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X. J. Jing, Z. Q. Lang, and S. A. Billings

Department of Automatic Control and Systems Engineering
The University of Sheffield
Mappin Street, Sheffield
S1 3JD, UK

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Nonlinear Influence in the Frequency Domain: Alternating Series

Xing Jian Jing*, Zi Qiang Lang, and Stephen A. Billings
Department of Automatic Control and Systems Engineering, University of Sheffield
Mappin Street, Sheffield, S1 3JD, U.K.
{X.J.Jing, Z.Lang & S.Billings}@sheffield.ac.uk

Abstract: The nonlinear influence on system output spectrum is studied for a class of nonlinear systems which have Volterra series expansion. It is shown that system output spectrum can be expressed into an alternating series with respect to some model nonlinear parameters under certain conditions. This alternating series has some interesting properties by which system output spectrum can be suppressed easily. The sufficient (and necessary) conditions in which the output spectrum can be transformed into an alternating series are studied. These results reveal a novel characteristic of the nonlinear influence on a system in the frequency domain, and provide a novel insight into the analysis and design of a class of nonlinear systems. Examples are given to illustrate the results.

Keywords: Nonlinear systems, Volterra series, Alternating series, Frequency domain

1 Introduction

It is known that, the transfer function of a linear system provides a coordinate-free and equivalent description for system characteristics, by which it is convenient to conduct the system analysis and design. Thus frequency domain methods are quite usual to engineers and widely applied in engineering practice. However, although the analysis and design of linear systems in the frequency domain have been well established, the frequency domain analysis for nonlinear systems is not straightforward. Nonlinear systems usually have very complicated output frequency characteristics and dynamic behaviour such as harmonics, inter-modulation, chaos and bifurcation. Investigation and understanding of these nonlinear phenomena in the frequency domain are far from full development. Frequency domain methods for nonlinear systems have also been investigated for many years. There have already been several different approaches to the analysis and design for nonlinear systems, such as describing functions (Graham and McRuer 1961, Nuij et al 2006), harmonic balance (Solomou et al 2002), and frequency domain methods developed from the absolute stability theory (Leonov et al 1996), for example the well-known Popov circle theorem, and so on.

Investigation of nonlinear systems in the frequency domain can also be done based on Volterra functional series expansion theory (Volterra 1959, Rugh 1981). There are a quite large class of nonlinear systems which have a convergent Volterra series expansion (Boyd and Chua 1985). For this class of nonlinear systems, referred to as Volterra systems, the generalized frequency response function (GFRF) was defined in George (1959), which is similar to the transfer function of linear systems. To obtain the GFRFs for Volterra systems described by nonlinear differential equations, the probing method can be used (Rugh 1981). Once the GRFRs are obtained for a practical system, system output spectrum can then be evaluated (Lang 1996). These form a fundamental basis for the analysis of nonlinear Volterra systems in the frequency domain.

In this study, understanding of nonlinearity in the frequency domain is investigated from a novel viewpoint for Volterra systems. The system output spectrum is shown to be an alternating series with respect to some model nonlinear parameters under certain conditions. This property has great significance in that the system output spectrum can therefore be suppressed easily. This also provides a novel insight into the understanding of nonlinear influence on a system. The sufficient (and necessary) conditions in which the output spectrum can be transformed into an alternating series are studied. These results are illustrated by two example studies which investigate a SDOF spring-damping system with a cubic nonlinear damping. The results established in this study reveal a novel characteristic of the nonlinear influence on a system in the frequency domain, and provide a novel insight into the analysis and design of nonlinear systems.

The paper is organised as follows. Section 2 provides a detailed background of this study. The novel nonlinear characteristic and its influence are discussed in Section 3. Section 4 gives a sufficient and necessary condition under which system output spectrum can be transformed into an alternating series. A conclusion is given in Section 5.

2 Frequency response functions of nonlinear systems

Nonlinear systems can be approximated by a Volterra series up to a maximum order N around the zero equilibrium (Boyd and Chua 1985) as

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{1}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is a real valued function of $\tau_1,\cdots,\tau_n$ called the nth-order Volterra kernel. Consider this class of nonlinear systems described by the following nonlinear differential equation (NDE) model

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}=0 \tag{2}$$

where $\left.\frac{d^k x(t)}{dt^k}\right|_{k=0}=x(t)$, $p+q=m$, $\sum_{k_1,k_{p+q}=0}^{K}(\cdot)=\sum_{k_1=0}^{K}(\cdot)\cdots\sum_{k_{p+q}=0}^{K}(\cdot)$, M is the maximum degree of nonlinearity in terms of y(t) and u(t), and K is the maximum order of the derivative. In this model, the parameters such as $c_{0,1}$(.) and $c_{1,0}$(.) are referred to as linear parameters corresponding to coefficients of linear terms in the model, *i.e.*, $\frac{d^k y(t)}{dt^k}$ and $\frac{d^k u(t)}{dt^k}$ for k=0,1,…,L, and $c_{pq}(\cdot)$ for p+q>1 are referred to as nonlinear parameters corresponding to nonlinear terms in the model of the form $\prod_{i=1}^{p}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=p+1}^{p+q}\frac{d^{k_i}u(t)}{dt^{k_i}}$, *e.g.*, $y(t)^p u(t)^q$. $p+q$ is referred to as nonlinear degree of parameter $c_{pq}(\cdot)$.

By using the probing method (Rugh 1981), a recursive algorithm for the computation of the *n*th-order generalized frequency response function (GFRF) for the NDE model (2) is provided in Billings and Peyton-Jone (1990). Therefore, the output spectrum of model (2) can be evaluated as (Lang and Billings 1996)

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \quad (3)$$

where,

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \quad (4)$$

is known as the *n*th-order GFRF defined in George (1959). When the system input is a multi-tone function described by

$$u(t)=\sum_{i=1}^{K}|F_i|\cos(\omega_i t+\angle F_i) \quad (5)$$

The system output frequency response function can be described as (Lang and Billings 1996):

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \quad (6)$$

where $F(\omega_{k_i})$ can explicitly be written as $F(\omega_{k_i})=\left|F_{|k_i|}\right|e^{j\angle F_{|k_i|}\cdot\mathrm{sig}(k_i)}$ for $k_i\in\{\pm1,\cdots,\pm\overline{K}\}$ in stead of the form in Lang and Billings (1996), $\mathrm{sgn}(a)=\begin{cases}1 & a\ge 0\\ -1 & a<0\end{cases}$, and $\omega_{k_i}\in\{\pm\omega_1,\cdots,\pm\omega_{\overline{K}}\}$.

In order to explicitly reveal the relationship between these frequency response functions above and model parameters, the parametric characteristics of the GFRFs and output spectrum are studied in Jing et al (2006). The *n*th-order GFRF can then be expressed into a more straightforward polynomial form as

$$H_n(j\omega_1,\cdots,j\omega_n)=CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot f_n(j\omega_1,\cdots,j\omega_n) \quad (7)$$

where, $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is referred to as the parametric characteristic of the *n*th-order GFRF $H_n(j\omega_1,\cdots,j\omega_n)$, which can recursively be determined as

$$CE(H_n(j\omega_1,\cdots,j\omega_n))=C_{0,n}\oplus\left(\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}C_{p,q}\otimes CE(H_{n-q-p+1}(\cdot))\right)\oplus\left(\mathop{\oplus}_{p=2}^{n}C_{p,0}\otimes CE(H_{n-p+1}(\cdot))\right) \quad (8)$$

and $f_n(j\omega_1,\cdots,j\omega_n)$ is a complex valued vector with the same dimension as $CE(H_n(j\omega_1,\cdots,j\omega_n))$. In Jing et al (2008), a mapping $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$ from the parametric characteristic $CE(H_n(j\omega_1,\cdots,j\omega_n))$ to its corresponding correlative function $f_n(j\omega_1,\cdots,j\omega_n)$ is established as

$$\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})$$
$$= \sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{pq}(\cdot) \text{ and } p>0}} \Big\{ f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) \cdot \sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{pq}(\cdot)}} \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{s_{x_1},\cdots,s_{x_p}\}}} \Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})$$
$$\cdot \prod_{i=1}^{p} \varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot)))}(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))})\Big]\Big\} \quad (9a)$$

where the terminating condition is k=0 and $\varphi_1(1;\omega_i) = H_1(j\omega_i)$ (which is the transfer function when all nonlinear parameters are zero), $\{s_{\bar{x}_1},\cdots s_{\bar{x}_p}\}$ is a permutation of $\{s_{x_1},\cdots s_{x_p}\}$, $\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}$ represents the frequency variables involved in the corresponding functions, $l(i)$ for i=1… $n(\bar{s})$ is a positive integer representing the index of the frequency variables, $\bar{s} = c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, $n(s_x(\bar{s})) = \sum_{i=1}^{x}(p_i+q_i) - x + 1$, x is the number of the parameters in s_x, $\sum_{i=1}^{x}(p_i+q_i)$ is the summation of the subscripts of all the parameters in s_x.

Moreover, $\bar{X}(i) = \sum_{j=1}^{i-1} n(s_{\bar{x}_j}(\bar{s}/c_{pq}(\cdot)))$, $L_n(j\omega_1+\cdots+j\omega_n) = -\sum_{k_1=0}^{K} c_{1,0}(k_1)(j\omega_1+\cdots+j\omega_n)^{k_1}$, and

$$f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) = (\prod_{i=1}^{q}(j\omega_{l(n(\bar{s})-q+i)})^{k_{p+i}} \Big/ L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_{l(i)}) \quad (9b)$$

$$f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)}) = \prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))})^{k_i} \quad (9c)$$

The mapping function $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$ enables the complex valued function $f_n(j\omega_1,\cdots,j\omega_n)$ to be analytically and directly determined in terms of the first order GFRF and model nonlinear parameters. Therefore, the nth-order GFRF can directly be written into a more straightforward and meaningful polynomial function in terms of the first order GFRF and model parameters by using the mapping function $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$ as

$$H_n(j\omega_1,\cdots,j\omega_n) = CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n) \quad (10)$$

Using equation (20), equations (3) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot\bar{F}_n(j\omega) \quad (11a)$$

where $\bar{F}_n(j\omega) = \frac{1}{\sqrt{n}(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} \varphi_n(CE(H_n(\cdot);\omega_1,\cdots,\omega_n)\cdot\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega$. Similarly, equation (6) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} CE(H_n(j\omega_{k_1},\cdots,j\omega_{k_n}))\cdot\tilde{F}_n(\omega) \quad (11b)$$

where $\tilde{F}_n(j\omega) = \frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}\varphi_n(CE(H_n(\cdot));\omega_{k_1},\cdots,\omega_{k_n})\cdot F(\omega_{k_1})\cdots F(\omega_{k_n})$.

As discussed in Jing et al (2008), it can be seen from equations (10) and (11) that the mapping function $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$ can facilitate the frequency domain analysis of nonlinear systems such that the relationship between the frequency response functions and model parameters, and the relationship between the frequency response functions and $H_1(j\omega_{l(1)})$ can be demonstrated explicitly, and some new properties of the GFRFs and output spectrum can be revealed.

In this study, a novel property of the nonlinear influence on system output spectrum is revealed by using the new mapping function $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$ and frequency response functions defined in Equations (10-11). It is shown that the nonlinear terms in a system can drive the system output spectrum to be an alternative series at certain frequencies when the system subjects to a sinusoidal input. This provides a novel insight into the nonlinear effect on the system output spectrum from a specific nonlinear term.

3 Alternating phenomenon in the output spectrum and its influence

For any specific nonlinear parameter *c* in model (2), the output spectrum (11a,b) can be expanded with respect to this parameter into a power series as

$$Y(j\omega)=F_0(j\omega)+cF_1(j\omega)+c^2F_2(j\omega)+\cdots+c^\rho F_\rho(j\omega)+\cdots \tag{12}$$

Note that when *c* represents a pure input nonlinearity, (12) may be a finite series; in other cases, it is definitely an infinite series, and if only the first ρ terms in the series (12) are considered, there is a truncation error denoted by $o(\rho)$. $F_i(j\omega)$ for i=0,1,2,… can be obtained from $\overline{F}_i(j\omega)$ or $\tilde{F}_i(j\omega)$ in (11a,b) by using the mapping $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$. Clearly, $F_i(j\omega)$ dominate the property of this power series. Thus the property of this power series can be revealed by studying the property of $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$. This will be discussed in detail in the next section. In this section, the alternating phenomenon of this power series and its influence are discussed.

For any $\upsilon\in\mathbb{C}$, define an operator as

$$\operatorname{sgn}_c(\upsilon)=\left[\operatorname{sgn}_r(\operatorname{Re}(\upsilon))\quad \operatorname{sgn}_r(\operatorname{Im}(\upsilon))\right] \tag{13}$$

where $\operatorname{sgn}_r(x)=\begin{cases}+1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$ for $x\in\mathbb{R}$.

Definition 1 (Alternating series). Consider a power series of form (12) with c>0. If $\operatorname{sgn}_c(F_i(j\omega))=-\operatorname{sgn}_c(F_{i+1}(j\omega))$ for i=0,1,2,3,…, then the series is an alternating series.

The series (12) can be written into two series as

$$\begin{aligned}Y(j\omega)&=\operatorname{Re}(Y(j\omega))+j(\operatorname{Im}(Y(j\omega)))\\&=\operatorname{Re}(F_0(j\omega))+c\operatorname{Re}(F_1(j\omega))+c^2\operatorname{Re}(F_2(j\omega))+\cdots+c^\rho\operatorname{Re}(F_\rho(j\omega))+\cdots\\&\quad+j(\operatorname{Im}(F_0(j\omega))+c\operatorname{Im}(F_1(j\omega))+c^2\operatorname{Im}(F_2(j\omega))+\cdots+c^\rho\operatorname{Im}(F_\rho(j\omega))+\cdots)\end{aligned} \tag{13}$$

From definition 1, if $Y(j\omega)$ is an alternating series, then $\mathrm{Re}(Y(j\omega))$ and $\mathrm{Im}(Y(j\omega))$ are both alternating. When (12) is an alternating series, there are some interesting properties summarized in Proposition 1. Denote

$$Y(j\omega)_{1\to\rho} = F_0(j\omega) + cF_1(j\omega) + c^2F_2(j\omega) + \cdots + c^\rho F_\rho(j\omega) \tag{14}$$

Proposition 1. Suppose (12) is an alternating series for c>0, then:

(1) if there exist T>0 and R>0 such that for i>T

$$\min\left\{-\frac{\mathrm{Re}(F_i(j\omega))}{\mathrm{Re}(F_{i+1}(j\omega))}, -\frac{\mathrm{Im}(F_i(j\omega))}{\mathrm{Im}(F_{i+1}(j\omega))}\right\} > R$$

then (12) has a radius of convergence R, the truncation error for a finite order $\rho > T$ is $|o(\rho)| \le c^{\rho+1}|F_{\rho+1}(j\omega)|$, and for n>0,

$$|Y(j\omega)_{1\to T+2}| < \cdots < |Y(j\omega)_{1\to T+2n}| < |Y(j\omega)| < |Y(j\omega)_{1\to T+2n+1}| < \cdots < |Y(j\omega)_{1\to T}|;$$

(2) $|Y(j\omega)|^2 = Y(j\omega)Y(-j\omega)$ is also an alternating series with respect to parameter c; Furthermore, $|Y(j\omega)|^2 = Y(j\omega)Y(-j\omega)$ is alternating only if $\mathrm{Re}(Y(j\omega))$ is alternating;

(3) there exists a $\bar{c} > 0$ such that $\frac{\partial|Y(j\omega)|}{\partial c} < 0$ for $0 < c < \bar{c}$. □

The proof is omitted. Proposition 1 shows that once the system output spectrum can be expressed into an alternating series with respect to a specific parameter c, it is always easier to find c such that the output spectrum is convergent, and its magnitude can always be suppressed by a properly designed c. Moreover, it is also shown that the low limit of the magnitude of the output spectrum that can be reached is larger than $|Y(j\omega)_{1\to T+2}|$ and the truncation error can be easily evaluated if the output spectrum can be expressed into an alternating series.

An example is given to illustrate these results.

Example 1. Consider a SDOF spring-damping system with a cubic nonlinear damping which can be described by the following differential equation,

$$m\ddot{y} = -k_0 y - B\dot{y} - c\dot{y}^3 + u(t) \tag{15}$$

Note that k_0 represents the spring characteristic, B the damping characteristic and c is the cubic nonlinear damping characteristic. This system is a simple case of NDE model (2) and can be written into the form of NDE model with M=3, K=2, $c_{10}(2) = m$, $c_{10}(1) = B$, $c_{10}(0) = k_0$, $c_{30}(111) = c$, $c_{01}(0) = -1$ and all the other parameters are zero.

Note that there is only one output nonlinear term in this case, the nth-order GFRF for system (15) can be derived according to the algorithm in Billings and Peyton-Jone (1990), which can recursively be given as

$$H_n(j\omega_1,\cdots,j\omega_n) = \frac{c_{3,0}(1,1,1)H_{n,3}(j\omega_1,\cdots,j\omega_n)}{L_n(j\omega_1+\cdots+j\omega_n)}$$

$$H_{n,3}(\cdot) = \sum_{i=1}^{n-2} H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,2}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n) = H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)$$

Proceeding with the recursive computation above, it can be seen that $H_n(j\omega_1,\cdots,j\omega_n)$ is a polynomial of $c_{30}(111)$, and substituting these equations above into (11) gives another polynomial for the output spectrum. By using the relationship (10) and the mapping function $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$, these results can be obtained directly as follows.

For simplicity, let $u(t)=F_d\sin(\Omega t)\quad(F_d>0)$. Then $F(\omega_{k_l})=-jk_lF_d$, for $k_l=\pm1, \omega_{k_l}=k_l\Omega$, and $l=1,\cdots,n$ in (11b). By using (8) or Proposition 5 in Jing et al (2006), it can be obtained that

$$CE(H_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1}))=(c_{3,0}(1,1,1))^n \text{ and } CE(H_{2n}(j\omega_1,\cdots,j\omega_{2n}))=0 \text{ for n=0,1,2,3,…} \quad (16)$$

Therefore, for n=0,1,2,3,…

$$H_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})=c^n\cdot\varphi_{2n+1}(CE(H_{2n+1}(\cdot));\omega_1,\cdots,\omega_{2n+1}) \text{ and } H_{2n}(j\omega_1,\cdots,j\omega_{2n})=0 \quad (17)$$

Then the output spectrum at frequency Ω can be computed as

$$Y(j\Omega)=\sum_{n=0}^{\lfloor N-1/2\rfloor}c^n\cdot\tilde{F}_{2n+1}(\Omega) \quad (18)$$

where $\tilde{F}_{2n+1}(j\Omega)$ can be computed as

$$\begin{aligned}\tilde{F}_{2n+1}(j\Omega)&=\frac{1}{2^{2n+1}}\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega}\varphi_{2n+1}(CE(H_{2n+1}(\cdot));\omega_{k_1},\cdots,\omega_{k_{2n+1}})\cdot(-jF_d)^{2n+1}\cdot k_1k_2\cdots k_{2n+1}\\&=\frac{1}{2^{2n+1}}\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega}\varphi_{2n+1}(CE(H_{2n+1}(\cdot));\omega_{k_1},\cdots,\omega_{k_{2n+1}})\cdot(-1)^{n+1}j(F_d)^{2n+1}\cdot(-1)^n \quad (19)\\&=-j(\frac{F_d}{2})^{2n+1}\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega}\varphi_{2n+1}(CE(H_{2n+1}(\cdot));\omega_{k_1},\cdots,\omega_{k_{2n+1}})\end{aligned}$$

and $\varphi_{2n+1}(CE(H_{2n+1}(\cdot));\omega_1,\cdots,\omega_{2n+1})=\varphi_{2n+1}(c_{30}(1,1,1)^n;\omega_1,\cdots,\omega_{2n+1})$ can be obtained according to equations (9a-c). For example,

$$\varphi_3(c_{30}(111);\omega_1,\omega_2,\omega_3)=\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}(j\omega_i)\cdot\prod_{i=1}^{3}H_1(j\omega_i)=\frac{\prod_{i=1}^{3}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}H_1(j\omega_i)$$

$$\begin{aligned}&\varphi_5(c_{3,0}(111)c_{3,0}(111);\omega_1,\cdots,\omega_5)\\&=f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot\sum_{\substack{\text{all the 3-partitions}\\\text{for } c_{3,0}(111)}}\sum_{\substack{\text{all the different}\\\text{permutations of }\{0,0,1\}}}\Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(111));\omega_1\cdots\omega_5)\\&\qquad\cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(111));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]\\&=f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot\begin{pmatrix}f_{2a}(s_0s_0s_1(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(111);\omega_3\cdots\omega_5)\\+f_{2a}(s_0s_1s_0(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(111);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\+f_{2a}(s_1s_0s_0(c_{30}(111));\omega_1\cdots\omega_5)\varphi_3(c_{30}(111);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)\end{pmatrix}\end{aligned}$$

$$=\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\left(\frac{(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\right)\cdot\prod_{i=1}^{5}H_1(j\omega_i)$$

where $\omega_i \in \{\Omega, -\Omega\}$, and so on. Substituting these results into Equation (18), the output spectrum is clearly a power series with respect to the parameter c. When there are more nonlinear terms, it is obvious that the computation process above can directly result in a straightforward multivariate power series with respect to these nonlinear parameters. To check the alternating phenomenon of the output spectrum, consider the following values for each linear parameter: m=240, k_0=16000, B=296, F_d=100, and $\Omega = 8.165$. Then it is obtained that

$$
\begin{aligned}
Y(j\Omega) &= \tilde{F}_1(\Omega) + c\tilde{F}_3(\Omega) + c^2\tilde{F}_5(\Omega) + \cdots \\
&= -j(\frac{F_d}{2})H_1(j\Omega) + 3(\frac{F_d}{2})^3 \frac{\Omega^3 |H_1(j\Omega)|^2 H_1(j\Omega)}{L_1(j\Omega)} \\
&\quad + 3(\frac{F_d}{2})^5 \frac{\Omega^5 |H_1(j\Omega)|^4 H_1(j\Omega)}{L_1(j\Omega)} (\frac{j6\Omega}{L_1(j\Omega)} + \frac{j3\Omega}{L_1(j3\Omega)} + \frac{-j3\Omega}{L_1(-j\Omega)}) + \cdots \\
&= (-0.02068817126756 + 0.00000114704116i) \\
&\quad + (5.982851578532449e\text{-}006 - 6.634300276113922e\text{-}010i)c \\
&\quad + (-5.192417616715994e\text{-}009 + 3.323565122085705e\text{-}011i)c^2 + \ldots
\end{aligned} \tag{20a}
$$

The series is alternating. In order to check the series further, computation of $\varphi_{2n+1}(c_{30}(1,1,1)^n; \omega_1, \cdots, \omega_{2n+1})$ can be carried out for higher orders. It can also be verified that the magnitude square of the output spectrum (20a) is still an alternating series, *i.e.*,

$$
\begin{aligned}
|Y(j\Omega)|^2 &= (4.280004317115985e\text{-}004) - (2.475485177721052e\text{-}007)c \\
&\quad + (2.506378395908398e\text{-}010)c^2 - \ldots
\end{aligned} \tag{20b}
$$

As pointed in Proposition 1, it is easy to find a c such that (20a-b) are convergent and their limits are decreased. From (20b) and according to Proposition 1, it can be computed that $0.01671739 < |Y(j\Omega)| < 0.0192276 < 0.0206882$ for c=600. This can be verified by Figure 1. Figure 1 is a result from simulation tests, and shows that the magnitude of the output spectrum is decreasing when c is increasing. This property is of great significance in practical engineering systems for output suppression through structural characteristic design or feedback control.

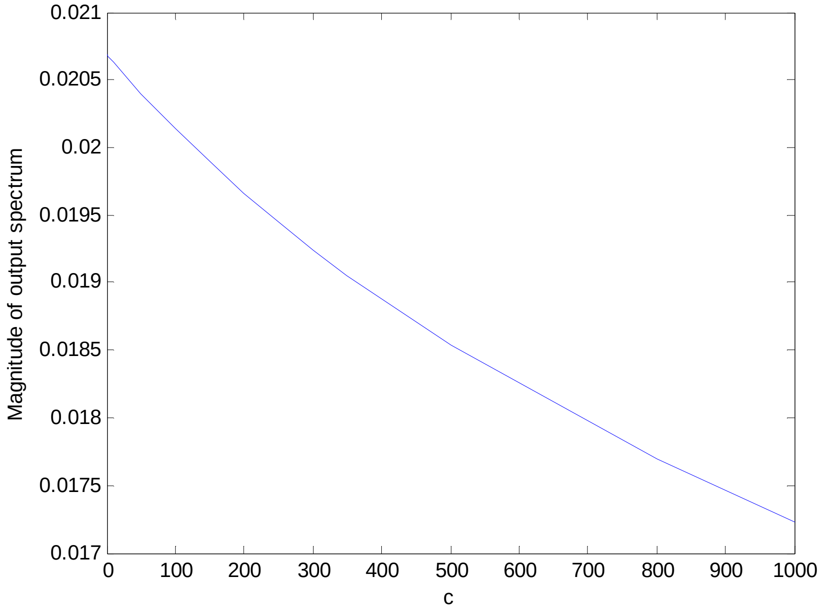

Figure 1. Magnitude of output spectrum

4 Alternating conditions

In this section, the conditions under which the output spectrum of Equation (12) can be expressed into an alternating series with respect to a specific nonlinear parameter are studied. Suppose the system subjects to a harmonic input $u(t) = F_d \sin(\Omega t) \quad (F_d > 0)$ and only the output nonlinearities (*i.e.*, $c_{p,0}(.)$ with $p \geq 2$) are considered. For convenience, assume that there is only one nonlinear parameter $c_{p,0}(.)$ in model (2) and all the other nonlinear parameters are zero. The results for this case can be extended to the general one.

Under the assumptions above, it can be obtained from the parametric characteristic analysis in Jing et al (2006) as demonstrated in Example 1 and Equation (11b) that

$$\begin{aligned} Y(j\Omega) &= Y_1(j\Omega) + Y_p(j\Omega) + \cdots + Y_{(p-1)n+1}(j\Omega) + \cdots \\ &= \tilde{F}_1(\Omega) + c_{p,0}(\cdot)\tilde{F}_p(\Omega) + \cdots + c_{p,0}(\cdot)^n \tilde{F}_{(p-1)n+1}(\Omega) + \cdots \\ &= \tilde{F}_1(\Omega) + c_{p,0}(\cdot)\tilde{F}_p(\Omega) + \cdots + c_{p,0}(\cdot)^n \tilde{F}_{(p-1)n+1}(\Omega) + \cdots \end{aligned} \tag{21a}$$

where $\omega_{k_i} \in \{\pm\Omega\}$, $\tilde{F}_{(p-1)n+1}(j\Omega)$ can be computed from (11b), and n is positive integer. Noting that $F(\omega_{k_l}) = -jk_l F_d$, $k_l = \pm 1, \omega_{k_l} = k_l\Omega$, and $l = 1, \cdots, n$ in (11b),

$$\tilde{F}_{(p-1)n+1}(j\Omega) = \frac{1}{2^{(p-1)n+1}} \sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega} \varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_{k_1}, \cdots, \omega_{k_{(p-1)n+1}}) \cdot (-jF_d)^{(p-1)n+1} \cdot k_1 k_2 \cdots k_{(p-1)n+1} \tag{21b}$$

If p is an odd integer, then (p-1)n+1 is also an odd integer. Thus there should be (p-1)n/2 frequency variables being $-\Omega$ and (p-1)n/2+1 frequency variables being Ω such that $\omega_{k_1} + \cdots + \omega_{k_{(p-1)n+1}} = \Omega$. In this case,

$$(-jF_d)^{(p-1)n+1} \cdot k_1 k_2 \cdots k_{(p-1)n+1} = (-1) \cdot j \cdot \left(j^2\right)^{(p-1)n/2} \cdot (F_d)^{(p-1)n+1} \cdot (-1)^{(p-1)n/2} = -j(F_d)^{(p-1)n+1}$$

If p is an even integer, then (p-1)n+1 is an odd integer for n=2k (k=1,2,3,…) and an even integer for n=2k-1 (k=1,2,3,…). When n is an odd integer, $\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}\neq\Omega$ for $\omega_{k_i}\in\{\pm\Omega\}$. This gives that $\tilde{F}_{(p-1)n+1}(j\Omega)$ =0. When n is an even integer, (p-1)n+1 is an odd integer. In this case, it is similar to that p is an odd integer. Therefore, for n>0

$$\tilde{F}_{(p-1)n+1}(j\Omega)=\begin{cases}-j\left(\dfrac{F_d}{2}\right)^{(p-1)n+1}\displaystyle\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}}) & \text{if p is odd or n is even}\\ 0 & \text{else}\end{cases} \tag{21c}$$

From Equations (21a-c) it is obvious that the property of the new mapping $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$ plays a key role in the series. To develop the alternating conditions for series (21a), the following results can be obtained.

Lemma 1. That $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$ is symmetric or asymmetric has no influence on $\tilde{F}_{(p-1)n+1}(j\Omega)$.

This lemma is obvious since $\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}(\cdot)$ includes all the possible permutations of $(\omega_{k_1},\cdots,\omega_{k_{2n+1}})$. Although there are many choices to obtain the asymmetric $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$ which may be different at different permutation $(\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$, they have no different effect on the analysis of $\tilde{F}_{(p-1)n+1}(j\Omega)$.

Lemma 2. Consider parameter $c_{p,q}(k_1,k_2,\ldots,k_{p+q})$.

(1) If $p\geq 2$ and q=0, then

$$\varphi_{n(\bar{s})}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$

$$=\frac{(-1)^{n-1}\prod_{i=1}^{(p-1)n+1}H_1(j\omega_{l(i)})}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\leq\bar{x}_i\leq n-1}}\left[\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right.\ \text{w}$$

$$\left.\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_k^*(k_1,\cdots,k_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}\right]$$

here,

$$\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$

$$=\frac{-1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\left[\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right.$$

$$\left.\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_k^*(k_1,\cdots,k_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}\right]$$

the termination is $\varphi'_1(1;\omega_i)=1$. $n_k^*(k_1,\cdots,k_p)=\frac{p!}{n_1!n_2!\cdots n_e!}$, $n_1+\ldots+n_e=p$, e is the number of distinct differentials k_i appearing in the combination, n_i is the number of repetitions of k_i, and a similar definition holds for $n_x^*(\bar{x}_1,\cdots,\bar{x}_p)$.

(2) If $p\ge 2$, q=0 and $k_1=k_2=\ldots=k_p=k$, then

$$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$

$$=\frac{(-1)^{n-1}\prod_{i=1}^{(p-1)n+1}[(j\omega_{l(i)})^k H_1(j\omega_{l(i)})]}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}$$

$$\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}} n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

where, if $\bar{x}_i=0$, $\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})=1$, otherwise,

$$\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

$$=\frac{(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^k}{-L_{(p-1)\bar{x}_i+1}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})}\cdot$$

$$\sum_{\substack{\text{all the different combinations}\\ \text{of }\{x_1,x_2,\ldots,x_p\}\text{ satisfying}\\ x_1+\cdots+x_p=\bar{x}_i-1,\,0\le x_j\le\bar{x}_i-1}} n_x^*(x_1,\cdots,x_p)\cdot\prod_{j=1}^{p}\varphi''_{(p-1)x_j+1}(c_{p,0}(\cdot)^{x_j};\omega_{l(\bar{X}'(j)+1)}\cdots\omega_{l(\bar{X}'(j)+(p-1)x_j+1)})$$

The recursive terminal of $\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$ is $\bar{x}_i=1$.

Proof.

$$\varphi_{n(\bar{s})}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\varphi_{(p-1)n+1}(c_{p,0}(\cdot)c_{p,0}(\cdot)\cdots c_{p,0}(\cdot);\omega_{l(1)}\cdots\omega_{(l(p-1)n+1)})$$

$$=\sum_{\substack{\text{all the 2-partitions}\\ \text{for }\bar{s}\text{ satisfying}\\ s_1(\bar{s})=c_{p,0}(\cdot)}}\Big\{f_1(c_{p,0}(\cdot),(p-1)n+1;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})\cdot\sum_{\substack{\text{all the p-partitions}\\ \text{for }\bar{s}/c_{3,0}(\cdot)}}\sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of }\{s_{x_1},\cdots,s_{x_p}\}}}$$

$$\Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{p0}(\cdot)^{n-1});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\cdot\prod_{i=1}^{p}\varphi_{n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1}))}(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1});\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1})))})\Big]\Big\}$$

$$= \frac{1}{L_{(p-1)n+1}(j\omega_{l(1)} + \cdots + j\omega_{l((p-1)n+1)})} \cdot \sum_{\substack{\text{all the p-partitions} \\ \text{for } \bar{s}/c_{p,0}(\cdot)}} \sum_{\substack{\text{all the different} \\ \text{permutations} \\ \text{of } \{s_{x_1},\cdots,s_{x_p}\}}} \left[\prod_{i=1}^{p} (j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1})))})^{k_i} \right.$$

$$\left. \cdot \prod_{i=1}^{p} \varphi_{n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1}))}(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1}); \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1})))}) \right]$$

$$= \frac{1}{L_{(p-1)n+1}(j\omega_{l(1)} + \cdots + j\omega_{l((p-1)n+1)})} \cdot \sum_{\substack{\text{all the different combinations} \\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying} \\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}} \sum_{\substack{\text{all the different} \\ \text{permutations of} \\ \text{each combination}}} \left[\prod_{i=1}^{p} (j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i} \right.$$

$$\left. \cdot \prod_{i=1}^{p} \varphi_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)}) \right]$$

Note that different permutations in each combination have no difference to $\prod_{i=1}^{p} \varphi_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$, thus $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_1 \cdots \omega_{(p-1)n+1})$ can be written as

$$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_1 \cdots \omega_{(p-1)n+1})$$

$$= \frac{1}{L_{(p-1)n+1}(j\omega_{l(1)} + \cdots + j\omega_{l((p-1)n+1)})} \cdot \sum_{\substack{\text{all the different combinations} \\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying} \\ \bar{x}_1+\cdots+\bar{x}_3=n-1,\ 0\le\bar{x}_i\le n-1}} \prod_{i=1}^{p} \varphi_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

$$\cdot \sum_{\substack{\text{all the different} \\ \text{permutations of} \\ \text{each combination}}} \prod_{i=1}^{p} (j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}$$

$$= \frac{1}{L_{(p-1)n+1}(j\omega_{l(1)} + \cdots + j\omega_{l((p-1)n+1)})} \cdot \sum_{\substack{\text{all the different combinations} \\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying} \\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}} \prod_{i=1}^{p} \varphi_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

$$\cdot \frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_k^*(k_1,\cdots,k_p)} \cdot \sum_{\substack{\text{all the different} \\ \text{permutations of} \\ \{k_1,\cdots,k_p\}}} \prod_{i=1}^{p} (j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}$$

$n_x^*(\bar{x}_1,\cdots,\bar{x}_p)$ and $n_k^*(k_1,\cdots,k_p)$ are the numbers of the corresponding combinations involved, which can be obtained from the combination theory and can also be referred to Peyton-Jones (2007). Inspection of the recursion in the equation above, it can be seen that there are $(p\text{-}1)n + 1$ $H_1(j\omega_i)$ with different frequency variable at the end of the recursion. Thus they can be brought out as a common factor. This gives

$$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_{l(1)} \cdots \omega_{l((p-1)n+1)}) = (-1)^n \prod_{i=1}^{(p-1)n+1} H_1(j\omega_{l(i)}) \cdot \varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_{l(1)} \cdots \omega_{l((p-1)n+1)}) \quad (22a)$$

where,

$$\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$

$$=\frac{-1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}}\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

$$\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_k^*(k_1,\cdots,k_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}\qquad(22b)$$

the termination is $\varphi'_1(1;\omega_i)=1$. Note that when $\bar{x}_i=0$, there is a term $(j\omega_{l(\bar{X}(i)+1)})^{k_i}$ appearing from $\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_k^*(k_1,\cdots,k_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}$. It can be verified that in each recursion of $\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$, there may be some frequency variables appearing individually in the form of $(j\omega_{l(\bar{X}(i)+1)})^{k_i}$, and these variables will not appear individually in the same form in the subsequent recursion. At the end of the recursion, all the frequency variables should have appeared in this form. Thus these terms can also be brought out as common factors if $k_1=k_2=\ldots=k_p$. In the case of $k_1=k_2=\ldots=k_p=k$,

$$\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_k^*(k_1,\cdots,k_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}$$

$$=n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}$$

Therefore (22ab) can be written, if $k_1=k_2=\ldots=k_p$, as

$$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$
$$=(-1)^n\prod_{i=1}^{(p-1)n+1}[(j\omega_{l(i)})^k H_1(j\omega_{l(i)})]\cdot\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})\qquad(23a)$$

$$\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$

$$=\frac{-1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}}\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

$$\cdot n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i(1-\delta(\bar{x}_i))}\qquad(23b)$$

(23b) can be further written as

$$\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$
$$=\frac{-1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}$$
$$\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}} n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\qquad(24a)$$

where, if $\bar{x}_i=0$, $\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})=1$, otherwise,

$$
\begin{aligned}
&\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\\
&=(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)x_i+1)})^k\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\\
&=\frac{(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^k}{-L_{(p-1)\bar{x}_i+1}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{x_1,x_2,\ldots,x_p\}\text{ satisfying}\\ x_1+\cdots+x_p=\bar{x}_i-1,\ 0\le x_i\le\bar{x}_i-1}} n_x^*(x_1,\cdots,x_p)\\
&\qquad\cdot\prod_{i=1}^{p}(j\omega_{l(\bar{X}'(i)+1)}+\cdots+j\omega_{l(\bar{X}'(i)+(p-1)x_i+1)})^{k_i(1-\delta(x_i))}\varphi'_{(p-1)x_i+1}(c_{p,0}(\cdot)^{x_i};\omega_{l(\bar{X}'(i)+1)}\cdots\omega_{l(\bar{X}'(i)+(p-1)x_i+1)})\\
&=\frac{(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^k}{-L_{(p-1)\bar{x}_i+1}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})}\cdot\\
&\qquad\sum_{\substack{\text{all the different combinations}\\ \text{of }\{x_1,x_2,\ldots,x_p\}\text{ satisfying}\\ x_1+\cdots+x_p=\bar{x}_i-1,\ 0\le x_i\le\bar{x}_i-1}} n_x^*(x_1,\cdots,x_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)x_i+1}(c_{p,0}(\cdot)^{x_i};\omega_{l(\bar{X}'(i)+1)}\cdots\omega_{l(\bar{X}'(i)+(p-1)x_i+1)})
\end{aligned}
\tag{24b}
$$

The recursive terminal of (24b) is $\bar{x}_i=1$. Replacing (22b) into (22a) and replacing (24ab) into (23a), the lemma can be obtained. This completes the proof. □

For convenience, define an operator "*" for $\mathrm{sgn}_c(.)$ satisfying

$$\mathrm{sgn}_c(\upsilon_1)*\mathrm{sgn}_c(\upsilon_2)=[\mathrm{sgn}_r(\mathrm{Re}(\upsilon_1\upsilon_2))\quad \mathrm{sgn}_r(\mathrm{Im}(\upsilon_1\upsilon_2))]$$

for any $\upsilon_1,\upsilon_2\in\mathbb{C}$. It is obvious $\mathrm{sgn}_c(\upsilon_1)*\mathrm{sgn}_c(\upsilon_2)=\mathrm{sgn}_c(\upsilon_1\upsilon_2)$.

The following lemma is straightforward.

Lemma 3. For $\upsilon_1,\upsilon_2,\nu\in\mathbb{C}$, suppose $\mathrm{sgn}_c(\upsilon_1)=-\mathrm{sgn}_c(\upsilon_2)$. If $\mathrm{Re}(\nu)\mathrm{Im}(\nu)=0$, then $\mathrm{sgn}_c(\upsilon_1\nu)=-\mathrm{sgn}_c(\upsilon_2\nu)$. If $\mathrm{Re}(\nu)\mathrm{Im}(\nu)=0$ and $\nu\neq 0$, then $\mathrm{sgn}_c(\upsilon_1/\nu)=-\mathrm{sgn}_c(\upsilon_2/\nu)$. □

Proposition 2. The output spectrum in (21a-c) is an alternating series with respect to any specific parameter $c_{p,0}(k_1,k_2,\ldots,k_p)$ satisfying $c_{p,0}(.)>0$ and $p=2r+1$ for r=1,2,3,...

(1) if and only if

$$\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}(-1)^{n-1}\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})\right)=const\text{, i.e.,}$$

$$
\begin{aligned}
&\mathrm{sgn}_c\left(\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)}\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\ \sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}}\left[\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right.\right.\\
&\qquad\left.\left.\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_k^*(k_1,\cdots,k_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{k_i}\right]\right)\\
&=const
\end{aligned}
\tag{25}
$$

(2) if $k_1=k_2=\ldots=k_p=k$ in $c_{p,0}(.)$, $\mathrm{Re}(\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)})\mathrm{Im}(\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)})=0$, and

$$\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\ \sum_{\substack{\text{all the different combinations}\\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}}\left[n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right]\right)=const \tag{26}$$

where *const* is a two-dimensional constant vector whose elements are +1, 0 or -1. □

The proof is completed. Proposition 2 provides a sufficient and necessary condition for the output spectrum series (21a-c) to be an alternating series with respect to a specific nonlinear parameter $c_{p,0}(k_1,k_2,\ldots,k_p)$ satisfying $c_{p,0}(.)>0$ and $p=2r+1$ for r=1,2,3,.... Similar results can also be established for any other nonlinear parameters. Regarding nonlinear parameter $c_{p,0}(k_1,k_2,\ldots,k_p)$ satisfying $c_{p,0}(.)>0$ and $p=2r$ for r=1,2,3,...., it can be obtained from (21a-c) that

$$Y(j\Omega)=\tilde{F}_1(\Omega)+c_{p,0}(\cdot)^2\tilde{F}_{2(p-1)+1}(\Omega)+\cdots+c_{p,0}(\cdot)^{2n}\tilde{F}_{2(p-1)n+1}(\Omega)+\cdots$$

$\tilde{F}_{2(p-1)n+1}(\Omega)$ for n=1,2,3,… should be alternating so that $Y(j\Omega)$ is alternating. This yields that

$$\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2(p-1)n+1}}=\Omega}\varphi_{2(p-1)n+1}(c_{p,0}(\cdot)^{2n};\omega_{l(1)}\cdots\omega_{l(2(p-1)n+1)})\right)$$
$$=-\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2(p-1)(n+1)+1}}=\Omega}\varphi_{2(p-1)(n+1)+1}(c_{p,0}(\cdot)^{2(n+1)};\omega_{l(1)}\cdots\omega_{l(2(p-1)(n+1)+1)})\right)$$

Clearly, this is completely different from the conditions in Proposition 2. It may be more difficult for the output spectrum to be alternating with respect to $c_{p,0}(.)>0$ with $p=2r$ than $c_{p,0}(.)>0$ with $p=2r+1$.

Note that Equation (21a) is based on the assumption that there is only nonlinear parameter $c_{p,0}(.)$ and all the other nonlinear parameters are zero. If the effects from the other nonlinear parameters are considered, Equation (21a) can be written as

$$Y(j\Omega)=\tilde{F}'_1(\Omega)+c_{p,0}(\cdot)\tilde{F}'_p(\Omega)+\cdots+c_{p,0}(\cdot)^n\tilde{F}'_{(p-1)n+1}(\Omega)+\cdots \tag{27a}$$

where

$$\tilde{F}'_{(p-1)n+1}(\Omega)=\tilde{F}_{(p-1)n+1}(\Omega)+\delta_{(p-1)n+1}(\Omega;C_{p',q'}\setminus c_{p,0}(.)) \tag{27b}$$

$C_{p',q'}$ includes all the nonlinear parameters in the system. Based on the parametric characteristic analysis in Jing et al (2006) and the new mapping function $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$ defined in Jing et al (2008), (27b) can be determined easily. For example, suppose *p* is an odd integer larger than 1, then $\tilde{F}_{(p-1)n+1}(j\Omega)$ is given in (21c), and $\delta_{(p-1)n+1}(\Omega;C_{p',q'}\setminus c_{p,0}(.))$ can be computed as

$$\delta_{(p-1)n+1}(\Omega; C_{p',q'} \setminus c_{p,0}(.)) = \sum_{\substack{\text{all the monomails consisting of the parameters in } C_{p',q'}\setminus c_{p,0}(\cdot) \\ \text{satisfying } np+\sum(p_i'+q_i') \text{ is odd and less than N}}} \left[-j\left(\frac{F_d}{2}\right)^{n(c_{p,0}{}^n s(\cdot))} \right.$$

$$\left. \cdot \sum_{\omega_{k_1}+\cdots+\omega_{k_{n(c_{p,0}{}^n s(\cdot))}}=\Omega} \varphi_{n(c_{p,0}{}^n s(\cdot))}(c_{p,0}{}^n s(C_{p',q'} \setminus c_{p,0}(\cdot)); \omega_{k_1} \cdots \omega_{k_{n(c_{p,0}{}^n s(\cdot))}}) \right]$$

where $s(C_{p',q'} \setminus c_{p,0}(\cdot))$ denotes a monomial consisting of some parameters in $C_{p',q'} \setminus c_{p,0}(\cdot)$.

It is obvious that if (21a) is an alternating series, then (27a) can still be alternating under a proper design of the other nonlinear parameters (for example the other parameters are sufficiently small). Moreover, from the discussions above, it can be seen that whether the system output spectrum is an alternating series or not with respect to a specific nonlinear parameter is greatly dependent on the system linear parameters.

Example 2. To demonstrate the theoretical results above, consider again model (15) in Example 1. Let $u(t) = F_d \sin(\Omega t) \quad (F_d > 0)$. The output spectrum at frequency Ω is given in (18-19). From Lemma 2, it can be derived for this case that

$$\varphi_{2n+1}(c_{3,0}(\cdot)^n; \omega_{l(1)} \cdots \omega_{l(2n+1)}) = \frac{(-1)^{n-1} \prod_{i=1}^{2n+1} [(j\omega_{l(i)})^k H_1(j\omega_{l(i)})]}{L_{2n+1}(j\omega_{l(1)} + \cdots + j\omega_{l(2n+1)})}$$

$$\cdot \sum_{\substack{\text{all the different combinations} \\ \text{of } \{\bar{x}_1, \bar{x}_2, \bar{x}_3\} \text{ satisfying} \\ \bar{x}_1+\bar{x}_2+\bar{x}_3=n-1,\ 0\le \bar{x}_i \le n-1}} n_x^*(\bar{x}_1, \bar{x}_2, \bar{x}_3) \cdot \prod_{i=1}^{3} \varphi''_{2\bar{x}_i+1}(c_{3,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+2\bar{x}_i+1)}) \tag{28a}$$

where, if $\bar{x}_i = 0$, $\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)}) = 1$, otherwise,

$$\varphi''_{2\bar{x}_i+1}(c_{3,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+2\bar{x}_i+1)})$$

$$= \frac{(j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+2\bar{x}_i+1)})^k}{-L_{2\bar{x}_i+1}(j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+2\bar{x}_i+1)})} \cdot \tag{28b}$$

$$\sum_{\substack{\text{all the different combinations} \\ \text{of } \{x_1, x_2, x_3\} \text{ satisfying} \\ x_1+x_2+x_3=\bar{x}_i-1,\ 0\le x_j \le \bar{x}_i-1}} n_x^*(x_1, x_2, x_3) \cdot \prod_{j=1}^{3} \varphi''_{2x_j+1}(c_{3,0}(\cdot)^{x_j}; \omega_{l(\bar{X}'(j)+1)} \cdots \omega_{l(\bar{X}'(j)+2x_j+1)})$$

Note that the terminal condition for (39) is

$$\varphi''_{2\bar{x}_i+1}(c_{3,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+2\bar{x}_i+1)})\Big|_{\bar{x}_i=1} = \varphi''_3(c_{3,0}(\cdot); \omega_{l(1)} \cdots \omega_{l(3)}) = \frac{(j\omega_{l(1)} + \cdots + j\omega_{l(3)})^k}{-L_3(j\omega_{l(1)} + \cdots + j\omega_{l(3)})} \tag{28c}$$

Therefore, from (28a-c) it can be easily shown that $\varphi_{2n+1}(c_{3,0}(\cdot)^n; \omega_1 \cdots \omega_{2n+1})$ can be written as

$$\varphi_{2n+1}(c_{3,0}(\cdot)^n; \omega_1 \cdots \omega_{2n+1})$$

$$= \frac{(-1)^{n-1} \prod_{i=1}^{2n+1} j\omega_i H_1(j\omega_i)}{L_{2n+1}(j\omega_1 + \cdots + j\omega_{2n+1})} \cdot \sum_{\substack{\text{all the combination } (x_1, x_2, \ldots, x_{n-1}) \\ \text{satisfying } x_i \in \{2j+1 \mid 1\le j\le n-1\} \\ x_1 \ge x_2 \ge .. \ge x_{n-1}, \text{ and} \\ \text{"=" happens only if } x_i + x_{i+1} \le 2n-2}} r_X(x_1, x_2, \ldots, x_{n-1}) \prod_{i=1}^{n-1} \frac{j\omega_{l(1)} + \cdots + j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)} + \cdots + j\omega_{l(x_i)})} \tag{29}$$

where $r_X(x_1, x_2, \ldots, x_{n-1})$ is a positive integer which can be explicitly determined by (28ab) and represents the number of all the involved combinations which have the same

$\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}$. Therefore, according to Proposition 2, it can be seen from (29) that the output spectrum (18) is an alternating series only if the following two conditions hold:

(a1) $\mathrm{Re}(\frac{H_1(j\Omega)}{L_{2n+1}(j\Omega)})\mathrm{Im}(\frac{H_1(j\Omega)}{L_{2n+1}(j\Omega)})=0$

(a2) $\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega}\ \sum_{\substack{\text{all the combination }(x_1,x_2,\ldots,x_{n-1})\\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\}\\ x_1\ge x_2\ge\ldots\ge x_{n-1},\text{ and}\\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}} r_X(x_1,x_2,\ldots,x_{n-1})\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\right)=const$

Suppose $\Omega=\sqrt{\frac{k_0}{m}}$ which is a natural resonance frequency of model (15). It can be derived that

$$L_{2n+1}(j\Omega)=-\sum_{k_1=0}^{K}c_{1,0}(k_1)(j\Omega)^{k_1}=-(m(j\Omega)^2+B(j\Omega)+k_0)=-jB\Omega$$

$$H_1(j\Omega)=\frac{-1}{L_1(j\Omega)}=\frac{1}{jB\Omega}$$

It is obvious that condition (a1) is satisfied if $\Omega=\sqrt{\frac{k_0}{m}}$. Considering condition (a2), it can be derived that

$$\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=\frac{j\varepsilon(x_i)\Omega}{-L_{x_i}(j\varepsilon(x_i)\Omega)} \tag{30a}$$

where $\varepsilon(x_i)\in\{\pm(2j+1)|0\le j\le\lceil n+1\rceil\}$, and $\lceil n+1\rceil$ denotes the odd integer not larger than n+1. Especially, when $\varepsilon(x_i)=\pm1$, it yields that

$$\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=\frac{\pm j\Omega}{-L_{x_i}(\pm j\Omega)}=\frac{\pm j\Omega}{\pm jB\Omega}=\frac{1}{B} \tag{30b}$$

when $|\varepsilon(x_i)|>1$,

$$\begin{aligned}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}&=\frac{j\varepsilon(x_i)\Omega}{-L_{x_i}(j\varepsilon(x_i)\Omega)}=\frac{j\varepsilon(x_i)\Omega}{m(j\varepsilon(x_i)\Omega)^2+B(j\varepsilon(x_i)\Omega)+k_0}\\&=\frac{j\varepsilon(x_i)\Omega}{(1-\varepsilon(x_i)^2)k_0+j\varepsilon(x_i)\Omega B}=\frac{1}{B+j(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})\sqrt{k_0m}}\end{aligned} \tag{30c}$$

If $B<<\sqrt{k_0m}$, then it gives

$$\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\approx\frac{1}{j(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})\sqrt{k_0m}} \tag{30d}$$

Note that in all the combinations involved in the summation operator in (29) or condition (a2), *i.e.*,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega}\ \sum_{\substack{\text{all the combination }(x_1,x_2,\ldots,x_{n-1})\\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\}\\ x_1\ge x_2\ge\ldots\ge x_{n-1},\text{ and}\\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}}(\cdot)$$

There always exists a combination such that

$$\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=\frac{1}{B^{n-1}} \tag{31}$$

Note that (30b) holds both for $\varepsilon(x_i)=\pm 1$, thus there is no combination such that

$$\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=-\frac{1}{B^{n-1}}$$

Noting that $B<<\sqrt{k_0 m}$, these show that

$$\max_{\substack{\text{all the involved}\\ \text{combinations}}}\left(\left|\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\right|\right)=\frac{1}{B^{n-1}}$$

which happens in the combination where (31) holds.

Because there are n+1 frequency variables to be $+\Omega$ and n frequency variables to be $-\Omega$ such that $\omega_1+\cdots+\omega_{2n+1}=\Omega$ in (18-19), there are more combinations where $\varepsilon(x_i)>0$ that is $(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})\sqrt{k_0 m}$ >0 in (30c-d). Thus there are more combinations where $\mathrm{Im}(\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})})$ is negative. Using (30b) and (30d), it can be shown under the condition that $B<<\sqrt{k_0 m}$,

$$\max_{\substack{\text{all the involved}\\ \text{combinations}}}\left(\left|\mathrm{Im}\left(\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\right)\right|\right)\approx\frac{1}{B^{n-2}(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})\sqrt{k_0 m}}\Bigg|_{\varepsilon(x_i)=3}=\frac{1}{2.7B^{n-2}\sqrt{k_0 m}}$$

This happens in the combinations where the argument of $\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}$ is either -90^0 or +90^0. Note that there are more cases in which the arguments are -90^0. If the argument is -180^0, the absolute value of the corresponding imaginary part will be not more than

$$\max_{\substack{\text{the combination}\\ \text{whose argument is}\\ -180^0}}\left(\left|\mathrm{Im}\left(\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\right)\right|\right)\approx\frac{1}{B^{n-4}(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})^3\sqrt{k_0 m}^3}\Bigg|_{\varepsilon(x_i)=3}=\frac{1}{2.7^3 B^{n-4}\sqrt{k_0 m}^3}$$

which is much less than $\frac{1}{2.7B^{n-2}\sqrt{k_0 m}}$.

Therefore, if B is sufficiently smaller than $\sqrt{k_0 m}$, the following two inequalities can hold for n>1

$$\mathrm{Re}\Bigg(\sum_{\substack{\text{all the combination }(x_1,x_2,...,x_{n-1})\\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\}\\ x_1\ge x_2\ge..\ge x_{n-1}\text{, and}\\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}} r_X(x_1,x_2,...,x_{n-1})\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\Bigg)>0$$

$$\mathrm{Im}\Bigg(\sum_{\substack{\text{all the combination }(x_1,x_2,...,x_{n-1})\\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\}\\ x_1\ge x_2\ge..\ge x_{n-1}\text{, and}\\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}} r_X(x_1,x_2,...,x_{n-1})\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\Bigg)<0$$

That is, condition (a2) holds for n>1 under $B<<\sqrt{k_0 m}$ and $\Omega=\sqrt{\frac{k_0}{m}}$. Hence, (18) is an alternating series if the following two conditions hold:

(b1) B is sufficiently smaller than $\sqrt{k_0 m}$,

(b2) The input frequency is $\Omega = \sqrt{\frac{k_0}{m}}$.

In Example 1, note that $\Omega = \sqrt{\frac{k_0}{m}} \approx$ 8.165, B=296<< $\sqrt{k_0 m}$ =1959.592. These are consistent with and verify the theoretical results established here. □

5 Conclusions

Nonlinear influence on system output spectrum is revealed in this study from a novel perspective based on Volterra series expansion in the frequency domain. For a class of system nonlinearities, it is shown for the first time that system output spectrum can be expanded into an alternating series with respect to a specific nonlinear coefficient under certain conditions and this alternating series has some interesting properties which are of significance to engineering practices. Further study will be focused on detailed design and analysis based on these results for practical systems.

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