Linckelmann, M. (2011). Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero. Bulletin of the London Mathematical Society, 43(5), 871 - 885. doi: 10.1112/blms/bdr024 <http://dx.doi.org/10.1112/blms/bdr024>

FINITE GENERATION OF HOCHSCHILD COHOMOLOGY OF HECKE ALGEBRAS OF FINITE CLASSICAL TYPE IN CHARACTERISTIC ZERO

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Abstract. We show that the Hochschild cohomology $HH^*(\mathcal{H})$ of a Hecke algebra $\mathcal{H}$ of finite classical type over a field k of characteristic zero and a non-zero parameter q in k is finitely generated, unless possibly if q has even order in $k^\times$ and $\mathcal{H}$ is of type B or D.

Mathematics Subject Classification (2000): 20C08, 16E40.

1. Introduction

The Hochschild cohomology of an algebra A over a field k is the graded k-algebra $HH^*(A) = \mathrm{Ext}^*_{A\otimes_k A^0}(A, A)$, where A^0 is the opposite algebra of A and where A is viewed as a left $A \otimes_k A^0$-module via multiplication in A. By a result of Gerstenhaber in [16], $HH^*(A)$ is graded commutative; in particular, $HH^*(A)$ is commutative if $\mathrm{char}(k) = 2$. If char $\neq 2$, the Krull dimension of $HH^*(A)$ is, by definition, that of the even part of $HH^*(A)$. For any A-A-bimodule M, viewed as $A\otimes_k A^0$-module in the obvious way, the graded k-module $\mathrm{Ext}^*_{A\otimes_k A^0}(A, M)$ becomes a right module over $HH^*(A)$ via the Yondeda product.

Theorem 1.1. *Let $\mathcal{H}$ be a Hecke algebra of a finite Coxeter group (W, S) over a field k of characteristic zero with non-zero parameter q in k. Suppose that all irreducible components of W are of type A, B, D, and suppose in addition that if W involves a component of type B or D then the order of q in $k^\times$ is not even. Then, for any finitely generated $\mathcal{H}$-$\mathcal{H}$-bimodule M, the $HH^*(\mathcal{H})$-module $\mathrm{Ext}^*_{\mathcal{H}\otimes_k\mathcal{H}^0}(\mathcal{H}, M)$ is Noetherian. In particular, $HH^*(\mathcal{H})$ is finitely generated as a k-algebra.*

If $q = 1$ or if q has infinite order then $\mathcal{H} \otimes_k \mathcal{H}^0$ is semi-simple, in which case $HH^*(\mathcal{H}) = HH^0(\mathcal{H})$ is finite-dimensional, and so we may assume that q is a primitve ℓ-th root of unity for some integer $\ell \geq 2$. We can be more precise regarding the Krull dimension of the Hochschild cohomology:

Theorem 1.2. *Let $\mathcal{H}$ be a Hecke algebra of type A_{n-1} or B_n $(n \geq 2)$, or of type D_n $(n \geq 4)$ over a field k of characteristic zero and a parameter q of finite order $\ell \geq 2$ in $k^\times$; if $\mathcal{H}$ is of type B_n or D_n suppose in addition that ℓ has odd order. Let m, a be the non-negative integers satisfying $n = \ell m + a$ and $0 \leq a \leq \ell - 1$. The Krull dimension of $HH^*(\mathcal{H})$ is equal to m.*

The above results have been motivated by work of Benson, Erdmann and Mikaelian [1], describing the cohomology $H^*(\mathcal{H}) = \mathrm{Ext}^*_{\mathcal{H}}(k, k)$ for Hecke algebras of type A, B , D with the same restrictions on the parameter q as in 1.1 above, in terms of stable elements in the cohomology of a maximal ℓ-parabolic subalgebra. As in [1], we make use of the transfer maps for Hecke algebras from [6], interpreted as a special case of the transfer maps for symmetric

Date: May 9, 2011.

algebras in [2], [3]. A key result is J. Du's Theorem 2.7 in [9], which we use to show that the theory of vertices for modules over Hecke algebras in [9] admits a bimodule version, which then allows us to play the problem back to maximal ℓ-parabolic subalgebras of $\mathcal{H}$. Since these are tensor products of Brauer tree algebras and semi-simple algebras, the result follows from well-known properties of the Hochschild cohomology of self-injective algebras of finite representation type. We refer to [15, §4.4, §8] for general background material and further references on Hecke algebras. It would certainly be desirable to describe $HH^*(\mathcal{H})$ more explicitly, possibly using the stable elements methods in Hochschild cohomology in [20]). The Hochschild cohomology of tame Hecke algebras is described in [13] and [25]. The main obstacle to a generalisation of the above results to Hecke algebras of type B and D with even ℓ, Hecke algebras of exceptional types, Hecke algebras over fields of positive characteristic, or Hecke algebras with unequal parameters, is that we do not have appropriate versions of [9, Theorem 2.7] in these cases.

2. Traces for symmetric algebras

The trace maps used in the context of Hecke algebras in various sources such as [6], [9], as well as Higman's criterion extended to modules over Hecke algebras in [19], can be interpreted as special cases of the trace maps and Higman's criterion for symmetric algebras associated with certain bimodules in [2], [3]. These trace maps are the degree zero components of transfer maps for the Hochschild cohomology of symmetric algebras in [20]. They are special cases of transfer maps defined by Chouinard in [5, §2] associated with a functor which has both a left and a (possibly different) right adjoint. Higman's criterion in the above mentioned cases arise as special cases of Chouinard's proposition [5, 3.2] and lemma [5, 3.3] (we will not need this degree of generality in the present paper). We review this material in the special case of restrictions to subalgebras - detailed proofs can be found in Broué [3]. Let k be a commutative ring. We adopt the usual convention that if A, B are k-algebras, an A-B-bimodule is the same as an $A \otimes_k B^0$-module, where B^0 is the algebra opposite to B; equivalently, we always assume that the left and right k-module structure of an A-B-bimodule coincide. A k-algebra A is called symmetric if A is isomorphic, as an A-A-bimodule, to its k-dual $A^* = \mathrm{Hom}_k(A, k)$ and if A is finitely generated projective as a k-module. The image s of 1_A under a bimodule isomorphism $A \cong A^*$ is called a *symmetrising form* for A; it has the property that $s(ab) = s(ba)$ for all $a, b \in A$ and that the bimodule isomorphism $A \cong A^*$ sends $a \in A$ to the map $s_a \in A^*$ defined by $s_a(b) = s(ab)$ for all $a, b \in A$. Since the automorphism group of A as an A-A-bimodule is canonically isomorphic to $Z(A)^\times$, any other symmetrising form of A is of the form s_z for some $z \in Z(A)^\times$. Given two symmetric algebras A, B and an A-B-bimodule M which is finitely generated projective as a left A-module and as a right B-module, the functors $M \otimes_B -$ and $M^* \otimes_A -$ are left and right adjoint to each other. More precisely, any choice of symmetrising forms s for A and t for B induces adjunction isomorphisms as follows. Composition with s and t induces B-A-bimodule isomorphisms $\mathrm{Hom}_A(M, A) \cong M^* \cong \mathrm{Hom}_{B^0}(M, B)$; similarly for M^* instead of M. The counit of $M^* \otimes_A -$ as left adjoint to $M \otimes_B -$ is given by the composition of B-B-bimodule homomorphisms $M^* \otimes_A M \cong \mathrm{Hom}_{B^0}(M, B) \otimes_A M \to B$, where the first isomorphism is induced by the isomorphism $M^* \cong \mathrm{Hom}_{B^0}(M, B)$ and the second map sends $\mu \otimes m$ to $\mu(m)$, where $\mu \in \mathrm{Hom}_{B^0}(M, B)$ and $m \in M$. The counit of M as left adjoint to M^* is obtained similarly, and the units of the two adjunctions are obtained by dualising the counits, exploiting the symmetry of A, B and the fact that $(M \otimes_B N)^* \cong N^* \otimes_B M^*$, where N is a B-A-bimodule which is finitely generated projective as a left B-module and as a right

A-module. If A is free as a k-module and X a k-basis of A, then the symmetrising form s on A determines a dual basis $X^\vee = \{x^\vee \mid x \in X\}$ satisfying $s(xx^\vee) = 1$ for $x \in X$ and $s(xy^\vee) = 0$ for $x, y \in X$, $x \neq y$. Let now A be a symmetric k-algebra with symmetrising form s and let B be a unitary symmetric subalgebra of A such that A is finitely generated projective as a left (or equivalently, as a right) B-module and such that the restriction of s to B is a symmetrising form on B (or equivalently, as a B-B-bimodule, B has a complement in A contained in $\ker(s)$). (Such a subalgebra is called in [3, 5.1] a *parabolic* subalgebra of A; it is well-known that finite-dimensional Hecke algebras are symmetric and that a parabolic subalgebra of a Hecke algebra in the sense of [15, 4.4.7] is also parabolic in the sense of [3, 5.1], but the converse need not be true.) Denote by $\mu : A \otimes_B A \to A$ the A-A-bimodule homomorphism induces by multiplication in A. Dualising yields an A-A-bimodule homomorphism $\mu^* : A^* \to (A \otimes_B A)^*$. Denote by $\tau_{A/B} : A \to A \otimes_B A$ the A-A-bimodule homomorphism given by composing the bimodule homomorphisms

$$A \cong A^* \to (A \otimes_B A)^* \cong A^* \otimes_B A^* \cong A \otimes_B A$$

The image $c_B^A = \tau_{A/B}(1_A)$ in $A \otimes_B A$ is called the *relative Casimir element.* Since $\tau_{A/B}$ is an A-A-bimodule homomorphism, we have $a \cdot c_B^A = c_B^A \cdot a$ for all $a \in A$. Write $c_B^A = \sum_{y \in Y} y \otimes y'$ for some finite subset Y of A and elements $y' \in A$ for each $y \in Y$. For an A-A-bimodule M write $M^A = \{m \in M \mid am = ma\ (\forall a \in A)\}$; similarly for B-B-bimodules. In particular, c_B^A belongs to $(A \otimes_B A)^A$. Note that an A-A-bimodule homomorphism $M \to N$ sends M^A to N^A, and hence any A-A-bimodule homomorphism $A \otimes_B A \to M$ sends c_B^A to an element in M^A. There is a trace map

$$\mathrm{tr}_B^A : M^B \to M^A$$

sending $m \in M^B$ to $\sum_{y \in Y} ymy'$. This map depends on the choice of s; since s is unique up to an element in $Z(A)^\times$, this is true for tr_B^A as well. The image of tr_B^A is a $Z(A)$-submodule of M^A; in particular, if $M = A$ the image of the trace map $\mathrm{tr}_B^A : A^B \to A^A = Z(A)$ is an ideal in $Z(A)$. A quick way to see that tr_B^A defined in this way sends indeed M^B to M^A and does not depend on any choice (other than that of the symmetrising form s) is as follows: any element $m \in M^B$ determines a unique B-B-bimodule homomorphism $B \to M$, sending 1_B to m. Through a standard adjunction, this corresponds to the A-A-bimodule homomorphism $A \otimes_B A \to M$ sending $a \otimes a'$ to ama', where $a, a' \in A$. Thus the image of the relative Casimir element c_B^A is equal to $\mathrm{tr}_B^A(m)$, hence contained in M^A and independent of the choice of the elements y, y'. In particular, if U, V are A-modules then $\mathrm{Hom}_A(U, V)$ becomes an A-A-module, and the map tr_B^A applied to this bimodule is a map from $\mathrm{Hom}_B(U, V)$ to $\mathrm{Hom}_A(U, V)$ (and [1, Lemma 2.10] is a special case of this observation). We have the following bimodule version of Higman's criterion in [3, 6.8]:

Lemma 2.1. *Let A be a symmetric k-algebra with symmetrising form s and B a unitary symmetric subalgebra of A such that A is finitely generated projective as a left B-module and such that the restriction to B of s is a symmetrising form of B. Then A is isomorphic to a direct summand of $A \otimes_B A$ as an A-A-bimodule if and only if the map $\mathrm{tr}_B^A : A^B \to Z(A)$ is surjective. In particular, if $\mathrm{tr}_B^A(1_A)$ is invertible in $Z(A)$ then A is isomorphic to a direct summand of $A \otimes_B A$ as an A-A-bimodule.*

Proof. By a standard argument for relative projectivity (see e.g. [3, Theorem 6.8]), A is isomorphic to a direct summand of $A \otimes_B A$ if and only if the map $A \otimes_B A \to A$ induced by multiplication in A splits as an A-A-bimodule homomorphism (because this map represents the counit of the adjunction between $A \otimes_B -$ and the restriction from A to B). Thus the dual $\tau_{A/B}$ of this map is split injective if and only if A is isomorphic to a direct summand of $A \otimes_B A$. Every A-A-bimodule homomorphism $A \otimes_B A \to A$ is of the form $a \otimes a \mapsto aca'$ for some uniquely determined element $c \in A^B$, and precomposing such a homomorphism with $\tau_{A/B}$ yields an endomorphism of A sending 1_A to $\mathrm{tr}_B^A(c)$. Thus A is isomorphic to a direct summand of $A \otimes_B A$ if and only if there is an element $c \in A^B$ such that $\mathrm{tr}_B^A(c)$ is invertible in $Z(A)$. Since the image of tr_B^A is an ideal in $Z(A)$, this is equivalent to tr_B^A being surjective. □

If A is free over k and X a k-basis for A with dual basis denoted as above by $X^\vee$, the relative Casimir element c_k^A is equal to the sum $\sum_{x \in X} x \otimes x^\vee$ in $A \otimes_k A$. Slightly more generally we have the following:

Lemma 2.2. *Let A be a k-free symmetric k-algebra with symmetrising form s and let B be a k-free unitary symmetric subalgebra of A such that A is free as a right B-module and such that the restriction to B of s is a symmetrising form of B. Let X be a k-basis of A, with dual basis $X^\vee = \{x^\vee \mid x \in X\}$. Suppose that X contains a basis Y of A as a right B-module such that for $b \in B$ and $y, y' \in Y$ we have $s(y'by^\vee) = s(b)$ and $s(y'by^\vee) = 0$ if $y' \neq y$. Then $Y^\vee = \{y^\vee \mid y \in Y\}$ is a basis of A as a left B-module, we have $c_B^A = \sum_{y \in Y} y \otimes y^\vee$ and $\mathrm{tr}_B^A(a) = \sum_{y \in Y} yay^\vee$ for all $a \in A^B$.*

Proof. Since A is free as right B-module, A^* is free as left B-module, with basis dual to Y, and thus, since A is symmetric, A is free as left B-module with basis $Y^\vee$. The isomorphism $A \cong A^*$ sends 1_A to the symmetrising form s. The map μ^* sends $s \in A^*$ to $s \circ \mu$, where $\mu : A \otimes_B A \to A$ is induced by multiplication in A, as above. For any $y, z \in Y$ such that $y \neq z$ we have $(s \circ \mu)(y \otimes y^\vee) = s(yy^\vee) = 1$ and $(s \circ \mu)(y \otimes z^\vee) = s(yz^\vee) = 0$. Note that the canonical isomorphism $(A \otimes_B A)^* \cong A^* \otimes_B A^*$ exchanges the order of the two copies of A. By elementary linear algebra, the image of $s \circ \mu$ in $A^* \otimes_B A^*$ is equal to $\sum_{y \in Y} s_y \otimes s_{y^\vee}$. Since the isomorphism $A^* \cong A$ sends s_y to y, the image in $A \otimes_B A$ of the above element yields the formula for c_B^A as claimed. The formula for tr_B^A follows immediately. □

This lemma shows that the trace maps considered in the context of parabolic subalgebras of Hecke algebras are indeed special cases of the general construction of trace maps for symmetric algebras. In particular, as a consequence of the quoted result [9, Theorem 2.7] of Du, 2.1 can be applied to maximal ℓ-local parabolic subalgebras of Hecke algebras of type A over a field of characteristic zero.

3. Separably equivalent algebras

Definition 3.1. Two algebras A and B over a commutative ring k are called *separably equivalent* if there is an A-B-bimodule M which is finitely generated projective as a left A-module and as a right B-module and a B-A-bimodule N which is finitely generated projective as a left B-module and as a right A-module, such that A is isomorphic to a direct summand of $M \otimes_B N$ as an A-A-bimodule and such that B is isomorphic to a direct summand of $N \otimes_B M$ as a B-B-bimodule.

The terminology is motivated by the fact that a finite-dimensional algebra A over a field k is separable (that is, projective as an $A \otimes_k A^0$-module) if and only if it is separably equivalent to k. Morita equivalent algebras are trivially separably equivalent. If A, B are symmetric algebras and there is a derived equivalence or a stable equivalence of Morita type between them then A and B are separably equivalent. A finite group algebra kG over a field of positive characteristic

p is separably equivalent to the group algebra kP of a Sylow-p-subgroup P of G. Any block algebra A of kG is separably equivalent to the group algebra kD of a defect group D of A. If A and B are indecomposable algebras over a complete local commutative Noetherian ring k, then as a consequence of the Krull-Schmidt theorem, the bimodules M and N in 3.1 can be chosen to be indecomposable. If A and B are symmetric k-algebras one can always choose M and N such that $N \cong M^*$, simply by replacing M by $M \oplus N^*$, but then M is no longer indecomposable. Here is how one reunites both properties:

Proposition 3.2. *Let A, B be indecomposable symmetric separably equivalent algebras over a complete local commutative Noetherian ring k. There is an indecomposable A-B-bimodule M which is finitely generated projective as a left A-module and as a right B-module such that A is isomorphic to a direct summand of $M \otimes_B M^*$ and B is isomorphic to a direct summand of $M^* \otimes_A M$.*

Proof. This is again a special case of a standard argument for relative projectivity. Since A, B are indecomposable there are indecomposable bimodules M, N satisfying the properties in the definition 3.1 of separably equivalent algebras. View $M \otimes_B -$ as a functor from $\mathrm{Mod}(B \otimes_k A^0)$ to $\mathrm{Mod}(A \otimes_k A^0)$. This functor has $M^* \otimes_A -$ as a left and right adjoint. Since A is isomorphic to a direct summand of $M \otimes_B N$, it follows from the implication $(ii) \Rightarrow (v)$ in [3, Theorem 6.8] that A is isomorphic to a direct summand of $M \otimes_B M^* \otimes_A A \cong M \otimes_B M^*$. A similar argument applied to the functor $- \otimes_A M$ and its left and right adjoint $- \otimes_B M^*$ concludes the proof. □

Proposition 3.3. *Let A be a symmetric algebra over a commutative ring k with symmetrising form s and B a unitary symmetric subalgebra of A such that A is finitely generated projective as a left B-module and such that the restriction to B of s is a symmetrising form of B. If $\mathrm{tr}_B^A(1_A)$ is invertible in $Z(A)$ then A and B are separably equivalent.*

Proof. Set $M = A$, viewed as an A-B-bimodule. Then $M^* \otimes_A M \cong A$, viewed as a B-B-bimodule, and $M^* \otimes_B M \cong A \otimes_B A$, since A is symmetric. By [3, 5.2], B is isomorphic to a direct summand of A as a B-B-bimodule, and by 2.1, A is isomorphic to a direct summand of $A \otimes_B A$. □

Let G be a finite group and B a G-algebra over a commutative ring k. The action of G on B induces an action on $Z(B)$, hence on $Z(B)^\times$. Let $\alpha \in Z^2(G; Z(B)^\times)$, a 2-cocycle of G with coefficients in $Z(B)^\times$, with respect to the induced action of G on $Z(B)^\times$. That is, α is a map from $G \times G$ to $Z(B)^\times$ satisfying the 2-cocycle identity $\alpha(x, y)\alpha(x, yz) = ({}^x\alpha(y, z))\alpha(x, yz)$ for x, y, z in G. Set $A = B_\alpha G$; that is, A is the crossed product equal to the free B-module $\oplus_{x\in G} B\hat{x}$ with a B-basis $\{\hat{x} \mid x \in G\}$ indexed by the elements of G and multiplication induced by $(b\hat{x})(c\hat{y}) = \alpha(x, y)b({}^xc)\widehat{xy}$, for $x, y \in G$ and $b, c \in B$. The 2-cocycle identity ensures that this multiplication is associative. Up to an isomorphism preserving the image of B, the algebra A depends only on the image of α in $H^2(G; Z(B)^\times)$. If we choose $\hat{1} = 1_B$ then α is normalised; that is, $\alpha(x, 1) = 1 = \alpha(1, x)$ for all $x \in G$, and then $\alpha(x, x^{-1}) = \alpha(x^{-1}, x)$ for all $x \in G$. It is well-known (and easy to check) that if B is symmetric with a G-invariant symmetrising form t then A is symmetric with symmetrising form s extending t to A by zero on the subspaces $B\hat{x}$ for $x \in G - \{1\}$.

Proposition 3.4. *Let G be a finite group, B a symmetric G-algebra over a commutative ring k having a G-invariant symmetrising form, and let $\alpha \in Z^2(G; Z(B)^\times)$. Set $A = B_\alpha G$. If $|G| \cdot 1_k$ is invertible in k then A and B are separably equivalent.*

Proof. Clearly B is a direct summand of A as a B-B-bimodule. The homomorphism of left A-modules σ from A to $A \otimes_B A$ sending $a \in A$ to $a \sum_{x\in G} a\hat{x} \otimes (\hat{x})^{-1}$ is in fact a homomorphism of A-A-bimodules. If B is free as a k-module one could show this using 2.2, but one can show this also in general by a direct calculation. Clearly σ is a homomorphism of A-B-bimodules, and so we only need to check that for $y \in G$ this map commutes with the right action of $\widehat{y}$. The map σ sends $\hat{y}$ to $\sum_{x\in G} \hat{y}\hat{x} \otimes (\hat{x})^{-1} = \sum_{x\in G} \hat{y}\hat{x} \otimes (\hat{y}\hat{x})^{-1}\hat{y} = \sum_{x\in G} \alpha(y,x)\widehat{yx} \otimes (\widehat{yx})^{-1}\alpha(y,x)^{-1}\hat{y} = \sum_{x\in G} \widehat{yx}\otimes(\widehat{yx})^{-1}\hat{y}$. Since x runs over G, so does yx, and so this sum is equal to $\sum_{x\in G} \hat{x}\otimes(\hat{x})^{-1}\hat{y}$ as required. Thus σ is indeed a homomorphism of bimodules from A to $A\otimes_B A$. Composed with the homomorphism $A\otimes_B A \to A$ induced by multiplication in A this yields the endomorphism of A given by multiplication with $|G|$. Since the image of $|G|$ is invertible in k, the homomorphism σ is split injective, and hence A is isomorphic to a direct summand of $A \otimes_B A$ as required. □

For the sake of completeness we include the following consequence of a result of Erdmann and Nakano [12]:

Proposition 3.5. *Let A, B be separably equivalent symmetric algebras over a field k. Then A has finite (resp. wild) representation type if and only if B has finite (resp. wild) representation type. In particular, if k is algebraically closed then A and B have the same representation type.*

Proof. Let M be an A-B-bimodule such that M is finitely generated projective as a left A-module, as a right B-module and such that A is isomorphic to a direct summand of $M \otimes_B M^*$ and B is isomorphic to a direct summand of $M^* \otimes_A M$. Suppose that A has wild representation type. The functors $M \otimes_B -$ and $M^* \otimes_A -$ satisfy the hypotheses (hence the conclusion) of [12, §2, Proposition], implying that B has wild representation type. Suppose next that A has finite representation type. Let V be an indecomposable B-module. Then since B is isomorphic to a direct summand of $M^* \otimes_A M$, V is isomorphic to a direct summand of $M^* \otimes_A M \otimes_B V$. Thus V is isomorphic to a direct summand of $M \otimes_A U$ for some indecomposable A-module U. Since A has only finitely many isomorphism classes of indecomposable modules, the same is true for B. Since (by a result of Drozd) a finite-dimensional algebra over an algebraically closed field k has a uniquely determined representation type which is either wild, tame, or finite, the proposition follows. □

The stable categories $\mathrm{stmod}(A)$, $\mathrm{stmod}(B)$ of finitely generated modules of separably equivalent symmetric algebras A, B over a field have the same dimension as triangulated categories, in the sense of [24]. A similar statement holds for bounded derived categories. This will be a consequence of an obvious extension of the notion of separable equivalence to triangulated categories in the next proposition, for which we will need the following notation. Let $(\mathcal{C}, \Sigma)$ be a triangulated category and let U be an object in $\mathcal{C}$. We denote by $\langle U\rangle_1$ the full additive subcategory of $\mathcal{C}$ consisting of all objects isomorphic to finite direct sums of summands of the objects $\Sigma^n(U)$, with $n \in \mathbb{Z}$. For $i \geq 2$ we define inductively $\langle U\rangle_i$ as the full additive subcategory of $\mathcal{C}$ consisting of all objects isomorphic to direct summands of objects Z for which there exists an exact triangle $X \to Y \to Z \to \Sigma(X)$ with X in $\langle U\rangle_{i-1}$ and Y in $\langle U\rangle_1$. Following [24, 3.6], the *dimension of* $\mathcal{C}$, denoted $\dim(\mathcal{C})$, is the smallest positive integer d for which there exists an object U in $\mathcal{C}$ such that $\langle U\rangle_{d+1} = \mathcal{C}$, provided there is such an integer. If no such integer exists, $\mathcal{C}$ is said to have infinite dimension.

Proposition 3.6. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be exact functors between triangulated categories $\mathcal{C}$, $\mathcal{D}$, such that $\mathrm{Id}_{\mathcal{C}}$ is a direct summand of the functor $\mathcal{G} \circ \mathcal{F}$ and $\mathrm{Id}_{\mathcal{D}}$ is a direct* *summand of the functor $\mathcal{F} \circ \mathcal{G}$. Let U be an object in $\mathcal{C}$. Then $\mathcal{F}$ sends $\langle U \rangle_i$ to $\langle \mathcal{F}(U) \rangle_i$, for any positive integer i. Moreover, if d is a positive integer such that $\langle U \rangle_{d+1} = \mathcal{C}$ then $\langle \mathcal{F}(U) \rangle_{d+1} = \mathcal{D}$; in particular, $\mathcal{C}$ and $\mathcal{D}$ have the same dimension.*

Proof. The first statement follows by induction over i from the fact that $\mathcal{F}$ is a an exact functor of triangulated categories. Suppose that $\langle U \rangle_{d+1} = \mathcal{C}$ and let W be an object in $\mathcal{D}$. Then W is isomorphic to a direct summand of $\mathcal{F}(\mathcal{G}(W))$. Since $\mathcal{G}(W)$ belongs to $\langle U \rangle_{d+1}$ it follows that $\mathcal{F}(\mathcal{G}(W))$, and hence W, belongs to $\langle \mathcal{F}(U) \rangle_{d+1}$, as required. □

Corollary 3.7. *Let A, B be separably equivalent symmetric algebras over a field k. Then* $\dim(\mathrm{stmod}(A)) = \dim(\mathrm{stmod}(B))$ *and* $\dim(D^b(\mathrm{mod}(A)) = \dim(D^b(\mathrm{mod}(B))$.

Proof. Let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module such that A is isomorphic to a direct summand of $M \otimes_B M^*$ and B is isomorphic to a direct summand of $M^* \otimes_A M$. The functors $M \otimes_B -$ and $M^* \otimes_A -$ between $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ are exact and preserve projectives, hence induce exact functors of triangulated categories between $\mathrm{stmod}(A)$ and $\mathrm{stmod}(B)$ satisfying the assumptions of 3.6. Similarly for the bounded derived categories. Thus 3.7 is a special case of 3.6. □

4. Separability and finite generation of Hochschild cohomology

The purpose of this section is to show that finite generation of Hochschild cohomology carries through separable equivalences. Let k be a commutative ring. If H^* is a graded k-algebra, we denote by $Z(H^*)$ its center in the graded sense; that is, the degree n component of $Z(H^*)$ consists of all $a \in H^n$ satisfying $ab = (-1)^{nm} ba$ for all $m \geq 0$ and all $b \in H^m$. Thus $Z(H^*)$ is graded-commutative, hence has a commutative quotient $Z(H^*)/I$ modulo an ideal I generated by nilpotent elements. If $Z(H^*)$ is left or right Noetherian then I is finitely generated, hence nilpotent, and the Krull dimension of $Z(H^*)$ is, by definition, that of $Z(H^*)/I$ or, equivalently, if $1 \neq -1$ in k, the Krull dimension of $Z(H^*)$ is defined as that of the (necessarily commutative) even part of $Z(H^*)$.

Theorem 4.1. *Let A, B be separably equivalent symmetric k-algebras. Then* $\mathrm{Ext}_{A \otimes_k A^0}(A, U)$ *is Noetherian as an $HH^*(A)$-module for any finitely generated A-A-bimodule U if and only if* $\mathrm{Ext}_{B \otimes_k B^0}(B, V)$ *is Noetherian as an $HH^*(B)$-module for any finitely generated B-B-bimodule V. In that case, the Krull dimensions of $HH^*(A)$ and of $HH^*(B)$ are equal.*

Remark 4.2. Let A be a finite-dimensional algebra over an algebraically closed field k. The property that $HH^*(A, U)$ is Noetherian as an $HH^*(A)$-module for any finitely generated $A \otimes_k A^0$-module U is by [10, 2.4], equivalent to the property that $HH^*(A)$ is Noetherian and that $\mathrm{Ext}^*_A(V, W)$ is Noetherian as an $HH^*(A)$-module for all finitely generated A-modules V, W. By [10, Theorem 2.5] this property forces A to be Gorenstein (that is, of finite injective dimension as a left and right A-module).

The proof of 4.1 uses the following formal observations:

Lemma 4.3. *Let A be a k-algebra, let U, V be A-modules and let U' be a nonzero direct summand of U. If* $\mathrm{Ext}^*_A(U, V)$ *is Noetherian as a module over the graded k-algebra* $\mathrm{Ext}^*_A(U, U)$ *then* $\mathrm{Ext}^*_A(U', V)$ *is Noetherian as a module over the graded k-algebra* $\mathrm{Ext}^*_A(U', U')$.

Proof. Let $e \in \mathrm{End}_A(U) = \mathrm{Ext}^0_A(U, U)$ be an idempotent corresponding to a projection of U onto U'. Then $\mathrm{Ext}^*_A(U', U') \cong e \cdot \mathrm{Ext}^*_A(U, U) \cdot e$ and $\mathrm{Ext}^*_A(U', W) \cong \mathrm{Ext}^*_A(U, V) \cdot e$. The result follows from standard properties of Noetherian modules and rings. □

Given two symmetric k-algebras A, B and an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module, the functor $M \otimes_B -$ has as left and right adjoint the functor $M^* \otimes_B -$. Applying this to the algebras $A \otimes_k A^0$ and $B \otimes_k B^0$ yields immediately the following statement:

Lemma 4.4. *Let A, B be symmetric k-algebras. Let M be an A-B-bimodule which is finitely generated as a left A-module and as a right B-module. The functor $M^* \otimes_A - \otimes_A M$ from $\mathrm{Mod}(A \otimes_k A^0)$ to $\mathrm{Mod}(B \otimes_k B^0)$ is left adjoint to the functor $M \otimes_B - \otimes_B M^*$.*

Lemma 4.5. *Let A, B be symmetric k-algebras. Let M be an A-B-bimodule which is finitely generated as a left A-module and as a right B-module. Let V be a B-B-bimodule. Consider $\mathrm{Ext}^*_{B\otimes_k B^0}(M^* \otimes_A M, V)$ as a right $HH^*(A)$-module by resticting its right $\mathrm{Ext}_{B\otimes_k B^0}(M^* \otimes_A M, M^* \otimes_A M)$-module structure via the algebra homomorphism $HH^*(A) \to \mathrm{Ext}_{B\otimes_k B^0}(M^* \otimes_A M, M^* \otimes_A M)$ induced by the functor $M^* \otimes_A - \otimes_A M$. The canonical adjunction isomorphism*

$$\mathrm{Ext}^*_{B\otimes_k B^0}(M^* \otimes_A M, V) \cong \mathrm{Ext}^*_{A\otimes_k A^0}(A, M \otimes_B V \otimes_B M^*)$$

is an isomorphism of right $HH^*(A)$-modules.

Proof. The adjunction in 4.4 extends to an adjunction between the bounded derived categories $D^b(A \otimes_k A^0)$ and $D^b(B \otimes_k B^0)$ of finitely generated $A \otimes_k A^0$-modules and $B \otimes_k B^0$-modules, respectively. For any integer $n \geq 0$, the elements in $HH^n(A)$ are morphisms $A \to A[n]$ in $D^b(A \otimes_k A^0)$. The naturality of the adjunction isomorphism in the first argument yields the compatibility with the $HH^*(A)$-module structure as stated. □

Proof of 4.1. By 3.2, there is an A-B-bimodule M which is finitely generated projective as a left A-module and as a right B-module such that A is isomorphic to a direct summand of $M \otimes_B M^*$ and such that B is isomorphic to a direct summand of $M^* \otimes_A M$, as bimodules. Suppose that $\mathrm{Ext}_{A\otimes_k A^0}(A, U)$ is Noetherian as a right $HH^*(A)$-module for any finitely generated A-A-bimodule U. Let V be a finitely generated B-B-bimodule. It follows from 4.5 that $\mathrm{Ext}_{B\otimes_k B^0}(M^* \otimes_A M, V)$ is Noetherian as a right $HH^*(A)$-module. Thus $\mathrm{Ext}_{B\otimes_k B^0}(M^* \otimes_A M, V)$ is Noetherian as a right $\mathrm{Ext}_{B\otimes_k B^0}(M^* \otimes_A M, M^* \otimes_A M)$-module. Since B is isomorphic to a direct summand of $M^* \otimes_A M$, it follows from 4.3 that $\mathrm{Ext}_{B\otimes_k B^0}(B, V)$ is Noetherian as a right $HH^*(B)$-module. Exchanging the roles of A and B shows the equivalence in the statement. Suppose now that the two equivalent statements hold. In order to prove the equality of the Krull dimensions we consider the adjunction isomorphisms $\mathrm{Ext}^*_{A\otimes_k A^0}(A, M \otimes_B M^*) \cong \mathrm{Ext}^*_{A\otimes_k B^0}(M, M) \cong \mathrm{Ext}^*_{B\otimes_k B^0}(B, M^* \otimes_A M)$ (cf. [21]). The functors $- \otimes_A M$ and $M \otimes_B -$ induce algebra homomorphisms from $HH^*(A)$ and $HH^*(B)$ to $\mathrm{Ext}^*_{A\otimes_k B^0}(M, M)$. These homomorphisms are injective since A is a summand of $M \otimes_B M^*$ and B is a summand of $M^* \otimes_A M$. By a result of Snashall and Solberg [26, Theorem 1.1], the images of these algebra homomorphisms are contained in the center $Z(\mathrm{Ext}^*_{A\otimes_k B^0}(M, M))$ as graded algebra, and by the assumptions, this center is finitely generated as a module over both $HH^*(A)$ and $HH^*(B)$. Thus the Krull dimensions of $HH^*(A)$ and $HH^*(B)$ are both equal to that of $Z(\mathrm{Ext}^*_{A\otimes_k B^0}(M, M))$. □

If G is a finite group and k a Noetherian ring it is well-known (as a consequence of a theorem of Evens and Venkov) that $\mathrm{Ext}_{k(G\times G)}(kG, U)$ is Noetherian as an $HH^*(kG)$-module, and hence 4.1 has the following immediate consequence:

Corollary 4.6. *Let G be a finite group, k a commutative Noetherian ring and A a symmetric k-algebra. If A and kG and separably equivalent then $\mathrm{Ext}^*_{A\otimes A^0}(A, U)$ is Noetherian as a right*

$HH^*(A)$-module for any finitely generated A-A-bimodule U; in particular, $HH^*(A)$ is finitely generated as a k-algebra.

In order to check the equivalent properties in 4.1 it suffices to verify them for simple modules:

Proposition 4.7. *Let A be a finite-dimensional algebra over a field k and U a finitely generated A-module. The following are equivalent:*

(i) For any finitely generated A-module V the $\mathrm{Ext}_A^*(U,U)$-module $\mathrm{Ext}_A^*(U,V)$ is Noetherian.

(ii) For any simple A-module S the $\mathrm{Ext}_A^*(U,U)$-module $\mathrm{Ext}_A^*(U,S)$ is Noetherian.

Proof. Suppose that (ii) holds. We show (i) by induction over the composition length of V. If V is simple there is nothing to prove. Otherwise there is a short exact sequence of A-modules of the form

$$0 \longrightarrow W \longrightarrow V \longrightarrow S \longrightarrow 0$$

for some simple A-module S. This induces a long exact sequence of the form

$$\cdots \longrightarrow \mathrm{Ext}_A^n(U,W) \xrightarrow{\alpha^n} \mathrm{Ext}_A^n(U,V) \xrightarrow{\beta^n} \mathrm{Ext}_A^n(U,S) \xrightarrow{\delta^n} \mathrm{Ext}_A^{n+1}(U,W) \longrightarrow \cdots$$

The direct sum $\alpha = (\oplus_{n\geq 0}\, \alpha^n)$ is a homomorphism of $\mathrm{Ext}_A^*(U,U)$-modules from $\mathrm{Ext}_A^*(U,W)$ to $\mathrm{Ext}_A^*(U,V)$, and similarly for the direct sum $\beta = \oplus_{n\geq 0}\beta^n$. By the assumptions and induction, $\mathrm{Ext}_A^*(U,S)$ and $\mathrm{Ext}_A^*(U,W)$ are Noetherian as $\mathrm{Ext}_A^*(U,U)$-modules. Thus $\mathrm{Ext}_A^*(U,V)$ is filtered by a submodule of $\mathrm{Ext}_A^*(U,S)$ and a quotient of $\mathrm{Ext}_A^*(U,W)$, both of which are again Noetherian, and hence so is $\mathrm{Ext}_A^*(U,V)$. This shows that (ii) implies (i); the converse is trivial. □

In what follows, graded modules and algebras are graded in non-negative degrees. If H^*, K^* are graded k-modules, we consider $H^* \otimes_k K^*$ as a graded k-module with degree n component $\oplus_{i+j=n}\, H^i \otimes_k K^j$. If H^*, K^* are graded k-algebras, we consider $H^* \otimes_k K^*$ as graded k-algebra with the multiplication $(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)} ac \otimes bd$, for homogeneous elements $a, c \in H^*$ and $b, d \in K^*$. (This sign convention implies that the tensor product of two graded commutative k-algebras is again graded commutative.) A graded commutative algebra H^* over a field k with finite-dimensional degree zero component H^0 is finitely generated as a k-algebra if and only if it is left and right Noetherian. Thus the tensor product of two graded commutative (left or right) Noetherian algebras H^*, K^* over a field with finite-dimensional degree zero components is again (left or right) Noetherian, and the Krull dimension of $H^* \otimes_k K^*$ is the sum of the Krull dimensions of H^* and K^*. Note that the Krull dimension of the direct product $H^* \times K^*$ is the maximum of the Krull dimensions of H^* and K^*. If A, B are two finite-dimensional algebras over a field k it is well-known that, with the above sign convention, we have $HH^*(A \otimes_k B) \cong HH^*(A) \otimes_k HH^*(B)$ and hence, if the Krull dimensions of $HH^*(A)$, $HH^*(B)$ are finite then the Krull dimension of $HH^*(A \otimes_k B)$ is finite and equal to the sum of the Krull dimensions of $HH^*(A)$ and $HH^*(B)$; see for instance the proof of [4, Proposition 7.4]. Similarly, $HH^*(A \times B) \cong HH^*(A) \times HH^*(B)$, and hence the Krull dimension of $HH^*(A \times B)$ is the maximum of the Krull dimensions of $HH^*(A)$, $HH^*(B)$ if these are finite. Thus finite generation of Hochschild cohomology passes on to tensor products and direct products. We will need a slightly more precise version of this fact for modules over Hochschild cohomology algebras.

Proposition 4.8. *Let A, B be finite-dimensional algebras over a field k having separable semi-simple quotients. Set $C = A \otimes_k B$. Suppose that $\mathrm{Ext}^*_{A\otimes_k A^0}(A,U)$ is Noetherian as an $HH^*(A)$-module for any finitely generated A-A-bimodule U and that $\mathrm{Ext}^*_{B\otimes_k B^0}(B,V)$ is Noetherian as* *an $HH^*(B)$-module for any finitely generated B-B-bimodule V. Then $\mathrm{Ext}^*_{C\otimes_k C^0}(C,W)$ is Noetherian as an $HH^*(C)$-module for any finitely generated C-C-bimodule W.*

Proof. By 4.7 it suffices to show this if W is a simple $C\otimes_k C^0$-module. Since A, B have separable semi-simple quotients, every simple $C\otimes_k C^0$-module is of the form $S\otimes_k T$ for a simple $A\otimes_k A^0$-module S and a simple $B\otimes_k B^0$-module T. The appropriate versions of Künneth's theorem imply that

$$\mathrm{Ext}^*_{C\otimes_k C^0}(C, S\otimes_k T) \cong \mathrm{Ext}^*_{A\otimes_k A^0}(A,S)\otimes_k \mathrm{Ext}^*_{B\otimes_k B^0}(B,T)$$

and through the isomorphism $HH^*(C)\cong HH^*(A)\otimes_k HH^*(B)$ this is an isomorphism of right $HH^*(C)$-modules. Since $HH^*(A)$, $HH^*(B)$ are graded commutative and Noetherian, the same is true for $HH^*(C)$. Thus finitely generated modules over $HH^*(C)$ are Noetherian. Since the tensor product of two finitely generated modules over $HH^*(A)$ and $HH^*(B)$, respectively, is finitely generated as an $HH^*(C)$-module, hence Noetherian, the result follows. □

Given a finite-dimensional self-injective algebra A over a field k and a finitely generated A-A-bimodule U we denote as usual by $\Omega_{A\otimes_k A^0}(U)$ the kernel of a projective cover $P_U\to U$ of U; this is unique up to unique isomorphism in the stable category of $A\otimes_k A^0$-modules. The following observation is well-known; we include a proof for the convenience of the reader.

Proposition 4.9. *Let A be a finite-dimensional self-injective algebra over a field k such that $\Omega^n_{A\otimes_k A^0}(A)\cong A$ for some positive integer n. Then $\mathrm{Ext}^*_{A\otimes_k A^0}(A,U)$ is Noetherian as an $HH^*(A)$-module for any finitely generated A-A-bimodule U; in particular, $HH^*(A)$ is finitely generated.*

Proof. A bimodule isomorphism $\Omega^n_{A\otimes_k A^0}(A)\cong A$ represents an element in $HH^n(A)$. Thus $HH^*(A)$ is generated, as a k-algebra, by a k-basis of $\oplus_{i=0}^n HH^i(A)$, hence $HH^*(A)$ is Noetherian (as $HH^*(A)$ is graded commutative). Similarly, $\mathrm{Ext}^*_{A\otimes_k A^0}(A,U)$ is generated, as an $HH^*(A)$-module, by a k-basis of the finite-dimensional space $\oplus_{i=0}^n \mathrm{Ext}^i_{A\otimes_k A^0}(A,U)$, hence the $HH^*(A)$-module $\mathrm{Ext}^*_{A\otimes_k A^0}(A,U)$ is Noetherian. □

Remark 4.10. A finite-dimensional algebra A over a field k satisfying $\Omega^n_{A\otimes_k A^0}(A)\cong A$ for some positive integer n is automatically self-injective, by a result of Butler (see [17, 1.5] for a more general result). Self-injective Nakayama algebras have this property by [11, §4.2, Lemma], and hence so do Brauer tree algebras because a Brauer tree algebra is derived equivalent to a symmetric Nakayama algebra (cf. [23]).

5. Proof of Theorem 1.1 and Theorem 1.2

Let k be a field of characteristic zero and q a non-zero element in k. Let $\mathcal{H}=\mathcal{H}(W,q)$ be a Hecke algebra over k with parameter q of the finite Coxeter group (W,S). That is, $\mathcal{H}$ has a k-basis $\{T_w \mid w\in W\}$ indexed by the elements of W such that T_1 is the unit element of $\mathcal{H}$, with multiplication given by $T_wT_{w'}=T_{ww'}$ if $w,w'\in W$ such that the length of ww' is the sum of the length of w, w', and the quadratic relations $(T_s)^2=qT_1+(1-q)T_s$ for $s\in S$. If S is a disjoint union of two non-empty subsets S_1, S_2 such that any element in S_1 commutes with any element in S_2, then the subgroups W_i of W generated by S_i, for $i=1,2$, can be identified with the Coxeter groups (W_i,S_i), and $W=W_1\times W_2$. If we denote by $\mathcal{H}_i$ the corresponding Hecke algebra of W_i, for $i=1,2$, then $\mathcal{H}\cong\mathcal{H}_1\otimes_k\mathcal{H}_2$. Thus, by 4.8, we may assume that (W,S) is irreducible, hence of type A_{n-1} or B_n or D_n ($n\geq 4$). Assume first that $\mathcal{H}=\mathcal{H}(S_n,q)$ is of type A_{n-1}. If $q=1$ or if q has infinite order in $k^\times$ then $\mathcal{H}$ is semi-simple (by [8, 4.3]), hence separable as $\mathrm{char}(k) = 0$. Thus, in that case, $\mathcal{H} \otimes_k \mathcal{H}^0$ is again semi-simple, and hence $\mathrm{Ext}^*_{\mathcal{H}\otimes_k\mathcal{H}^0}(\mathcal{H}, M) = \mathrm{Ext}^0_{\mathcal{H}\otimes_k\mathcal{H}^0}(\mathcal{H}, M)$ is finite-dimensional for any finitely generated $\mathcal{H} \otimes_k \mathcal{H}^0$-module, so 1.1 and 1.2 hold trivially. Assume that q is a primitive ℓ-th root of unity for some integer $\ell \geq 2$. It is well-known (cf. [15, Proposition 8.1.1]) that $\mathcal{H}$ is symmetric, with a canonical symmetrising form, such that the restriction of this form to the Hecke algebra $\mathcal{H}'$ of a parabolic subgroup of S_n is the canonical symmetrising form of $\mathcal{H}'$; in other words, parabolic subalgebras in the context of Hecke algebras (cf. [15, 4.4.7]) are indeed parabolic subalgebras in the sense of [3, 5.1]. As in [1], denote by $\mathcal{B}$ a maximal ℓ-parabolic subalgebra $\mathcal{H}(\lambda, q)$ of $\mathcal{H}$, where λ is the partition $(\ell^m, 1^a)$, with $n = \ell m + a$ and $0 \leq a \leq \ell - 1$. An immediate consequence of a result of Du [9, Theorem 2.7] in conjunction with 2.1 is the following observation:

Proposition 5.1. *With the notation above, the symmetric algebras $\mathcal{H}$ and $\mathcal{B}$ are separably equivalent. More precisely, $\mathcal{B}$ is a direct summand of $\mathcal{H}$ as a $\mathcal{B}$-$\mathcal{B}$-bimodule, and $\mathcal{H}$ is isomorphic to a direct summand of $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{H}$ as an $\mathcal{H}$-$\mathcal{H}$-bimodule.*

Proof. The fact that $\mathcal{B}$ is a direct summand of $\mathcal{H}$ as a $\mathcal{B}$-$\mathcal{B}$-bimodule is a general fact of parabolic subalgebas (as an immediate consequence of the distinguished double coset representatives [15, 2.1.7], [19, (2.28)]). Du's result [9, Theorem 2.7] says that $\mathrm{tr}^{\mathcal{H}}_{\mathcal{B}}(1)$ is invertible in $Z(\mathcal{H})$. It follows thus from 2.1 that $\mathcal{H}$ is isomorphic to a direct summand of $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{H}$, completing the proof. □

It follows therefore from 4.1 that in order to prove 1.1 and 1.2 for the Hecke algebra $\mathcal{H}$ of type A_{n-1} it suffices to prove the conclusion for the maximal ℓ-parabolic subalgebra $\mathcal{B}$ instead. The algebra $\mathcal{B}$ in turn is the tensor product of m copies of the Hecke algebra $\mathcal{H}(S_\ell, q)$, and thus, by 4.8, it suffices to prove 1.1 and 1.2 for $\mathcal{H}(S_\ell, q)$. This is a product of a Brauer tree algebra and a semi-simple algebra (cf. [14], [27]), and so both results hold as the Hochschild cohomology of a Brauer tree algebra is periodic (cf. [11] and 4.9, 4.10). This concludes the proof of 1.1 and 1.2 in the case of type A_{n-1}. For the types B and D we proceed as in [1, §6], playing the problem back to type A. Suppose that $\mathcal{H}$ is of type B_n and that either the order of q is infinite or that its order ℓ is not even. By [7, Theorem 4.17], $\mathcal{H}$ is Morita equivalent to the direct product of tensor products of Hecke algebras of type A of the form

$$\prod_{j=0}^{n} \mathcal{H}(S_j, q) \otimes_k \mathcal{H}(S_{n-j}, q)$$

where S_0 is the trivial group, by convention. As a consequence of 4.8, the theorems 1.1 and 1.2 follow in this case from the fact that they hold in type A. Suppose that $\mathcal{H}$ is of type D_n for some odd integer $n \geq 5$. Then by [22, Theorems 3.6, 3.7] (made explicit in [18]), the algebra $\mathcal{H}$ is Morita equivalent to the algebra

$$\prod_{j=(n+1)/2}^{n} \mathcal{H}(S_j, q) \otimes_k \mathcal{H}(S_{n-j}, q)$$

and so the results in type A imply again both 1.1 and 1.2. Suppose finally that $\mathcal{H}$ is of type D_n for some even integer $n \geq 4$. Then, by the main result in [18], $\mathcal{H}$ is Morita equivalent to the algebra

$$A(n/2) \times \prod_{j=(n+1)/2}^{n} \mathcal{H}(S_j, q) \otimes_k \mathcal{H}(S_{n-j}, q)$$

where $A(n/2)$ is a subalgebra of $\mathcal{H}(S_n, q)$ generated by $\mathcal{H}(S_{n/2}, q) \otimes_k \mathcal{H}(S_{n/2}, q)$ and an invertible element in $\mathcal{H}(S_n, q)$ which exchanges the two factors in this tensor product and whose square is in the center of this subalgebra. In other words, the explicit description of $A(n/2)$ in [18, Remark 2.4] shows that $A(n/2)$ is a crossed product of the form $(\mathcal{H}(S_{n/2}, q) \otimes_k \mathcal{H}(S_{n/2}, q))_\alpha C_2$, where C_2 is a cyclic group of order 2 and α a 2-cocycle of C_2 with values in $Z(\mathcal{H}(S_{n/2}, q) \otimes_k \mathcal{H}(S_{n/2}, q))^\times$. It follows from 3.4 that $A(n/2)$ and $\mathcal{H}(S_{n/2}, q) \otimes_k \mathcal{H}(S_{n/2}, q)$ are separably equivalent. Thus 1.1 and 1.2 follow yet again from the corresponding results in type A. This concludes the proof of both theorems.

6. Further remarks

The purpose of this section is to sketch some arguments which may be used to show the finite generation of $HH^*(A)$ in some situations in which it is not known that $\mathrm{Ext}^*_{A \otimes_k A^0}(A, U)$ is Noetherian for all finitely generated A-A-bimodules U. Let k be a commutative Noetherian ring and let A be a k-algebra. An A-module U is called *relatively k-injective* if every A-homomorphism from U to another A-module V which is split injective as k-homomorphism, is split injective as A-homomorphism. An injective module is relatively k-injective, and if k is a field the converse holds as well. Dually, U is *relatively k-projective* if every A-homomorphism $V \to U$ which is split surjective as k-homomorphism is split surjective as A-homomorphism. If A is symmetric, the classes of relatively k-projective A-modules and relatively k-injective A-modules coincide (and the content of this section can be generalised to the class of not nexessarily symmetric algebras with this property). Slightly generalising earlier notation, we denote now by $\mathrm{stmod}(A)$ the k-stable category of the category $\mathrm{mod}(A)$ of finitely generated A-modules; that is, $\mathrm{stmod}(A)$ has the same objects as $\mathrm{mod}(A)$, and for any two finitely generated A-modules U, V, the homomorphism space in $\mathrm{stmod}(A)$ from U to V is the quotient space $\underline{\mathrm{Hom}}_A(U, V) = \mathrm{Hom}_A(U, V)/\mathrm{Hom}^{pr}_A(U, V)$, where $\mathrm{Hom}^{pr}_A(U, V)$ is the space of all A-homomorphisms from U to V which factor through a relatively k-projective A-module. If A is symmetric then the category $\mathrm{stmod}(A)$ is triangulated, with suspension functor Σ sending a finitely generated A-module U to the cokernel of a k-split embedding $U \to I_U$ of U into a relatively k-injective A-module I_U. Such a module I_U always exists; for instance, one could take $I_U = \mathrm{Hom}_k(A, U)$ with the map $U \to I_U$ sending $u \in U$ to the map $a \mapsto au$. For two k-algebras A, B we denote by $\mathrm{perf}(A, B)$ the category of A-B-bimodules which are finitely generated projective as left A-modules and as right B-modules. If A and B are finitely generated projective as k-modules then the category $\mathrm{perf}(A, B)$ contains the finitely generated projective A-B-bimodules. If A, B are symmetric k-algebras then all relatively k-projective modules in $\mathrm{perf}(A, B)$ are actually projective A-B-bimodules and the category $\mathrm{stperf}(A, B)$ is a thick subcategory of the k-stable category $\mathrm{stmod}(A \otimes_k B^0)$; in particular, $\mathrm{stperf}(A, B)$ is again triangulated, with suspension functor, denoted abusively again by Σ, sending a bimodule M in $\mathrm{stperf}(A, B)$ to the cokernel of a relatively k-injective envelope $M \to I_M$ of M in the category of A-B-bimodules.

Proposition 6.1. *Let A, B be symmetric algebras over a commutative Noetherian ring k and let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. Suppose that B is isomorphic to a direct summand of $M^* \otimes_A M$ as a B-B-bimodule and that $M \otimes_B M^*$ belongs to the thick subcategory of the k-stable category of A-A-bimodules generated by A. If $HH^*(A)$ is finitely generated as a k-algebra then so is $HH^*(B)$.*

The proof of 6.1 uses the following two lemmas.

Lemma 6.2. *Let A be a symmetric algebra over a commutative Noetherian ring k and U, V finitely generated A-modules such that $\mathrm{Ext}_A^*(U,V)$ is Noetherian as an $\mathrm{Ext}_A^*(U,U)$-module. Then $\mathrm{Ext}_A^*(U,W)$ is Noetherian as an $\mathrm{Ext}_A^*(U,U)$-module for any A-module W belonging to the thick subcategory $\langle V\rangle$ of the stable module category $\mathrm{stmod}(A)$ generated by V.*

Proof. Clearly $\mathrm{Ext}_A^*(U,V')$ is Noetherian as an $\mathrm{Ext}_A^*(U,U)$-module for any direct summand V' of V. We observe next that $\mathrm{Ext}_A^*(U,\Sigma^n(V))$ is Noetherian as an $\mathrm{Ext}_A^*(U,U)$-module, for any integer n. Indeed, we have $\mathrm{Ext}_A^n(U,V) = \underline{\mathrm{Hom}}_A(U,\Sigma^n(V))$ if n is a positive integer and $\mathrm{Ext}_A^n(U,V) = \{0\}$ if n is a negative integer. Therefore, if i and n are integers such that both i and $i-n$ are positive then $\mathrm{Ext}_A^i(U,V) \cong \mathrm{Ext}_A^{i-n}(U,\Sigma^n(V))$. Thus the kernel and cokernel of the obvious graded map of degree $-n$ from $\mathrm{Ext}_A^*(U,V)$ to $\mathrm{Ext}_A^*(U\Sigma^n(U))$ extending the above isomorphisms are finitely generated as k-modules, and hence $\mathrm{Ext}_A^*(U,\Sigma^n(V))$ is Noetherian. Let $X \to Y \to Z \to \Sigma(X)$ be an exact triangle in $\underline{\mathrm{mod}}(A)$. Suppose that $\mathrm{Ext}_A^*(U,X)$ and $\mathrm{Ext}_A^*(U,Y)$ are Noetherian as $\mathrm{Ext}_A^*(U,U)$-modules. The maps in the exact triangle induce a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_A^n(U,X) \xrightarrow{\alpha^n} \mathrm{Ext}_A^n(U,Y) \xrightarrow{\beta^n} \mathrm{Ext}_A^n(U,Z) \xrightarrow{\delta^n} \mathrm{Ext}_A^{n+1}(U,X) \longrightarrow \cdots$$

The direct sum $\beta = (\oplus_{n\geq 0}\ \beta^n)$ is a homomorphism of $\mathrm{Ext}_A^*(U,U)$-modules from $\mathrm{Ext}_A^*(U,Y)$ to $\mathrm{Ext}_A^*(U,Z)$, and similarly for the direct sum $\delta = \oplus_{n\geq 0}\delta^n$. Thus, by the assumptions, both the image and the kernel of δ are Noetherian $\mathrm{Ext}_A^*(U,U)$-modules, hence so is $\mathrm{Ext}_A^*(U,Z)$. The result follows. □

Lemma 6.3. *Let A be a symmetric algebra over a commutative Noetherian ring k. Denote by $\mathcal{T}$ the thick subcategory of $\mathrm{stperf}(A,A)$ generated by the A-A-bimodule A. If M, N belong to $\mathcal{T}$ then so does $M \otimes_A N$.*

Proof. Every module in the subcategory $\mathcal{T}$ of $\mathrm{stperf}(A,A)$ is obtained from applying a finite number of times the shift functor Σ with respect to $A \otimes_k A^0$, taking direct summands, direct sums, and completing triangles. Thus we may filter $\mathcal{T}$ by full additive subcategories $\mathcal{T}_m$, $m \geq 0$, defined inductively as follows: we denote by $\mathcal{T}_0$ the full additive subcategory of $\mathcal{T}$ consisting of all finite direct sums of summands of the A-A-bimodules $\Sigma^n(A)$, where $n \in \mathbb{Z}$. If $\mathcal{T}_m$ is defined for some $m \geq 0$, we define $\mathcal{T}_{m+1}$ as the full additive subcategory of finite direct sums of summands of bimodules W for which there exists an exact triangle $M \to N \to W \to \Sigma(M)$ such that M, N belong to $\mathcal{T}_m$. Clearly every module in $\mathcal{T}$ belongs to $\mathcal{T}_m$ for some $m \geq 0$. What we will show is that if M belongs to $\mathcal{T}_t$ and N belongs to $\mathcal{T}_s$ then $M \otimes_A N$ belongs to $\mathcal{T}_{t+s}$. We do this by induction over $t+s$. For $t+s=0$ this is clear because $\mathcal{T}_0$ is closed under taking tensor products over A; indeed, $\Sigma^m(A) \otimes_A \Sigma^n(A) \cong \Sigma^{m+n}(A)$ in $\mathrm{stperf}(A,A)$ for any two integers m, n. Suppose $t+s \geq 0$. Let again $M \to N \to W \to \Sigma(M)$ be an exact triangle in $\mathrm{stperf}(A,A)$ such that M, N belong to $\mathcal{T}_t$. Let X be a bimodule in $\mathcal{T}_s$. Then the triangle $M \otimes_A X \to N \otimes_A X \to W \otimes_A X \to \Sigma(M) \otimes_A X$ is exact, and by the assumptions, $M \otimes_A X$, $N \otimes_A X$ are in $\mathcal{T}_{t+s}$. Thus $W \otimes_A X$ is in $\mathcal{T}_{t+s+1}$. This holds then clearly for any W in $\mathcal{T}_{t+1}$ and any X in $\mathcal{T}_s$. By reversing the roles of t and s one concludes the proof. □

Proof of 6.1. By the assumptions, $M \otimes_B M^*$ is in the thick subcategory of $\mathrm{stperf}(A,A)$ generated by A. Thus, by 6.3, $M \otimes_B M^* \otimes_A M \otimes_B M^*$ is in the thick subcategory of $\mathrm{stperf}(A,A)$. It follows from 6.2 that $\mathrm{Ext}_{A\otimes_k A^0}(A, M \otimes_B M^* \otimes_A M \otimes_B M^*)$ is Noetherian as an $HH^*(A)$-module. Therefore, by 4.5, $\mathrm{Ext}_{B\otimes B^0}(M^* \otimes_A M, M^* \otimes_A M)$ is Noetherian as an $HH^*(A)$-module, hence

Noetherian itself because any left or right ideal in this algebra is, in particular, an $HH^*(A)$-submodule. Since B is isomorphic to a direct summand of $M^* \otimes_A M$ it follows from 4.3 that $HH^*(B)$ is Noetherian, hence finitely generated as a k-algebra. $\square$

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