is odd. Hence D has order 2 or 4. The group N has two classes of involutions, with representatives t_1 and t_4. Suppose first that u is conjugate to t_1. Then $C_N(u) \cong 2.(\mathrm{PSL}_2(q) \times \mathrm{PSp}_6(q)).2$. Hence by Corollary 6.4 we cannot have $|D| = 2$ or $D \leq L$, as otherwise c_Q would have defect greater than that of c. Hence D possesses an involution acting as an inner-diagonal but non-inner automorphism of $\overline{H}$. But then by Lemma 12.1 and Proposition 11.1 the block c_Q, and so b, is real. Suppose that u is conjugate to t_4. Then $C_N(u) \cong \mathrm{Spin}_9(q)$. Every 2-block of $\mathrm{Spin}_9(q)$ is real by Proposition 11.1, and hence c_Q, and so b, is real. □

Lemma 12.5. *If $N/Z(N)$ is of type $E_6(q)$ and $D \neq 1$, then $P \leq N$ and b is automorphically dual.*

Proof. Here $Z(N)$ has order dividing $\gcd(3, q-1)$, and it suffices to consider the case $|Z(N)| = \gcd(3, q-1)$. We are assuming that D is non-trivial, so it contains an involution u. There are two classes of involutions in N, with representatives t_1 and t_2. Let τ be an automorphism of order 2 of N induced by the automorphism of order 2 of the Dynkin diagram.

(i) Suppose that u is conjugate to t_1. Then $C_N(u)$ has a cyclic central subgroup of order $\gcd(4, q-1)$, so $q \equiv 3 \bmod 4$ since D has exponent 2. Hence we have $C_N(u) \cong 2.\mathrm{P}\Omega_{10}(q).2 \times \mathbb{Z}_{(q-1)/2} = (\mathrm{Spin}_{10}(q).2) * \mathbb{Z}_{q-1}$. By Corollary 6.4 we cannot have $|D| = 2$ or $D \leq L$, as otherwise c_Q would have defect greater than that of c. Hence in particular $G = N$. Write $H = H_1\langle x \rangle$, where $H_1 \cong \mathrm{Spin}_{10}(q).2$ and x has order $q-1$, with $x^{(q-1)/2} = u$, so that $H = H_1 \times \langle x^2 \rangle$. We have $H_1' = O^{r'}(H)$, where r is the prime dividing q. It follows that H_1 is a characteristic subgroup of H. Now x is semi-simple, so by Proposition 6.2 there is $h \in G$ such that $x^{-1} = {}^h({}^\tau x)$. With a slight abuse of notation, write $\alpha = h\tau$, an automorphism of G. In particular $\alpha(u) = u$, $\alpha(H) = H$ and $\alpha(H_1) = H_1$. By Proposition 6.8, α induces a product of an inner-diagonal and a graph automorphism on H_1', and further α does not induce an inner-diagonal automorphism. Write $c_Q = c_1c_2$, where c_1 is a block of H_1 and c_2 is a block of $\langle x^2 \rangle$. We have chosen α so that $\alpha(c_2) = c_2^\vee$. By Proposition 11.1 either $c_1^\vee = c_1$ (and graph automorphisms fix c_1) or c_1 is automorphically dual by a graph automorphism. By Corollary 6.4 the inner and diagonal parts of α fix c_1 and $c_1^\vee$ (noting that c_1 is a block idempotent for H_1' since H_1 fixes the block of H_1' covered by c_1 and c_1 is the unique block of H_1 covering it, and also noting that in our case the image of such an automorphism has order dividing 2 in $\operatorname{Out}(H_1')$). By Proposition 6.9 the image of every element of H_1 under a graph automorphism is H_1-conjugate to its inverse, and so in particular such an automorphism takes every block to its dual (noting that H_1 is indeed $\mathrm{SO}_{10}(q)$ in this case). Hence $\alpha(c_1) = c_1^\vee$, and so $\alpha(c_Q) = \alpha(c_1)\alpha(c_2) = c_1^\vee c_2^\vee = c_Q^\vee$. Therefore we have that

$$\tau(b) = \alpha(b) = \alpha(c_Q^G) = (\alpha(c_Q))^G = (c_Q^\vee)^G = (c_Q^G)^\vee = b^\vee,$$

and b is automorphically dual.

(ii) Now suppose that u is conjugate to t_2. Then $H \cong 2.(\mathrm{PSL}_2(q) \times Z(N).\,\mathrm{PSL}_6(q)).2$. By Corollary 6.4 we cannot have $|D| = 2$ or $D \leq L$, as otherwise c_Q would have defect greater than that of c. In particular we have $G = N$. Since u is the unique conjugacy class with the given centralizer type, $\tau(u)$ is G-conjugate to $u^{-1} = u$, say $\tau(u) = {}^h u$ for some $h \in G$. Write $\alpha = h^{-1}\tau$, so $\alpha(u) = u$ and $\alpha(H) = H$. By Proposition 6.8 α induces a non-inner-diagonal inner-diagonal-graph automorphism on H. Hence α induces an inner-diagonal-graph automorphism on $H'/O_2(Z(H)) \cong \mathrm{PSL}_2(q) \times Z(N).\,\mathrm{PSL}_6(q)$. Since c_Q covers an H-stable block of H', and c_Q is the unique block of H covering that block of H', it follows that c_Q corresponds uniquely to a block f of defect 0 of $H'/O_2(Z(H))$. Write $H'/O_2(Z(H)) = H_1 \times H_2$, where $H_1 \cong \mathrm{PSL}_2(q)$ and $H_2 \cong Z(N).\,\mathrm{PSL}_6(q)$. Let α_i be the automorphism of H_i induced by α (which may also be regarded as an automorphism of H'). By the discussion above the image $\bar{\alpha}_i$ of α_i in $\mathrm{Out}(H_i/Z(H_i))$ lies in $\mathrm{Outdiag}(H_i/Z(H_i)).\langle\tau_i\rangle$, where τ_i is the inverse-transpose automorphism. Write $f = f_1 f_2$, where f_i is a block of H_i. By Lemma 12.1, f_1 is real, and using Corollary 6.4 it follows that f_1 is fixed by α_1. Now by [34, Lemma 3.2] simple modules of H_2 are automorphically dual via automorphisms in $Z(N)\,\mathrm{PGL}_6(q)\langle\tau_2\rangle$. Since it may be the case that $3 \mid (q-1)$ we must consider the possibility that f_2 is automorphically dual via an automorphism whose image in $\mathrm{Out}(H_2)$ has order divisible by 3, which does not occur in H. Hence we consider X with $G \leq X$ and $X/Z(X) \cong \mathrm{Inndiag}(G/Z(G))$ in order to provide such an automorphism. It follows from [34, Lemma 3.2] that $\alpha_2(f_2) = \psi(f_2^\vee)$ for some $\psi \in \mathrm{Inndiag}(\mathrm{PSL}_6(q))$. By [22, 4.5.1] there is $\beta \in C_X(u)$ such that β induces ψ on H_2 and an inner-diagonal automorphism on H_1. Since $|\,\mathrm{Outdiag}(\mathrm{PSL}_2(q))| = 2$, it follows (using Corollary 6.4) that $\beta(f_1) = f_1$. Hence

$$\beta^{-1}\alpha(f) = f_1(\beta^{-1}\alpha_2(f_2)) = f_1(f_2^\vee) = f_1^\vee f_2^\vee.$$

Hence f, and so c_Q, is automorphically dual via $\beta^{-1}\alpha$. So

$$\beta^{-1}\tau(b) = \beta^{-1}\alpha(b) = \beta^{-1}\alpha(c_Q^G) = (\beta^{-1}\alpha(c_Q))^G = (c_Q^\vee)^G = (c_Q^G)^\vee = b^\vee.$$

□

Lemma 12.6. *Suppose that $N/Z(N)$ is of type $E_6(q)$ and that $D = 1$. Then there exists a group $\tilde{G}$ containing G as a subgroup of odd index and a block $\tilde{b}$ of $k\tilde{G}$ such that kGb and $k\tilde{G}\tilde{b}$ are source-algebra equivalent and $\tilde{b}$ is Galois dual.*

Proof. Note that (by for example [12]) $\mathrm{Out}(N) \cong \mathbb{Z}_{\gcd(3,q-1)} \rtimes (\mathbb{Z}_a \times \mathbb{Z}_2)$, where $q = r^a$ for r a prime. By hypothesis, $\mathrm{Out}(N)$ contains an elementary abelian group of order 4, so $q = q'^2$ is a square and there is a simple algebraic group $\tilde{\mathbf{N}}$ of type E_6, and endomorphisms, $F, F_1 : \tilde{\mathbf{N}} \to \tilde{\mathbf{N}}$ such that F_1 is a Frobenius endomorphism of $\tilde{\mathbf{N}}$ with respect to an $\mathbb{F}_{q'}$-structure on $\tilde{\mathbf{N}}$, $\tilde{\mathbf{N}}^{F_1}$ is of type $E_6(q')$, $F = F_1^2$ and $N = O^{r'}(\tilde{\mathbf{N}}^F)$. Further, there exists an algebraic group automorphism $\tau : \tilde{\mathbf{N}} \to \tilde{\mathbf{N}}$ of order 2, commuting with F_1 and inducing an automorphism of order 2 on the underlying Dynkin diagram (as in Proposition 6.2). Denote by σ the restriction of F_1 to N and by τ the restriction of τ to N. The image of $\langle\sigma, \tau\rangle$ in $\mathrm{Out}(N)$ is an elementary abelian group of order 4. But $\mathrm{Out}(N)$ has, up to conjugacy, a unique elementary abelian subgroup of order

4. So, by [34, Lemma 4.4], $G \cong N \rtimes (\langle\sigma\rangle \times \langle\tau\rangle)$, through an isomorphism which preserves N. Thus replacing b with its image in $kN \rtimes (\langle\sigma\rangle \times \langle\tau\rangle)$, we may assume that $G = N \rtimes (\langle\sigma\rangle \times \langle\tau\rangle)$. Note that under this replacement b still remains a block of defect 0 of kN. We first reduce to the case that N is simple; indeed, suppose not. Then $Z(N) = \langle x\rangle$ has order 3, $\tilde{\mathbf{N}}$ is simply connected and $N = \tilde{\mathbf{N}}^F$. By Proposition 6.2, applied to the central element x, one sees that $\tau(x) = x^{-1} \neq x$. On the other hand, b is G-stable, hence τ-stable. Since x is central we have $xb = \zeta b$ for some cube root of unity ζ. So, we have $\tau(xb) = \zeta\tau(b) = \zeta b$ and $\tau(x)b = x^{-1}b = \zeta^{-1}b$, which implies that $\zeta = 1$, hence x acts as the identity on b. Thus, by replacing b by its image under the canonical surjection $kN \to kN/Z(N)$, we may assume that N is simple. In other words, $Z(\tilde{\mathbf{N}}) = 1$ and $\tilde{N} := \tilde{\mathbf{N}}^F$ contains N as a normal subgroup of index 3. Set $\tilde{G} = \tilde{N} \rtimes (\langle\sigma\rangle \times \langle\tau\rangle)$. The index of N in $\tilde{N}$ being prime, it follows from Lemma 3.7 that there is a $(\langle\sigma\rangle \times \langle\tau\rangle)$-stable block $\tilde{b}$ of $k\tilde{N}$ of defect 0 such that $k\tilde{G}\tilde{b}$ is source-algebra equivalent to kGb. The block $\tilde{b}$ is Galois dual by Proposition 6.5, and the result is proved. □

Lemma 12.7. *If $N/Z(N)$ is of type ${}^2E_6(q)$, then $P \leq N$ and b is automorphically dual.*

Proof. Here $Z(N)$ has order dividing $\gcd(3, q+1)$, and $|G : N| \leq 2$. Hence D has order 2 or 4. Now N has two conjugacy classes of involutions, with representatives t_1 and t_2.

(i) Suppose that u is conjugate to t_1. Then $C_N(u)$ has a cyclic central subgroup of order $\gcd(4, q+1)$, so $q \equiv 1 \bmod 4$. Hence $C_N(u) \cong 2.\mathrm{P}\Omega^-_{10}(q).2 \times \mathbb{Z}_{(q+1)/2}$. By Corollary 6.4 we cannot have $|D| = 2$ or $D \leq L$, otherwise c_Q would have defect greater than that of c. Hence in particular $G = N$. Let τ be a graph automorphism of G. Write $H = H_1\langle x\rangle$, where $H_1 \cong \mathrm{P}\Omega^-_{10}(q).2$ and x has order $q+1$, with $x^{(q+1)/2} = u$, so that $H = H_1 \times \langle x^2\rangle$. We have $H_1' = O^{r'}(H)$, where r is the prime dividing q. It follows that H_1 is a characteristic subgroup of H. Since x is semi-simple, by Proposition 6.2 we see that x^{-1} is G-conjugate to $\tau(x)$, say $\tau(x) = {}^h(x^{-1})$. Write $\alpha = h^{-1}\tau \in \mathrm{Aut}(G)$. So $\alpha(x) = x^{-1}$, and in particular $\alpha(u) = u$, $\alpha(H) = H$ and $\alpha(H_1) = H_1$. By Proposition 6.8, α induces a product of an inner, a diagonal, and a graph automorphism on H_1'. Write $c_Q = c_1c_2$, where c_1 is a block of H_1 and c_2 is a block of $\langle x^2\rangle$. We have $\alpha(c_2) = c_2^\vee$. By Proposition 6.9 either $c_1^\vee = c_1$ (and graph automorphisms fix c_1) or c_1 is automorphically dual via a graph automorphism. By Corollary 6.4 inner and diagonal automorphisms fix c_1 and $c_1^\vee$ (noting that c_1 is a block idempotent for H_1' since H_1 fixes the block of H_1' covered by c_1 and c_1 is the unique block of H_1 covering it, and also noting that the image of such an automorphism has order 1 or 2 in $\mathrm{Out}(H_1')$). It follows from Proposition 6.9 that graph automorphisms take blocks to their duals, so $\alpha(c_1) = c_1^\vee$. Hence $\alpha(c_u) = \alpha(c_1)\alpha(c_2) = c_1^\vee c_2^\vee = c_u^\vee$. Hence

$$\tau(b) = \alpha(b) = \alpha(c_u^G) = (\alpha(c_u))^G = (c_u^\vee)^G = (c_u^G)^\vee = b^\vee,$$

and b is automorphically dual.

(ii) Suppose that u is conjugate to t_2. Then $H \cong 2.(\mathrm{PSL}_2(q) \times (3, q+1).\mathrm{PSU}_6(q)).2$. Now c_Q' has defect group of order 2. By Corollary 6.4 we cannot have $|D| = 2$ or $D \leq L$, as otherwise c_Q would have defect greater than that of c. In particular we have $G = N$. Since u is the unique conjugacy class with the given centralizer type, $\tau(u)$ is G-conjugate to $u^{-1} = u$, say $\tau(u) = {}^hu$. Write $\alpha = h^{-1}\tau$, so $\alpha(u) = u$ and $\alpha(H) = H$. By Proposition 6.8, α induces a product of an inner-diagonal and a graph automorphism on $H'/O_2(Z(H)) \cong \mathrm{PSL}_2(q) \times Z(N).\mathrm{PSU}_6(q)$. Since c_Q covers an H-stable block of H', and c_Q is the unique block of H covering that block of H', it follows that c_Q corresponds uniquely to a block f of defect 0 of $H'/O_2(Z(H))$. Write

$H'/O_2(Z(H)) = H_1 \times H_2$, where $H_1 \cong \mathrm{PSL}_2(q)$ and $H_2 \cong Z(N).\mathrm{PSU}_6(q)$. Let α_i be the automorphism of H_i induced by α (which may also be regarded as an automorphism of H'). The image $\bar{\alpha_i}$ of α_i in $\mathrm{Out}(H_i/Z(H_i))$ lies in $\mathrm{Outdiag}(H_i/Z(H_i)).\langle\tau_i\rangle$, where τ_1 is the inverse-transpose automorphism and τ_2 is the automorphism sending every matrix entry to its qth power. Write $f = f_1 f_2$, where f_i is a block of H_i. By Lemma 12.1, f_1 is real, and using Corollary 6.4 it follows that f_1 is fixed by α_1. By [34, Lemma 3.2] simple modules of H_2 are automorphically dual via automorphisms in $Z(N)\,\mathrm{PGU}_6(q)\langle\tau_2\rangle$. Hence we consider X with $G \leq X$ and $X/Z(X) \cong \mathrm{Inndiag}(G/Z(G))$. It follows from [34, Lemma 3.2] that $\alpha_2(f_2) = \psi(f_2^\vee)$ for some $\psi \in \mathrm{Inndiag}(\mathrm{PSU}_6(q))$. By [22, 4.5.1] there is $\beta \in C_X(u)$ such that β induces ψ on H_2 and an inner-diagonal automorphism on H_1. Since $|\mathrm{Outdiag}(\mathrm{PSL}_2(q))| = 2$, it follows that $\beta(f_1) = f_1$. Hence

$$\beta^{-1}\alpha(f) = f_1\beta^{-1}\alpha_2(f_2) = f_1(f_2^\vee) = f_1^\vee f_2^\vee.$$

Hence f, and so c_Q, is automorphically dual via $\beta^{-1}\alpha$. Thus

$$\beta^{-1}\tau(b) = \beta^{-1}\alpha(b) = \beta^{-1}\alpha(c_Q^G) = (\beta^{-1}\alpha(c_Q))^G = (c_Q^\vee)^G = (c_Q^G)^\vee = b^\vee.$$

□

Lemma 12.8. *If $N/Z(N)$ is of type $E_7(q)$ and $D \neq 1$, then $P \leq N$ and b is real.*

Proof. In this case $Z(N) = 1$. Suppose that D is non-trivial, so that it contains an involution u. Write $E = \mathrm{Inndiag}(N)$, so that $|E : N| = 2$. Table 4.5.1 of [22] gives us the centralizer of u in E. There are five potential conjugacy classes of involutions of E to consider, with representatives t_1, t_4, t_4', t_7 and t_7'. Note that if involutions are conjugate in E but not in N, then the centralizers are isomorphic, and so the same argument will deal with both involutions.

(i) Suppose that u is conjugate to t_1. Then $C_E(u) \cong 2.(\mathrm{PSL}_2(q) \times \mathrm{P}\Omega_{12}(q))(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Hence $|D| \neq 2$ and also c_Q' cannot have defect group inside $\mathrm{PSL}_2(q) \times \mathrm{P}\Omega_{12}(q)$, else c would be covered by a block of defect 4 in E by Corollary 6.4. By Lemma 12.1 and Proposition 11.1 c_u, and so b is real.

(ii) Suppose that u is conjugate to t_4. In this case $q \equiv 1 \bmod 4$ (else t_4 is non-inner), and $C_E(u) \cong 2.\mathrm{PSL}_8(q).\mathbb{Z}_{\gcd(8,q-1)/2}.\gamma$, where γ is a graph automorphism. By considering the preimage of u in $2.E_7(q)$ and making use of Table 4.5.2 of [22], we see that $C_N(u) \cong 2.\mathrm{PSL}_8(q).\mathbb{Z}_{\gcd(8,q-1)/2}$. By Corollary 6.4 we cannot have $|D| = 2$, and we must have $q \equiv 5 \bmod 8$. Let $\langle y\rangle \leq \overline{H}$ be a defect group for c_Q'. If y induces an automorphism of type t_1 in $\overline{H}$, then we have $C_{\overline{H}}(y) \cong (\mathrm{SL}_7(q).\mathbb{Z}_{\gcd(7,q-1)}) * \mathbb{Z}_{q-1}$. Since we have $q \equiv 1 \bmod 4$, we have a normal subgroup of order 4, a contradiction. A similar argument applies in case y is of type t_2, t_3 or t_4 in $\overline{H}$. Suppose that y is of type t_4'. Then $C_{\overline{H}}(y) \cong \mathrm{PSL}_4(q^2).4 * \mathbb{Z}_{q+1}$. In this case too we obtain a contradiction by Corollary 6.4.

(iii) Suppose that u is conjugate to t_4'. In this case $q \equiv 3 \bmod 4$ (otherwise t_4' is non-inner), and $C_E(u) \cong 2.\mathrm{PSU}_8(q).\gamma$. As in case (ii), $C_N(u) \cong 2.\mathrm{PSU}_8(q)$, and the same argument applies as for type t_4.

(iv) Suppose that $u = t_7$. Then $q \equiv 1 \bmod 4$ and $C_E(u) \cong (\gcd(3, q-1).E_6(q).3 * \mathbb{Z}_{q-1}).2$. By consideration of [22, Table 4.5.2] we see that $C_N(u) \cong (\gcd(3, q-1).E_6(q).3 * \mathbb{Z}_{q-1})$. Hence H, and so D, contains a cyclic normal subgroup of order 4, a contradiction.

(v) Suppose that $u = t'_7$. Then $q \equiv 3 \bmod 4$ and $C_E(u) \cong (\gcd(3, q+1).{}^2E_6(q).3 * \mathbb{Z}_{q+1}).2$. As above, we see that $C_N(u) \cong (\gcd(3, q+1).{}^2E_6(q).3 * \mathbb{Z}_{q+1})$. Hence H, and so D, contains a cyclic normal subgroup of order 4, a contradiction. □

Lemma 12.9. *If $N/Z(N)$ is of type $E_7(q)$ and $D = 1$, then b is Galois dual.*

Proof. The group N is simple in this case and $\mathrm{Out}(N) \cong \mathbb{Z}_2 \times \mathbb{Z}_a$, where $q = r^a$ for r a prime. Since, by hypothesis, $\mathrm{Out}(N)$ has an elementary abelian subgroup of order 4, $q = q'^2$ for some positive integer q' and there is a simple algebraic group $\tilde{\mathbf{N}}$ of type E_7, with $Z(\tilde{\mathbf{N}}) = 1$ and endomorphisms, $F, F_1 : \tilde{\mathbf{N}} \to \tilde{\mathbf{N}}$ such that F_1 is a Frobenius endomorphism of $\tilde{\mathbf{N}}$ with respect to an $\mathbb{F}_{q'}$-structure on $\tilde{\mathbf{N}}$, $\tilde{\mathbf{N}}^{F_1}$ is of type $E_7(q')$, $F = F_1^2$ and $N = O^{r'}(\tilde{\mathbf{N}}^F)$ is of index 2 in $\tilde{\mathbf{N}}^F$. Denote by σ the restriction of F_1 to N and by τ the restriction of τ to N. Since $\mathrm{Out}(N)$ has up to conjugacy a unique elementary subgroup of order 4, by [34, Lemma 4.4] we may assume that $G = \tilde{N} \rtimes \langle\sigma\rangle$, where $\tilde{N} = \tilde{\mathbf{N}}^F$. Now b is a block of defect 0 of kN, hence a block of $k\tilde{N}$, such that $k\tilde{N}b$ has cyclic defect groups of order 2. In particular, every ordinary irreducible character of $k\tilde{N}b$ has 2-defect 1. Also, by [24, Theorem 3.1] there is an ordinary irreducible character, say χ in b such that $\chi \in \mathcal{E}(\tilde{N}, [s])$ for some odd s. Since χ extends to a character of kG, χ is σ-stable. The result now follows from Lemma 6.7. □

Lemma 12.10. *If $N/Z(N)$ is of type $E_8(q)$, then $P \leq N$ and b is Galois dual.*

Proof. Here we have $Z(N) = 1$. Also $|G : N| \leq 2$, so D has order 2 or 4. The group N possesses two conjugacy classes of involutions, with representatives t_1 and t_8.

(i) Suppose that u is conjugate to t_1. Then $C_N(u) \cong 2.\mathrm{P}\Omega_{16}(q).2$. Let $\langle y\rangle \leq \overline{H}$ be a defect group for c'_Q. Note that by Corollary 6.4 we cannot have $|D| = 2$, and y cannot lie inside $\mathrm{P}\Omega_{16}(q)$, otherwise c'_Q would have defect 2. The conjugacy classes of involutions in $\mathrm{Inndiag}(\mathrm{P}\Omega_{16}(q))$ have representatives t_1, t'_1, t_2, t'_2, t_3 and t'_3, t_4, t'_4, t''_4, and t'''_4, t_7, t'_7, t_8, and t'_8. Suppose first that y is conjugate to t_1 in $\overline{H} \cong \mathrm{P}\Omega_{16}(q).2$. Note that $q \equiv 3 \bmod 4$ since y is not inner. Then $C_{\overline{H}}(y)/\langle y\rangle \cong (\mathrm{P}\Omega_{14}(q).\mathrm{Outdiag}(\mathrm{P}\Omega_{14}(q)) \times \mathbb{Z}_{(q-1)/2})$, and $\mathrm{Outdiag}(\mathrm{P}\Omega_{14}(q))$ has even order, so this case cannot occur by Corollary 6.4. A similar argument applies in the case y is of type t'_1, t_2, t'_2, t_3 and t'_3, t_4, t'_4, t''_4, and t'''_4, t_7, t'_7, t_8, and t'_8.

(ii) Suppose that $u = t_8$. Then $C_G(u) \cong 2.(\mathrm{PSL}_2(q) \times E_7(q)).2$. By Corollary 6.4 we cannot have $|D| = 2$ or $D \leq L$, as otherwise c_Q would have defect greater than that of c. By Lemmas 12.9 and 12.1, it follows that c_u is Galois dual, i.e., there is a field automorphism σ of k such that $\sigma(c_u) = c_u^\vee$. Hence $\sigma(b) = \sigma(c_u^G) = (\sigma(c_u))^G = (c_u^\vee)^G = b^\vee$, and b is also Galois dual. □

Collecting the results of this section, we have:

Proposition 12.11. *Let b be a block with Klein four defect group P of a finite group G with $N \lhd G$, where N is a quasi-simple group with odd centre and $N/Z(N)$ a simple exceptional group of Lie type, and suppose that $C_G(N) = Z(N)$. Then b is real, or Galois dual, or automorphically dual, or there exists a finite group L containing G as a subgroup of odd index and a block f of kL source-algebra equivalent to kGb and such that f is Galois dual.*

Remark 12.12. We briefly note that Lemma 12.5 contradicts [42, Theorem 4], which states that a finite simple group of type $E_6(q)$ possesses a block of defect 1 when $q \equiv 3 \bmod 4$. The error in the proof of [42, Theorem 4] appears to derive from a small misprint in [28] concerning the existence of even-order diagonal automorphisms for $\mathrm{Spin}_{10}(q)$.

We now prove Theorems 1.4 and 1.1:

Proof. Theorem 1.4 is immediate from Propositions 8.6, 8.9, 9.8, 10.4 and 12.11, and Corollary 11.2. Now Theorem 1.1 is equivalent to the formulation (3) given in the introduction and hence follows from Theorem 4.1, Theorem 1.4 and Corollary 5.4. □

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