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Evaluating Power-Law Properties from Data Using a Wavelet Transform Correlation Method with Applications to Foreign Exchange Rates

H. L. Wei and S. A. Billings

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Department of Automatic Control and Systems Engineering
The University of Sheffield
Mappin Street, Sheffield,
S1 3JD, UK

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Evaluating Power-Law Properties from Data Using a Wavelet Transform Correlation Method with Applications to Foreign Exchange Rates

H. L. Wei and S. A. Billings

Department of Automatic Control and Systems Engineering, the University of Sheffield,

Mappin Street, Sheffield, S1 3JD, UK.

s.billings@shef.ac.uk, w.hualiang@shef.ac.uk

Abstract: Numerous studies in the literature have shown that the dynamics of many time series including observations in foreign exchange markets exhibit scaling behaviours. A novel statistical method, derived from the concept of the continuous wavelet transform correlation function (WTCF), is proposed for the evaluation of power-law properties from observed data. The new method reveals that foreign exchange rates obey power-laws and thus belong to the class of self-similarity processes.

Keywords: Continuous wavelet transform, correlation function, foreign exchange rates, scaling law, power law, self-similarity.

1. Introduction

The wavelet transform (Daubechies 1992) provides a powerful tool for analyzing and synthesizing signals. The wavelet transform has the property of localisation both in time and frequency. In wavelet analysis, the scale that can be used to look at data at different resolution levels plays a special role, because wavelet algorithms process data at different scales or resolutions. At a coarse resolution level, one would notice gross features. Similarly, at a fine resolution level, one would get detailed features. This enables us to see both the 'forest' and the 'trees', so to speak, and makes wavelets very useful (Amara 1995) for data modelling and analysis in diverse fields including dynamical system modelling (Billings and Wei 2005a, 2005b, Wei and Billings 2004a, 2004b, 2006, 2007), as well as random signal processing and analysis for example statistical self-similarity detection and fractal property characterization (Argoul 1990, Flandrin 1992, Wornell 1995, Arneodo et al. 1996, Abry and Veitch 1998, Soltania et al. 2004).

Numerous studies in the literature have shown that the dynamics of many time series in foreign exchange markets exhibit scaling behaviours (Muller et al. 1990, Mantegna and Stanley 1995, Dacorogna et al. 1996, Guillaume1 et al. 1997, Vandewallea and Ausloos, 1998, Gopikrishnan et al. 1999, Mulligan 2000, Gencay et al. 2001, Muniandya et al. 2001, Xu and Gencay 2003, Jiang et al. 2007). For example, Muller et al. (1990) and Guillaume1et al. (1997) have shown that the mean absolute price changes over certain time intervals for foreign exchange rates obey scaling laws. Recently, Xu and Gencay (2003) have shown that US dollar to Deutschemark (USD-DEM) returns present scaling and multifractal properties.

The objective of this paper is to introduce a new wavelet transform based method to detect and evaluate the fractal self-similarity properties from observed time series. The new method involves the calculation of a continuous wavelet transform correlation function (WTCF), which plays key a role in linking the time-domain data with the associated scaling law properties that are explicitly presented by the wavelet scale (revolution) parameter.

2. The Wavelet Transform Correlation Method

(A) The wavelet transform

Let *f*(*t*) be a square integrable function defined in $L^2(R)$. For a given *mother wavelet* ψ, the continuous wavelet transform (CWT) of the function *f*(*t*) is defined as (Daubechies 1992)

$$W_f^{\psi}(b,a) = \int_{-\infty}^{+\infty} f(t)\overline{\psi_{b,a}(t)}dt = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}dt \qquad (1)$$

where $\psi_{b,a}(t) = a^{-1/2}\psi((t-b)/a)$, $a \in R^+$ and $b \in R$ are the dilation (scale) and translation (shift) parameters, respectively. The over-bar above the function $\psi(\cdot)$ indicates the complex conjugate. In order to guarantee (1) is invertible so that f can be reconstructed from W_f^ψ, the following admissible condition is required

$$C_\psi = \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty \tag{2}$$

where $\hat{\psi}$ is the Fourier transform of the function ψ.

For a stochastic process *f*(*t*), the wavelet transform $W_f^\psi(b,a)$ can be viewed as a random field on the upper (positive) half plane. For a given scale parameter *a*, the transform $W_f^\psi(b,a)$ contains a piece of information of the original process at this given scale. Extensive research has been done to exploit the wavelet transform, to analyze and determine the characteristics of random processes (Mallat and Hwang 1992, Flandrin 1992, Masry 1993, Wornell 1995, Abry and Veitch 1998).

(B) The wavelet transform correlation function

Let $x(t)$ be a wide-sense (weak-sense) stationary random process that is square integrable in $L^2(R)$. For a chosen wavelet ψ, the *wavelet transform correlation function* (WTCF) of the signal $x(t)$, with respect to the locations b_1 and b_2 at scales a_1 and a_2, is defined as below

$$\begin{aligned}\Phi_{x,x}^\psi(b_1,b_2;a_1,a_2) &= E[W_x^\psi(b_1,a_1)\overline{W_x^\psi(b_2,a_2)}] \\ &= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^\infty\int_{-\infty}^\infty E[x(\tau_1)\overline{x(\tau_2)}]\psi\left(\frac{\tau_1-b_1}{a_1}\right)\overline{\psi\left(\frac{\tau_2-b_2}{a_2}\right)}d\tau_1 d\tau_2\end{aligned} \tag{3}$$

Note that $R_{x,x}(\tau_1,\tau_2) = E[x(\tau_1)\overline{x(\tau_2)}]$ is the correlation function of $x(t)$. Using the property that $R_{x,x}(\tau_1,\tau_2) = R_{x,x}(\tau_2-\tau_1,0) = R_x(\tau_2-\tau_1) = E[x(t)\overline{x(t+\tau_2-\tau_1)}]$, it can be derived from equation (3) that

$$\begin{aligned}\Phi_{x,x}^\psi(b_1,b_2;a_1,a_2) &= \int_{-\infty}^\infty\int_{-\infty}^\infty R_{x,x}(\tau_1,\tau_2)\psi_{b_1,a_1}(\tau_1)\overline{\psi_{b_2,a_2}(\tau_2)}d\tau_1 d\tau_2 \\ &= \int_{-\infty}^\infty\left[\int_{-\infty}^\infty R_x(\tau_2-\tau_1)\psi_{b_1,a_1}(\tau_1)d\tau_1\right]\overline{\psi_{b_2,a_2}(\tau_2)}d\tau_2 \\ &= \int_{-\infty}^\infty\left[(R_x * \psi_{b_1,a_1}(\tau_2)\right]\overline{\psi_{b_2,a_2}(\tau_2)}d\tau_2\end{aligned} \tag{4}$$

where the symbol "$*$" indicates the convolution of two functions.

Assume that the power spectrum $P_x(\omega)$ of the signal $x(t)$ exits. From the Parseval's theorem, which states that the inner product of two functions is equal to the inner product of the Fourier transforms of the two individual functions, as well as the convolution theorem that states that under suitable conditions the Fourier transform of a convolution is the pointwise product of the Fourier transforms of the two individual functions, it can then be further derived that

$$\Phi^{\psi}_{x,x}(b_1,b_2;a_1,a_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_x(\omega)\hat{\psi}_{b_1,a_1}(\omega)\overline{\hat{\psi}_{b_2,a_2}(\omega)}d\omega$$

$$= \frac{\sqrt{a_1 a_2}}{2\pi}\int_{-\infty}^{\infty} P_x(\omega)\hat{\psi}(a_1\omega)\overline{\hat{\psi}(a_2\omega)}e^{-i(b_1-b_2)\omega}d\omega \tag{5}$$

This shows that for the wide-sense stationary process $x(t)$, the wavelet transform correlation function $\Phi^{\psi}_{x,x}(b_1,b_2;a_1,a_2)$, with respect to the locations b_1 and b_2 at given scales a_1 and a_2, is a function of b_1 and b_2 only through the difference (b_2-b_1). Here, for the first time, we have derived, by means of the Parseval's theorem and the convolution theorem, the relation between the time-domain signal and the frequency-domain behaviour presented by the spectra of the signal and the wavelet.

(C) The power-law case

As a special case of the wavelet transform correlation function, the *wavelet transform autocorrelation function* (WTAF) of the signal $x(t)$, at scale a, can be calculated from (5) by letting $a_1 = a_2 = a$ and $b_2 = b_1 = b$, that is,

$$\Phi^{\psi}_{x}(a) = E[|W^{\psi}_{x}(b,a)|^2] = \frac{a}{2\pi}\int_{-\infty}^{\infty} P_x(\omega)|\hat{\psi}(a\omega)|^2 d\omega \tag{6}$$

Now assume that the dynamics of the process $x(t)$ exhibits a power-law behaviour, that is, the power spectral density of the process has a power-law dependence in frequency as given below

$$P_x(\omega) \propto |\omega|^{-\beta} \tag{7}$$

It can then be obtained from (6) that

$$\Phi^{\psi}_{x}(a) = \frac{a}{2\pi}\int_{-\infty}^{\infty}\frac{|\hat{\psi}(a\omega)|^2}{|\omega|^{\beta}}d\omega = \frac{a^{\beta}}{2\pi}\int_{-\infty}^{\infty}\frac{|\hat{\psi}(\omega)|^2}{|\omega|^{\beta}}d\omega = Ca^{\beta} \tag{8}$$

where $C = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\omega|^{-\beta}|\hat{\psi}(\omega)|^2 d\omega < \infty$. Equation (8) suggests that for a power-law signal $x(t)$ that obeys the power-law given by (7), the wavelet transform autocorrelation function $\Phi^{\psi}_{x}(a)$ also obeys a power-law with respect to the wavelet scale parameter a, and the value of the scaling exponent is exactly the same as in the original power-law presentation but with an opposite symbol, that is $-\beta$ in (7) becomes $+\beta$ in (8). Therefore, the new introduced formulas (8) can be used to estimate the power-law exponent of the signal $x(t)$. Note that the relationship between the power-law exponent β, the Hurst exponent H, and the fractal dimension D is given by $\beta = 2H + 1 = 5 - 2D$ (Voss 1988). For a self-affine process, $0 \leq H \leq 1$, $1 \leq D \leq 2$ and $1 < \beta < 3$ (Malamud and Turcotte 1999a). For a Brownian motion, $H = 0.5$, $D = 1.5$ and $\beta = 2$ (Malamud and Turcotte 1999b).

3. Results for foreign exchange rates

Monthly average dollar exchange rates, taken from the Federal Reserve Bank of St. Louis for a selection of twenty countries, were considered in this study, and these are shown in Table 1. Monthly average exchange rates are of more interest than daily exchange rates for at least four groups of investors (Mulligan 2000): program traders, investors who follow deterministic rules, investors who routinely accept exposure approximately one month or longer, and currency hedgers.

Table 1 The power-law exponents estimated using the wavelet transform correlation function for the monthly average dollar exchange rates of twenty countries. The data came from the Federal Reserve Bank of St. Louis.

Country	Observation period of the foreign exchange rates (dd/mm/yy)	Data length	Lowest and highest rates	Power-law exponent β
Austria / U.S.	01/01/1971—01/12/2001	372	9.72 / 25.873	2.0113
Belgium / U.S.	01/01/1971—01/12/2001	372	27.96 / 66.31	2.1235
Brazil / U.S.	01/01/1995—01/12/2007	156	0.8412 / 3.7966	2.4642
Canada / U.S.	01/01/1971—01/12/2007	444	0.9623 / 1.5997	2.0544
Denmark / U.S.	01/01/1971—01/12/2007	444	5.0766 / 11.8071	2.0630
Finland / U.S.	01/01/1971—01/12/2001	372	3.4926 / 6.9645	2.2413
France / U.S.	01/01/1971—01/12/2001	372	4.0048 / 10.0933	2.2331
Germany / U.S.	01/01/1971—01/12/2001	372	1.3812 / 3.637	2.0137
Greece / U.S.	01/01/1981—01/12/2000	237	53.18 / 398.29	2.8873
India / U.S.	01/01/1973—01/12/2007	420	7.27 / 49.02	2.5634
Italy / U.S.	01/01/1971—01/12/2001	372	265.26 / 2271.28	2.5347
Japan / U.S.	01/01/1971—01/12/2007	444	83.69 / 358.02	2.0193
Mexico / U.S.	01/01/1993—01/12/2007	170	3.108 / 11.52	2.3566
Netherlands / U.S.	01/01/1971—01/12/2001	372	1.5474 / 3.7387	2.0589
Norway / U.S.	01/01/1971—01/12/2007	444	4.8167 / 9.4695	2.1079
Portugal / U.S.	01/01/1973—01/12/2001	348	22.41 / 235.17	2.6922
Spain / U.S.	01/01/1973—01/12/2001	348	55.8 / 195.17	2.4333
Sweden / U.S.	01/01/1971—01/12/2007	444	3.9166 / 10.793	2.2444
Switzerland / U.S.	01/01/1971—01/12/2007	444	1.1233 / 4.3053	1.9271
U.S. / U.K.	01/01/1971—01/12/2007	444	1.0931 / 2.6181	2.0470

The proposed wavelet transform auto correlation function was used to analyze the twenty datasets. The calculation procedure is as below:

- For each dataset, apply the continuous wavelet transform to calculate the wavelet coefficient $W_x^{\psi}(b,a)$, where the Daubechies' wavelet of order 20 ('db20') was used; the scale parameter a was allowed to vary from 1 to 16, and the shift parameter b was allowed to vary from 1 to N (N is the data length).
- For each single value of the scale (or resolution level) parameter a, calculate the expectation $\Phi_x^{\psi}(a) = E[|W_x^{\psi}(b,a)|^2] = \text{var}[W_x^{\psi}(b,a)] + \langle W_x^{\psi}(b,a) \rangle^2$, where 'var' indicates calculating the variance and '< >' indicates taking the average of the relevant signal; $\Phi_x^{\psi}(a)$ is a function of the scale parameter a.
- Plot the graph of $\log_2[\Phi_x^{\psi}(a)]$ (vertical axis) versus $\log_2(a)$ (horizontal axis).
- Calculate the slope of the plot formed by $\log_2[\Phi_x^{\psi}(a)]$ and $\log_2(a)$; the value of the slope can be viewed as an estimate of the power-law exponent β .

The graphs for the twenty datasets are shown in Figure 1, where graphs are displayed, from left to right and from top to bottom, in the order that is exactly the same as in Table 1. Values of the estimated power-law exponent β are given in Table 1. The above calculation procedure was also performed over some daily average dollar exchange rates for some countries and it has been observed that results are very similar to those that were obtained for the associated monthly average cases.

Note that in the above calculation the original datasets were directly used to test and evaluate the power-law properties of the foreign exchange rates; no data pre-processing procedure has been performed. Data normalization for example mean-removal might very slightly affect the estimation results.

4. Conclusions

The proposed wavelet transform correlation analysis method can be used to detect and evaluate the fractal scaling-law behaviours from observed time series. Compared with existing approaches, the new method has several advantages, for example, it is not necessary for this method to use a large number of observations to obtain accurate estimates of the power-law exponent; unlike traditional power spectral density estimation methods which require data smoothing (windowing) and which are sensitive to the window 'shapes', the new method does not need any windowing techniques. Moreover, this new non-parametric method can be performed speedily and efficiently using existing tools for continuous wavelet transform calculation in Matlab. The presented results have shown that the foreign exchange rates, for the twenty countries considered, exhibit power-laws and thus belong to the class of fractal self-similarity processes.

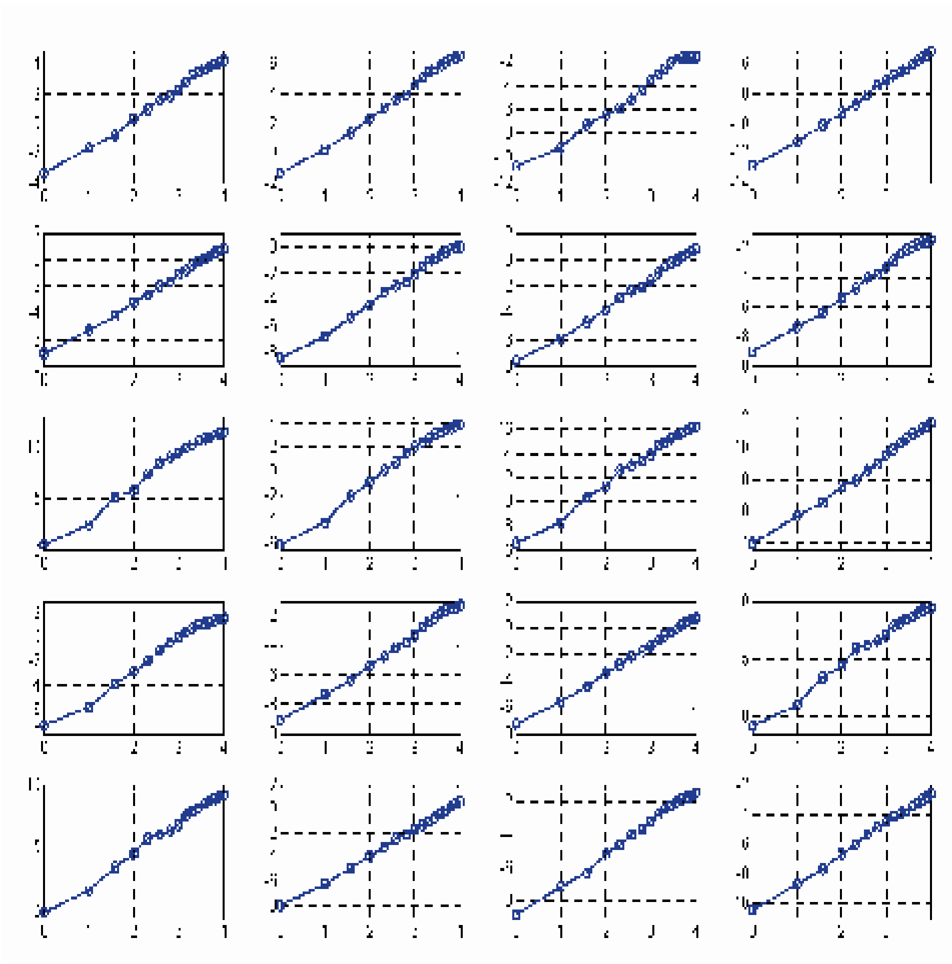

Figure 1 Graphs of the wavelet transform correlation function defined by (8) for the twenty datasets of foreign exchange rates listed in Table 1. From left to right and from top to bottom, these are displayed in order that is exactly the same as in Table 1. In these graphs, the vertical axis is $\log_2[\Phi_x^\psi(a)]$ and the horizontal axis is $\log_2(a)$.

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