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Nonlinear Output Frequency Response Functions of MDOF Systems with Multiple Nonlinear Components

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Abstract: In engineering practice, most mechanical and structural systems are modeled as Multi-Degree-of-Freedom (MDOF) systems. When some components within the systems have nonlinear characteristics, the whole system will behave nonlinearly. The concept of Nonlinear Output Frequency Response Functions (NOFRFs) was proposed by the authors recently and provides a simple way to investigate nonlinear systems in the frequency domain. The present study is concerned with investigating the inherent relationships between the NOFRFs for any two masses of nonlinear MDOF systems with multiple nonlinear components. The results reveal very important properties of the nonlinear systems. One significant application of the results is to detect and locate faults in engineering structures which make the structures behave nonlinearly.

1 Introduction

In engineering practice, for many mechanical and structural systems, more than one set of coordinates are needed to describe the system behaviour. This implies a MDOF model is needed to represent the system. In addition, these systems may also behave nonlinearly due to nonlinear characteristics of some components within the systems. For example, a beam with breathing cracks behaves nonlinearly because of the cracked elements inside the beam [1]. For nonlinear systems, the classical Frequency Response Function (FRF) cannot achieve a comprehensive description for the system dynamical characteristics, which, however, can be fulfilled using the Generalised Frequency Response Functions (GFRFs) [2]. The GFRFs, which are extension of the FRFs to the nonlinear case, are defined as the Fourier transforms of the kernels of the Volterra series [3]. GFRFs are powerful tools for the analysis of nonlinear systems and have been widely studied in the past two decades.

If a differential equation or discrete-time model is available for a nonlinear system, the GFRFs can be determined using the algorithm in [4]~[6]. However, the GFRFs are much more complicated than the FRF. GFRFs are multidimensional functions [7][8], which can be difficult to measure, display and interpret in practice. Recently, the novel concept known as Nonlinear Output Frequency Response Functions (NOFRFs) was proposed by the authors [9]. The concept can be considered to be an alternative extension of the FRF to the nonlinear case. NOFRFs are one dimensional functions of frequency, which allow the analysis of nonlinear systems in the frequency domain to be implemented in a manner similar to the frequency domain analysis of linear systems and which provide great insight into the mechanisms which dominate important nonlinear behaviours.

The present study is concerned with the analysis of the inherent relationships between the NOFRFs for any two masses of MDOF systems with multiple nonlinear components. The results reveal, for the first time, very important properties of the nonlinear systems, and can be applied to detect and locate faults in engineering structures which make the structures behave nonlinearly.

2. MDOF Systems with Multiple Nonlinear Components

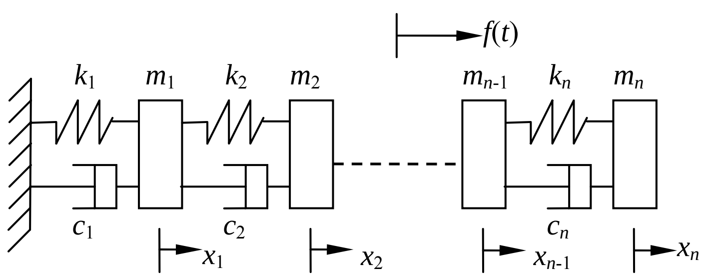

Figure 1, a multi-degree freedom oscillator

A typical multi-degree-of-freedom oscillator is shown as Figure 1, the input force is added on the *J*th mass.

If all springs and damping have linear characteristics, then this oscillator is a MDOF linear system, and the governing motion equation can be written as

$$M\ddot{x} + C\dot{x} + Kx = F(t) \tag{1}$$

where

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

is the system mass matrix, and

$$C=\begin{bmatrix} c_1+c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2+c_3 & -c_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -c_{n-1} & c_{n-1}+c_n & -c_n \\ 0 & \cdots & 0 & -c_n & c_n \end{bmatrix} K=\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{n-1} & k_{n-1}+k_n & -k_n \\ 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

are the system damping and stiffness matrix respectively. $x=(x_1,\cdots,x_n)'$ is the displacement vector, and

$$F(t)=(\overbrace{0,\cdots,0}^{J-1}, f(t),\overbrace{0,\cdots,0}^{n-J})'$$

is the external force vector acting on the oscillator.

Equation (2) is the basis of the modal analysis method, which is a well-established approach for determining dynamic characteristics of engineering structures [10]. In the linear case, the displacements $x_i(t)$ ($i=1,\cdots,n$) can be expressed as

$$x_i(t)=\int_{-\infty}^{+\infty}h_{(i)}(t-\tau)f(\tau)d\tau \tag{2}$$

where $h_{(i)}(t)$ ($i=1,\cdots,n$) are the impulse response functions that are determined by equation (1), and the Fourier transform of $h_{(i)}(t)$ is the well-known FRF.

Assume there are $\overline{L}$ nonlinear components, which have nonlinear stiffness and damping, in the MDOF system, and they are the $L(i)$th ($i=1,\cdots,\overline{L}$) components respectively, and the corresponding restoring forces $FS_{L(i)}(\Delta)$ and $FD_{L(i)}(\dot{\Delta})$ are the polynomial functions of the deformation Δ and $\dot{\Delta}$, i.e.,

$$FS_{L(i)}=\sum_{l=1}^{P}r_{(L(i),l)}\Delta^l\ ,\ \ FD_{L(i)}(\dot{\Delta})=\sum_{l=1}^{P}w_{(L(i),l)}\dot{\Delta}^l$$

where P is the degree of the polynomial. Without loss of generality, assume $L(i)-1$ and $L(i)\neq 1,J,n$ ($1\leq i\leq \overline{L}$) and $k_{L(i)}=r_{(L(i),1)}$ and $c_{L(i)}=w_{(L(i),1)}$. Then the motion of the MDOF oscillator in Figure 1 can be determined by equations (3)~(8) below.

For the masses that are not connected to the $L(i)$th ($i=1,\cdots,\overline{L}$) spring, the governing motion equations are

$$m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=0 \tag{3}$$

$$m_i\ddot{x}_i+(c_i+c_{i+1})\dot{x}_i-c_i\dot{x}_{i-1}-c_{i+1}\dot{x}_{i+1}+(k_i+k_{i+1})x_i-k_ix_{i-1}-k_{i+1}x_{i+1}=0$$

$$(i\neq L(l)-1,L(l),J\ ;\ 1\leq l\leq \overline{L}) \tag{4}$$

$$m_J\ddot{x}_J+(c_J+c_{J+1})\dot{x}_J-c_J\dot{x}_{J-1}-c_{J+1}\dot{x}_{J+1}+(k_J+k_{J+1})x_J-k_Jx_{J-1}-k_{J+1}x_{J+1}=f(t) \tag{5}$$

$$m_n\ddot{x}_n+c_n\dot{x}_n-c_n\dot{x}_{n-1}+k_nx_n-k_nx_{n-1}=0 \tag{6}$$

For the mass that is connected to the left of the $L(i)$th spring, the governing motion equation is

$$m_{L(i)-1}\ddot{x}_{L(i)-1} + (k_{L(i)-1} + k_{L(i)})x_{L(i)-1} - k_{L(i)-1}x_{L(i)-2} - k_{L(i)}x_{L(i)} + (c_{L(i)-1} + c_{L(i)})\dot{x}_{L(i)-1}$$
$$- c_{L(i)-1}\dot{x}_{L(i)-2} - c_{L(i)}\dot{x}_{L(i)} + \sum_{l=2}^{P} r_{(L(i),l)}(x_{L(i)-1} - x_{L(i)})^l + \sum_{l=2}^{P} w_{(L(i),l)}(\dot{x}_{L(i)-1} - \dot{x}_{L(i)})^l = 0$$
$$(1 \le i \le \overline{L}) \quad (7)$$

For the mass that is connected to the right of the $L(i)$ th spring, the governing motion equation is

$$m_{L(i)}\ddot{x}_{L(i)} + (k_{L(i)} + k_{L(i)+1})x_{L(i)} - k_{L(i)}x_{L(i)-1} - k_{L(i)+1}x_{L(i)+1} + (c_{L(i)} + c_{L(i)+1})\dot{x}_{L(i)}$$
$$- c_{L(i)}\dot{x}_{L(i)-1} - c_{L(i)+1}\dot{x}_{L(i)+1} - \sum_{l=2}^{P} r_{(L(i),l)}(x_{L(i)-1} - x_{L(i)})^l - \sum_{l=2}^{P} w_{(L(i),l)}(\dot{x}_{L(i)-1} - \dot{x}_{L(i)})^l = 0$$
$$(1 \le i \le \overline{L}) \quad (8)$$

Denote

$$NonF_{L(i)} = \sum_{l=2}^{P} r_{(L(i),l)}(x_{L(i)-1} - x_{L(i)})^l + \sum_{l=2}^{P} w_{(L(i),l)}(\dot{x}_{L(i)-1} - \dot{x}_{L(i)})^l \quad (9)$$

$$NF = \left(nf(1)\cdots nf(n)\right)' \quad (10)$$

where

$$nf(l) = \begin{cases} 0 & \text{if } l \ne L(i)-1, L(i),\ 1 \le i \le \overline{L} \\ -NonF_{L(i)} & \text{if } l = L(i)-1,\ 1 \le i \le \overline{L} \\ NonF_{L(i)} & \text{if } l = L(i),\ 1 \le i \le \overline{L} \end{cases} \quad (1 \le l \le n) \quad (11)$$

Equations (3)~(8) can be rewritten in a matrix form as

$$M\ddot{x} + C\dot{x} + Kx = NF + F(t) \quad (12)$$

Equations (9)~(12) are the motion governing equations of nonlinear MDOF systems with multiple nonlinear components. The $\overline{L}$ nonlinear components can lead the whole system to behave nonlinearly. In this case, the Volterra series [2] can be used to describe the relationships between the displacements $x_i(t)$ ($i = 1, \cdots, n$) and the input force $f(t)$ as below

$$x_i(t) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{(i,j)}(\tau_1, ..., \tau_j) \prod_{i=1}^{j} f(t - \tau_i) d\tau_i \quad (13)$$

under quite general conditions [2]. In equation (13), $h_{(i,j)}(\tau_1, ..., \tau_j)$, ($i = 1, \cdots, n$, $j = 1, \cdots, N$), represents the jth order Volterra kernel for the relationship between $f(t)$ and the displacement of m_i.

When a system is linear, its dynamical properties are easily analyzed using the FRFs defined as the Fourier transform of $h_{(i)}(t)$ ($i = 1, \cdots, n$) in equation (2), however, as equation (13) shows, the dynamical properties of a nonlinear system are determined by a series of Volterra kernels, such as $h_{(i,j)}(\tau_1, ..., \tau_j)$, ($i = 1, \cdots, n$, $j = 1, \cdots, N$) for the MDOF nonlinear systems considered in the present study. The objective of this paper is to study the nonlinear MDOF systems using the concept of Nonlinear Output Frequency Response

Functions (NOFRFs), which is an alternative extension of the FRF to the nonlinear case and is derived based on the Volterra series approach of nonlinear systems.

3. Nonlinear Output Frequency Response Functions

3.1 Nonlinear Output Frequency Response Functions under General Inputs

The definition of NOFRFs is based on the Volterra series theory of nonlinear systems. The Volterra series extends the well-known convolution integral description for linear systems to a series of multi-dimensional convolution integrals, which can be used to represent a wide class of nonlinear systems [2].

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,...,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \qquad (14)$$

where $y(t)$ and $u(t)$ are the output and input of the system, $h_n(\tau_1,...,\tau_n)$ is the nth order Volterra kernel, and N denotes the maximum order of the system nonlinearity. Lang and Billings [2] derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega) \quad \text{for } \forall\omega \\ Y_n(j\omega)=\frac{1/\sqrt{n}}{(2\pi)^{n-1}}\int_{\omega_1+,...,+\omega_n=\omega}H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega} \end{cases} \qquad (15)$$

This expression reveals how nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In (15), $Y(j\omega)$ is the spectrum of the system output, $Y_n(j\omega)$ represents the nth order output frequency response of the system,

$$H_n(j\omega_1,...,j\omega_n)=\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}h_n(\tau_1,...,\tau_n)e^{-(\omega_1\tau_1+,...,+\omega_n\tau_n)j}d\tau_1...d\tau_n \qquad (16)$$

is the nth order Generalised Frequency Response Function (GFRF) [2], and

$$\int_{\omega_1+,...,+\omega_n=\omega}H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)$ over the n-dimensional hyper-plane $\omega_1+\cdots+\omega_n=\omega$. Equation (15) is a natural extension of the well-known linear relationship $Y(j\omega)=H(j\omega)U(j\omega)$, where $H(j\omega)$ is the frequency response function, to the nonlinear case.

For linear systems, the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by equation (14), however, the relationship between the input and output frequencies is more complicated. Given the frequency range of an input, the output frequencies of system (14) can be determined using the explicit expression derived by Lang and Billings in [2].

Based on the above results for the output frequency response of nonlinear systems, a new concept known as the Nonlinear Output Frequency Response Function (NOFRF) was recently introduced by Lang and Billings [9]. The NOFRF is defined as

$$G_n(j\omega)=\frac{\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+,\ldots,+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}} \tag{17}$$

under the condition that

$$U_n(j\omega)=\int_{\omega_1+,\ldots,+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}\neq 0 \tag{18}$$

Notice that $G_n(j\omega)$ is valid over the frequency range of $U_n(j\omega)$, which can be determined using the algorithm in [2].

By introducing the NOFRFs $G_n(j\omega)$, $n=1,\cdots N$, equation (15) can be written as

$$Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega)=\sum_{n=1}^{N}G_n(j\omega)U_n(j\omega) \tag{19}$$

which is similar to the description of the output frequency response for linear systems. The NOFRFs reflect a combined contribution of the system and the input to the system output frequency response behaviour. It can be seen from equation (17) that $G_n(j\omega)$ depends not only on H_n (n=1,…,N) but also on the input $U(j\omega)$. For a nonlinear system, the dynamical properties are determined by the GFRFs H_n (n= 1,…,N). However, from equation (16) it can be seen that the GFRF is multidimensional [7][8], which can make the GFRFs difficult to measure, display and interpret in practice. According to equation (17), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1,\ldots,j\omega_n)$ over $\omega_1+\cdots+\omega_n=\omega$ with the weights depending on the test input. Therefore $G_n(j\omega)$ can be used as an alternative representation of the dynamical properties described by H_n. The most important property of the NOFRF $G_n(j\omega)$ is that it is one dimensional, and thus allows the analysis of nonlinear systems to be implemented in a convenient manner similar to the analysis of linear systems. Moreover, there is an effective algorithm [9] available which allows the estimation of the NOFRFs to be implemented directly using system input output data.

3.2 Nonlinear Output Frequency Response Functions under Harmonic Input

When system (14) is subject to a harmonic input

$$u(t) = A\cos(\omega_F t + \beta) \tag{20}$$

Lang and Billings [2] showed that equation (14) can be expressed as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N}\left(\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} H_n(j\omega_{k_1},\cdots,j\omega_{k_n})A(j\omega_{k_1})\cdots A(j\omega_{k_n})\right) \tag{21}$$

where

$$A(j\omega_{k_i}) = \begin{cases} |A|e^{\mathrm{jsign}(k)\beta} & if \quad \omega_{k_i} \in \{k\omega_F, k=\pm 1\}, i=1,\cdots,n \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

Define the frequency components of the *n*th order output of the system as Ω_n, then according to equation (21), the frequency components in the system output can be expressed as

$$\Omega = \bigcup_{n=1}^{N} \Omega_n \tag{23}$$

where Ω_n is determined by the set of frequencies

$$\left\{\omega = \omega_{k_1} + \cdots + \omega_{k_n} \mid \omega_{k_i} = \pm\omega_F, i=1,\cdots,n\right\} \tag{24}$$

From equation (24), it is known that if all $\omega_{k_1},\cdots,\omega_{k_n}$ are taken as $-\omega_F$, then $\omega = -n\omega_F$. If *k* of these are taken as ω_F, then $\omega = (-n+2k)\omega_F$. The maximal *k* is *n*. Therefore the possible frequency components of $Y_n(j\omega)$ are

$$\Omega_n = \left\{(-n+2k)\omega_F, k=0,1,\cdots,n\right\} \tag{25}$$

Moreover, it is easy to deduce that

$$\Omega = \bigcup_{n=1}^{N} \Omega_n = \{k\omega_F, k=-N,\cdots,-1,0,1,\cdots,N\} \tag{26}$$

Equation (26) explains why superharmonic components are generated when a nonlinear system is subjected to a harmonic excitation. In the following, only those components with positive frequencies will be considered.

The NOFRFs defined in equation (17) can be extended to the case of harmonic inputs as

$$G_n^H(j\omega) = \frac{\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} H_n(j\omega_{k_1},\cdots,j\omega_{k_n})A(j\omega_{k_1})\cdots A(j\omega_{k_n})}{\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} A(j\omega_{k_1})\cdots A(j\omega_{k_n})} \qquad n = 1,\ldots,N \tag{27}$$

under the condition that

$$A_n(j\omega) = \frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} A(j\omega_{k_1})\cdots A(j\omega_{k_n}) \neq 0 \tag{28}$$

Obviously, $G_n^H(j\omega)$ is only valid over Ω_n defined by equation (25). Consequently, the output spectrum $Y(j\omega)$ of nonlinear systems under a harmonic input can be expressed as

$$Y(j\omega)=\sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n^H(j\omega)\,A_n(j\omega) \tag{29}$$

When k of the n frequencies of $\omega_{k_1},\cdots,\omega_{k_n}$ are taken as ω_F and the remainders are as $-\omega_F$, substituting equation (22) into equation (28) yields,

$$A_n(j(-n+2k)\omega_F)=\frac{1}{2^n}|A|^n\, e^{j(-n+2k)\beta} \tag{30}$$

Thus $G_n^H(j\omega)$ becomes

$$G_n^H(j(-n+2k)\omega_F)=\frac{\frac{1}{2^n}H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k})|A|^n\, e^{j(-n+2k)\beta}}{\frac{1}{2^n}|A|^n\, e^{j(-n+2k)\beta}}$$

$$=H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k}) \tag{31}$$

where $H_n(j\omega_1,...,j\omega_n)$ is assumed to be a symmetric function. Therefore, in this case, $G_n^H(j\omega)$ over the nth order output frequency range $\Omega_n=\{(-n+2k)\omega_F, k=0,1,\cdots,n\}$ is equal to the GFRF $H_n(j\omega_1,...,j\omega_n)$ evaluated at $\omega_1=\cdots=\omega_k=\omega_F$, $\omega_{k+1}=\cdots=\omega_n=-\omega_F$, $k=0,\cdots,n$.

4. Analysis of MDOF Systems with Multiple Nonlinear Components Using NOFRFS

4.1 GFRFs of MDOF Systems with Multiple Nonlinear Components

From equations (3)~(8), the GFRFs $H_{(i,j)}(j\omega_1,...,j\omega_j)$, ($i=1,\cdots,n$, $j=1,\cdots,N$) can be determined using the harmonic probing method [5][6].

First consider the input $f(t)$ is of a single harmonic

$$f(t)=e^{j\omega t} \tag{32}$$

Substituting (30) and

$$x_i(t)=H_{(i,1)}(j\omega)e^{j\omega t} \qquad (i=1,\cdots,n) \tag{33}$$

into equations (3)~(8) and extracting the coefficients of $e^{j\omega t}$ yields, for the first and nth masses,

$$\left(-m_1\omega^2+j(c_1+c_2)\omega+(k_1+k_2)\right)H_{(1,1)}(j\omega)-\left(jc_2\omega+k_2\right)H_{(2,1)}(j\omega)=0 \tag{34}$$

$$\left(-m_n\omega^2+jc_n\omega+k_n\right)H_{(n,1)}(j\omega)-\left(jc_n\omega+k_n\right)H_{(n-1,1)}(j\omega)=0 \tag{35}$$

for other masses excluding the Jth mass

$$\left(-m_i\omega^2+j(c_i+c_{i+1})\omega+k_i+k_{i+1}\right)H_{(i,1)}(j\omega)-\left(jc_i\omega+k_i\right)H_{(i-1,1)}(j\omega)$$

$$-\left(jc_{i+1}\omega+k_{i+1}\right)H_{(i+1,1)}(j\omega)=0 \qquad (i\neq 1,J,n) \quad (36)$$

for the Jth mass

$$\left(-m_J\omega^2+j(c_J+c_{J+1})\omega+k_J+k_{J+1}\right)H_{(J,1)}(j\omega)-\left(jc_J\omega+k_J\right)H_{(J-1,1)}(j\omega)$$
$$-\left(jc_{J+1}\omega+k_{J+1}\right)H_{(J+1,1)}(j\omega)=1 \quad (37)$$

Equations (34)~(37) can be written in matrix form as

$$\left(-M\omega^2+jC\omega+K\right)H_1(j\omega)=(\overbrace{0\cdots0}^{J-1}\quad 1\quad \overbrace{0\cdots0}^{n-J})^T \quad (38)$$

where

$$H_1(j\omega)=\left(H_{(1,1)}(j\omega)\quad\cdots\quad H_{(n,1)}(j\omega)\right)^T \quad (39)$$

From equation (39), it is known that

$$H_1(j\omega)=\left(-M\omega^2+jC\omega+K\right)^{-1}(\overbrace{0\cdots0}^{J-1}\quad 1\quad \overbrace{0\cdots0}^{n-J})^T \quad (40)$$

Denote

$$\Theta(j\omega)=-M\omega^2+jC\omega+K \quad (41)$$

and

$$\Theta^{-1}(j\omega)=\begin{pmatrix} Q_{(1,1)}(j\omega) & \cdots & Q_{(1,n)}(j\omega) \\ \vdots & \ddots & \vdots \\ Q_{(n,1)}(j\omega) & \cdots & Q_{(n,n)}(j\omega) \end{pmatrix} \quad (42)$$

It is obtained from equations (40)~(42) that

$$H_{(i,1)}(j\omega)=Q_{(i,J)}(j\omega) \qquad (i=1,\cdots,n) \quad (43)$$

Thus, for any two consecutive masses, the relationship between the first order GFRFs can be expressed as

$$\frac{H_{(i,1)}(j\omega)}{H_{(i+1,1)}(j\omega)}=\frac{Q_{(i,J)}(j\omega)}{Q_{(i+1,J)}(j\omega)}=\lambda_1^{i,i+1}(\omega) \qquad (i=1,\cdots,n-1) \quad (44)$$

The above procedure used to analyze the relationships between the first order GFRFs can be extended to investigate the relationship between the $\overline{N}$th order GFRFs with $\overline{N}\geq 2$. To achieve this, consider the input

$$f(t)=\sum_{k=1}^{\overline{N}}e^{j\omega_k t} \quad (45)$$

Substituting (45) and

$$x_i(t)=H_{(i,1)}(j\omega_1)e^{j\omega_1 t}+\cdots+H_{(i,1)}(j\omega_{\overline{N}})e^{j\omega_N t}+\cdots$$
$$+\overline{N}!H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})e^{j(\omega_1+\cdots+\omega_{\overline{N}})t}+\cdots \qquad (i=1,\cdots,n) \quad (46)$$

into equations (3)~(8) and extracting the coefficients of $e^{j(\omega_1+\cdots+\omega_{\overline{N}})t}$ yields

$$\left(-m_1(\omega_1+\cdots+\omega_{\overline{N}})^2+j(c_1+c_2)(\omega_1+\cdots+\omega_{\overline{N}})+(k_1+k_2)\right)H_{(1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})$$
$$-\left(jc_2(\omega_1+\cdots+\omega_{\overline{N}})+k_2\right)H_{(2,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=0 \quad (47)$$

$$
\begin{aligned}
&\left(-m_n(\omega_1+\cdots+\omega_{\overline{N}})^2+jc_n(\omega_1+\cdots+\omega_{\overline{N}})+k_n\right)H_{(n,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\\
&-\left(jc_n(\omega_1+\cdots+\omega_{\overline{N}})+k_n\right)H_{(n-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=0
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
&\left(-m_i(\omega_1+\cdots+\omega_{\overline{N}})^2+j(c_i+c_{i+1})(\omega_1+\cdots+\omega_{\overline{N}})+k_i+k_{i+1}\right)H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\\
&-\left(jc_i(\omega_1+\cdots+\omega_{\overline{N}})+k_i\right)H_{(i-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\\
&-\left(jc_{i+1}(\omega_1+\cdots+\omega_{\overline{N}})+k_{i+1}\right)H_{(i+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=0\\
&\qquad (i\neq 1,L(l)-1,L(l),n,1\le l\le \overline{L})
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
&\begin{pmatrix}-m_{L(i)-1}(\omega_1+\cdots+\omega_{\overline{N}})^2+j(c_{L(i)-1}+c_{L(i)})(\omega_1+\cdots+\omega_{\overline{N}})\\ +k_{L(i)-1}+k_L(i)\end{pmatrix}H_{(L(i)-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\\
&-\left(jc_{L(i)-1}(\omega_1+\cdots+\omega_{\overline{N}})+k_{L(i)-1}\right)H_{(L(i)-2,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\\
&-\left(jc_{L(i)}(\omega_1+\cdots+\omega_{\overline{N}})+k_{L(i)}\right)H_{(L(i),\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})+\Lambda_{\overline{N}}^{L(i)-1,L(i)}(j\omega_1,\cdots,j\omega_{\overline{N}})=0\\
&\qquad (1\le i\le \overline{L})
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&\begin{pmatrix}-m_{L(i)}(\omega_1+\cdots+\omega_{\overline{N}})^2+j(c_{L(i)}+c_{L(i)+1})(\omega_1+\cdots+\omega_{\overline{N}})\\ +k_{L(i)}+k_{L(i)+1}\end{pmatrix}H_{(L(i),\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\\
&-\left(jc_{L(i)}(\omega_1+\cdots+\omega_{\overline{N}})+k_{L(i)}\right)H_{(L(i)-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\\
&-\left(jc_{L(i)+1}(\omega_1+\cdots+\omega_{\overline{N}})+k_{L(i)+1}\right)H_{(L(i)+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})-\Lambda_{\overline{N}}^{L(i)-1,L(i)}(j\omega_1,\cdots,j\omega_{\overline{N}})=0\\
&\qquad (1\le i\le \overline{L})
\end{aligned} \tag{51}
$$

In equations (50) and (51), $\Lambda_{\overline{N}}^{L(i)-1,L(i)}(j\omega_1,\cdots,j\omega_{\overline{N}})$ represents the extra terms introduced by $NonF_{L(i)}$ for the $\overline{N}$ th order GFRFs, for example, for the second order GFRFs,

$$
\begin{aligned}
&\Lambda_2^{L(i)-1,L(i)}(j\omega_1,j\omega_2)=\left(-w_{(L(i),2)}\omega_1\omega_2+r_{(L(i),2)}\right)\Big(H_{(L(i)-1,1)}(j\omega_1)H_{(L(i)-1,1)}(j\omega_2)\\
&+H_{(L(i),1)}(j\omega_1)H_{(L(i),1)}(j\omega_2)-H_{(L(i)-1,1)}(j\omega_1)H_{(L(i),1)}(j\omega_2)-H_{(L(i)-1,1)}(j\omega_2)H_{(L(i),1)}(j\omega_1)\Big)\\
&\qquad (1\le i\le \overline{L})
\end{aligned} \tag{52}
$$

Denote

$$
H_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}})=\left(H_{(1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\quad\cdots\quad H_{(n,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\right)^T \tag{53}
$$

and

$$
\overline{A}_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}})=\left[\overline{A}_{\overline{N}}(1)\cdots\overline{A}_{\overline{N}}(n)\right]^T \tag{54}
$$

where

$$
\overline{A}_{\overline{N}}(l)=\begin{cases}0 & \text{if } l\neq L(i)-1,L(i),\ 1\le i\le\overline{L}\\ -\Lambda_{\overline{N}}^{L(i)-1,L(i)}(j\omega_1,\cdots,j\omega_{\overline{N}}) & \text{if } l=L(i)-1,\ 1\le i\le\overline{L}\\ \Lambda_{\overline{N}}^{L(i)-1,L(i)}(j\omega_1,\cdots,j\omega_{\overline{N}}) & \text{if } l=L(i),\ 1\le i\le\overline{L}\end{cases}\qquad (1\le l\le n) \tag{55}
$$

then equations (47)~(51) can be written in a matrix form as

$$
\Theta(j(\omega_1+\cdots+\omega_{\overline{N}}))H_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}})=\overline{A}_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}}) \tag{56}
$$

so that

$$H_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}})=\Theta^{-1}(j(\omega_1+\cdots+\omega_{\overline{N}}))\overline{A}_{\overline{N}}(j\omega_1,\cdots,j\omega_{\overline{N}}) \quad (57)$$

Therefore, for each mass, the $\overline{N}$ th order GFRF can be calculated as

$$H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=\sum_{l=1}^{\overline{L}}\begin{pmatrix}Q_{i,L(l)-1}(j(\omega_1+\cdots+\omega_{\overline{N}}))\\ Q_{i,L(l)}(j(\omega_1+\cdots+\omega_{\overline{N}})\end{pmatrix}^T\begin{pmatrix}-\Lambda_{\overline{N}}^{L(l)-1,L(l)}(j\omega_1,\cdots,j\omega_{\overline{N}})\\ \Lambda_{\overline{N}}^{L(l)-1,L(l)}(j\omega_1,\cdots,j\omega_{\overline{N}})\end{pmatrix}$$
$$(i=1,\cdots,n) \quad (58)$$

and consequently, for two consecutive masses, the $\overline{N}$ th order GFRFs have the following relationships

$$\frac{H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(i+1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}=\frac{\sum_{l=1}^{\overline{L}}\begin{pmatrix}Q_{i,L(l)-1}(j(\omega_1+\cdots+\omega_{\overline{N}}))\\ Q_{i,L(l)}(j(\omega_1+\cdots+\omega_{\overline{N}})\end{pmatrix}^T\begin{pmatrix}-\Lambda_{\overline{N}}^{L(l)-1,L(l)}(j\omega_1,\cdots,j\omega_{\overline{N}})\\ \Lambda_{\overline{N}}^{L(l)-1,L(l)}(j\omega_1,\cdots,j\omega_{\overline{N}})\end{pmatrix}}{\sum_{l=1}^{\overline{L}}\begin{pmatrix}Q_{i+1,L(l)-1}(j(\omega_1+\cdots+\omega_{\overline{N}}))\\ Q_{i+1,L(l)}(j(\omega_1+\cdots+\omega_{\overline{N}})\end{pmatrix}^T\begin{pmatrix}-\Lambda_{\overline{N}}^{L(l)-1,L(l)}(j\omega_1,\cdots,j\omega_{\overline{N}})\\ \Lambda_{\overline{N}}^{L(l)-1,L(l)}(j\omega_1,\cdots,j\omega_{\overline{N}})\end{pmatrix}}$$
$$=\lambda_{\overline{N}}^{i,i+1}(\omega_1,\cdots,\omega_{\overline{N}}) \qquad (i=1,\cdots,n-1) \quad (59)$$

Equations (44) and (59) give a comprehensive description for the relationships between the GFRFs of any two consecutive masses for the nonlinear MDOF system (12).

In addition, denote

$$\lambda_{\overline{N}}^{0,1}(j\omega_1,\cdots,j\omega_{\overline{N}})=0 \qquad (\overline{N}=1,\cdots,N) \quad (60)$$

$$\Lambda_{(J-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=0 \qquad (\overline{N}=1,\cdots,N) \quad (61)$$

$$\Lambda_{(J,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=\begin{cases}1 & \text{if } \overline{N}=1\\ 0 & \text{if } \overline{N}=2,\cdots,N\end{cases} \quad (62)$$

$$\Lambda_{(L(i)-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=-\Lambda_{\overline{N}}^{L(i)-1,L(i)}(j\omega_1,\cdots,j\omega_{\overline{N}}) \quad (\overline{N}=1,\cdots,N, 1\le i\le\overline{L}) \quad (63)$$

$$\Lambda_{(L(i),\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})=\Lambda_{\overline{N}}^{L(i)-1,L(i)}(j\omega_1,\cdots,j\omega_{\overline{N}}) \quad (\overline{N}=1,\cdots,N, 1\le i\le\overline{L}) \quad (64)$$

and $L(0)=J$. Then, for the first two masses, from equations (34) and (47), it can be known that

$$\lambda_{\overline{N}}^{1,2}(\omega_1,\cdots,\omega_{\overline{N}})=\frac{H_{(1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(2,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}=\frac{jc_2(\omega_1+\cdots+\omega_{\overline{N}})+k_2}{\begin{bmatrix}-m_1(\omega_1+\cdots+\omega_{\overline{N}})^2+k_2+jc_2(\omega_1+\cdots+\omega_{\overline{N}})\\ +\left(1-\lambda_{\overline{N}}^{0,1}(\omega_1,\cdots,\omega_{\overline{N}})\right)\left(jc_1(\omega_1+\cdots+\omega_{\overline{N}})+k_1\right)\end{bmatrix}}$$
$$(\overline{N}=1,\cdots,N) \quad (65)$$

Starting with equation (65), and iteratively using equations (36) and (49) from the first mass, it can be deduce that, for the masses that aren't connected to nonlinear components and the Jth spring, the following relationships hold for the GFRFs.

$$\lambda_{\bar{N}}^{i,i+1}(\omega_1,\cdots,\omega_{\bar{N}})$$

$$=\frac{H_{(i,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}{H_{(i+1,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}=\frac{jc_{i+1}(\omega_1+\cdots+\omega_{\bar{N}})+k_{i+1}}{\left[\begin{array}{l} j\left(\left(1-\lambda_{\bar{N}}^{i-1,i}(\omega_1,\cdots,\omega_{\bar{N}})\right)c_i+c_{i+1}\right)(\omega_1+\cdots+\omega_{\bar{N}}) \\ +\left(1-\lambda_{\bar{N}}^{i-1,i}(\omega_1,\cdots,\omega_{\bar{N}})\right)k_i+k_{i+1}-m_i(\omega_1+\cdots+\omega_{\bar{N}})^2 \end{array}\right]}$$

$$(1\le i<n,\ i\ne J, L(l)-1, L(l),\ l=0,\cdots,\bar{L},\ \bar{N}=1,\cdots,N) \quad (66)$$

For the masses that are connected to nonlinear components and the *J*th spring, from equations (37), (50) and (51), it can be known that the following relationships hold for the GFRFs.

$$\lambda_{\bar{N}}^{i,i+1}(\omega_1,\cdots,\omega_{\bar{N}})=\frac{H_{(i,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}{H_{(i+1,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}$$

$$=\bar{\lambda}_{\bar{N}}^{i,i+1}(\omega_1,\cdots,\omega_{\bar{N}})\left(1+\frac{1}{k_i+jc_i(\omega_1+\cdots+\omega_{\bar{N}})}\frac{\Lambda_{(i,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}{H_{(i+1,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}\right)$$

$$(i=L(l)-1, L(l),\ l=0,\cdots,\bar{L},\ \bar{N}=1,\cdots,N) \quad (67)$$

where

$$\bar{\lambda}_{\bar{N}}^{i,i+1}(\omega_1,\cdots,\omega_{\bar{N}})=\frac{jc_{i+1}(\omega_1+\cdots+\omega_{\bar{N}})+k_{i+1}}{\left[\begin{array}{l} j\left(\left(1-\lambda_{\bar{N}}^{i-1,i}(\omega_1,\cdots,\omega_{\bar{N}})\right)c_i+c_{i+1}\right)(\omega_1+\cdots+\omega_{\bar{N}}) \\ +\left(1-\lambda_{\bar{N}}^{i-1,i}(\omega_1,\cdots,\omega_{\bar{N}})\right)k_i+k_{i+1}-m_i(\omega_1+\cdots+\omega_{\bar{N}})^2 \end{array}\right]}$$

$$(i=L(l)-1, L(l),\ l=0,\cdots,\bar{L},\ \bar{N}=1,\cdots,N) \quad (68)$$

Moreover, denote $\lambda_{\bar{N}}^{n+1,n}(\omega_1,\cdots,\omega_{\bar{N}})=1$, ($\bar{N}=1,\cdots,N$), $c_{n+1}=0$ and $k_{n+1}=0$. Then, for the last two masses, from equations (35) and (48) it is can be deduced that

$$\lambda_{\bar{N}}^{n,n-1}(\omega_1,\cdots,\omega_{\bar{N}})=\frac{1}{\lambda_{\bar{N}}^{n-1,n}(\omega_1,\cdots,\omega_{\bar{N}})}=\frac{H_{(n,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}{H_{(n-1,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}$$

$$=\frac{jc_n(\omega_1+\cdots+\omega_{\bar{N}})+k_n}{\left[\begin{array}{l} -m_n(\omega_1+\cdots+\omega_{\bar{N}})^2+\left(1-\lambda_{\bar{N}}^{n+1,n}(\omega_1,\cdots,\omega_{\bar{N}})\right)k_{n+1}+k_n \\ +j\left(\left(1-\lambda_{\bar{N}}^{n+1,n}(\omega_1,\cdots,\omega_{\bar{N}})\right)c_{n+1}+c_n\right)(\omega_1+\cdots+\omega_{\bar{N}}) \end{array}\right]} \quad (\bar{N}=1,\cdots,N) \quad (69)$$

Starting with equation (69), and iteratively using equations (36) and (49), it can be deduce that, for the masses that aren't connected to nonlinear components and the *J*th spring, the following relationships hold for the GFRFs.

$$\lambda_{\bar{N}}^{i,i-1}(\omega_1,\cdots,\omega_{\bar{N}})=\frac{1}{\lambda_{\bar{N}}^{i-1,i}(\omega_1,\cdots,\omega_{\bar{N}})}=\frac{H_{(i,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}{H_{(i-1,\bar{N})}(j\omega_1,\cdots,j\omega_{\bar{N}})}$$

$$=\frac{jc_i(\omega_1+\cdots+\omega_{\bar{N}})+k_i}{\left[\begin{array}{l} -m_i(\omega_1+\cdots+\omega_{\bar{N}})^2+\left(1-\lambda_{\bar{N}}^{i+1,i}(\omega_1,\cdots,\omega_{\bar{N}})\right)k_{i+1}+k_i \\ +j\left(\left(1-\lambda_{\bar{N}}^{i+1,i}(\omega_1,\cdots,\omega_{\bar{N}})\right)c_{i+1}+c_i\right)(\omega_1+\cdots+\omega_{\bar{N}}) \end{array}\right]}$$

$$(2 \le i \le n,\ i \ne L(l)-1, L(l),\ l = 0,\cdots,\overline{L},\ \overline{N} = 1,\cdots,N) \quad (70)$$

For the masses that are connected to nonlinear components and the Jth spring, from equations (37), (50) and (51), it can be known that the following relationships hold for the GFRFs.

$$\lambda_{\overline{N}}^{i,i-1}(\omega_1,\cdots,\omega_{\overline{N}}) = \frac{H_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(i-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}$$

$$= \overline{\lambda}_{\overline{N}}^{i,i-1}(\omega_1,\cdots,\omega_{\overline{N}})\left(1 + \frac{1}{k_i + jc_i(\omega_1+\cdots+\omega_{\overline{N}})}\frac{\Lambda_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}{H_{(i-1,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})}\right)$$

$$(i = L(l)-1, L(l),\ l = 0,\cdots,\overline{L},\ \overline{N} = 1,\cdots,N) \quad (71)$$

where

$$\overline{\lambda}_{\overline{N}}^{i,i-1}(\omega_1,\cdots,\omega_{\overline{N}}) = \frac{jc_i(\omega_1+\cdots+\omega_{\overline{N}}) + k_i}{\begin{bmatrix} -m_i(\omega_1+\cdots+\omega_{\overline{N}})^2 + \left(1-\lambda_{\overline{N}}^{i+1,i}(\omega_1,\cdots,\omega_{\overline{N}})\right)k_{i+1} + k_i \\ + j\left(\left(1-\lambda_{\overline{N}}^{i+1,i}(\omega_1,\cdots,\omega_{\overline{N}})\right)c_{i+1} + c_i\right)(\omega_1+\cdots+\omega_{\overline{N}}) \end{bmatrix}}$$

(72)

From different perspectives, equations (65)~(68) and equations (69)~(72) give two alternative descriptions for the relationships between the GFRFs of any two consecutive masses for the nonlinear MDOF system (12).

4.2 NOFRFs of MDOF Systems with Multiple Nonlinear Components

According to the definition of NOFRF in equation (17), the $\overline{N}$ th order NOFRF of the ith mass can be expressed as

$$G_{(i,\overline{N})}(j\omega) = \frac{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega} H_{(i,\overline{N})}(j\omega_1,\ldots,j\omega_{\overline{N}})\prod_{q=1}^{\overline{N}} F(j\omega_q)d\sigma_{\overline{N}\omega}}{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega} \prod_{q=1}^{\overline{N}} F(j\omega_q)d\sigma_{\overline{N}\omega}} \quad (1 \le \overline{N} \le N,\ 1 \le i \le n) \quad (73)$$

where $F(j\omega)$ is the Fourier transform of $f(t)$.

According to equation (66), for the masses that aren't connected to nonlinear components and the Jth spring, equation (73) can be rewritten as

$$G_{(i,\overline{N})}(j\omega) = \frac{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega} \lambda_{\overline{N}}^{i,i+1}(\omega_1,\cdots,\omega_{\overline{N}})H_{(i+1,\overline{N})}(j\omega_1,\ldots,j\omega_{\overline{N}})\prod_{q=1}^{\overline{N}} F(j\omega_q)d\sigma_{\overline{N}\omega}}{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega} \prod_{q=1}^{\overline{N}} F(j\omega_q)d\sigma_{\overline{N}\omega}}$$

$$= \frac{jc_{i+1}\omega + k_{i+1}}{\left[-m_i\omega^2 + \left(1-\lambda_{\overline{N}}^{i-1,i}(\omega)\right)\left(jc_i\omega + k_i\right) + jc_{i+1}\omega + k_{i+1}\right]} G_{(i+1,\overline{N})}(j\omega)$$

$$(1 \le i < n\,,\ i \ne L(l)-1, L(l)\,,\ l = 0,\cdots,\overline{L}\,,\ \overline{N} = 1,\cdots,N\,) \quad (74)$$

Therefore, for two consecutive masses that aren't connected to these nonlinear components and the Jth spring, the NOFRFs have the following relationship

$$\frac{G_{(i,\overline{N})}(j\omega)}{G_{(i+1,\overline{N})}(j\omega)} = \lambda_{\overline{N}}^{i,i+1}(\omega) = \frac{jc_{i+1}\omega + k_{i+1}}{\left[-m_i\omega^2 + \left(1-\lambda_{\overline{N}}^{i-1,i}(\omega)\right)\left(jc_i\omega + k_i\right) + jc_{i+1}\omega + k_{i+1}\right]}$$

$$(1 \le i < n\,,\ i \ne L(l)-1, L(l)\,,\ l = 0,\cdots,\overline{L}\,,\ \overline{N} = 1,\cdots,N\,) \quad (75)$$

where $\lambda_{\overline{N}}^{0,1}(\omega) = 0$.

Similarly, for the masses that are connected to nonlinear components and the Jth spring, from equations (67) and (68), it can be deduced that

$$\lambda_{\overline{N}}^{i,i+1}(\omega) = \frac{G_{(i,\overline{N})}(j\omega)}{G_{(i+1,\overline{N})}(j\omega)} = \overline{\lambda}_{\overline{N}}^{i,i+1}(\omega)\left(1 + \frac{1}{k_i + jc_i\omega}\frac{\Gamma_{(i,\overline{N})}(j\omega)}{G_{(i+1,\overline{N})}(j\omega)}\right)$$

$$(\,i = L(l)-1, L(l)\,,\ l = 0,\cdots,\overline{L}\,,\ \overline{N} = 1,\cdots,N\,) \quad (76)$$

where

$$\overline{\lambda}_{\overline{N}}^{i,i+1}(\omega) = \frac{jc_{i+1}\omega + k_{i+1}}{\left[-m_i\omega^2 + \left(1-\lambda_{\overline{N}}^{i-1,i}(\omega)\right)\left(jc_i\omega + k_i\right) + jc_{i+1}\omega + k_{i+1}\right]} \quad (77)$$

and

$$\Gamma_{(i,\overline{N})}(j\omega) = \frac{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega} \Lambda_{(i,\overline{N})}(j\omega_1,\cdots,j\omega_{\overline{N}})\prod_{q=1}^{\overline{N}} F(j\omega_q)d\sigma_{\overline{N}\omega}}{\int_{\omega_1+,\ldots,+\omega_{\overline{N}}=\omega} \prod_{q=1}^{\overline{N}} F(j\omega_q)d\sigma_{\overline{N}\omega}} \quad (78)$$

Equations (75)~(78) give a comprehensive description for the relationships between the NOFRFs of two consecutive masses of the nonlinear MDOF system (12).

Using the same procedure, from equations (69)~(72), an alternative description can be established for the following relationships between the NOFRFs of two consecutive masses. For the masses that aren't connected to nonlinear components and the Jth spring

$$\lambda_{\overline{N}}^{i,i-1}(\omega) = \frac{1}{\lambda_{\overline{N}}^{i-1,i}(\omega)} = \frac{G_{(i,\overline{N})}(j\omega)}{G_{(i-1,\overline{N})}(j\omega)} = \frac{jc_i\omega + k_i}{\left[-m_i\omega^2 + \left(1-\lambda_{\overline{N}}^{i+1,i}(\omega)\right)\left(jc_{i+1}\omega + k_{i+1}\right) + jc_i\omega + k_i\right]}$$

$$(\,2 \le i \le n\,,\ i \ne L(l)-1, L(l)\,,\ l = 0,\cdots,\overline{L}\,,\ \overline{N} = 1,\cdots,N\,) \quad (79)$$

For the masses that are connected to nonlinear components and the Jth spring

$$\lambda_{\overline{N}}^{i,i-1}(\omega) = \frac{1}{\lambda_{\overline{N}}^{i-1,i}(\omega)} = \frac{G_{(i,\overline{N})}(j\omega)}{G_{(i-1,\overline{N})}(j\omega)} = \overline{\lambda}_{\overline{N}}^{i,i-1}(\omega)\left(1 + \frac{1}{k_i + jc_i\omega}\frac{\Gamma_{(i,\overline{N})}(j\omega)}{G_{(i-1,\overline{N})}(j\omega)}\right)$$

$$(\,i = L(l)-1, L(l)\,,\ l = 0,\cdots,\overline{L}\,,\ \overline{N} = 1,\cdots,N\,) \quad (80)$$

where $\lambda_{\overline{N}}^{n+1,n}(\omega) = 1$ ($\overline{N} = 1,\cdots,N$) and

$$\overline{\lambda}_{\overline{N}}^{i,i-1}(\omega) = \frac{jc_i\omega + k_i}{\left[-m_i\omega^2 + \left(1-\lambda_{\overline{N}}^{i+1,i}(\omega)\right)\left(k_{i+1} + jc_{i+1}\omega\right) + k_i + \omega c_i\right]} \quad (81)$$

From different perspectives, both equations (75)~(78) and equations (79)~(81) give a comprehensive description for the relationships between the NOFRFs of any two consecutive masses of the nonlinear MDOF system (12).

4.3 The Properties of NOFRFs of the Locally Nonlinear System

Without loss of generality, assume $L(1) < \cdots < L(\overline{L})$. Then, from equations (75)~(81), the following important properties of the NOFRFs of MDOF systems with multiple nonlinear components can be obtained.

i) If $J \le L(1)$, then for the masses ($1 \le i \le J-1$ and $L(\overline{L}) \le i < n$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (1 \le i \le J-1 \text{ and } L(\overline{L}) \le i < n) \quad (82)$$

for the masses ($J \le i < L(1)-1$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (J \le i < L(1)-1) \quad (83)$$

for the masses ($L(1) \le i < L(\overline{L})$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} \ne \cdots \ne \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (L(1)-1 \le i < L(\overline{L})) \quad (84)$$

ii) If $L(1) \le J \le L(\overline{L})$, then for the masses ($1 \le i < L(1)-1$ and $L(\overline{L}) \le i < n$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (1 \le i < L(1)-1 \text{ and } L(\overline{L}) \le i < n) \quad (85)$$

for the masses ($L(1) \le i < L(\overline{L})$), the following relationships hold

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} \ne \cdots \ne \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (L(1)-1 \le i < L(\overline{L})) \quad (86)$$

iii) If $J \ge L(\overline{L})$, then for the masses ($1 \le i < L(1)-1$ and $J \le i < n$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (1 \le i < L(1)-1 \text{ and } J \le i < n) \quad (87)$$

for the masses ($L(\overline{L}) \le i < J$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (L(\overline{L}) \le i < J) \quad (88)$$

for the masses ($L(1)-1 \le i < L(\bar{L})$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} \neq \cdots \neq \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \qquad (L(1)-1 \le i < L(\bar{L})) \quad (89)$$

iv) For the masses ($1 \le i \le \min(J, L(1)-1)-1$ and $\max(L(\bar{L}), J) \le i < n$), the following relationships of the output frequency responses hold

$$x_i(j\omega) = \lambda^{i,i+1}(\omega) x_{i+1}(j\omega)$$

$$(1 \le i \le \min(J, L(1))-1 \text{ and } \max(L(\bar{L}), J) \le i < n) \quad (90)$$

where

$$\lambda^{i,i+1}(\omega) = \frac{jc_{i+1}\omega + k_{i+1}}{\left[-m_i\omega^2 + \left(1-\lambda^{i-1,i}(\omega)\right)\left(jc_i\omega + k_i\right) + jc_{i+1}\omega + k_{i+1}\right]} \quad (91)$$

The first property is straightforward. For the masses on the left of the *J*th mass, substituting $\lambda_{\bar{N}}^{0,1}(\omega) = 0$ ($\bar{N} = 1, \cdots, N$) into equation (75), it is obtained that

$$\lambda_1^{1,2}(\omega) = \cdots = \lambda_N^{1,2}(\omega) = \frac{k_2 + jc_2\omega}{\left(-m_1\omega^2 + j\omega(c_1 + c_2) + k_1 + k_2\right)} = \lambda^{1,2}(\omega) \quad (92)$$

Subsequently, substituting (92) into equation (76) yields

$$\lambda_1^{2,3}(\omega) = \cdots = \lambda_N^{2,3}(\omega) = \frac{jc_3\omega + k_3}{\left[-m_2\omega^2 + \left(1-\lambda^{1,2}(j\omega)\right)\left(jc_2\omega + k_2\right) + jc_3\omega + k_3\right]} = \lambda^{2,3}(\omega) \quad (93)$$

Iteratively using above procedure until *i*=(*J*-1), for the masses ($1 \le i \le J-1$), property (82) can be proved.

Similarly, substituting $\lambda_{\bar{N}}^{n+1,n}(\omega) = 1$ ($\bar{N} = 1, \cdots, N$) into equation (79), it is known that

$$\lambda_1^{n,n-1}(\omega) = \frac{1}{\lambda_1^{n-1,n}(\omega)} = \cdots = \lambda_{\bar{N}}^{n,n-1}(\omega) = \frac{1}{\lambda_{\bar{N}}^{n-1,n}(\omega)} = \frac{jc_n\omega + k_n}{\left[-m_n\omega^2 + jc_n\omega + k_n\right]} = \lambda^{n,n-1}(\omega) \quad (94)$$

Subsequently, substituting (94) into equation (79), it can be deduced that

$$\lambda_1^{n-1,n-2}(\omega) = \frac{1}{\lambda_1^{n-2,n-1}(\omega)} = \cdots = \lambda_{\bar{N}}^{n-1,n-2}(\omega) = \frac{1}{\lambda_{\bar{N}}^{n-2,n-1}(\omega)} =$$
$$\frac{jc_{n-1}\omega + k_{n-1}}{\left[-m_{n-1}\omega^2 + \left(1-\lambda^{n,n-1}(\omega)\right)\left(jc_n\omega + k_n\right) + jc_{n-1}\omega + k_{n-1}\right]} = \lambda^{n-1,n-2}(\omega) \quad (95)$$

Iteratively using above procedure until $i = L(\bar{L})$, for the masses ($L(\bar{L}) \le i < n$), property (82) can be proved.

Obviously, from equations (62) and (78), it is known that

$$\Gamma_{(J,\bar{N})}(j\omega) = \begin{cases} 1 & \text{if } \bar{N} = 1 \\ 0 & \text{if } \bar{N} = 2, \cdots, N \end{cases} \quad (96)$$

Substituting $\lambda_1^{J-1,J}(\omega) = \cdots = \lambda_N^{J-1,J}(\omega)$ and equation (96) into (76), it can be deduced that

$$\lambda_2^{J,J+1}(\omega) = \cdots = \lambda_N^{J,J+1}(\omega) = \frac{(k_J + jc_J\omega)G_{(J+1,\overline{N})}(j\omega)}{1+(k_J + jc_J\omega)G_{(J+1,\overline{N})}(j\omega)}\lambda_1^{J,J+1}(\omega) \tag{97}$$

Obviously,

$$\lambda_1^{J,J+1}(\omega) \neq \lambda_2^{J,J+1}(\omega) = \cdots = \lambda_N^{J,J+1}(\omega) \tag{98}$$

Substituting $\lambda_1^{J,J+1}(\omega) \neq \lambda_2^{J,J+1}(\omega) = \cdots = \lambda_N^{J,J+1}(\omega)$ into equation (75), it can be proved that

$$\lambda_1^{J+1,J+2}(\omega) \neq \lambda_2^{J+1,J+2}(\omega) = \cdots = \lambda_N^{J+1,J+2}(\omega) \tag{99}$$

Iteratively using this procedure until $i = L(1)-2$, for the masses ($J \leq i < L(1)-1$), property (83) can be proved.

Then, substituting $\lambda_1^{L(1)-2,L(1)-1}(\omega) \neq \lambda_2^{L(1)-2,L(1)-1}(\omega) = \cdots = \lambda_N^{L(1)-2,L(1)-1}(\omega)$ into equation (77), it can be known that

$$\overline{\lambda}_1^{L(1)-1,L(1)}(\omega) \neq \overline{\lambda}_2^{L(1)-1,L(1)}(\omega) = \cdots = \overline{\lambda}_N^{L(1)-1,L(1)}(\omega) \tag{100}$$

Moreover, generally,

$$\frac{\Gamma_{(L(1)-1,1)}(j\omega)}{G_{(L(1),1)}(j\omega)} \neq \cdots \neq \frac{\Gamma_{(L(1)-1,\overline{N})}(j\omega)}{G_{(L(1),\overline{N})}(j\omega)} \tag{101}$$

Substituting (100) and (101) into equation (77), it can be deduced that

$$\lambda_1^{L(1)-1,L(1)}(\omega) \neq \lambda_2^{L(1)-1,L(1)}(\omega) \neq \cdots \neq \lambda_N^{L(1)-1,L(1)}(\omega) \tag{102}$$

Iteratively using the procedure until $i = L(\overline{L})-1$, for the masses ($L(1)-1 \leq i < L(\overline{L})$), property (84) can be proved.

Following the same procedure, the second and third properties can be proved. The details are omitted here.

The fourth property is also straightforward since, according to equation (19), the output frequency response of the *i*th mass can be expressed as

$$x_{i+1}(j\omega) = \sum_{k=1}^{N} G_{(i+1,k)}(j\omega)\, F_k(j\omega) \tag{103}$$

Equation (103) can be rewritten as

$$x_{i+1}(j\omega) = \sum_{k=1}^{N} \lambda_k^{i,i+1}(\omega) G_{(i,k)}(j\omega)\, F_k(j\omega) \tag{104}$$

Using the first three properties, it can be known that,

$$\lambda_1^{i,i+1}(\omega) = \cdots = \lambda_N^{i,i+1}(\omega) = \lambda^{i,i+1}(\omega)$$

$$(1 \leq i \leq \min(J, L(1))-1 \text{ and } \max(L(\overline{L}), J) \leq i < n) \tag{105}$$

Substituting (105) into (104) yields

$$x_{i+1}(j\omega) = \sum_{k=1}^{N} \lambda^{i,i+1}(\omega) G_{(i,k)}(j\omega)\, F_k(j\omega) \tag{106}$$

Obviously, $x_{i+1}(j\omega) = \lambda^{i,i+1}(\omega) x_i(j\omega)$, then the fourth property is proved.

Above fourth properties can be easily extended to a more general case, as the following.

v) For any two masses whose positions are $i,k \subseteq [1, \min(J, L(1))-1]$, or $i,k \subseteq [\max(J, L(\bar{L})), n-1]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} = \lambda^{i,k}(\omega)$$

$$(i,k \subseteq [1, \min(J, L(1))-1], \text{ or } i,k \subseteq [\max(J, L(\bar{L})), n-1]) \quad (107)$$

and

$$\lambda^{i,k}(\omega) = \prod_{d=0}^{k-i-1} \lambda^{i+d,i+d+1}(\omega) \quad (108)$$

Moreover, the following relationships of their output frequency responses hold

$$x_i(j\omega) = \lambda^{i,k}(\omega) x_k(j\omega) \quad (109)$$

vi) If $J \le L(1)$, for any two masses whose positions are $i,k \subseteq [J, L(1)-1]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} = \lambda^{i,k}(\omega)$$

$$(i,k \subseteq [J, L(1)-1]) \quad (110)$$

vii) If $J \ge L(\bar{L})$, then for any masses whose positions are $i,k \subseteq [L(\bar{L}), J]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} = \lambda^{i,k}(\omega)$$

$$(i,k \subseteq [L(\bar{L}), J]) \quad (111)$$

viii) For any two masses whose positions are $i,k \subseteq [L(1)-1, L(\bar{L})]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} \ne \cdots \ne \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} \quad (112)$$

ix) For any two masses whose positions are $i \subseteq [1, L(1)-1]$ and $k \subseteq [L(1), n]$ or $i \subseteq [1, L(\bar{L})-1]$ and $k \subseteq [L(\bar{L}), n]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} \ne \cdots \ne \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} \quad (113)$$

The proof of the above five properties only needs some simple calculations. The details are therefore omitted here.

5 Numerical Study

To verify above analysis results, a damped 10-DOF oscillator was adopted, in which the fourth and sixth spring were nonlinear. The damping was assumed to be proportional damping, e.g., $C = \mu K$. The values of the system parameters are

$$m_1 = \cdots = m_{10} = 1, \ k_1 = \cdots = k_5 = k_{10} = 3.6 \times 10^4, \ k_6 = k_7 = k_8 = 0.8 \times k_1,$$

$$k_9 = 0.9 \times k_1,\ \ \mu = 0.01,\ \ w_{(4,2)} = w_{(4,3)} = w_{(6,2)} = w_{(6,3)} = 0$$

$$r_{(4,2)} = 0.8 \times k_1^2,\ \ r_{(4,3)} = 0.4 \times k_1^3,\ r_{(6,2)} = 0.5 \times r_{(4,2)},\ r_{(6,3)} = 0.1 \times r_{(4,3)}$$

and the input is a harmonic force, $f(t) = A\sin(2\pi \times 20t)$.

If only the NOFRFs up to the 4th order is considered, according to equations (29) and (30), the frequency components of the outputs of the 10 masses can be written as

$$\begin{aligned}
x_i(j\omega_F) &= G_{(i,1)}^H(j\omega_F)F_1(j\omega_F) + G_{(i,3)}^H(j\omega_F)F_3(j\omega_F) \\
x_i(j2\omega_F) &= G_{(i,2)}^H(j2\omega_F)F_2(j2\omega_F) + G_{(i,4)}^H(j2\omega_F)F_4(j2\omega_F) \\
x_i(j3\omega_F) &= G_{(i,3)}^H(j3\omega_F)F_3(j3\omega_F) \\
x_i(j4\omega_F) &= G_{(i,4)}^H(j4\omega_F)F_4(j4\omega_F) \qquad (i = 1,\cdots,10)
\end{aligned} \tag{114}$$

From equation (115), it can be seen that, using the method in [9], two different inputs with the same waveform but different strengths are sufficient to estimate the NOFRFs up to 4th order. Therefore, in the numerical studies, two different inputs were *A*=0.8 and *A*=1.0 respectively. The simulation studies were conducted using a fourth-order *Runge–Kutta* method to obtain the forced response of the system.

Case 1. Input Force Acting on the Eighth Mass (*J* = 8)

In this case, the position of the input force is on the right of the two nonlinear components. The evaluated results of $G_1^H(j\omega_F)$, $G_3^H(j\omega_F)$, $G_2^H(j2\omega_F)$ and $G_4^H(j2\omega_F)$ for all masses are given in Table 1 and Table 2. According to Property iii) in the previous section, it can be known that the following relationships should be tenable.

$$\begin{aligned}
\lambda_1^{i,i+1}(j\omega_F) &= \frac{G_{(i,1)}^H(j\omega_F)}{G_{(i+1,1)}^H(j\omega_F)} = \frac{G_{(i,3)}^H(j\omega_F)}{G_{(i+1,3)}^H(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) && \text{for } i = 1,2,8,9 \\
\lambda_1^{i,i+1}(j\omega_F) &= \frac{G_{(i,1)}^H(j\omega_F)}{G_{(i+1,1)}^H(j\omega_F)} \neq \frac{G_{(i,3)}^H(j\omega_F)}{G_{(i+1,3)}^H(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) && \text{for } i = 3,4,5,6,7 \\
\lambda_2^{i,i+1}(j2\omega_F) &= \frac{G_{(i,2)}^H(j2\omega_F)}{G_{(i+1,2)}^H(j2\omega_F)} = \frac{G_{(i,4)}^H(j2\omega_F)}{G_{(i+1,4)}^H(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) && \text{for } i = 1,2,6,7,8,9 \\
\lambda_2^{i,i+1}(j2\omega_F) &= \frac{G_{(i,2)}^H(j2\omega_F)}{G_{(i+1,2)}^H(j2\omega_F)} \neq \frac{G_{(i,4)}^H(j2\omega_F)}{G_{(i+1,4)}^H(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) && \text{for } i = 3,4,5
\end{aligned} \tag{115}$$

Table 1, the evaluated results of $G_1^H(j\omega_F)$ and $G_3^H(j\omega_F)$

	$G_1^H(j\omega_F)$ ($\times 10^{-6}$)	$G_3^H(j\omega_F)$ ($\times 10^{-9}$)
Mass 1	0.7415267278199+1.816751164684i	-1.483934590902 -3.186259401277i
Mass 2	0.9686508822055+3.482999245151i	-2.034574952226-6.147775997808i
Mass 3	0.2866237674289+4.763897212055i	-0.9248301066482-8.498481928587i
Mass 4	-1.462317904435+ 5.295827553965i	2.239865303216-7.667003443243i
Mass 5	-4.094410876431+4.614528872429i	6.580908764112-6.986590863164i
Mass 6	-7.746867339414+1.688023192826i	-9.325179952009+ 6.177740218363i
Mass 7	-10.20345149262-3.666962508263i	-3.099783998060+6.466785756735i
Mass 8	-9.510413275294-10.96839663935i	2.056977729641+4.552141996250i
Mass 9	-16.46061750950-2.605973256906i	5.171041280935+2.478570894858i
Mass 10	-19.35855072740+1.845265232929i	6.564490607347+1.295984511655i

Table 2, the evaluated results of $G_2^H(j2\omega_F)$ and $G_4^H(j2\omega_F)$

	$G_2^H(j2\omega_F)$ ($\times 10^{-8}$)	$G_4^H(j2\omega_F)$ ($\times 10^{-10}$)
Mass 1	1.202748566964+0.4681292667667i	0.01177472689947-0.6145473240625i
Mass 2	1.834979003885+1.548888270576i	0.3496457869565-1.088809469915i
Mass 3	1.093072003742+3.364850755945i	1.283912193671-1.091281155712i
Mass 4	-2.686267598254+1.553092622255i	2.608199029485+1.794043257583i
Mass 5	-6.011457034430+1.227859095376i	2.184292191061+3.969798930410i
Mass 6	5.106241229081-1.766725681899i	-1.893210085076-4.132529294142i
Mass 7	1.627889631587-2.693885701039i	-2.387756928966-1.144878010356i
Mass 8	-0.3076562316709-1.585358322807i	-1.302621593522+0.3861114911929i
Mass 9	-0.8845095711763-0.3838344220964i	-0.2497346565579+0.7717822225892i
Mass 10	-0.9601836703663+0.2564373781398i	0.2925531660608+0.7832618217179i

From the NOFRFs in Table 1 and Table 2, $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ ($i=1,\cdots,9$) can be calculated. The results are given in Table 3 and Table 4. It can be seen that the results shown in Table 3 and Table 4 have a strict accordance with the relationships in (115).

Table 3, the evaluated values of $\lambda_1^{i,i+1}(j\omega_F)$ and $\lambda_3^{i,i+1}(j\omega_F)$

	$\lambda_1^{i,i+1}(j\omega_F)$	$\lambda_3^{i,i+1}(j\omega_F)$
i=1	0.539116771645 -0.062966074902i	0.539114740172 -0.062960206148i
i=2	0.740676347710-0.158768210839i	0.740678235075 -0.158801705108i
i=3	0.821942385548-0.281082572848i	0.988815929108 -0.409500872246i
i=4	0.799438168470-0.392436688044i	0.741488963114 -0.377838915193i
i=5	0.628478650790-0.458719915806i	-0.835410927162 +0.195775222459i
i=6	0.619740592208-0.388161663940i	1.338885322270 +0.800231343587i
i=7	0.651280377150-0.365550780088i	0.924193400529 +1.098566179383i
i=8	0.666553636485+0.560815879662i	0.666591724409 +0.560805259874i
i=9	0.829929766156+0.213725389799i	0.829923787276 +0.213726030642i

Table 4, the evaluated values of $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$

	$\lambda_2^{i,i+1}(j2\omega_F)$	$\lambda_4^{i,i+1}(j2\omega_F)$
i=1	0.508497253663 -0.174103444440i	0.508509862998 -0.174110405319i
i=2	0.576617917017 -0.358023041530i	0.576584821679-0.357963201543i
i=3	0.237807435696 -1.115121138339i	0.138793754874-0.513873036792i
i=4	0.479618528466 -0.160391839157i	0.624392965910-0.313452235669i
i=5	-1.125713938165 -0.149027160197i	-0.994124084547+0.073128688720i
i=6	1.319440301221+1.098167617907i	1.319394960471+1.098094611355i
i=7	1.445523160511+1.307347247702i	1.445520238610+1.307371222544i
i=8	0.947244483068+1.381300241262i	0.947248244745+1.381297530888i
i=9	0.760202911490 +0.602779323827i	0.760201968688+0.602779474567i

Case 2. Input Force Acting on the Fifth Mass (*J* = 5)

In this case, the input force is located between the two nonlinear components. The evaluated results of $G_1^H(j\omega_F)$, $G_3^H(j\omega_F)$, $G_2^H(j2\omega_F)$ and $G_4^H(j2\omega_F)$ for all masses are given in Table 5 and Table 6. According to Property ii) in the previous section, it can be shown that the following relationships should be tenable.

$$\lambda_1^{i,i+1}(j\omega_F) = \frac{G_{(i,1)}^H(j\omega_F)}{G_{(i+1,1)}^H(j\omega_F)} = \frac{G_{(i,3)}^H(j\omega_F)}{G_{(i+1,3)}^H(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) \quad \text{for } i = 1,2,6,7,8,9$$

$$\lambda_1^{i,i+1}(j\omega_F) = \frac{G_{(i,1)}^H(j\omega_F)}{G_{(i+1,1)}^H(j\omega_F)} \neq \frac{G_{(i,3)}^H(j\omega_F)}{G_{(i+1,3)}^H(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) \quad \text{for } i = 3,4,5$$

$$\lambda_2^{i,i+1}(j2\omega_F) = \frac{G^H_{(i,2)}(j2\omega_F)}{G^H_{(i+1,2)}(j2\omega_F)} = \frac{G^H_{(i,4)}(j2\omega_F)}{G^H_{(i+1,4)}(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) \qquad \text{for } i = 1,2,6,7,8,9$$

$$\lambda_2^{i,i+1}(j2\omega_F) = \frac{G^H_{(i,2)}(j2\omega_F)}{G^H_{(i+1,2)}(j2\omega_F)} \neq \frac{G^H_{(i,4)}(j2\omega_F)}{G^H_{(i+1,4)}(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) \qquad \text{for } i = 3,4,5$$

(116)

Table 5, the evaluated results of $G_1^H(j\omega_F)$ and $G_3^H(j\omega_F)$

	$G_1^H(j\omega_F)$ ($\times 10^{-6}$)	$G_3^H(j\omega_F)$ ($\times 10^{-8}$)
Node1	-6.004311435381+0.5980810166970i	0.1295081340997+1.556090308409i
Node2	-11.11526321870 -0.1888307567420i	-0.09555232191420+2.875063343638i
Node3	-14.29538892099 -3.319252673742i	-0.9190827822761+3.685438469193i
Node4	-14.33495756714 -8.940411424246i	-8.147538091115+0.9042498552996i
Node5	-10.02484499526-16.10488682654i	-8.409614928777-1.003954297577i
Node6	-11.66774855007 -7.396738122696i	5.405178957847+5.050193617028i
Node7	-8.853271381015 -0.2333059825496i	4.635482228708+1.001306089351i
Node8	-4.094410864503+ 4.614528943238i	2.612399846945-2.021844501402i
Node9	-0.1863928697550+7.079375693477i	0.8006120355434 -3.706665572718i
Node10	1.849477584057+8.053867260589i	-0.1739577595525-4.421473593117i

Table 6, the evaluated results of $G_2^H(j2\omega_F)$ and $G_4^H(j2\omega_F)$

	$G_2^H(j2\omega_F)$ ($\times 10^{-8}$)	$G_4^H(j2\omega_F)$ ($\times 10^{-10}$)
Node1	-6.909233458613-1.949535672491i	6.139860082511+20.00249601942i
Node2	-10.98761266955-7.595710148583i	-1.246535839654+38.91203058802i
Node3	-7.846059575930-18.04611666444i	-31.80550973916+47.72222810926i
Node4	-3.958911551108+2.542004603355i	-6.895109772109 -9.745118814136i
Node5	11.36534576060-11.30577344782i	-51.06352947352-33.38863242228i
Node6	-0.07046569941447+21.52523824551i	51.82672918796-1.461435366305i
Node7	7.990357154274+9.664979250636i	22.66182418325-19.97094746422i
Node8	6.366778422385+0. 9277715851552i	1.750248731000-15.39837288857i
Node9	2.606698134275-2.821689966157i	-6.991103479880-6.061394620598i
Node10	0.2982812100639-3.948280315643i	-9.528102406109 -0.4183544781386i

From the NOFRFs in Table 5 and Table 6, $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ ($i=1,\cdots,9$) can be calculated. The results are given in Table 7 and Table 8. It can be seen that the results shown in Tables 7~8 have a strict accordance with the relationships in (116).

Table 7, the evaluated values of $\lambda_1^{i,i+1}(j\omega_F)$ and $\lambda_3^{i,i+1}(j\omega_F)$

	$\lambda_1^{i,i+1}(j\omega_F)$	$\lambda_3^{i,i+1}(j\omega_F)$
i=1	0.539116475689 -0.062965921274i	0.539144288494-0.062963698909i
i=2	0.740677219089 -0.158768963590i	0.740525663557-0.158746936134i
i=3	0.821938417409 -0.281076168130i	0.161023987718-0.434466523754i
i=4	0.799439023625 -0.392471262929i	0.942565996984-0.220050864920i
i=5	1.237062417913 +0.596058446920i	-0.923341628835+0.676961804824i
i=6	1.338989660357 +0.800195037459i	1.338905575647 +0.800248847486i
i=7	0.924175246088 +1.098555946898i	0.924193246937 +1.098561205039i
i=8	0.666592586075 +0.560806903225i	0.666591608191+0.560805538524i
i=9	0.829923646648 +0.213725654399i	0.829923826262+0.213725968290i

Table 8, the evaluated values of $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$

	$\lambda_2^{i,i+1}(j2\omega_F)$	$\lambda_4^{i,i+1}(j2\omega_F)$
i=1	0.508479126951-0.174080070890i	0.508467502449 -0.174076832929i
i=2	0.576625221246 -0.358158874336i	0.576653888587 -0.358202785310i
i=3	-0.669148901810 + 4.128694684056i	-1.724486070038 -4.483888942345i
i=4	-0.286911040971-0.061744415025i	0.182002416343 + 0.071838101924i
i=5	-0.526956204410 -0.526275867142i	-0.966339150393 -0.671485065289i
i=6	1.319350502086 +1.098041407803i	1.319244199748 +1.098107594434i
i=7	1.445521775923 +1.307390436574i	1.445556127111+1.307394070261i
i=8	0.947249859811 +1.381294198466i	0.947246001989 +1.381289963097i
i=9	0.760201320744 +0.602779988797i	0.760201704191+0.602781182293i

The two numerical case studies verify the properties of the NOFRFs of MDOF systems with multiple nonlinear components derived in the present study. These properties can provide a convenient method to detect the positions of the nonlinear components in a MDOF system by analyzing the relationships between the NOFRFs.

6 Conclusions

In this paper, significant relationships between the NOFRFs of MDOF systems with multiple nonlinear components have been derived and verified by numerical studies. The results reveal, for the first time, important properties of this general class of MDOF nonlinear systems and can be used to detect and locate faults in engineering structures which make the structures behave nonlinearly.

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References

1. T.G. Chondros, A.D. Dimarogonas, J. Yao, Vibration of a beam with breathing crack, *Journal of Sound and Vibration*, **239** (2001) 57-67
2. Z. Q. Lang, S. A. Billings, Output frequency characteristics of nonlinear system, *International Journal of Control* **64 (**1996**)** 1049-1067.
3. K. Worden, G. Manson, G.R. Tomlinson, A harmonic probing algorithm for the multi-input Volterra series. *Journal of Sound and Vibration* **201**(1997) 67-84
4. S.A. Billings, K.M. Tsang, Spectral analysis for nonlinear system, part I: parametric non-linear spectral analysis. Mechanical Systems and Signal Processing, **3** (1989) 319-339
5. S.A. Billings, J.C. Peyton Jones, Mapping nonlinear integro-differential equations into the frequency domain, *International Journal of Control* 52 (1990**)** 863-879.
6. J.C. Peyton Jones, S.A. Billings, A recursive algorithm for the computing the frequency response of a class of nonlinear difference equation models. *International Journal of Control* **50** (1989**)** 1925-1940.
7. H. Zhang, S. A. Billings, Analysing non-linear systems in the frequency domain, I: the transfer function, *Mechanical Systems and Signal Processing* **7** (1993) 531–550.
8. H. Zhang, S. A. Billings, Analysing nonlinear systems in the frequency domain, II: the phase response, *Mechanical Systems and Signal Processing* **8** (1994) 45–62.
9. Z. Q. Lang, S. A. Billings, Energy transfer properties of nonlinear systems in the frequency domain, *International Journal of Control* **78 (**2005**)** 354-362.
10. J He, Z Fu, Modal Analysis, Butterworth-Heinemann press, 2001