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Constraints on Four Dimensional Effective Field Theories From String and F-Theory

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Abstract: This thesis is a study of string theory compactifications to four dimensions and the constraints the Effective Field Theories must exhibit, exploring both the closed and open sectors. In the former case, we focus on axion monodromy scenarios and the impact the backreaction of the energy density induced by the vev of an axion has on its field excursions. For all the cases studied, we find that the backreaction is small up to a critical value, and the proper field distance is flux independent and at most logarithmic in the axion vev.

We then move to the open sector, where we use the framework of F-theory. We first explore the relation between the spectra arising from F-theory GUTs and those coming from a decomposition of the adjoint of E_8 to $SU(5) \times U(1)^n$. We find that extending the latter spectrum with new $SU(5)$-singlet fields, and classifying all possible ways of breaking the Abelian factors, all the spectra coming from smooth elliptic fibration constructed in the literature fit in our classification. We then explore generic properties of the spectra arising when breaking $SU(5)$ to the Standard Model gauge group while retaining some anomaly properties. We finish by a study of F-theory compactifications on a singular elliptic fibration via Matrix Factorisation, and find the charged spectrum of two non-Abelian examples.

Zusammenfassung: Diese Arbeit befasst sich mit Stringtheorie Kompaktifizierungen und ihren Bedingungen an effektive Feldtheorien in vier Dimensionen, wobei sowohl der Sektor geschlossener als auch offener Strings berücksichtigt wird. Im Fall geschlossener Strings liegt unser Fokus auf Axion Monodromie Szenarien und dem Einfluss der Rückkopplung der durch den Axion Vakuumerwartungswert induzierten Energiedichte auf die entsprechenden Feldwerte. In allen untersuchten Szenarien finden wir, dass die Rückkopplung bis zu einem kritischen Wert klein ist, wobei die Feldreichweite unabhängig vom Fluss ist und höchstens logarithmisch vom Axion Vakuumerwartungswert abhängt.

Den Sektor offener Strings betrachten wir im Rahmen von F-Theorie. Zunächst untersuchen wir den Zusammenhang zwischen den Spektren von F-Theorie GUTs und den entsprechenden Spektren aus Zerlegungen der adjungierten Darstellung von E_8 nach $SU(5) \times U(1)^n$. Wir finden heraus, dass durch Erweiterung letzteren Spektrums durch zusätzliche $SU(5)$ Singlet Felder und durch eine Klassifizierung aller Möglichkeiten, die zusätzlichen Abelschen Faktoren zu brechen, alle bereits in der Literatur konstruierten Spektren von glatten elliptischen Faserungen in unsere Klassifizierung fallen. Weiterhin untersuchen wir generische Eigenschaften dieser Spektren nach Brechung von $SU(5)$ auf die Eichgruppe des Standardmodells, während wir einige Anomalie Eigenschaften beibehalten. Abschliessend befassen wir uns mit F-Theorie Kompaktifizierungen auf einer singulären elliptischen Faserung mittels Matrix Faktorisierung und berechnen das geladene Spektrum für zwei nicht-Abelsche Beispiele.

A la mémoire de l'Abner et du Raymond.

Contents

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Chapter 1

Introduction

In the late 19th century, most of the observed physical phenomena were described to good accuracy by Newtonian Physics. The motion of objects, from solids to stars, were governed by Newton's laws of motion, charged particles by Maxwell's electrodynamics, and the collective behaviour of complex systems could be described by the laws of thermodynamics. This led many members of the scientific community to think that they were close to the Holy Grail of Physics: a complete description of Nature in terms of first principles. In fact, an apocryphal quote often misattributed to William Thomson, 1st Baron Kelvin epitomises this school of thought:

> *There is nothing new to be discovered in physics now. All that remains is more and more precise measurement.*

This way of thinking was in fact quite wrong, as within a few years, new experimental methods led to observations that could not be explained by Newtonian Physics, demanding a new framework to explain them. Two paradigms arose simultaneously to try to understand the discrepancy between the theoretical and experimental worlds. On the one hand, Quantum Mechanics began explaining phenomena occurring at short distances, while Special Relativity arose as a description of those having a high velocity.

These two new theories were not completely *ex nihilo*, in the sense that they did not replace Newtonian Physics, but rather built upon it to generalise it. Indeed, when quantum fluctuations are small compared to the size of the observed object, it can be described by the rules of Newtonian Physics. Similarly, if the velocity of an object is slow with respect to the speed of light, the Newtonian framework is sufficient to characterise it to good accuracy.

These two new theories were soon combined into Quantum Field Theory, describing objects of high velocity exhibiting a quantum behaviour. This framework has been one of the most celebrated theoretical achievements of the past century, and is one of the two cornerstones on which our current understanding of the Universe is built. The other, General Relativity, is a generalisation of Special Relativity and Newton's law of gravitation. All of these theories supersede Newtonian Physics, and are characterised by a fundamental constant. For Quantum Mechanics, it is the reduced Planck constant $\hbar$, while Special and General Relativity are characterised by the speed of light c and Newton's constant G_N.

These three constants have been experimentally measured to a high degree of accuracy [1]

$$\begin{aligned} \hbar &= 1.05 \times 10^{-34}\,\mathrm{Js}, \\ c &= 3.00 \times 10^{8}\,\mathrm{ms}^{-1}, \\ G_N &= 6.67 \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}, \end{aligned} \tag{1.1}$$

and define expansion parameters measuring the deviations from Newtonian Physics. For observable quantities where the zeroth order of the expansion dominates, the Newtonian framework is enough to describe the phenomenon up to small corrections, and can be pictorially represented by Okun's cube, see figure 1.1. Quantum Field Theory corresponds to deviations into both the c and $\hbar$ directions, as it is associated to a unified description of Special Relativity and Quantum Mechanics, and General Relativity deviates from the Newtonian framework in the c and G_N directions.

We have so far not discussed two corners of the cube: one is corresponding to Quantum Gravity, while the other encompasses theories where the deviations from classical mechanics are done in all directions, called Theories of Everything. In the former case, such theories are necessary to understand systems that intrinsically exhibit both quantum and gravitational features, such as black holes, the Big Bang or the Cosmological Microwave Background. Much of the research effort in the past thirty to forty years has been devoted to better understand Quantum Gravity and aspects related to Theories of Everything. Unlike for

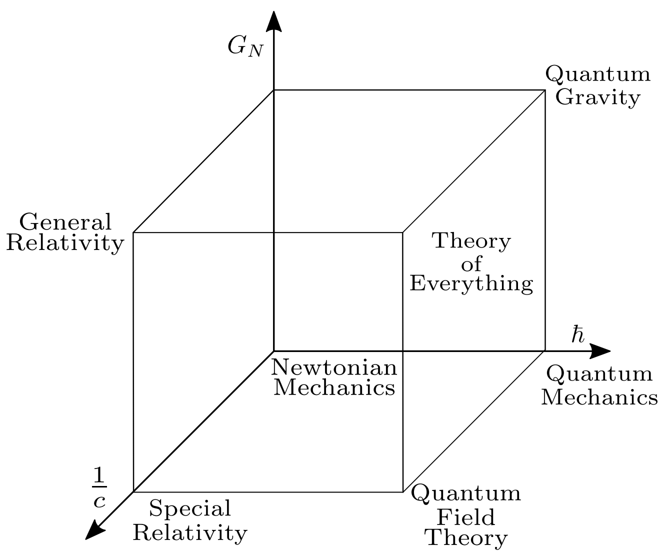


Figure 1.1: Okun's cube. Each direction represents a deformation away from Newtonian Physics.

Quantum Field Theory with Quantum Mechanics and Special Relativity, it is not possible to unify General Relativity and Quantum Field Theory in a simple manner, as physical quantities diverge in a way that cannot be controlled using the standard tools [2]. To this day, the question of how to merge these two frameworks into a mathematically self-consistent theory has yet to be settled. There are however candidates for such a description, of which the most successful is arguably string theory.

String theory was first introduced in the late nineteen-sixties as a formulation of the strong force. While it naturally had a spin-2 field necessary to Quantum Gravity in its spectrum, it also had a number of problems, such as being gauge anomalous and requiring spacetime to have more than four dimensions, and string theory slowly fell out of fashion, to be abandoned in favour of Quantum Chromodynamics in 1973. It was however brought back to life about a decade later during the epoch now known as the first string revolution, when Green and Schwarz discovered an anomaly cancellation mechanism [3], combined to the possibility of obtaining spin one-half states by introducing fermionic strings. Since then, string theory has grown into a subfield of Physics standing on its own, and has had influences in all of theoretical physics.

One of the great successes of string theory has been a deeper understanding of Quantum Field Theories in various dimensions. For instance, string theory is naturally invariant under supersymmetry, a transformation relating bosonic states and their fermionic partners. Supersymmetry is a priori not related to string theory, but has shown to greatly constrain the properties of, for instance, gauge theories, and string theory has been the designated laboratory for their study. As an example, it has been proven by Seiberg and Witten [4, 5] that certain supersymmetric gauge theories are related to each other by dualities, in a way explained very naturally by string theory (see [6] for a review). This discovery started yet another second string revolution, as it was realised that these properties were not only relating gauge theories, but all regimes of string theory in many ways through a web of dualities, and the proposal of a theory superseding string theory, called M-theory, followed.

This crucial observation by Witten [7] has surprisingly enabled many results in mathematics as well. It was shown that superstring theory is only consistent in ten spacetime dimensions, and if it had any connection to Nature, the extra six dimensions must be small and form a compact space to have escaped experimental detection. We will be more precise in chapter 2 on why this is the case, but this compact manifold is generally required to be of Calabi–Yau type, and the extra dimensions have to be compactified, a procedure called Kaluza-Klein reduction, to obtain a four dimensional Quantum Field Theory. It was found that two different manifolds could give the same Quantum Field Theory, in yet another form of duality. Armed with that observation, mathematicians started to investigate the source of this intriguing fact and realised that these pairs, called mirror duals, were obtained by exchanging some topological quantities. String theory provided numerous "experimental data" to mathematicians by constructing physically motivated Calabi–Yau manifolds, or obtaining results using physical arguments alone. For instance, demanding the absence of gauge anomalies prompted a physicist's proof of the Atiyah-Singer index theorem, and enabled a generalisation of the original proof [8].

Interestingly, from Newton's *Principia* to the dawn of Quantum Mechanics, progress in theoretical physics has mainly been in response to the discovery of a new phenomenon. Quantum Mechanics has indeed been developed in the early twentieth century in response to the Ultraviolet Catastrophe, where the equipartition theorem failed to predict the spectral radiance of a black body at short wavelength. Another example is Special Relativity, developed to explain the absence of luminiferous aether after the famous Michelson-Morley experiment. Ironically, the quote above is misattributed to Lord Kelvin's 1900 speech [9] at the Royal Institution of Great Britain. He actually argued quite the opposite, pointing out that those two experiments were two clouds obscuring the skies of Newtonian Physics, and needed more attention from the scientific community.

Since then, theoretical physics has however known a faster development than its experimental sibling, and started branching out to a more independent subfield of physics. Theories were developed as generalisations or unifications of others, and emerged as solutions to theoretical problems rather than experimental conundra. For instance, while observations of the perihelion of the planet Mercury contradicting the predictions of Newtonian gravity were a well-known result in Einstein's time, the infancy of General Relativity arose from a need to generalise Special Relativity and provide a unified description of gravity and the concept of spacetime, rather than to explain these experimental results. Even the Standard Model, the theory explaining our current understanding of Nature at small distances, started as an attempt to better classify baryons and mesons, and unify three of the fundamental forces into a common description, and it was not until 2012 that its missing keystone, the Higgs boson, was discovered [10, 11]. The source of the discrepancy between the rapidly evolving theoretical realm and the comparatively slower experimental world stems from the evermore complexity of experiments. While Classical Physics was developed in small collaborations, if not by people working alone in their study, experiments nowadays necessitate the participation of hundreds or thousands of scientists and engineers.

Together, the Standard Model and the ΛCDM model of cosmology are the state-of-the-art description of our understanding of the Universe. They however both have shortcomings: neither elucidate the nature of dark matter or dark energy, explain why the Universe looks the same in all directions and expands (horizon and flatness problems), or hint to why the electroweak scale is so small compared to the Planck scale (the hierarchy problem). There are many elegant models explaining diverse phenomena, but there is not yet an experiment to confirm or infirm their validity, and no consensus on a model explaining all of them simultaneously. In the case of string theory, the matter is even worse, as the characteristic energies involved are of orders of magnitude far beyond the current experimental capabilities.

Confronted with that thought, it seems hopeless to verify the validity of string theory in our lifetime. However, there is a glimmer of hope coming from effective theories. This concept arose from Wilson's work on the Renormalisation Group [12–14], but is in fact more general. When one first encounters Quantum Field Theory, one expects to describe phenomena for all energy scales. It however turns out that physical observables are functions of that energy scale. In some cases, called non-renormalisable theories, the physical quantities are well-behaved in the infrared regime (IR) corresponding to low energies, but start to diverge and run out of control at high energies, called the ultraviolet regime (UV). On the other hand, another class of theories, called renormalisable, are well-behaved and do not exhibit singularities at higher energies. In the early days of Quantum Field Theory, the condition that a theory should be renormalisable was sacrosanct and any theory that was not was considered non-physical. Following Wilson, it was however soon realised that this point of view is incorrect and that the singularities are all but unphysical: they signal a breakdown of the description at high energy, as new degrees of freedom become relevant. A non-renormalisable theory is then simply an *effective* description valid in a given regime, where the characteristic energies are below a certain breakdown scale. Knowledge about the effective theory can however be powerful when used in conjunction to experimental data. A celebrated example is the charm quark, which had been conjectured to exist as an explanation of Kaon oscillations [15], for which the effective description was shown to break down at a scale of at most $10\,\mathrm{GeV}$, predicting that the mass of this new particle was of at most that magnitude [16]. A few months later, collaborations at the Stanford Linear Accelerator Center and the Brookhaven National Laboratory discovered the charm quark, with a mass $m_c = 1.29\,\mathrm{GeV}$ [17, 18].

This point of view is very powerful: It enables one to focus on a particular regime adapted to a given energy scale of a more complicated theory without worrying about all the details of its UV features that can be highly non-trivial. Such theories are typically simpler, as they have fewer degrees of freedom, and exhibit many more symmetries than their UV parents.

An analogy is that of cattle grazing in a field, looked upon by distant hikers. From their point of view, these cows appear as spherically symmetric points at first, but as they come closer (corresponding to an increase in the energy scale), they will start to observe horns, tails, and much more complicated patterns, corresponding in this analogy to the details of the UV theory and the appearance of new degrees of freedom. From this effective perspective, a very interesting question arises: the proverbial cows themselves are a complicated systems made of cells, who in turn are made of atoms, etc...Where does it stop? In this thesis, we shall take the easy way out and invoke Occam's razor: String theory is a candidate for a Theory of Everything, and will therefore be the ultimate destination of the hiker's voyage.

Indeed, featuring properties such as UV finiteness, the presence of a graviton and a robust mathematical framework, it is by far the most studied and well-established of the candidates. Moreover, it is very easy to engineer Quantum Field Theories with various features from string theory, such as non-perturbative gauge or conformal theories, and it therefore acts as a sort of experimental laboratory for mathematics. This is one reason among many to study string theory even if it is not realised in Nature, as it has already taught us so much about Mathematics and Physics, and cannot be too easily discarded.

The possibilities for compactifying the extra six dimensions required by string theory down to a four dimensional effective theory are enormously vast, and form the so-called landscape of string theory. However, as numerous as they are, they do not encompass all the effective theories, and this has led to the notion of swampland of string theory [19], containing effective theories that cannot be obtained from string theory. For instance, there is some evidence that theories of Quantum Gravity do not allow the presence of global symmetries [20, 21], and field theories exhibiting global symmetries that cannot be gauged in the UV to be consistent lie in this swampland. Taking string theory as a candidate for a Theory of Everything, we can reverse the argument and ask what properties an effective description descending from string theory must display to lie in the landscape.

This is the main motivation for this thesis: We want to perform a study of some of the constraints that a four dimensional effective Quantum Field Theory descending from string theory must exhibit. Our analysis will cover different aspects of string theory, involving two kinds of sector coming from the boundary condition a one dimensional object, the string, can have. They can be either closed or open, and depending on which sector one considers, one will find different phenomena. For instance, closed strings give rise to gravity, while open strings mainly involve gauge theories in the effective description.

The first part of our analysis will be dedicated to the closed sector. The compactification procedure for this sector gives rise to an important number of massless scalars in four dimensions that have to obtain a large mass to have avoided detection already. Some of these scalars are endowed with a shift symmetry, and are called axions. Their shift symmetry is a very powerful tool, as it shields them against perturbative and non-perturbative quantum corrections, and is a key ingredient to probe Quantum Gravity phenomena, such as those present during the infancy of the universe, where quantum fluctuations were as strong as gravity and can be used to describe models of inflation [22]. String theory axions were however initially thought to be poor candidates for inflation, despite having a precious shift symmetry, as their decay constant was generally too small [23], but several mechanisms have since been found to enhance it up to the desired scales, and are are currently under great scrutiny [24–26].

We will focus our attention on quantifying the difficulty to displace axions away from their minimum by studying the gravitational backreaction on their vacuum expectation values in string theoretic setups. More precisely, we will study them in the context of axion monodromy scenarios, where the shift symmetry is broken by turning on fluxes in such a way that they induce a mass term. We will find that the backreaction makes it difficult to travel super-Planckian distances. As the Lyth bound [27] relates the scalar-to-tensor ratio to the displacement of the inflaton away from its minimum, our results puts constraints on the possibility of having large values of the ratio if the inflaton is a stringy axion, which is currently being tested experimentally by several collaborations.

We will then turn to the open sector of string theory. As we have already mentioned, this sector corresponds in the IR regime to gauge theories. The most celebrated example of a gauge theory is the Standard Model, where the gauge group is $SU(3) \times SU(2) \times U(1)$, corresponding to the strong, weak, and hypercharge forces respectively. While we saw that the Standard Model has shortcomings, it also gives indications to what is the first step in its UV completion: a study of the running of the gauge coupling reveals that they are almost intersecting at energies of the order of 10^{15} GeV: if one requires supersymmetry, hinting at a common origin. As the Standard Model spectrum fits into larger groups, such as $SU(5)$, a realistic possibility is that these three forces merge into a single force in the UV. This description in terms of a single gauge group is called a Grand Unified Theory (GUT), and was first theorised by Georgi and Glashow [28].

To study these Grand Unified theories, we will use F-theory, a description of Type IIB string theory in a non-perturbative regime, introduced by Morrison and Vafa in 1996 [29–31], which proved to be a particularly adapted framework to study gauge theories. In this context, the physical data of the effective theory is encoded in the singularity structure of a singular elliptically fibered Calabi–Yau. For instance, the type IIB D7-branes—giving the gauge group in the IR—are associated to codimension one singularities, while higher codimension loci, such as curves and points, give some information about the spectrum and interactions of the lower dimensional theory.

After experiencing a revival in 2008, when novel mechanisms that could be applied to phenomenology [32–34] were discovered, F-theory proved to be a successful tool to study systems usually difficult to analyse, including both phenomenological and formal setups. For instance, a classification of most six dimensional Supersymmetric Conformal Field Theories (SCFTs) has been achieved using F-theory [35]. Progress has also been made towards a better comprehension of Abelian [36–38] and discrete symmetries [39,40], and their relations to the geometry of the elliptic fibration. Using them as a stepping stone, we will explore the relations between $SU(5)$ F-theory GUTs and the exceptional group E_8, in order to classify a class of GUT theories.

We will then pursue our analysis of Grand Unified Theories by exploring the possibilities of breaking the GUT group to that of the Standard model using Hypercharge flux. We will find that if we want to avoid some undesirable features, we will need to introduce new exotic states not lying in the spectrum of the Standard Model. We will then explore a new mechanism to solve the μ-problem and give a large mass to these exotics by using an additional symmetry.

Finally, we will explore a formal aspect of F-theory. The most common way of obtaining the low energy data in F-theory compactifications on a singular space is to first resolve its singularities. However, doing so obscures data of a large portion of some theories. A proposal appropriate to study these theories was recently introduced by Collinucci and Savelli [41] using Matrix Factorisation. This proposal has the advantage that it encompasses naturally the data about fluxes in the description, but the authors did not make explicit global constructions involving non-abelian gauge groups. We will study two examples involving the group $SU(2)$, and explain how the spectrum charged under the gauge group arises in that case and show how the flux data can be obtained. We will then check our results by taking a limit where one can use the perturbative regime, and compare with the results obtained in F-theory.

This thesis is organised as follows: We begin the study of effective theory and string theory by reviewing their basic features in chapter 2, in order to show that the geometry is a language particularly adapted to discussing Effective Field Theory, and set the ground for F-theory. In chapter 3, we discuss the constraints of string theory on axion field ranges in the context of Type IIA supergravity. As it explores the closed sector of string theory while the rest of the thesis focuses on the open sector, it is therefore more standalone than the other chapters.

In chapter 4, we turn ourselves to F-theory by giving the mathematical basics of elliptic fibrations, and explain how to extract the physical data from a given geometry. This chapter is also introductory in nature, and can be skipped by the *cognoscenti*. Chapter 5 starts our analysis of Grand Unified Theories, focussing on their relation with the group E_8, while chapter 6 focuses on breaking $SU(5)$ GUTs to the Standard Model gauge group, and explores some mechanisms associated to hypercharge flux breaking. Finally, chapter 7 then studies examples of non-Abelian F-theory models via Matrix Factorisation methods, and we give our conclusions in chapter 8. In the appendix, we give a mathematical glossary that could be useful to the reader.

The study of the consequence of gravitational backreaction on axion field ranges discussed in chapter 3 is part of published work in collaboration with Eran Palti [42], while the role of E_8 in F-theoty GUTS (chapter 5), in collaboration with Eran Palti and Sebastian Schwieger, has first been discussed in [43]. Moreover, chapters 6 and 7 are part of work in progress at the time of the submission of this thesis. During the elaboration of this thesis, a study of Renormalisation Group flows and the a-theorem has been published in collaboration with Boaz Keren-Zur, Riccardo Rattazzi, and Lorenzo Vitale [44], but will not be discussed in this thesis.

Chapter 2

Four Dimensional Effective Actions and Geometry

In order to establish the various constraints that will be explored throughout this thesis, we start by reviewing the deep and beautiful relations between σ-models and various areas of mathematics. We will argue that in such models, everything can be associated with geometric quantities related to either spacetime or the space of physical configurations, with observables being differential-geometric invariants. This remarkable observation, dubbed the *Geometric Principle* by Cecotti [45], will have far reaching implications and make clearer the origin of the constraints established throughout this thesis.

Effective Field Theories (EFTs) describe a physical system up to a certain energy scale, Ω, called the cut-off of the theory. If the various observables of this system can be described as a formal expansion of a set of small parameters, we say that it admits a *Lagrangian description* in terms of a Lagrangian $\mathcal{L}$. Such a Lagrangian consists of an a priori infinite series, in terms of the fields describing the degrees of freedom of the system and their derivatives whose coefficients are called coupling constants. If the processes considered have a characteristic energy reaching order Ω, the effective description breaks down, as one might need to consider new degrees of freedom.

Using naive dimensional analysis, coupling constants of terms in the Lagrangian with more than two derivatives have to be suppressed by an adequate power of the cut-off. If we stay at characteristic energies far beneath that of the cut-off, these terms contribute only negligibly amount to observables, and can be ignored in an effective description. This arguments simplifies the Lagrangian description of EFTs as we can separate between terms containing up to two derivatives, called *kinetic terms* and terms depending only on the field, the *potential*. It turns out that in the (perturbative) effective approach to physics, matter can always be described by this class of theories. By matter, we mean degrees of freedom whose spin is at most one-half, i.e. scalar or fermions.

2.1 Sigma-models and their Symmetries

We now turn our attention to the geometric properties of such EFTs. As we argued above the scalar sector of such theories is described by a two-derivative kinetic term and a potential. Such a description is called a *non-linear σ-model*. Ignoring possible symmetries for now, it is described by a collection of scalar fields φ^i, whose dynamics are controlled by the action

$$S_\sigma = \int_\Sigma d^d x \sqrt{-h}\mathcal{L}_\sigma \qquad \mathcal{L}_\sigma = \frac{1}{2} g_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j - V(\phi), \tag{2.1}$$

where (Σ, h) is a (pseudo-)Riemannian manifold of dimension d describing spacetime. In this section we have adopted the conventions of [45]. Physical observables should not depend on the fields used to parameterise a given configuration, and observables should therefore be invariant under a reparametrisation of the fields $\phi^i \to \varphi^i(\phi)$. In particular, to preserve the form of the two-derivative term in (2.1), the tensor g_{ij} must transform as $g_{ij} \to \partial_i \varphi^k \partial_j \varphi^l g_{kl}$. This transformation is reminiscent of that of the metric of a manifold $\mathcal{M}$ under local diffeomorphisms. Moreover, if we want to have any meaningful sense of probabilities, we need the theory to be unitary, forcing g_{ij} to be symmetric positive-definite.

This leads us to interpret the scalar fields as local coordinates of a manifold $\mathcal{M}$ equipped with a Riemannian metric g. The full σ-model then comes endowed with two manifolds: A *spacetime* Σ which can a priori be curved, and the *target manifold* $\mathcal{M}$. A classical field configuration is then a smooth map $\Phi : \Sigma \to \mathcal{M}$, which in local coordinates is given by $\phi^i(x^\mu)$ and whose kinetic term is then simply the trace of the pull-back metric $\Phi^* g$.

As hinted at the beginning of this section, the invariance of physical quantities under field redefinition in such models will translate to differential-geometric invariants of the target manifold. This simple observation is the first building block of the *Geometric Principle*.

This geometric interpretation says even more: Let us imagine the Lagrangian to be invariant under the global action of a continuous group G acting on the fields ϕ^i. Such an action in particular requires to leave the kinetic term, and therefore the target space metric invariant. In other words, G must be a subgroup of the isometry group $\mathrm{Iso}(\mathcal{M})$ of $\mathcal{M}$. The presence of symmetries therefore restricts the set of possible manifolds. For instance, symmetric Riemannian spaces, whose isometry groups are Lie groups, have been completely classified [46, 47].

In many of the relevant physical cases, at least part of G is gauged, and one requires the presence of spin-1 vector fields $A_\mu^A,\ A = 1, \cdots, \dim(G)$, in the spectrum. The coupling of these fields to the σ-model is achieved by considering the associated Killing vectors $K_A^i,\ A = 1, \cdots, \dim(G)$, generating the gauge transformation. One then simply promotes the usual derivative to a covariant one: $\partial_\mu \phi^i \to D_\mu \phi^i = \partial_\mu \phi^i - A_\mu^A K_A^i$. The *gauged σ-model*

$$\mathcal{L} = \frac{1}{2} g_{ij} D_\mu \phi^i D^\mu \phi^j - V(\phi), \tag{2.2}$$

is then guaranteed to be invariant under gauge transformations. Infinitesimally, they are given by

$$\delta\phi^i = \varepsilon^A K_A^i \qquad \delta A_\mu^A = D_\mu \varepsilon^A = \partial_\mu + f^A{}_{BC} A_\mu^B \varepsilon^C, \tag{2.3}$$

with $f^A{}_{BC}$ the structure constants of the Lie algebra $\mathfrak{g}$ of G. In a local chart of $\mathcal{M}$, the vector fields transform as both spacetime vectors and as elements of the Lie algebra, i.e. they are elements of $T\mathcal{M} \otimes \mathfrak{g}$. Globally, we must allow for possible twists of the topology, which in mathematical terms is to consider them as sections[1] of a principal G-bundle over $\mathcal{M}$.

Note that the full dynamics involve terms in the Lagrangian containing derivatives of the gauge fields. While these are not part of the σ-model per se, they have important physical implications, such as anomaly terms, and should not be discarded when considering a full theory.

So far, we have argued that everything in equation (2.2) has a geometric interpretation. However, any model expected to have any connection with the real world has to contain fermions, and we are thus lead to wonder how they are connected to the target manifold point of view. In the following, we focus on the four dimensional, case where the minimal fermions (the smallest fermionic representation) are of Weyl type, but the results can be obtained *mutatis mutandis* to other dimensions by finding the corresponding minimal spinors.

The most general one derivative term[2] for Weyl fermions is

$$\mathcal{L}_\psi = i f_{ab}(\phi) \overline{\chi}^a_{\dot{\alpha}} \overline{\sigma}^{\dot{\alpha}\alpha\mu} \partial_\mu \chi^b_\alpha. \tag{2.4}$$

Locally, the right- and left-handed fermions χ_α, $\overline{\chi}_{\dot{\alpha}}$ transform as functions of $\mathcal{S}_\pm \times \Sigma$ with $\mathcal{S}_\pm$ the vector space associated to the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representation of $SO(1,3)$ respectively. We are therefore in a similar situation to that of vector fields, albeit for the fact we consider spacetime transformations and not gauge ones. We must also account for non-trivial global topologies of the spacetime manifold and are led to rather interpret fermions as sections of vector bundles over Σ with fibre $\mathcal{S}_\pm$. Moreover as f_{ab} is again positive-definite by unitarity, and is interpreted as a fibre metric. This means that the bundles are in turn pull-backs of bundles $\mathcal{V}_\pm$ over $\mathcal{M}$ with fibre metric f.

The interpretation of fermions as sections of vector bundles was established using solely the fermionic kinetic term. One may wonder about potentials involving both fermions and scalars. Consider for instance a potential of the Yukawa type $V \supset y_{ab}(\phi)\chi^a \cdot \chi^b$. For this term to be reparametrisation invariant, y_{ab} must transform in such a way that the transformation of both fermions is cancelled. Almost by definition, the coupling must be an element of the (pull-back) of the dual bundles $\mathcal{V}_-^\vee \otimes \mathcal{V}_-^\vee$, see appendix A for definitions.

This argument is straightforwardly generalised to arbitrary potential terms as well as arbitrary spins and leads us to the General Lesson:

General Lesson 2.1. *The physics of the matter sector of an Effective Field Theory is encoded in the geometry of the target manifold $\mathcal{M}$ describing scalar fields, and a collection of vector bundles $\mathcal{V}_\mathcal{R}$ over $\mathcal{M}$ describing higher spin fields. These higher spin fields are sections of*

$$\mathcal{S}_\mathcal{R} \otimes \Phi^*\mathcal{V} \longrightarrow \Sigma, \tag{2.5}$$

[1]The mathematical term for the elements of a bundle is section, see appendix A.

[2]To ensure covariance under field parameterisation, one should consider a covariant derivative with a metric connection associated to g_{ij}. For brevity, we will not consider such a term here as it does not change the spirit of the analysis. Similar terms are discussed in [45, 48].

with $\mathcal{S}_{\mathcal{R}} \to \Sigma$ the vector bundle associated with the appropriate spin representation $\mathcal{R}$. The couplings in the potential between scalar and higher spin fields can be seen as maps between these bundles and their duals. Any physical quantity then corresponds to differential-geometric invariants of the target manifold and its bundles.

This collection of observations is called the Geometric Principle *[45].*

To show the power of the Geometric Principle, let us consider a σ-model with only a scalar kinetic term for simplicity. The β-function of the metric, $\beta_{ij} = \frac{\partial}{\partial \ln \mu} g_{ij}$, must be a covariant symmetric tensor made out of the metric and its Riemann tensor. At first order in R_{ijkl}—corresponding to a one-loop computation—it can only be written as[3]

$$\beta_{ij} = c_G R_{ij} + (\text{higher orders}) \tag{2.6}$$

to respect differential-geometric properties. As we will see in section 2.3, this very simple result obtained only by geometric arguments will have interesting consequences for string theory.

For a general QFT, the vector bundles over $\mathcal{M}$ are a priori unrelated and arbitrary, as interactions between fields of different spins are also unrelated. In that context, the geometric approach to effective actions seems not to be the most convenient, and one might ponder over the usefulness of what we have achieved so far. However, if a symmetry were to relate fields or different spins, the vector bundles would in turn be related to the geometry of the target space, which can be as we saw very constrained.

It has been known for a long time [49,50] that the only such possibility is supersymmetry (SUSY). Originally, the motivation for supersymmetry was to alleviate problems in Renormalisation Theory and partly address the hierarchy problem. Indeed supersymmetry leads to non-renormalisation theorems [51,52] curing the problems of potential infinities occurring e.g. in corrections to the Higgs mass. While having very interesting consequences for low energy model building, this feature is not why supersymmetry is of such importance in this thesis. Starting with the works of Seiberg and Witten [4,5], it was realised, as we shall argue shortly, that supersymmetric theories exhibit a deep relation with geometry, constraining the geometry of $\mathcal{M}$ so much that it allows one to make non-perturbative exact statements about the structure of some EFTs. Furthermore, it made clear the existence of a rich network of dualities relating a priori unrelated theories, which have nowadays been extended to non-supersymmetric cases, see [53] and references therein for an historical account.

The precise structure of the SUSY algebra depends on its representation on spacetimes of different dimensions and signatures, and we will focus here on four dimensional Minkowski spaces[4], where in that case the algebra is generated by [51]

$$\left\{Q^I_\alpha, \overline{Q}_{\dot\alpha J}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^I_J \qquad \left\{Q^I_\alpha, Q^J_\beta\right\} = \varepsilon_{\alpha\beta} Z^{IJ}, \tag{2.7}$$

where P_μ is the generator of translation, Z^{IJ}, $I, J = 1, \ldots \mathcal{N}$, the so-called central charge and we left the possibility for multiple supersymmetries.

[3] A term proportional to R is identified as zero by noting that the result must be independent of $\mathcal{M}$, and in particular must vanish for flat space.

[4] For a discussion in various dimensions see [45].

The fields of the theory then organise themselves into multiplets according to their spin. In this work we will be mainly interested in *chiral multiplets*, containing a complex scalar ϕ^i and a left-handed Weyl fermion χ^i_α, whose infinitesimal SUSY transformation is

$$\delta\phi^i = \sqrt{2}\varepsilon^\alpha\chi^i_\alpha \qquad \delta\chi^i_\alpha = i\sqrt{2}\bar{\varepsilon}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi^i + (\text{non-derivative terms}). \tag{2.8}$$

Under a field redefinition $\phi^i \to \varphi^i(\phi)$, it appears that the fermions must transform as $\chi^i_\alpha \to \partial_j\varphi^i\chi^j_\alpha$ to be consistent with (2.8). This transformation rule is that of a vector, or in more mathematical terms, a section of the tangent bundle of $\mathcal{M}$. Supersymmetry thus forces the vector bundles $\mathcal{V}_\pm$ to be identified with the tangent bundle of the target space (twisted by the appropriate spacetime spin bundle)! This result has an immediate consequence: By the Geometric Principle, if the theory contains only scalars and fermions, interactions between fields are uniquely fixed by the scalar potential $V(\phi)$.

Supersymmetry does even better: In four dimensions, the invariance of the supercharges (2.7) under $U(\mathcal{N})$ transformations, and the fact that the minimal spinors are either left- or right-handed forces $\mathcal{M}$ be a Kähler manifold[5].

This has two consequences: First, $\mathcal{M}$ has to have even (real) dimensions, and we can therefore switch to complex coordinates $i, \bar{\jmath} = 1, \ldots, \dim_{\mathbb{C}}(\mathcal{M})$, and the metric $g_{i\bar{\jmath}}$ locally descends from an Hermitian Kähler potential $K(\phi, \overline{\phi})$:

$$g_{i\bar{\jmath}} = \partial_i\overline{\partial}_{\bar{\jmath}}K(\phi, \overline{\phi}) \qquad K(\phi, \overline{\phi})^\dagger = K(\phi, \overline{\phi}) \tag{2.9}$$

Secondly, while a bit more involved, it is straightforward to show that any deformation of the supercharges, which are equivalent to turning on a potential term, must descend from an holomorphic function $W(\phi)$:

$$V(\phi, \overline{\phi}) = \sum_i \partial_i W \overline{\partial_i W}. \tag{2.10}$$

There is also the possibility of having local supersymmetry, called *supergravity (SUGRA)*. This happens when there is an invariance of the σ-model under supersymmetry with an infinitesimal transformation depending on the coordinates of spacetime. It forces the spacetime metric to be part of the representation of the SUSY algebra called the *gravity supermultiplet.*

For $\mathcal{N} = 1$, it contains the spin 2 metric, as well as a spin-$\frac{3}{2}$ fermionic partner called the gravitino ψ^μ_α. For brevity, we will not go into the details of constructing supergravity invariants, but rather illustrate its geometric constraints on the target manifold, and how quantities useful in the rest of this thesis arise.

The presence of a gravity supermultiplet restricts further the set of admissible spacetime manifolds, and leads to the condition that the target manifold must be Hodge Kähler [54]. Such a manifold is characterised by the existence of a line bundle over $\mathcal{M}$ whose first Chern class is represented by the Kähler form and whose fibre metric is the exponential of the Kähler potential e^K (see appendix A for more details). By the Geometric Principle, the scalar potential has to be $\mathcal{M}$-geometric invariant, and keeping the four dimensional scalar

[5] This can be seen by observing that one can define a complex structure on spacetime [45].

potential to be of the form (2.10) is not tenable anymore: To take a well-defined derivative of the superpotential, one needs a connection on the line bundle. It is given in this case by $K_i = \partial_i K$. Similarly, to take the norm one needs both the fibre e^K and target space $g_{i\bar{\jmath}}$ metrics. This reasoning leads to the following form of the scalar potential in supergravity [45, 48]:

$$V(\phi, \overline{\phi}) = e^K \left(g^{i\bar{\jmath}} D_i W \overline{D_{\bar{\jmath}} W} - 3W\overline{W} \right), \qquad D_i = \partial_i + K_i \tag{2.11}$$

where the second term comes from the additional gravity supermultiplet. We see again the power of geometry in dealing with EFTs, as it allowed us to guess correctly the potential by knowing the structure of the target space.

Supergravity has one more interesting consequence: As fermions are sections of bundles over Σ, they transform non-trivially under spacetime coordinate redefinitions. We should then require that the fermions have vanishing vev to avoid any spontaneous breaking of spacetime symmetries. In the supergravity case, this means that the variation of the gravitino given by $\delta_\varepsilon \psi^I_\mu \sim \nabla_\mu \varepsilon^I$ must vanish. The presence of such covariantly constant spinors restricts the possible holonomy $\mathrm{Hol}_0(\Sigma)$ of the spacetime manifold. Luckily, the holonomy of irreducible Riemannian manifolds have long been classified by Berger [55], and the possible cases, while depending on the dimension of Σ and the number of supersymmetries, are all known [45, 56].

In section 2.3, we shall see that in string theory, spacetime is required to be 10 dimensional. For phenomenological reasons, we will have to require that it splits into the usual four dimensional Minkowski space and a compact six dimensional compact manifold with $SU(3)$ holonomy, called a *Calabi–Yau* manifold. This particular feature will be one of the keystones of the constraints on the four dimensional low energy effective actions that will be derived in this thesis, and they will depend directly on the properties of the Calabi–Yau manifold. These characteristics are expanded upon in the followings sections, as well as in appendix A.

The results we obtained in this section can be summarised by the following general lesson:

General Lesson 2.2. *Supersymmetry selects the bundles in which the fermionic superpartners and the scalar fields ϕ^i live. It follows that for supersymmetric EFTs, the couplings of the Lagrangian are given by canonical geometric objects of $\mathcal{M}$. In particular, in four dimensions the target space is Kähler, and the scalar potential* (2.10) *(or* (2.11) *for supergravity) descends from an holomorphic superpotential.*

Furthermore, the possible spacetime manifolds compatible with supergravity are classified by their holonomy group.

2.2 Compactification of Field Theories

In the last sections, we saw that very general arguments on symmetries impose stringent restrictions on the geometry of the target manifold as well as the spacetime manifold. A natural question is then if there can be relations between them beyond symmetries, and in particular their topology. Considering string theory will lead us to spacetimes that have a compact component, constraining the 4D effective actions.

To illustrate the implications of having a compact component in Σ, we start with the simplest five dimensional toy-model: Spacetime is taken to be $\Sigma = \mathbb{R}^{1,3} \times S^1$ with a real field ϕ. The compact coordinate y of the circle has the equivalence relation $y \sim y + 2\pi R$, where R is the radius of the circle, and we choose the Lagrangian to be that of a free field:

$$\mathcal{L}_{5D}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 \tag{2.12}$$

As the compact dimension is periodic, we can expand the field in a Fourier basis, $\phi(x, y) = \sum_k \phi_k(x) e^{ik\cdot y/R}$. Integrating over the S^1, we get a tower of increasingly massive fields, called a Kaluza-Klein (KK) tower:

$$S = \int_{\mathbb{R}^{1,3}\times S^1} d^4x dy \mathcal{L}_{5D} = \int_{\mathbb{R}^{1,3}} d^4x \mathcal{L}_{4D} = \int_{\mathbb{R}^{1,3}} d^4x \frac{1}{2} \sum_{k\geq 0} \left((\partial_\mu \phi_k)^2 - \frac{k^2}{R^2}\phi_k^2 \right) \tag{2.13}$$

The compactification procedure thus starts with a one dimensional target space $\mathcal{M}_5$ in five dimensions to an infinite dimensional one, $\mathcal{M}_4$, from the point of view of a four dimensional σ-model. It seems that we started from a very simple system and the compactification procedure made it unnecessarily complicated. If we take the Wilsonian approach however, the compactification introduced a scale R in a previously scale invariant Lagrangian. If we are probing energies much lower than the inverse radius of the S^1, we can then discard the whole Kaluza-Klein tower, getting back to a one dimensional target manifold. For more complicated systems, we will be able to discard interactions, and reduce the number of degrees of freedom in the effective description.

In string theoretical cases, this procedure will allow us to go from a theory of various fields and spins to a more manageable four dimensional EFT with fields of maximum spin two, and with interactions described by a σ-model whose couplings are geometrical quantities related to those of the Calabi–Yau.

While the case of a single S^1 is straightforward, it is natural to ask how to generalise to the case of an arbitrary compact n-dimensional manifold Y, and how many massless fields one obtains upon compactification. What we are after are fields that are massless, and therefore annihilated by the Laplacian[6], $\Delta\phi(x) = 0$. The solution to this problem is cohomological in nature, and can be solved in an abstract way. Denoting the set of n-forms valued on a module[7] R as $\Omega^n(Y, R)$, there are two important maps relating these sets: the differential d, raising the degree of a form by one, and the codifferential δ, decreasing it by one. It is then possible to define the *de Rham cohomology group* $H^n(Y, \mathbb{R})$ as the set of n-forms that are closed but not exact. In the sequel, we will sometime need to differentiate between cohomology groups defined over the real numbers $\mathbb{R}$ or a different module R, and it

[6]Note that for curved spacetimes, the Laplacian involves non-trivial terms involving the metric.
[7]A module is a generalisation of a vector space, see appendix A.

is useful to define them in the language of homological algebra, as cokernels[8] of the map d:

$$H^n(Y,R) = \mathrm{Coker}(d) := \mathrm{Ker}(d)/\mathrm{Im}(d), \qquad d:\Omega^n(Y,R)\to\Omega^{n+1}(Y,R). \tag{2.14}$$

This definition of the cohomology groups in terms of the cokernel of a map can be generalised and will be reappear in chapter 7, and is explained in appendix A. The Laplacian on Y, defined as $\Delta = d\delta + \delta d$, does not change the degree of a form in $H^n(Y,R)$, and can therefore be considered as a map from the groups to themselves, called an *automorphism*. When $R = \mathbb{R}$, it is possible to prove *Hodge's theorem*, stating that the cohomology group $H^n(Y,\mathbb{R})$ of *smooth* manifolds is isomorphic to the group of harmonic n-forms

$$H^n(Y,\mathbb{R}) \cong \mathcal{H}^n = \{\omega \in \Omega^n(Y,\mathbb{R}) | \Delta\omega = 0\}\,. \tag{2.15}$$

It has the consequence that each element of the de Rham cohomology group is represented by a unique harmonic form. We thus reduced our problem of finding harmonic forms to an algebraic one: that of finding the cohomology groups of Y.

For Kähler manifolds, the presence of a complex structure leads to simplifications: One can decompose the differential into the Dolbeault operators $\partial, \overline{\partial}$, acting on the sets of degree (p,q)-forms $\Omega^{(p,q)}(Y)$, and it can be proved that the de Rham cohomology groups are related to those of Dolbeault in the following way:

$$H^n(Y) = \bigoplus_{p+q=n} H^{(p,q)}_{\overline{\partial}}(Y), \qquad H^{(p,q)}_{\overline{\partial}} = \mathrm{Coker}(\overline{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)}). \tag{2.16}$$

The dimensions of the cohomology groups $h^{(p,q)} = \dim_{\mathbb{C}} H^{(p,q)}(Y)$ are called the *Hodge numbers* and satisfy various symmetry properties, recalled in appendix A.

As we will see in the next section, some of the Hodge numbers control part of the numbers of scalar fields in the compactified IR theory. In that sense, we already see that the choice of the spacetime manifold Σ constrains the possible structure of the target manifold, as it selects at least part of the scalars and their possible symmetries.

The advantage of this somewhat more abstract approach is that it opens us to the technology developed in the area of algebraic geometry. Such techniques are very powerful and allow computations without the knowledge of the metric of Y, and enables one to construct Calabi–Yau manifolds in an elegant fashion [57]. Moreover, we emphasise again that (co)-homological groups are topological quantities that do not require an explicit knowledge of the metric. As we will soon see, string theory will require Y to be of Calabi–Yau type, for which the metric is not known in all but a few simple cases. The number of fields of a given type in the reduced theory is then simply given by the dimension of the associated cohomology groups of the compact space.

There are many applications of these methods to physics. Another important example is the *Dirac quantisation condition*: let us consider a $(p-1)$-form C_{p-1}, i.e. an element

[8] To avoid cluttered notation, we will never differentiate between an element of $H^n(Y,R)$ and a representative, in the sense of equivalence class. Moreover, when the module is unambiguous, we will omit it.

of the pull-back of $T\mathcal{M}$ on Σ. If Σ has a non-trivial topology and admits the presence of non trivial $(p-1)$-cycles $\Sigma_{p-1} \subset \Sigma$, there will be line operators $\exp(iq \int_{\Sigma_{p-1}} C_{p-1})$ in the spectrum. Using Stokes theorem, it is possible to write the operator in terms of the field strength $F_p = dC_{p-1}$ integrated over a p-cycle Π_p whose boundary is Σ_{p-1}. On an oriented manifold, there are two possible choices, as one can choose Π_p to be the either the "interior" or the "exterior" of Σ_p. The two choices must lead to equivalent physics, and thus force the field strengths to satisfy the *Dirac quantisation condition*:

$$\frac{F_p}{2\pi} \in H^p(\Sigma, \mathbb{Z}). \tag{2.17}$$

The simplest example being the Wilson line, where a 1-form is integrated over a circle. This circle can then be thought of as the border of either the northern or southern hemisphere (see figure 2.1). In the familiar example of electrodynamics this is the statement of the quantisation of the electric and magnetic charges. This illustrates the power of homological algebra in physics, and this section can be summarised thusly:

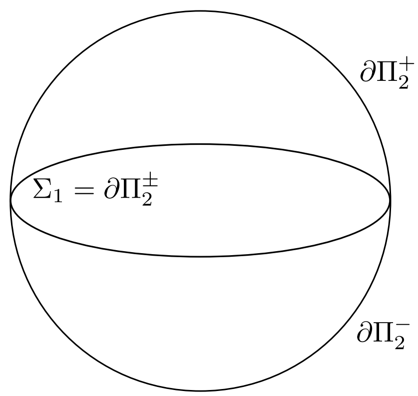


Figure 2.1: Example of how a circle can be written as the border of two different 2-cycles, the hemispheres.

General Lesson 2.3. *Finding the low energy spectrum of a higher dimensional theory on a compact manifold Y is a cohomology problem. Massless scalar fields are determined by topological invariants whose study is well suited for algebraic geometry. If Y is Kähler, the massless fields are counted by the Hodge numbers of Y.*

Moreover, the presence of non-trivial cycles on Y leads to the quantisation condition (2.17).

2.3 String Theory: Space-time as a Target Space

So far we have argued that couplings of a σ-model have to behave properly under diffeomorphisms and bundle morphisms of the target manifold. The Geometric Principle 2.1 can indeed be interpreted as seeing the couplings between fields of various spins as local fields having general covariance, similar to General Relativity and gauge transformations if we were discussing spacetime.

String theory takes this deep insight seriously and considers the target space to be *spacetime itself*! The spacetime manifold of the σ-model is then only a mere auxiliary space parametrising the whole theory. For General Relativity, this paradigm is known as the worldline formalism, where a point-particle takes a one-dimensional trajectory Σ across spacetime [58]. A question one may ask is what happens the point-like elementary particles are replaced with higher dimensional objects. The next natural case is to replace the worldline by a two-dimensional *world-sheet*. This choice offers a richer realm of possibilities as there are two kinds of possible strings: It can be either open or closed—as Σ can be a closed or open surface—and leads to different spectra upon quantisation.

The two dimensional case turns out to be incredibly constrained. The σ-model describing the propagation of the string through spacetime is a Conformal Field Theory (CFT). Such theories are more manageable than usual QFTs and the conformal group is infinite dimensional in two dimensions, making the quantisation of a string in a flat background completely solvable.

Another motivation for considering strings rather than points is that strings naturally come with a length scale ℓ_s. It offers a natural ultraviolet cutoff to the quantum theory, and is in principle the only dimensionful parameter of string theory. The presence of such a natural cutoff indicates that string theory can be used to describe all energy scales, as it is not plagued by the infinities encountered in usual Quantum Field Theories, and makes it a potential candidate for a UV complete Theory of Everything.

In order to set the notation and lingo that shall be used extensively throughout this thesis, we now shortly review the well-known properties of string theory. To avoid confusion with the now familiar idea of a target manifold as a collection of fields representing particles, as described in the previous sections, we denote the target space of the string as the d-dimensional pseudo-Riemannian manifold $(\mathcal{X}, G)$, with local coordinates $X^M,\ M = 1, \ldots, d$, and the 2-dimensional world-sheet as a Riemannian manifold (Σ, h) with local coordinates $x^a,\ a = 1, 2$.

As we will require the presence of fermions in the spectrum of the string, for phenomenological reasons among others, we will directly consider supersymmetric strings, whose bosonic part, the *Polyakov action*, is nothing other than the linear σ-model

$$S_P = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2x\sqrt{-h} G_{MN} h^{ab} \partial_a X^M \partial_b X^N. \tag{2.18}$$

where $\alpha' = (\ell_s/2\pi)^2$—the so-called Regge slope—is the only dimensionful parameter of the theory. As hinted previously, the Polyakov action has two very important symmetries in addition to general covariance: it is invariant under an $SO(1, d-1)$ transformation of X^M, and under a Weyl rescaling of the metric $h_{ab} \to \Omega(x) h_{ab}$, making it a CFT. The latter is of paramount consequence: Classically, the conservation law associated to Weyl symmetry is the tracelessness of the world-sheet energy momentum tensor $T^a_a = 0$. Quantum dynamically, it can in principle get anomalous contributions[9]

$$\langle T^a_a(x)\rangle = cR(x) + \beta_{MN} h^{ab}\partial_a X^M \partial_b X^N. \tag{2.19}$$

However, anomalies associated to gauge symmetries are well-known to be inconsistent and we must therefore ensure that they vanish, leading to the conditions $c = 0 = \beta_{MN}$. The vanishing of the metric β-function is astonishing: From the Geometric Principle, we argued that it is given at leading order by (2.6), and therefore naturally incorporates Einstein vacuum equations.

The c-function on the other hand is a topological invariant and is known to be proportional to $(d-10)$ in the supersymmetric case [60]. This is a remarkable result: The dimension of the physical spacetime is a *consequence* of considering a quantum modelling of strings! It means that, whereas in usual Quantum Field Theories the physical spacetime is chosen arbitrarily, panning to the string theoretical paradigm we are no longer free to start with four dimensional models. This result might be a drawback of string theory rather than a success, as we observe only four dimensions. However, we saw in the last section that if the extra six dimensions are sufficiently small, we can use compactification techniques to find an EFT valid at distances where the resulting KK tower can be neglected.

For a flat spacetime $\mathcal{X} = \mathbb{R}^{1,9}$, the spectrum of the Polyakov action (2.18) can be found exactly, and one finds a tower of fields of higher and higher spins and masses. Moreover, in the presence of world-sheet fermions, boundary conditions lead to two possible sectors for either end of the string, called the Neveu-Schwarz (NS) and Ramond (R) sectors. In the presence of all four possible combinations, the theory contains a tachyon, which has to be removed in order to avoid truncating the spectrum, in a procedure called the *GSO projection*. To make a rather long story short, we will only state that there are two types of projection with a supersymmetric massless spectrum, named Type I and II, after the number of supersymmetries their spectrum exhibits. In the latter case, there are then two possibilities to make the GSO projection, dubbed A and B.

In this thesis, we will focus on Type II theories, as they can be shown to be dual to type I theories, in a sense that shall be explained in section 2.4.2. Both Type II theories share a common feature: Their massless spectrum are $\mathcal{N} = 2$ supergravities, whose bosonic sectors contain a scalar field, the dilaton Φ, a symmetric 2-tensor G_{MN}, a 2-form B_{MN} called the Kalb-Ramond field, as well as so-called RR p-forms of various degrees, named after the sector from which they originate. The spectra are summarised in table 2.1.

We started with a flat background and an immediate question is that of more arbitrary backgrounds. Arguments about scattering amplitudes of strings [61] show that a modification of the Polyakov action is necessary, but that the massless spectrum holds and that G_{MN} has to be identified with the metric of $\mathcal{X}$. This means that string theory is a theory of quantum gravity, which is in part what prompted the phenomenal body of work in string theory since the eighties. For phenomenological reasons, we are led to demand that spacetime factors into four large dimensions (which we take to be Minkowski space)) and another compact

[9]This is valid only in a particular renormalisation scheme. For a discussion of scheme independent quantities in two and four dimensions, see [44, 59] and references therein.

Type IIA field	Name	Type IIB field
Φ	Dilaton	Φ
G_{MN}	Metric	G_{MN}
B_{MN}	Kalb-Ramond 2-form	B_{MN}
C_1, C_3	RR p-forms	C_0, C_2, C_4

Table 2.1: Massless bosonic spectrum of the Type II superstring theories.

six dimensional manifold Y_3 that has to have a small volume to have eluded experimental constraints so far:

$$\mathcal{X} = \mathbb{R}^{1,3} \times Y_3 \tag{2.20}$$

From General Lesson 2.2, the possible manifolds Y_3 are restricted as we need to preserve supergravity. A careful inspection of the gravitini SUSY variation shows that the holonomy group of spacetime has to be $SU(3)$ [60]. Such manifolds are of the Calabi–Yau type and have been the focus of a great body of research of both mathematics and physics (see e.g. [57] for reviews).

So far we have only discussed the closed sector of string theory, where the world-sheet is a compact surface. If it admits a boundary, the string spectrum admits a richer zoology of fields. Pictorially, the string is open-ended, and its ends can span a $p+1$-dimensional submanifold of $\mathcal{X}$, called a *Dp-brane*. Upon quantisation, the spectrum admits a $U(1)$ vector field A_M. We can also imagine stacks of n Dp-branes on top of each other, enhancing the $U(1)$ gauge group to $U(n)$. In the semi-classical limit, they can be understood as the degrees of freedom associated to the fluctuating strings "pushing and pulling" the branes.

One may wonder if it is possible to obtain gauge fields charged under other groups than $U(n)$. A possibility is to consider $\mathcal{X}$ to be an orientifold, a manifold quotiented by a discrete group. The invariant points in the space time manifold under the discrete groups are called *O-planes* and lead to $SO(n)$ or $Sp(n)$ groups. In fact, the presence of O-planes is required to have a consistent theory. Branes carry a positive tension —a generalisation of the charge for one dimensional objects—which has to be cancelled so that the total charge in the compact space vanishes as a consequence of Gauss's law, and generically leads to only one supersymmetry in four dimensions and therefore offers a nice way to reduce supercharges to a more phenomenologically desirable number.

String theory computations on an arbitrary spacetime manifold $\mathcal{X}$ are two-dimensional CFTs that, while very constrained, are often ill-suited for practical purposes. It however admits two expansions: The first is an expansion in α', which allows us to discard the tower of increasingly massive particles. This leaves us with an effective description in terms of fields falling in supergravity multiplets, but we are still left with an a priori non-perturbative description. However, when computing partition functions, one notices that there exists a second expansion controlled by the vacuum expectation value of the dilaton, through a quantity called the *string coupling* $g_s = e^{\langle\Phi\rangle}$. In the limit where g_s is small, an inspection of the correlation functions shows that there is a 10D supergravity description in terms of a Lagrangian.

For instance, the Lagrangian for Type IIB admits the so-called democratic formulation [62]. The bosonic part of the closed sector is given by

$$\begin{aligned}\frac{\ell_s^8}{2\pi}S_{IIB} \supset \int_{\mathcal{X}} d^{10}x e^{-2\Phi}\sqrt{-G}\left(R+4\partial_M\Phi\partial^M\Phi\right) - \frac{1}{2}e^{-2\Phi}\int_{\mathcal{X}} H_3\wedge *H_3 \\ -\frac{1}{4}\sum_{p=0}^{4}\int_{\mathcal{X}} F_{2p+1}\wedge *F_{2p+1} - \int_{\mathcal{X}} C_4\wedge H_3\wedge F_3,\end{aligned} \tag{2.21}$$

where the field strengths are defined as

$$\begin{aligned}H_3 = dB_2, \qquad F_1 = dC_0, \qquad F_3 = dC_2 - C_0 dB_2, \\ F_5 = dC_4 - \frac{1}{2}C_2\wedge dB_2 + \frac{1}{2}B_2\wedge dC_2.\end{aligned} \tag{2.22}$$

This Lagrangian however does not quite reproduce correlation function computations obtained from path integrating the string σ-model, and has to be supplemented by the relations $F_9 = *F_1$, $F_7 = -*F_3$, $F_5 = *F_5$ at the level of the equations of motion. The open sector can also be described by an action through the Chern-Simons and DBI terms, omitted for brevity

We saw that string theory is a very powerful framework encapsulating various features of field theory naturally, which are summed up by:

General Lesson 2.4. *String theory uniquely sets the possible dimensions and supersymmetry forces the compact extra dimensions to be a Calabi–Yau manifold. Upon quantisation, the massless excitations of the strings contain a graviton, making string theory a candidate for Quantum Gravity. We note that in the presence of fluxes, this condition can be modified, see section 2.4.1.*

Moreover, the massless spectra admit a supergravity Lagrangian description in the limit where g_s is small. The action for Type IIB is given in equation (2.21).

We note that the literature is quite cavalier regarding the terminology and often uses Type II string theory and Type II supergravity interchangeably, as the massive tower can be omitted in most of the semi-realistic constructions, as well as sometimes blurring the lines between perturbative and non-perturbative statements. In this thesis, we attempted to differentiate the terms as best as we could, in order to allow one to understand the conditions where a given statement is valid.

2.4 Compactification of Type II Supergravity

In the previous section, we saw that the presence of gravitini in Type II string theories requires the compact dimensions to be a Calabi–Yau 3-fold (Y_3, g). We can now combine this result with General Lesson 2.3 to Type II supergravities compactified to four dimensions. We are interested in quantum regimes, and the metric will hence fluctuate from its background value $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \mu = 1, \cdots, 6$. We thus have to require that it does not change the Calabi–Yau condition, which is equivalent to Ricci flatness. After having fixed a gauge —in our case the Lorenz gauge $\nabla^\mu g_{\mu\nu} = 0$—to avoid overcounting degrees of freedom, one is led to the condition

$$R(g + \delta g) = 0 \longrightarrow \Delta \delta g_{\mu\nu} + 2R_{\mu\ \nu}^{\ \rho\ \sigma} \delta g_{\mu\nu} = 0. \tag{2.23}$$

Switching to complex coordinates $x^\mu \to (z^i, \overline{z}^{\overline{\imath}})$ we have to separate between two cases:

1. Deformations with one index of each type, corresponding to a deformation of the Kähler $(1,1)$-form $J = g_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}$, named *Kähler deformations.* The condition (2.23) is then that $\delta g = \delta g_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}$ is an harmonic form and by Hodge's theorem, we can decompose it on a basis $\{\omega_a\}$ of $H^{(1,1)}(Y_3, \mathbb{R})$

$$\delta g \sum_{a=1}^{h^{1,1}} t^a \omega_a, \qquad \omega_a = \omega_{ai\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}. \tag{2.24}$$

2. The part of the metric $\delta g_{\bar{\imath}\bar{\jmath}}$ that has only anti-holomorphic indices, for which (2.23) is equivalent to

$$\Delta_{\overline{\partial}} \delta g^i = \Delta_{\overline{\partial}} \delta g^i_{\bar{\jmath}} d\overline{z}^{\bar{\jmath}} = 0.$$

   This means that $\delta g^i \in H^{(0,1)}(Y_3, TY_3)$, as the coefficients carry an index i. We cannot directly use Hodge's theorem as in the previous case, as the ring is TY_3 and not $\mathbb{R}$ or $\mathbb{C}$. Using the holomorphic $(3,0)$-form Ω however, one can define an isomorphism between this cohomology group and $H^{(2,1)}(Y_3, \mathbb{C})$ that can be expanded over a basis $\{b^\alpha\}$ of harmonic $(2,1)$-forms:

$$\Omega_{ijk} \delta g^k_{\bar{l}} = \sum_{\alpha=1}^{h^{2,1}} U^\alpha b_{\alpha ij\bar{l}}. \tag{2.25}$$

Now that we have catalogued the possible deformations, what happens from the point of view of the four dimensional EFT? Considering the full spacetime $\mathbb{R}^{1,3} \times Y_3$, the deformation of the Calabi–Yau metric will give rise to two types of scalar fields, called moduli, in four dimensions. We will have $h^{2,1}$ complex structure moduli coming from the holomorphic three-form, as well as $h^{1,1}$ real fields from the Kähler form.

In the closed string sector, we however also have the Kalb-Ramond two-form B. We can thus combine it with the Kähler form into a complexified two-form

$$J_c = B + iJ. \tag{2.26}$$

The argument above generalises straightforwardly, and we are left with $h^{1,1}$ *complex* scalar fields U_α. We have thus found the following property of the target manifold of the four dimensional EFT:

General Lesson 2.5. *The target manifold of the effective field theory of Type II supergravity compactified on a Calabi–Yau three-fold Y_3 contains a sector controlled by the possible deformations of the metric of the compact dimensions called moduli. In the absence of O-planes, there will be $h^{1,1}$ Kähler moduli $T^a = b^a + it^a$, as well as $h^{2,1}$ complex structure moduli $U^\alpha = u^\alpha + i\nu^\alpha$. In the presence of O-planes, this number is smaller.*

In chapter 3, we will consider a concrete example of Type IIA compactification in the presence of O-planes, but before doing so, we will however need an additional ingredient.

2.4.1 Flux Compactification

By construction, the moduli are massless and therefore appear at first to be ill-suited for semi-realistic models as the Hodge numbers for many Calabi–Yau 3-fold can be of order one hundred, which would induce a significant number of fields that would be observed at low energies. The may however acquire a mass when the field strengths associated to p-forms have a non-trivial vacuum expectation value, called *fluxes*.

In Type II compactifications, which are the ones that are of interest in this work, we will encounter two types of fluxes: Those associated to the Kalb-Ramond three-form, $H_3 = dB_2$, and those coming from the Ramond sector, p-forms $F_p = dC_{p-1}$. We will refer to them as NS and RR fluxes respectively, from the sector they arise. One may also encounter metric fluxes, but those can be shown to be dual to NS-fluxes. As there are terms of the schematic form $\sqrt{-G}(F_p)^2$ in the ten dimensional supergravity action, see e.g. (2.21) for Type IIB, the moduli will acquire a potential upon reduction—and therefore a mass—in the effective four dimensional theory.

Fluxes can have vacuum expectation values that depend on the coordinates of the internal manifold Y_3 as they do not induce a breaking of four-dimensional Poincaré invariance and satisfy a quantisation condition (see General Lesson 2.3). However, the presence of spatial coordinates will modify the supersymmetric variation of the gravitini (see e.g. [63] for a review). In General Lesson 2.2, we saw the possible spacetime manifold supporting supergravity are related to the presence of covariantly constant spinors. The presence of non-vanishing fluxes therefore modifies this condition and restricts the structure group of the internal manifold, i.e. the group of transformations needed to patch the orthonormal frame bundles together[10], rather than its holonomy group. For Calabi–Yau manifolds, the structure group is $SU(3)$, i.e. equal to its holonomy group. It has to be noted fluxes will generically preserve only $\mathcal{N} = 1$ supersymmetry. This is however not problematic, as we have already seen that the presence of O-planes already reduces the number of spacetime supersymmetries.

[10]For a Riemannian d-manifold, the group is $SO(d)$, as the respective coordinates of two overlapping patches are related by an $SO(d)$ transformation.

2.4.2 Unification of String Supergravities

As hinted in section 2.1, it was realised in the mid-nineties that there exist theories that are a priori unrelated yet describe the same physical processes. An illuminating example is to consider coupling a σ-model to $(p-1)$-forms C_{p-1} that appears only in the Lagrangian via its kinetic term in a covariant derivative $F_p = dC_{p-1}$:

$$\mathcal{L} = \mathcal{L}_\sigma + \frac{1}{2}\tau(\phi^i) F_p \wedge *F_p. \tag{2.27}$$

The equations of motion depend only on the field strength

$$dG_p = 0, \qquad G_p = \frac{\partial \mathcal{L}}{\partial F_p} = \tau * F_p, \qquad \text{(Equation of motion)} \tag{2.28}$$

$$dF_p = 0. \qquad \text{(Bianchi identity)} \tag{2.29}$$

These equations are clearly invariant under the exchange $F_p \leftrightarrow G_p$. Furthermore, any linear combination also leaves these equations invariant. We conclude that classically, this system has *modularity*, namely an invariance of the dynamics under an $SL(2,\mathbb{R})$ redefinition of the degrees of freedom. Quantum mechanically, a consequence of General Lesson 2.3 is that only linear combinations with integer coefficients are allowed, reducing the duality group to $SL(2,\mathbb{Z})$. In the sequel, we will always consider duality groups of quantum systems.

It is furthermore possible to show that under such a duality transformation, the function τ transforms as

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{2.30}$$

A particular example is the S-duality, where we make the changes

$$\begin{pmatrix} F_p \\ G_p \end{pmatrix} \longrightarrow \begin{pmatrix} \widetilde{F}_p \\ \widetilde{G}_p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F_p \\ G_p \end{pmatrix} \qquad \tau \longrightarrow \widetilde{\tau} = -\frac{1}{\tau}. \tag{2.31}$$

The duality sends a system described by the degree of freedom F_p with a coupling τ to a system described by $\widetilde{F}_p$ with a coupling $\widetilde{\tau}$. If we choose τ to be a small constant, the S-duality sends it to a system that does not admit a perturbative description. This is one of the simplest example of the so-called weak-strong duality.

The arguments presented here generalise to any Lagrangian involving only field strengths. In that case it is possible that the dual $d\widetilde{G}_p = 0$ does not admit a solution descending from a Lagrangian. The duality thus relates the non-perturbative, non-Lagrangian sector of theory space to the more manageable sector consisting of σ-models coupled to p-forms. Such dualities are however very difficult to find in practice, as one does not only need to find the dual degrees of freedom, but also establish the duality for the whole dynamics.

Type IIB string theory exhibits a similar duality, where the role of τ is given by the *axio*-dilaton

$$\tau = C_0 + ie^{-\Phi} = C_0 + \frac{i}{g_s} \tag{2.32}$$

We have already argued in the last section that if g_s is small, then the massless fields can be described in terms of ten-dimensional supergravity. The duality group will then take Type IIB supergravity to system with large g_s, indescribable by supergravity. In chapter 4, we shall see that the appropriate description for the non-perturbative dual of Type IIB supergravity is F-theory, where all the physical information is encoded in terms of a torus fibration over the compact extra dimensions Y_3.

There is another type of duality through compactification of string theories. It can be shown that the reduction of Type IIB string theory on a circle of radius R produces the same spectrum as that of Type IIA on a circle of radius α'/R. This non-trivial result attracted a lot of attention and it was soon realised that such dualities were in fact ubiquitous in the supergravity regime and beyond, relating Type II, type I and heterotic supergravities—another type of string theory containing a mix of bosonic and superstrings—as well, depending on the type of compactification.

In fact, all supergravities coming from string theory can be obtained via compactification of eleven dimensional supergravity on a one-dimensional compact space, see [60] for explicit computations. Eleven is the highest dimension that can host supergravity and has led string theorists to think that this is the sign of something deeper. Its bosonic content is extremely simple, as it contains only a metric and a 3-form $\mathcal{C}_3$, leading to the simple bosonic Lagrangian[11]:

$$S_{11} = \frac{1}{\kappa_{11}^2} \int d^{11}x\sqrt{-g}\left(R - \frac{1}{2}\left|G_4\right|^2\right) \qquad G_4 = d\mathcal{C}_3, \tag{2.33}$$

where G_4 is the field strength associated to the 3-form and is commonly referred to as the G_4-flux. The different supergravities can be obtained from this action. However we know that the supergravity regime is appropriate only when the string coupling g_s is small. The question is then whether this regime is the low-energy limit of a theory of 11-dimensional quantum gravity. The precise structure of this conjectured *M-theory* is yet unknown. Despite the tremendous progress made since its inception [7, 64, 65], most of what is known has been discovered via duality arguments similar to those discussed above, and a full non-perturbative description is still missing.

Now that we have explained the relationship between the geometry of the Effective Field Theories target manifold and the structure of the compact dimensions, we are ready to explore the constraints given by string theory on the geometry of $\mathcal{M}$. In the next chapter, we will study the closed sector of type IIA, while the rest of the thesis will focus on the open sector of type IIB.

[11] There is also a topological Chern-Simons term that we can safely ignore in our discussion.

Chapter 3

Constraints From Field Ranges

In the previous section we have seen in string theory, there two are different sectors depending on whether the string is closed or opened. In this section we will explore constraints on four dimensional Effective Field Theories (EFTs) associated to the closed sector. As described in section (2.4) the dimensional reduction of the massless excitations of such strings give rise to a myriad of massless fields that need to be given a mass in a procedure called moduli stabilisation. The moduli can then be used to build semi-realistic model of inflation, see [26] for a review of inflationary models in the context of string theory.

Field theoretic models of inflation can be categorised in two main classes, depending on whether the inflaton ϕ is displaced over distances that are super-Planckian, called large field inflation, or sub-Planckian, dubbed small field inflation. From a purely effective perspective, large field inflation is more attractive than its small field field counterpart, as it can be realised by extremely simple potentials and does not require any fine tuning of the initial value of the field.

Such models however predict significant primordial tensor modes and as of the time of writing, no such modes have been observed [66]. A possible explanation for this lack of experimental results could be that the effective approach is misleading because there are Quantum Gravity effects actually obstructing this naively natural possibility, by for instance limiting the available field range over which the potential remain flat enough to be sub-Planckian. Through their connections to primordial tensors, possible field excursions therefore offer a interesting area to explore the constraints on four dimensional EFTs from string theory.

A possible objection to super-Planckian field displacements from an effective point of view is that after UV degrees of freedom have been integrated out, one expects the potential of the inflaton ϕ to be an infinite tower of Planck scale suppressed operators

$$V_{\text{eff}}(\phi) = \sum_{n>0} c_n \frac{\phi^n}{M_{\text{Pl}}^{n-4}}, \tag{3.1}$$

and thus when the field displacement is of order M_{Pl}, control over the theory is lost and the EFT breaks down. A possible way around it could be that the UV physics is invariant under an additional symmetry controlling the tower beyond naive dimensional analysis. On the other hand, it is expected that there are no global symmetries in Quantum Gravity [21], and at this point, the way to proceed is not clear as the UV theory of gravity is not known and we cannot therefore perform an analysis of quantum corrections.

A way to tackle the problem is to use general expectations of properties of Quantum Gravity such as the Weak Gravity Conjecture (WGC) [20, 67–75] or analyses of entropy bounds [76,77]. In this chapter we shall use string theory as the UV completion and explicitly study these questions in such a framework.

We will be mainly interested in fields arising from the closed sector of string theory compactification as candidates for fields that can support super-Planckian excursions and therefore possibly large field inflation. Such fields split in two categories, moduli and axions differing in a number of ways: The most important is that axions do not appear non-perturbatively as an expansion in the moduli vacuum expectation value (vev) in the Kähler potential due to a shift symmetry. Moduli fields have been considered as possible inflaton candidates that could support super-Planckian excursions, see [26] for a review, [78] for earlier work.

Models where the role of the inflaton is played by moduli however have two problems: First, controlling Planck suppressed operators remains an obstacle in the absence of a UV symmetry, and second, moduli values are also parameters controlling the EFT and their possible range is bounded within the controlled setup. If their values are too small there are large corrections, while for too large values the cut-off scale of the theory—the string or Kaluza-Klein scale for instance—becomes too low.

Even worse, the Kähler potential depends on the moduli and thus so does their target space metric, in such a way that the canonically normalised fields are logarithmic in the moduli value. Schematically, for a model with only one modulus c, we have

$$g_{cc}(c) = \partial_c \partial_c K(c) \sim c^2 \qquad \Rightarrow \qquad \phi \sim \int \frac{dc}{\sqrt{g_{cc}}} \sim \log c. \tag{3.2}$$

The displacement distance of the canonical field is exponentiated as a modulus variation and therefore one is fighting for control against an exponential. This fight could be won if the coefficient in the exponent can be controlled in some way, but it typically depends on geometric invariants of the Calabi–Yau, rather than flux numbers which could be tuned more easily. It is thus hard to have full control over super-Planckian field excursions in those cases while keeping a high cutoff scale, although we are not aware of a general proof against such scenarios.

Conversely closed string axions—fields coming from the reduction of RR of Kalb-Ramond p-forms—do not naively appear to suffer from such difficulties: Their vevs do not control any EFT expansions and they do not appear in the (perturbative) Kähler potential, and therefore their target space metric is also independent of their vevs. Moreover these axions come from p-forms and keep a remnant continuous shift symmetry broken to a discrete one by non-perturbative effects.

This symmetry makes them attractive candidates to get control over super-Planckian displacements. On the other hand, this periodicity appear at the same time to make super-Planckian periods impossible, see [23] for the key original paper and [79] for the most recent analysis (while for example [80, 81] study possible exceptions). A way around it might be to consider mixing of two (or more) axions in such a way that their combined contribution reaches an effective super-Planckian decay constant, called the *alignment mechanism* [24]. This mechanism has been extensively studied for string theory axions, see [26] for a review and [82–88] for recent work. More relevant for us is the study by Palti [87] of such mechanism in the context of Type IIA supergravity compactifications, where it was shown that when taking backreaction effects into account, the target space metric of the axions is modified so as to precisely cancel any enhancement of the axion periodicity.

The main focus of this chapter will be on a different idea for realising super-Planckian displacements with axion, termed *axion monodromy* [25,89]. The basic idea is to induce terms in the superpotential breaking the shift symmetry, by for instance turning on a mass term, and "decompactify" the axionic target manifold, therefore enabling large field excursions as the metric is constant from point of view of the axions. For an axion field ρ we have by opposition to the modulus case

$$g_{\rho\rho} = \partial_\rho \partial_\rho K = \text{const} \qquad \Rightarrow \qquad \phi \sim \rho, \tag{3.3}$$

and excursions of the canonical field are only linear in the axion variation, allowing for greater control.

In the case of closed-string, this effect can be generetaed with D6-branes in the Type IIA case [90], D5-branes in Type IIB [25], or with background fluxes. Background fluxes have long been known to induce shift breaking potentials, but have only recently applied to large field inflation [91–103]. Recent work has also studied axion monodromy induced by non-flux effects [90, 104].

Both mechanisms share the feature that there exists an integer parameter we shall call N such that for $N = 1$ the field remain sub-Planckian and is then parametrically increased to super-Planckian values by tuning N to be large. We shall focus here on the backreaction of N on the target manifold metric, as it exhibits the right properties to form part of censorship mechanism for large field displacements in string theory. The backreaction involves gravitational physics and modify the metric of the axion target manifold, thus determining the distance it can travel. Further, from the EFT point of view we are free in choosing the target space metric, and it is typically considered to be constant, but the target manifold of string theory—or more precisely supergravity—is highly structured and constrained, e.g. the Kähler potential has a logarithmic behaviour due to supergravity, and its singularity structure captures highly non-trivial non-perturbative quantum gravitational physics. The cancellation of N found in [87] for alignment scenarios are examples of this behaviour for which the form of the Kähler potential plays a key role, and blindly taking the target manifold metric to be constant, or even arbitrary, one would not be made aware of those effects.

For axion monodromy the parameter N is associated with the number of periods that the axion traverses and its backreaction effects thus the backreaction of the energy density induced by its vev as it moves along its potential. For sufficiently smooth and diluted flux background, the backreaction of N is captured at leading order by its effect on other moduli[1]. To study this effect and determine its consequences on the possible axion excursions, one must be careful to account for any Quantum Gravity feature of string theory that affect this calculation.

We therefore restrict ourselves to a framework as clean and simple as possible and choose to study moduli stabilisation in Type IIA flux compactification on a Calabi–Yau manifold [106,107] or its twisted torus cousin [108,109] (we again refer to [63] for a more complete list of references on these topics).

In this setting, moduli stabilisation is realised by a perturbative process, and one only need to consider tree-level expressions for the Kähler potential and superpotential. This will enable us to solve the backreaction of N explicitly and precisely for several semi-realistic examples and is moreover a setting for which the uplift to 10-dimensional supergravity solutions is well-understood, adding a further level of control.

In particular, we will focus on the backreaction of the axion vev on its target space metric and thereby its excursion distance. We note that there are several studies of backreaction of axions vevs for axion monodromy scenarios in the literature for different settings focusing instead on the flattening effect of the potential or on possible destabilisation of the vacuum. See for example [100, 101, 110–114]. Perhaps most similar in a technical sense to our Type IIA settings are the non-geometric compactifications studied in [96–98, 102] which appear to share some features with our constructions.

This chapter is structured as follows: After a short review of Type IIA flux compactification in section 3.1, we study backreaction effects in axion monodromy scenarios in section 3.2. Our results are summarised in section 3.3.

3.1 Type IIA Flux Compactification

We have argued in chapter 2 that Type II supergravity compactifications lead to an $\mathcal{N} = 1$ supersymmetric effective action in four dimensions, characterised by a superpotential K and a superpotential W depending on the moduli of the three-fold. However, while General Lesson 2.5 gave us information on the field content of the four dimensional EFT, it did not tell us anything about its dynamics, namely the functional form of K and W. We here shortly review how to obtain these quantities in a Type IIA setting, starting with Calabi–Yau and then orientifold compactifications, and finally turning on fluxes.

For a Calabi–Yau without any fluxes or O-planes, the compactification preserves enough supercharges to obtain an $\mathcal{N} = 2$ supersymmetric four dimensional Effective Field Theory, and the relevant quantities can be constructed by counting the massless fields coming from the IIA supergravity spectrum. We already know from General Lesson 2.5 that there are $h^{1,1}$ Kähler and $h^{2,1}$ complex structure moduli. Additionally, we have to take into account the fields coming from the dimensional reduction of the two RR forms, which admit a

[1]See [105] for an analysis of backreaction in a non-dilute region within the context of large field inflation.

Type	number	Bosonic Content
Gravity multiplet	1	$(g_{\mu\nu}, A_0)$
Vector multiplets	$h^{1,1}$	(A_a, b_a, t_a)
Hypermultiplets	$h^{2,1}$	$(u_\alpha, \nu_\alpha, \xi^\alpha, \tilde{\xi}_\alpha)$
Tensor multiplet	1	$(B_{\mu\nu}, \Phi, \xi^0, \tilde{\xi}_0)$

Table 3.1: Supermultiplets arising from the closed sector of Type IIA compactified on a Calabi–Yau three-fold. Note that the bosonic content is given in terms of *real* fields.

decomposition into harmonic forms using Hodge's decomposition[2] (2.16) :

$$C_0 = A^0(x), \qquad C_3 = A^a(x) \wedge \omega_a + \xi^K(x)\alpha_K - \tilde{\xi}_K(x)\beta^K, \tag{3.4}$$

where $\{\omega^a\}$ is the basis of $H^{1,1}(Y_3)$ already used in section 2.4 and $\left\{\alpha_K, \beta^K\right\}$, $K = (0, \alpha) = 0, 1, \cdots, h^{2,1}$ forms a symplectic basis basis of $H^3(Y_3)$, satisfying $\int_Y \alpha_K \wedge \beta^L = \delta^L_K$. The real fields appearing in this decomposition can be combined into $\mathcal{N} = 2$ supermultiplets, as summarised in table 3.1. Note that in four dimensions, one can use Poincaré duality to transform the 2-form $B_{\mu\nu}$ into a 1-form, giving rise to an additional hypermultiplet.

In the absence of fluxes, there is no potential and the target space splits into a direct product $\mathcal{M}_{\text{hyper}} \times \mathcal{M}_{\text{vector}}$. As a consequence of General Lesson 2.2, $\mathcal{N} = 2$ supersymmetry imposes constraints on the two factors and therefore their Kähler potentials. For a Calabi–Yau three-fold, it can be shown that they take the form [115]

$$K_{CS} = -\ln\left(i\int_{Y_3} \Omega \wedge \bar{\Omega}\right), \qquad K_K = -\ln\left(\frac{4}{3}\int_{Y_3} J \wedge J \wedge J\right). \tag{3.5}$$

In the absence of a superpotential, this totally fixes the effective field theory coming from the closed sector. We however want to now consider an orientifold, to generically get $\mathcal{N} = 1$ supersymmetry, which projects out some of the fields. The Kähler potential will therefore be inherited from the extended supersymmetry of the original theory and will be of the form (3.5).

In Type IIA, the orientifold projection is chosen to be a combination of world-sheet parity, the fermion number $(-1)^F$ and an anti-holomorphic involution[3] σ of the three-fold Y_3 [106]. The involution is a map that acts on the local coordinates of Y_3 as $\sigma : z^i \to \bar{z}^{\bar{\imath}}$, and globally acts on the Kähler and holomorphic forms as

$$\sigma^* J = -J, \qquad \sigma^*\Omega = \overline{\Omega}, \tag{3.6}$$

An invariance of the massless fields of table 2.1 under the orientifold projection requires the involution to act on them as

$$\sigma^* B_2 = -B_2 \qquad \sigma^*\Phi = -\Phi \qquad \sigma^* C_1 = -C_1 \qquad \sigma^* C_3 = C_3 \qquad . \tag{3.7}$$

[2] We follow the usual notation of [106, 107]

[3] An involution on X is a map $\sigma : X \to X$ satisfying $\sigma^2 = Id_X$.

One of the results of General Lesson 2.5 was that Kähler moduli arise from the complexified Kähler form $J_c = B_2 + iJ$ by expanding it on a basis of harmonic $(1,1)$-forms. The involution has no effect on this space, as there have the same numbers of holomorphic and anti-holomorphic coordinates, so we can separate the cohomology groups in even and odd subspaces $H^{1,1}(Y_3) = H^{1,1}_+(Y_3) \oplus H^{1,1}_-(Y_3)$ of respective dimensions $h^{1,1}_+$ and $h^{1,1}_-$. The expansion of J_c is therefore performed only over odd forms, and give rise to $h^{1,1}_- < h^{1,1}$ complex scalars $T^a = b^a + it^a$. The Kähler potential (3.5) for that sector is thus given by

$$K_K = -\ln\left(\frac{4}{3}\kappa_{abc}t^a t^b t^c\right), \qquad \kappa_{abc} = \int_{Y_3} \omega^-_a \wedge \omega^-_b \wedge \omega^-_c \tag{3.8}$$

where κ_{abc} is called the triple intersection number of Y_3.

The hypermultiplet sector, on the other hand, can be shown to be summarised by the reduction of the complexified 3-form [106]

$$\Omega_c = C_3 + 2i\mathrm{Re}(\mathcal{C}\Omega), \qquad \mathcal{C} = \sqrt{8}e^{-(\Phi+K_K)/2}, \tag{3.9}$$

and an argument similar to that of the Kähler sector can made, as we can also separate H^3 into an even and odd part, each of dimension $h^{2,1}+1$. This can be seen from the Hodge diamond of a Calabi–Yau three-fold, see appendix A, and using equation (2.16). Ω_c being even under the involution, we deduce that the orientifold projection projects out half of the $h^{2,1}+1$ $\mathcal{N}=2$ hypermultiplets, containing two complex scalars each. The surviving scalars reorganise with their fermionic partners to form $h^{2,1}+1$ $\mathcal{N}=1$ supermultiplets. A careful analysis reveals that the Kähler potential for that sector is given by the simple formula

$$K_{CS} = -2\ln\mathcal{V}' - \ln(S+\overline{S}) + \mathcal{O}(e^{-u}), \tag{3.10}$$

with $S = s + i\sigma$ a complex scalar field coming from a hypermultiplet and

$$\mathcal{V}' = \frac{d_{\alpha\beta\gamma}}{6}v^\alpha v^\beta v^\gamma, \qquad u_\alpha = \frac{\mathcal{V}'}{\partial v^\alpha}. \tag{3.11}$$

v^α are fields related to the local coordinates of $\mathcal{M}_{CS}$. From now on, we will work in the large complex structure limit $u^\alpha \gg 1$ and we can therefore neglect the exponential corrections. One may wonder about the striking resemblance between the first term of K_{CS} and equation (3.8). This is in fact not an accident: We have seen that Type IIA string theory compactified on a circle is dual to its Type IIB counterpart compactified on a circle of inverse radius. A generalisation of this duality, called *mirror duality*, relates Type IIA compactified on a Calabi–Yau Y_3 to Type IIB compactified on a different Calabi–Yau Y_3'. This manifold, called the *mirror manifold* of Y_3, has the property that its Hodge diamond is the mirror of the original manifold: $h^{p,q}(Y_3) = h^{3-p,q}(Y_3')$. This has for effect to exchange the Kähler and complex structure moduli, and $\mathcal{V}'$ can be interpreted as the volume of Y_3'.

At this stage of the discussion, we have not yet turned on any fluxes, and the moduli remain massless. The fluxes can also be written in terms of harmonic p-forms compatible with the orientifold projection:

$$\langle H_3\rangle = h_\alpha\alpha^\alpha + h_0\beta^0, \qquad \langle F_2\rangle = -m^a\omega^-_a, \qquad \langle F_4\rangle = e_a\tilde{\omega}^a_+, \tag{3.12}$$

where $H_3 = dB_3$, $F_i = dC_i$, and we used Poincaré duality to define a basis $\left\{\tilde{\omega}_+^a\right\}$ of $H_+^{2,2}$. The expectation value is taken over the coordinates of the compact manifold, and the flux numbers h^α, m^a, e_0 are integers by General Lesson 2.3. The superpotential is then obtained by reducing the Type IIA ten dimensional action. We note that the presence of RR fluxes, the Kalb-Ramond field acquires a mass m_0 through a Stückelberg-like mechanism and one has to reduce the ten dimensional action of *massive* supergravity. Its precise form is not relevant to our purpose, and the resulting superpotential is found to be given by [106]

$$W = e_0 + ih_0 S - ih_\alpha U^\alpha + ie_a T^a - \frac{1}{2} q_{ab} T^a T^b + i\frac{m_{abc}}{6} T^a T^b T^c. \tag{3.13}$$

where we defined $q_{ab} = \kappa_{abc} m^c$, $m_{abc} = m_0 \kappa_{abc}$. To summarise, we have found that

General Lesson 3.1. *The compactification of Type IIA supergravity on a Calabi–Yau orientifold Y_3 gives rise to $h_-^{1,1}$ Kähler moduli $T^a = b^a + it^a$ and additional $h^{2,1} + 1$ complex scalar fields $U^i = u^i + i\nu^i$ and $S = s + i\sigma$ coming from the dilaton and complex structure moduli. The full Kähler potential associated to the scalar manifold $\mathcal{M}$ in the large complex structure limit is given by*

$$K = K_{CS} + K_K = -\ln\left(S + \overline{S}\right) - 2\ln \mathcal{V}' - \ln\left(\frac{4}{3}\kappa_{abc} t^a t^b t^c\right) \tag{3.14}$$

with $\mathcal{V}'$ is defined in equation (3.11)*. Turning on fluxes, a perturbative superpotential*

$$W = e_0 + ih_0 S - ih_\alpha U^\alpha + ie_a T^a - \frac{1}{2} q_{ab} T^a T^b + i\frac{m_{abc}}{6} T^a T^b T^c \tag{3.15}$$

is generated.

Additionally, there are non-perturbative effects, coming in particular from Euclidean 2-branes[4] wrapping even three-cycles. The induced superpotential looks like

$$W_{NP} = \sum_I A_I e^{-a_0^I - a_\alpha^I U^\alpha} \tag{3.16}$$

with I running over the different instantons, while the constants a_0^I, a_λ^I refer to combinations of 3-cycles wrapped by the instanton. Notice that in the absence of fluxes, the imaginary part of the moduli have a discrete shift symmetry, and are named axions after their QCD counterpart proposed by Peccei and Quinn [116]. In the presence of fluxes, this shift symmetry is broken by the perturbative potential. If the real part of the moduli has large enough values, the non-perturbative terms are exponentially suppressed and the perturbative superpotential dominates.

[4] A Euclidean 2-brane, named E2 brane, is a membrane filling only the compact dimensions, and looking like an instanton from the four dimensional point of view.

3.2 Axion Monodromy and Backreaction

In this section we study the backreaction on the axion target space in different models of Type IIA string theory. As reviewed in the last section, flux compactification on a Calabi–Yau orientifold includes two types of axions, the RR axions ImU^α coming from the complex structure sector, as well as the NS axions ImT^a from the Kähler sector. Turning on RR fluxes induces a potential for the NS axions and vice versa, as can be seen from (3.15) and the definition of the fluxes (3.12).

The superpotential (3.15) makes them so-called monodromy axions, as the original shift symmetry is now completely broken. The parameter N introduced at the beginning of the chapter describes in this case the number of times they traverse their original period and is measured by their vev. Therefore, in order to take into account the effect of N on the target space metric, we need to study the backreaction on the moduli of the axion vev.

Before quantifying this backreaction for various models, we expand on the methodology used throughout this chapter. The notion of a bound on a single field variation—tracing a path in the target space—is not well-defined in the presence of multiple fields since such a target manifold supports a path of infinite length. In principle, it might be possible to formulate a constraint on the volume of $\mathcal{M}$, but not knowing the full structure of the target manifold, we adopt a different approach: Once a potential has been turned on, we can consider the dimensionality of the vacuum space of the theory rather than the field space $\mathcal{M}$.

At the non-perturbative level, one expects all fields to get a potential and therefore the space of vacua is a collection of points (if not empty). We can also consider the perturbative vacuum space which can be continuous, but still typically has a smaller dimensionality than the target space space $\mathcal{M}$.

Within this framework there are two natural ways to identify a one-dimensional submanifold on which we can test field excursions: The first is the lightest direction in field space, and can be thought of in Wilson's approach as integrating out all the heavier fields to obtain an EFT for a single field. One can then use the knowledge of its UV origin to see how the effective field range is constrained, and has been studied in the context of axion alignment in [42].

Conversely, another possibility is to drop the requirement for one field to lie in the vacuum, displace it by hand away from its minimum, and ask how far one can go. This is the approach we shall adopt in this section, with the field displaced away from its vacuum being the massive axion. In this setting, one should not think of integrating out the other fields to get an EFT with only field, but rather keeping them in the theory and carefully tracking their continuously changing minima as a function of the displaced field, capturing the backreaction effect. If the other fields are much more massive than the field which is displaced, they will not change much over the traveled distance, but this does not have to be imposed for the procedure to be well-defined. We note that there might be cases where after being displaced far away from its stable vacuum, one reaches a point where the minima for the other fields disappear and the whole system becomes unstable. In this case, it undergoes a phase transition to a different, more stable vacuum. While we will show examples of such behaviour, they will not be the of key interest in our study, and we shall focus on cases where the other fields have well-defined minima.

For a space spanned by local coordinates given by axions ν^i and their moduli partners c^i, and endowed with a metric g_{ij}, we will denote the path γ taken during the excursion of an axion as the embedding $\gamma: \ \rho \to \nu^i(\rho)$, whose proper path length is then given by

$$\Delta\phi \equiv \int_\gamma \sqrt{g_{ij}\frac{\partial \nu^i}{\partial \rho}\frac{\partial \nu^j}{\partial \rho}} d\rho. \tag{3.17}$$

Here, ρ takes the role of a world-line element along the path γ and, as discussed above, the backreaction effect of the axion vev is is captured by studying how the vev of other fields react to a displacement of ρ, that is to say they satisfy the following system of equations, we will call the *stabilisation equation*:

$$\frac{\partial V}{\partial c^i} = 0, \qquad \frac{\partial V}{\partial \psi^j} = 0, \tag{3.18}$$

where ψ^j are the directions orthogonal to the combination ρ. After imposing these equations, a key point is that the metric g_{ij} is a function of ρ, and this has to be taken into account when computing the proper length. Applying this to the Type IIA string theory setting we will be interested in cases where the path in (axion) field space is along a certain linear combination of fields

$$\rho = \sum_i h_i \, \nu^i. \tag{3.19}$$

In such cases, the integral (3.17) simplifies and is written only in terms of the inverse metric and the coefficients h_i related to flux number in our setup:

$$\Delta\phi = \int_{\rho_i}^{\rho_f} \left(h_i g^{ij} h_j\right)^{-\frac{1}{2}} d\rho. \tag{3.20}$$

This is seen to be the integration of the canonical normalisation factor[5] for the field ρ from its initial values ρ_i to its final one ρ_f.

General Lesson 3.2. *The backreaction of an axion vev on the target manifold is captured by studying the stabilisation equations* (3.18) *on the vacuum space. The backreaction introduces a dependency of the axions in the metric, which in turn modifies the proper field length, given in Type IIA flux compactification by an equation of the form* (3.20).

3.2.1 Ramond-Ramond Axions

We begin our analysis with the displacement of a massive combination of RR axions from their minima. We will initially study a simplified version of General Lesson 3.1 where each

[5]Note that in this section we work for convenience in conventions where a real scalar field has canonical kinetic terms $(\partial\phi)^2$, with no $\frac{1}{2}$ prefactor.

sector (complex structure, Kähler, and dilaton) have only one representative. This captures the behaviour of the system under a universal scaling of the moduli, and can be thought of as restricting the moduli values to be equal. We will see that the important physics is already manifest in this simple setting. In section 3.2.1 we will then generalise the results to realistic Calabi–Yau systems of moduli in the RR axions sector. In section 3.2.1 we further generalise the setting to the case of a twisted torus.

Single Field Models

The starting point is the model studied in [87], which consists of a simplified version of General Lesson 3.1

$$K = -\ln s - 3\ln u - 3\ln t, \qquad W = e_0 + ih_0 S - ih_1 U + \frac{i}{6} m T^3, \tag{3.21}$$

$$S = s + i\sigma, \qquad U = u + i\nu, \qquad T = t + iv. \tag{3.22}$$

There are two simplifications which enter this construction. The first is that, as discussed above, we have taken only a single modulus in the Kähler and complex-structure moduli sector. The second simplification is that at this point we have turned off some fluxes, but we shall come back to the case with all fluxes shortly. From (3.21), it is easy to see that there is one combination ρ of RR axions which becomes massive due to the fluxes and therefore suffers a monodromy effect

$$\rho = e_0 - h_0\sigma + h_1\nu. \tag{3.23}$$

From General Lesson 3.2, the backreaction effect of this axion combination is encoded in the stabilisation equations (3.18), which in this case reduces to solving

$$\partial_T V = \partial_u V = \partial_s V = 0 \tag{3.24}$$

as a function of the vev of ρ. Here V is the scalar potential descending from the superpotential (3.21) using the supergravity formula (2.11). We do not impose the stabilisation equations for the two axion combinations of σ and ν, as one combination is perturbatively massless and satisfies them trivially, while the other is the one we would like to displace from its minimum.

The solution to these equations was presented in [87] and reads

$$s = \alpha\frac{\rho}{h_0}, \qquad u = -3\alpha\frac{\rho}{h_1}, \qquad t = 1.96\left(\frac{\rho}{m}\right)^{\frac{1}{3}}, \qquad v = 0, \tag{3.25}$$

with $\alpha \simeq 0.38$. Note that this solution is only valid for sufficiently large values of ρ. In fact the solution does not flow to a physical minimum for any value of ρ, which can be seen by noting that there is no physical supersymmetric vacuum for the system (due to the restriction on the superpotential). It nonetheless serves as a useful example capturing the axion backreaction, as the crucial point is that the axion field space metric $\sqrt{g_{\rho\rho}} \sim s^{-1} \sim \rho^{-1}$ depends on the axion combination. This means that the metric on the field space of the axion is modified so that the canonically normalised field distance is only logarithmic in the vev of ρ [87]. More precisely we obtain from (3.20)

$$\Delta\phi = \int_{\rho_i}^{\rho_f} \frac{1}{2}\left((h_0 s)^2 + \frac{1}{3}(h_1 u)^2\right)^{-\frac{1}{2}} d\rho \simeq 0.7 \ln\left(\frac{\rho_f}{\rho_i}\right). \tag{3.26}$$

There are two qualitative features of (3.26) to highlight: The most important is that the proper field distance is only logarithmic in the axion variation, which by comparing (3.2) and (3.3) is the type of behaviour we expect from moduli fields and not axions. In this sense, once backreaction is accounted for, inducing a superpotential makes an axion behave like a modulus. The second important feature is that the prefactor 0.7 does not depend on any flux parameter. There is therefore no possible way to adjust the model to make the field excursion large while keeping the logarithmic term small.

While formally the field distance is unbounded, as discussed earlier, it is not likely to be possible to obtain super-Planckian displacements in such a setting. Indeed it is precisely this logarithmic behaviour of moduli which is attempted to be avoided when working with axions. An exponentially large variation of the moduli is difficult to support in a controlled EFT and from (3.25) we see that indeed the moduli scale exponentially with the proper axion field distance.

Let us consider the generality of the result of $\Delta\phi \sim \ln\rho$. To do this, it is useful to notice that the equations encoding the backreaction (3.24) are invariant under a rescaling of the variables

$$\rho \to \lambda\rho, \qquad s \to \lambda s, \qquad u \to \lambda u, \qquad T \to \lambda^{\frac{1}{3}} T. \tag{3.27}$$

When we solve for the moduli in terms of ρ and the fluxes, the only parameter carrying a non-trivial weight under rescaling is the field ρ, and therefore s and u, carrying weight one, must be proportional to it. This argument is sufficient to establish the behaviour (3.26) up to a constant of proportionality factor. Now consider a general Calabi–Yau compactification with an arbitrary number of complexified moduli. The system is still invariant under the scale symmetry (3.27), with the weight choices being obvious due to the logarithmic behaviour of the Kähler potential (3.14). Therefore, there must be an overall proportionality of all the u_α moduli to ρ which is sufficient to establish the logarithmic behaviour of (3.26). The proportionality factor 0.7 in the universal behaviour (3.26) can however in principle depend on dimensionless fluxes, such as the h_i, in the case of an arbitrary Calabi–Yau manifold. In section 3.2.1, we study this problem and show that it is flux independent and of order one.

For the superpotential (3.21), the only flux parameter which carries weight under the scaling symmetry (3.27) is e_0. This case is particularly simple, since it can just be absorbed into the definition of ρ. Yet, if we consider the most general superpotential for Type IIA on a Calabi–Yau with one field of each type, there are additional fluxes breaking the scaling symmetry (3.27):

$$W = e_0 + ih_0 S - ih_1 U + ie_1 T - qT^2 + \frac{i}{6} m T^3. \tag{3.28}$$

In the spirit of dimensional analysis, we can restore the scaling symmetry by assigning a spurious weight to each additional fluxes

$$e_1 \to \lambda^{\frac{2}{3}} e_1, \qquad q \to \lambda^{\frac{1}{3}} q\,. \tag{3.29}$$

The introduction of those additional parameters implies a number of important changes: First, as we will show, the theory now develops physical minima. Secondly, they imply a modification to the general argument leading to the behaviour (3.26) to account for the dimensionful fluxes. The solutions for the moduli in terms of ρ now fall into two classes: those for which the limit $\{e_1, q\} \to 0$ reduce to the previous solution, and those breaking down in this limit.

The solutions will be studied in detail soon, but let us first make some general statements about the first class, for which $\{e_1, q\} \to 0$ reduces to (3.26). These solutions must have some functional form for the moduli in terms of ρ, such that when ρ is larger than some critical value $\rho_{\rm crit}$ set by the magnitude of the fluxes which break the symmetry

$$\rho_{\rm crit} \sim \left(e_1^{\frac{3}{2}} + q^3\right), \tag{3.30}$$

they converge to (3.25). The first thing to observe is therefore that by taking large fluxes, one can delay the onset of the scaling behaviour (3.25) arbitrarily far in ρ distance. The fluxes therefore naively seem to shield the moduli from the axion vev backreaction. However, even though the variation in ρ can be extended parametrically far through this method, it does not necessarily imply an arbitrarily large proper field distance.

The rough argument is that if we assume that the vevs of the moduli controlling the axion field space metric—s and u—remain approximately constant up to $\rho \sim \rho_{\rm crit}$, then at $\rho_{\rm crit}$, their values go like $s(\rho_{\rm crit}) \sim \rho_{\rm crit}$ and this will be their approximate value over the regime $\rho \lesssim \rho_{\rm crit}$. The proper axion field distance $\Delta\phi$ up to $\rho_{\rm crit}$ will therefore behave as

$$\Delta\phi \sim \frac{\Delta\rho}{\{s,u\}|_{\rho_{\rm crit}}} \sim \frac{\rho_{\rm crit}}{\rho_{\rm crit}} \sim 1. \tag{3.31}$$

While $\rho_{\rm crit}$ may be arbitrarily large, controlling the backreaction for arbitrarily large field distances, its value cancels in the proper field distance. The argument presented is quite general but imprecise, and the rest of this section will be dedicated to essentially filling in the missing details. There will be two steps to improving the argument just presented for the structure of the proper length field variation: we would like to make it more quantitative by keeping track of the relevant coefficients, and then determine the actual values of these coefficients.

It is useful to introduce some coordinate redefinitions. We first absorb some of the fluxes into the definition of the moduli and other fluxes

$$\begin{gathered} \tilde{s} = h_0 s, \qquad \tilde{u} = -h_1 u, \qquad \tilde{T} = T m^{\frac{1}{3}}, \\ \tilde{e}_1 = e_1 m^{-\frac{1}{3}}, \qquad \tilde{q} = q m^{-\frac{2}{3}}, \end{gathered} \tag{3.32}$$

and introduce the flux combination

$$f \equiv -\tilde{e}_1 - 2\tilde{q}^2\,. \tag{3.33}$$

An important role in our analysis will be played by the particular moduli values which correspond to a physical supersymmetric vacuum of the system. In this vacuum, the fields take the values

$$\tilde{s}_0 = \frac{\tilde{u}_0}{3} = \frac{\tilde{t}_0^3}{15} = \frac{2}{9}\sqrt{\frac{10}{3}} f^{\frac{3}{2}}, \qquad \rho_0 = \frac{2}{3}\tilde{q}\left(3f + 2\tilde{q}^2\right), \qquad \tilde{v}_0 = -2\tilde{q}\ , \tag{3.34}$$

where we take $h_0 > 0$, $h_1 < 0$ and $m > 0$. In analysing the potential it is convenient to shift the axion definitions by their supersymmetric minimum values

$$\rho' \equiv \rho - \rho_0\ , \qquad v' \equiv \tilde{v} - \tilde{v}_0\ . \tag{3.35}$$

The reason for introducing those shifted and rescaled quantities is that, recasting the potential in terms of those new variables, one finds that, up to an overall constant factor, it depends explicitly on the single flux parameter f

$$V = h_0 h_1^3 m \tilde{V}\left(\tilde{s}, \tilde{u}_1, \tilde{t}, v', \rho', f\right). \tag{3.36}$$

Therefore solutions to the equations (3.24) will depend on only ρ' and f. The dependence of the potential on only one flux parameter f can be understood as follows: Three flux parameters in the superpotential can be absorbed into a rescaling of the moduli as in (3.32), leaving three parameters e_0, e_1 and q. As we saw in section (3.1), the theory respected two shift symmetries, one for the RR axions and one for the NS axion broken by fluxes for which e_0 and q play the role of order parameters. They can also be assigned a spurious shift transformation which can be used to absorb two more flux parameters, leaving only one flux parameter, which is f in our notation.

Coming back to the quantity of interest to us, the proper distance traversed by the massive axion field ρ up to its critical value as given in (3.20), we find that

$$\Delta\phi = \int_{\rho_i}^{\rho_f} \left(h_i g^{ij} h_j\right)^{-\frac{1}{2}} d\rho = \int_0^{\rho'_{\rm crit}} \frac{1}{2}\left(\tilde{s}^2 + \frac{1}{3}\tilde{u}^2\right)^{-\frac{1}{2}} d\rho' = G\left(\frac{\rho'_{\rm crit}}{f^{\frac{3}{2}}}\right) = k, \tag{3.37}$$

where G is some arbitrary function depending only on the shown ratio of $\rho'_{\rm crit}$ and f, and k is a flux-independent number. The important non trivial step is the third equality: while $\tilde{s}$ and $\tilde{u}$ are some complicated functions of ρ', they have to scale properly under the transformation (3.27), under which f carries the weight $\frac{2}{3}$. As $\Delta\phi$ must have a trivial weight, it can only be constructed out of the unique dimensionless combination of $\rho'_{\rm crit}$ and f. Finally, $\rho'_{\rm crit}$ must be proportional to $f^{\frac{3}{2}}$ as it is the only parameter breaking the original symmetry. We are thus left with a flux independent coefficient which is expected to be of order one.

To find its precise value, we proceed to an analysis of the structure of the scalar potential in detail to determine the precise value of r in (3.37). Consider the following combination of (3.24)

$$-\frac{3}{4}e^{-K}\left(3\tilde{s}\frac{\partial\tilde{V}}{\partial\tilde{s}} - \tilde{u}\frac{\partial\tilde{V}}{\partial\tilde{u}}\right) = \left(3\tilde{s} - \tilde{u}\right)\left(6\tilde{s} - \tilde{t}^3 + 2\tilde{u}\right) = 0\ . \tag{3.38}$$

The factorisation shows that turning points of the potential split into two branches

$$\begin{aligned} \text{Branch 1}: \qquad & \tilde{u} = 3\tilde{s}, \\ \text{Branch 2}: \qquad & \tilde{u} = -3\tilde{s} + \frac{1}{2}\tilde{t}^3. \end{aligned} \tag{3.39}$$

From (3.34), we see that only the first branch supports a supersymmetric minimum, and only for $f > 0$. However a turning point of the potential occurs for both branches, for either sign of f, at the point $\rho' = v' = 0$. These correspond to non-supersymmetric minima in general. Analysing the Hessian at these turning points shows that for the supersymmetric turning point of branch 1 there is one negative eigenvalue, while for the non-supersymmetric minima of branch 2 all the eigenvalues are positive. In the case of a negative eigenvalue, it lies above the Breitenlohner-Freedman bound and so all these turning points are stable minima. These minima will form the starting points for our axion excursions in ρ'. As we move away from the minimum in ρ', the stability with respect to the other directions continues to hold. For clarity, we will henceforth restrict ourselves to $f > 0$ so that the minimum of branch 1 is supersymmetric, while the branch 2 minimum is non-supersymmetric.[6]

As we move ρ away from its supersymmetric minimum the other moduli will adapt according to the stabilisation equations (3.24). Let us consider branch 1 of equation (3.39). The system is quite complicated but we could solve it numerically and match the result onto a function. We find that to good accuracy, the following function matches the numerical analysis

$$\tilde{s} = \left[(\alpha\rho')^4 + \beta f^3 (\rho')^2 + \tilde{s}_0^4 \right]^{\frac{1}{4}} \ . \tag{3.40}$$

Here α is as in (3.25), $\alpha \simeq 0.38$, and $\beta \simeq 0.05$.[7] This shows the interpolating behaviour between the supersymmetric minimum value for ρ' and the large vev limit (3.25). We can therefore define ρ'_{crit} as the value of ρ' for which the first term in (3.40) becomes equal in magnitude to the sum of the other two, i.e.

$$\rho'_{\text{crit}} \simeq 1.7 f^{\frac{3}{2}} \ . \tag{3.41}$$

Evaluating the critical proper field excursion by plugging back this value in (3.37) yields

$$\Delta\phi \simeq 0.9. \tag{3.42}$$

This gives the precise numerical evaluation of the general structure discussed previously. The key result is that the canonically normalised field distance is independent of any fluxes and is of order one.

[6] We have also analysed the cases for $f < 0$. We find similar behaviour with the only key difference being that for this sign of flux excursions for large ρ' along branch 2 destabilise the potential such that the turning points in the moduli disappear and the theory then undergoes a phase transition to a new vacuum. While interesting, this limits the excursion distances in field space and is not the focus of this work.

[7] For $f < 0$ we find a similar fit but with $\beta \simeq 0.03$.

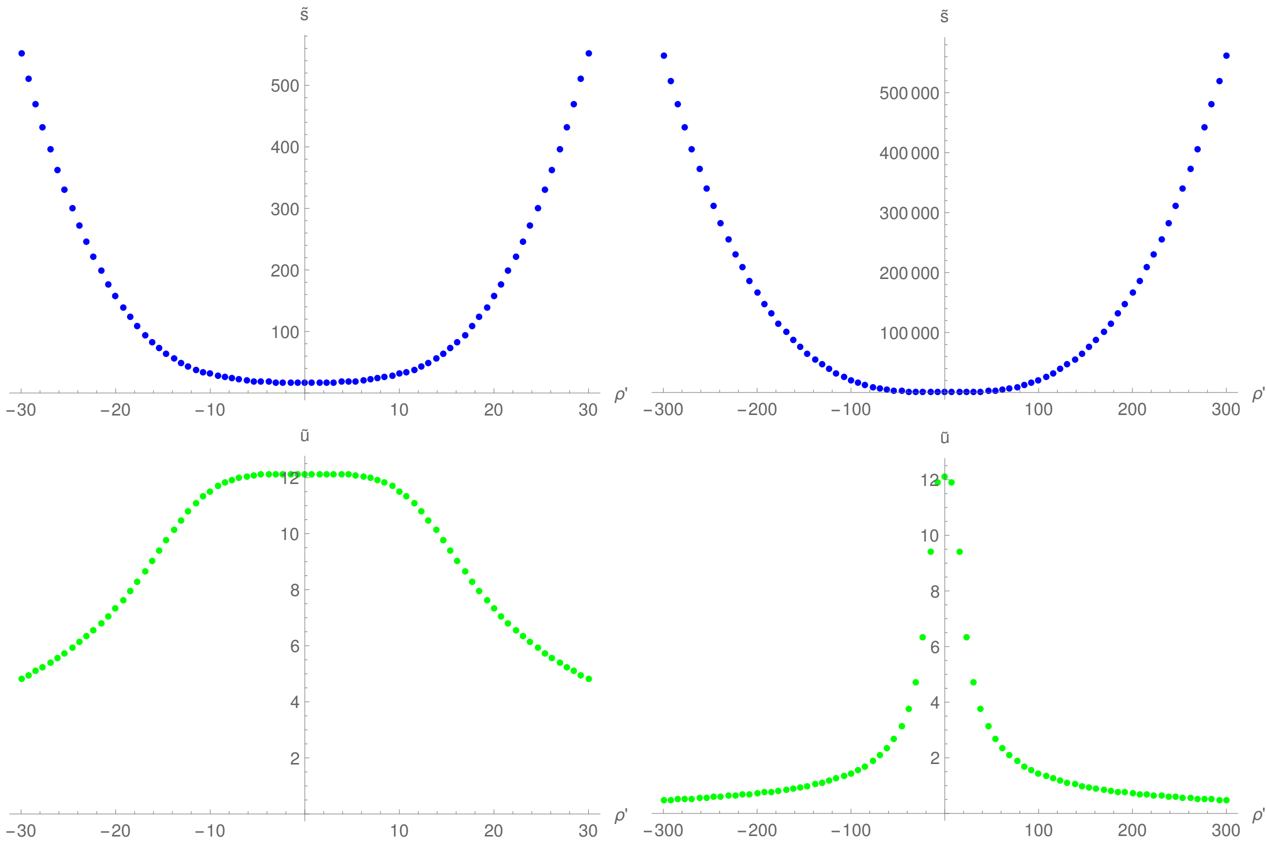


Figure 3.1: Plots showing the moduli $\tilde{s}$ and $\tilde{u}$ as a function of $\rho' = \rho - \rho_0$ for displacement of ρ' along branch 2 of (3.39). The plots are for flux value $f = 6$ and show the same function over two different ranges so as to show the behaviour up to $\rho_{\rm crit} \simeq f^{\frac{3}{2}} \simeq 15$, and the asymptotic behaviour of $\tilde{u}$.

We performed a similar evaluation for the non-supersymmetric branch 2 in (3.39). While we did not derive an analytic expression for the moduli as a function of ρ', we studied it numerically and found that for $\tilde{s}$, the large ρ' scaling regime takes the form

$$\tilde{s} \simeq 1.7 \left(\frac{\rho'}{f}\right)^3 . \tag{3.43}$$

The critical value of ρ' where this regime begins is again around $f^{\frac{3}{2}}$, while the modulus $\tilde{u}$ instead asymptotes to zero for values of $\rho' \gg \rho_{\rm crit}$, as shown in figure 3.1. We can approximate to a decent accuracy the distance traveled by the canonical field up to $\rho'_{\rm crit}$ by taking $\tilde{s}$ and $\tilde{u}$ to be constants over that distance, giving $\Delta\phi \simeq 0.5$.

Note that because $\tilde{s}$ scales with a cubic power of ρ' after the critical point, the distance up to $\rho' \to \infty$ is not even logarithmically divergent, but finite. It gets cut off very quickly, giving an increase in $\Delta\phi$ of order a percent.

General Lesson 3.3. *The stabilisation equations defined by the super- and Kähler potential* (3.21) *are homogeneous under the rescaling* (3.27), (3.29). *Upon field redefinitions, they reduce to a one parameter system. It is possible to show that the proper field distance up to a certain critical value are flux independent and of order one. After the critical value, the proper field distance receives corrections that are at most logarithmic in the axion vev.*

The cancellation of the flux parameters of the proper length in General Lesson 3.3 is strikingly similar to a flux cancellation found in the case of axion alignment in [87]. There is in fact a natural relation between the two: We can approximate the moduli to be independent of ρ' for values $\rho' < \rho'_{\rm crit}$ and write the distance that the axion traverses before strong backreaction as

$$\Delta\phi \sim \rho'_{\rm crit} f_{\rho'} \sim N f_{\rho'} \qquad N = f^{\frac{3}{2}} \tag{3.44}$$

Here $f_{\rho'}$ is the normalisation factor appearing in the proper field distance (3.20) evaluated at $\rho' < \rho'_{\rm crit}$, which can be defined as the axion decay constant for ρ'.

Now consider an axion alignment scenario between two axions, labeled by ν_1 and ν_2. The effective massless axion combination ψ appears in two instantons of the form (3.16), one of them with an effective decay constant which, before accounting for backreaction effects, is enhanced by N so that $f^1_\psi \sim N f_{\nu_2}$. For the other instanton, the effective decay constant is not enhanced $f^2_\psi \sim f_{\nu_2}$. Here f_{ν_i} are the fundamental decay constants for the two axions. After accounting for backreaction, as in [87], one finds that $f_{\nu_2} \sim \frac{f_{\nu_1}}{N}$ so that f^1_ψ does not enhance.

This scaling with the parameter N is reminiscent of that of (3.44), and indicates an identification of f^1_ψ with $\Delta\phi$, and the second axion decay constant with the axion decay constant of ρ', and we therefore have the map

$$f^1_\psi \leftrightarrow \Delta\phi, \qquad f^2_\psi \leftrightarrow f_{\rho'}. \tag{3.45}$$

This identification can be thought as focusing on the origin of the potential in an axion alignment scenario, where in the large axion constant decay limit, the potential looks quadratic rather that sinusoidal. This is regime is therefore similar to that of axion monodromy. On top of the quadratic potential, there is an oscillating term coming from the sub-leading instantons, and after N such periods we reach a critical axion value. In the alignment scenario, this is where the quadratic approximation breaks down and the periodic nature of the system kicks in to censure the excursion distance, in the monodromy setting at the same axion value instead the strong backreaction kicks in and serves as the cutoff mechanism. In both cases although there are N oscillations before the cutoff mechanism, the oscillation period scales as $\frac{1}{N}$ thereby ensuring a cancellation in the proper excursion length.

Calabi–Yau Models

So far we studied the simplest of models, where we considered only one moduli representative of each sector. In this section we study how the result obtained for this toy model transpose to a more involved Calabi–Yau with more than three moduli. We start by considering the same toy model, but augment the complex-structure sector with an additional field. One of the simplest extensions is to consider the mirror of the $\mathbb{P}_{[1,1,2,2,6]}$ Calabi–Yau studied in [115], see [106, 117] for details of the mirror map in Type IIA orientifolds:

$$\begin{aligned} K &= -\log s - 2\log\sqrt{u_1}\left(u_2 - \frac{2}{3}u_1\right) - 3\log t\ , \\ W &= e_0 + ih_0 S - ih_1 U_1 - ih_2 U_2 + ie_1 T - qT^2 + \frac{i}{6}mT^3\ . \end{aligned} \tag{3.46}$$

In a similar vein to the previous case, we want to displace the massive axion combination

$$\rho = e_0 - h_0 s + h_1\nu_1 + h_2\nu_2. \tag{3.47}$$

The proper length (3.20) for the combination is easily found to be

$$\begin{aligned} \Delta\phi &= \int_{\rho_i}^{\rho_f}\sqrt{\frac{3}{2}}\left[6h_0^2 s^2 + 6h_1^2 u_1^2 + 8h_1h_2u_1^2 + h_2^2\left(4u_1^2 - 4u_1u_2 + 3u_2^2\right)\right]^{-\frac{1}{2}} d\rho, \\ &= \int_{\rho_i}^{\rho_f}\sqrt{\frac{3}{2}}\left[6\tilde{s}^2 + \left(6 + 8r + 4r^2\right)\tilde{u}_1^2 - 4r\tilde{u}_1\tilde{u}_2 + 3\tilde{u}_2^2\right]^{-\frac{1}{2}} d\rho, \end{aligned} \tag{3.48}$$

where we defined the rescaled fields $\tilde{s} = h_0 s$, $\tilde{u}_i = -h_i u_i$, and the ratio $r = \frac{h_2}{h_1}$.

An important qualitative difference from the proper length of General Lesson 3.3 is that there is now a flux parameter r in the expression. This parameter does not carry any weight under the symmetry (3.27) and may therefore appear arbitrarily in the evaluation of the $\Delta\phi$. Moreover, it is also not possible to define new flux numbers such that the potential can be written with only a one-parameter explicit dependence as in (3.36). We instead find that it must also depend explicitly on r and we find ourselves in a two-parameter system:

$$V \propto \tilde{V}\left(\tilde{s}, \tilde{u}_1, \tilde{u}_2, \tilde{t}, v', \rho', f, r\right) \tag{3.49}$$

To see how the new parameter r affects the field distance, we need to proceed as before and study the stabilisation equations. The supersymmetric vacuum is now at

$$\begin{aligned} \tilde{s}_0 = \frac{3+2r}{3}, \qquad \left(\tilde{u}_1\right)_0 = \frac{3+2r}{6\left(1+r\right)}, \qquad \left(\tilde{u}_2\right)_0 = \frac{\tilde{t}_0^3}{15} = \frac{2}{9}\sqrt{\frac{10}{3}}f^{\frac{3}{2}}, \\ \rho_0 = \frac{2}{3}\tilde{q}\left(3f + 2\tilde{q}^2\right), \qquad \tilde{v}_0 = -2\tilde{q}m, \end{aligned} \tag{3.50}$$

where we defined ρ' and v' as in (3.35). The stabilisation equations (3.24) again exhibit a factorisation structure, but with now four different branches determining both $\tilde{u}_1$ and $\tilde{u}_2$ in terms of $\tilde{s}$ and $\tilde{t}$:

$$\begin{aligned}
&\text{Branch 1}: \quad && \tilde{u}_1 = \frac{3\tilde{s}}{3+2r}\,, \quad && \tilde{u}_2 = \frac{6\,(1+r)\,\tilde{s}}{3+2r},\\
&\text{Branch 2}: \quad && \tilde{u}_1 = \frac{3\tilde{s}}{3+2r}\,, \quad && \tilde{u}_2 = \frac{1}{3}\left(-\frac{6\,(3+r)\,\tilde{s}}{3+2r} + \tilde{t}^3\right),\\
&\text{Branch 3}: \quad && \tilde{u}_1 = \frac{-6\tilde{s}+\tilde{t}^3}{2\,(3+2r)}\,, \quad && \tilde{u}_2 = \frac{6\,(3+r)\,\tilde{s}+r\tilde{t}^3}{9+6r},\\
&\text{Branch 4}: \quad && \tilde{u}_1 = \frac{-6\tilde{s}+\tilde{t}^3}{2\,(3+2r)}\,, \quad && \tilde{u}_2 = -\frac{(1+r)\left(6\tilde{s}-\tilde{t}^3\right)}{3+2r}.
\end{aligned} \tag{3.51}$$

When lying on any of those branches, the dependence on r drops out in the remaining three equations (3.24) and are only function of $\tilde{s}$, $\tilde{t}$, v', ρ' and f, which means that the solutions for $\tilde{s}$ and $\tilde{t}$ will only depend on f and ρ'. Furthermore, once we impose a branch from (3.51), r happens to also drop out of the proper field length (3.48). As an example, the simplest case of branch 1, which does not involve $\tilde{t}$, yields

$$\Delta\phi = \int_0^{\rho'_{\text{crit}}} \frac{1}{4\tilde{s}} d\rho', \tag{3.52}$$

which reduces to the same expression as branch 1 for the one modulus case. After imposing branch 1 of (3.51), the equations determining $\tilde{s}$ are also the same as the one modulus case and the result is identical to (3.42).

Similarly, the case of branch 4 in equation (3.51) reproduces the one modulus branch 2 case. The other new branches for the two modulus case give different results, but the key property that the integrand in $\Delta\phi$ depends on only one flux parameter f is guaranteed by the previous reasoning using the scaling symmetry, and the result is flux independent.

In summary, we find that for this example of a more complicated Calabi–Yau setting with multiple complex-structure moduli, there is additional structure in the relations between the u_i and s, and an additional flux parameter in the potential. In the proper field length, this parameter however drops out and the result is qualitatively—and for some cases even quantitatively—the same as General Lesson 3.3.

We can also consider another Calabi–Yau whose complex-structure moduli Kähler potential is the mirror of the $\mathbb{P}_{[1,1,1,6,9]}$ Calabi–Yau studied in [118].

$$K = -\log s - 2\log\left(u_1^{\frac{3}{2}} - u_2^{\frac{3}{2}}\right) - 3\log t. \tag{3.53}$$

The massive combination we want to displace is the same as that of the previous model (3.47), and its proper path length again depends on the fluxes:

$$\Delta\phi = \int_{\rho_i}^{\rho_f} \sqrt{\frac{3}{4}}\left[3h_0^2 s^2 + 6h_1h_2u_1u_2 + h_2^2\sqrt{u_2}(2u_1^{\frac{3}{2}} + u_2^{\frac{3}{2}}) + h_1^2(u_1^2 + 2\sqrt{u_1}u_2^{\frac{3}{2}})\right]^{-\frac{1}{2}} d\rho \tag{3.54}$$

Here, the matter seems worst than before, as it depends not only on the ratio of the two fluxes but both of them explicitly. We are again led to study the stabilisation equations and define the rescaled fields and fluxes

$$\tilde{s} = h_0 s, \qquad \tilde{T} = T m^{\frac{1}{3}}, \qquad \tilde{e}_1 = e_1 m^{-\frac{1}{3}}, \qquad \tilde{q} = q m^{-\frac{2}{3}} \ , \tag{3.55}$$

and f as in (3.33). This model has again a supersymmetric vacuum, located at

$$\begin{gathered} \tilde{s}_0 = -\frac{h_1^3 + h_2^3}{3h_2^2}, \qquad (u_2)_0 = -\frac{h_1^3 + h_2^3}{3h_1^2}, \qquad (u_1)_0 = \frac{\tilde{t}_0^3}{15} = \frac{2}{9}\sqrt{\frac{10}{3}} f^{\frac{3}{2}}, \\ \rho_0 = \frac{2}{3}\tilde{q}\left(3f + 2\tilde{q}^2\right), \qquad \tilde{v}_0 = -2\tilde{q} \ . \end{gathered} \tag{3.56}$$

Note that the physical domain of fluxes is at $h_0 > 0$, $h_1 < 0$, $h_2 > 0$, and $|h_1| > |h_2|$. As before we shift the axions by their supersymmetric values (3.35) to define ρ' and v', and the ratio $r = \frac{h_2}{h_1}$. The solutions to the stabilisation equations (3.24) in that case only exhibit two physical branches

$$\begin{aligned} &\text{Branch 1}: \qquad u_1 = -\frac{3\tilde{s}}{h_1\left(1 + r^3\right)}, \qquad u_2 = -\frac{3r^2\tilde{s}}{h_1\left(1 + r^3\right)}, \\ &\text{Branch 2}: \qquad u_1 = \frac{6\tilde{s} - \tilde{t}^3}{2h_1\left(1 + r^3\right)}, \qquad u_2 = \frac{r^2\left(6\tilde{s} - \tilde{t}^3\right)}{2h_1\left(1 + r^3\right)}. \end{aligned} \tag{3.57}$$

Restricting to these branches leads to equations independent of the h_i that exactly match the corresponding equations for the one modulus case in the respective branches and the solutions for $\tilde{s}$ are once more the same, and an evaluation of the integrand (3.54) leads to a cancellation of the h_i fluxes. The result is once more flux independent and analogous to the one modulus case.

We have found the same behaviour also for other examples for Calabi–Yau Kähler potentials, and given that the cancellation of the fluxes in the final result appears to be very intricate, it seems reasonable to expect that there is an underlying reason or symmetry behind this which holds for any Calabi–Yau.

Twisted Torus Models

In the previous subsection, we studied a modification of the simple one modulus model (3.21) by considering additional RR axions, changing the structure of the Kähler potential. This changed the structure of the canonical field distance by introducing an additional independent flux parameter.

In this section, we go further and consider another possible modification of the one modulus model through a change of the superpotential which will again lead to a two parameters system, by considering a compactification of Type IIA string theory on a twisted torus[8], and

[8] More generally these can be considered as compactifications on a manifold with $SU(3)$-structure as in [119, 120]. In particular coset spaces are very tractable cases with few moduli.

consider the setup studied in [109]. The manifold has intrinsic torsion which means it has a set of non-closed 1-forms

$$d\eta^P = -\frac{1}{2} w^P_{MN} \eta^M \wedge \eta^N, \tag{3.58}$$

where $M = 1, .., 6$ are the six toroidal directions. The structure constants w^P_{MN} have the following properties

$$w^P_{MN} = w^P_{[MN]}, \qquad w^P_{PN} = 0, \qquad w^P_{[MN} w^S_{L]P} = 0, \tag{3.59}$$

where the last equation follows from the nilpotency of the exterior derivative d, and is called Jacobi identity. It is convenient to introduce labels for the non-vanishing components of the torsion

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \omega^1_{56} \\ \omega^2_{64} \\ \omega^3_{45} \end{pmatrix}, \qquad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} -\omega^1_{23} & \omega^4_{53} & \omega^4_{26} \\ \omega^5_{34} & -\omega^2_{31} & \omega^5_{61} \\ \omega^6_{42} & \omega^6_{15} & -\omega^3_{12} \end{pmatrix}. \tag{3.60}$$

The Jacobi identities imply the twelve constraints

$$\begin{aligned} b_{ij} a_j + b_{jj} a_i &= 0, \qquad i \neq j, \\ b_{ik} b_{kj} + b_{kk} b_{ij} &= 0, \qquad i \neq j \neq k, \end{aligned} \tag{3.61}$$

with indices $i, j = \{1, 2, 3\}$. The resulting superpotential takes the form [109]

$$\begin{aligned} W =& e_0 + i h_0 S + \sum_{i=1}^{3} \left[\left(i e_i - a_i S - b_{ii} U_i - \sum_{j \neq i} b_{ij} U_j \right) T_i - i h_i U_i \right] \\ & - q_1 T_2 T_3 - q_2 T_1 T_3 - q_3 T_1 T_2 + i m T_1 T_2 T_3 \ , \end{aligned} \tag{3.62}$$

and the Kähler potential is

$$K = -\log s - \sum_{i=1}^{3} \log u_i - \sum_{i=1}^{3} \log t_i. \tag{3.63}$$

We will restrict ourselves to models with only one modulus of each type, therefore considering

$$K = -\log s - 3 \log u - 3 \log t, \tag{3.64}$$

$$W = e_0 + i a l S - i b l U + i e_1 T - q T^2 + \frac{i}{6} m T^3 + a S T - b T U. \tag{3.65}$$

The key difference with the models studied previously is a coupling of the complex structure and dilaton sector to the Kähler sector. Due to this interaction, the two RR axions gain a mass for generic fluxes, and we will restrict ourselves to fluxes for which only one combination gains a perturbative mass, while the other remains massless. The massive direction yields a monodromy axion along which we can displace it and study its backreaction effect.

To do so, we have defined the NS flux numbers h_i to be of the form $h_0 = al$ and $h_1 = bl$ for some free parameter l constrained only by quantisation of the NS flux. Again, we shift the axions and fluxes to get rid of e_0 and q:

$$\begin{aligned} v &= v' - \frac{2q}{m}, \qquad \sigma = \sigma' + \frac{3e_0 m^2 + 6e_1 mq + 8q^3}{3am\,(lm-2q)}, \\ l' &= l - \frac{2q}{m}, \qquad e_1' = \frac{3e_0 m^2 + 3e_1 lm^2 + 6lmq^2 - 4q^3}{3lm^2 - 6mq}, \end{aligned} \tag{3.66}$$

and define the rescaled fields as

$$T' = \frac{\tilde{T}}{m^{\frac{1}{3}}}, \qquad U = -\frac{\tilde{l} m^{\frac{1}{3}} \tilde{U}}{b}, \qquad S' = \frac{\tilde{l} m^{\frac{1}{3}} \tilde{S}}{a}, \tag{3.67}$$

$$e_1' = \tilde{e}_1 m^{\frac{1}{3}}\ , \qquad l' = \frac{1}{\tilde{l} m^{\frac{1}{3}}}. \tag{3.68}$$

In those coordinates, the super- and Kähler potential (3.65) (up to an unimportant constant shift) take the form

$$K = -\log \tilde{s} - 3\log \tilde{u} - 3\log \tilde{t}, \tag{3.69}$$

$$W = i\left(\tilde{S} + \tilde{U}\right) + i\tilde{e}_1 \tilde{T} + \frac{i}{6}\tilde{T}^3 + \tilde{l}\tilde{T}\left(\tilde{S} + \tilde{U}\right). \tag{3.70}$$

In those variables, we easily see that there are only two independent flux parameters, $\tilde{l}$ and $\tilde{e}_1$, and that the massive RR axion combination is $\rho = \tilde{\sigma} + \tilde{\nu}$. Moreover we can see that in the torus limit $\tilde{l} \to 0$, the superpotential (3.70) goes to the one modulus model studied earlier[9].

The system has a supersymmetric vacuum, which in the torus limit is given by (3.34). For non-vanishing $\tilde{l}$, it was shown [109] that it is given by

$$\tilde{s}_0 = \frac{1}{3}\tilde{u}_0 = \frac{\tilde{v}_0 \tilde{t}_0}{\tilde{l}}, \qquad \tilde{t}_0^2 = \frac{15}{\tilde{l}}\tilde{v}_0\left(1 + \tilde{v}_0 \tilde{l}\right), \qquad \rho_0 = -\frac{9\tilde{v}_0 + 8\tilde{v}_0^2 \tilde{l} + 2\tilde{e}_1 \tilde{l}}{2\tilde{l}^2}, \tag{3.71}$$

and $\tilde{v}_0$ satisfies the cubic equation

$$160\tilde{v}_0^3 \tilde{l}^2 + 186\tilde{v}_0^2 \tilde{l} + 27\tilde{v}_0 + 6\tilde{e}_1 \tilde{l} = 0. \tag{3.72}$$

As before, the shifted axion field $\rho' = \rho - \rho_0$ proves to be a useful quantity when studying the stabilisation equations (3.24). One again finds that the solutions come in two different branches:

$$\begin{aligned} &\text{Branch 1}: \qquad \tilde{u} = 3\tilde{s}, \\ &\text{Branch 2}: \qquad \tilde{u} = \frac{-6\tilde{s} - 12\tilde{v}\tilde{l}\tilde{s} - 6\tilde{v}^2\tilde{l}^2\tilde{s} - 2\tilde{l}^2\tilde{s}\tilde{t}^2 + \tilde{t}^3}{2\left(1 + 2\tilde{v}\tilde{l} + \tilde{v}^2\tilde{l}^2 - \tilde{l}^2\tilde{t}^2\right)}. \end{aligned} \tag{3.73}$$

[9] Note that in terms of the original fluxes this is taking a,$b \to 0$, $l \to \infty$ with al and bl finite.

These branches are reminiscent of the one modulus model, and we will focus our analysis to the first branch[10]. The resulting potential has three turning points when $\tilde{e}_1 < 0$ and $\rho' = 0$, and only one of them is a supersymmetric minimum. In the rest of this section, we will restrict ourselves to excursions away from this point.

Conversely to the previously studied cases, the field $\tilde{s}$ as a function of ρ', shown in figure 3.2, is not an even function anymore, and exhibits a second minimum. Studying this for different flux values, we find that it is at positive values of ρ' for $\tilde{l} > 0$ and vice versa and that outside the region between the two minima, $\tilde{s}$ quickly enters a linear scaling regime asymptoting to $\tilde{s} = \alpha\rho'$ as in (3.25). Inside that region, $\tilde{s}$ remains approximately constant and we find that the distance between the two minima is

$$\Delta\rho' \simeq 2\left(-\frac{\tilde{e}_1}{\tilde{l}^2}\right)^{\frac{3}{4}}. \tag{3.74}$$

We can thus approximate the proper field excursion length by taking $\tilde{s}$ to be approximately constant along $\Delta\rho'$ for a value given by that of the supersymmetric minimum. This can be easily solved for analytically and we find that

$$\tilde{s}_0 \simeq \left(-\frac{\tilde{e}_1}{\tilde{l}^2}\right)^{\frac{3}{4}} \tag{3.75}$$

We therefore conclude that to a good approximation the proper field length $\Delta\phi$ is flux independent. As a measure of this, we scanned over flux ranges $-100 \leq \tilde{e}_1 \leq -3$ and $1 \leq \tilde{l} \leq 100$ finding $2 \leq \Delta\phi \leq 3.5$. Such a small variation over such a large variation in $\Delta\rho'$ presents good evidence that $\Delta\phi$ is flux independent also for this setting[11].

3.2.2 Neveu-Schwarz Axions

In section 3.2.2, we performed a study of the excursions of the massive RR combination away from its minimum for different models. We would like now to perform a similar analysis for different directions of the axion field space by displacing massive combination of of Neveu-Schwarz (NS) axion. Contrary to the case of the RR sector where only one combination of the axion was given a perturbative mass by the superpotential (3.15), the cubic nature of the superpotential for the Kähler sector implies that generically, all NS axions will gain a mass from the fluxes. The NS field space therefore does not have a preferred direction along which one can displace the axion, with the exception of the case where all the moduli and axions are set to equal value, where the displacement is made along this universal value. We shall therefore focus our analysis on this case, which coincides with the one modulus model (3.28). The stabilisation equations are in that case given by

$$\partial_t V = \partial_U V = \partial_S V = 0, \tag{3.76}$$

[10] We have also performed a study of the second branch (3.73) and found it behaves similarly to that of the one modulus, shown in figure 3.1, but with the behaviour of $\tilde{t}$ and $\tilde{u}$ reversed.

[11] Note that the fact that it is possible to reach values such as $\Delta\phi \simeq 3.5$ does not imply super-Planckian excursions, since these are approximate values and are not reliable up to order one factors.

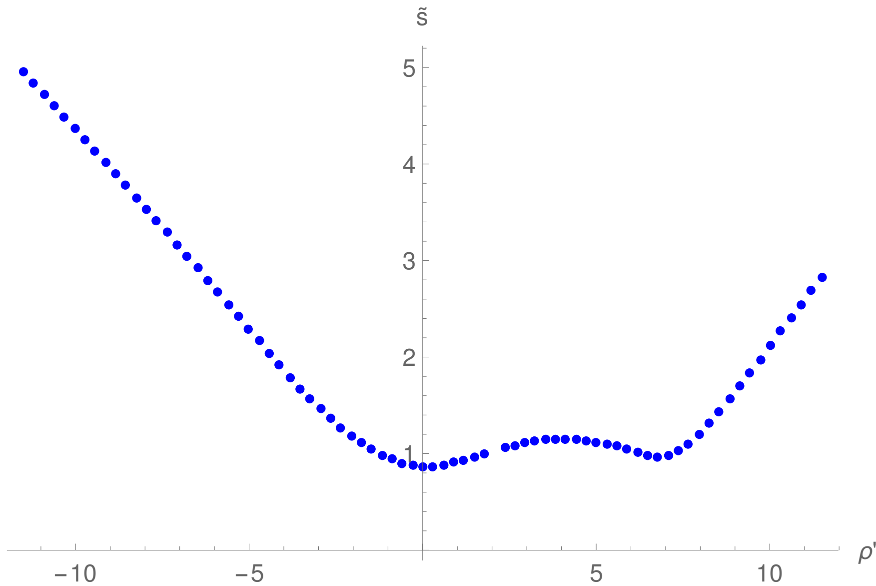


Figure 3.2: Plot showing $\tilde{s}$ as a function of ρ' for the case of a twisted torus compactification with fluxes $\tilde{e}_1 = -6$ and $\tilde{l} = 1$.

while keeping the NS axion v unconstrained and free to displace from the minimum. Most of the analysis of section 3.2.1 such as the minima of the potential, the branch structure (3.39), and the scaling symmetry (3.27) continues to hold and the potential is still a one-parameter model, see (3.36) and the related discussion.

However, unlike for the RR sector, both branches (3.39) now support physical solutions for $f = 0$ which read

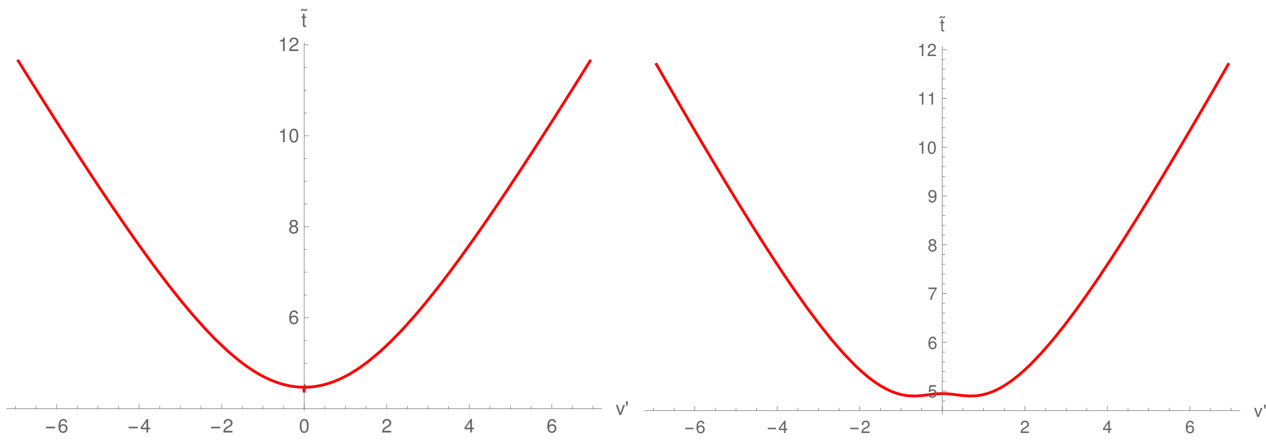
$$\begin{aligned} &\text{Branch 1}: \quad \tilde{s} = 0.36 v'^3, \qquad \tilde{u} = 1.07 v'^3, \qquad \tilde{t} = 1.57 v', \qquad \rho' = -0.17 v'^3, \\ &\text{Branch 2}: \quad \tilde{s} = 0.26 v'^3, \qquad \tilde{u} = 1.19 v'^3, \qquad \tilde{t} = 1.58 v', \qquad \rho' = -0.17 v'^3. \end{aligned} \tag{3.77}$$

As before, they do not flow to a physical minimum due to the restriction $f = 0$ but as expected, the proper field distance exhibits a similar logarithmic behaviour in the axion distance v'.

Turning on $f \neq 0$, the stabilisation equations become more complicated, but are simpler than the RR case and an analytic expression can be found for the moduli. As a function of v', $\tilde{t}$ is given by a root of the following polynomials, depending on the branch:

$$\begin{aligned} \text{Branch 1}: \quad & 25v'^6 + 35\tilde{t}^2 v'^4 + 8f(33\tilde{t}^4 v'^2) + 8f^2(33\tilde{t}^4 + 35\tilde{t}^2 v'^2 + 75v'^4) \\ & + 25v'^8 + 70\tilde{t}^2 v'^6 + 115\tilde{t}^4 v'^4 + 6\tilde{t}^6 v'^2 + 800 f^3 v'^2 - 27\tilde{t}^8 + 400 f^4 \\ \text{Branch 2}: \quad & 25v'^8 + 10(20f + 7\tilde{t}^2)v'^6 + (600f^2 + 280f\tilde{t}^2 + 43\tilde{t}^4)v'^4 \\ & + 2(400f^3 + 140f^2\tilde{t}^2 - 12f\tilde{t}^4 - 15\tilde{t}^6)v'^2 + 8f^2(50f^2 - 3\tilde{t}^4) \end{aligned} \tag{3.78}$$

The relevant root for each branch is a known complicated function of v' and f. For clarity's sake we will not give it explicitly here, but we have instead plotted it in 3.3. For large values



(a) Branch 1 (b) Branch 2

Figure 3.3: Plots showing the moduli $\tilde{t}$ as a function of $v' = v - v_0$ for displacement of v' along both branches (3.39). They are given for flux value $f = 6$. The range is chosen such that it is possible to see the asymptotic linear behaviour is reached after v' reaches its critical value (3.79).

of v', one can show that we enter the regime found in equation (3.77), happening after a critical value

$$v'_{\text{crit}} \simeq \sqrt{2f}. \tag{3.79}$$

The proper distance travelled by the NS axions up to that critical value can be computed in complete analogy with what has been done in section 3.2.1. In particular the argument that it is expected to be independent of flux values and carries through due to the scaling symmetry. We indeed find it is the case in both branches:

$$\Delta\phi = \frac{\sqrt{3}}{2}\int_0^{v'_{\text{crit}}} \frac{dv'}{\tilde{t}} \simeq \begin{cases} 0.57 & \text{Branch 1} \\ 0.55 & \text{Branch 2} \end{cases}. \tag{3.80}$$

As before, any further excursions will add only logarithmic corrections to those values. One might once again be worried about the stability of the potential when moving away from the turning point $v' = \rho' = 0$. However, similarly to the RR sector, an analysis of the Hessian matrix with respect to the other directions shows that despite one of the eigenvalues picking up a negative sign for some values of v', it always lies above the Breitenlohner-Freedman bound. This stability holds for both branches, as well as both signs of f. Moreover plugging back the fields satisfying (3.76) in the potential such that it only depends on v' and the flux numbers, this expression has a local minimum at $v' = 0$ for both signs of f in branch 2. For branch 1, it has a global minimum for positive f and local maximum for negative f.

Finally, we also studied excursions along the NS axions in the twisted torus setting of section 3.2.1. We find a similar structure to the RR axion case with a range over which $\tilde{t}$ is approximately constant bounded by linear scaling regimes. We find that a good fit for the length of the approximately constant region is $\Delta v' \simeq 3(-\frac{\tilde{e}_1}{\tilde{l}})^{\frac{1}{3}}$. This seems to be a good fit to the supersymmetric value $\tilde{t}_0$. Scanning over the flux ranges $-100 \leq \tilde{e}_1 \leq -3$ and $1 \leq \tilde{l} \leq 100$ we find $1.8 \leq \Delta\phi \leq 2.0$, which presents good evidence that it is flux independent.

3.3 Summary

In this chapter, we have studied axion monodromy scenarios in Type IIA string theory compactified on Calabi–Yau manifolds and a twisted torus, where the monodromy is induced through fluxes. In particular, we calculated how the backreaction of the axion vev modifies its proper field length in field space. We found that there is a universal behaviour in all the settings we studied: the backreaction is small up to certain critical value controlled by the flux numbers, and becomes strong once it has been crossed, in such a way that the proper field distance increases logarithmically in the axion vev with a flux independent prefactor. While the critical value can be made arbitrarily large by an appropriate choice of fluxes, thereby allowing for large changes in the axion vev, the backreaction imposes an exact cancellation of the fluxes such that the distances travelled by the proper field is independent of this choice and sub-Planckian.

More precisely, our starting point was the axion monodromy model of [87], where one considers one type of modulus coming from each sector, and with some of the flux numbers turned off. We find that in that case, the backreaction is very strong and the axion enters directly a logarithmic regime. We argued that this behaviour can be attributed to a scaling symmetry that can be used to write the stabilisation in a flux independent way, and can be generalised to arbitrary Calabi–Yau, as long as the flux numbers which were turned off remain so.

We next considered the case where all the fluxes where turned back on. Despite breaking the scaling symmetry, the flux numbers can be thought of as spurions and assigned a non-trivial weight such that their are treated as order parameters. One can rewrite the stabilisation equations as a system depending on a unique combination of the fluxes, in terms of which the logarithmic regime can be arbitrarily delayed by strongly breaking the scale symmetry. However, the combination shielding the moduli from the strong backreaction in turn backreacted themselves on the axionic target space metric, in a way leading to an exact cancellation of any flux number in the proper distance up to the critical value. This result was generalised to two non-trivial Kähler potentials descending from realistic Calabi–Yau manifolds. In those cases, the stabilisation equations can be reduced to systems depending on two parameters rather than one, but the stabilisation equations split into branches, which reduced to system analogous to the one-parameter case, and the proper distance up to the critical value was once again flux independent.

We then considered compactifications on a twisted torus, which introduced new terms in the superpotential mixing the RR and NS sectors. Again, the stabilisation equations could be made functions of only two parameters by an appropriate change of variables. Solving the equations numerically, we found that the backreaction of the moduli as a function of the axion vev was linear beyond some critical values defining an approximately constant region in-between. Scanning over values of the flux parameters numerically, we showed that the proper field distance up to the critical value is once again flux independent and have the same qualitative behaviour as Calabi–Yau compactifications.

In this chapter, we have therefore found a class of new mechanisms coming from string theory censuring super-Planckian excursions in axion monodromy scenarios. We have chosen to study then in the context of Type IIA, as it the simplest and best understood framework to calculate such effects, because moduli stabilisation can be achieved without the need for non-perturbative effects. As the censorship mechanisms appear in various cases, it is reasonable to believe that they could also occur in other setups as well. It would therefore be quite interesting to see how the backreaction of axion vevs affects its proper distance in other string theory constructions, and gain a more general understanding of this effect.

Interestingly, the logarithmic regime of the axion proper field distance beyond some critical value matches the swampland conjectures of maximum logarithmic growth for any field asymptoting infinite field values [121]. The non-trivial cancellations in the proper field distance before the logarithmic regime thus lends some weight to a sharpened swampland conjecture. At least as the same evidence level the original conjecture had, we could conjecture that the logarithmic growth rate of the proper distance must occur at a sub-Planckian proper path distance.[12] The scenarios studied in this work present non-trivial tests of such a sharpened statement. There is a case that is relevant for the possibility of such a conjecture: If the moduli has axion superpartners, where we can apply the Weak Gravity Conjecture to an axion a, and find a relation between its decay constant f_a and the magnitude of its associated instanton, the modulus vev u [20, 69]:

$$S_{\mathrm{Inst}} f_a = u f_a \leq M_{\mathrm{Pl}}. \tag{3.81}$$

One generally uses this relation to bound the decay constant, because if $f_a \geq M_{\mathrm{Pl}}$, then $u < 1$ and control over the instanton expansion is lost. Conversely, one can also use it to bound the magnitude of u, as supersymmetry demands $f_a = \sqrt{g_{uu}}$ giving a measure on the modulus field space. We therefore have

$$\sqrt{g_{uu}} u \leq M_{\mathrm{Pl}}. \tag{3.82}$$

For super-Planckian vev $u > M_{\mathrm{Pl}}$, this condition imposes that $\sqrt{g_{uu}}$ must decay at least with a power of $\frac{1}{u}$, and therefore establishes that the proper field distance is at best logarithmic at this point.

We note that in [42], we have also studied an axion alignment model descending from Type IIA string theory compactified on a twisted torus, where up to four fundamental axions mix. Neglecting the backreaction, we found that there is a particular combination of axions that can be enhanced to arbitrarily large values by tuning the fluxes. However, taking the backreaction into account, the enhancement is cancelled and the effective axion decay constant remains sub-Planckian. Our results are also in agreement with the proposed conjecture.

[12] There is a finite size transition region around the critical axion value between the small backreaction region and the linear scaling strong backreaction regime. This is the transition region to the logarithmic growth regime. The conjecture is that it begins at sub-Planckian values, and that the point where logarithmic growth is a better description than linear growth is also sub-Planckian.

Chapter 4

From Type IIB Supergravity to F-theory and Back Again

In the previous chapter, we found constraints on four dimensional EFTs from the closed string sector of F-theory. While they have have interesting phenomenological consequences, for instance models describing inflation, they were related to matter neutral under gauge symmetries and we have yet to explore charged sectors and constraints from string theory. The rest of this thesis is dedicated to the study of such constraints on gauge theories, and some of their potential implication for model building. To do so, we will use the framework of *F-theory* [29–31], which as we will see offers a very powerful mapping between the gauge symmetry data of an EFT and the geometry of a Calabi–Yau four-fold. For instance, the non-perturbative nature of F-theory will allow access to a richer set of terms in the superpotential that are forbidden in the supergravity description, and have interesting phenomenological applications.

We will follow the usual approach to F-theory of [122–124] by studying the relation between the profile of the Type IIB axio-dilaton in the vicinity of a D7-brane and $SL(2,\mathbb{Z})$ invariance. We will see that the invariance can be "geometrised" by introducing two extra auxiliary compact dimensions and working with a Calabi–Yau four-fold. This bigger space is often singular, meaning many of the usual tools we used so far—such as Hodge's theorem—are no longer applicable. However, their singularity structure will exhibit a very beautiful and intricate relation to Lie algebra and is the keystone of the dictionary we are interested to explore. The results can then be checked by "desingularising" the space and use dualities to verify that the data indeed correspond to physical quantities.

In the Type IIB supergravity approach, we are in a limit where we can neglect the backreaction of the branes on the background geometry. For Dp-branes with $p < 7$, it can be shown that the backreaction is suppressed by a factor $1/r^{7-p}$ as ones moves away from the brane, and can be safely ignored as long as one keeps sufficiently far from these extended objects. For D7-branes on the other hand, the backreaction goes logarithmically with the distance and has to be taken care of. Taking four of the eight directions spanned by the D7-brane to completely fill the large Minkowski dimensions, we are left with two orthogonal dimensions we parameterise by a complex coordinate z. The equation of motion for the axio-dilaton in presence of a stack of n coincident D7-branes (and thus an $SU(n)$ gauge group in the IR) is of the form [123]

$$\tau(z) = \frac{1}{2\pi i} \log \frac{z - z_0}{\lambda}. \tag{4.1}$$

where z_0 is the position of the stack—viewed as a point from the perspective of the orthogonal dimensions—and $\lambda \in \mathbb{R}$ a constant. One might worry that varying the axio-dilaton along a path γ centered on z_0 (on which the stack of D7-branes is located), there is a monodromy

$$\oint_\gamma dz\, \tau(z) = n. \tag{4.2}$$

This apparent problem is solved by the $SL(2,\mathbb{Z})$ invariance, as this is nothing more than a transformation (2.30) with parameters $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. This result suggests a relation between the gauge theory arising in the EFT described by the stack and the monodromy of the axio-dilaton, and one might ask if this generalises further. Another possibility is to consider the additional presence of an O7-plane on top of the stack, where it can be shown that the lower dimensional theory is endowed with an $SO(2n)$ gauge symmetry with a monodromy matrix $\left(\begin{smallmatrix} -1 & 4-n \\ 0 & -1 \end{smallmatrix}\right)$. This connection seems very geometric and feels incomplete: the Lie algebra $\mathfrak{su}(n)$ and $\mathfrak{so}(2n)$ are only the first algebras in Dynkin's famous ADE classification (see [125] for a physicist's approach and [126] for a more mathematical description) and this relation therefore begs for two questions: What is its origin, as the ADE classification is ubiquitous in mathematics, and is it possible to engineer other gauge groups, such as the exceptional group E_8. The answer to the latter is unfortunately no in Type IIB supergravity, and one has to turn to non-perturbative effects—such as multi-pronged strings [32,127–129]—to describe such gauge theories.

Concerning the former, Vafa [29] realised in 1996 that the action of the $SL(2,\mathbb{Z})$ symmetry of Type IIB supergravity could be identified with the set of transformations leaving a torus invariant, its mapping class group. To see this, we recall that a torus is usually defined as a quotient $T^2 = \mathbb{R}^2/\Lambda$ where Λ is a two dimensional lattice. The mapping class group is therefore the set of transformations leaving Λ invariant and in this case is $SL(2,\mathbb{Z})$, as reviewed in appendix A. The axio-dilaton of the Type IIB description is then identified with the complex structure of the torus: To each point of the internal manifold of Type IIB, we can associate a torus whose complex structure modulus is given by the value of the axio-dilaton. In the presence of a D7-brane, the torus becomes singular as τ diverges, and one of its cycles pinches, as illustrated in figure 4.1. Two points of the internal space related by an $SL(2,\mathbb{Z})$ monodromy are thus described by the same torus, which means that the duality is embedded into the formalism, and therefore "geometrised".

The rest of this chapter is dedicated to explain the relation between this geometrisation of the axio-dilaton and the gauge data of the EFT, and is structured as follows: We introduce the mathematical concepts required to extract information about the low energy regime in section 4.1. We will then have the necessary tools to define F-theory as the dual description of M-theory in section 4.2. We review how the physical data arises in section 4.3, exemplifying

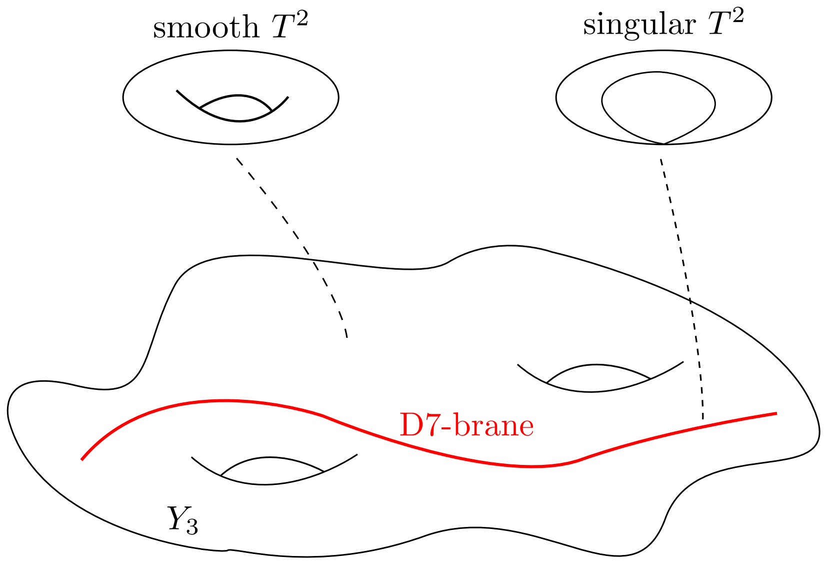


Figure 4.1: Illustration of the geometrisation of the axio-dilaton of Type IIA. To each point of the Calabi–Yau Y_3, one associates a torus, which is singular at points where a D7-branes is present.

with the gauge group $SU(5)$, as it will be the focus of the next chapters. Finally, we will come full circle in section 4.4 and see recover the perturbative Type IIB supergravity limit of F-theory.

4.1 Elliptic Fibrations

Before exploring the mathematics and physics of F-theory, let us pause a moment and explore for a bit the geometry of the torus. In algebraic geometry, tori are also referred to as elliptic curves, and it turns out this field of mathematics is the one that is the natural framework of F-theory. There are various different ways to describe elliptic curves, but the most relevant to us is as a hypersurface equation in the weighted projective space

$$\mathbb{P}_{[2,3,1]} = \frac{\mathbb{C}^3 \setminus \{0\}}{\sim} \qquad (x, y, z) \sim (\lambda^2 x, \lambda^3 y, \lambda z),\ \lambda \in \mathbb{C}^*. \tag{4.3}$$

An elliptic curve is then defined as the vanishing locus of a degree six polynomial in the ambiant space $\mathbb{P}_{[2,3,1]}$. After a suitable coordinate redefinition, such a polynomial can always be written in the so-called *Weierstrass equation* [130]

$$P_W = y^2 - x^3 - fxz - gz^6. \tag{4.4}$$

The coefficients f and g of this polynomial specify the form of the elliptic curve. Given a torus with a complex structure τ, it can be shown that the Weierstrass coefficients are given by the infinite Eisenstein series [130]

$$f = -4^{\frac{1}{3}}60 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^4}, \qquad g = -4\cdot 140 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^6}. \tag{4.5}$$

Conversely, given a Weierstrass form (4.4), one can find the complex structure parameter of the torus by inverting Klein's j-invariant (or simply j-function)

$$j(\tau) = \frac{4(24f(\tau))^3}{4f(\tau)^3 + 27g(\tau)^2}. \tag{4.6}$$

The j-function is an $SL(2,\mathbb{Z})$ invariant and admits an expansion $j(\tau) = e^{-2\pi i\tau} + 744 + 196884e^{2\pi i\tau} + \cdots$. This expansion has attracted a lot of attention since 1978 when John McKay noticed that the coefficients where related to the dimensions of irreducible representation of the Monster group [131]. This relation, named Moonshine, led to unexpected results bridging seemingly unrelated fields of mathematics and physics. For a pedagogical review see [132].

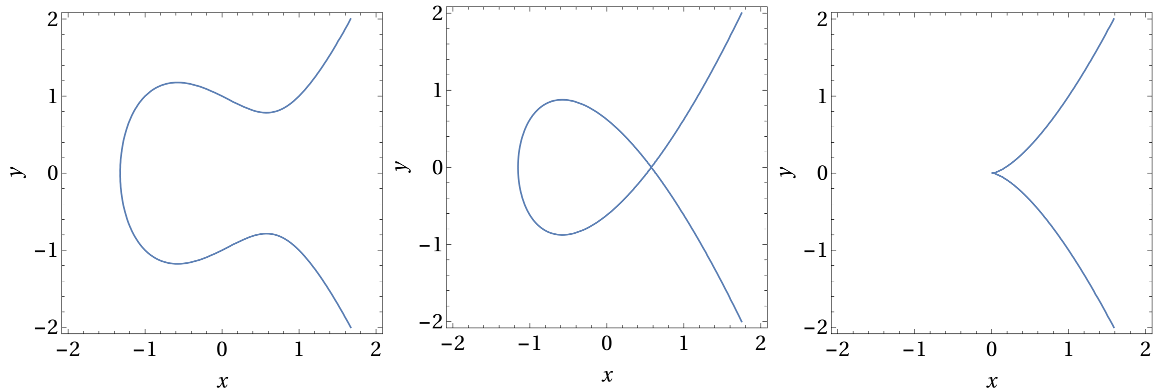
For our purpose however, it suffices to say that in the limit $\tau \to i\infty$ the j-function diverges, signalling that the discriminant

$$\Delta = 4f(\tau)^3 + 27g(\tau)^2 \tag{4.7}$$

vanishes. In that limit, we therefore expect the torus to be singular, signalling the presence of a D7-brane in the Type IIB context. It is straightforward to show that a vanishing discriminant (4.7) happens precisely when the Weierstrass polynomial P_W vanishes along with its derivatives $\partial_{x,y,z}P_W$. Δ thus encodes whether the elliptic curve given by the Weierstrass equation is singular or not, but in that case there is not much structure: As $f, g \in \mathbb{C}$ we have only one possibility of getting a singular curve, happening when $f \sim g^{\frac{2}{3}}$, where the curve has a self intersection point (or a cusp if $f = 0 = g$). Figure 4.2 shows different smooth elliptic curves, in the chart where we have used the scaling of the projective space to set $z = 1$, and restricted to the real planes for the other two.

Now that we have an elegant way of describing the axio-dilaton in terms of an elliptic curve, we want to assign one to each point of Type IIB Calabi–Yau three-fold we shall henceforth call the *base* B_3, for reasons that will be obvious shortly. This is achieved by letting f and g to be a *varying* function of B_3, in the mathematical lingo this is described by considering an *elliptic fibration* Y_{n+1}, where we let the dimension to be arbitrary. An elliptic fibration comes equipped with a projection $\pi : Y_{n+1} \to B_n$ such that for each points b of the base B_n, the inverse map $\pi^{-1}(b)$ is generically an elliptic curve. This quantity encodes much of the data of the dimensionally reduced theory, and among others describes the gauge group of the target manifold and the charged matter spectrum.

Before exploring the various singularities that an elliptic fibration may exhibit, let us discuss the Calabi–Yau condition: in Type IIB supergravity or M-theory, we saw that to keep a number of supersymmetries in lower dimensions, we need the extra dimensions to be Calabi–Yau. However, lifting to F-theory, the strong backreactions will modify the space,



(a) $f=-1,\ g=1$ (b) $f=-1,\ g=\frac{2}{3\sqrt{3}}$ (c) $f=0=g$

Figure 4.2: Examples of smooth and singular elliptic curves. Note that the complex coordinates (x,y) have been restricted to the real plane.

potentially ruining the Calabi–Yau condition for the base. The singular torus fibres can however acquire a curvature in such a way that the full elliptic fibration Y_4 is Calabi–Yau.

In a global setting, f and g are not functions globally defined on the base, but the Calabi–Yau condition rather forces them to be sections of an appropriate power of the anti-canonical bundle[1] K_B^{-1}

$$\begin{aligned} x \in H^0(B_n, K_{B_n}^{-2}), \qquad y \in H^0(B_n, K_{B_n}^{-3}), \qquad z \in H^0(B_n, \mathcal{O}), \\ f \in H^0(B_n, K_{B_n}^{-4}), \qquad g \in H^0(B_n, K_{B_n}^{-6}). \end{aligned} \tag{4.8}$$

A *Weierstrass model* is then an elliptic fibration defined through a Weierstrass equation (4.4).

Having a more rigorous definition of the quantities we are dealing with, let us come back to the discussion of singularities of elliptic fibration. It becomes singular along a locus $\{\Delta = 0\}$ defining a hypersurface in the base, and can present a variety of structure. For instance it can factorise into various components $\Delta = \prod_i \Delta_i$. By analysing the vanishing orders of f, g and Δ, Kodaira [133] and Néron [134] managed to first, classify all the possible singular fibres, and second to associate to each of them a simple Lie algebra giving raise to an ADE type classification. This classification is shown in table 4.1, where the additional information will be explained in section 4.3 where we will expand on it.

[1] In algebraic geometry, the set of sections of a line bundle $\mathcal{L}$ over a base B is denoted $H^0(B,\mathcal{L})$. The notation comes from sheaf cohomology, where this set is part of series of groups $H^i(B,\mathcal{L})$ defined in a similar way we defined de Rham cohomology in section 2.2, and should therefore not be confused with 0-forms. For our purpose, these sections can be thought of as holomorphic functions of the coordinates of the base with a given scaling property.

type	G	ord(Δ)	ord(a_1)	ord(a_2)	ord(a_3)	ord(a_4)	ord(a_6)
$\mathbf{I_0}$	—	0	0	0	0	0	0
$\mathbf{I_1}$	—	0	0	0	1	1	1
$\mathbf{I_2}$	$SU(2)$	2	0	0	1	1	2
I_{2n}^{ns}	$Sp(n)$	$2n$	0	0	n	n	$2n$
$\mathbf{I_{2n}^{s}}$	$SU(2n)$	$2n$	0	1	n	n	$2n$
I_{2n+1}^{ns}	unconven.	$2n+1$	0	0	$n+1$	$n+1$	$2n+1$
$\mathbf{I_{2n+1}^{s}}$	$SU(2n+1)$	$2n+1$	0	1	n	$n+1$	$2n+1$
II	—	2	1	1	1	1	1
III	$SU(2)$	3	1	1	1	1	2
IV^{ns}	unconven.	4	1	1	1	2	2
$\mathbf{IV^{s}}$	$SU(3)$	4	1	1	1	2	3
$I_0^{*\,ns}$	G_2	6	1	1	2	2	3
$I_0^{*\,ss}$	$SO(7)$	6	1	1	2	2	4
$\mathbf{I_0^{*\,s}}$	$SO(8)$	6	1	1	2	2	4
$I_1^{*\,ns}$	$SO(9)$	7	1	1	2	3	4
$\mathbf{I_1^{*\,s}}$	$SO(10)$	7	1	1	2	3	5
$I_2^{*\,ns}$	$SO(11)$	8	1	1	3	3	5
$\mathbf{I_2^{*\,s}}$	$SO(12)$	8	1	1	3	3	5
$I_{2n-3}^{*\,ns}$	$SO(4n+1)$	$2n+3$	1	1	n	$n+1$	$2n$
$\mathbf{I_{2n-3}^{*\,s}}$	$SO(4n+2)$	$2n+3$	1	1	n	$n+1$	$2n+1$
$I_{2n-2}^{*\,ns}$	$SO(4n+3)$	$2n+4$	1	1	$n+1$	$n+1$	$2n+1$
$\mathbf{I_{2n-2}^{*\,s}}$	$SO(4n+4)$	$2n+4$	1	1	$n+1$	$n+1$	$2n+1$
$IV^{*\,ns}$	F_4	8	1	2	2	3	4
$\mathbf{IV^{*\,s}}$	E_6	8	1	2	2	3	5
$\mathbf{III^*}$	E_7	9	1	2	3	3	5
$\mathbf{II^*}$	E_8	10	1	2	3	4	5
non-min	—	12	1	2	3	4	6

Table 4.1: Refined Kodaira–Néron classification, where ord(b) indicates the vanishing order of b. The order of f and g are the same as those of b_4 and b_6 respectively. To distinguish the split (s), non-split (ns), and semi-split (ss), an extra condition might be needed [135]. The bolden cases are those of the original Kodaira–Néron classification. In the case of the infinite series, $n \geq 2$. The last row are non-minimal singularities, which cannot be resolved crepantly in codimension one.

4.2 Defining F-theory from M-theory

Until now our motivation of F-theory has only been a bookkeeping device for D7-branes in Type IIB string theory, but we are yet to discuss how the structure of the target space of the effective action is found beyond its isometry group. To do so, it is useful to explore for a moment the duality chains between Type II string theories and M-theory through their ten and eleven dimensional supergravity description.

Let us consider 11D supergravity compactified on a torus $T^2 = S^1_a \times S^1_b$, down to a nine dimensional space $\mathcal{X}_9$. The total space—away from singularities of the torus—is thus parameterised by the metric

$$ds^2_M = \frac{v}{\tau_b}\left((dx + \tau_a dy)^2 + \tau_b^2 dy^2\right) + ds^2_{\mathcal{X}_9}, \tag{4.9}$$

where x and y are coordinates of the torus T^2 with complex structure $\tau = \tau_a + i\tau_b$ and volume v. As we argued in section 2.4.2, the effective action will be that of Type IIA supergravity on a circle S^1_b or radius R_b, or via T-duality to Type IIB on a circle of radius α'/R_b. Let us suppose further than $\mathcal{X}_9$ factors into a product of a compact complex n-fold B_n space with Minkowski space $\mathbb{R}^{1,8-2n}$. In the limit where $R_b \to 0$, the Type IIB circle will "decompactify" to an additional real line, restoring the full ten dimensional Type II supergravity in a spacetime $\mathbb{R}^{1,9-2n} \times B_n$. From the M-theory point of view this limit can be achieved by keeping the complex structure of the torus fixed while sending the volume to zero.

Similarly the Kalb-Ramond 2-form and its RR counterpart can be obtained by reducing the M-theory 3-form

$$\mathcal{C}_3 = C_3 + B_2 \wedge dx_a + C_2 \wedge dx_b + B_1 \wedge dx_a \wedge dx_b. \tag{4.10}$$

After T-dualising along S^1_b, B_2 and C_2 become the RR and Kalb-Ramond 2-forms, C_4 arises from $C_3 \wedge dx_b$, and B_1 becomes part of the reduced metric.

While we considered a constant torus, this reasoning generalises straightforwardly to an elliptic fibration $\pi:\ Y_{n+1} \to B_n$. In this case the vanishing limit has to be done fibre by fibre and is called the F-theory limit. This way of compactifying M-theory on a torus (fibration) and matching to Type IIB will be taken as the definition of F-theory:

General Lesson 4.1. *An F-theory compactification on an elliptic fibration $\pi : Y_{n+1} \to B_n$ is defined as the Type IIB string theory compactification on B_n that is dual to M-theory compactified on Y_{n+1} in the limit where the torus volume vanishes.*

Note that in contrast to the Type IIB supergravity approach, where the torus was merely an auxiliary space to keep track of the axio-dilaton, it has now become an integral part of spacetime. The advantage of this approach is that there is that the duality of F-theory opens the door of M-theory compactification technologies and enable us to get a clear picture of the physical data we can extract from the geometry. In fact, compactifying both M- and F-theory on the same elliptic four-fold, one will obtain an effective description in three or four dimensions respectively. It is known [45] that quantum field theories with $\mathcal{N} = 1$ in four dimensions are equivalent to theories with $\mathcal{N} = 2$ extended supersymmetry in three dimensions, and we therefore have a dictionary at hand to compare the corresponding physical data.

4.3 Extracting Target Manifold Data From Singularities

If we want gauge symmetries in the lower dimensional theory, we have seen that the elliptic fibration has to be singular. However the compactification of M-theory on singular spaces is not well-understood, and the usual procedure to study F-theory compactifications is to modify the geometry of the elliptic fibration Y_n to a smooth Calabi–Yau $\hat{Y}_n$ by resolving the singularities. In this thesis, we will not be interested in the particular details of the resolution process, and will only review how matter arises in that context without delving into the details of the resolution procedure. To get a working knowledge of the procedure, let us consider a toy example characterised by the following Weierstrass model

$$y^2 = x^3 + f\omega x + g\omega^2, \tag{4.11}$$

defining the elliptically fibered $(n+1)$-fold Y_{n+1}. For simplicity, we will work locally, taking (x, y, ω) as coordinates of $\mathbb{C}^3$. The discriminant, $\Delta = \omega^3(4f^3 + 27g^3\omega)$, indicates that Y_{n+1} is singular along the divisor[2] $\mathcal{S} : \ \{\omega = 0\}$. Moreover, looking at the Kodaira–Néron classification, it is a type III singularity, see table 4.1. Following [136], this singularity can be resolved by a so-called blowup. In this procedure, one introduces an additional homogeneous coordinate s along with transforms $\tilde{x}$, $\tilde{y}$, $\tilde{\omega}$ of the original coordinates:

$$(x, y, \omega) = (\tilde{x}s, \tilde{y}s, \tilde{\omega}s), \qquad (\tilde{x}, \tilde{y}, \tilde{\omega}, s) \sim (\tilde{x}\lambda, \tilde{y}\lambda, \tilde{\omega}\lambda, s\lambda^{-1}), \tag{4.12}$$

where the degrees of the scaling by $\lambda \in \mathbb{C}^*$ have been chosen such that the original coordinates do not transform. The coordinate s defines a larger ambient space, where the proper coordinates cannot vanish simultaneously. The singular point $x = 0 = y = \omega$ is therefore not part of this space, and the resulting hypersurface is smooth. This new Calabi–Yau $\hat{Y}_{n+1}$ comes equipped with a new projection map $\hat{\pi} : \hat{Y}_{n+1} \to B_n$, such that the fibre at the original singularities, $\hat{\pi}^{-1}(p), p \in \mathcal{S}$), are now smooth, and in effect, we have replaced the singular point of a torus by a sphere (a $\mathbb{P}^1$), as depicted in figure 4.3. This defines a divisor of the fibration $\hat{\pi} : \mathcal{D}_1 \to \mathcal{S}$, called an exceptional divisor. The procedure can be generalised to more complicated Weierstrass models and depending on the severity of the singularity, one needs to introduce more exceptional divisors $\mathcal{D}_i$ with there own projection π_i such that their respective fibres are isomorphic to different $\mathbb{P}^1$'s.

For elliptic surfaces ($n = 2$), Kodaira and Néron [133, 134] showed that the fibres of the divisors $\mathcal{D}_i$ intersect like an affine Dynkin diagram of a Lie algebra of rank r in the ADE

[2] A divisor is a generalisation of a hypersurface in algebraic geometry, corresponding to a codimension one subvariety.

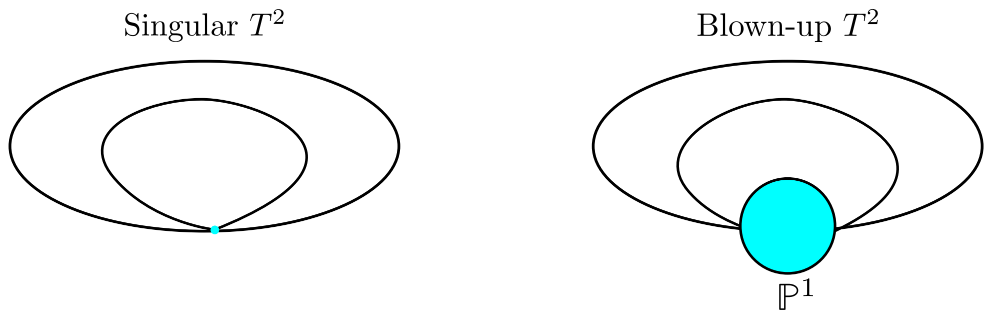


Figure 4.3: Resolution of a singularity by blow up. The singular point (blue) is blown up to a $\mathbb{P}^1$.

classification, where the role of affine node is played by the original component and the others by the $\mathbb{P}^1$'s. While for elliptic surfaces only the algebras $\mathfrak{su}(r)$, $\mathfrak{so}(2r)$ and $\mathfrak{e}_{6,7,8}$ are accessible, this classification was generalised to higher dimensional elliptic fibrations, where all other simple Lie algebras are allowed [137]. Note that there are I_1 singularities that leads to a degeneration of the fibres, but do not render the whole four-fold singular and do not introduce additional exceptional divisors.

Gauge symmetry from codimension 1 singularities

We motivated F-theory by observing the monodromy behaviour of the axio-dilaton around branes, and we therefore expect a relation between the singularities and a gauge group. However, contrary to Type IIB supergravity, we now have access to a richer set of possibilities to engineer a gauge group!

As we are ultimately interested to extract constraints on the low energy effective theory, let us see how the gauge bosons associated to an algebra $\mathfrak{g} = \mathrm{Lie}(G)$ arise from the dual M-theory compactification on a resolved Calabi–Yau. Consider the reduction of the M-theory 3-form:

$$\mathcal{C}_3 = A_i \wedge \omega^i + \cdots \tag{4.13}$$

where $\omega^i \in H^{1,1}(\hat{Y}_4)$ are Poincaré dual to the exceptional divisors $\mathcal{D}_i$. After compactification one is left with r gauge bosons in three dimensions associated to a $U(1)^r$ gauge group that are part of the Cartan subalgebra[3] of $\mathfrak{g}$. The remaining degrees of freedom arise from M2-branes—three dimensional extended objects coupling to the M-theory 3-form—wrapping oriented chains of $\mathbb{P}^1$'s in the fibre. Computing the charges of those states under the Cartan $U(1)$ in terms of intersection numbers, one finds that they have the correct charges to be embedded into a decomposition of the full adjoint of $\mathfrak{g}$ into its Cartan subalgebra. These states are however massive if the compactification is done on $\hat{Y}_4$. However taking the F-theory limit and shrinking the size of the $\mathbb{P}^1$, they become massless, as it is expected of

[3]The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is defined as the maximal commuting subgroup of $\mathfrak{g}$, i.e. the set $\mathfrak{h} = \{T \in \mathfrak{g} | [T, U] = 0,\ \forall U \in \mathfrak{g}\}$. Its dimension $r = \dim(\mathfrak{h})$ is called the rank of $\mathfrak{g}$.

gauge bosons. In the F-theory limit, the $U(1)^r$ gauge symmetry therefore enhances to the full group G.

Due to the important relation between the singularity structure of the elliptic fibration and the low energy data, we would like to have a systematic way to engineer the gauge symmetry we desire in a simple fashion. This can be achieved with Tate's algorithm, which is applicable when the Weierstrass model can be written in *Tate form* [135]

$$P_T = x^3 - y^2 + a_1 xyz + a_2 x^2 z^2 + a_3 yz^3 + a_4 xz_4 + a_6 z^6 = 0. \tag{4.14}$$

The a_i depend on the coordinates of the base and are can be easily be shown to be elements of $H^0(B_n, K_B^{-i})$, and encode the singularity structure of the elliptic fibration. Given a Tate form, we can recover a Weierstrass model by defining

$$b_2 = a_1^2 + 4a_2, \qquad b_4 = a_1 a_3 + 2a_4, \qquad b_6 = a_3^2 + 4a_6, \tag{4.15}$$

and shifting the coordinates. One finds that the parameter of the Weierstrass equation are then given by

$$f = -\frac{1}{48}(b_2^2 - 24b_4), \qquad g = -\frac{1}{864}\left(-b_2^3 + 36b_2b_4 - 216b_6\right). \tag{4.16}$$

The usefulness of the Tate form is that the gauge algebra associated to a singularity is determined by the vanishing order of the a_i. This defines a way of obtaining models with singularities in the Kodaira–Néron classification, first developed in [135], called Tate's algorithm. For practical purposes we have reproduced it in in table 4.1.

As expected its associated gauge symmetry is $\mathfrak{su}(2)$. As an example that will be again useful throughout the rest of this thesis, let us engineer an I_5 singularity, corresponding to an $SU(5)$ gauge group, along a divisor $\mathcal{S}: \ \{\omega = 0\}$. It is achieved by extracting the appropriate factor of ω from the a_i given by table 4.1:

$$a_1 \to a_1, \qquad a_2 \to a_2\omega, \qquad a_3 \to a_3\omega^2, \qquad a_4 \to a_3\omega^3, \qquad a_6 \to a_6\omega^5. \tag{4.17}$$

The coefficients a_i may still depend on ω, but cannot be factored further. It is then straightforward to show that the discriminant is written as

$$\Delta = -\omega^5\left(P_{10}^4 P_5 + \omega P_{10}^2(8a_2P_5 + b_5R) + \mathcal{O}(\omega^2)\right), \tag{4.18}$$

where we defined the coefficients

$$P_{10} = a_1, \qquad P_5 = a_2a_3^2 + a_1a_3a_4 + a_6a_1, \qquad R = -a_3^3 - a_4^2a_1 + 4a_6a_2a_1. \tag{4.19}$$

The discriminant indeed exhibits an I_5 singularity associated with a $\mathfrak{su}(5)$ gauge algebra in the low energy description.

Matter at codimension 2 singularities

From the supergravity description, we expect that strings stretching between two branes—or rather stacks of branes—support matter transforming in the bifundamental representation of the gauge groups supported by each brane. These fields will have a mass proportional to the tension of the string. If the branes intersect however, these fields will become massless and localise at the intersection point. In F-theory, this happens when two singular loci intersect and the singularity enhances over that locus. For a four-fold, two divisors $\mathcal{S}_a$ and $\mathcal{S}_b$ intersect along a co-dimension 2 curve $\mathcal{C}_{ab} = \mathcal{S}_a \cap \mathcal{S}_b$. Note that in principle a divisor can intersect itself, as is the case if one has a single gauge group in the effective theory.

Much as for codimension one singularities, it is easier to determine what happens in the low energy description by studying the dual M-theory on the resolved Calabi–Yau. The $\mathbb{P}^1$'s over the divisors $\mathcal{S}_a$, $\mathcal{S}_b$ intersect as dictated by their associated gauge algebra $\mathfrak{g}_a$ and $\mathfrak{g}_b$ respectively. At the intersection curve $\mathcal{C}_{ab}$, their number increases and now intersect according to the Dynkin diagram of an enhanced ADE algebra $\mathfrak{g}_{ab}$. Again, In the dual picture, the M2-branes will wrap chains of $\mathbb{P}^1$'s, giving copies of the adjoint of $\mathfrak{g}_{ab}$. This enhanced gauge group is however not physical and has to be decomposed into representations of true physical algebra $\mathfrak{g}_a \oplus \mathfrak{g}_b$, always following the pattern

$$\begin{aligned} &G_{ab} \longrightarrow G_a \times G_b, \\ &\mathrm{Adj} \longrightarrow (\mathrm{Adj}_a, 1) \oplus (1, \mathrm{Adj}_b) \oplus \sum_i \left[\left(\mathcal{R}_a^i, \mathcal{R}_b^i \right) \oplus \mathrm{c.c} \right], \end{aligned} \tag{4.20}$$

with $\mathcal{R}^i_{a,b}$ corresponding to irreducible representations of the groups. Note that the decomposition may include singlets which group theoretically are associated to fields charged under the Cartan subalgebra of G_{ab} but not $G_a \times G_b$.

For the I_5 singularity, the discriminant (4.18) can enhance in two different ways, depending on whether P_{10} or P_5 vanish. For the curve $\mathcal{C}_5 : \ \{\omega = 0\} \cap \{P_5 = 0\}$, Tate's algorithm predicts an I_5 singularity while the curve $\mathcal{C}_{10} : \ \{\omega = 0\} \cap \{P_{10} = 0\}$ is associated to a I_1^*—or $\mathfrak{so}(10)$ singularity. Their decomposition back to $SU(5)$ are given by [138]:

$$\begin{aligned} SO(10): \ & \mathbf{45} \longrightarrow \mathbf{24} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1} \\ SU(5): \ & \mathbf{35} \longrightarrow \mathbf{24} \oplus \mathbf{5} \oplus \overline{\mathbf{5}} \oplus \mathbf{1} \end{aligned} \tag{4.21}$$

Yukawa coupling and codimension three singularities

For a Calabi–Yau four-fold, there is one further possibility of enhancement at codimension three. One could naively expect the appearance of new matter at these points, but it was shown in [32, 33] that no such behaviour occurs. Instead, these further enhancements are associated to Yukawa couplings and give information about the form of the superpotential. From the M-theory perspective, these coupling can again be seen when wrapping M2-branes on the $\mathbb{P}^1$'s. In our $SU(5)$ example, there are three such enhancements, at the points

$$\begin{aligned} \mathcal{P}_{\mathfrak{e}_6}: \ \{\omega = P_{10} = a_2 = 0\}, \qquad \mathcal{P}_{\mathfrak{so}(12)}: \ \{P_{10} = a_3 = 0\}, \\ \mathcal{P}_{\mathfrak{su}(7)}: \ \{\omega = P_5 = R = 0\}, \end{aligned} \tag{4.22}$$

Point	coupling	GUT interpretation
$\mathcal{P}_{\mathfrak{e}_6}$	$\mathbf{5}\,\mathbf{10}\,\mathbf{10}$	Up type Yukawa
$\mathcal{P}_{\mathfrak{so}_{12}}$	$\overline{\mathbf{5}}\,\overline{\mathbf{5}}\,\mathbf{10}$	Down type Yukawa
$\mathcal{P}_{\mathfrak{su}_7}$	$\overline{\mathbf{5}}\,\mathbf{5}\,\mathbf{1}$	μ-term like operator

Table 4.2: Yukawa points arising in codimension three for an I_5 singularity engineered through Tate's algorithm and their interpretation in GUT models.

corresponding to enhancement to $\mathfrak{e}_6$, $\mathfrak{so}(12)$ and $\mathfrak{su}(7)$ respectively. A decomposition of the adjoint of these enhanced algebras then give the Yukawa coupling. For the simplest case of the point $\mathcal{P}_{\mathfrak{su}(7)}$, the decomposition is given by

$$\begin{aligned} SU(7) &\longrightarrow SU(5)\times U(1)^2 \\ \mathbf{48} &\longrightarrow \mathbf{24}_{(0,0)} \oplus (\mathbf{5}\oplus\overline{\mathbf{5}})_{(-6,0)} \oplus (\mathbf{5}\oplus\overline{\mathbf{5}})_{(0,6)} \oplus \mathbf{1}_{(6,-6)} \oplus \mathbf{1}_{(-6,6)} \oplus 2\cdot\mathbf{1}_{(0,0)}. \end{aligned} \tag{4.23}$$

Here, we have displayed the charges under the Cartan $U(1)$ not fitting inside $SU(5)$. From that we see that there is an $SU(5)$ invariant triplet $\overline{\mathbf{5}}\,\mathbf{5}\,\mathbf{1}$, corresponding to a μ-term like operator in Grand Unified Theories (we shall be more explicit about this terminology when discussing phenomenological constraints in chapter 6). For the others, the result of the decomposition and their interpretation in GUT models are summarised in table 4.2.

The presence of an $\mathfrak{e}_6$-point in F-theory attracted a lot of attention and sparked a renewed interest in the field. Indeed, this type of coupling is not possible perturbatively in Type IIB intersecting brane models, and can only be generated non-perturbatively through an E3-instanton [139–141], and are therefore strongly suppressed.

Extra Sections and Abelian Gauge Symmetries

We will need one more ingredient to generate the effective field theories that we are interested in. In addition to non-abelian gauge groups, we also want the spectrum to be endowed with extra $U(1)$ gauge symmetries. These abelian factors are very useful when doing model building, as they can prevent the presence of some operators in the superpotential, such as mass or proton decay operators.

In F-theory, unlike their non-abelian cousins these symmetries do not arise from codimension one singularities of the elliptic fibration, but are rather associated to global properties. Here we will discuss elliptic fibration containing extra global sections. For a fibration $\pi:\ Y_{n+1}\to B_n$, a section is a holomorphic map $\sigma:\ B_n\to Y_{n+1}$ satisfying the condition that $\pi\circ\sigma=\mathbb{1}_B$. For each point in the base, a section therefore assigns one point in the fibre and defines a divisor of Y.

For a Weierstrass model a section at a point $b\in B$, $\sigma(b)$, is defined by a holomorphic function $[x(b):y(b):z(b)]$ and the Weierstrass equation. Every Weierstrass model always has one such a section defined by $[x:y:z]=[1:1:0]$ called the zero section. Depending on the functional form of the coefficient of the Weierstrass model, the elliptic fibration is better described as a hypersurface in another ambient space than $\mathbb{P}_{[2,3,1]}$. Computations are generally simpler in those other ambient spaces, but can be mapped back to Weierstrass. For more we refer to [36].

An example of a model having an extra section can be engineered through table 4.1 by setting $a_6 = 0$, in a procedure called $U(1)$-*restriction* [142]. Mapping the Tate model to a Weierstrass equation, one obtains

$$(y - \frac{a_3 z}{2})(y + \frac{a_3 z}{2}) = \left(x - \frac{b_2}{3} z^2\right)\left(x^2 + \frac{b_2}{3} x z^2 + z^4 \left(b_4 - \frac{b_2^2}{9}\right)\right). \tag{4.24}$$

This model clearly has a global section defined by the holomorphic map[4] $[x : y : z] = \left[\frac{b_2}{3} : \frac{a_3}{2} : 1\right]$. Resolving the singularity to obtain a smooth Calabi–Yau $\hat{Y}_4$, we can use Poincaré duality to define a $(1,1)$-form $[\sigma]$, which in the M-theory dual gives rise to a $U(1)$ gauge field through the expansion of the 3-form $\mathcal{C}_3 = A \wedge [\sigma] + \cdots$.

This is straightforwardly generalised to a higher number of sections: For each section σ_i $r = 1, \cdots, r$, there is a dual $(1,1)$-form giving rise to a $U(1)$ gauge boson, and the total gauge group in the effective theory is enhanced to $G \times U(1)^r$, G arising from codimension 1 singularities of the elliptic fibration.

The presence of extra abelian factors will give an additional $U(1)$ charge to states charged under the non-abelian group G, which is found by calculating the intersection numbers between the exceptional divisors and $[\sigma_i]$, see e.g. [36]. We note that this introduces an additional subtlety due to the fact that each state in a given non-abelian representation $\mathcal{R}$ must have the same charge, but we shall not expand on the issue for brevity and refer to [143].

In absence of any non-abelian symmetry, we still expect matter charged under the $U(1)$'s. Like their non-abelian counterparts, they are localised on curves of the base. These curves exhibit an I_2 singularity and are in general rather difficult to identify, and require more advanced tools coming from algebraic geometry. These are beyond the scope of this thesis and we refer to [36] in the case of a single $U(1)$ and [38, 144, 145] for models with more than one extra section. We note that there are also elliptic fibrations with multi-sections, i.e. maps from the base to multiple points of the fibre that may coincide. Such multi-sections give rise to discrete abelian symmetries of the type $\mathbb{Z}_n$, see [39, 40]. We note that there are elliptic fibrations that admit multi-sections, but no zero section. These models cannot be written in Weierstrass form, but nonetheless lead to interesting physics [146] involving discrete symmetries.

To conclude this section, we summarise the dictionary between the geometry of the elliptic fibration and the field space of the effective field theory:

General Lesson 4.2. *In F-theory compactifications on an elliptically fibered Calabi–Yau Y_n over a base B_{n-1}, the data defining the target space $\mathcal{M}$ of the lower dimensional effective field theory is given by the geometry of Y_n:*

[4] Note that we could also have defined the map by asking $y = -\frac{a_3}{2}$. It can be shown that the two maps are homologically equivalent and give rise to the same section.

- *The non-abelian sector arises from the divisor $\mathcal{S}: \{\Delta = 0\}$ over which Y_n is singular. The gauge group can be read off the Kodaira–Néron classification, see table 4.1. A given singularity can be engineered using table 4.1.*

- *The matter spectrum is localised on curves over the base where the singularity locus enhances to a higher rank algebra, and can be read off by a group theoretical decomposition of the adjoint.*

- *For four-fold ($n = 4$), there are additional points where the singularity enhances further. A study of the decomposition of the adjoint provides the structure of the superpotential.*

- *If the elliptic fibration has additional global sections, the gauge group has extra abelian $U(1)$ factors.*

Applying Tate's algorithm to obtain an effective theory with an $SU(5)$ gauge group, the matter spectrum contains fields transforming in the **5** *and* **10** *representations, with an associated superpotential generically containing operators of the type given in table 4.2.*

4.4 Going Back to Type IIB Supergravity: Sen's Limit

At the beginning of this chapter, we have motivated F-theory as a non-perturbative description of Type IIB supergravity by examining the behaviour of the axio-dilaton. Now that we have defined a framework for F-theory that is independent of the Type IIB perspective, we can reverse the argument and ask: given an elliptic fibration $Y_{n+1} \to B_n$, is it possible to find a regime there we recover Type IIB supergravity? This limit is realised when $g_s \sim (\mathrm{Im}\tau) \to 0$, except possibly at the location of D7 branes. Recalling the definition of the j-function (4.6), this happens when the discriminant vanishes, which generically happens when

$$f \sim -3b_2^2 + \mathcal{O}(\varepsilon), \qquad g \sim 2b_2^3 + \mathcal{O}(\varepsilon). \tag{4.25}$$

In the limit $\varepsilon \to 0$, called *Sen's limit*, the axio-dilaton blows up and we recover a perturbative regime. The parameterisation in terms of b_2 is there to be reminiscent of the Tate polynomial coefficients (4.15). Indeed, we can parameterise f and g in terms of the coefficients b_i as in equation (4.16). Sen's original limit is then to demand the b_i vanish in ε to an order given by

$$b_2 \to b_2, \qquad b_4 \to b_4\varepsilon, \qquad b_6 \to b_6\varepsilon^2. \tag{4.26}$$

Plugging this ansatz back into the definition of the discriminant and the j-function, we deduce that in the limit of infinitesimal ε we have

$$\Delta = -\frac{1}{4}\varepsilon^2 b_2^2 \left(b_2 b_6 - b_4^2\right) + \mathcal{O}(\varepsilon^3), \qquad \tau \sim \frac{b_2^4}{\varepsilon^2(b_2 b_6 - b_4^2)}. \tag{4.27}$$

The string coupling is therefore small almost everywhere, except possibly at the locus $\{b_2 = 0\}$. A study of the monodromies around the singular loci [147, 148] shows that they are associated with the following monodromy matrices

$$\{b_2 = 0\} : \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \qquad \{b_2 b_6 - b_4^2 = 0\} : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.28}$$

These are precisely the monodromies associated to an O7-plane and a D7-brane respectively that we found when discussing the axio-dilaton at the beginning of this chapter. In particular, a D7-brane wrapping Whitney's umbrella $b_2 b_6 - b_4^2 = 0$ is commonly called a Whitney brane. The n-fold X_n on which Type IIB is compactified is then given by a double cover of the base B_n defined by

$$X_n : \ \xi^2 = b_2, \tag{4.29}$$

where the orientifold involution acts as $\sigma^* : \xi \to -\xi$.

Sen's limit thus offers a very simple dictionary between a Weierstrass model and the Type IIB data, and has been extensively used in the literature, see e.g. [149] and references therein. Sen's original study of the monodromies has also been extended to a stable limit [150], as well to the case of elliptic fibrations that do not admit a Weierstrass model [151].

General Lesson 4.3. *Given a Tate polynomial, it is possible to recover the Type IIB supergravity description by making its coefficients vanish to the order defined in equation* (4.27)*. The result is a discriminant factorising in two parts, corresponding to the loci describing O7-planes and D7-branes inside a Calabi–Yau defined by the locus $\xi^2 = b_2$ in the base of the elliptic fibration.*

In order to better see how it works in practice, let us take the limit for a model similar to the one that we will study in more details in chapter 7 and consider Tate's algorithm for an I_2 singularity with a $U(1)$-restriction and take the case where $a_4 = 1$. In shifted coordinates, the Weierstrass model reads:

$$y^2 = x^3 + b_2 x^2 + b_4 \omega x + \omega^2. \tag{4.30}$$

Taking Sen's limit, we obtain the discriminant

$$\Delta = -\frac{1}{4}\varepsilon^2 b_2^2 \omega^2 \left(b_2 - b_4^2\right) = -\frac{1}{4}\varepsilon^2 b_2^2 \omega^2 (\xi + b_4)(\xi - b_4), \tag{4.31}$$

where we plugged back the definition of the Type IIB orientifold (4.29). Using the dictionary (4.28) we have, in addition of the O7 plane on the locus $\{\xi = 0\}$, a stack of 2 branes on $\{\omega = 0\}$ and two branes on the loci $\mathcal{D}_\pm = \{\xi \pm b_4\}$. The two branes $\mathcal{D}_\pm$ intersect only on the orientifold plane and are image of one another, as depicted in figure 4.4.

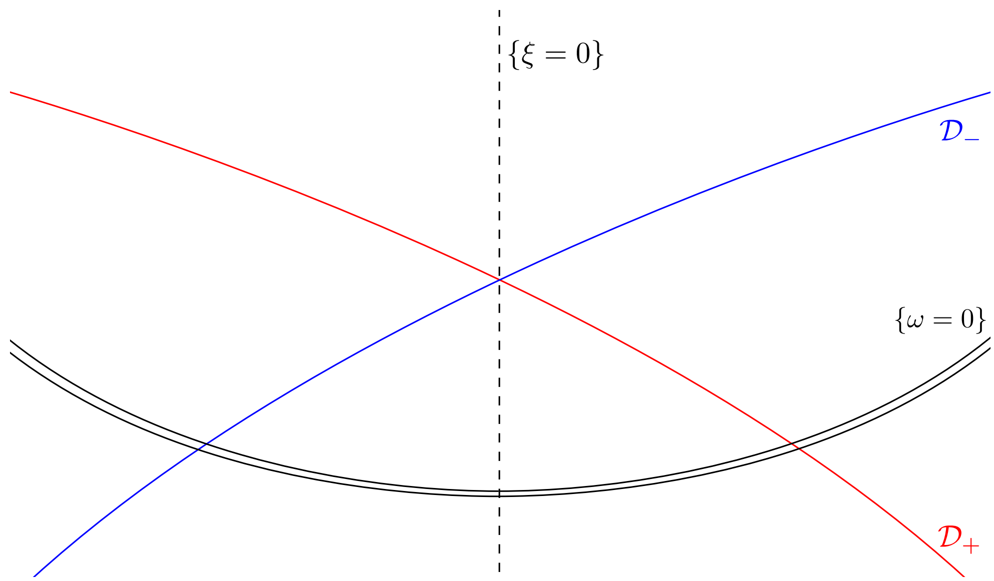


Figure 4.4: Pictorial representation of Sen's limit in the case of the model (4.30). The branes $\mathcal{D}_\pm$ are image of one another. The orientifold plane is shown as a dashed line.

Chapter 5

The Role of E_8 in F-theory GUTs

We have seen in the last chapter that F-theory is a framework particularly appropriate to the study of gauge theory, and can be applied to the study of Grand Unified Theories (GUTs). Such theories are gauge theories that embed the gauge group of the Standard model $SU(3) \times SU(2) \times U(1)$ and its spectrum into a bigger group, such as $SU(5)$ or $SO(10)$.

In recent years, there have been important efforts to study how to realise the minimal GUT group, $SU(5)$, with possible extra Abelian symmetries in F-theory [37,39,136,142,144–146,149,152–166]. While many examples have been studied, there are so far no systematic understanding of the possible symmetries and spectra that can be realised in such models.

This can be contrasted with early F-theory model building, where local models were build on the spectral-cover construction [32,33,167–171]. In this cases, one focuses on a patch of the base B_n, by definition isomorphic to $\mathbb{C}^n$. The geometry can then be described as a Higgs bundle[1] over the codimension one locus giving rise to the GUT group, where the Higgs field takes value inside the commutant of the GUT group under a decomposition of the **248** adjoint representation of

$$E_8 \to G_{\mathrm{GUT}} \times G_{\perp}. \tag{5.1}$$

In the case of $SU(5)$, the commutant is another $SU(5)_{\perp}$, and the possible spectra arising from these fields can be easily classified, as they arise by breaking E_8 when giving a vev to elements of its adjoint. This classification says of course nothing about the remaining of the effective theory, such as its massless spectrum and the values of operators, which depends on the detail of the background geometry and fluxes. However all the possible Abelian symmetries for any such models were embeddable inside E_8, as the matter charged under the GUT group sat inside the **248** adjoint representation.

In this chapter, we will explore the possibility of a similar role of the exceptional group E_8 by giving constraints on the possible symmetries and matter charges in global models, with the aim of a better systematic understanding of F-theory GUT constructions. The role that E_8 may play is directly limited: It is known that there can be gauge group that are

[1] A Higgs bundle is a holomorphic vector bundle equipped with a 1-form. This 1-form must satisfy conditions reminiscent of that of the Higgs field in gauge theories, hence the name.

far larger than E_8, and may in particular contain thousands of such factors [172]. However each non-Abelian gauge group will be localised on a separate divisor in the geometry, and considering the matter spectrum on the specific $SU(5)_{\mathrm{GUT}}$ divisor we in some sense decouple from the other non-Abelian sectors.

Another limitation is that given an $SU(5)$ symmetry on a single divisor, it is still possible that it enhances further over higher codimension subloci [31]. Kodaira's classification indeed demands the discriminant to vanish to order 10 to obtain an $\mathfrak{e}_8$ singularity, and it is not difficult to construct geometries where it vanishes to higher orders inside the singularity divisor. Such loci however are usually associated to tensionless strings and lead to an infinite tower of massless degrees of freedom [173–178]. In the context of F-theory, these arise because the singular limit, the size of some 4-cycle will shrink to zero over the loci, the associated M5-branes wrapping them lead to such strings. This means that if one requires the absence of such infinite tower, e.g. for phenomenological reasons, an extension of E_8 in such a way is forbidden.

Furthermore, a gauge group not part of the exceptional branch of Lie group such as $SU(5)$ is part of an infinite $SU(n)$ series, and one would therefore expect the possibility for matter charged under an infinite number of representations[2]. We can again appeal to phenomenological reasoning to demand the GUT group not to originiate from one of the infinite series such as $SU(n)$ or $SO(2n)$. Indeed, as we have seen in section 4.3, the presence of an up type Yukawa require an exceptional codimension three enhancement, which is not possible for the classical groups.

Notice in particular that the requirement of such a Yukawa point does not necessarily means the gauge group should be embeddable in an exceptional group. The interplay between the matter charged under it and its interaction operators in F-theory construction is not yet understood at a level necessary to systematically classify lower co-dimension singularity data from higher ones. There are however explicit global constructions where the spectrum cannot descend from that of a broken E_8 theory (see [144], following hints from [37, 154]). This then raises the question of whether E_8 plays a role at all in classifying and constraining possible F-theory GUT models.

In an effort to better understanding these models, we will classify and study an extension of the set of theories obtained by Higgsing down an E_8 theory which can account for the global models found in [144] and others in the literature. This class of theories go beyond E_8 while still being closely tied to it, showing that this exceptional group still might have a role to play in the understanding and classification of possible GUT models in F-theory.

It could be that the set of theory we obtain are a complete classification of possible GUT models in F-theory including an exceptional $\mathfrak{e}_6$-point, no infinite tower of massless states and are generic in a sense defined in section 5.1.2. While we have no rigorous proof and cannot make such claim, our analysis forms a first step towards a complete classification. In particular, we do not include theories obtained by breaking the group with a chiral singlet rather than a vector-like pair, which should correspond to geometric gluing modes

[2]There are mild constraints making this finite due to the tadpole cancellation condition, but still allow for the rank of $SU(n)$ to be larger than E_8.

[41, 179–183].

The set of theories we will consider are constructed as follows: Starting from the decomposition of the **248** adjoint representation of E_8 to $SU(5)_{\text{GUT}} \times U(1)^4$, one obtains 20 fields $\mathbf{1}_i$ neutral under the GUT group but charged under the Abelian factors. Group theoretically, they descend from a Cartan decomposition of the adjoint of $SU(5)_\perp$ (four being neutral under the whole remnant group). Therefore a theory coming from the Higgsing of E_8 is described by the singlets getting a vacuum expectation value (vev)[3], and each singlet acquiring a vev will therefore break one additional $U(1)$ factor, eventually leaving no more than $SU(5)_{\text{GUT}}$.

Our proposal is to extend this set of theories by adding 15 new singlet fields not coming from a decomposition of the adjoint of $SU(5)_\perp$—therefore having different charges under $U(1)^4$—and construct the extended set of spectra that can be reached via Higgsing. The addition of these new singlets is detailed in section 5.1, but boils down to the fact that generally, we expect that for any pair of **5** fields, there is a gauge invariant coupling $\mathbf{1}\,\overline{\mathbf{5}}\,\mathbf{5}$.

The result is a classification of the spectrum that can appear under any possible additional Abelian symmetry group (including discrete symmetries). We stress that we do not construct explicit F-theory geometries associated to all of these spectra, but rather perform a group theoretic analysis. In section 5.1.3, we compare our classification with explicit GUT models in the literature and find that of the 30 $SU(5)$ elliptic fibration we study, only three could not be made flat or generic enough (in a sense defined below), and of the remaining 27, only one could be embedded into a Higgsed E_8. However, all of the others find their place in our classification. We summarise our results in section 5.2.

5.1 Global F-theory Models and E_8

This chapter will focus on the minimal GUT gauge group $SU(5)$ and possible Abelian factors. This means that the total gauge group is of the form

$$G = SU(5) \times U(1)^n, \qquad n = 0, \cdots, 4. \tag{5.2}$$

In section 5.1.2, we also incorporate discrete Abelian of the form $\mathbb{Z}_m$. As hinted previously, a key role will be played by representations arising from the decomposition of the adjoint of E_8 into $SU(5)\times U(1)^n$. To see how the charges arise, it is convenient to consider an intermediate embedding $E_8 \supset SU(5)_{\text{GUT}} \times SU(5)_\perp$. The $SU(5)_{\text{GUT}}$ factor then stays untouched while its commutant is broken to its Cartan subalgebra. Under this intermediate embedding, the adjoint of E_8 decomposes according to (4.20) and one finds

$$\begin{aligned} E_8 &\longrightarrow \; SU(5)_{\text{GUT}} \times SU(5)_\perp, \\ \mathbf{248} &\longrightarrow \; (\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}) \oplus (\mathbf{10},\mathbf{5}) \oplus \left(\overline{\mathbf{5}},\mathbf{10}\right) \oplus \left(\overline{\mathbf{10}},\overline{\mathbf{5}}\right) \oplus \left(\mathbf{5},\overline{\mathbf{10}}\right). \end{aligned} \tag{5.3}$$

[3]The GUT singlets come in pairs with opposite Abelian charges. In this chapter, we will only consider backgrounds where the pair has the same vev.

The GUT **10**-multiplets are therefore in the fundamental representation of $SU(5)_\perp$ and the GUT **5**-multiplets are in its anti-symmetric representation. An embedding of the Cartan sub-algebra into $SU(5)_\perp$ is then specified by 5 parameters a_i satisfying a tracelessness constraint $\sum_i a_i = 0$. Our notation will be to write a particular $U(1)$ as

$$U(1)_A = \sum_{i=1}^{5} a_i^A t^i, \tag{5.4}$$

where the coefficients t^i give the charge under the $U(1)$ and are chosen as follows: Given a representation of $SU(5)_{\mathrm{GUT}}$, we introduce a parameter t_i uniquely labelling it:

$$\mathbf{10}_i : t_i, \qquad \overline{\mathbf{5}}_{ij} : t_i + t_j, \qquad \mathbf{1}_{ij} : t_i - t_j, \qquad i \neq j \tag{5.5}$$

The t_i's also correspond to the $U(1)$ charges of the representations, in the sense that for a given $U(1)$ specified by (5.4), the charges are simply given by the contraction of the t_i and t^i using $t_i t^j = \delta_i^j$. Note that this way of parametrising the charges makes it clear that there are two types of gauge invariant operators that can be be constructed: Operators whose t_i sum to zero, such as those of the type $\mathbf{5}\,\mathbf{10}\,\mathbf{10}$, and those whose t_i sum to $t_1+t_2+t_3+t_4+t_5$ and are neutral by the tracelessness constraint, such as $\overline{\mathbf{5}}\,\overline{\mathbf{5}}\,\mathbf{10}$.

Up to this point, the discussion has only been about group theory, and we now consider elliptically fibered four-fold $Y_4 \to B_3$ that realise an $SU(5)$ gauge group on a divisor projecting on a surface $\mathcal{S} \subset B_3$. From General Lesson 4.2, we know that matter will localise on curves and the superpotential will contain term where three curves intersects and are of the type given in table 4.2. We are interested in exploring the interaction between this class of F-theory models and the group E_8, and at this point it will be useful to introduce some nomenclature. We define:

- A *network* as the data of the collection of $SU(5)$-charged matter curves on $\mathcal{S}$ and their intersections.
- A *partially complete network* as a network where any pair of curves intersect each other at least once.
- A *complete network* as a partially complete network where additionally, any pair of **5** or **10** matter curves have a cubic coupling with a GUT singlet at some point.
- A *flat network* as a network where for any point of intersection of two curves there is an associated cubic gauge-invariant coupling.

These definitions correspond to properties of the F-theory elliptic fibrations and the data of the matter curves is completely captured by the Weierstrass equation (or an equivalent fiber equation). For instance, a partially complete network maps to geometries where the base of the fibration is sufficiently generic, since a non-generic base might have certain matter curves not intersecting. The notion of flat network comes about in F-theory construction with extra sections, where one might find intersection points with non-minimal singularities. We argued at the beginning of this chapter that such singularities lead to tensionless strings [178] which we want to avoid. Finally the difference between complete and partial networks stem from the fact that there can exist geometries where a pair of **5** form gauge neutral operator when coupled to a **10** but not a GUT singlet, and lead to specific geometries [166]. We will mainly be interested in the relation between E_8 and complete flat network[4].

Coming back to the group theory analysis, consider fibrations with matter curved charges given by (5.5). A natural question that can be raised is whether these charges form a complete network.

It is easy to see that in the case of $\mathfrak{e}_6$- and $\mathfrak{so}(12)$-points, there is always a gauge invariant cubic interaction for each pairs. For the $\mathfrak{su}(7)$-point however, not every pair of **5** will have a charged singlet to make it invariant, and therefore the E_8 decomposition does not have enough matter to provide a complete network. We note that this behaviour is specific to taking the GUT group to be $SU(5)$, as in the case of $SO(10)$ or larger gauge groups the Abelian charges are such that one can always form a complete network.

A natural conclusion is therefore that F-theory compactifications leading to complete flat networks can have more singlet fields that those coming from a decomposition of the adjoint of E_8, and we are led to the following extension of the spectrum of fields: For each pair of **5** curves, we require that there should be a singlet field such that there is a gauge invariant operator $\mathbf{1}\,\overline{\mathbf{5}}\,\mathbf{5}$. Doing so provides an additional set of 15 singlets to those coming from E_8 and offer a richer way of Higgsing down the $U(1)$ factors. The set of theory we will obtain will include theories coming from E_8 as a subset of our larger classification and will form complete flat networks based on E_8 but extend it.

The remainder of this section is dedicated to classify these theories. In section 5.1.1, we give an example of an elliptic fibration leading to a spectrum not lying in the E_8 classification, and in section 5.1.2 we construct the full set of theories coming from Higgsing down beyond E_8. In section 5.1.3 we compare the obtained classification with explicit F-theory realisations found in the literature.

5.1.1 An Example of Higgsing Beyond E_8

To illustrate the main ideas of our extension, let us consider the case of breaking E_8 to $SU(5)\times U(1)$, for which a decomposition of the adjoint shows the spectrum to be constituted of the following fields:

$$\mathbf{10}^1_{-2}\,, \quad \mathbf{10}^2_3, \quad \mathbf{5}^1_{-6}, \quad \mathbf{5}^2_4, \quad \mathbf{5}^3_{-1}, \quad \mathbf{1}^1_5, \tag{5.6}$$

where the subscripts denote the charge under the remnant $U(1)$. This spectrum alone cannot form a complete network, as it would require the presence of an additional $SU(5)$ singlet with $U(1)$ charge 10 we denote $\mathbf{1}^2_{10}$ to produce a gauge invariant operator $\mathbf{1}^2_{10}\overline{\mathbf{5}}^2_{-4}\mathbf{5}^1_{-6}$.

[4]We expect that for less generic configurations, the relation with E_8 becomes more complicated. For example, one could consider a network which splits into two factors that do not share any intersections, and that the point of $\mathfrak{e}_6$ enhancement lies in only one factor. Then it is not clear why the other factor in the network of curves should be tied to the exceptional groups at all.

The global realisation of this breaking of E_8 has been constructed in [37], where the idea was to take spectral cover description of the Higgsing process amounting to a certain factorisation [169], and restrict the coefficients of the Tate form such as to match those of the spectral cover. The elliptic fibration was given both as a Tate form and a fibration in $\mathbb{P}_{[1,1,2]}$. As explained in 4.3, the latter can be mapped into the former, and for the sake of brevity, we shall only discuss the Tate polynomial, obtained by demanding its coefficients take the form

$$\begin{gathered} a_1 = e_2 d_3, \qquad a_2 = (e_2 d_2 + \alpha\delta d_3)\,\omega, \qquad a_3 = (\alpha\delta d_2 + \alpha\beta d_3 - e_2\delta\gamma)\,\omega^2, \\ a_4 = \left(\alpha\beta d_2 + \beta e_2\gamma - \alpha\delta^2\gamma\right)\omega^3, \qquad a_6 = \alpha\beta^2\gamma\omega^5, \end{gathered}$$

with $\alpha, \beta, \gamma, \delta, e_2, d_3$ sections of the base, and $\omega = 0$ is the $SU(5)$ divisor. One finds that the matter curves are localised along [37, 169]

$$\begin{gathered} \mathbf{10}^1_{-2} : \ \omega = 0 = d_3, \qquad \mathbf{10}^2_3 : \ \omega = 0 = e_2, \\ \mathbf{5}^1_{-6} : \ \omega = 0 = \delta, \qquad \mathbf{5}^2_4 : \ \omega = 0 = \beta d_3 + d_2\delta, \\ \mathbf{5}^3_{-1} : \ \omega = 0 = \alpha^2 c_2 d_2^2 + \alpha^3\beta d_3^2 + \alpha^3 d_2 d_3\delta - 2\alpha c_2^2 d_2\gamma - \alpha^2 c_2 d_3\delta\gamma + c_2^3\gamma^2. \end{gathered} \tag{5.7}$$

In addition, this model has two GUT singlets, one of them being precisely the one lying outside of E_8, $\mathbf{1}^2_{10}$. It is localised on $\{\beta = 0 = \delta\}$ and intersect the GUT divisor at the point $\{\omega = \delta = \beta = 0\}$, where $\mathbf{5}^1_{-6}$ and $\mathbf{5}^2_4$ intersect as well. General Lesson 4.2 tells us there is indeed a gauge invariant interaction between them in the effective theory. We conclude that in constructing the global F-theory geometry based on this Higgsing of E_8, we automatically obtain a complete network over a generic base[5] B_3.

The example shows a realisation motivating the inclusion of singlets beyond those coming from E_8. They can moreover be used to break the Abelian factors further, realised in F-theory by performing a deformation of the elliptic fibration and obtaining a new spectrum. In the case at hand, the deformation corresponds to giving a vev to $\mathbf{1}^1_{10}$ in effective description. It has been studied [39, 146, 184–186] and is done by shifting the following terms to the Tate coefficients.

$$\begin{aligned} a_4 &\longrightarrow \ a_4 - c_{4,1}\alpha\left(\alpha d_3^2 + 4\gamma w\right) w^3, \\ a_6 &\longrightarrow \ a_6 + c_{4,1}\left(-\alpha d_2 + e_2\gamma\right)^2 w^5. \end{aligned} \tag{5.8}$$

Notice that these deformations of a_4, a_6 do not change their vanishing order in ω, and therefore do not modify the gauge group. The presence of higher order term however changes the structure of the codimension 2 loci, and the matter curves corresponding to $\mathbf{5}^1_{-6}$ and $\mathbf{5}^2_4$ recombine into a single curve localised at

$$\tilde{\mathbf{5}}^1 : \ \omega = 0 = \delta\left(\beta d_3 + d_2\delta\right) + e_2 c_{4,1} d_3^2, \tag{5.9}$$

[5] This fibration has a non-minimal singularity at $\alpha = 0 = e_2$. A resolution of this model has been presented [37], but a singularity remains at $\alpha = 0 = \gamma$. These two issues can be bypassed by specialising to a model where α is a non-vanishing constant [37, 144].

while the other matter loci are unchanged. The presence of the higher order terms also guarantees the presence of a Yukawa point[6] associated to $\mathbf{1}_1^1\,\tilde{\mathbf{5}}_0^1\,\overline{\mathbf{5}}_1^3$, but however destroys the extra section associated to the $U(1)$ into a so-called bisection. We have already briefly discussed multi-section at the end of section 4.3, and shall not expand here, but suffice to say that in this case the $U(1)$ of the effective field theory is broken to a $\mathbb{Z}_2$. The resulting spectrum corresponds to

$$\mathbf{10}_0^1, \quad \mathbf{10}_1^2, \quad \tilde{\mathbf{5}}_0^1, \quad \mathbf{5}_1^3, \quad \mathbf{1}_1^1, \tag{5.10}$$

where the subscript now denotes the $\mathbb{Z}_2$ charges. This model, having a discrete symmetry, does not lie in the possible way to break E_8, and we have reached it precisely through the process described above. As an aside, note that this is the first example of such a $\mathbb{Z}_2$ model with two **10**-matter curves.

Interstingly, giving a vev to the singlets lying in the adjoint of E_8 corresponds by contrast to geometries obtained deforming only the leading order of the Tate coefficients, which can be can be seen in the so-called factorised 4-1 model [144]. In the case of a model based on $S\left[U(4)\times U(1)\right]\supset SU(5)_\perp$ realised via a $U(1)$-restriction (see equation (4.24)), one finds only two **5**-matter curves and the singlet at their intersection lies inside the E_8 decomposition. The Higging process is then realised by taking $a_6\omega^3\neq 0$ and leaving the other coefficients unchanged.

5.1.2 Classifying Higgsing Beyond E_8

The Higgsing chain away from E_8 studied in the previous section is but a small portion of the possible chains we will now classify. Recall that the E_8 singlets are defined through their charges (5.5). We can therefore define the additional 15 GUT singlets in terms of the t_i. As our motivation for their introduction is to guarantee the presence of a coupling $\mathbf{1}\,\overline{\mathbf{5}}\,\mathbf{5}$ for each pair of fundamental representation, we take their charges to be given by

$$\mathbf{1}_{ijkl}:\; t_i+t_j-t_k-t_l, \qquad i\neq j\neq k\neq l. \tag{5.11}$$

The set of theories we wish to study are defined by starting from the maximal decomposition

$$E_8\longrightarrow SU(5)_{\mathrm{GUT}}\times U(1)^4, \tag{5.12}$$

and then breaking the Abelian factors one by one by giving a vev to an increasing number of all possible GUT singlets. The remnant group G will be the commuting subgroup of $SU(5)\times U(1)^4$ with all the Higgsed singlets, and the matter representation will correspond to the representation of G descending from E_8, and the additional GUT singlets.

[6] As pointed out in [144], $\mathbf{15}\overline{\mathbf{5}}$ couplings generally depend on higher order terms in ω. Note that this is not inconsistent with the fact that the Yukawa points can be determined purely from the leading order terms as it corresponds to the intersection of two **5**-matter curves. Indeed the global aspect of a section ensures that the sub-leading parts are such that there is an appropriate discriminant enhancement at the intersection of two **5**-matter curves.

Let us denote the set of N singlets acquiring a vev by $\mathbf{1}_\alpha$, with the subscript α ranging from 1 to N, and their charges under $U(1)^5$—before imposing the tracelessness condition—as $Q_{i\alpha}$, with $i = 1, \dots, 5$, who can be seen as an integer $5 \times N$ matrix. The tracelessness condition can be implemented in this framework by including an additional singlet with charge $(1, 1, 1, 1, 1)$, effectively breaking $U(1)^5 \to S\left[U(1)^5\right]$ and making α range from 0 to N. To obtain the charges of the spectrum under the remnant group, one must first go to a basis of $U(1)^5$ where all the singlets are uncharged under the remnant group. Following the methodology of [187], we go to the so-called *Smith form* D of $Q_{i\alpha}$. For any integer matrix $Q \in \mathrm{Mat}(N, 5, \mathbb{Z})$, it is indeed possible to find two unimodular matrices $U \in \mathrm{SL}(N, \mathbb{Z})$ and $V \in \mathrm{SL}(5, \mathbb{Z})$ such that

$$UQV = D = \mathrm{diag}(d_1, \dots, d_r, 0, \dots, 0), \tag{5.13}$$

where the integer entry d_{i-1} divides d_i for all $i = 2, \dots, r = \mathrm{Rank}(\mathcal{D})$. This decomposition guarantees that the matrix V determines the appropriate change of basis such that the charge vector q_i of any state transforms to $q'_i = (qV)_i$ and keeps the charges integer. In particular, the charges under the unbroken combinations are therefore given by q'_i for $i = r+1, ..., 5$.

Note that some of the $U(1)$s may be broken to a remnant discrete symmetry. Indeed let us imagine a spectrum consisting of only three fields $\varphi_m,\ m = 1, 2, 3$ charged m under an hypothetical $U(1)$. Then, upon giving a vev to φ_3, any interaction involving the other fields are left invariant under a $\mathbb{Z}_3$ symmetry:

$$\varphi_n \to e^{2\pi i n/3}\varphi_n, \qquad n = 1, 2. \tag{5.14}$$

This reasoning is easily generalised to arbitrary number of fields and $U(1)$'s. The possible remnant discrete gauge groups are encoded in the Smith form, and are simply

$$G_{\mathrm{Discrete}} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}. \tag{5.15}$$

If any $d_i = 1$, the factor is of course trivial. Notice that the discrete part G_{Discrete} may be part of the $\mathbb{Z}_5$ centre of $SU(5)$ in which case the physical discrete subgroup is given by $G_{\mathrm{Discrete}}/\mathbb{Z}_5$.

As an example of a breaking encompassing the different features encoded in the Smith form, let us consider the giving a vev to the singlets $\mathbb{1}_{1234}$ and $\mathbb{1}_{1324}$. The associated charge matrix and associated Smith form are

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}. \tag{5.16}$$

As the rank of the matrix is 3 and $d_3 = 2$, the smith form signals that the remnant symmetry is $U(1)^2 \times \mathbb{Z}_2$. Rotating to the basis defined by V, one finds that e.g. $\mathbf{10}^1$ has charge $(1, 0)_1$, and similarly for the other matter fields.

The above procedure must be performed for all possible combinations of singlets. The initial spectrum containing 25 charged singlets, we have 25 possibilities to break $SU(5)_{\mathrm{GUT}} \times$

$U(1)^4 \to SU(5)_{\text{GUT}} \times U(1)^3$ with one singlet, 300 possibilities of breaking an additional $U(1)$ with two singlets, 2300 possibilities for Higgsing with 3 singlets and finally 12650 possibilities to break all the $U(1)$'s to $SU(5) \times G_{\text{Discrete}}$. There are of course a significant equivalent possibilities: For instance, it is obvious giving a vev to the singlet $\mathbf{1}_{12}$ or $\mathbf{1}_{13}$ lead to the same spectrum up to relabeling, and only $\mathbf{1}_{ij}$ and $\mathbf{1}_{ijkl}$ give rise to two different $SU(5) \times U(1)^3$ spectra. A similar analysis of redundancies can be performed to the case of two singlets analytically but is much harder to do for more singlets. We therefore resolved to classify all possibilities through a computer scan and some analytic checks.

The final result is shown in tables 5.1 and 5.2. The various models are labeled by three number denoting how many differently charged **10**, **5**, and **1** the spectrum contains. The number of physically distinct models with 3, 2, 1 and 0 $U(1)$s is 2, 6, 11, 6 respectively. The paths which can be taken to reach each of the models as a Higgsing process are shown in figure 5.1.

The final set of models with no $U(1)$ symmetry are differentiated purely by their discrete symmetries. Indeed it is interesting to note that models with discrete symmetries lie outside the Higgsed E_8 subset, marked in bold in the tables, so discrete symmetries are only induced by Higgsing non-E_8 singlets. However, there is a set of discrete symmetries which is not captured by our analysis: From the perspective of a Higgsed E_8 theory these arise from symmetries lying in $SU(5)_\perp$ which are not embedded in its Cartan subgroup. They occur when the Higgs vev is restricted beyond just which components are non-vanishing but there are relations between the non-vanishing components.[7] These non-generic Higgs backgrounds can lead to models with discrete symmetries that come from a Higgsed E_8. We have not attempted to implement these in our classification because it is not clear what the prescription should be to extend these beyond the Higgsed E_8 picture. It would be interesting to understand such symmetries better from a global perspective and thereby gain some intuition as to how they may be implemented beyond a Higgsed E_8.

General Lesson 5.1. *$SU(5)$ GUT spectra coming from a decomposition of the* **248** *of E_8 do not form complete networks. To do so, they have to be extended by a set of 15 GUT singlets, that can in turn be used to break the Abelian factors one by one. The result is the Higgsing tree in figure 5.1 and the spectra of tables 5.1 and 5.2. Some spectra obtained by Higgsing away from E_8 contain discrete symmetries, which is not possible when the extra singlets are absent.*

5.1.3 Embedding Known Models

In the previous section, the classification we obtained was constructed by finding the gauge group and representations of a given model, and was therefore purely group theoretic. The

[7] In terms of the spectral cover approach such symmetries occur when the Galois group of the roots of the spectral cover is not a product of permutation groups (dictated by the $U(1)$ factorisation) but subgroups of them. Or using earlier terminology when the monodromy group is not the full permutation group. See [168, 188–191] for studies of this.

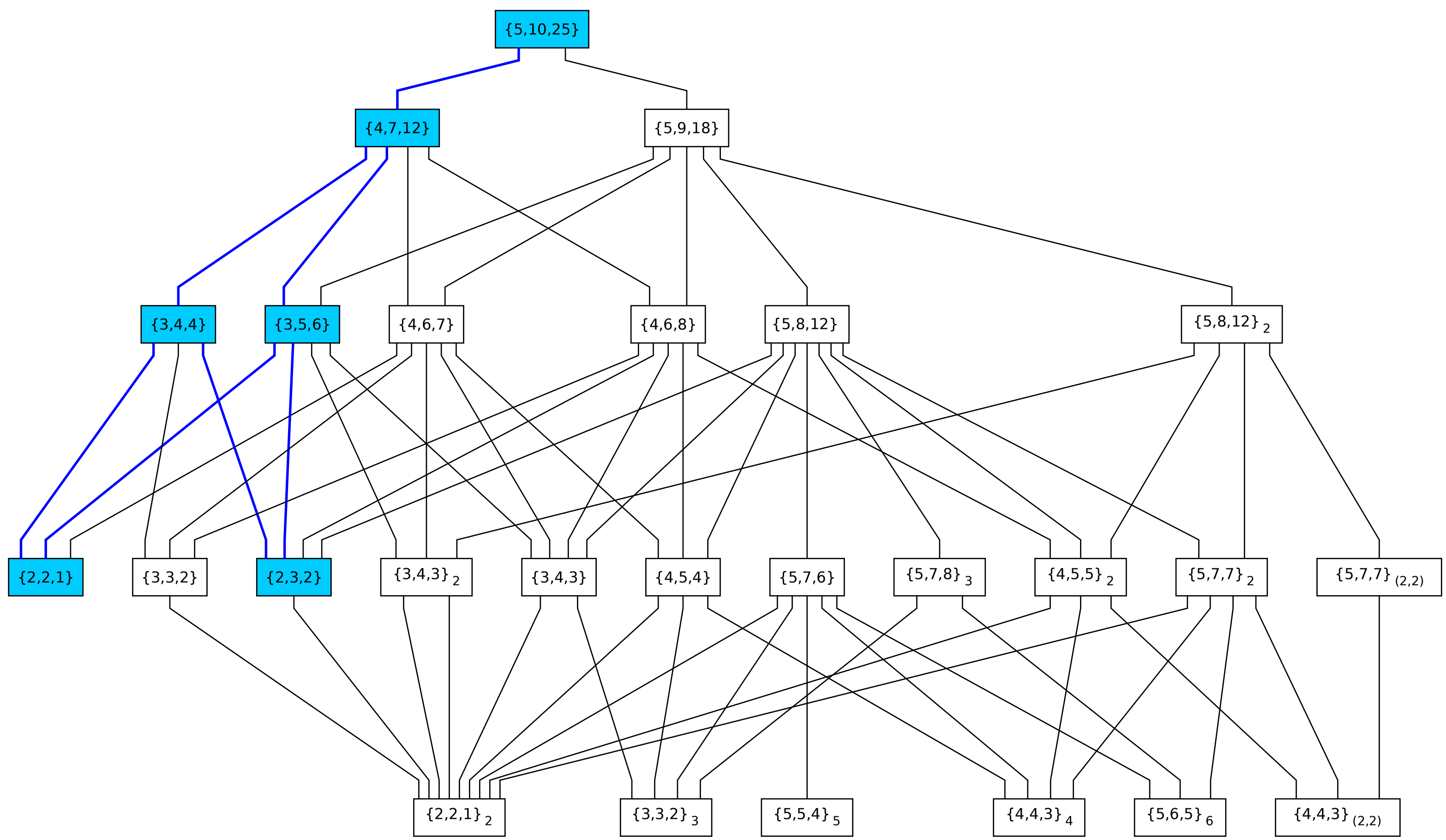


Figure 5.1: Set of theories that can be reached by Higgsing down from $SU(5) \times U(1)^4$ using the GUT singlets. Each rectangle denotes a spectrum of fields, with the numbers giving the number of differently charged **10**, **5** and **1** representations, and the subscript denoting the order of any discrete group present. The spectra of all the models are given in tables 5.1 and 5.2. The paths connecting the models denote Higgsing of GUT singlets. The blue nodes and paths correspond to the set of theories reached from adjoint Higgsing of E_8, though the spectrum of singlets in these models is extended beyond E_8. The decreasing levels denote a decreasing number of $U(1)$s.

Model	$\mathbf{10}_1$	$\mathbf{10}_2$	$\mathbf{10}_3$	$\mathbf{10}_4$	$\mathbf{10}_5$	$\overline{\mathbf{5}}_1$	$\overline{\mathbf{5}}_2$	$\overline{\mathbf{5}}_3$	$\overline{\mathbf{5}}_4$	$\overline{\mathbf{5}}_5$	$\overline{\mathbf{5}}_6$	$\overline{\mathbf{5}}_7$	$\overline{\mathbf{5}}_8$	$\overline{\mathbf{5}}_9$
	Three $U(1)$'s models													
$\{\mathbf{4},\mathbf{7},\mathbf{12}\}$	$(-2,-1,-1)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	—	$(-2,-1,0)$	$(-2,0,-1)$	$(-1,-1,-1)$	$(1,1,0)$	$(1,0,1)$	$(0,1,1)$	$(2,0,0)$	—	—
	$(0,1,-1)$, $(1,-1,0)$, $(1,0,-1)$,$(4,1,0)$, $(4,0,1)$, $(2,2,1)$, $(2,1,2)$, $(1,2,2)$, $(2,-1,-1)$, $(3,2,0)$, $(3,0,2)$, $(3,1,1)$													
$\{5,9,18\}$	$(-2,-2,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(1,1,-1)$	$(-2,-2,1)$	$(-2,-1,0)$	$(-1,-2,0)$	$(1,1,0)$	$(1,0,1)$	$(0,1,1)$	$(-1,-1,-1)$	$(1,2,-1)$	$(2,1,-1)$
	$(4,3,-2)$, $(4,2,-1)$, $(3,4,-2)$, $(3,3,-1)$, $(3,2,0)$, $(3,1,1)$, $(2,4,-1)$, $(2,3,0)$, $(2,2,1)$, $(2,1,2)$, $(2,0,-2)$, $(1,3,1)$, $(1,2,2)$, $(1,1,-2)$, $(1,0,-1)$, $(1,-1,0)$, $(0,2,-2)$, $(0,1,-1)$													
	Two $U(1)$'s models													
$\{\mathbf{3},\mathbf{4},\mathbf{4}\}$	$(-3,-1)$	$(1,0)$	$(0,1)$	—	—	$(-3,0)$	$(-2,-1)$	$(1,1)$	$(2,0)$	—	—	—	—	—
	$(1,-1)$, $(3,2)$, $(4,1)$, $(5,0)$													
$\{\mathbf{3},\mathbf{5},\mathbf{6}\}$	$(-2,-2)$	$(1,0)$	$(0,1)$	—	—	$(-2,-1)$	$(-1,-2)$	$(1,1)$	$(2,0)$	$(0,2)$	—	—	—	—
	$(1,-1)$, $(1,4)$, $(2,-2)$, $(2,3)$, $(3,2)$, $(4,1)$													
$\{4,6,7\}$	$(-1,2)$	$(0,-4)$	$(1,0)$	$(0,1)$	—	$(-1,-2)$	$(-1,3)$	$(0,-3)$	$(0,2)$	$(1,-4)$	$(1,1)$	—	—	—
	$(0,5)$, $(1,-6)$, $(1,-1)$, $(1,4)$, $(2,-7)$, $(2,-2)$, $(2,3)$													
$\{4,6,8\}$	$(-2,-2)$	$(0,1)$	$(1,0)$	$(3,3)$	—	$(-4,-4)$	$(-2,-1)$	$(-1,-2)$	$(1,1)$	$(3,4)$	$(4,3)$	—	—	—
	$(1,-1)$, $(2,3)$, $(3,2)$, $(4,6)$, $(5,5)$, $(6,4)$, $(7,8)$, $(8,7)$													
$\{5,8,12\}$	$(-4,6)$	$(-1,1)$	$(0,1)$	$(2,-4)$	$(3,-4)$	$(-5,7)$	$(-4,7)$	$(-2,2)$	$(-1,2)$	$(1,-3)$	$(2,-3)$	$(3,-3)$	$(5,-8)$	—
	$(1,0)$, $(2,-5)$, $(2,0)$, $(3,-5)$, $(4,-5)$, $(5,-10)$, $(5,-5)$, $(6,-10)$, $(7,-10)$, $(8,-10)$, $(9,-15)$, $(10,-15)$													
$\{5,8,12\}_2$	$(-2,-2)_0$	$(1,0)_0$	$(0,1)_0$	$(1,0)_1$	$(0,1)_1$	$(-2,-1)_0$	$(-1,-2)_0$	$(1,1)_0$	$(-2,-1)_1$	$(-1,-2)_1$	$(1,1)_1$	$(2,0)_1$	$(0,2)_1$	—
	$(1,-1)_0$, $(1,4)_0$, $(2,-2)_0$, $(2,3)_0$, $(3,2)_0$, $(4,1)_0$, $(0,0)_1$, $(1,-1)_1$, $(1,4)_1$, $(2,3)_1$, $(3,2)_1$, $(4,1)_1$													
	One $U(1)$ models													
$\{\mathbf{2},\mathbf{2},\mathbf{1}\}$	-4	1	—	—	—	-3	2	—	—	—	—	—	—	—
	5													
$\{\mathbf{2},\mathbf{3},\mathbf{2}\}$	-3	2	—	—	—	-6	-1	4	—	—	—	—	—	—
	5, 10													
$\{3,3,2\}$	-1	0	1	—	—	-1	0	1	—	—	—	—	—	—
	1, 2													

Table 5.1: First part of the summary of the $SU(5)$-charged spectra for the models reached by Higgsing $SU(5) \times U(1)^4$. The numbers indicate charges under the $U(1)$s present, with subscripts indicating a discrete charge. The second row for each model lists the GUT singlets present. Models in bold are models accessible by Higgsing only E_8 singlets and therefore have charged matter spectra arising from the adjoint of E_8 but with additional singlets.

Model	$\mathbf{10}_1$	$\mathbf{10}_2$	$\mathbf{10}_3$	$\mathbf{10}_4$	$\mathbf{10}_5$	$\overline{\mathbf{5}}_1$	$\overline{\mathbf{5}}_2$	$\overline{\mathbf{5}}_3$	$\overline{\mathbf{5}}_4$	$\overline{\mathbf{5}}_5$	$\overline{\mathbf{5}}_6$	$\overline{\mathbf{5}}_7$	$\overline{\mathbf{5}}_8$	$\overline{\mathbf{5}}_9$
	One $U(1)$ models (Continued)													
$\{3,4,3\}$	-4	1	6	—	—	-8	-3	2	7	—	—	—	—	—
	5, 10, 15													
$\{4,5,4\}$	-8	-3	2	7	—	-11	-6	-1	4	9	—	—	—	—
	5, 10, 15, 20													
$\{5,7,6\}$	-2	-1	0	1	2	-3	-2	-1	0	1	2	3	—	—
	1, 2, 3, 4, 5, 6													
$\{3,4,3\}_2$	-4_0	1_0	1_1	—	—	-3_0	-3_1	2_0	2_1	—	—	—	—	—
	0_1, 5_0, 5_1													
$\{4,5,5\}_2$	-3_0	-3_1	2_0	2_1	—	-6_1	-1_0	-1_1	4_0	4_1	—	—	—	—
	0_1, 5_0, 5_1, 10_0, 10_1													
$\{5,7,7\}_2$	-4_0	-4_1	1_0	1_1	6_0	-8_1	-3_0	-3_1	2_0	2_1	7_0	7_1	—	—
	0_1, 5_0, 5_1, 10_1, 10_0, 15_0, 15_1													
$\{5,7,8\}_3$	-3_1	-3_2	2_0	2_1	2_2	-6_0	-1_0	-1_1	-1_2	4_0	4_1	4_2	—	—
	0_1, 0_2, 5_0, 5_1, 5_2, 10_0, 10_1, 10_2													
$\{5,7,7\}_{2,2}$	$-4_{0,0}$	$1_{0,0}$	$1_{0,1}$	$1_{1,0}$	$1_{1,1}$	$-3_{0,0}$	$-3_{0,1}$	$-3_{1,0}$	$-3_{1,1}$	$2_{0,1}$	$2_{1,0}$	$2_{1,1}$	—	—
	$0_{0,1}$, $0_{1,0}$, $0_{1,1}$, $5_{0,0}$, $5_{0,1}$, $5_{1,0}$, $5_{1,1}$													
	Zero $U(1)$ models													
$\{2,2,1\}_2$	0	1	—	—	—	0	1	—	—	—	—	—	—	—
	1													
$\{3,3,2\}_3$	0	1	2	—	—	0	1	2	—	—	—	—	—	—
	1, 2													
$\{4,4,3\}_{(2,2)}$	(0, 0)	(0, 1)	(1, 0)	(1, 1)	—	(0, 0)	(0, 1)	(1, 0)	(1, 1)	—	—	—	—	—
	(0, 1), (1, 0), (1, 1)													
$\{4,4,3\}_4$	0	1	2	3	—	0	1	2	3	—	—	—	—	—
	1, 2, 3													
$\{5,5,4\}_5$	0	1	2	3	4	0	1	2	3	4	—	—	—	—
	1, 2, 3, 4													
$\{5,6,5\}_6$	0	1	2	4	5	0	1	2	3	4	5	—	—	—
	1, 2, 3, 4, 5													

Table 5.2: Second part of the summary of the $SU(5)$-charged spectra for the models reached by Higgsing $SU(5) \times U(1)^4$. The numbers indicate charges under the $U(1)$s present, with subscripts indicating a discrete charge. For the pure discrete symmetry models the discrete charges are not sub-scripted. The second row for each model lists the GUT singlets present. Models in bold are models accessible by Higgsing only E_8 singlets and therefore have charged matter spectra arising from the adjoint of E_8 but with additional singlets.

correspondence between the obtained spectra and F-theory compactifications should be essentially that the massless gauge fields and matter modes match the singularity structure of an elliptic fibration and that the matter curve intersect accordingly on codimension three loci. In particular the GUT group has to match an I_5 singularity in F-theory and have singular curve with the desired enhancement, while the Abelian factor should be associated to (multi-)section of the elliptic fibration.

There are however some differences we desire to outline: Abelian symmetries could also in principle be related to massive Abelian symmetries, namely gauge symmetries whose gauge bosons have been made massive by a Higgs mechanism. If such symmetries correspond to a complex-structure deformation of the fibration, then we expect this to be the same as giving a vev to a GUT singlet in our analysis, and it should therefore flow to a theory with less $U(1)$ in our classification.

From the Type IIB perspective, there are two other possible ways to make a $U(1)$ massive that do not correspond to a Higgsing by an open string mode. This can be achieved either through some background fluxes, or through a geometric mass [192–194]. In F-theory these are expected to uplift to backgrounds supporting a G_4-flux leading to a massive $U(1)$, see for instance [195], and to backgrounds which include a particular set of non-closed forms, as studied in [142, 194]. While the possibility of considering geometries with fluxes will be studied following chapters, we have restricted ourselves here to geometries with no background flux, and the first possibility for a mass term should therefore be absent. We expect the same for the other class of mass terms—at least in the geometric constructions we will consider—but as these models as not yet fully understood in F-theory, we cannot rule out this possibility.

When trying to fit models that have been constructed in the literature in our classification, we will therefore take the most constrained embedding, where the Abelian sector is completely massless. One should however keep in mind that with the lack of a formal proof, their is always the possibility that a construction on a given elliptic fibration can still be embedded inside of one of our theories with a larger Abelian sector. The charges of the matter fields would then be under some subgroup of this large Abelian symmetry group.

Moreover the matter representations should correspond to matter curves, and in a given F-theory geometry it is to be expected that not all possible representations of our theories will be present. Embedding a geometric model, one should therefore show that the massless matter forms a subset of the representation in our model. The additional data of whether a representation corresponds to actual massless matter is purely geometric and can be turned off by appropriate choices of fibrations

Now that we have shed light on these differences, we can turn ourselves on which class of F-theory geometries we might expect to be captured by our classification. Using the definitions in section 5.1, it is natural to study embeddings of F-theory geometries which form *flat* complete networks in our classification. Demanding flatness affects most of the models in the literature as many have non-flat points. A given construction can however be restricted by setting some parameters of the elliptic fibration to be constants therefore turning off non-flat points. Doing so might also turn off some matter loci as well, and it is this restricted fibration that we then attempt to fit in our classification. The criterion of a partially complete network simply amounts to considering a generic base for the fibration which we therefore assume in our embeddings. Finally, although most of the constructions in the literature form complete networks, there are a few which only form partially complete networks. We will discuss these special cases below.

In table 5.3 we show the possible embeddings of models in the literature in our classification. Of the 30 elliptic fibrations we have considered, 8 of these had an $SU(5)$ spectrum embeddable in a Higgsed E_8 theory, but apart from one (the $4-1$ factorised Tate model of [37]), all of them also had GUT singlets which were not embeddable in the adjoint of E_8, and therefore 29 were in fact not lying in E_8.

One model could not be made flat over a generic base, and of the remaining 29, once the they were constrained to be flat, 27 could be embedded into our classification. We present the analysis of restricting the fibrations to be flat in appendix B. We did not list the four models constructed in [37] with more that one $U(1)$, which were based on a global extension of Higgsed E_8 theories because no smooth resolution was presented. There, however, the results are known by construction: the charged matter spectrum can be embedded in a Higgsed E_8 theory, while the GUT singlet spectrum cannot.

General Lesson 5.2. *When extending an $SU(5)$ spectrum according to 5.1, we find that out of 30 models, all 27 forming flat networks (after imposing the absence of non-flat points) can be embedded in our classification. One of them could be restricted to be flat and two could not be embedded. The results are summarised in table 5.3.*

There are two models, constructed in [166], which were not embeddable in our classification. They contain non-flat points but the analysis in appendix B shows that in principle, for a restricted class of bases of the fibration, it is possible to turn them off by an appropriate choice of fibration. This also turns off some of the matter curves, but still the remaining spectrum is not embeddable. There are two features of these models which may be related to this property. The first is that they do not form complete networks, but only partially complete ones, i.e. there are **5** matter curves which do not have a $\mathbf{1}\,\mathbf{5}\,\overline{\mathbf{5}}$ coupling. If one attempts to restrict the fibration so as to turn off these **5** matter curves then also the single **10** matter curve must be turned off and there is no E_6 Yukawa point which places them outside our classification. They are the only models which have this feature.

Additionally, they exhibit codimension three points located at the intersection of matter curves, where the discriminant would in principle expected to enhance to a higher order, but there is no coupling associated to the point. For instance, the discriminant of the model labeled by $I_5^{s(0|12)}$ takes the schematic form

$$\Delta \sim \sigma_2 \Delta_5 \omega^5 + \Delta_6 \omega^6 + \mathcal{O}\left(w^7\right). \tag{5.17}$$

Here $\sigma_2 = \omega = 0$ corresponds to a **5**-matter curve [166], while the component Δ_5 is some function which does not vanish over $\sigma_2 = 0$. The unusual property is that

$$\Delta_6|_{\sigma_2=0} = s_{3,1}\tilde{\Delta}_6, \tag{5.18}$$

Model	spectrum embedded in
No $U(1)$ models	
[39, 146]	$\{2,2,2\}_2$
[39]	$\{2,2,2\}_2$
One $U(1)$ models	
[154]	$\{3,4,3\}$
[144], [163] fiber type $I_5^{(01)}$	$\{3,3,2\}$
[163] fiber type $I_{5,ncnc}^{(01)}$	$\{3,3,2\}$
[144], [163] fiber type $I_5^{(0\|1)}$	$\{4,5,4\}$ or $\{\mathbf{2},\mathbf{3},\mathbf{2}\}$
[144], [163] fiber type $I_{5,nc}^{(0\|1)}$	$\{\mathbf{2},\mathbf{3},\mathbf{2}\}$
[144], [163] fiber type $I_{5,nc}^{(0\|\|1)}$	$\{3,4,3\}$
Two $U(1)$'s models	
[37] $4-1$ split	$\{\mathbf{2},\mathbf{2},\mathbf{1}\}$
[37] $3-2$ split	$\{\mathbf{2},\mathbf{3},\mathbf{2}\}$
Top 1	$\{\mathbf{3},\mathbf{5},\mathbf{6}\}$
Top 2	$\{5,8,12\}$
Top 3	$\{4,6,7\}$
Top 4	$\{4,6,8\}$
[165]	$\{5,8,12\}$
$I_5^{s(0\|1\|\|2)}$ $(2,2,2,0,0,0,0,0)$	$\{\mathbf{3},\mathbf{4},\mathbf{4}\}$, $\{4,6,7\}$, $\{5,8,12\}*$
$I_5^{s(0\|1\|2)}$ $(2,1,1,1,0,0,1,0)$	$\{\mathbf{3},\mathbf{5},\mathbf{6}\}$
$I_5^{s(0\|1\|\|2)}$ $(2,1,1,1,0,0,1,0)$	$\{5,8,12\}$
$I_5^{s(1\|0\|2)}$ $(3,2,1,1,0,0,0,0)$	$\{5,8,12\}$
$I_5^{s(01\|2)}$ $(3,2,1,1,0,0,0,0)$	$\{4,6,8\}$
$I_5^{s(0\|12)}$ $(4,2,0,2,0,0,0,0)$	Not embeddable
$I_5^{s(012)}$ $(5,2,0,2,0,0,0,0)$	Not embeddable
$I_5^{s(01\|\|2)}$ $(2,2,2,0,0,0,0,0)$	$\{4,6,7\}$
$I_5^{s(0\|1\|\|2)}$ $(2,1,1,1,0,0,0,0)$	$\{\mathbf{3},\mathbf{5},\mathbf{6}\}$ *
$I_5^{s(01\|\|2)}$ $(2,1,1,1,0,0,0,0)$	$\{4,6,7\}$
$I_5^{s(1\|0\|2)}$ $(2,1,1,1,0,0,0,0)$	$\{5,8,12\}$
$I_5^{s(0\|2\|\|1)}$ $(1,1,1,1,0,0,1,0)$	$\{5,8,12\}$
$I_5^{s(0\|1\|\|2)}$ $(1,1,1,0,0,0,0,0)$	No consistent way to turn off non-flat points.
[156] 2 Fibrations	Any of the 2 $U(1)$ models

Table 5.3: Known models and the spectrum they are embeddable in. The two U(1) models come from [144] and [166]. An asterisk means that one needs to turn off the non-flat points to find an embedding. The models marked in bold have $SU(5)$ charged matter which is associated to the E_8 part of the tree, see figure 5.1, though all such embeddings, with the exception of $\{\mathbf{2},\mathbf{2},\mathbf{1}\}$, require beyond E_8 singlets.

where $s_{3,1}$ is some section of the fibration. However there is no intersection of matter curves at the locus $\sigma_2 = s_{3,1} = \omega = 0$. Now the vanishing order of the discriminant at this point can be either 6 or 7 depending on the vanishing order of σ_2. If σ_2 vanishes to order one, the discriminant vanishes to order 6, as it does over the rest of the matter curve, and there is no enhancement. However if the vanishing order of σ_2 is higher, then there is an enhancement of the vanishing order of the discriminant over this locus, but no known associated physics. It can be checked that it is not possible to turn off all such points where this feature occurs in the fibration consistently. We do not know if the fact that these models are not embeddable is related to this feature or not.[8]

$SO(10)$ **models**

The introduction of singlets not embeddable into the adjoint of E_8 was motivated by the fact that for an $SU(5)$ GUT group, it was not possible to form complete network without them. As mentioned in the beginning of this chapter, this not the case for higher rank groups. In the case of $G_{\text{GUT}} = SO(10)$, matter arise from the decomposition of the adjoint of E_8 as follows

$$\begin{aligned} E_8 &\longrightarrow \quad SO(10) \times SU(4), \\ \mathbf{248} &\longrightarrow \ (\mathbf{45},\mathbf{1}) \oplus (\mathbf{16},\mathbf{4}) \oplus \left(\overline{\mathbf{16}},\overline{\mathbf{4}}\right) \oplus (\mathbf{10},\mathbf{6}) \oplus (\mathbf{1},\mathbf{15})\,. \end{aligned} \tag{5.19}$$

In terms of charges of the Cartan of $SU(4)$, the antisymmetric **16** representation has an associated charge given by a parameter t_i, in a similar way to what we have done for $SU(5)$ in equation (5.5). On the other hand the fundamental **10** representations have charges given by $t_i + t_j$, while charges $t_i - t_j$, with $i = 1,..,4$ are associated to $SO(10)$ singlets. We again have the tracelessness condition $\sum_i t_i = 0$. The difference with $SU(5)$, is that there is no need for additional singlet to make pairs of **5** neutral here: The reduced number of available Cartan vectors is such that each pair of **10** is neutralised by a singlet coming from the decomposition of E_8, and we therefore always generically have a complete network in $SO(10)$ GUT theories.

It is then interesting to consider F-theory compactifications over $SO(10)$ geometries and Abelian sector and their possible embedding in a Higgsed E_8. One could expect that if the fact that the $SU(5)$ geometries were not embeddable in E_8 is attributed to the missing singlets, we will not have the same problem with $SO(10)$. In [43], we constructed $SO(10)$ geometries as $\mathbb{P}_{[1,1,2]}$ and $\mathbb{P}_{[1,1,1]}$ fibrations corresponding to the most general fibrations for one and to two $U(1)$ respectively [36, 145, 155] following [196]. We have found that there are two main differences with $SU(5)$: The number of matter curves is very small in the case of $SO(10)$ (a maximum of two for both **10** and **16**), and there are an important number of non-flat loci. We were able to conclude that the full set of models constructed fibrations

[8] It is also interesting to note that this dependence of the vanishing order of the discriminant on the vanishing order of some sections occurs in other models in the literature. In particular for top 2 in [144] one finds this over the full matter curve $c_{2,1} = w = 0$. This curve also happens to exhibit a non-flat point. It would be interesting to study this feature of fibrations further.

were embeddable in a Higgsed E_8 theory, which is consistent with the fact that the singlets coming from the **248** of E_8 are sufficient to form a complete network.

5.2 Summary

In this chapter, we studied the relation between global F-theory GUTs and the exceptional group E_8. We proposed an extension of the set of theories that can be reached from a breaking of E_8 by introducing additional GUT singlets that do not arise from a decomposition of the adjoint of E_8. These singlets can then be used to break the original spectrum and reach new theories. We gave an explicit specific global realisation of this process, the so-called global $3-2$ Factorised Tate model, that includes such an additional singlet, and deformed it to a different elliptic fibration. In the effective theory, this deformation amounts to giving the extra singlet a vev leading to a $\mathbb{Z}_2$ remnant symmetry. We then classified the full set of the possible spectra that could be reached by giving a vev to more and more singlets, extending the 6 Higgsed E_8 spectra by an additional 20. We presented the full set of representations and Abelian charges for these theories, see figure 5.1 and tables 5.1 and 5.2.

We went on to compare this classification of spectra with explicit F-theory realisations constructed in the literature. We considered the 30 resolved $SU(5)$ fibrations listed in table 5.3, and four more given as factorised Tate models in [37], for which no resolution was presented. Of these 34 fibrations, one could not be made flat, and two did not form a complete network, as defined in section 5.1. The remaining 27 resolved fibration could all be embedded into our extended set of spectra. Of these, only one, the global $4-1$ factorised Tate Model, sits into a Higgsed E_8 theory, as no new singlet is needed to form all cubic operators between the fields.

We note that in [43], we also considered 10 $SO(10)$ fibrations that all fit inside a broken E_8, the reason being that the no additional singlets are needed to form complete networks. In that publication, we also explored the heterotic duals to the F-theory fibrations lying outside the original classification. We found that sometimes—but not always—there can be a correlation between singlets outside of E_8 and singularities of the heterotic dual geometry. Additionally, we considered some phenomenological applications, by identifying the $\mathbb{Z}_2$ symmetries with matter-parity. We found that it was not possible to find models where each generation of matter was arising from a different curve.

This work is far from exhaustive, and is just an initial inspection of the relation between E_8 and global F-theory GUTs. The most obvious direction to follow is to continue to check for more F-theory geometries that are either embeddable in E_8, in our classification, or in neither. This would be another step towards a better geometric understanding of codimension three singularities in elliptically fibered four-fold, as it is likely that the intersection structure of matter curves play a role in their classification. We have also restricted ourselves to complete networks, and it would be interesting to study models where that are partially complete and some intersection points are missing. Another obvious direction is to extend the analysis to theories with fluxes, and the next chapter will be a step in that direction.

Chapter 6

Hypercharge Flux Breaking and Anomaly Unifications

In the last chapter, we have explored $SU(5)$ F-theory GUTs, but have yet to relate them to the Standard Model (SM) gauge group $G_{\mathrm{SM}} = SU(3) \times SU(2) \times U(1)$. As already hinted previously, from the field theory perspective, $SU(5)$ is attractive because the spectrum of the Standard Model can be embedded into the fundamental **5** and antisymmetric **10** representations of $SU(5)$. Beyond this group theoretical nicety, there are other phenomenological motivations to embed the SM into a larger single group: performing a running of the gauge couplings of the Minimal Supersymmetric Standard Model (MSSM) shows that they intersect at one-loop order around 10^{15} TeV. Taking a top-down approach and starting with an $SU(5)$ gauge theory, one need to break the GUT group to that of the SM, usually achieved in EFTs through a Higgs-like mechanism.

F-theory offers a natural alternative, by inducing the breaking through non-trivial flux background along one of the Cartan $U(1)$ of $SU(5)$, which in the M-theory picture appears from a non-trivial background of the G_4-flux. From the Type IIB perspective, giving a non-trivial vev to a field strength element $F_A \in \mathfrak{su}(5)$ breaks the group to its commutant, similarly to a Higgs mechanism. Such a process will induce a chirality in the spectrum, meaning that there will be a different number of left- and right-handed spinors. It is the source of the breaking, as it is no longer possible to arrange all the fields in complete multiplets. Choosing carefully this non-trivial background, it is then possible to take the unbroken group to be G_{SM}. This way of obtaining a lower rank group is very desirable because the Standard Model is a chiral theory, and any semi-realistic realisation must arrange for such a possibility.

However, a consequence of inducing chirality in presence of an extra $U(1)$ factor is the appearance of fields that are not associated to Standard Model fields, called exotics. The exotics couple to the rest of the spectrum and generically lead to undesirable operators, for instance responsible for fast proton decay. To avoid experimentally ruled out phenomena, these operators must either be strongly suppressed to have avoided detection so far, or forbidden. Moreover, the MSSM is plagued by the so-called μ-problem, the question of why the parameter of the Higgs mass term, μ, is so small compared to the cut-off of the theory. This problem can be cured by introducing some additional symmetry forbidding the μ-term, and generate it by spontaneous symmetry breaking, see [197] for a review of the problems associated to the MSSM.

In the F-theory framework, the option of having a GUT group and extra Abelian factors give natural candidates to forbid these unwished-for operators. This possibility has been extensively studied in the literature for gauge symmetries. A particularly interesting consequence of these models is that if we require the extra symmetry to protect the μ-term, the vanishing of anomalies requires the presence of additional quasi-vector-like states[1], [169, 170, 189, 198] for original works, and [199] for recent applications.

Conversely, the possibility of global symmetries has not been studied as much. There are a several of different reasons for that: The main one being that once the symmetry has been broken, there will be a pseudo Nambu–Goldstone boson (pNGB). This field, an axion, gets a mass through QCD instanton effects, but it is too light and has been experimentally ruled out, see e.g. [200] and references therein. In string theory similar instanton terms allow however for a larger range of masses.

In this chapter, we will study a possibility arising naturally from string and F-theory, where the symmetry protecting unwanted operators is global. Such behaviour arise by requiring that the GUT breaking does not induce new anomalies, and arise naturally within F-theory because the hypercharge flux is globally trivial.

In section 6.1, we recall the general properties associated to hypercharge flux and their induced anomalies. In section 6.2, we present a class of spectra motivated by F-theory and propose a mechanism protecting the μ-term. In section 6.2.1 we explore the feature of the pseudo-Nambu–Goldstone bosons and exemplify how it is lifted in a particular class of models. In section 6.3, we summarise our findings.

6.1 General Properties of Hypercharge Flux GUT Breaking

In an F-theory compactification over an elliptic fibration Y_4, gauge fluxes are described by considering the $G_4 = d\mathcal{C}_3$ field strength associated to the dual M-theory compactification over the resolved four-fold $\hat{Y}_4$. It must meet some constraints, such as an (half)-integer quantisation condition [201] and, similarly to the case of Type IIA discussed in chapter 3, the fluxes will induce a mass to the gauge bosons on which the fluxes are turned on.

In F-theory and in absence of fluxes, a matter curve $\mathcal{C}$ leads to an equal number of massless fields in a given representation $\mathcal{R}$ and its conjugate $\overline{\mathcal{R}}$. When turning on fluxes, it has been shown [32, 34] that the chiral modes are counted by cohomology groups $H^i(\mathcal{C}, \mathcal{L} \otimes K_{\mathcal{C}}^{1/2})$, where $\mathcal{L}$ is the line bundle associated to the flux, and a factor of the canonical bundle of the curve $K_{\mathcal{C}}$ has been factored out for convenience. The net chirality is then be obtained by

[1] By quasi-vector-like, we mean states that are vector-like under the Standard Model gauge group, but not the Abelian factor.

the Hirzebruch-Riemann-Roch theorem (see appendix A).

$$\chi_{\mathcal{C}} = \#\mathcal{R} - \#\overline{\mathcal{R}} = \int_{\mathcal{C}} c_1(\mathcal{L}), \tag{6.1}$$

where $c_1(\mathcal{L})$ is the Chern class of the line bundle, which is up to a numerical factor the field strength.

In this chapter, we will be interested in the case of $SU(5)$ GUTs with Abelian factors, and therefore we expect the presence of $\mathbf{5}$ and $\mathbf{10}$ matter curves, as we have now seen several times in the previous chapters. If we turn on a flux along the hypercharge generator, whose commutant is the standard model gauge group, we will induce a chirality to Standard Model representation, and we are no longer able to fit them into complete $SU(5)$ representations, therefore breaking $SU(5) \to G_{\text{SM}}$. Note that while this gives us a way to break the GUT group, it a priori does not seem very useful, as the hypercharge gauge boson gets a mass and therefore would lead to only a global $U(1)_Y$ in the effective at low energy. Upon inspection of the dimensional reduction in the M-theory dual, one finds that the mass is proportional to [202]

$$\int_{\mathcal{S}} c_1(\mathcal{L}) \wedge i^*\omega, \qquad \forall \omega \in H^2(\hat{Y}_4), \tag{6.2}$$

with $i^* : H^2(\hat{Y}_4) \to H^2(\mathcal{S})$ the pull-back of ω on the GUT divisor. The condition to get a massless hypercharge gauge boson is then than the pushforward $i_* : H^2(\mathcal{S}) \to H^2(\hat{Y}_4)$ of the Chern class vanishes. A way to achieve this is to consider the 2-cycle dual $\Pi \in H_2(\mathcal{S})$ Poincaré dual to $c_1(\mathcal{L})$, for which an equivalent condition is that Π is trivial in $\mathcal{S}$, but not in B_3. One then chooses a 3-chain Γ in B_3 such that $\Pi = \partial\Gamma$. We will not delve here in specific realisations of F-theory geometries allowing massless fluxes, but they are possible to engineer by either lifting them from Type IIB [149], or by an appropriate factorisation of Tate's coefficients [203].

6.1.1 The General Spectrum

From the decomposition of the adjoint of $SU(5)$ down to that of the Standard model

$$\begin{aligned} SU(5) &\longrightarrow\ SU(3) \times SU(2) \times U(1), \\ \mathbf{24} &\longrightarrow (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{3},\mathbf{2})_{-\frac{5}{6}} \oplus (\overline{\mathbf{3}},\mathbf{2})_{\frac{5}{6}} \oplus (\mathbf{1},\mathbf{1})_0, \end{aligned} \tag{6.3}$$

we see that there are fields in the $(\mathbf{3},\mathbf{2})_{-\frac{5}{6}}$ representation, called lepto-quarks in that context. These are not in the spectrum of the MSSM, and we have to choose the line bundle along the hypercharge generator in such a way that their chirality vanishes, obtaining to the condition $c_1^2(\mathcal{L}_Y^{\frac{5}{6}}) = -2$ [32, 34].

For the sake of completeness, we also recall how (MS)SM matter fields fit in the fundamental and antisymmetric representations of $SU(5)$. The down type quarks and leptons (or rather their conjugates) can be embedded into a $\mathbf{5}$ representation, while the other states are embedded into a **10**:

$$\begin{aligned}\mathbf{5} &\longrightarrow \overbrace{(\mathbf{3},\mathbf{1})_{-\frac{1}{3}}}^{d_R} \oplus \overbrace{(\mathbf{1},\overline{\mathbf{2}})_{\frac{1}{2}}}^{L_L^c},\\ \mathbf{10} &\longrightarrow \underbrace{(\mathbf{3},\mathbf{2})_{\frac{1}{6}}}_{Q_L} \oplus \underbrace{(\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}}_{u_R^c} \oplus \underbrace{(\mathbf{1},\mathbf{1})_{1}}_{e_R^c}.\end{aligned} \tag{6.4}$$

The Higgs field can also embedded into a anti-fundamental $\overline{\mathbf{5}}$ representation but requires the introduction of an exotic triplet. This field not present in the MSSM spectrum has to be given a high mass upon breaking. This conundrum, called the doublet-triplet splitting problem has to be solved to get a realistic spectrum. Moreover, because of anomaly cancellation (see below) conditions, there must be two Higgs fields, denoted H_u and H_d, in fundamental and anti-fundamental representation respectively.

Let us consider models where the GUT group is accompanied by an additional $U(1)$ factor along which we can also turn on some flux. This will modify the chirality (6.1), as we need to consider another line bundle. An analysis of the index associated to the matter curves shows that the chiralities of fields coming from the fundamental of $SU(5)$ are given by [124]

$$\begin{aligned}\#\,(\mathbf{3},\mathbf{1})^{Q_{\mathbf{5}}^a}_{-\frac{1}{3}} - \#\left(\overline{\mathbf{3}},\mathbf{1}\right)^{-Q_{\mathbf{5}}^a}_{\frac{1}{3}} &= M_{\mathbf{5}}^a,\\ \#\,(\mathbf{1},\mathbf{2})^{Q_{\mathbf{5}}^a}_{\frac{1}{2}} - \#\,(\mathbf{1},\mathbf{2})^{-Q_{\mathbf{5}}^a}_{-\frac{1}{2}} &= M_{\mathbf{5}}^a + N_{\mathbf{5}}^a,\end{aligned} \tag{6.5}$$

while those from the antisymmetric have chirality

$$\begin{aligned}\#\,(\mathbf{3},\mathbf{2})^{Q_{\mathbf{10}}^i}_{\frac{1}{6}} - \#\left(\overline{\mathbf{3}},\mathbf{2}\right)^{-Q_{\mathbf{10}}^i}_{-\frac{1}{6}} &= M_{\mathbf{10}}^i,\\ \#\left(\overline{\mathbf{3}},\mathbf{1}\right)^{Q_{\mathbf{10}}^i}_{-\frac{2}{3}} - \#\,(\mathbf{3},\mathbf{1})^{-Q_{\mathbf{10}}^i}_{\frac{2}{3}} &= M_{\mathbf{10}}^i - N_{\mathbf{10}}^i,\\ \#\,(\mathbf{1},\mathbf{1})^{Q_{\mathbf{10}}^i}_{+1} - \#\,(\mathbf{1},\mathbf{1})^{-Q_{\mathbf{10}}^i}_{-1} &= M_{\mathbf{10}}^i + N_{\mathbf{10}}^i.\end{aligned} \tag{6.6}$$

The superscripts $Q_{\mathbf{5}}^a$, $Q_{\mathbf{10}}^i$ denote the $U(1)$ charges and the flux parameters M_{10}^i, M_5^a, N_{10}^i, N_5^a are integers related to the degrees the line bundles. The indices i and a run over the different $U(1)$ charges of the matter curves **5** and **10** respectively. We see that hypercharge flux breaking offers an elegant solution to the doublet triplet problem, as we can choose the flux parameters such that we do not get any $(\mathbf{3},\mathbf{1})_{\frac{1}{3}}$ [34, 170].

From the four dimensional point of view, the non-zero chirality induces either global or gauge chiral anomalies, depending on the nature of the symmetry. Anomalies can be seen as a non-perturbative effect arising by a non-trivial transformation of the path integral measure under redefinitions of the local coordinates of the target manifold $\mathcal{M}$, and can be computed from correlation functions of 3 gauge bosons (in four dimensions). Each representation $\mathcal{R}_\alpha$ of a gauge group G_i gives a contribution to the total anomaly. One can differentiate two cases: the first is anomalies coming from correlation functions of three gauge bosons associated to non-Abelian groups that we denote $\mathcal{A}_{G_i-G_j-G_k}$. The other is when at least one of the gauge bosons is associated with the group $U(1)$, denoted e.g. $\mathcal{A}_{G_i-G_j-U(1)}$. It turns out that these quantities depends solely on the group theoretical data, and not the particular details of the theory [204]:

$$\mathcal{A}_{G_i^2-U(1)} = \sum_\alpha C^{(2)}(\mathcal{R})(\mathcal{R}_\alpha)q(\mathcal{R}_\alpha)\chi(\mathcal{R}), \qquad \mathcal{A}_{G_\alpha-U(1)^2} = \sum_\alpha C^{(1)}(\mathcal{R}_\alpha)q(\mathcal{R}_\alpha), \tag{6.7}$$

where $C^{(s)}(\mathcal{R})$ is the s-th Casimir of the representation $\mathcal{R}$. For $SU(n)$, $C^{(1)}(\mathcal{R})$ vanishes, and there is always a normalisation for which $C^2(\mathbf{n}) = \frac{1}{2}$. In the case of hypercharge flux breaking, the anomalies of the Standard Model must be proportional to those of the GUT group [124], called anomaly unification:

$$\begin{aligned} \mathcal{A}_{SU(3)^2-U(1)} \propto \mathcal{A}_{SU(2)^2-U(1)} \propto \mathcal{A}_{U(1)_Y-U(1)} \propto \mathcal{A}_{SU(5)^2-U(1)}, \\ \mathcal{A}_{U(1)_Y-U(1)^2} \propto \mathcal{A}_{SU(5)-U(1)^2} = 0. \end{aligned} \tag{6.8}$$

Notice that the anomaly (6.8) must always vanish, the first Casimir of $SU(n)$ being always trivial. Knowing the chirality (6.5) and (6.6), we see that the parameters controlling the chirality are constrained by the anomalies. As we want the group of the Standard Model to be a gauge symmetry, the anomalies $A_{G^3_{\text{SM}}}$ must be trivial, as they would otherwise lead to to a violation of unitarity, and allow unphysical states to appear. Requiring so, one must impose the sum of the parameters associated to both the hypercharge and extra $U(1)$ fluxes to vanish:

$$\sum_i M^i_{\mathbf{10}} + \sum_i M^a_{\mathbf{5}} = 0, \qquad \sum_i N^i_{\mathbf{10}} = 0 = \sum_i N^a_{\mathbf{5}} = 0 \tag{6.9}$$

As we will consider in the sequel a global $U(1)$, its anomalies from correlation functions of its gauge boson with two gauge bosons of the Standard Model are not required to vanish. They are however required to be proportional by anomaly unifications, the flux numbers are still constrained. One finds that (6.1.1) and (6.8) lead to the relations

$$\begin{aligned} \sum_i N^i_{\mathbf{10}} Q^i_{\mathbf{10}} + \sum_a N^a_{\mathbf{5}} Q^a_{\mathbf{5}} = 0, \\ 3\sum_i N^i_{\mathbf{10}} (Q^i_{\mathbf{10}})^2 + \sum_a N^a_{\mathbf{5}} (Q^a_{\mathbf{5}})^2 = 0. \end{aligned} \tag{6.10}$$

The first immediate consequence of the constraints (6.9) and (6.1.1) is that if one assumes that the MSSM matter states—i.e. excluding Higgs fields H_u, H_d—have the same $U(1)$ charges by originating from the same GUT curves, then one needs to require $Q_5^{H_u} = Q_5^{H_d}$ in order to get only the MSSM spectrum. This implies that the Higgs fields form a vector-like pair and the $U(1)$ symmetry cannot forbid the presence of a μ-term $\mu H_u H_d$ in the superpotential. A corollary is that if do want to prevent the apparition of the μ-term by demanding the Higgs fields are quasi-vector-like, we are led to include additional states as well. They must also be quasi-vector-like, and we will refer to them as exotics. Historically, these constraints were first discovered as an observation in specific local examples [170], and only later related to anomalies in local [198, 205] and global [203] constructions.

Note that if we relax the condition that MSSM matter states must come from the same GUT matter curves, the prediction of exotics appears to be lost. However, it can be checked that the neutral up and down Yukawa couplings requires that the $U(1)$ charges are the same for all generations is enough to show that the MSSM matter should come from complete GUT multiplets (or at least have the same charges with respect to the $U(1)$). We could still ask the charges to be different from one generation to the other, but this would require to break it at a high scale to induce the proper Yukawa operators, and we cannot expect it to protect the μ-term.

Moreover, one could ask what happens if we allow for another flux parameter, call it $L^i_{\mathbf{10}}$, such that $\chi(\mathbf{1},\mathbf{1})^{Q^i_{\mathbf{10}}}_{+1} = M^i_{\mathbf{10}} + N^i_{\mathbf{10}} + L^i_{\mathbf{10}}$, and all the chiralities are independent. One can check that this does not change the anomaly constraints (6.9) and (6.1.1), but requires an additional conditions on this new set of parameters: $\sum_i L^i_{\mathbf{10}} = 0$. This means that this class of spectra are not affected in the case we require to make the chiralities unrelated as we have one parameter for each field. On the other hand, one can also consider the setup where the global symmetry is taken to be discrete, and of the type $\mathbf{Z}_N$. As these symmetry have a $U(1)$ origin, they lead to anomaly constraints [206,207]. In our case, we find that the constraints (6.9), (6.1.1) are modify so as to vanish modulo N.

Finally, the constraints (6.9)–(6.1.1) can further restrict the spectrum of exotics if we want to preserve gauge coupling unification. Indeed, they could change the behaviour of the β-functions and ruin the unification of the three forces at the GUT scale $\sim 10^{15}\,\mathrm{GeV}$. We therefore require that they must either form complete GUT multiplets (but with possible different $U(1)$ charges), or combinations that contribute to the β-function as if they were GUT multiplets. As first noticed in [170], while the latter does not generically arise for all F-theory Hypercharge flux breaking, they do appear rather naturally in many examples. For instance, one can consider the following combinations to mimic the behaviour of antisymmetric and fundamental representations

$$
\begin{aligned}
\mathbf{10} &\sim \left[(\mathbf{3},\mathbf{2})_{\frac{1}{6}} \oplus (\overline{\mathbf{3}},\mathbf{2})_{\frac{1}{3}} \oplus 2\cdot(\mathbf{1},\mathbf{1})_1\right], \\
\mathbf{10}\oplus\mathbf{5} &\sim \left[(\mathbf{3},\mathbf{2})_{\frac{1}{6}} \oplus 2\cdot(\mathbf{3},\mathbf{1})_{\frac{2}{3}} \oplus (\mathbf{1},\mathbf{2})_{\frac{1}{2}}\right].
\end{aligned}
\tag{6.11}
$$

In the second case, notice that the exotics are acting as a $\mathbf{10}\oplus\mathbf{5}$ simultaneously, but not individually.

6.2 A General Mechanism

In light of the discussions of the previous chapters, and the general properties of hypercharge flux breaking in F-theory, we find the following general lesson:

General Lesson 6.1. *The spectrum of F-theory $SU(5)$ GUTs with Abelian factors broken to the Standard Model through hypercharge flux breaking naturally leads to a class of low energy spectra with the following properties:*

1. *Appropriately charged singlets S_i are present in the spectrum to form neutral cubic couplings with the quasi-vector-like states.*

2. *There are additional exotic states beyond those required for anomaly unification if the μ-term is to be protected.*

3. *The spectrum satisfies the anomaly unification* (6.1.1)–(6.1.1).

4. *The spectrum of states maintains gauge coupling unification at 1-loop.*

The first property is akin to ask for geometries inducing complete networks, while the second and third are motivated by hypercharge flux breaking, as discussed at the beginning of this chapter. The fourth is related to running couplings and do not generically happen. However, if we wish to maintain gauge coupling unification at the GUT scale, we have to impose this phenomenologically motivated property. Notice that this class of spectra, while obtained from F-theory, can be treated in its own right as Effective Field Theory spectra with a well motivated UV origin.

Since the charges of the quasi-vector-like states are constrained by the chiral anomaly, and therefore there are also relations between the charges of their singlets necessary to make any given pair neutral; we define their charges as $Q_{ij} = Q_i^{\mathbf{10}} - Q_i^{\mathbf{10}}$ and $Q_{ab} = Q_a^{\mathbf{5}} - Q_a^{\mathbf{5}}$. We take without loss of generality the singlet S_0 protecting the μ-term to have charge $Q_{S_0} = Q_{12}$. We can then take two non-zero reference flux numbers $N_1^{\mathbf{10}}, N_1^{\mathbf{5}}$ and combine the anomaly unification equations (6.9) and (6.1.1) to show that they satisfy the relation

$$\sum_{i=2} N_{\mathbf{10}}^1 Q_{1i} = N_{\mathbf{5}}^1 Q_{S_0} + N_{\mathbf{5}}^1 \sum_{a=2} Q_{1a} \tag{6.12}$$

It is therefore possible to classify the spectra by the number of differently charged singlets. For simplicity, we will focus on spectra requiring precisely two singlets to lift the exotics, whose most general superpotential takes the general form

$$W = \lambda_0 S_0 H_u H_d + S_1 \sum_{\alpha=1}^{n_1} \lambda_1^\alpha E_1^\alpha \overline{E}_1^\alpha + S_2 \sum_{\alpha=1}^{n_2} \lambda_2^i E_2^\alpha \overline{E}_2^\alpha, \tag{6.13}$$

where $n_{1,2}$ count the number of quasi-vector-like pairs of exotics E_1^α and E_2^α respectively. We again stress that E_1^α and $\overline{E}_1^\alpha$ are different fields transforming in conjugate representations, and the superpotential is holomorphic, as expected.

It can then be shown that there are three minimal cases satisfying General Lesson 6.1 with three different singlets:

$$\text{Case 1} \quad : \; E_1 = (\mathbf{3},\mathbf{1})_{-\frac{1}{3}}, \quad E_2 = (\mathbf{1},\mathbf{2})_{\frac{1}{2}}, \tag{6.14}$$

$$\text{Case 2} \quad : \; E_1^1 = (\mathbf{3},\mathbf{2})_{\frac{1}{6}}, \quad E_1^2 = (\mathbf{1},\mathbf{1})_1, \quad E_1^3 = (\mathbf{1},\mathbf{1})_1, \quad E_2 = (\mathbf{3},\mathbf{1})_{-\frac{1}{3}}, \tag{6.15}$$

$$\text{Case 3} \quad : \; E_1^1 = (3,2)_{\frac{1}{6}}, \quad E_1^2 = (\mathbf{3},\mathbf{1})_{\frac{2}{3}}, \quad E_1^3 = (\mathbf{3},\mathbf{1})_{\frac{2}{3}}, \quad E_2 = (\mathbf{1},\mathbf{2})_{\frac{1}{2}} \; . \tag{6.16}$$

In all the cases, the spectrum indeed satisfies the fourth property of General Lesson 6.1. This first case of course forms a **5** representation, albeit for different charges, while the second case acts as an antisymmetric representation. In the third case, they mimic the combined contribution of a **5** and a **10** to the β-function. The three minimal cases can be extended by adding more exotics acting as a fundamental representation. While depending on the charges, one can still need only three singlets to form the superpotential (6.13), one generically needs additional singlets

Therefore, from the general properties of General Lesson 6.1, we showed that the minimal cases where the exotics are lifted by precisely two singlets are given by the very constrained and potentially predictive spectra (6.14)–(6.15). In the next section, we will explore more how the breaking of the global symmetry is achieved and how the associated pNGB is lifted.

6.2.1 The Pseudo-Nambu–Goldstone Boson

As we argued in the previous section, the $U(1)$ we introduced to protect the μ-term is a global symmetry, as we do not demand the anomalies to vanish. While such global symmetries are believed to be forbidden in a theory of Quantum Gravity [21], they can arise from massive gauge theories. In those cases, the $U(1)$ is anomalous but the total contribution is cancelled by an axion through the Green-Schwarz mechanism, see e.g. [208] for a review. The axion is then eaten by the $U(1)$ gauge boson and acquires a mass, leaving behind an effective global symmetry that is violated only by non-perturbative effect. This mechanism is prolific in string theory, where the axion descends from the Kalb-Ramond 2-form or the RR sector, and the non-perturbative terms involving the axion are D-brane or string instantons [141]. The mass of the axion is therefore of the order of the string scale, and terms violating the residual global symmetry are highly suppressed. The exact form of the suppression factor depends on the details of the UV theory, and we will hence take it as a free parameter. Moreover, we assume that the instantons are weak enough to protect the μ-term and the mass of the exotics down to the TeV scale.

The global symmetry being only broken by non-perturbative effects, there will be an associated perturbatively massless pseudo-Nambu–Goldstone boson (pNGB). Notice that it will be lifted by QCD effects, but those are ruled experimentally as they need a decay constant above the TeV scale to obtain a mass above the MeV scale [209, 210]. In string theory, the instanton effects due to branes enhance the mass range and offers a wider variety of effects.

Notice that the superpotential (6.13) has three further accidental symmetries, and we thus must expect more pNGBs. One of them can be broken by a perturbative trilinear operators of the form $S_i S_j S_k$. One may want to break another accidental symmetry with a further cubic operator, but it generically ensues that at least one of the singlets must be neutral under the $U(1)$ when combined with the constraint (6.12), which we want to avoid, as the Higgs or the exotics would be vector-like. The remaining two pNGBs have then to be lifted through non-perturbative effects, generically of the form

$$W \supset e^{-nU} S_i S_j. \tag{6.17}$$

Coupl.	Q_{S_0}	Q_{S_1}	Q_{S_2}	$\mathcal{I}_{00}$	$\mathcal{I}_{01}$	$\mathcal{I}_{02}$	$\mathcal{I}_{11}$	$\mathcal{I}_{12}$	$\mathcal{I}_{22}$
$S_1S_2^2$	3	-2	1	6	1	4	-4	-1	2
$S_1^2S_2$	3	-1	2	6	2	5	-2	1	4
$S_0^2S_2$	1	-3	-2	2	-2	-1	-6	-5	-4
$S_0S_2^2$	2	-3	1	4	-1	1	-6	-4	-2

Table 6.1: Singlet and instanton charges $\mathcal{I}_{ij}$ for the quadratic terms S_iS_j, depending on the cubic term used to lift one accidental symmetry (up to integer normalisation). As $\mathcal{I}_{ij}$ can be take both signs, we always need at least two different instantons to produce all mass terms. The minimal charges are marked in blue.

Here U is a complex field whose imaginary part is an axion, cancelling the $U(1)$ anomaly via the Green-Schwarz mechanism. Generically, we must take $n \in \mathbb{N}$ because this effect has to be suppressed. This is guaranteed from the string perspective, as the axion field arises when branes are wrapping n-times some cycles of the internal manifold. Under the symmetry, the axion shifts as $\mathrm{Im}(U) \to \mathrm{Im}(U) + Q_U$, and the instanton transforms as

$$e^{-nU} \longrightarrow e^{-nU} e^{-i\mathcal{I}}, \qquad \mathcal{I} = nQ_U. \tag{6.18}$$

To ensure that there exists a neutral coupling, the instanton charge must of course satisfy the condition $\mathcal{I} = (Q_{S_i} + Q_{S_j})$. One might be tempted to introduce an additional singlet to introduce two more perturbative operators in the superpotential, but this would lift only one combination, the other needed to be lifted through instantons. These instantons generally acquire a large vev, and will be taken as an input parameter in the sequel.

To exemplify this quite general mechanism and show how the pNGB is lifted more precisely, we will consider a spectrum where the charges of the three singlets satisfy the relation

$$Q_{S_0} + Q_{S_1} = Q_{S_2}. \tag{6.19}$$

It is satisfied by considering e.g. models where all the exotics come from **5** GUT multiplets such as Case 1 (6.14). To lift one of the pNGB, we have to choose a cubic term among the list $S_1^2S_2,\ S_2^2S_0,\ S_0^2S_2,\ S_2^2S_1$, as any other choice consistent with (6.19) leads to vector-like singlets, and therefore defeats the purpose of using the $U(1)$ symmetry to protect the μ-term and lift the exotics. The two conditions fix the charges of the singlets up to an integer normalisation, as summarised in table 6.1 for all four possible trilinear operators. In each case, the instanton charges $\mathcal{I}$ come in both signs, and we therefore require minimally two instantons we denote $U_1,\ U_2$, to get control over all possible mass terms.

As the procedure is very similar in all cases, we will focus on the spectrum where we lift one of the pNGBs with the cubic coupling $S_1S_2^2$. We then need two instantons, U_1 and U_2, with charge $+1$ and -1 respectively to lift the two remaining pNGBs. The resulting superpotential, including the leading non-perturbative effects is given by

$$
\begin{aligned}
W =& \lambda_0 S_0 H_u H_d + S_1 \sum_{\alpha=1}^{N_1} \lambda_1^\alpha E_1^\alpha \overline{E}_1^\alpha + S_2 \sum_{i=1}^{N_2} \lambda_2^i E_2^\alpha \overline{E}_2^\alpha + \eta S_1 S_2^2 \\
&+ m e^{-U_1} S_0 S_1 + m e^{-U_2} S_1 S_2 + \mathcal{O}(e^{-2U_i})
\end{aligned} \tag{6.20}
$$

The parameter η is dimensionless, while m is mass dimension one. As it is commonly the case when dealing with pNGBs, we parametrise the real components of the GUT scalars in term of their modulus and complex phase

$$
S_i = \rho_i e^{i\theta_i}. \tag{6.21}
$$

The phase is an axion, as it encodes the $U(1)$ transformation through a shift by Q_{S_i}. The vev of the scalars moreover sets the scale of the $U(1)$ breaking, and we therefore take them of the same order $\rho_i \sim \rho$ and larger than the electroweak scale. Note that after symmetry breaking, the canonically normalised fields are θ_i/ρ. If we have any hope of reproducing at least semi-realistically the Standard Model, we need to induce supersymmetry breaking. We include this in the mechanism by also considering a soft a-term

$$
V_{\text{soft}} = a\left(S_i S_j^2 + \text{c.c.}\right) = 2a\rho_1\rho_2^2 \cos(\theta_1 + 2\theta_2), \tag{6.22}
$$

whose precise value of a measures SUSY breaking and sets the mass of the pNGB by minimising the potential. The scalar potential is then given by the sum of the soft terms and of the one induced by supersymmetry using equation (2.10)[2]

$$
V = V_{\text{SUSY}} + V_{\text{soft}}. \tag{6.23}
$$

Without the instanton corrections, The potential has a minimum at $\theta_1 = -2\theta_2 + \pi$, and the mass matrix takes the form

$$
M_{ij}^2 = \frac{1}{2} \frac{\partial V}{\partial\theta_i \partial\theta_j}\bigg|_{\text{min}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a\rho^3 & 2a\rho^3 \\ 0 & 2a\rho^3 & 4a\rho^3 \end{pmatrix} + \mathcal{O}(e^{-U_i}) \tag{6.24}
$$

There are, as expected, two perturbatively massless combinations, while the heavy eigenstate $\varphi = \rho(\theta_1 + 2\theta_2)$ has mass $\sqrt{5a\rho}$. Taking into account instanton corrections, the eigenvalues of the matrix (6.24) are roots of a complicated degree three polynomial. Assuming that the axions are stabilised at different scales, we can consider the limits where one instanton is comparatively far larger than the other, e.g. $e^{-U_1} \gg e^{-U_2} \gg e^{-2U_1}$, or vice versa.

[2]From the point of view of string theory, the charged field arise from the open sector, which do not contain a graviton, and we are therefore decoupled from the supergravity regime. Moreover, in this effective description, the Kähler metric is approximatively flat, as terms inducing a curvature will be suppressed by powers of the Planck scale.

Int.	$H_u H_d \varphi$	$H_u H_d \varphi^2$	$\widetilde{E}_1^\alpha \widetilde{\overline{E}}_1^\alpha \varphi$	$\tilde{E}_2^\alpha \widetilde{\overline{E}}_2^\alpha \varphi$
coupl.	$-im\lambda_0 v_0 e^{-U_1}$	$-\lambda_0 \frac{m}{2\rho} v_0^2 e^{-U_1}$	$i\lambda_1^\alpha v_1$	$i\lambda_2^\alpha v_2$
$e^{-U_1} \gg e^{-U_2}$	$-\frac{i}{\sqrt{2}}\lambda_0 m e^{-U_1}$	$-\frac{m}{4\rho}\lambda_0 e^{-U_1}$	$-\frac{i}{\sqrt{2}}\lambda_1^\alpha + \mathcal{O}(e^{-U_2})$	$\frac{i}{2\sqrt{2}}\lambda_2^\alpha + \mathcal{O}(e^{-U_2})$
$e^{-U_2} \gg e^{-U_1}$	$-i\lambda_0 m e^{-U_1}$	$-\frac{m}{2\rho}\lambda_0 e^{-U_1}$	$\frac{2i}{3}\lambda_1^\alpha \frac{m}{\eta} e^{-U_1}$	$-\frac{i}{3}\lambda_2^\alpha \frac{m}{\eta} e^{-U_1}$

Table 6.2: Coupling constant (at leading order in the instanton) associated to interaction between Higgs scalars H_u, H_d and exotics fermions, denoted by $\tilde{E}_i^\alpha$, $\tilde{\overline{E}}_i^\alpha$. v_i is the prefactor of the eigenstate in the original field basis: $\varphi = \rho v_i \theta_i$.

In the first case, $e^{-U_1} \gg e^{-U_2}$, we find that the eigenstates have—at leading order in the instantons—squared masses that are given by

$$5a\rho^2 + \frac{16}{5} m\eta\rho e^{-U_1}, \qquad \frac{1}{3} m\eta\rho e^{-U_2}, \qquad \frac{9}{5} m\eta\rho e^{-U_1}, \tag{6.25}$$

The lightest states, whose mass is proportional to e^{-U_2}, is found to be

$$\begin{aligned}\varphi = &\left(\frac{1}{\sqrt{2}}\rho + \frac{m(7a - 6\eta^2\rho)}{36\sqrt{2}a\eta} e^{-U_2}\right)\theta_0 + \left(-\frac{1}{\sqrt{2}}\rho + \frac{m(7a - 6\eta^2\rho)}{36\sqrt{2}a\eta} e^{-U_2}\right)\theta_1 \\ &+ \left(\frac{1}{2\sqrt{2}}\rho - \frac{m(7a + 6\eta^2\rho)}{72\sqrt{2}a\eta} e^{-U_2}\right)\theta_2\end{aligned} \tag{6.26}$$

In the opposite regime $e^{-U_2} \gg e^{-U_1}$, the eigenvalues are

$$5a\rho^2 + \frac{12}{5} m\eta\rho e^{-U_2}, \qquad e^{-U_1} m\eta\rho, \qquad \frac{3}{5} e^{-U_2} m\eta\rho, \tag{6.27}$$

and the eigenvector associated with the lightest field is given by

$$\varphi = \rho\theta_0 + \frac{2}{3}\frac{m\rho}{\eta} e^{-U_1}\theta_1 - \frac{1}{3}\frac{m\rho}{\eta} e^{-U_1}\theta_2 \tag{6.28}$$

The eigenstate φ will then couple to the Standard Model via its interaction with the Higgs field. As it is made out of the three axions θ_i, it will also couple to all the exotics. Depending on the value of the instantons, it can predict a set of new states around the TeV scale, which could in principle be detected as a resonance at the Large Hadron Collider decaying through loops of quasi-vector-like fermionic states, and therefore suppressed enough to have evaded detection so far. Assuming a dominant gluon fusion production, the principal decay channel would be to diphtons, i.e. an event of the form $pp \to \varphi \to \gamma\gamma$. In table 6.2, we give for future reference the various coupling constants of the interactions between φ and the Higgs scalar and the fermionic components of the exotics.

6.3 Summary

In this section, we have proposed a new mechanism inspired by the general consequences of hypercharge flux breaking in F-theory compactifications forming a complete network having the other generic properties described in General Lesson 6.1. Despite being inspired by string theoretic arguments, it can nonetheless be defined solely in the Quantum Field Theory picture. If one desires to retain GUT anomaly unification, but does not necessarily want to enforce cancellation as to allow for global symmetries, one can use the $U(1)$ symmetry to forbid a mass term for the Higgs fields. It is then to use the selection rules to alleviate the μ-problem, but new quasi-vector-like states are needed to satisfy the anomaly constraints. A possible global symmetry is then broken only by suppressed non-perturbative terms in the superpotential, giving rise to a light pseudo-Nambu–Goldstone boson that could be in principle detected by collider experiments around the TeV scale. In section 6.2.1, we showed how to obtain the lightest combinations, and its coupling to the Higgs fields and the exotics in a particular class of spectra where the charges of the GUT singlets are related.

There are numerous possible new directions: An obvious extension of this work would be a more in-depth analysis of the phenomenological implications of this class of spectra, in particular its relation to di-photon signatures and their tension with current LHC data. Another route would be to study the consequences of this mechanism to the spectra obtained in chapter 5. As already mentioned previously, a short phenomenological analysis of the spectra with an Abelian factor $U(1) \times \mathbb{Z}_2$ performed in [43] revealed that it was also not possible to put all the Standard Model spectrum on a single **5** or **10** curves. As the charges are all known in these models and so it the number of singlets, these spectra offer a nice set of data to further study the mechanism. Some first steps in that direction have already been started in [199], but it would be interesting to see if some of the spectra of the extended E_8 tree have intrinsically more natural phenomenological properties related to gauge coupling unification than others, and study their origin. It would also be interesting to study General Lesson 6.1 when extended to discrete symmetries and how the pNBG behaves in that case.

Chapter 7

F-theory and Matrix Factorisation

In the previous chapters, we have discussed the singularity structure of some F-theory geometries and the lower dimensional effective theories they give rise to. Except for the codimension one singularity of example (4.11), we have never given an explicit resolution. This procedure is however very important, as one needs to get control over the geometry to use the duality with M-theory and argue for the presence of the physical degrees of freedom in the EFT. Working with a smooth space also enables the use of very powerful results of differential geometry, such as for instance Hodge's theorem, to relate the cohomology groups of the Calabi–Yau with massless modes in the IR.

One the other hand, we have seen in chapter 2 that when desingularising the elliptic fibration only the gauge bosons associated to the Cartan subalgebra of the gauge group G are massless and the gauge group is therefore broken as $G \to U(1)^r$. It is not until one takes the F-theory limit that the W-like bosons become massless and the full gauge symmetry is restored. This breaking to the Abelian subgroup, called going to the Coulomb branch of the theory, is unfortunate as it renders difficult the study of a larger class of theories. One could desire to go to the Higgs branch, corresponding to breaking the gauge group to a subgroup not arising from a Cartan decomposition. In string theory, these systems are realised by so-called T-branes (see [211, 212] for the original works and [179, 180] for a more recent take on those systems), and have been an active field of research in recent years, see [213–217] and reference therein for a non-exhaustive list.

The simplest example of a T-brane is to consider Type IIB string theory with two coincident D7-branes at position $z = 0$ in the coordinates orthogonal to the branes. The spectrum contains a complex adjoint field $\Phi = \varphi_i T_i$ parametrising the fluctuations of the brane in the transverse directions. If this field obtains a vacuum expectation value, we expect the brane to be separated at positions φ_i. The position of the branes is in fact given by the roots of the characteristic polynomial of $\langle\Phi\rangle$, i.e. the solution to the equation $\det(z\mathbb{1} - \langle\Phi\rangle) = 0$. However, there are some breakings that the geometry cannot see: for instance, for upper Triangular vacuum expectation values (giving T-branes their name), the stack recombines as a bound state, but the characteristic polynomial carries no information about it since $\det[z\mathbb{1} - \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)] = z^2$, and the geometry misrepresents the effective theory, as one would still naively expect an $SU(2)$ gauge group. In F-theory, when the singularities are removed, the data of these models are therefore obscured and one would need to go the singular space. We note that for T-branes, there is a proposal to obtain the T-brane information by taking a particular singular limit of the resolved Calabi–Yau [182] preserving at least part of the T-brane data. The framework we will work on in this paper is, however, more general and not only applicable to T-branes.

We have already encountered cases where the geometry does not capture all the information. If one desires to introduce chirality by turning on fluxes, the data about the associated line bundles has to be added *ad hoc*, and is not encoded in the elliptic fibration. For Type IIB orientifolds, it turns out that the fluxes of D7-branes is naturally embedded in Sen's tachyon condensation picture [218]. The idea is that D7-branes can be obtained as a bound state left after putting a pair of brane anti-brane on top of each other. The condensation is parameterised by a so-called tachyon map T, whose domain is related to the flux data. This framework is very useful in F-theory as it pairs very well with Sen's limit to Type IIB, and offers a nice way of comparing one's results with the perturbative limit expectations [219].

A proposal by Collinucci and Savelli [183] is that the tachyon condensation picture is the appropriate framework to the treatment of T-branes. In particular, they showed that the spectrum of states charged under the remnant gauge group was also encoded in the tachyon map, and could be explicitly computed using homological algebra techniques.

In fact, in a companion paper [41], this description of D7-branes as tachyon condensation could be uplifted to F-theory, in a sense that will made be clearer in section 7.1: instead of associating a tachyon map to a set of D7-branes, one can associated a *Matrix Factorisation* (MF) corresponding to a pair of matrices, whose product is the Weierstrass model (times the identity matrix) of the associated singular elliptic fibration. Their proposal is that the tools used to compute the charged spectrum of T-branes can be applied in a similar fashion to obtain the charged spectrum of the effective theory, in a way that does not require resolving the singularities. This method hence completely bypasses the need for blowups, as one works *directly* on the singular space, and therefore gives a definition of F-theory independent of M-theory[1].

Despite applying their strategy to the study of a class of global F-theory compactifications, there is not yet an example involving a non-abelian gauge group in the literature. As a first step towards a better understanding of Matrix Factorisation methods in global models, we will study two examples exhibiting a I_2 singularity. For simplicity, we will start with a fibration having an extra section and therefore an additional $U(1)$ factor. We will then check the spectrum we obtain using MF technology by taking Sen's limit, and compare it with the spectrum found through tachyon condensation. Finally, we will move on to a fibration with an $SU(2)$ singularity where the computations—while following the same methodology—are more involved and make the picture less clear. We find the groups counting the physical degrees of freedom have a higher dimension that one expects, and supplement the original proposal with a further condition. We note that this extra condition could be a consequence of the proposal rather than an additional constraint, although we have been unable to prove

[1]We note that there has been another proposal to deal directly on singular spaces before [159], but we will not discuss it in this chapter.

it.

In section 7.1, we motivate Matrix Factorisation by reviewing the tachyon condensation picture and how to obtain the matter spectrum in Type IIB, and give the mathematical background necessary to do the computations. We then move to specific models with an I_2 singularity in section 7.2. In both cases, we compute the relevant groups counting the degrees of freedom in both the F-theory picture and its weak coupling limit. We summarise our findings in section 7.3. This chapter is based on forthcoming work.

7.1 A First Encounter with Matrix Factorisation

It is known that in Type IIB string theory, all Dp-branes can be obtained by placing pairs of D9- and anti-D9-branes on top of each other, in a phenomenon called tachyon condensation [218]. In that picture, each stack of anti-D9-branes or D9-branes carries a gauge bundle $\mathcal{E}$, respectively $\mathcal{F}$, associated to the gauge theory living on the stack. We will take the two bundles to have the same rank $\mathrm{rk}(\mathcal{E}) = r = \mathrm{rk}(\mathcal{F})$, so as to avoid a net D9-charge after condensation, which would break supersymmetry in an orientifold background.

Under some constraints, if the two stacks are placed on top of each other there will be tachyonic open string modes stretching between the two stacks that will condensate to form a new supersymmetric system. These tachyonic modes can be thought as a linear map

$$T:\ \mathcal{E} \longrightarrow \mathcal{F}. \tag{7.1}$$

To preserve some supersymmetry in the effective theory, we also demand that this map is holomorphic. If it is an isomorphism at each point of the bundles, the two stacks will completely annihilate, leaving nothing behind. If it is not the case, the map will leave a residual D-brane with an associated gauge bundle $\mathcal{G} = \mathcal{F}/(T\mathcal{E})$. For the case where both bundles are of the same rank r, the tachyon map can be seen as a square matrix, and is therefore an isomorphism everywhere, except at points where the determinant vanishes. This defines a complex codimension one surface $\mathcal{S}$: det $T = 0$ which in our case is an 8 dimensional submanifold of the total space, i.e. a D7-brane.

Note that at these points the dimension of the fibre may jump and thus $\mathcal{G}$ is not a proper bundle, but rather a sheaf. In order to not introduce a potentially confusing extra layer of abstract concepts, we will not dive deeper into a fascinating field of mathematics, but will only mention the necessary results and definitions when needed. For our purpose, sheaves can be regarded as bundles where the dimension of the fibers may jump at some points.

In the case of an orientifold, one has, in addition to the holomorphicity condition, to take into account the projection, corresponding to [220] the condition

$$T = -\sigma^* T^t, \qquad \mathcal{E} = \mathcal{F}^{-1}, \tag{7.2}$$

where $\mathcal{F}^{-1}$ is the inverse bundle, defined such that the Whitney sum of $\mathcal{F}$ and its inverse is the trivial bundle $\mathcal{F} \oplus \mathcal{F}^{-1} = \mathcal{O}$, and σ is the involution action.

Let us consider a simple example and find the tachyon condensation associated to Sen's limit of the Weierstrass model (4.30) discussed in section 4.4. We have found that it corresponds to branes at a loci $\mathcal{S}: \ \{\omega^2\xi_+\xi_- = 0\}$. We must therefore find a matrix T satisfying $\det(T) = \omega^2\xi_+\xi_-$ and (7.2). The discriminant factorising into four components, it is not difficult to find that in this case, up to a change of basis, it is given by

$$T = \begin{pmatrix} 0 & \xi_+ & 0 & 0 \\ \xi_- & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}. \tag{7.3}$$

In the language of algebraic homology, that picture of a D7-brane is then simply described by a cochain complex:

$$0 \longrightarrow \mathcal{E} \xrightarrow{T} \mathcal{F} \longrightarrow \mathcal{G} = \mathrm{Coker}(T) \longrightarrow 0. \tag{7.4}$$

If we want to multiply a section s of $\mathcal{G}$ by $\det(T)$, then automatically $s \det(T) = 0$ [41]. This implicitly introduces a map $\tilde{T}: \ \mathcal{F} \to \mathcal{E}$ satisfying the condition

$$\tilde{T} \circ T = \det(T)\mathbb{1} = T \circ \tilde{T}. \tag{7.5}$$

This means that D7-brane on a locus $P_{D7} = 0$ inside the compact dimension is described in the tachyon condensation picture by a pair of two matrices $[T, \tilde{T}]$ satisfying $T\tilde{T} = P_{D7}\mathbb{1}$. This is our first encounter with a *Matrix Factorisation*.

A Hint of Homological Algebra

After having motivated the relation between Matrix Factorisations and D7-branes in Type IIB, we proceed by shortly reviewing some mathematical concepts, following [183, 221], and use this opportunity to establish the notation that will be useful in the remainder of this chapter.

A *Matrix Factorisation* (MF) for a polynomial P over a given ring is a pair of square matrices $[\Phi, \Psi]$ such that

$$\Phi \cdot \Psi = P \cdot \mathbb{1} = \Psi \cdot \Phi. \tag{7.6}$$

The set of all Matrix Factorisations of P is denoted $\mathrm{MF}(P)$. We can immediately see that there are two very simple examples, namely the 1×1 cases $[1, P]$ and $[P, 1]$, called *trivial* and *non-reduced* respectively. A Matrix Factorisation that can be written in a block diagonal form, e.g.

$$\left[\begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix}, \begin{pmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{pmatrix} \right] = [\Phi_1, \Psi_1] \oplus [\Phi_2, \Psi_2], \tag{7.7}$$

is called *reducible*. Moreover, two Matrix Factorisations $[\Phi_1, \Psi_1]$ and $[\Phi_2, \Psi_2]$ are *equivalent* if there exists a change of basis such that $\Phi_1 = U^{-1}\Phi_2 U$. Note that we will be working with global F-theory models, which means that the entries of the matrices are sections of a given bundle, and the coefficients can therefore vanish on some locus, severely restraining the possible matrices as one needs to require that their inverse exists and are holomorphic everywhere.

The simplest example of a MF is that of the conifold $P = xy - uv$, where in addition to the 1×1 ones, there is only one non-reducible matrix $\left[\left(\begin{smallmatrix} x & u \\ v & y \end{smallmatrix}\right), \left(\begin{smallmatrix} y & -u \\ -v & x \end{smallmatrix}\right)\right]$. This case will be the prototypical example when considering elliptic fibrations, as in the factorised $U(1)$ model can be put in such a form. In fact, in affine space, a hypersurface equation admits non-trivial Matrix Factorisations only if it is singular, and each of them carries some information about the singularity. In the cases we will study, they will correspond to D7-brane data.

Homological algebra is a very adequate framework to use with Matrix Factorisations, as we have seen with the example (7.3). A particular choice of MF defines intrinsically the two sheaves as the domain and codomain of the matrices.

$$0 \longrightarrow \mathcal{E} \xrightarrow{\Phi} \mathcal{F} \longrightarrow \mathcal{G} = \mathrm{Coker}(\Phi) \longrightarrow 0 \tag{7.8}$$

Our main interest in this chapter will be focused on the so-called Ext groups between two complexes. In order to define them, we will first need the notion of cochain maps $\rho_\bullet : \ A_\bullet \to B_\bullet$, a collection of maps ρ_i that can be depicted in the following way:

$$\begin{tikzcd}
\cdots \arrow[r, "d^A_{i-2}"] & A_{i-1} \arrow[r, "d^A_{i-1}"] \arrow[d, "\rho_{i-1}"] & A_i \arrow[r, "d^A_i"] \arrow[d, "\rho_i"] & A_{i+1} \arrow[r, "d^A_{i+1}"] \arrow[d, "\rho_{i+1}"] & \cdots \\
\cdots \arrow[r, "d^B_{i-2}"] & B_{i-1} \arrow[r, "d^B_{i-1}"] & B_i \arrow[r, "d^B_i"] & B_{i+1} \arrow[r, "d^B_{i+1}"] & \cdots
\end{tikzcd} \tag{7.9}$$

By definition these maps must satisfy that each square commutes, i.e. $d^B_i \circ \rho_i = \rho_{i+1} \circ d^A_i$. Two cochain maps $\rho_\bullet$, $\tilde{\rho}_\bullet$ between $A_\bullet$ and $B_\bullet$ are said equivalent if there is a cochain homotopy map $h_\bullet$ between the two complexes, defined by a set of diagonal maps $h_i : \ A_i \to B_{i-1}$ such that $\rho_i - \tilde{\rho}_i = d^B_{i-1} \circ h_i + h_{+1} \circ d^A_i$.

$$\begin{tikzcd}
\cdots \arrow[r, "d^A_{i-2}"] & A_{i-1} \arrow[r, "d^A_{i-1}"] \arrow[d, "\rho_{i-1}"'] & A_i \arrow[r, "d^A_i"] \arrow[d, "\rho_i"'] \arrow[dl, dashed, "h_i"'] & A_{i+1} \arrow[r, "d^A_{i+1}"] \arrow[d, "\rho_{i+1}"] \arrow[dl, dashed, "h_{i+1}"'] & \cdots \\
\cdots \arrow[r, "d^B_{i-2}"] & B_{i-1} \arrow[r, "d^B_{i-1}"] & B_i \arrow[r, "d^B_i"] & B_{i+1} \arrow[r, "d^B_{i+1}"] & \cdots
\end{tikzcd} \tag{7.10}$$

The maps are then denoted $\rho_\bullet \sim \tilde{\rho}_\bullet$. The extension group $\mathrm{Ext}^1(A_\bullet, B_\bullet)$ is then defined as the set of vertical maps where the lower complex is shifted from one to the left, which means that $\rho_i : A_i \to B_{i+1}$ and $h_i : A_i \to B_i$. Note that this definition of the Ext group can be generalised to higher degrees groups Ext^n by shifting the lower complex n times to the left. They will however not be of interest to us in this chapter, and we refer the interested reader to [221] for a complete treatment of these quantities.

As we are interested in Matrix Factorisation, we will restrict to complexes of the form (7.8) for the rest of this thesis. The group $\mathrm{Ext}^1(A_\bullet, B_\bullet)$ between two complexes $A_\bullet$, $B_\bullet$ defined by two MF $[\Phi_1, \Psi_1]$ and $[\Phi_2, \Psi_2]$ is therefore the set of maps ρ, $\tilde{\rho}$ satisfying $\Psi_2 \rho = \tilde{\rho} \Phi_1$.

Pictorially, this is represented as

$$\begin{tikzcd}
 & \mathcal{E}_1 \arrow[r, "\Phi_1"] \arrow[dl, dashed, "g"'] \arrow[d, "\rho"] & \mathcal{F}_1 \arrow[r, "\Psi_1"] \arrow[dl, dashed, "h"'] \arrow[d, "\tilde{\rho}"] & \mathcal{G}_1 \arrow[dl, dashed, "l"'] \\
\mathcal{E}_2 \arrow[r, "\Phi_2"] & \mathcal{F}_2 \arrow[r, "\Psi_2"] & \mathcal{G}_2 &
\end{tikzcd} \tag{7.11}$$

As in this case the complexes are unambiguously defined by the matrix Φ_i and $\tilde{\rho}$ is fixed in terms of ρ since the square must commute, we shall henceforth use the abuse of notation $\rho \in \mathrm{Ext}^1(\Phi_1, \Phi_2)$.

Furthermore, one can use the cochain homotopy relations (7.10) to define an equivalence

$$\begin{aligned}
\rho &\sim \rho + \Phi_2 g + h\Phi_1, \\
\tilde{\rho} &\sim \tilde{\rho} + \Psi_2 h + l\Psi_1.
\end{aligned} \tag{7.12}$$

The group obtained by modding out this equivalence relation is called the *reduced Ext group*, and is denoted $\underline{\mathrm{Ext}}^1(\Phi_1, \Phi_2)$.

Chiral Matter from Ext Groups

After having established the mathematical formalism appropriate to Matrix Factorisation, let us come back to the tachyon condensation interpretation of D7-branes. There, to each stack of D7-branes is associated a complex, and one can raise the question of the physical interpretation of cochain maps. In that picture, they should be related to chiral matter coming from massless excitation of strings stretching from one stack to another.

It was proposed by Collinucci and Savelli in [183] that these degrees of freedom correspond to deformations $\delta T \in \underline{\mathrm{Ext}}(T, T)$ of the tachyon map. Note that the charged matter spectrum missed in the supergravity approach corresponds to the reduced group $\underline{\mathrm{Ext}}^1(T, T)$ rather than $\mathrm{Ext}^1(T, T)$.

7.2 Matrix Factorisation of Non-Abelian F-theory Models

Simultaneously to the proposition that Ext groups of the tachyon map encode the physical charged degrees of freedom in the tachyon condensation picture, Collinucci and Savelli [41] also proposed a way to find the charged degrees of freedom in F-theory, from Matrix Factorisation of the Weierstrass polynomial.

In some sense, this lift of tachyon condensation to F-theory seems very natural: the elliptic fibration of the F-theory compactification is defined by the zero locus of a Weierstrass polynomial P_W and, as argued in General Lesson 4.2, the charged matter is encoded into singular loci of codimension one and two. One therefore expects to find a certain number of Matrix Factorisations $[\Phi_i, \Psi_i]$, of *a priori* various dimensions associated to each locus.

The Collinucci–Savelli proposal is then that there is a Matrix Factorisation $[\Phi_{\rm tot}, \Psi_{tot}]$ such that the chiral spectrum is encoded in Ext groups $\underline{\rm Ext}^1(\Phi_{\rm tot}, \Phi_{\rm tot})$. In terms of matrices, the chiral modes are then similar to turning on off-diagonal elements of the matrix

$$\begin{pmatrix} \Phi_{\rm tot} & \rho \\ 0 & \Psi_{\rm tot} \end{pmatrix}, \qquad \rho \in \underline{\rm Ext}^1(\Phi_{\rm tot}, \Phi_{\rm tot}). \tag{7.13}$$

To accompany their proposal, they gave several local realisations, as well as testing it for a compact $U(1)$-restricted model. There is however no global examples involving non-Abelian singularities so far in the literature. In section 7.2.1, we build on their example to construct the simplest non-trivial extension: an $SU(2)\times U(1)$ model where the extra section is engineered via $U(1)$-restriction [142]. This case is particularly simple because the conifold form involves only 2×2 matrices, making computations quite manageable. In section 7.2.2, we consider the model by Morrison and Park [36] and set one of the coefficients to zero. This enhances the $U(1)$ back to a geometry with only an $SU(2)$ singularity. There, the MFs are more complicated and introduce additional subtleties. We argue why they should lead to the correct description. In both cases we also study Sen's limit and find the charged degrees of freedom by using the tachyon condensation picture.

7.2.1 An $SU(2)\times U(1)$ Model

Let us consider a specialised version of the model introduced in section 4.4 for which we already found the tachyon map (7.3): a model with an I_2 singularity and an extra section given by the Tate model

$$y^2 = x^3 + \frac{b_2}{4}x^2 z + \frac{a_1}{2}a_3 x z^2 + \frac{a_3^2}{4}z^6, \tag{7.14}$$

where we have shifted the coordinates from its Weierstrass form for clarity and set $a_4 = 0$. As described in section 4.1 we recall that the Calabi–Yau condition imposes the coefficients to be sections of the base, namely $x \in H^0(B_n, K_B^{-2})$, $y \in H^0(B_n, K_B^{-3})$ and $a_i \in H^0(B_n, K_B^{-i})$. The discriminant indicates that there an I_2 singularity along the base divisor $\mathcal{S}:\ \{a_3 = 0\}$ and that there is an I_3 enhancement along the curve $\mathcal{C}:\ \{a_3 = 0 = a_2 b_2^2\}$. Introducing the variables $y_\pm = y \pm \frac{z}{2}(a_1 x + a_3 z^2)$ the equation can be recasted into a conifold form

$$y_+ y_- = x^2(x + a_2 z^2). \tag{7.15}$$

In this form, the extra global section is at $[a_3, 0, 1]$. All the interesting phenomena occurs at the fibre point $x = 0 = y$, and we will hence work without loss of generality in a chart where $z = 1$. For such a simple case, there are only two non-trivial irreducible Matrix Factorisations given by

$$\begin{aligned} [\varphi_1, \psi_1] &= \left[\begin{pmatrix} y_+ & x^2 \\ x + a_2 & y_- \end{pmatrix}, \begin{pmatrix} y_- & -x^2 \\ -(x+a_2) & y_+ \end{pmatrix}\right], \\ [\varphi_2, \psi_2] &= \left[\begin{pmatrix} y_+ & x \\ x(x + a_2) & y_- \end{pmatrix}, \begin{pmatrix} y_- & -x \\ -x(x+a_2) & y_+ \end{pmatrix}\right]. \end{aligned} \tag{7.16}$$

From here on, the computation could be simplified by using Knörrer's periodicity [222] to reduce the computations on the hypersurface given by the Weierstrass model to a simpler one having only 1×1 matrices. While computationally advantageous, doing so would lose the pedagogical insights that we will gain using 2×2 matrices, as we will build on this example when considering the more complicated case with no $U(1)$. For more on Knörrer periodicity applied to F-theory, we refer to [41].

We propose that the chiral matter spectrum of this system can be constructed out of the 6×6 Matrix Factorisation

$$[\Phi, \Psi] = \left[\begin{pmatrix} \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \\ 0 & 0 & \varphi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & \psi_2 \end{pmatrix} \right]. \tag{7.17}$$

In order to fully specify the data, we need to provide a domain and codomain to the matrix. The codomain of each matrix φ_i is parameterised by an arbitrary line bundle $\mathcal{L}_i$ and the matrix Φ depends therefore on three different line bundles. However, they can only be fixed up to an overall twist by an arbitrary line bundle that can be used to eliminate one of them. For later convenience we choose it such that the domain and codomain of Φ is

$$\begin{array}{ccc} \mathcal{L}_1 \otimes K_B \otimes (\mathcal{O} \oplus K_B^{-1}) & & \mathcal{L}_1 \otimes K_B^{-2} \otimes (\mathcal{O} \oplus K_B^{-1}) \\ \oplus & & \oplus \\ \mathcal{L}_2 \otimes (K_B^2 \oplus K_B^1) & \xrightarrow{\Phi} & \mathcal{L}_2 \otimes (K_B^{-1} \oplus K_B^{-2}) \\ \oplus & & \oplus \\ \mathcal{L}_2^{-1} \otimes (K_B^{-1} \oplus K_B^{-2}) & & \mathcal{L}_2^{-1} \otimes (K_B^{-4} \oplus K_B^{-6}) \end{array} , \tag{7.18}$$

where the third factor in each entry corresponds to the domain of 2×2 matrices up to the line bundle. We will show that a computation of the Ext groups leads to the expected spectrum and then compare to the Type IIB description. The computation will also make cleat that—modulo a change of basis—this choice is the only one that works.

As this MF is reducible, the computation of the Ext group decomposes into the computation of Ext groups of the 2×2 matrices (7.16). Parametrising ρ as in (7.13) we have

$$\rho = \begin{pmatrix} * & A_1 & A_2 \\ B_1 & * & C_1 \\ B_2 & C_2 & * \end{pmatrix}, \tag{7.19}$$

with $A_{1,2}$, $B_{1,2}$ and $C_{1,2}$ respectively part of $\underline{\mathrm{Ext}}^1(\varphi_1, \varphi_2)$, $\underline{\mathrm{Ext}}^1(\varphi_2, \varphi_1)$, and $\underline{\mathrm{Ext}}^1(\varphi_2, \varphi_2)$. Notice that A_1 are A_2 are not really in the same Ext group, as the complexes are parameterised by different line bundles. In our abuse of notation we will not make the distinction, as computing them will not depend on the precise structure of the (co)domains, but only the functional forms of the matrices. Once they have been obtained, we will be more careful and give each degree of freedom as a section of the appropriate line bundle. We will not consider the diagonal elements of ρ, are they correspond to neutral degrees of freedom. Indeed, ρ has the same domain and codomain as Φ, and the block diagonal elements are section of certain powers of the canonical bundle and not the line bundles $\mathcal{L}_i$. As we will see later that the line bundles are associated to $U(1)$ fluxes.

Let us find the most general form of $C_1 \in \underline{\mathrm{Ext}}^1(\varphi_2, \varphi_2)$. Taking the domains and codomains into account it is defined through the complexes

$$\begin{tikzcd}
 & \mathcal{E} \arrow[r, "\varphi_2"] \arrow[dl, dashed, "g"'] \arrow[d, "\rho"] & \mathcal{E} \otimes K_B^{-3} \arrow[r, "\psi_2"] \arrow[dl, dashed, "h"'] \arrow[d, "\tilde{\rho}"] & \mathcal{E} \otimes K_B^{-6} \arrow[dl, dashed, "l"'] \\
\mathcal{F} \arrow[r, "\varphi_2"] & \mathcal{F} \otimes K_B^{-3} \arrow[r, "\psi_2"] & \mathcal{F} \otimes K_B^{-6} &
\end{tikzcd} \tag{7.20}$$

where $\mathcal{E} = \mathcal{L}_2 \otimes K_B^2 \otimes (K_B^{-1} \oplus \mathcal{O})$ and $\mathcal{F} = \mathcal{L}_2^{-1} \otimes K_B^{-1} \otimes (K_B^{-1} \oplus \mathcal{O})$ are the domain of φ_2 with the appropriate line bundle. ρ must satisfy two conditions: in order for the square to commute we must have $\psi_2\rho = \tilde{\rho}\varphi_2$, and it is defined up to homotopies $\rho \sim \rho + \varphi_2 g + h\varphi_2$. Let us start from the latter. It is possible to find a combination of g and h such that

$$\rho \sim \rho + \begin{pmatrix} \mu_1 x + \mu_2(y + a_3) & \nu_1 x + \nu_2 y + \nu_3 a_3 \\ * & \kappa_1 x + \kappa_2(y - a_3) \end{pmatrix}, \tag{7.21}$$

where $*$ depends on a combination of all the coefficients above. This means that up to homotopies ρ_{12} does not depend on either $x,\ y,\ a_3$ and that ρ_{11} and ρ_{22} do not depend on s and cannot factor $y \pm a_3$, respectively. All their dependencies is then transferred into the coefficient ρ_{21}. Performing a similar analysis for $\tilde{\rho}$ reveals that both $\tilde{\rho}_{12},\ \tilde{\rho}_{22}$ do not depend on s and cannot factor $y + a_3$. In particular, that means that ρ_{12} is forced to be localised on the divisor of the base $\{x = y = a_3 = 0\}$, and therefore a section of $H^0(\mathcal{S}, \mathcal{L}'|_{\mathcal{S}})$, where $\mathcal{L}'$ is a line bundle depending on the domains and codomains of the matrices.

After having used all homotopies to get rid of the dependences on some coefficients, or in more homological language, used the equivalence to force the coefficients to be sections of a bundle restricted on a particular locus, we can turn our attention to the commutation constraint. Consider the entry $1,2$ of $\psi_2\rho = \tilde{\rho}\varphi_2$

$$(y - a_3)(\rho_{12} - \tilde{\rho}_{12}) = x(\tilde{\rho}_{11} + \rho_{22}). \tag{7.22}$$

From the homotopies, only $\tilde{\rho}_{11}$ can depend on s and one must have $\tilde{\rho}_{12} = \rho_{12},\ \tilde{\rho}_{11} = -\rho_{22}$. Performing a similar analysis for the rest of the entries one finds that the most general solution in that particular homotopy class is

$$\underline{\mathrm{Ext}}^1(\varphi_2, \varphi_2) = \left\{ \begin{pmatrix} 0 & \tau_1 \\ -(x + a_2)\tau_1 & 0 \end{pmatrix} \middle|\ \tau_1 \in H^0(\mathcal{S}, \mathcal{L}'|_{\mathcal{S}}) \right\}. \tag{7.23}$$

Indeed, we have at that point not needed to assume anything about the entries and the analysis carries similarly for C_2. For the complexes (7.20), we find that ρ_{21} must be a section $H^0(\mathcal{S}, \mathcal{L}_2^{-2} \otimes K_B^{-5}|_{\mathcal{S}})$ where the restriction to $\mathcal{S}$ is due to the homotopies.

A similar analysis can be performed for $\underline{\mathrm{Ext}}^1(\varphi_2, \varphi_1)$. There, the procedure carries in a comparable fashion. The only difference is that it is possible to use the homotopies to localise the degree of freedom on the curve $\mathcal{C}$, and one finds that its most general form is given by

$$\underline{\mathrm{Ext}}^1(\varphi_2, \varphi_1) = \left\{ \begin{pmatrix} 0 & \alpha \\ -s\alpha & 0 \end{pmatrix} \middle| \; \alpha \in H^0(\mathcal{C}, \mathcal{L}'|_{\mathcal{C}}) \right\} \tag{7.24}$$

The other Ext groups are found in a similar fashion, and all depend on only one physical degree of freedom:

$$A_i = \alpha_i \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \qquad B_i = \beta_i \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \qquad C_1 = \tau_i \begin{pmatrix} 0 & 1 \\ x + a_2 & 0 \end{pmatrix}. \tag{7.25}$$

The coefficients α_i, β_i, and τ_i have localisation properties following from (7.23) and (7.24). The line bundles are easily inferred from equation (7.18), and one finds that they are sections of:

$$\begin{aligned} \alpha_1 &\in H^0(\mathcal{C}, \mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \otimes K_B^{-3}\big|_{\mathcal{C}}) && (+1, -1) \\ \alpha_2 &\in H^0(\mathcal{C}, \mathcal{L}_1 \otimes \mathcal{L}_2\big|_{\mathcal{C}}) && (+1, +1) \\ \\ \beta_1 &\in H^0(\mathcal{C}, \mathcal{L}_1^{-1} \otimes \mathcal{L}_2 \otimes K_B^{-1}\big|_{\mathcal{C}}) && (-1, +1) \\ \beta_2 &\in H^0(\mathcal{C}, \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1} \otimes K_B^{-4}\big|_{\mathcal{C}}) && (-1, -1) \\ \\ \tau_1 &\in H^0(\mathcal{S}, \otimes \mathcal{L}_2^{2} \otimes K_B\big|_{\mathcal{S}}) && (0, +2) \\ \tau_2 &\in H^0(\mathcal{S}, \mathcal{L}_2^{-2} \otimes K_B^{-5}\big|_{\mathcal{S}}) && (0, -2) \end{aligned} \tag{7.26}$$

All of the sections of these line bundles are reminiscent of equation (6.1), i.e. a group counting chiral fields over which two $U(1)$ fluxes have been turned on. The last column (7.26) gives their charges, obtained by looking at the degree of the two line bundles. By inspecting the charges, it ensues that the second $U(1)$ seems to correspond to the Cartan subalgebra of $SU(2)$. Indeed $\tau_{1,2}$ have both the correct charges and localisation properties to correspond fields coming from the adjoint, while the pairs $(\alpha_i, \beta_i)_{i=1,2}$ have the Cartan charges and localisation of chiral/anti-chiral matter coming from the fundamental representation. This is confirmed by looking at the index of the two pairs, using the Hirzebruch-Riemann-Roch theorem (see A):

$$I_1 = \#\alpha_1 - \#\beta_1 = \int_{\mathcal{C}} (a_1(\mathcal{L}_1) - c_1(\mathcal{L}_2) + c_1(B_3))|_{\mathcal{C}}\,, \tag{7.27}$$

$$I_2 = \#\alpha_2 - \#\beta_2 = \int_{\mathcal{C}} (c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) - 2c_1(B_3))|_{\mathcal{C}}\,. \tag{7.28}$$

Notice that in a special case where $\mathcal{L}_2|_{\mathcal{S}} = K_B^{-\frac{3}{2}}|_{\mathcal{S}}$, the index of the pairs $\alpha_1,\ \alpha_2$, and $\beta_1,\ \beta_2$ coincide, and the $SU(2)$ symmetry is restored. Indeed, $\tau_1,\ \tau_2$ recombine into a full adjoint of $SU(2)$, and α_i and β_i recombined into $SU(2)$ doublets of charge ± 1.

The Matrix Factorisation (7.17) therefore reproduces correctly the degrees of freedom one expects, without the need for any blowup or other resolution process. Moreover, at no point have we introduced flux data *ad hoc*, as the line bundles appear as constituents of the domain and therefore data of the Matrix Factorisation.

Sen's Limit

As a consistency check, let us compare these results with the matter spectrum obtained via tachyon condensation at weak coupling. Sen's limit in this case is almost the same as the one used to examplify the procedure in section 4.4. Let us recap it briefly: we have $\Delta \sim \varepsilon^2 a_2^2 a_3^2 (a_1^2 - 4a_2)$. Plugging back in the definition of the orientifold, $a_2 = \xi^2$, the locus of the D7-branes is $\{a_3^2 \xi_+ \xi_- = 0\}$ with $\xi_\pm = \xi \pm \frac{a_1}{2}$, and we have the following picture: there is a stack of two coindident D7-branes on the locus $\mathcal{S} : \{a_3 = 0\}$, and a brane-anti-brane image $\mathcal{D}_\pm : \{\xi_\pm = 0\}$ intersecting only on the orientifold plane $\xi = 0$, as shown in figure 7.1.

We thus expect six different charged degrees of freedom: two states from strings stretching from one brane of the stack to the other that are image of one another, and four additional degrees of freedom associated to strings stretching from $\mathcal{D}_\pm$ to either branes of the stack. Notice that a brane stretching from $\mathcal{D}_+$ to the first brane of the stack is the image of the string going from the second brane of the stack and ending on $\mathcal{D}_-$. In figure 7.1, we have shown only the string going from or ending on $\mathcal{D}_-$.

The tachyon map is the same as that of (7.3) up to redefinition of the coefficients. Taking into account that the presence of the orientifold restricts the domain of the map via equation (7.2), one finds that the condensation is described by the following data:

$$\mathcal{E}^{-1} = \begin{array}{c} \mathcal{L}_1^{-1} \\ \oplus \\ \mathcal{L}_1 \otimes K_B \\ \oplus \\ \mathcal{L}_2^{-1} \\ \oplus \\ \mathcal{L}_2 \otimes K_B^3 \end{array} \xrightarrow{\begin{pmatrix} 0 & \xi_+ & 0 & 0 \\ \xi_- & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & -a_3 & 0 \end{pmatrix}} \mathcal{E} = \begin{array}{c} \mathcal{L}_1 \\ \oplus \\ \mathcal{L}_1^{-1} \otimes K_B^{-1} \\ \oplus \\ \mathcal{L}_2 \\ \oplus \\ \mathcal{L}_2^{-1} \otimes K_B^{-3} \end{array} \tag{7.29}$$

It defines two complexes from which we can compute the Ext group:

$$\begin{tikzcd} & \mathcal{E}^{-1} \arrow[r, "T"] \arrow[d, "\delta T"] \arrow[dl, dashed, "g"'] & \mathcal{E} \arrow[r, "\tilde{T}"] \arrow[d, "\delta\tilde{T}"] \arrow[dl, dashed, "h"'] & \mathcal{E}^{-1} \arrow[dl, dashed, "l"'] \\ \mathcal{E}^{-1} \arrow[r, "T"'] & \mathcal{E} \arrow[r, "\tilde{T}"'] & \mathcal{E}^{-1} & \end{tikzcd} \tag{7.30}$$

We then follow the same strategy we used in the F-theory context: first one uses the homotopies to get rid of the dependencies on the coefficients, localising them along a given locus, and then imposing the commutation of the square. In this case, there are only two non-trivial equations, which are of the form $-a_3 \xi_+ \delta T_{21} = \xi_- \delta\tilde{T}_{12}$. Choosing the homotopy maps adequately, the dependency on ξ_- of δT_{21} can be removed and it ensues that $\delta T_{21} = 0 = \delta\tilde{T}_{12}$.

The same reasoning can be applied to the other equation and one is left with 14 degrees of freedom. However, like the tachyon maps, δT must also be orientifold invariant and one must impose the condition $\delta T = -\sigma^*(\delta T)^t$, yielding

$$\rho = \begin{pmatrix} 0 & 0 & \alpha_2(\xi_-) & \alpha_1(\xi_-) \\ 0 & 0 & \beta_1(\xi_+) & \beta_2(\xi+) \\ -\alpha_2(-\xi_+) & -\beta_1(-\xi_-) & \tau_1\xi & \tau_3(\xi) \\ -\alpha_1(-\xi_+) & -\beta_2(-\xi_-) & -\tau_3(-\xi) & \tau_2\xi \end{pmatrix}. \tag{7.31}$$

None of the coefficients depend on a_3, while the coefficients α_i, β_i additionally do not depend on ξ_+, ξ_- respectively. They have therefore the same properties as the degrees of freedom in F-theory. We can interpret α_i as a string stretching from the i-th brane of the stack to $\mathcal{D}_+$, while $\beta_i(\xi_+)$ is the brane stretching from the i-th brane to $\mathcal{D}_-$. Similarly, it can be viewed as its image $-\beta_i(-\xi_-)$, stretching from the i-th brane of the stack and ending in $\mathcal{D}_+$, i.e. with the opposite orientation to that of α_i.

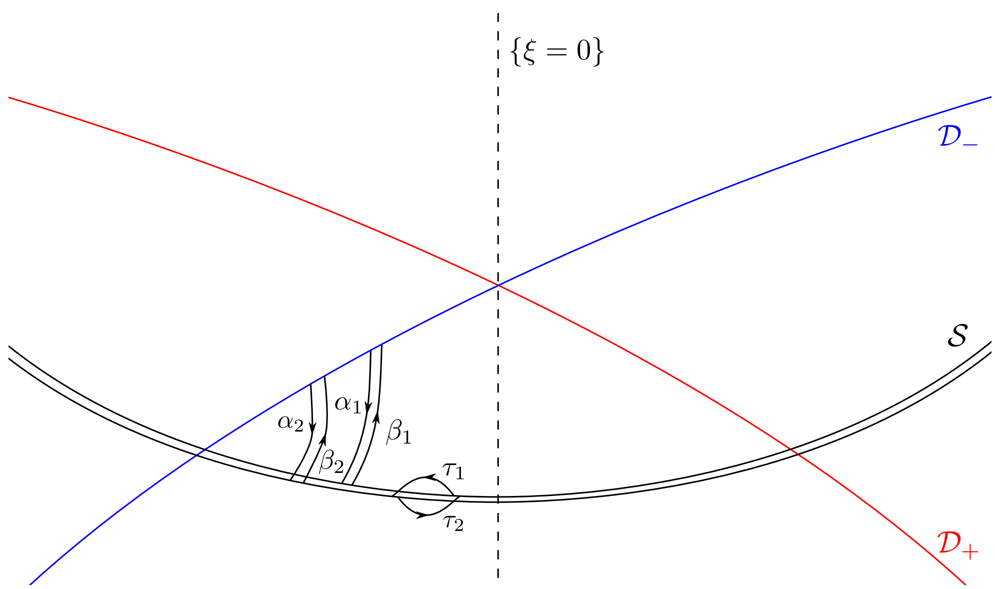


Figure 7.1: Summary of the degrees of freedom associated to the fluctuations (7.31). We show only the states associated to strings stretching from or to $\mathcal{D}_-$. Their image end or start on $\mathcal{D}_+$ with opposite orientations.

The coefficient τ_3 corresponds to a deformation of a_3, and can be understood as modifying the center of mass of the stack, or splitting it in a brane-image brane system. This degree of freedom is therefore neutral, and of no particular interest to us.

Defining the fluxes associated to the Cartan of the $SU(2)$ stack and the $U(1)$ of the brane image-brane system by extracting the usual factor of the canonical bundle

$$F_{SU(2)} = \big(c_1(\mathcal{L}_2) - \frac{3}{2}c_1(B_3)\big)\big|_{\mathcal{S}}, \qquad F^{\pm}_{U(1)} = \pm\big(c_1(\mathcal{L}_1) - \frac{1}{2}c_1(B_3)\big)\big|_{\mathcal{D}_\pm}, \tag{7.32}$$

the chiral indices found in the F-theory picture (7.27) can easily be verified to agree, using the relations

$$I_1 = \int_{\mathcal{C}} \left(F^{+}_{U(1)} - F_{SU(2)} \right), \qquad I_2 = \int_{\mathcal{C}} \left(F^{+}_{U(1)} + F_{SU(2)} \right), \tag{7.33}$$

where the integrand is understood to be restricted to the matter curve $\mathcal{C} = \mathcal{S} \cap \mathcal{D}_+$.

The tachyon condensation picture thus confirms the results we obtained on the singular space via Matrix Factorisation and lends more weight to the Collinucci–Savelli proposal. In this point of view, the flux data again arises from the definition of the tachyon map, and is not put in by hand, but comes from the restriction of the allowed line bundles from the orientifold projection condition.

7.2.2 Towards an Example Without Abelian Factors

In the previous example, the $U(1)$-restriction enabled us to write the Weierstrass model as conifold form, and therefore the Matrix Factorisations had a very simple form. In this section we will consider an example where there is only an $SU(2)$ singularity. This model is based on that of Morrison and Park [36], the most general model for a $U(1)$ with fields of charge one and two. Setting one of the coefficients to be trivial leads to an "unHiggsing" to a full $SU(2)$ model, where the charge one and two fields enhance to fundamental and adjoint representations respectively. The Weierstrass equation for this elliptic fibration, again in shifted coordinates for simplicity, is given by

$$P_W = y^2 - x^3 - a_2 x^2 - a_1 a_3 x - a_0 a_3^2 = 0. \tag{7.34}$$

This hypersurface equation is similar to that of our $SU(2) \times U(1)$ model (7.14), but cannot be put in conifold form, as a_0 is generically not a square. Its discriminant exhibits an I_2 singularity along the divisor $\mathcal{S}: \ \{a_3 = 0\}$:

$$\Delta = a_3^2 \left(a_2^2(-a_1^2 + 4a_0 a_2) + a_3(4a_1^3 - 18 a_0 a_1 a_2) + 27 a_0^2 a_3^2 \right). \tag{7.35}$$

It enhances to an I_3 singularity along the curve $\mathcal{C}: \ \{a_3 = 0 = a_2^2(4a_0 a_2 - a_1^2)\}$ signalling the presence of a fundamental representation. The Calabi–Yau condition forces the coefficients to be sections of:

$$\begin{aligned} a_0 &\in H^0(B_n, \mathcal{J}^2), & a_1 &\in H^0(B_n, \mathcal{J} \otimes K_B^{-1}), \\ a_2 &\in H^0(B_n, K_B^{-2}), & a_3 &\in H^0(B_n, \mathcal{J}^{-1} \otimes K_B^{-3}), \end{aligned} \tag{7.36}$$

where $\mathcal{J}$ is a line bundle introduced because the coefficients a_0, a_1, a_3 appear only in products in (7.34) we must take into account the possibility of a twist.

This absence of a conifold form introduces additional subtleties with respect to the previous model, of which the rest of this section is dedicated to exploring. Our strategy is the following: To get a grip on the novelties coming with the absence of $U(1)$, we will start by studying Sen's limit via the condensation, and doing so we will already encounter that the line bundle $\mathcal{J}$ requires particular attention. Then, moving to the F-theory picture, we will find that the generalisation of the MF (7.17) to this case has more degrees of freedom than we expect from Sen's limit. We perform some consistency checks and propose that for adjoint fields, the correct number of degrees of freedom is found by imposing an additional condition.

Type IIB Tachyon Condensation

Let us take Sen's limit for the Weierstrass model of (7.34). Extracting the correct factor of ε according to (4.26), the discriminant takes the form

$$\Delta = \varepsilon^2 a_2^2 a_3^2 (a_1^2 - 4\xi^2 a_0) + \mathcal{O}(\varepsilon^3), \tag{7.37}$$

where we readily used the definition of the orientifold on which the Type IIB theory is compactified, given by $X_3: \ \xi^2 = a_2$. We therefore have an O7-plane on the locus $\{\xi = 0\}$, and D7-branes on $\{a_3^2(a_1^2 - 4\xi^2 a_0) = 0\}$. This locus factorises and we have a stack of two branes on the divisor $\mathcal{S}: \ \{a_3 = 0\}$, and a Whitney brane on $\mathcal{W}: \ \{W = a_1^2 - 4\xi^2 a_0 = 0\}$. Notice that unlike the previous example, the two image branes are recombined into a single brane due to a_0 being a full-fledged section, and not a mere coefficient anymore. The setup is summarised pictorially in figure 7.2.

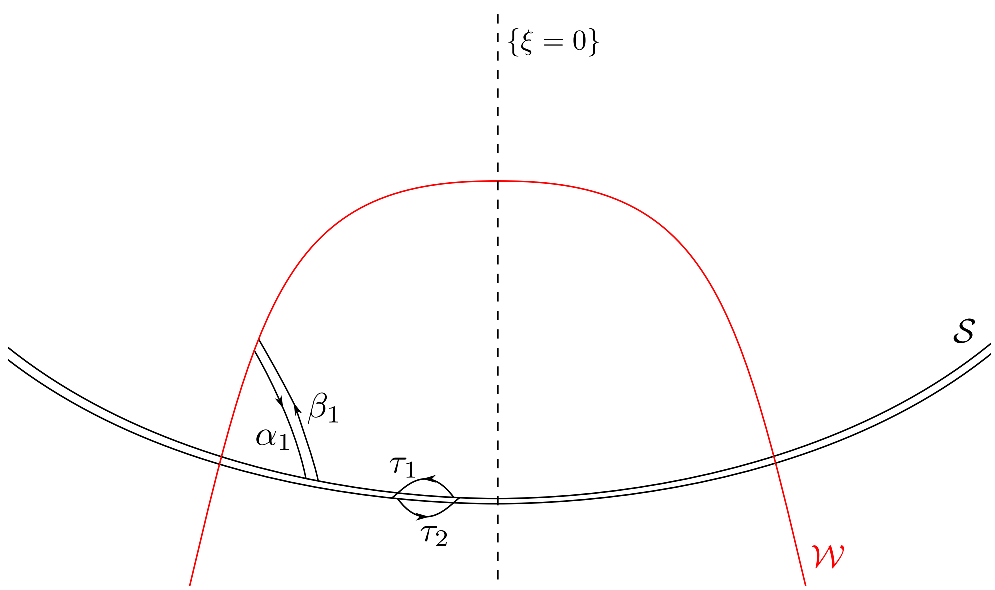


Figure 7.2: Sen's limit of the $SU(2)$ model (7.34). The dashed line represents the orientifold plane $\{\xi = 0\}$. The Whitney brane (in red) is located on $a_1^2 - \xi^2 a_0$. The image of the two degrees of freedom end and start from the second brane of the $SU(2)$ stack.

This system can again be view as a tachyon condensation, where the associated tachyon map and its partner are easily found remembering the orientifold projection:

$$T = \begin{pmatrix} -4a_0\xi & a_1 & 0 & 0 \\ -a_1 & \xi & 0 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & -a_3 & 0 \end{pmatrix}, \qquad \tilde{T} = \begin{pmatrix} a_3^2\xi & -a_1a_3^2 & 0 & 0 \\ a_1a_3^2 & -4a_0a_3^2\xi & 0 & 0 \\ 0 & 0 & 0 & -a_3W \\ 0 & 0 & a_3W & 0 \end{pmatrix}. \tag{7.38}$$

We again need to specify the domain $\mathcal{E}^{-1}$ of T, remembering to take into account the orientifold condition $\mathcal{E}^{-1} \xrightarrow{T} \mathcal{E}$. It is straightforward to show that contrary to the $SU(2) \times U(1)$ case, the sheaves are completely fixed up to only one arbitrary line bundle $\mathcal{L}$:

$$\mathcal{E}^{-1} = \begin{matrix} \mathcal{J}^{-1}\otimes K_B^{\frac{1}{2}} \\ \oplus \\ K_B^{\frac{1}{2}} \\ \oplus \\ \mathcal{L}^{-1} \\ \oplus \\ \mathcal{L}\otimes K_B^{3}\otimes\mathcal{J} \end{matrix} \xrightarrow{\quad T \quad} \mathcal{E} = \begin{matrix} \mathcal{J}\otimes K_B^{-\frac{1}{2}} \\ \oplus \\ K_B^{-\frac{1}{2}} \\ \oplus \\ \mathcal{L} \\ \oplus \\ \mathcal{L}^{-1}\otimes K_B^{-3}\otimes\mathcal{J}^{-1} \end{matrix} \tag{7.39}$$

The group counting the charged spectrum, $\underline{\mathrm{Ext}}(T,T)$, is then found by considering the complexes (7.30) using the input data above. The computation proceeds in the same way as that of the $SU(2)\times U(1)$ model, with the difference that one can remove the dependency on the off-diagonal elements on the Whitney brane locus, rather than that of the brane or its image as was the case for the $SU(2)\times U(1)$ model. After imposing the orientifold condition, one is left with the following result

$$\delta T = \begin{pmatrix} 0 & 0 & \alpha_2(\xi) & \alpha_1(\xi) \\ 0 & 0 & \beta_1(\xi) & \beta_2(\xi) \\ -\alpha_2(-\xi) & -\beta_1(-\xi) & \tau_1\xi & \tau_3(\xi) \\ -\alpha_1(-\xi) & -\beta_2(-\xi) & -\tau_3(-\xi) & \tau_2\xi \end{pmatrix}. \tag{7.40}$$

Again, τ_3 corresponds to a neutral degree of freedom associated to a deformation of the stack, while defining $\mathcal{C} = \mathcal{S} \cap \mathcal{W}$, the remaining elements are sections of:

$$\begin{aligned} \alpha_1 &\in H^0(\mathcal{C}, \mathcal{L}^{-1} \otimes K_B^{-\frac{7}{2}}|_{\mathcal{C}}) & \alpha_2 &\in H^0(\mathcal{C}, \mathcal{L}^{+1} \otimes K_B^{-\frac{1}{2}} \otimes \mathcal{J}|_{\mathcal{C}}) \\ \beta_1 &\in H^0(\mathcal{C}, \mathcal{L}^{+1} \otimes K_B^{-\frac{1}{2}}|_{\mathcal{C}}) & \beta_2 &\in H^0(\mathcal{C}, \mathcal{L}^{-1} \otimes K_B^{-\frac{7}{2}} \otimes \mathcal{J}^{-1}|_{\mathcal{C}}) \\ \tau_1 &\in H^0(\mathcal{S}, \mathcal{L}^{+2} \otimes K_B^{1}|_{\mathcal{S}}) & \tau_2 &\in H^0(\mathcal{S}, \mathcal{L}^{+2} \otimes K_B^{-5} \otimes \mathcal{J}^{-2}|_{\mathcal{S}}) \end{aligned} \tag{7.41}$$

It seems that there are more degrees of freedom in the off-diagonal blocs than we would expect from the naive intersecting branes picture, where one would assume that there are only strings stretching from each brane of the stack to the Whitney and their image, totalling three different physical degrees of freedom. However, notice that the pairs α_1, β_2 and α_2, β_1 have the same degree under the line bundle we associated to the Cartan $U(1)$ flux. If the line bundle $\mathcal{J}$ was trivial when restricted on the curve, $\mathcal{J}|_{\mathcal{C}} = \mathcal{O}$, both pairs would be sections of the same line bundle, and we would be left with two line bundles in addition to those localised on the divisor $\mathcal{S}$, which is what we expect. While we were not able to find a rigorous proof, we suspect it to be the case, as away from the orientifold plan, we can use Gauss operations to divide by $\xi \neq 0$ and recast the upper left bloc into the form $\mathrm{diag}(1, W)$. Performing again the computation of the ext group, we find only two degrees of freedom with charge ± 1 under the Cartan $U(1)$.

F-theory Description

Doing the uplift to F-theory, we would therefore expect to have 2 degrees of freedom, one localised on the I_2 divisor and the other on the I_3 curve, coming from the two Ext groups $\underline{\mathrm{Ext}}^1(\varphi_i, \varphi_j)$, as well as their conjugates associated to $\underline{\mathrm{Ext}}^1(\varphi_j, \varphi_i)$. As we have already pointed out at the beginning of this section, there are no 2×2 matrices and we must look for higher dimensional matrices. In [219], it was found that there is a 4×4 MF, which in our case is

$$[\varphi_{4\times4}, \psi_{4\times4}] = \left[\begin{pmatrix} y & x(x+a_2)+a_1a_3 & a_3 & 0 \\ x & y & 0 & -a_3 \\ a_0a_3 & 0 & y & x(x+a_2)+a_1a_3 \\ 0 & -a_0a_3 & x & y \end{pmatrix}, \det(\varphi_{4\times4})\varphi_{4\times4}^{-1}\right]. \tag{7.42}$$

Its domain $\mathcal{E}_{4\times4}$ is rapidly observed to depend on only one arbitrary line bundle, and takes the form

$$\mathcal{E}_{4\times4}(\mathcal{L}) = \begin{matrix} \mathcal{L}\otimes K_B^{-1} \\ \oplus \\ \mathcal{L} \oplus \\ \mathcal{L}\otimes\mathcal{J}\otimes K_B^{-1} \\ \oplus \\ \mathcal{L}\otimes\mathcal{J} \end{matrix} \xrightarrow{\Phi_{4x4}} \mathcal{E}_{4\times4}(\mathcal{L}) \otimes K_B^{-3}. \tag{7.43}$$

We will now argue that this Matrix Factorisation encodes the information about the adjoint representation, and compute its Ext group $\underline{\mathrm{Ext}}^1(\varphi_{4\times4}, \varphi_{4\times4})$. We do so by considering the two complexes

$$\begin{tikzcd} & \mathcal{E}(\mathcal{L}) \arrow[r, "\varphi_{4\times4}"] \arrow[dl, dashed, "g"'] \arrow[d, "\rho"] & \mathcal{E}(\mathcal{L}) \otimes K_B^{-3} \arrow[r, "\psi_{4\times4}"] \arrow[dl, dashed, "h"'] \arrow[d, "\tilde{\rho}"] & \mathcal{E}(\mathcal{L}) \otimes K_B^{-6} \arrow[dl, dashed, "l"] \\ \mathcal{E}(\mathcal{L}') \arrow[r, "\varphi_{4\times4}"] & \mathcal{E}(\mathcal{L}') \otimes K_B^{-3} \arrow[r, "\psi_{4\times4}"] & \mathcal{E}(\mathcal{L}) \otimes K_B^{-6} & \end{tikzcd} \tag{7.44}$$

For the sake of clarity, we will not give the details of the computation here, and only state the result. Details of the computations have however been gathered in appendix C. We find that $\rho \in \underline{\mathrm{Ext}}^1(\varphi_{4\times4}, \varphi_{4\times4})$ depends on four different components that are all localised on the I_2 divisor of the base $\mathcal{S}$

$$\rho = \begin{pmatrix} -a_1\rho_{23} - \rho_{33} & -(s+a_2)\rho_{43} & \rho_{13} & -(s+a_2)\rho_{23} \\ \rho_{43} & a_1\rho_{23} + \rho_{33} & \rho_{23} & -\rho_{13} \\ -a_0\rho_{13} - a_1\rho_{43} & -a_0(s+a_2)\rho_{23} & \rho_{33} & -(s+a_2)\rho_{43} \\ a_0\rho_{23} & a_0\rho_{13} + a_1\rho_{43} & \rho_{43} & -\rho_{33} \end{pmatrix} = \tilde{\rho}. \tag{7.45}$$

An inspection of the domains shows that the four parameters are sections of

$$\begin{aligned} \rho_{13} &\in H^0(\mathcal{S}, \mathcal{L}^{-1} \otimes \mathcal{L}' \otimes K^{-3} \otimes \mathcal{J}^{-1}\big|_{\mathcal{S}}) & \rho_{23} &\in H^0(\mathcal{S}, \mathcal{L}^{-1} \otimes \mathcal{L}' \otimes K^{-2} \otimes \mathcal{J}^{-1}\big|_{\mathcal{S}}) \\ \rho_{33} &\in H^0(\mathcal{S}, \mathcal{L}^{-1} \otimes \mathcal{L}' \otimes K^{-3}\big|_{\mathcal{S}}) & \rho_{43} &\in H^0(\mathcal{S}, \mathcal{L}^{-1} \otimes \mathcal{L}' \otimes K^{-2}\big|_{\mathcal{S}}) \end{aligned} \tag{7.46}$$

The presence of four different degrees of freedom is unexpected, as we argued that there should be only one localised on $\mathcal{S}$, and it seems that there are degrees of freedom that go beyond those found in the tachyon condensation picture. As a consistency check, we can restrict to the case where $a_0 = 1$. Doing so, there it is possible to find a conifold form, and therefore this specialised system can be described in terms of 2×2 matrices. For the Matrix Factorisation (7.42), one can then perform Gauss operation to recast it into a block diagonal matrix

$$\varphi_{4\times 4} \rightsquigarrow \begin{pmatrix} y+a_3 & s(s+a_2)+a_1a_3 & 0 & 0 \\ s & y-a_3 & 0 & 0 \\ 0 & 0 & y-a_3 & s(s+a_2)+a_1a_3 \\ 0 & 0 & s & y+a_3 \end{pmatrix}. \tag{7.47}$$

The Ext group should therefore decompose into Ext group of 2×2 matrices that can be computed almost exactly as those of the $SU(2) \times U(1)$ example. Comparing the restriction of (7.45) in that basis, one finds that they must satisfy

$$\rho_{12} = 0 = \rho_{43} \qquad \rho_{33} = -\frac{a_1}{2}\rho_{23} \tag{7.48}$$

If our proposal that this Ext group indeed counts fields in the adjoint, as their localisation indicates, their numbers should not change and it seems that an additional constraint is needed. We conjecture that, at least in that case, the condition is that if one takes two elements $\rho, \rho' \in \underline{\mathrm{Ext}}^1(\varphi_{4\times 4}, \varphi_{4\times 4})$, they must in turn define a larger Matrix Factorisation $\rho\rho' = P_{\mathrm{def}}\mathbb{1}$. Our motivation for this conjecture is the following: let us consider the 8×8 matrices:

$$\Phi_8 = \begin{pmatrix} \varphi_{4\times 4} & \rho \\ \rho' & \varphi_{4\times 4} \end{pmatrix} \qquad \Psi_8 = \begin{pmatrix} \psi_{4\times 4} & \rho \\ -\rho' & \psi_{4\times 4} \end{pmatrix} \tag{7.49}$$

If we consider the product of the two, the off-diagonal block vanish by definition of the Ext group. The upper left bloc on the other hand reads $P_W \mathbb{1}_{4\times 4} - \rho\rho'$, while the lower-right is $P_W \mathbb{1}_{4\times 4} - \rho'\rho$. Here, ρ and ρ' must have opposite charges under the line bundles associated to both matrices $\varphi_{4\times 4}$ and their product is therefore neutral. As we saw in Sen's limit, such neutral degrees of freedom corresponds to deformations of the Tate model, and it is not a stretch to propose that physical degrees of freedom should reflect such a fact by also forming a Matrix Factorisation. It turns out that the only holomorphic solution reproduces (7.48).

At the time of writing, it is not clear if this condition is a constraint that has to be imposed ad hoc, or if it is embedded in the framework. Indeed, it is not unimaginable that as we are considering complexes and maps between them, this extra condition is a consequence of homological algebra.

So far, we have only discussed degrees of freedom localised on the I_2 divisor, but we have no candidate for a matter curve localised on a codimension two locus. We have found that in addition to 4×4 MF, there is another larger 6×6 MF $[\varphi_{6\times 6}, \psi_{6\times 6}]$ where

$$\varphi_{6\times 6} = \begin{pmatrix} y & \frac{1}{2}a_1 & 0 & a_3 & 0 & x+a_2 \\ xa_3 & y & \frac{1}{2}a_1a_3+x(x+a_2) & 0 & a_3^2 & 0 \\ 0 & x & y & 0 & 0 & -a_3 \\ -a_0a_3-\frac{1}{2}a_1x & 0 & \frac{1}{4}(a_1^2-4a_0(x+a_2)) & -y & \frac{1}{2}a_1a_3+x(x+a_2) & 0 \\ 0 & -a_0 & 0 & x & -y & -\frac{1}{2}a_1 \\ -x^2 & 0 & a_0a_3+\frac{1}{2}a_1x & 0 & -xa_3 & -y \end{pmatrix}, \tag{7.50}$$

and its partner $\psi_{6\times6} = (\varphi_{6\times6})^{-1}/\det(\varphi_{6\times6})^2$. The domain of $\varphi_{6\times6}$, similarly to that of $\varphi_{4\times4}$ is fixed up to arbitrary line bundle.

$$\mathcal{L}\otimes\mathcal{D} = \begin{array}{c} \mathcal{L}\otimes K_B^{-1}\otimes\mathcal{J}^{-1} \\ \oplus \\ \mathcal{L}\otimes K_B^{-3}\otimes\mathcal{J}^{-2} \\ \oplus \\ \mathcal{L}\otimes K_B^{-2}\otimes\mathcal{J}^{-2} \\ \oplus \\ \mathcal{L}\otimes K_B^{-1} \\ \oplus \\ \mathcal{L}\otimes\mathcal{O} \\ \oplus \\ \mathcal{L}\otimes K_B^{-2}\otimes\mathcal{J}^{-1} \end{array} \xrightarrow{\Phi_{6x6}} \mathcal{L}\otimes\mathcal{D}\otimes K_B^{-3} \tag{7.51}$$

A computation of the group $\underline{\mathrm{Ext}}^1(\varphi_{4\times4}, \varphi_{6\times6})$—where the vertical maps are 6×4matrices—can be achieved by considering the two complexes:

$$\begin{tikzcd} & \mathcal{L}\otimes\mathcal{C} \arrow[r, "\varphi_{4\times4}"] \arrow[dl, dashed, "g"'] \arrow[d, "\rho"] & \mathcal{L}\otimes\mathcal{C}\otimes K_B^{-3} \arrow[r] \arrow[dl, dashed, "h"'] \arrow[d, "\sigma"] & \mathcal{L}\otimes\mathcal{C}\otimes K_B^{-6} \arrow[dl, dashed, "l"'] \\ \mathcal{L}'\otimes\mathcal{D} \arrow[r, "\varphi_{6\times6}"] & \mathcal{L}'\otimes\mathcal{D}\otimes K_B^{-3} \arrow[r, "\psi_{6\times6}"] & \mathcal{L}'\otimes\mathcal{D}\otimes K_B^{-6} & \end{tikzcd} \tag{7.52}$$

Again, we do not show explicitly the computation here, but the main steps have been summarised in appendix C. We find that there is one degree of freedom ρ_{53} that is localised on $\mathcal{S}$, and a combination of ρ_{13} and ρ_{43} that is localised on the matter curve $\mathcal{C}$. We have, as expected, a degree of freedom that seems to correspond to an field part of the fundamental representation of $SU(2)$, with its conjugate coming from $\underline{\mathrm{Ext}}^1(\varphi_{6\times6}, \varphi_{4\times4})$, for which the computation proceeds in the exact same way.

The presence of another section localised on the I_2 divisor, one the other hand, is again quite unexpected. It would have the same degree under the Cartan line bundle as the combination localised on the matter curve, which cannot be. From General Lesson 4.2, we know that matter field with such a Cartan charge must come from a codimension two locus. Moreover, we were unable to find a condition similar to the 4×4 case. Indeed, the elements of $\underline{\mathrm{Ext}}^1(\varphi_{4\times4}, \varphi_{6\times6})$ are 6×4 matrices, and we cannot invoke the same argument as before, since they are not square matrices.

The case of the "unHiggsed" Morisson and Park model is therefore more subtle than that of a $U(1)$-restriction. Even though taking a 14×14 MF with a domain similar to (7.18) seems to lead to the correct charges to obtain a fundamental and adjoint representation, we cannot yet explain the extra degree of freedom coming from the Ext group between the 4×4 and 6×6 MF, and a deeper study is needed.

7.3 Summary

In this chapter, we have tested the Collinucci–Savelli proposals [41, 183] in both Type IIB supergravity and F-theory through a study of global non-Abelian models. For the specific model with a $U(1)$-restriction (7.14), the presence of a section ensures that the relevant Matrix Factorisations are 2×2, and computation of the charged spectrum is quite manageable. We find that we can obtain the whole data from a 6×6 MF without ever needing to blow up the singularity. Moreover, the flux data is completely encoded in the computation, and is part of the definition of the MF. We have then checked our results using Sen's limit and the tachyon condensation picture. Here we again find results in accordance with the Collinucci–Savelli proposal, and with the expectation from the naive intersecting branes point of view. There, the flux data is also encoded into a Matrix Factorisation, and matches what we have found in F-theory.

In the case of a model exhibiting only a I_2 singularity and no extra section, we find that the situation is much more subtle. In the tachyon condensation picture, while we a priori find more degrees of freedom than expected, by inspecting the line bundles these degrees of freedom are sections of, we have however good reasons to believe that the proposal is correct, but were unable to prove the triviality of the line bundle $\mathcal{J}$ over the matter curve, which is needed to get the correct matter. In F-theory, the situation is unexpected, as an analysis of the Ext group associated to the two MF we find gives more degrees of freedom than we want. In the case of the group associated to what we believe counts matter transforming in the adjoint representation, we conjecture a condition necessary to reduce the number of degrees of freedom to one. This additional constraint is quite natural, and we do not know if it is something that one needs to impose *ad hoc*, or if can be more directly deduced from the homological algebra framework Matrix Factorisation is naturally associated with.

These issues need to be resolved before pushing the Matrix Factorisation approach to F-theory forward. However, this framework is very attractive, as it completely bypass the resolution procedure and encompasses naturally the flux data, and therefore offers a number of possible future directions to explore. We have here only presented two examples involving a $\mathfrak{su}(2)$ algebra corresponding to an I_2 singularity in Kodaira's classification. When the issues involving the "unHiggsed" Morisson and Park model have been resolved, it should be straightforward to generalise it to the general I_2 singularity as they are similar. This would be the first step to generalise the procedure to the whole A-series, and hopefully enable a full understanding of all ADE singularities. In particular, it would be interesting to understand how the different representations under a given gauge group appear, and what is the equivalent to the chains of $\mathbb{P}^1$'s in the resolution procedure. In particular, in light of the recent achievements of constructing "exotic" representations, i.e. that are not in the fundamental, adjoint, or anti-symmetric representations of the gauge group [223, 224], and their difficulty in resolving the singularities, it would be interesting to see what Matrix Factorisation has to say about the matter. Moreover, as the Ext group also encodes information about complex structure deformations of an elliptic fibration, it also opens the way to a study of Higgsing chains, particularly those leading to T-branes. These could be used to find explicit realisations of the classification of the spectra discussed in chapter 5, and potentially give insights on the nature of E_8 in F-theory GUTs.

Another direction would be to apply the Collinucci–Savelli proposal to the study of discrete symmetries. There are now quite a few examples of explicit realisations of models with a $\mathbb{Z}_2$ or $\mathbb{Z}_3$ symmetry, as well as models with torsion, i.e. gauge group of the form $G/\mathbb{Z}_n$, see e.g. [39, 40, 164]. In these cases, the equation defining the elliptic fibration is not a Weierstrass equation, but rather a hypersurface in another projective space. The two descriptions are however related by a birational map and it begs the question of how the groups counting the physical quantities change under the map. Furthermore, we have seen that performing a $U(1)$ restriction of the $SU(2)$ model, we reduced the 4×4 MF to two 2×2 MF, and one can check a similar result for the 6×6 case. Breaking to a $U(1)$ therefore simplifies the MF's, and one may wonder what happens when we further break the $U(1)$ to $\mathbb{Z}_n$.

Chapter 8

Conclusions

String theory has vastly improved our understanding of the structure of Effective Field Theories (EFTs), and more specifically, F-theory has proven to be a great laboratory to study gauge theories in a non-perturbative regime. As candidates of a theory of Quantum Gravity, they also serve as a playground to test constraints on EFTs. Beyond the formal insights string theory has led to, it has also revealed to be an active field of research for phenomenology. Indeed, its close relation to Planck scale physics makes it an ideal regime to probe inflation and other cosmological phenomena of the early Universe. On the other hand, the natural description of gauge theories as brane fluctuations is closely tied to particle phenomenology.

In this thesis, we have explored some constraints of string theory on EFTs by considering a broad range of setups coming from both the closed and open sectors. We have first argued that the structure of EFTs is tightly related to geometry by reviewing the basic notions of σ-models, which led us to the Geometric Principle (General Lesson 2.1). In particular, we have argued that in presence of supersymmetry, the possible target manifolds are very constrained, and in the context of string theory, the relevant geometric quantities of the effective theory are connected to the structure of the extra dimensions.

We first started by exploring the closed sector of string theory in chapter 3. More precisely, we studied the allowed field ranges in the context of axion monodromy in Type IIA supergravity compactified on a Calabi–Yau orientifold. In those cases, the axion is given a potential induced by fluxes that breaks the shift symmetry, in principle allowing for large field excursions. The gravitational backreaction modified the axionic target space in such a way that after a certain critical value, the distance travelled by the canonically normalised field is logarithmic (see General Lesson 3.3). We moreover found that while one would naively expect that this logarithmic behaviour can be delayed arbitrarily late by tuning the flux parameters, a homogeneity property of the stabilisation equations forces the proper distance travelled until the critical value to be flux independent and of order one. We have tested this property for a realistic Calabi–Yau setup and twisted tori, and found good evidence that this property is satisfied by all of them. The mechanism we have discovered censures the super-Planckian axion excursions in axion monodromy, and lends some weight to the Weak Gravity and Swampland Conjectures [19, 20]. It would be interesting to study field excursions in other string theoretic realisations to find a better understanding of the origin of this mechanism.

We then moved on to the open sector of string theory, starting with a review of F-theory in chapter 4, where we explained the connection between gauge theories and the singularity structure of an elliptic fibration. This non-perturbative generalisation of Type IIB supergravity is particularly adapted to model building, as it allows one to engineer effects that can be forbidden in the perturbative regime of Type IIB, such as a top Yukawa coupling. In particular, F-theory allows to engineer elliptic fibrations that give rise to Grand Unified Theories (GUTs) in the IR.

In local F-theory models, all the spectra of F-theory GUTs with $U(1)$ factors can be embedded in a decomposition of the **248** adjoint representation of E_8. In global models with an $SU(5)$ GUT group however, we demonstrated in chapter 5 through an explicit example that all spectra cannot be embedded into such a decomposition of E_8. The $U(1)$ factor of this specific example could be broken by complex structure deformations to a model with a remnant $\mathbb{Z}_2$ symmetry, which cannot be obtain from E_8. In general lesson 5.1, we proposed an extension of the E_8 spectra by introducing additional GUT singlets charged under the Abelian sector such that there is a gauge invariant cubic operator between any three representations, spectra we called complete network. We then proceeded to a complete classification of the spectra that can be obtained by following the various Higgsing chains. When comparing this classification with the literature, we find that out of 30 models, all 27 forming flat networks could be embedded in our classification when turning off non-flat points. Of the remaining three models, one could not be made flat over a generic base, and the other two did not form complete networks. Our results are encouraging, and form an additional step to uncover a possible role of E_8 in F-theory GUTs. The classification has since been shown to form a strict subset of the charges allowed in the presence of $U(1)$ factors by a geometric analysis of the fiber structure [225], techniques that are a priori disconnected from ours.

In light of this classification, we then explored how to break the $SU(5)$ GUT group to that of the Standard Model, and summarised the generic features of hypercharge flux breaking with an extra $U(1)$ in General Lesson 6.1. These properties were however defining a larger class of spectra that can be studied in their own right, with a well-motivated UV origin. In particular, the presence of an extra $U(1)$ field could be used to forbid the Higgs mass term, and therefore palliate the μ-problem. The constraints coming from anomaly unification—asking that the anomaly of the MSSM be proportional to the ones related to $SU(5)$—then demand the presence of exotic states in addition to that of the MSSM that are quasi-vector-like and therefore cannot acquire a mass term. If the symmetry protecting these operators is taken to be global, which in the string theory setup can be realised by giving a Planck scale mass to the $U(1)$ gauge boson, there will be a pseudo-Nambu–Goldstone boson gaining a mass due to non-perturbative effects. As the breaking come from operators that are strongly suppressed, the pseudo-Nambu–Goldstone boson is naturally light and could potentially be of order of the TeV scale. We exemplified how to find it, and how it couples to the Higgs fields and the exotics for a minimal spectrum in which three singlets satisfying a charge relation are necessary. We have however not performed a thorough phenomenological analysis in this thesis, and it is only a first step towards a better understanding of this mechanism. Such a study would be very interesting to do, as the light scalar could be in principle have a dominant diphotons decay channel, and be measured by currently and future collider experiments.

Finally, we focussed our attention on how charged degrees of freedom arise in Type IIB supergravity and F-theory in a way that naturally encompasses the flux data in chapter 7. Following the two Collinucci–Savelli proposals [41, 183], we described a method to obtain the charged spectrum in F-theory without the need to go through the resolution procedure and M-theory duality to make sense of the degrees of freedom, and explored the weakly coupled limit of two models exemplifying the procedure. In the context of F-theory, one uses Matrix Factorisation, while in Type IIB, the charged spectrum is obtained from tachyon condensation. The charged states were obtained by computing the Ext groups of the MF or the tachyon map, depending on which picture one works with. This method had the enormous advantage that it embeds the flux data in a natural way as the domain of the maps, which is usually provided as extra information.

We applied the proposals to two different global examples involving an $SU(2)$ singularity. In the simpler case of a model with an extra section in addition to the $SU(2)$ gauge group, we found that Collinucci–Savelli proposals is satisfied, and a computation of the Ext groups reproduced correctly all degrees of freedom and the fluxes in F-theory. We then took Sen's limit to obtain the type IIB description of that system, and found that the result of this perturbative limit were in agreement with those of F-theory. In the case of the Morrison and Park model [36] enhanced to an I_2 singularity, the absence of a conifold form proved more challenging, as we had to deal with MFs of size 4×4 and 6×6. A computation of the associated Ext groups revealed that there are more degrees of freedom that one would expect in both pictures. For Type IIB, we however have good evidence that some of them describes the same physical states, but left a proof for future work. In F-theory, we conjectured a natural extra condition that leads to a correct result for fields transforming in the adjoint representation. We could on the other hand not determine a well-motivated condition getting rid of the extra parameter in the Ext group that we expect counts chiral fields in the fundamental representation. We would again like to stress that it is at this stage not clear if the extra condition necessary is really something that we have to impose by hand. It could indeed be a consequence of the description in terms of algebraic homology, and therefore be embedded into the framework we studied. This analysis was but a first step in the Matrix Factorisation program and can be extended in numerous ways. It would be for instance interesting to find a systematic way to find the particular MF associated to a given matter locus, and understand what is the analogue of the chains of $\mathbb{P}^1$ wrapped by M2-branes in the resolution process.

While we have described several ways in which string theory constrains four dimensional Effective Field Theories, this framework still has shortcomings, and we are still far from a situation where "*all that remains is more and more precise measurement*". Indeed, to paraphrase Lord Kelvin, there are many clouds obscuring the skies of twenty-first century physics: On the phenomenological side, there is not yet a satisfactory description of dark matter or dark energy. Similarly, there are still many theoretical questions that are to be addressed, such as a full non-perturbative description of string and M-theory including the whole massive tower of states. The web of duality relating the different regimes is also not completely understood. An open question is also on the existence of a description that is invariant under the dualities, in the same way that Special Relativity provided a description invariant under Lorentz transformations. There are also various constraints on effective theories that have yet to be extracted from string theory, such as the proof of the Weak Gravity Conjecture and a systematic exploration of the swampland, as well as a complete classification of F-theory GUTs, efforts towards which this thesis has contributed.

Appendix A

Mathematical Glossary

In this appendix, we compile definitions, theorems, and conventions that are sometimes silently assumed throughout this thesis. We note that this appendix is not intended to be neither self-contained nor exhaustive, but rather a glossary, and we refer to the standard references [126, 130, 221, 226, 227] for more details.

Definition 1. *A* manifold *M of real dimension n is a topological space with a choice of open sets $U_\alpha \subset \mathcal{M}$, $V_\alpha \subset \mathbb{R}^n$ and homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$, such that $\mathcal{M} = \cup_\alpha U_\alpha$, and whenever $U_\alpha \cap U_\beta$, $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a C^∞ map.*

Given two manifolds M_1, M_2, their Cartesian product $M_1 \times M_2$ is also a manifold. One might however want to consider objects that look like a Cartesian product locally, but not globally:

Definition 2. *A* fiber bundle *$\mathcal{E}$, or bundle for short, is a collection of spaces $(\mathcal{E}, B, F)$ with a projection map $\pi : \mathcal{E} \to B$, such that for each point $b \in B$ of the base, the inverse of the projection map is isomorphic to the fiber F: $\pi^{-1}(b) \cong F$.*

It is called a *vector bundle* $\mathcal{V}$ of rank r if the fibers are all isomorphic to $\mathbb{R}^r$, and a *section* of a bundle is a map $\sigma : B \to \mathcal{V}$, such that $\pi \circ \sigma = \mathrm{id}_B$. The set of smooth global sections on $\mathcal{E}$ is denoted $\Gamma(\mathcal{E})$. If a bundle $\mathcal{L}$ over a base B has one-dimensional fibers, it is called a *line bundle*.

The *dual bundle* $\mathcal{V}^\vee$ of a vector bundle $\mathcal{V}$ defined through the projection map $\pi : \mathcal{V} \to B$ is a bundle defined over the same base, but with a projection $\pi^\vee : \mathcal{V}^\vee \to B$, such that the fibers $F^\vee =$ are the duals of those of $\mathcal{V}$, in the sense that there exists a bilinear form $\langle .,. \rangle : F \times F^\vee \to \mathbb{R}$.

There will be two particularly relevant operations on bundles in this thesis: First, The Whitney sum $\mathcal{E} \oplus \mathcal{F}$ of two bundles $\mathcal{E}$, $\mathcal{F}$ over the same base B with respective fibers $F_\mathcal{E}$, $F_\mathcal{F}$ is a bundle over B with fiber $F_\mathcal{E} \oplus F_\mathcal{F}$. $\mathcal{F}$ is the inverse bundle of a rank r vector bundle $\mathcal{E}$ if their Whitney sum is the trivial bundle: $\mathcal{E} \oplus \mathcal{F} = \mathcal{O} \cong \mathbb{R}^r \times B$. Second, the tensor product $\mathcal{E} \otimes \mathcal{F}$ of those two bundles over the same base is again a bundle over B where the fibers are isomorphic to $F_\mathcal{E} \otimes F_\mathcal{F}$.

The most familiar bundle is the tangent bundle of a manifold M, defined over a base M, where the fibers are isomorphic to the tangent space T_xM at a reference point $x \in M$. Its dual is the cotangent bundle T^*M, where the fibers are isomorphic to the set of 1-forms at the reference point x, with a bilinear map in local coordinates $\left\langle \frac{\partial}{\partial x^\mu}, dx^\nu \right\rangle = \delta^\nu_\mu$. The set of differential p-forms on M, $\Omega^p(M)$, is then the p-th exterior derivative of the cotangent bundle.

Note that if the p-forms are valued over a particular ring R, e.g. matrices instead of the usual real numbers $\mathbb{R}$, we denote it $\Omega^p(M, R)$. We can then define the cohomology groups as done in chapter 4, or do so in a more abstract way, by using homological algebra. We first generalise the notion of vector spaces:

Definition 3. *A (left) R-*module *A over a ring R with identiry $1 \neq 0$ is an Abelian group together with a map $p : R \times A \to A$, called the product and written $p(r, a) = r \cdot a$, such that*

$$\begin{aligned} (r + r') \cdot a &= r \cdot a + r' \cdot a, & (rr') \cdot a &= r \cdot (r \cdot a), \\ r \cdot (a + a') &= r \cdot a + r \cdot a', & 1 \cdot a &= a. \end{aligned} \tag{A.1}$$

We can then consider a *chain complex* $(A_\bullet, d_\bullet)$, i.e. a sequence R-modules, A_i, connected by homomorphisms $d_n : A_n \to A_{n-1}$ called *boundary operators* having the property that $d_n \circ d_{n-1} = 0 \ \forall n$. They are usually represented in the following diagrammatical way:

$$\cdots \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \xrightarrow{d_{n-2}} \cdots . \tag{A.2}$$

Similarly, one can define a *cochain complex* $(A^\bullet, d^\bullet)$ as a sequence of modules A^i connected by homomorphisms $d^n : A^n \to A^{n+1}$ called *differentials* having the property $d^{n+1} \circ d^n = 0 \ \forall n$.

$$\cdots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} A^{n+2} \xrightarrow{d^{n+2}} \cdots . \tag{A.3}$$

The index n of the modules A_n or A^n is called its degree. Note that the only difference between chain and cochain complexes is that the degree of their homomorphisms is increasing in the case of a cochain complex, while decreasing for a chain complex.

A familiar example of a chain complex is the de Rham complex on a smooth manifold Y, where $A^n = \Omega^n(Y)$ is the set of n-forms, and where all homomorphisms are taken to be the exterior derivative d. Its cochain complex counterpart is $A_n = \Omega^n$ equipped with the codifferential δ:

$$\begin{aligned} &\Omega^0(Y) \xrightarrow{d} \Omega^1(Y) \xrightarrow{d} \Omega^2(Y) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d(Y) \\ &\Omega^d(Y) \xrightarrow{\delta} \Omega^{d-1}(Y) \xrightarrow{\delta} \Omega^{d-2}(Y) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^0(Y) \end{aligned} \tag{A.4}$$

This example motivates the difference between chain and cochain complexes as the exterior derivative increases the degree of a form while the codifferential decreases it.

One can then define the (co-)homology group[1] $H_n(A^\bullet, R)$ (resp. $H^n(A_\bullet, R)$) as the cokernel of d^n

$$H_n(Y, R) = \mathrm{Coker}(d_n) := \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1}) \qquad H^n(Y, R) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}) \tag{A.5}$$

[1] A note on notation: when there is an underlying space Y associated to the A_n or A^n, we write $H^n(Y, R)$, $H_n(Y, R)$, Moreover, if the ring of these modules is unambiguously $\mathbb{R}$ or $\mathbb{C}$, we omit it.

In the case of a manifold, the dimensions $b_n = \dim(H^n(M,R))$ are called the betti numbers. They are topological invariants, and satisfy the relation $\chi(M) = \sum_i (-1)^i b_i$, where χ is the Euler characteristic of M.

These definitions allow one to define most of the familiar related notions in an economical fashion: A *Riemannian manifold* (M,g) is a manifold equipped with a section, the metric, of $T^*M \otimes T^*M$ so that at each point there is a symmetric definite-positive bilinear form. In this thesis we work only with either Euclidean metrics, or Lorentzian metrics in the mostly minus signature. Similarly, a fiber metric k on $\mathcal{E}$ can be defined as a map $k : \mathcal{E} \times \mathcal{E} \to \times B\mathcal{R}$ such that it defines a metric at each point of B.

Moreover, a *complex* manifold is a manifold of dimension $2n$ with a complex structure, i.e. a globally defined map $J : TM \to TM$ such that $J^2 = -\mathrm{Id}$. If additionally it is endowed with a metric satisfying $g(Ju, Jv) = g(u,v)$, it is called a *Kähler* manifold. In a similar way, a fiber bundle over a complex manifold with a holomorphic projection map is called a *holomorphic vector bundle*. For complex manifold, the p-forms can written in terms of holomorphic and anti-holomorphic indices, i.e. $\Omega^p = \sum_{r+s=p} \Omega^{r,s}$. The *canonical bundle* on a real dimension n complex manifold M is then defined as $K_M = \Omega^{0,n}$. It is also possible to associate a chain complex with the groups $\Omega^{r,s}$, where the role of the differential maps are played by the Dolbeault operator $\overline{\partial}$, obtain by splitting the de Rham differential with holomorphic and anti-holomorphic indices $d = \partial + \overline{\partial}$. The cohomology group are then given by $H^{(r,s)} = \mathrm{Coker}(\overline{\partial} : \Omega^{(r,s)} \to \Omega^{(r,s+1)})$. There dimensions $h^{r,s} = \dim(H^{(r,s)})$ are called the Hodge numbers. For Kähler manifolds, they satisfy the two properties $h^{r,s} = h^{s,r}$ and $h^{r,s} = h^{n-r,n-s}$.

If the fibers of a holomorphic vector bundle over B are homeomorphic to a Lie group G, it is called a *principal G-bundle*. One can then associate a *connection*, a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(B, \mathfrak{g})$, and a curvature $F = dA + a \wedge A$, and define the total Chern class $c(E)$ by

$$c(E) = \det\left(\mathbb{1} + \frac{i}{2\pi}F\right) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E) \tag{A.6}$$

The elements of the series $c_r(E) \in \Omega^{2r}(B)$ are called the r-th *Chern class*. Any manifold M of dimension n has a natural $O(n)$-bundle, representing the change of coordinates in a local patch of M, whose endowed with curvature given by the Riemann tensor. Other related quantities are the Chern character $ch(\mathcal{E})$ and Todd class $Td(\mathcal{E})$

$$ch(\mathcal{E}) = \mathrm{Tr}\left(e^{\frac{i}{2\pi}F}\right) \qquad Td(\mathcal{E}) = \prod_i \frac{x_i}{1 - e^{-x_i}}, \qquad x_i \in \mathrm{Spec}(\frac{i}{2\pi}F). \tag{A.7}$$

These two quantities are important as they appear in the Hirzebruch-Riemann-Roch theorem:

Theorem 1. *Let $\mathcal{E}$ a holomorphic vector bundle over a compact complex manifold B. The holomorphic Euler characteristic is given by*

$$\chi(\mathcal{E}, B) := \sum_{i=0}^{dim_{\mathbb{C}} B} (-1)^i dim_{\mathbb{C}}(H^i(B, \mathcal{E})) = \int_B ch(\mathcal{E}) \wedge Td(TB) \tag{A.8}$$

In some cases, we are interested in applying this theorem to a submanifold $C \in B$, and the *adjunction formula* $K_B|_C = K_C \otimes N_{S/B}^{-1}$ can prove useful, where the $N_{S/B}$ is the normal bundle.

An important class of complex manifolds throughout this thesis are manifolds with restricted :

Definition 4. *A Calabi–Yau manifold is a Kähler manifold of complex dimension n satisfying one of the following equivalent conditions:*

- *Its first Chern class vanishes*
- *It has vanishing Ricci curvature*
- *the holonomy of its metric is contained in $SU(n)$*

The Hodge numbers of a Calabi–Yau have constraints in addition to those of a Kähler manifold. It is common to arrange them in an so-called *Hodge diamond*. For a three-fold, we have

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{3,1} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{A.9}$$

When dealing with F-theory, we will need generalisation of fiber bundles, as the fiber can be different:

Definition 5. *A* fibration *Y is a collection of spaces (Y, B) with a projection map $\pi : Y \to B$, satisfying the homotopy lifting property for any space X. For each point $b \in B$ of the base, the space $\pi^{-1}(b)$ is called the fiber.*

The homotopy lifting condition, see e.g. [130], ensures that every fiber is equivalent in the homotopic sense, but not necessarily isomorphic. For instance, in a singular elliptic fibration, singular and smooth tori are not isomorphic, as their topology is different.

Finally, the second part of this thesis focuses on tori. A torus T^2 is defined through the quotient $\mathbb{R}^2/\Lambda = \mathbb{C}/\Lambda$, where $\Lambda = \{\alpha v_1 + \beta v_2 | \alpha, \beta \in \mathbb{Z}\}$ is a lattice generated by $v_1, v_2 \in \mathbb{C}$. The quotient action is done by identifying $x \sim x + v_1 \sim x + v_2$. As we can always rescale the elements defining the lattice such that $v_1 = \tau$, $v_2 = 1$, any lattice is uniquely defined by a single complex number τ. Then $\tau + 1$ clearly generates the same lattice, as $\tau + 1 \sim \tau$. Indeed, thinking of v_1, v_2 as vectors in the plane, $v_1 + v_2$ gives the upper-right corner of the parallelogram defining the lattice, which can be thought of as defining the initial parallelogram "shifted by one to the right". Similarly, we could exchange the role of v_1 and v_2. Properly rescaling, this amounts to send $\tau \to -\frac{1}{\tau}$. Those two operations happen to form a group generating all the possible equivalent change of bases defining the same lattice, which can be thought of as sending $\tau \to \frac{a\tau+b}{c\tau+d}$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$.

Appendix B

Embedding in the Presence of Non-Flat Points

Some of the two $U(1)$ models studied in [166] are not directly embeddable in the E_8 Higgsing tree (figure 5.1). They however contain non-flat points that once turned off also turn off matter curves. These models are labeled by their Kodaira fiber I, and the two sets

$$\begin{gathered}(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8) \\ [d_{2,n_1}, d_{0,n_2}, b_{0,n_3}, d_{1,n_4}, b_{1,n_5}, c_{2,n_6}, b_{2,n_7}, c_{1,n_8}]\,.\end{gathered} \tag{B.1}$$

The integers n_i denote the leading non-vanishing order of the Tate model coefficients, while the terms in square brackets define a specialisation of the Tate form coefficients. The homology classes of the coefficients are combinations of three classes on the base B_3 denoted $\bar{\mathcal{K}}$, α, and β and are given in table B.1. There are five a priori non-embeddable models:

1. The first model is

$$I_5^{s(0|1||2)} \quad : \qquad \left\{ \begin{array}{c} (2,2,2,0,0,0,0,0) \\ [-,-,-,\sigma_2\sigma_5, \sigma_2\sigma_4+\sigma_3\sigma_5, \sigma_3\sigma_4, \sigma_1\sigma_2, \sigma_1\sigma_3] \end{array} \right\}. \tag{B.2}$$

   It has non flat points at the loci $\{\sigma_2 = \sigma_3 = 0\}$ and $\{\sigma_4 = \sigma_5 = 0\}$. From table B.1,

b_0	b_1	b_2	c_1	c_2	d_0	d_1	d_2
$\alpha - \beta + \bar{\mathcal{K}}$	$\bar{\mathcal{K}}$	$-\alpha + \beta + \bar{\mathcal{K}}$	$-\alpha + \bar{\mathcal{K}}$	$-\beta + \bar{\mathcal{K}}$	$\alpha + \bar{\mathcal{K}}$	$\beta + \bar{\mathcal{K}}$	$\alpha + \beta + \bar{\mathcal{K}}$

Table B.1: The classes of the sections in the fibration of [145,155,166]. $\bar{\mathcal{K}}$ is the anti-canonical class of the base B_3.

we can read off the classes of the sections σ_i to be:

$$
\begin{aligned}
[d_{2,2}] &= \alpha + \beta + \bar{\mathcal{K}} - 2\omega \\
[d_{0,2}] &= \alpha + \bar{\mathcal{K}} - 2\omega \\
[b_{0,2}] &= \alpha - \beta + \bar{\mathcal{K}} - 2\omega \\
[d_1] &= [\sigma_2] + [\sigma_5] = \beta + \bar{\mathcal{K}} \\
[b_1] &= [\sigma_2] + [\sigma_4] = [\sigma_3] + [\sigma_5] = \bar{\mathcal{K}} \\
[c_2] &= [\sigma_3] + [\sigma_4] = -\beta + \bar{\mathcal{K}} \\
[b_2] &= [\sigma_1] + [\sigma_2] = -\alpha + \beta + \bar{\mathcal{K}} \\
[c_1] &= [\sigma_1] + [\sigma_3] = -\alpha + \bar{\mathcal{K}}
\end{aligned}
$$

There are thus four possibilities to turn them off:

(a) $[\sigma_2] = [\sigma_4] = 0$: This implies that the anti-canonical bundle $\bar{\mathcal{K}} = 0$, which is inconsistent.

(b) $[\sigma_3] = [\sigma_5] = 0$: Same case as the previous one, hence inconsistent.

(c) $[\sigma_2] = [\sigma_5] = 0$: This implies $[d_1] = -\alpha$ and $[b_2] = \alpha$. At least one of those classes is not effective, which is inconsistent.

(d) $[\sigma_3] = [\sigma_4] = 0$. In that case, we must turn off two $\bar{\mathbf{5}}$ curves. The resulting spectrum is then embeddable in several models (see table 5.3).

2. Model

$$
I_5^{s(0|1||2)} \quad : \quad \left\{ \begin{array}{c} (2,1,1,1,0,0,0,0) \\ [-, \sigma_1\xi_3, \sigma_1\xi_2, -, \sigma_4\xi_3, \sigma_4\xi_2, \xi_3\xi_4, \xi_2\xi_4] \end{array} \right\} \tag{B.3}
$$

has three non-flat points at $\{\sigma_1 = \sigma_4 = 0\}$, $\{\sigma_4 = \xi_4 = 0\}$ and $\{\xi_2 = \xi_3 = 0\}$. Using a similar reasoning as before, one finds that the only consistent possibility to turn off these points is to set at least $[\xi_4]$ trivial, turning off a $\bar{\mathbf{5}}$ that then allow an embedding in a $\{3, 5, 6\}$ model.

3.

$$
I_5^{s(1|02)} \quad : \quad \left\{ \begin{array}{c} (4,2,0,2,0,0,0,0) \\ [-, -, \sigma_3\sigma_4, -, \sigma_2\sigma_4 + \sigma_3\sigma_5, \sigma_1\sigma_3, \sigma_2\sigma_5, \sigma_1\sigma_2] \end{array} \right\} \tag{B.4}
$$

has non flat points at the loci $\{\sigma_2 = \sigma_3 = 0\}$ and $\{\sigma_4 = \sigma_5 = 0\}$. There are therefore four consistent ways to turn off the non flat points. The first is to set $[\sigma_2] = [\sigma_5] = 0$. This constraints the classes to $\beta = \alpha - \bar{\mathcal{K}} \leq 0$, $\omega \leq \alpha/2$. The second possibility is to set $[\sigma_3] = 0 = [\sigma_4]$. This leads to the same constraints as before, with the role of α and β reversed. The two remaining possibilities lead to a vanishing anti-canonical class, which is inconsistent.

We however find that even with a reduced spectrum, there is still no possible embedding into the tree.

4.
$$I_5^{s(012)} \quad : \qquad \left\{ \begin{array}{c} (5,2,0,2,0,0,0,0) \\ [-,\sigma_1\sigma_2,\sigma_2\sigma_5,\sigma_1\sigma_3,\sigma_2\sigma_4+\sigma_3\sigma_5,-,\sigma_3\sigma_4,-] \end{array} \right\} \tag{B.5}$$

is similar to the previous one: It has non-flat points at the same loci, and there are two consistent ways to turn off the non-flat points. Either one sets the classes $[\sigma_2] = 0 = [\sigma_5]$. The classes are then constrained to $\alpha \leq 0$, $\beta = \alpha + \bar{\mathcal{K}} \geq 5\omega/2$. The other consistent possibility is to set $[\sigma_3] = 0 = [\sigma_4]$. This gives rise to the same constraints on the classes, with the role of α and β reversed.

As for the previous case, we find no possible embedding in the tree.

5. The last case,

$$I_5^{s(0|1||2)} \quad : \qquad \left\{ \begin{array}{c} (1,1,1,0,0,0,0,0) \\ [\xi_3\delta_3\delta_4,\delta_4(\delta_3\xi_2+\delta_2\xi_3),\xi_2\delta_2\delta_4,\xi_3\delta_1\delta_4, \\ \qquad \delta_1(\delta_2\xi_3+\delta_3\xi_2),\delta_1\delta_2\xi_2,\sigma_1\xi_3,\sigma_1\xi_2] \end{array} \right\} \tag{B.6}$$

has five non flat points:

$$\begin{array}{cc} \{\delta_1 = \delta_2 = 0\} & \{\delta_1 = \delta_4 = 0\} \\ \{\delta_1 = \sigma_1 = 0\} & \{\delta_2 = \delta_3 = 0\} \\ \multicolumn{2}{c}{\{\xi_2 = \xi_3 = 0\}} \end{array}$$

We find that there is no consistent way to turn them off by setting classes of the different sections to zero.

Appendix C

Ext Groups for an $SU(2)$ Model

In this appendix, we explain the procedure leading to the Ext group for the $SU(2)$ model. We find that there are two Matrix Factorisation. One is 4×4:

$$[\varphi_{4\times4}, \psi_{4\times4}] = \left[\begin{pmatrix} y & x(x+a_2)+a_1a_3 & a_3 & 0 \\ x & y & 0 & -a_3 \\ a_0a_3 & 0 & y & x(x+a_2)+a_1a_3 \\ 0 & -a_0a_3 & x & y \end{pmatrix}, P_W\varphi_{4\times4}^{-1} \right], \tag{C.1}$$

where P_W is given by equation (7.34). The other is a 6×6 MF, that one can find to be:

$$\varphi_{6\times6} = \begin{pmatrix} y & \frac{1}{2}a_1 & 0 & a_3 & 0 & s+a_2 \\ sa_3 & y & \frac{1}{2}a_1a_3+s(s+a_2) & 0 & a_3^2 & 0 \\ 0 & s & y & 0 & 0 & -a_3 \\ -a_0a_3-\frac{1}{2}a_1s & 0 & \frac{1}{4}(a_1^2-4a_0(s+a_2)) & -y & \frac{1}{2}a_1a_3+s(s+a_2) & 0 \\ 0 & -a_0 & 0 & s & -y & -\frac{1}{2}a_1 \\ -s^2 & 0 & a_0a_3+\frac{1}{2}a_1s & 0 & -sa_3 & -y \end{pmatrix}, \tag{C.2}$$

with $\psi_{6\times6} = P_W\varphi_{6\times6}^{-1}$. Let us start with the Ext group $\underline{\mathrm{Ext}}^1(\varphi_{4\times4}, \varphi_{4\times4})$. The associated complexes are:

$$\begin{tikzcd} & \mathcal{E}(\mathcal{L}) \arrow[r, "\varphi_{4\times4}"] \arrow[dl, dashed, "g"'] \arrow[d, "\rho"] & \mathcal{E}(\mathcal{L}) \otimes K_B^{-3} \arrow[r, "\psi_{4\times4}"] \arrow[dl, dashed, "h"'] \arrow[d, "\tilde{\rho}"] & \mathcal{E}(\mathcal{L}) \otimes K_B^{-6} \arrow[dl, dashed, "l"] \\ \mathcal{E}(\mathcal{L}') \arrow[r, "\varphi_{4\times4}"] & \mathcal{E}(\mathcal{L}') \otimes K_B^{-3} \arrow[r, "\psi_{4\times4}"] & \mathcal{E}(\mathcal{L}) \otimes K_B^{-6} & \end{tikzcd} \tag{C.3}$$

$\rho,\ \tilde{\rho}$ must satisfy the relation $\psi_{4\times4}\rho = \tilde{\rho}\varphi_{4\times4}$ so that the square commutes, and are defined up to homotopies

$$\rho \sim \rho + \Phi_{4\times4}g + h\Phi_{4\times4}, \qquad \tilde{\rho} \sim \tilde{\rho} + \Psi_{4\times4}h + l\Psi_{4\times4} \tag{C.4}$$

It is possible to use all of the homotopies to work in the class where the following dependences on the vertical maps have been cut:

$$\rho : \quad \begin{pmatrix} - & - & a_3, s, y & a_3 \\ - & - & a_3, s, y & a_3 \\ - & - & a_3, s, y & a_3 \\ - & - & a_3, s, y & a_3 \end{pmatrix} \qquad \tilde{\rho} : \quad \begin{pmatrix} - & - & a_3, s, y & a_3 \\ - & - & a_3, s & a_3 \\ - & - & a_3, s, y & a_3 \\ - & - & a_3, s & a_3 \end{pmatrix} \tag{C.5}$$

The symbol $-$ signifies that no dependence was cut and that the associated coefficient generically depend on all the variables. Let us come back to the relation imposed by the commuting square. One finds that they are all of the form

$$yA(y) + sB(y,s) + a_3 C(y,s,a_3) = 0 \tag{C.6}$$

As A, B cannot factor a_3, we deduce that the only way for this equation to hold is $C = 0$. The same argument for s ensures that $A = B = 0$. As an example, let us consider the entry $(2,3)$

$$a_3(\rho_{43} - \tilde{\rho}_{21}) + y(\rho_{23} - \tilde{\rho}_{23}) - s(\rho_{13} + \tilde{\rho}_{24}) = 0. \tag{C.7}$$

An inspection of (C.5) shows that $\rho_{13}, \rho_{23}, \rho_{43}, \tilde{\rho}_{23}$ and the line of argument of equation (C.6) reveal that $\tilde{\rho}_{23} = \rho_{23},\ \tilde{\rho}_{24} = -\rho_{13},\ \tilde{\rho}_{21} = \rho_{43}$. The third and fourth columns are enough to completely fix $\tilde{\rho}$ as a function of ρ. Using the remaining equations, one finds

$$\rho = \begin{pmatrix} -a_1\rho_{23} - \rho_{33} & -(s+a_2)\rho_{43} & \rho_{13} & -(s+a_2)\rho_{23} \\ \rho_{43} & a_1\rho_{23} + \rho_{33} & \rho_{23} & -\rho_{13} \\ -a_0\rho_{13} - a_1\rho_{43} & -a_0(s+a_2)\rho_{23} & \rho_{33} & -(s+a_2)\rho_{43} \\ a_0\rho_{23} & a_0\rho_{13} + a_1\rho_{43} & \rho_{43} & -\rho_{33} \end{pmatrix} = \tilde{\rho} \tag{C.8}$$

The elements are sections of

$$\begin{aligned} \rho_{13} &\in H^0(\mathcal{S}, \mathcal{L}^{-1} \otimes \mathcal{L}' \otimes K^{-3} \otimes \mathcal{J}^{-1}\big|_{\mathcal{S}}) \qquad & \rho_{23} &\in H^0(\mathcal{S}, \mathcal{L}^{-1} \otimes \mathcal{L}' \otimes K^{-2} \otimes \mathcal{J}^{-1}\big|_{\mathcal{S}}) \\ \rho_{33} &\in H^0(\mathcal{S}, \mathcal{L}^{-1} \otimes \mathcal{L}' \otimes K^{-3}\big|_{\mathcal{S}}) \qquad & \rho_{43} &\in H^0(\mathcal{S}, \mathcal{L}^{-1} \otimes \mathcal{L}' \otimes K^{-2}\big|_{\mathcal{S}}) \end{aligned} \tag{C.9}$$

where $\mathcal{S}$ is the surface defined by $\{a_3 = 0\}$. In a similar fashion, one can compute $\underline{\mathrm{Ext}}^1(\varphi_{4\times 4}, \varphi_{6\times 6})$, using the following complexes

$$\begin{tikzcd} & \dots \arrow[r] & \mathcal{L} \otimes \mathcal{C} \arrow[r, "\varphi_{4\times4}"] \arrow[dl, dashed, "g"'] \arrow[d, "\rho"] & \mathcal{L} \otimes \mathcal{C} \otimes K_B^{-3} \arrow[r] \arrow[dl, dashed, "h"'] \arrow[d, "\tilde{\rho}"] & \dots \arrow[dl, dashed, "l"'] \\ \dots \arrow[r] & \mathcal{L}' \otimes \mathcal{D} \arrow[r, "\varphi_{6\times6}"] & \mathcal{L}' \otimes \mathcal{D} \otimes K_B^{-3} \arrow[r, "\psi_{6\times6}"] & \mathcal{L}' \otimes \mathcal{D} \otimes K_B^{-6} \arrow[r] & \dots \end{tikzcd} \tag{C.10}$$

Commutation of the diagram imposes that one needs to satisfy $\psi_{6\times6} \circ \rho = \tilde{\rho} \circ \varphi_{4\times4}$, and the vertical maps are defined up the homotopies relations

$$\begin{aligned} \rho &\sim \rho + \varphi_{6\times6} g + h\varphi_{4\times4}, \\ \tilde{\rho} &\sim \tilde{\rho} + \psi_{6\times6} h + l\psi_{4\times4}. \end{aligned} \tag{C.11}$$

Using these relations, it is possible to remove the following dependencies:

$$\rho : \begin{pmatrix} - & - & a_3, y, x & a_3 \\ - & - & a_3, y, x & a_3 \\ a_3, y, x & - & a_3, y, x & a_3, x, y \\ - & - & a_3, y, x & a_3 \\ - & - & a_3, y, x & a_3, x, y \\ - & - & a_3, y, x & a_3 \end{pmatrix} \qquad \tilde{\rho} : \begin{pmatrix} - & - & a_3, x & a_3 \\ - & - & a_3, x & a_3 \\ - & - & a_3, x & a_3 \\ - & - & a_3, x & a_3 \\ - & - & a_3, x & a_3 \\ - & - & a_3, x & a_3 \end{pmatrix} \tag{C.12}$$

Using this parameterisation, one gets that vertical maps are given by:

$$\rho = \begin{pmatrix} \rho_{43} & \frac{a_1}{2}\rho_{13} - (a_2+s)\rho_{53} & \rho_{13} & 0 \\ -s\rho_{13} + a_3\rho_{53} & a_3\rho_{43} & 0 & 0 \\ 0 & s\rho_{13} & 0 & 0 \\ a_0\rho_{13} + \frac{a_1}{2}\rho_{53} & \frac{a_1}{2}\rho_{43} & \rho_{43} & \frac{a_1}{2}\rho_{13} + (s+a_2)\rho_{53} \\ 0 & a_1\rho_{53} & \rho_{53} & \rho_{43} \\ -s\rho_{53} & s\rho_{43} & 0 & s\rho_{13} \end{pmatrix} \tag{C.13}$$

The procedure to get this result is the following: Starting with the third and fourth columns of the commutation relation, and using the same arguments as for the 4×4 case, one can infer the value of $\tilde{\rho}_{i1}, \tilde{\rho}_{i2}$ from the fact that the various coefficients do not depend on a_3 after fixing part of the homotopy maps. Using the same argument with s and y to the third column to fix completely fix $\tilde{\rho}$.

The third column then gives three equations leading to $\rho_{23} = \rho_{33} = \rho_{63} = 0$. The remaining 6 equations can then be used to fix ρ_{i4} as function of $\rho_{13}, \rho_{43}, \rho_{53}$ in a holomorphic way. One can then use the fact that ρ_{31} doesn't depend on y, x, a_3 to set it to zero using the first column, and similarly for ρ_{52} in the second column.

One is left with the parameters $\rho_{13},\ \rho_{43},\ \rho_{53}$, and one can use the remaining homotopies to show that there is a combination of $\rho_{13},\ \rho_{43}$ that is localised on $\mathcal{C}$.

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