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On Lagrange's four squares theorem with almost prime variables

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Abstract. In 1994, Brüdern and Fouvry [1] initiated the investigation of Lagrange's four squares theorem with almost prime variables. In this paper, we prove that every sufficiently large integer, congruent to 4 modulo 24, can be represented as a sum of four squares of integers, each of which has at most four prime factors. Instead of the four-dimensional vector sieve developed by Brüdern and Fouvry [1], we establish this result by combining the three-dimensional sieve and the switching principle.

1. Introduction

We consider the equation of Lagrange

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = N \tag{1.1}$$

with multiplicative restrictions. It is expected that sufficiently large integers under certain necessary congruence condition can be written as sums of four squares of primes. This problem has not been solved so far. However Hua [10] proved that all large integers congruent to 5 modulo 24 are sums of five squares of primes by using Vinogradov's method for the ternary Goldbach problem.

Kloosterman [11] developed the circle method to study the asymptotic formula for the number of integer solutions of the following positive definite quadratic forms

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = N.$$

Estermann [5] investigated the indefinite quadratic forms via the circle method and the Kloosterman refinement. The classical circle method with mean value theorems (see [16] for the exposition) provides an asymptotic formula for quadratic forms with five or more variables only.

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Greaves [6] considered the solutions of (1.1) with two prime and two integral variables. Plaksin [12] and Shields [14] obtained an asymptotic formula for the number of solutions. Podsypanin [13] derived an asymptotic formula for the number of solutions of (1.1) in which x_1, x_2, x_3, x_4 are square-free.

In 1994, Brüdern and Fouvry [1] established that every sufficiently large integer, congruent to 4 modulo 24, can be written as the sum of four squares of integers, each of which has at most 34 prime factors.

In 2003, Heath-Brown and Tolev [9] managed to solve the equation

$$p^2 + x_1^2 + x_2^2 + x_3^2 = N \tag{1.2}$$

with multiplicative restrictions, where p denotes a prime number. Precisely, they established the solvability of (1.2) with each of x_i having at most 101 prime divisors. This was improved by Tolev [15] who showed that 101 can be replaced by 80, and then improved by Cai [3] who showed that 42 is acceptable. Our first result is as follows.

Theorem 1.1. *Every sufficiently large integer N, congruent to* 4 *modulo* 24*, can be represented in the form of*

$$N = p^2 + x_1^2 + x_2^2 + x_3^2, \tag{1.3}$$

where p is a prime and each of x_1, x_2, x_3 has at most five prime factors.

As in [9], the proof of the above theorem will be finished in two steps. In the first step, we combine the circle method with Kloosterman refinement and the square sieve to control the reminder term uniformly for certain level D. In the second step, we use the sieve method to produce the almost primes. For the uniform estimate, we shall develop the strategy of Heath-Brown and Tolev, and push their method further.

Comparing to the proof in [9], two things are different. We start from applying Cauchy's inequality for summation over k only rather than summations over k and $\mathbf{d}$ (see (5.2) in Section 5). This subtle difference contributes to a better result because it can be proved that the error term is dominated by diagonal contributions up to P^ε (see Lemma 12 below). To this end, we have to introduce the condition (2.8) at the beginning, and then we remove it before we close the proof. This obstacle is one of the reasons that force Brüdern and Fouvry to develop the vector sieve (cf. the discussion in [1, Section III]). Finally instead of the vector sieve which was used by Brüdern and Fouvry, and by Heath-Brown and Tolev, we appeal to the switching principle of Chen [4] to reduce the number of prime factors for each variable.

The square sieve of Heath-Brown [8] plays an important role in the proof. However we will choose the parameter R to be of type P^ε. Therefore, the square sieve is really responsible for the success of the proof, while it does not determine the quality of the level D.

Concerning the Lagrange equation with four almost prime variables, the value 34 due to Brüdern and Fouvry was sharpened by Heath-Brown and Tolev [9] to 25, by Tolev [15] to 20, and by Cai [3] to 13. We shall prove the following result.

Theorem 1.2. *Every sufficiently large integer N, congruent to* 4 *modulo* 24*, can be represented in the form of*

$$N = x_1^2 + x_2^2 + x_3^2 + x_4^2, \tag{1.4}$$

where each of x_1, x_2, x_3, x_4 has at most four prime factors.

In prior works [1, 9], in order to obtain a sharp result involving four almost prime variables, people considered the equation

$$N = d_1^2x_1^2 + d_2^2x_2^2 + d_3^2x_3^2 + d_4^2x_4^2.$$

Our approach to Theorem 1.2 is different. We consider the equation

$$N = q^2 + d_1^2x_1^2 + d_2^2x_2^2 + d_3^2x_3^2,$$

where q is an almost prime. Therefore, instead of the vector sieve developed by Brüdern and Fouvry [1], we shall combine the three-dimensional sieve and the switching principle to establish Theorem 1.2.

As usual, we write $e(z)$ for $e^{2\pi iz}$. We use ε to denote a sufficiently small positive number, and the letter A denotes a sufficiently large constant. We use $\ll$ to denote Vinogradov's well-know notation, while implicit constants may depend on ε and A.

2. A crucial proposition

Before giving the main propositions, we introduce some notations. Let N be a sufficiently large integer satisfying $N \equiv 4 \pmod{24}$. Set $P = N^{1/2}$. The letter p is reserved for prime numbers. As usual, $\mu(n)$, $\phi(n)$ and $\tau(n)$ denote the Möbius function, Euler's totient function, and the divisor function respectively. For the natural number q and real number α, we write $e_q(\alpha) = e(\alpha/q)$. We use $\sum_{x(q)}$ and $\sum_{x(q)^*}$ to denote sums with x running over a complete system, respectively reduced system of residues modulo q. The Gauss sums $S(q,m,n)$ and $T(q,a)$ are defined by

$$(2.1)\qquad S(q,m,n) = \sum_{x(q)} e_q(mx^2 + nx), \quad S(q,m) = S(q,m,0),$$
$$T(q,a) = \sum_{x(q)^*} e_q(ax^2),$$

and

$$S_{\mathbf{d}}(q,m,\mathbf{n}) = \prod_{j=1}^{3} S(q, md_j^2, n_j), \quad S_{\mathbf{d}}(q,m) = S_{\mathbf{d}}(q,m,\mathbf{0}),$$

where we use the bold style letter **d** to indicate the three-dimensional vectors (d_1, d_2, d_3). Now we define the singular series

$$(2.2)\qquad \Sigma_0(\mathbf{d}, N) = \sum_{q=1}^{\infty} h_{\mathbf{d}}(q) = \prod_{p>2}(1 + h_{\mathbf{d}}(p)),$$

where

$$h_{\mathbf{d}}(q) = q^{-3}\phi(q)^{-1} \sum_{a(q)^*} S_{\mathbf{d}}(q,a)T(q,a)e_q(-aN).$$

Let

$$\omega_0(t) = \begin{cases} \exp\left(\frac{1}{(20t-10)^2-1}\right) & \text{if } \frac{9}{20} < t < \frac{11}{20}, \\ 0 & \text{otherwise,} \end{cases}$$

and denote

$$\omega(x) = \omega_0(xP^{-1}), \quad \omega(\mathbf{x}) = \prod_{j=1}^{3} \omega(x_j).$$

We define

$$I(\beta, u) = \int_{-\infty}^{+\infty} \omega_0(x) e(\beta x^2 + ux) dx, \quad I(\beta) = I(\beta, 0),$$

and

$$I_{\mathbf{d}}(\beta, \mathbf{u}) = \prod_{j=1}^{3} I(\beta, u_j d_j^{-1}).$$

Let

$$H(t) = \int_{-\infty}^{+\infty} I^3(\beta) e(-t\beta) d\beta. \tag{2.3}$$

For any $j \leq 15$, let $\mathcal{A}_j$ denote the set consisting of integers $q \leq P$ satisfying two restrictions:

(i) the number of prime factors of q counting multiplicity is exactly j,

(ii) all prime factors of q are greater than $P^{1/16}$.

In order to apply the sieve method, we study

$$\mathcal{L}_{\mathbf{d}}(N) := \mathcal{L}_{\mathbf{d}}^{j}(N) = \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ q \in \mathcal{A}_j,\, d_i \mid x_i \ (1 \leq i \leq 3)}} \omega(q)\omega(\mathbf{x}). \tag{2.4}$$

Here we also attach the smooth weight $\omega(q)$ for the use of switching principle. The corresponding singular integral is defined by

$$\mathcal{N}_0(N) := \mathcal{N}_0^{j}(N) = P \int_0^P H\left(1 - \frac{x^2}{P^2}\right) \frac{\omega(x) C_j(x) dx}{\log x}, \tag{2.5}$$

where $C_1(x) = 1$ and for $j \geq 2$,

$$C_j(x) = \sum_{P^{1/16} < p_1 \leq \cdots \leq p_{j-1} \leq (x p_1^{-1} \cdots p_{j-2}^{-1})^{1/2}} \frac{\log x}{(\log x - \log(p_1 \cdots p_{j-1})) p_1 \cdots p_{j-1}}.$$

The expected main term for $\mathcal{L}_{\mathbf{d}}(N)$ is $\mathcal{N}_0(N)\Sigma_0(\mathbf{d}, N)(d_1 d_2 d_3)^{-1}$. (Here and after, we shall often suppress the dependence on j when the meaning is clear from the context.) We plan to investigate

$$\mathcal{H}(D) = \sum_{\substack{d_1, d_2, d_3 \\ [d_1, d_2, d_3] \leq D}} \beta(\mathbf{d}) \left(\mathcal{L}_{\mathbf{d}}(N) - \frac{\mathcal{N}_0(N)\Sigma_0(\mathbf{d}, N)}{d_1 d_2 d_3} \right),$$

where $\beta(\mathbf{d}) = \beta(d_1, d_2, d_3)$ is a real function satisfying

$$|\beta(\mathbf{d})| \leq \tau^2(d_1)\tau^2(d_2)\tau^2(d_3), \tag{2.6}$$

and $\beta(\mathbf{d}) = 0$ unless

$$\mu(2d_1)\mu(2d_2)\mu(2d_3) \neq 0. \tag{2.7}$$

Let

$$\mathcal{H}_0(D) = \sum_{d_1,d_2,d_3 \le D} \beta(\mathbf{d}) \left(\mathcal{L}_{\mathbf{d}}(N) - \frac{\mathcal{N}_0(N)\Sigma_0(\mathbf{d}, N)}{d_1 d_2 d_3} \right).$$

Heath-Brown and Tolev established the following result.

Proposition 1 ([9]). *If* $D = P^{2/69-\varepsilon}$, *then one has*

$$\mathcal{H}_0(D) \ll P^2 (\log P)^{-A}.$$

Define

$$B(d_1, d_2, d_3, d_4, N) = \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=N \\ x_i \equiv 0 \pmod{d_i}}} \prod_{i=1}^{4} \omega(x_i)$$

and

$$\Sigma_1(d_1, d_2, d_3, d_4, N) = \sum_{q=1}^{\infty} q^{-4} \sum_{a(q)^*} e_q(-aN) \prod_{i=1}^{4} S(q, a d_i^2).$$

Let

$$\mathcal{H}_0^*(D) = \sum_{d_1,d_2,d_3,d_4 \le D} \left| B(d_1, d_2, d_3, d_4, N) - \kappa_1 N \frac{\Sigma_1(d_1, d_2, d_3, d_4, N)}{d_1 d_2 d_3 d_4} \right|,$$

where the constant κ_1 is given by

$$\kappa_1 = \int_{-\infty}^{\infty} I^4(\beta) e(-\beta) d\beta.$$

It was proved by Heath-Brown and Tolev that:

Proposition 2 ([9]). *If* $D \le P^{1/8-\varepsilon}$, *then one has*

$$\mathcal{H}_0^*(D) \ll P^{2-\varepsilon}.$$

Proposition 2 improves upon the result of Brüdern and Fouvry [1] who essentially showed $\mathcal{H}_0^*(D) \ll P^{2-\varepsilon}$ provided that $D \le P^{1/11-\varepsilon}$.

Concerning $\mathcal{H}(D)$, we prove the following result.

Proposition 3. *Suppose that* $D < P^{1/2-\varepsilon}$. *Then we have*

$$\mathcal{H}(D) \ll P^2 (\log P)^{-A}.$$

Remark. (i) The function $\beta(\mathbf{d})$ may depend on N, but in view of (2.6), the absolute value of $\beta(\mathbf{d})$ is bounded from above by $\prod_{j=1}^{3} \tau^2(d_j)$ which is independent of N.

(ii) With the application to Lagrange's four squares theorem with almost prime variables in mind, Proposition 3 improves upon both Proposition 1 and Proposition 2.

In order to prove Proposition 1, Heath-Brown and Tolev investigated the following object:

$$\Omega_{\mathbf{d}}(n) = \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ d_j \mid x_j \ (1 \le j \le 3)}} \omega(\mathbf{x}).$$

The application of circle method suggests that the sum $\Omega_{\mathbf{d}}(n)$ may be approximated by

$$\mathcal{M}_{\mathbf{d},Q}(n) = \frac{PH(nN^{-1})}{d_1 d_2 d_3} \sum_{q \le Q} q^{-3} \sum_{a(q)^*} e_q(-an) S_{\mathbf{d}}(q,a).$$

Consider

$$\widetilde{\mathcal{E}}(D,Q) = \sum_{d_1,d_2,d_3 \le D} \tau(d_1)\tau(d_2)\tau(d_3) \sum_{k \le P} |\Omega_{\mathbf{d}}(N-k^2) - \mathcal{M}_{\mathbf{d},Q}(N-k^2)|.$$

Heath-Brown and Tolev established

$$\widetilde{\mathcal{E}}(D,Q) \ll P^{2-\varepsilon}$$

provided that $Q = P^{20/23}$ and $D = P^{2/69-10\varepsilon}$. Our purpose is to establish the following result.

Proposition 4. *Let*

$$\mathcal{G}(n; Q, D_1, D_2, D_3) = \sum_{d_i \le D_i \ (1 \le i \le 3)} \beta(d_1,d_2,d_3)(\Omega_{\mathbf{d}}(n) - \mathcal{M}_{\mathbf{d},Q}(n)),$$

where $\beta(d_1,d_2,d_3)$ *satisfies* (2.6), (2.7) *and*

$$(d_i, d_j) \le P^{\varepsilon} \quad \text{for } 1 \le i < j \le 3. \tag{2.8}$$

Suppose that $D_1 D_2 D_3 < P^{1/2-6\varepsilon}$ *and* $P^{1-4\varepsilon} < Q < P^{1-2\varepsilon}$. *Then we have*

$$\sum_{k \le P} |\mathcal{G}(N-k^2; Q, D_1, D_2, D_3)| \ll P^{2-\varepsilon}.$$

Proposition 4 will be proved in Sections 3–6. As an application of Proposition 4, we shall prove Proposition 3 in Section 7.

3. Basic assumptions

We first quote some lemmas which are well known.

Lemma 1. *The Gauss sum defined by* (2.1) *satisfies:*

(i) *If* $(q_1,q_2) = 1$, *then*

$$S(q_1 q_2, a_1 q_2 + a_2 q_1, n) = S(q_1, a_1 q_2^2, n) S(q_2, a_2 q_1^2, n).$$

(ii) *Suppose that* $(q,m) = k$. *Then we have*

$$S(q,m,n) = \begin{cases} kS(q/k, m/k, n/k) & \text{if } k \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If* $(q, 2m) = 1$, *then*

$$S(q,m,n) = e_q(-\overline{4m}n^2)\left(\frac{m}{q}\right)S(q,1),$$

where $\overline{x}$ *denotes the inverse of* x *modulo* q.

(iv) *If* q *is odd, then*

$$S(q,1) = \begin{cases} q^{1/2} & \text{if } q \equiv 1 \pmod 4, \\ iq^{1/2} & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

(v) *If* $(2, a) = 1$, *then*

$$|S(2^l, a, n)| \le 2^{1+l/2}.$$

(vi) *For any odd integer* q, *we have* $|\gamma(q)| \le q^{1/2}$, *where*

$$\gamma(q) = \sum_{x(q)^*} \left(\frac{x}{q}\right) e_q(x).$$

Throughout, by $(\frac{l}{q})$ *we denote the Jacobi symbol.*

Lemma 2. *Denote the Kloosterman sum by*

$$K(p,m,n) = \sum_{x=1}^{p-1} e_p(mx + n\overline{x}).$$

If $p \nmid (m,n)$, *then*

$$|K(p,m,n)| \le 2p^{1/2}.$$

Lemma 3. *Let* $c_q(n)$ *be the Ramanujan sum given by*

$$c_q(n) = \sum_{x(q)^*} e_q(xn).$$

Then we have

$$|c_q(n)| \le (q,n).$$

Lemma 4. *The following statements hold.*

(i) *Suppose that* $u = \max\{|u_1|, \dots, |u_6|\} > 0$. *Then we have*

$$\int_{-\infty}^{+\infty} |I(\beta,u_1)\cdots I(\beta,u_6)|d\beta \ll \min\{1, u^{-2+\varepsilon}\}.$$

(ii) *Suppose that* $\beta_0 > 0$. *Then we have*

$$\int_{|\beta|\ge\beta_0} |I(\beta)|^6 d\beta \ll \beta_0^{-2}.$$

Lemma 4 is a consequence of [9, Lemma 9 and Lemma 10]. Let

$$f_d(\alpha) = \sum_{\substack{x\in\mathbb{Z} \\ x\equiv 0 \pmod d}} \omega(x)e(\alpha x^2), \quad f_{\mathbf{d}}(\alpha) = \prod_{i=1}^{3} f_{d_i}(\alpha),$$

and let

$$\mathcal{W}_{\mathbf{d},Q}(\alpha) = \sum_{n\in\mathbb{Z}} \mathcal{M}_{\mathbf{d},Q}(n)e(\alpha n).$$

The following two lemmas provide useful approximations to $f_d(\alpha)$ and $\mathcal{W}_{\mathbf{d},Q}(\alpha)$ respectively.

Lemma 5 ([9, Lemma 12]). *Let $q, d, b, h \in \mathbb{N}$, let $\beta \in \mathbb{R}$, $q \le P$ with $|\beta| \le (qP)^{-1}$ and $d, b \le P^2$. Then we have*

$$f_d\left(\frac{h}{b} + \beta\right) = \frac{P}{bd} \sum_{|n|\le bdq^{-1}P^{\varepsilon}} S(b, hd^2, n) I\left(\beta N, -\frac{Pn}{bd}\right) + O(P^{-A}),$$

where the implicit constant depends on A and ε.

Lemma 6 ([9, Lemma 16]). *Suppose that $Q \le P^{1-\varepsilon}$, $|\beta| \le (qP)^{-1}$ and $(a, q) = 1$. Then we have*

$$\mathcal{W}_{\mathbf{d},Q}(\alpha) = \begin{cases} \frac{P^3}{q^3 d_1 d_2 d_3} S_{\mathbf{d}}(q, a) I^3(\beta N) + O(P^{-A}) & \text{if } 1 \le q \le Q, \\ O(P^{-A}) & \text{if } Q < q \le P, \end{cases}$$

where the implicit constants depend on A and ε.

Set

$$\sigma_0 = \int_{-\infty}^{+\infty} |I^6(\beta)| d\beta.$$

Lemma 7. *Let*

$$B(\alpha) = \sum_{n\in\mathbb{Z}} H^2\left(\frac{n}{N}\right) e(n\alpha),$$

where $H(t)$ is given in (2.3). We have

$$B(\alpha) = \begin{cases} \sigma_0 P^2 + O(P^{-A}) & \text{if } \alpha \in \mathbb{Z}, \\ O(P^{-A}) & \text{if } \|\alpha\| \ge P^{\varepsilon-2}, \end{cases}$$

where the implicit constants may depend on A and ε.

Lemma 7 is [9, formula (115)].

4. Some preparations

Let

$$\theta(q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, v, b_1, b_2) = \sum_{a(q)^*} e_q(\overline{ab_1} v) S_{\mathbf{d}}(q, ab_2^2, \mathbf{n}) S_{\mathbf{t}}(q, -ab_1^2, -\mathbf{l}), \tag{4.1}$$

and let

$$\eta = \eta(\mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, v, b_1, b_2) = b_1^2 b_2^2 \left(-4\frac{v}{b_1} + \sum_{\substack{j=1 \\ n_j \ne 0}}^{3} \frac{n_j^2}{d_j^2 b_2^2} - \sum_{\substack{i=1 \\ l_i \ne 0}}^{3} \frac{l_i^2}{t_i^2 b_1^2}\right) \times \text{lcm}, \tag{4.2}$$

where we use lcm to denote the least common multiple of d_j $(j \in J)$ and t_i $(i \in I)$ with the index sets $J = \{1 \le j \le 3 : n_j \ne 0\}$ and $I = \{1 \le i \le 3 : l_i \ne 0\}$. For $\mathbf{n} = \mathbf{l} = \mathbf{0} \in \mathbb{Z}^3$, we assume lcm $= 1$. For simplicity, in this section we always assume that $\mathbf{d}$ satisfies (2.7), (2.8) and $d_i \le D_i$ for $1 \le i \le 3$, and we also have analogous assumptions with $\mathbf{t}$ in place of $\mathbf{d}$. We use the notations $(q, \mathbf{k}) = (q, k_1)(q, k_2)(q, k_3)$ and $(q, \mathbf{k}^2) = (q, k_1^2)(q, k_2^2)(q, k_3^2)$ for $\mathbf{k} = \mathbf{d}$ and $\mathbf{t}$.

Lemma 8. *Let* $\theta(q) := \theta(q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, v, b_1, b_2)$ *be defined in formula* (4.1). *Suppose that* $(2, r) = 1$ *and* $(b_1 b_2, 2r) = 1$. *Then we have*

$$|\theta(2^u r)| \le \begin{cases} 2^{4u+6} r^3 (r, \eta)(r, \mathbf{d})(r, \mathbf{t}) & \text{if } \eta \ne 0 \text{ and } (\mathbf{n}, \mathbf{l}) \ne \mathbf{0} \in \mathbb{Z}^6, \\ 2^{4u+6} r^4 (r, \mathbf{d}^2)^{1/2} (r, \mathbf{t}^2)^{1/2} & \text{otherwise,} \end{cases}$$

where $\eta = \eta(\mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, v, b_1, b_2)$ *is given by* (4.2).

Proof. As a function of q, $\theta(q) = \theta(q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, v, b_1, b_2)$ is multiplicative. So it suffices to prove

$$|\theta(2^u)| \le 2^{4u+6}, \tag{4.3}$$

and

$$|\theta(p^\alpha)| \le \begin{cases} p^{3\alpha} (p^\alpha, \eta)(p^\alpha, \mathbf{d})(p^\alpha, \mathbf{t}) & \text{if } \eta \ne 0 \text{ and } (\mathbf{n}, \mathbf{l}) \ne \mathbf{0}, \\ p^{4\alpha} (p^\alpha, \mathbf{d}^2)^{1/2} (p^\alpha, \mathbf{t}^2)^{1/2} & \text{otherwise,} \end{cases} \tag{4.4}$$

for odd prime p. Plainly, (4.3) follows from Lemma 1 (v). By Lemma 1 (ii) and (iii),

$$S(p^\alpha, am^2, n) = \begin{cases} e\big(\frac{-\overline{4m^2a}n^2}{p^\alpha}\big) S(p^\alpha, a) & \text{if } p \nmid m, \\ e\big(\frac{-\overline{4s^2a}t^2}{p^{\alpha-2}}\big) p^2 S(p^{\alpha-2}, a) & \text{if } p \parallel m = ps,\ p^2 \,|\, n = p^2 t \text{ and } \alpha \ge 2, \\ 0 & \text{if } p \parallel m = ps,\ p^2 \nmid n \text{ and } \alpha \ge 2, \\ p & \text{if } p \,|\, m,\ p \,|\, n \text{ and } \alpha = 1, \\ 0 & \text{if } p \,|\, m,\ p \nmid n \text{ and } \alpha = 1, \end{cases}$$

where $p \parallel m$ means $p \,|\, m$ but $p^2 \nmid m$. Then for $\alpha \ge 2$, we have

$$\theta(p^\alpha) = \begin{cases} p^{\beta+3\alpha} \sum\limits_{a(p^\alpha)^*} e\Big(\frac{\overline{ab_1 v}}{p^\alpha}\Big) e\Big(-\overline{4a}\frac{h\overline{c}}{p^\alpha}\Big) & \text{if } (p^\alpha, d_j^2) \,|\, n_j \text{ and } (p^\alpha, t_j^2) \,|\, l_j \ (1 \le j \le 3), \\ 0 & \text{otherwise,} \end{cases} \tag{4.5}$$

where $p^\beta \parallel d_1 d_2 d_3 t_1 t_2 t_3$ and

$$\frac{h}{c} = \sum_{j=1}^{3} \bigg(\frac{n_j^2}{d_j^2 b_2^2} - \frac{l_j^2}{t_j^2 b_1^2} \bigg)$$

with $(h, c) = 1$. This proves (4.4) in the case $\alpha \ge 2$ by using Lemma 3. Hence we turn to the case $\alpha = 1$.

If $p \nmid d_1d_2d_3t_1t_2t_3$, then formula (4.5) still holds for $\alpha = 1$. We therefore confine to the case $p \mid d_1d_2d_3t_1t_2t_3$. Note that

$$|S(p, am^2, n)| \le p^{1/2}(p, m)^{1/2}$$

by Lemma 1 (ii), (iii) and (iv). The second estimate of (4.4) follows easily. Now we assume that $\eta \neq 0$ and $(\mathbf{n}, \mathbf{l}) \neq \mathbf{0}$. If $p \parallel d_1d_2d_3t_1t_2t_3$, then $\theta(p)$ is in the form

$$\theta(p) = pS(p,1)^5 \sum_{a(p)^*} \left(\frac{a}{p}\right) e\left(\frac{\overline{ab}c}{q}\right),$$

where $(b, p) = 1$. The summation over a is either 0 or a Gauss sum. So, by Lemma 1 (vi), we have

$$|\theta(p)| \le p^{3+1/2}(p, \mathbf{d})^{1/2}(p, \mathbf{t})^{1/2} = p^3(p, \mathbf{d})(p, \mathbf{t}).$$

If $p^2 \mid d_1d_2d_3t_1t_2t_3$, then we apply the trivial bound for summation over a to get

$$|\theta(p)| \le p^4(p, \mathbf{d})^{1/2}(p, \mathbf{t})^{1/2} \le p^3(p, \mathbf{d})(p, \mathbf{t}).$$

The proof is completed. □

From now on, we assume that p, p' are two different primes satisfying $R \le p, p' < 2R$ and $(pp', d_1d_2d_3t_1t_2t_3N) = 1$. For $\Delta \mid (pp')^2$, we have the unique decomposition $\Delta = \delta\delta'$, where $(p, \delta') = (p', \delta) = 1$. We use π and π' to denote a power of p, and respectively a power of p' (note that π and π' may be equal to 1).

Let us define

$$W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', v) = \sum_{s(pp')^*} \left(\frac{s}{pp'}\right) e_{pp'}(-sN) \sum_{\substack{a(q)^* \\ (app'+sq, qpp')=\Delta}} e_q(\overline{a}v)$$
$$\times S_{\mathbf{d}}(qpp'\Delta^{-1}, (app'+sq)\Delta^{-1}, \mathbf{n}) S_{\mathbf{t}}(q, -a, -\mathbf{l})$$

and

$$\mathcal{R}(\delta, p, \pi; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, m_1, m_2, n) = \sum_{c(p)^*} \left(\frac{c}{p}\right) e_p(-cn) \sum_{\substack{b(\pi)^* \\ (bp+c\pi, \pi p)=\delta}} e_\pi(\overline{bm_1}v)$$
$$\times S_{\mathbf{d}}\left(\frac{\pi p}{\delta}, \frac{bp+c\pi}{\delta} m_2^2, \mathbf{n}\right) S_{\mathbf{t}}(\pi, -bm_1^2, -\mathbf{l}).$$

Lemma 9. *Suppose that $q = \pi\pi' r$ with $(pp', r) = 1$. Then one has*

$$W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', v) = \left(\frac{p}{p'}\right)\left(\frac{p'}{p}\right) \theta(r; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, v, \pi\pi', \pi\pi' pp'\Delta^{-1})$$
$$\times \mathcal{R}\left(\delta, p, \pi; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, r\pi', \frac{rp'\pi'}{\delta'}, p'N\right)$$
$$\times \mathcal{R}\left(\delta', p', \pi'; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, r\pi, \frac{rp\pi}{\delta}, pN\right).$$

Proof. Let

$$a = \alpha\pi\pi' + b\pi' r + b'\pi r$$

with $(\alpha, r) = (b, p) = (b', p') = 1$ and

$$s = cp' + c'p$$

with $(c, p) = (c', p') = 1$. Note that

$$(app' + sq, qpp') = \Delta \iff (bp + c\pi, p\pi) = \delta \text{ and } (b'p' + c'\pi', p'\pi') = \delta'.$$

Obviously

$$\frac{app' + sq}{\Delta} = \alpha \frac{p\pi}{\delta} \frac{p'\pi'}{\delta'} + \frac{bp + c\pi}{\delta} \frac{p'\pi'}{\delta'} r + \frac{b'p' + c'\pi'}{\delta'} \frac{p\pi}{\delta} r.$$

With the help of Lemma 1 (i), the desired result can be obtained by changing variables. □

Lemma 10. *We have*

$$\left| \mathcal{R}\left(\delta, p, \pi; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, r\pi', \frac{rp'\pi'}{\delta'}, p'N\right) \mathcal{R}\left(\delta', p', \pi'; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, r\pi, \frac{rp\pi}{\delta}, pN\right) \right|$$
$$\leq \min\{\Delta^{-2} R^4 (4\pi\pi')^5 (1 + \Delta R^{-2})^{-1}, 2^6 R^4 (\pi\pi')^4\}.$$

Proof. We can assume that $\delta \mid p\pi$ and $\delta \mid p'\pi'$. Otherwise, the desired estimate holds trivially. We write

$$\mathcal{R}(\delta, p, \pi) := \mathcal{R}\left(\delta, p, \pi; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, r\pi', \frac{rp'\pi'}{\delta'}, p'N\right).$$

Obviously $(1 + \delta R^{-1})^{-1}(1 + \delta' R^{-1})^{-1} \leq (1 + \Delta R^{-2})^{-1}$. Thus it is enough to prove

$$|\mathcal{R}(\delta, p, \pi)| \leq \min\{\delta^{-2} R^2 (2\pi)^5 (1 + \delta R^{-1})^{-1}, 8R^2\pi^4\}.$$

Set $m_1 = r\pi'$ and $m_2 = rp'\pi'/\delta'$. It is clear that $p \nmid m_1 m_2$.

When $\delta = 1$, $(bp + c\pi, \pi p) = \delta$ is equivalent to $\pi = 1$. So one has

$$\mathcal{R}(1, p, \pi) = \sum_{c(p)^*} \left(\frac{c}{p}\right) e_p(-cp'N) S_{\mathbf{d}}(p, cm_2^2, \mathbf{n})$$
$$= S(p, 1)^3 \sum_{c(p)^*} e_p(-cp'N) e_p\left(-\overline{4cm_2^2} \sum_{j=1}^{3} n_j^2 \overline{d_j^2}\right).$$

Then by Lemma 1 (iv) and Lemma 2, we get

$$|\mathcal{R}(1, p, \pi)| \leq 2p^2.$$

When $\delta = p$, we have $(bp + c\pi, \pi p) = \delta$ if and only if $p \mid \pi$ and $(b + c\frac{\pi}{p}, p) = 1$. Hence

$$\mathcal{R}(\delta, p, \pi) = \sum_{c(p)^*} \left(\frac{c}{p}\right) e_p(-cp'N) \sum_{\substack{b(\pi)^* \\ (b + c\pi p^{-1}, \pi) = 1}} e_\pi(\overline{bm_1} v)$$
$$\times S_{\mathbf{d}}\left(\pi, \left(b + c\frac{\pi}{p}\right) m_2^2, \mathbf{n}\right) S_{\mathbf{t}}(\pi, -bm_1^2, -\mathbf{l}).$$

Using the trivial bound for summations over c and b, we get

$$|\mathcal{R}(\delta, p, \pi)| \leq p\pi^4.$$

When $\delta = p^2$, we have $(bp + c\pi, \pi p) = \delta$ if and only if $\pi = p$ and $b + c \equiv 0 \pmod p$. Therefore, one has

$$\begin{aligned}\mathcal{R}(\delta, p, \pi) &= \sum_{c(p)^*} \left(\frac{c}{p}\right) e_p(-cp'N) \sum_{\substack{b(p)^* \\ b+c\equiv 0 \pmod p}} e_p(\overline{bm_1}v) S_{\mathbf{t}}(p, -bm_1^2, -\mathbf{l}) \\ &= \sum_{c(p)^*} \left(\frac{c}{p}\right) e_p(-cp'N) e_p(\overline{-cm_1}v) S_{\mathbf{t}}(p, cm_1^2, -\mathbf{l}) \\ &= S^3(p,1) \sum_{c(p)^*} e_p(-cp'N) e_p(\overline{-cm_1}v) e_p\left(-\overline{4cm_1^2} \sum_{j=1}^{3} l_j^2 \overline{t_j^2}\right).\end{aligned}$$

Then we get

$$|\mathcal{R}(\delta, p, \pi)| \le 2p^2.$$

This completes the proof. □

In view of Lemma 8, we define

$$\xi(r, \eta, \mathbf{d}, \mathbf{t}) = \begin{cases} (r,\eta)(r,\mathbf{d})(r,\mathbf{t}) & \text{if } \eta \neq 0 \text{ and } (\mathbf{n},\mathbf{l}) \neq \mathbf{0}, \\ r(r,\mathbf{d}^2)^{1/2}(r,\mathbf{t}^2)^{1/2} & \text{otherwise.} \end{cases}$$

We summarize from Lemmas 8–10 that

Lemma 11. *Suppose that $q = 2^u \pi\pi' r$ with $(2pp', r) = 1$. Then we have*

$$\begin{aligned}&|W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', v)| \\ &\quad \le 2^{4u+6} r^3 \xi(r, \eta, \mathbf{d}, \mathbf{t}) \min\{\Delta^{-2} R^4 (4\pi\pi')^5 (1 + \Delta R^{-2})^{-1}, 2^6 R^4 (\pi\pi')^4\}.\end{aligned}$$

The following estimate is a key ingredient in our proof.

Lemma 12. *Let $H > 0$. Define*

$$\mathcal{U} = \sum_{|v|\le P} \frac{1}{1+|v|} \sum_{\mathbf{d}} \sum_{\mathbf{t}} \sum_{\substack{(\mathbf{n},\mathbf{l})\neq\mathbf{0}\in\mathbb{Z}^6 \\ |n_j|\le d_j H P^{\varepsilon},\, |l_i|\le t_i P^{\varepsilon} \\ \eta(\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,b_1,b_2)=0}} \max\left\{\frac{|n_j|}{d_j H}, \frac{|l_i|}{t_i}\right\}^{-1}.$$

Suppose that $(b_1 b_2, 2d_1 d_2 d_3 t_1 t_2 t_3) = 1$. Then we have

$$\mathcal{U} \ll P^{\varepsilon} (D_1 D_2 D_3)^2 (1 + H).$$

Remark. Recalling (4.2), when $b_1 = b_2 = 1$ and $v = 0$, one has the diagonal contributions from $\mathbf{d} = \mathbf{t}$ and $\mathbf{n} = \mathbf{l}$. Therefore in the case $b_1 = b_2 = 1$ and $H = 1$, we have the lower bound $\mathcal{U} \gg (D_1 D_2 D_3)^2$ which coincides with the upper bound (up to P^{ε}).

Proof. We decompose $\mathcal{U}$ to get

$$\mathcal{U} \ll P^{\varepsilon} \sup_{\substack{N_j \le D_j H P^{\varepsilon} \\ L_i \le D_i P^{\varepsilon} \\ |v|\le P}} \mathcal{U}(N_1, N_2, N_3, L_1, L_2, L_3),$$

where

$$\mathcal{U}(N_1,N_2,N_3,L_1,L_2,L_3)=\sum_{\mathbf{d}}\sum_{\mathbf{t}}\sum_{\substack{\mathbf{n},\mathbf{l}\\ N_j\le|n_j|\le 2N_j,\,L_i\le|l_i|\le 2L_i\\ (\mathbf{n},\mathbf{l})\ne\mathbf{0}\in\mathbb{Z}^6\\ \eta:=\eta(\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,b_1,b_2)=0}}\max\left\{\frac{|n_j|}{d_jH},\frac{|l_i|}{t_i}\right\}^{-1}.$$

Without loss of generality, we assume either $L_1\ne 0$ or $N_1\ne 0$.

We first consider the case $L_1\ne 0$. Let

$$e_i=\prod_{\substack{p\mid t_i\\ p\nmid d_1d_2d_3t_1t_2t_3t_i^{-1}}}p.$$

Since e_i^2 is a divisor of $\eta-b_1^2b_2^2l_i^2/(t_i^2b_1^2)\,\mathrm{lcm}$ and $(b_1^2b_2^2t_i^{-2}\,\mathrm{lcm},e_i^2)=1$, we have $e_i^2\mid l_i^2$ and then $e_i\mid l_i$. Hence for fixed $\mathbf{d}$ and $\mathbf{t}$, there are at most $2L_1e_1^{-1}(1+2L_2e_2^{-1})(1+2L_3e_3^{-1})$ possible choices for $\mathbf{l}$. Clearly

$$\max\left\{\frac{|n_j|}{d_jH},\frac{|l_i|}{t_i}\right\}^{-1}\le t_1L_1^{-1}.$$

Now fix $\mathbf{d}$, $\mathbf{t}$ and $\mathbf{l}$, we claim there are at most $O(P^\varepsilon+HP^\varepsilon)$ possible choices for n_1. Suppose that (n_1,n_2,n_3) and (n_1',n_2',n_3') are two solutions for $\eta(\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,b_1,b_2)=0$. Then

$$\sum_{j=1}^{3}\frac{n_j^2-n_j'^2}{d_j^2}=0.$$

We write it in the form

$$\frac{[d_1,d_2,d_3]^2}{d_1^2}(n_1^2-n_1'^2)=-\frac{[d_1,d_2,d_3]^2}{d_2^2}(n_2^2-n_2'^2)-\frac{[d_1,d_2,d_3]^2}{d_3^2}(n_3^2-n_3'^2).$$

Let

$$k=\prod_{\substack{p\mid d_1\\ p\nmid d_2d_3}}p.$$

We see that $n_1^2\equiv n_1'^2\pmod{k^2}$. So we can find $K\mid k^2$ with $K\ge k$ such that either $K\mid n_1+n_1'$ or $K\mid n_1-n_1'$. We deal with $K\mid n_1-n_1'$, and the other case can be handled similarly. Suppose that $n_1-n_1'=Km$ for some m with $0\le|m|\le d_1HP^\varepsilon K^{-1}$. Note that $K\ge d_1P^{-\varepsilon}$ due to $K\ge k$ and the condition (2.8). Then the number of possible choices for m is $O((1+H)P^\varepsilon)$. The number of K satisfying $K\mid k^2$ is at most $O(P^\varepsilon)$. Thus there are at most $O((1+H)P^\varepsilon)$ choices for n_1. Then n_2,n_3 can be determined (up to at most P^ε choices) by $\mathbf{d},\mathbf{t},\mathbf{l},n_1$ due to the equation $\eta=0$. Hence

$$\begin{aligned}\mathcal{U}(N_1,N_2,N_3,L_1,L_2,L_3)&\ll\sum_{\mathbf{d}}\sum_{\mathbf{t}}\frac{L_1}{e_1}\left(1+\frac{L_2}{e_2}\right)\left(1+\frac{L_3}{e_3}\right)\frac{t_1}{L_1}(1+H)P^\varepsilon\\ &\ll\sum_{\mathbf{d}}\sum_{\mathbf{t}}\prod_{i=1}^{3}\frac{t_i}{e_i}(1+H)P^\varepsilon.\end{aligned}$$

Note that

$$t_i \le e_i \prod_{j=1}^{3} (d_j, t_i)(t_1, t_2)(t_1, t_3)(t_2, t_3) \ll e_i \prod_{j=1}^{3} (d_j, t_i) P^{\varepsilon},$$

so

$$\begin{aligned}\mathcal{U}(N_1, N_2, N_3, L_1, L_2, L_3) &\ll \sum_{\mathbf{d}} \sum_{\mathbf{t}} \prod_{i=1}^{3} \prod_{j=1}^{3} (d_j, t_i)(1 + H) P^{\varepsilon} \\ &\ll (D_1 D_2 D_3)^2 (1 + H) P^{\varepsilon}.\end{aligned}$$

Now we assume $N_1 \ne 0$ and $\mathbf{l} = \mathbf{0}$. Let

$$h = \prod_{\substack{p \mid d_1 \\ p \nmid d_2 d_3 t_1 t_2 t_3}} p.$$

The similar argument as before implies $h \mid n_1$. So there are at most $4N_1 h^{-1}$ choices for n_1. Then n_2, n_3 will be determined. We arrive at

$$\begin{aligned}\mathcal{U}(N_1, N_2, N_3, L_1, L_2, L_3) &\ll \sum_{\mathbf{d}} \sum_{\mathbf{t}} (N_1 h^{-1}) d_1 H N_1^{-1} \\ &\ll \sum_{\mathbf{d}} \sum_{\mathbf{t}} \prod_{i} (d_1, t_i) H P^{\varepsilon} \\ &\ll (D_1 D_2 D_3)^2 H P^{\varepsilon}.\end{aligned}$$

The proof is completed. □

To handle the contribution from $\eta \ne 0$, we need the following.

Lemma 13. *Let*

$$\mathcal{V} = \sum_{|v| \le P} \frac{1}{1 + |v|} \sum_{\mathbf{d},\mathbf{t}} \sum_{r \le P} \frac{1}{r} \sum_{\substack{(\mathbf{n},\mathbf{l}) \ne \mathbf{0} \in \mathbb{Z}^6 \\ |n_j| \le d_j H P^{\varepsilon}, |l_i| \le t_i P^{\varepsilon} \\ (d_j^2, r) \mid n_j, (t_i^2, r) \mid l_i \\ \eta := \eta(\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,b_1,b_2) \ne 0}} (r, \eta)(r, \mathbf{d})(r, \mathbf{t}) \max\left\{\frac{n_j^2}{d_j^2 H^2}, \frac{l_i^2}{t_i^2}\right\}^{-1}$$

with $H > 0$. One has

$$\mathcal{V} \ll P^{\varepsilon} (D_1 D_2 D_3)^4 (1 + H)^3.$$

Proof. By changing variables, we get

$$\begin{aligned}\mathcal{V} &\le \sum_{|v| \le P} \frac{1}{1 + |v|} \sum_{\mathbf{d},\mathbf{t}} \sum_{\substack{h_j \mid d_j \\ k_i \mid t_i}} h_1 h_2 h_3 k_1 k_2 k_3 \sum_{\substack{(\mathbf{n},\mathbf{l}) \ne \mathbf{0} \in \mathbb{Z}^6 \\ |n_j| \le d_j H P^{\varepsilon} \\ |l_i| \le t_i P^{\varepsilon} \\ h_j \mid n_j, k_i \mid l_i \\ \eta \ne 0}} \max\left\{\frac{n_j^2}{d_j^2 H^2}, \frac{l_i^2}{t_i^2}\right\}^{-1} \sum_{r \le P} \frac{(r, \eta)}{r} \\ &\ll \sum_{\mathbf{d},\mathbf{t}} \sum_{h_j \mid d_j} \sum_{k_i \mid t_i} h_1 h_2 h_3 k_1 k_2 k_3 \sum_{\substack{(\mathbf{n},\mathbf{l}) \ne \mathbf{0} \in \mathbb{Z}^6 \\ |n_j| \le d_j H P^{\varepsilon} \\ |l_i| \le t_i P^{\varepsilon} \\ h_j \mid n_j, k_i \mid l_i}} \max\left\{\frac{n_j^2}{d_j^2 H^2}, \frac{l_i^2}{t_i^2}\right\}^{-1} P^{\varepsilon}.\end{aligned}$$

Without loss of generality, we assume

$$\max\left\{\frac{n_j^2}{d_j^2H^2}, \frac{l_i^2}{t_i^2}\right\} = \frac{n_1^2}{d_1^2H^2}.$$

First we handle the case that $(n_2, n_3, l_1, l_2, l_3) \in \mathbb{Z}^5$ is non-zero. We have

$$\sum_{h_j \mid n_j} 1 \le 1 + (|n_1| d_1^{-1} H^{-1}) d_j H h_j^{-1} \quad \text{for } 2 \le j \le 3$$

and

$$\sum_{k_i \mid l_i} 1 \le 1 + (|n_1| d_1^{-1} H^{-1}) t_i k_i^{-1} \quad \text{for } 1 \le i \le 3.$$

Since $(n_2, n_3, l_1, l_2, l_3) \in \mathbb{Z}^5$ is non-zero, at least one of the five inequalities above holds with 1 omitted. Hence

$$\begin{aligned}\mathcal{V} \ll \sum_{\mathbf{d},\mathbf{t}} \sum_{h_j \mid d_j} \sum_{k_i \mid t_i} h_1h_2h_3k_1k_2k_3 \sum_{\substack{h_1 \mid n_1 \\ 0<|n_1| \le d_1HP^{\varepsilon}}} \frac{d_1^2H^2P^{\varepsilon}}{n_1^2}\Bigg(\Big(1 + \frac{|n_1|d_2}{d_1h_2}\Big)\Big(1 + \frac{|n_1|d_3}{d_1h_3}\Big) \\ \times \Big(1 + \frac{|n_1|t_1}{d_1Hk_1}\Big)\Big(1 + \frac{|n_1|t_2}{d_1Hk_2}\Big)\Big(1 + \frac{|n_1|t_3}{d_1Hk_3}\Big) - 1\Bigg).\end{aligned}$$

A simple calculation reveals that

$$\mathcal{V} \ll P^{\varepsilon}(D_1D_2D_3)^4(1 + H)^3.$$

Now we turn to the case $n_2 = n_3 = l_1 = l_2 = l_3 = 0$. Note that η is independent of d_2, d_3 and $\mathbf{t}$ in this case. Switching the summations, we arrive at

$$\begin{aligned}\mathcal{V} &\ll \sum_{|v| \le P} \frac{1}{1 + |v|} \sum_{d_1} \sum_{r \le P} \frac{1}{r} \sum_{\substack{0<|n_1| \le d_1HP^{\varepsilon} \\ (r,d_1) \mid n_1 \\ \eta \ne 0}} \frac{d_1^2H^2(r,\eta)(r,d_1)}{n_1^2} \sum_{d_2,d_3,\mathbf{t}} (r,d_2)(r,d_3)(r,\mathbf{t}) \\ &\ll \sum_{|v| \le P} \frac{1}{1 + |v|} \sum_{d_1} \sum_{r \le P} \frac{1}{r} \sum_{\substack{0<|n_1| \le d_1HP^{\varepsilon} \\ \eta \ne 0}} d_1^2H^2(r,\eta)n_1^{-1}D_1(D_2D_3)^2P^{\varepsilon}.\end{aligned}$$

Hence we easily obtain $\mathcal{V} \ll D_1^4D_2^2D_3^2H^2P^{\varepsilon}$. The proof is thus completed. □

For $H > 0$, we define

$$\mathcal{N}_{\mathbf{d},q}(H) = \{\mathbf{n} \in \mathbb{Z}^3 : |n_i| \le d_iHP^{\varepsilon},\ n_i \equiv 0 \ (\mathrm{mod}\ (q, d_i^2)),\ i = 1,2,3\},$$

and write $\mathcal{N}_{\mathbf{d},q} = \mathcal{N}_{\mathbf{d},q}(1)$. Let

$$\begin{aligned}\mathcal{Y}_1 = \sum_{|v| \le P} \frac{1}{1 + |v|} \sum_{\mathbf{d},\mathbf{t}} \sum_{\Delta \mid (pp')^2} \sum_{q \le P} \frac{\Delta^3}{q^6} \\ \times \sum_{\substack{\mathbf{n} \in \mathcal{N}_{q,\mathbf{d}}(pp'/\Delta) \\ \mathbf{l} \in \mathcal{N}_{q,\mathbf{t}}}} |W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', v)| \mathcal{C}(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, pp'\Delta^{-1}),\end{aligned}$$

where

$$\mathcal{C}(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},H)=\begin{cases}\min & \text{if } (\mathbf{n},\mathbf{l})\neq\mathbf{0},\\ q^2P^{-2} & \text{if } (\mathbf{n},\mathbf{l})=\mathbf{0},\end{cases} \tag{4.6}$$

and

$$\min=\min\left\{\frac{q^2}{P^2}\left(\sum_{j=1}^{3}\left(\frac{n_j^2}{d_j^2H^2}+\frac{l_i^2}{t_i^2}\right)\right)^{-1},\frac{q}{P}\left(\sum_{j=1}^{3}\left(\frac{|n_j|}{d_jH}+\frac{|l_i|}{t_i}\right)\right)^{-1}\right\}.$$

Let

$$\mathcal{Y}_2=\sum_{|v|\le P}\frac{1}{1+|v|}\sum_{\mathbf{d}}\sum_{\mathbf{t}}\sum_{q\le P}\frac{1}{q^6}\sum_{\substack{(\mathbf{n},\mathbf{l})\neq\mathbf{0}\in\mathbb{Z}^6\\ \mathbf{n}\in\mathcal{N}_{q,\mathbf{d}}\\ \mathbf{l}\in\mathcal{N}_{q,\mathbf{t}}}}|\theta(q;\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,1,1)|$$
$$\times\min\left\{\frac{q^2}{P^2}\left(\sum_{j=1}^{3}\left(\frac{n_j^2}{d_j^2}+\frac{l_i^2}{t_i^2}\right)\right)^{-1},\frac{q}{P}\left(\sum_{j=1}^{3}\left(\frac{|n_j|}{d_j}+\frac{|l_i|}{t_i}\right)\right)^{-1}\right\}.$$

With the help of Lemmas 8–13, we establish the following result.

Lemma 14. *Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be defined as above. Then one has*

$$\mathcal{Y}_1\ll(D_1D_2D_3)^2P^{\varepsilon-1}R^6+(D_1D_2D_3)^4P^{-2+\varepsilon}R^{10},$$
$$\mathcal{Y}_2\ll(D_1D_2D_3)^2P^{-1+\varepsilon}+(D_1D_2D_3)^4P^{-2+\varepsilon}.$$

Proof. The proofs for $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are similar. Indeed, the argument for $\mathcal{Y}_2$ is easier, because there is no R involved. We only work out the details for $\mathcal{Y}_1$.

We write $\mathcal{Y}_1=\mathcal{Y}_1^{(1)}+\mathcal{Y}_1^{(2)}+\mathcal{Y}_1^{(3)}$, where $\mathcal{Y}_1^{(1)}$ is the contribution from $(\mathbf{n},\mathbf{l})=\mathbf{0}$, $\mathcal{Y}_1^{(2)}$ is the contribution from those terms with $(\mathbf{n},\mathbf{l})\neq\mathbf{0}$ but $\eta:=\eta(\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,\pi\pi',\pi\pi'pp'/\Delta)=0$, and $\mathcal{Y}_1^{(3)}$ is the contribution from the remaining terms. By Lemma 11, when $(\mathbf{n},\mathbf{l})=\mathbf{0}$ or $\eta=0$,

$$\begin{aligned}&W(\Delta,q;\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},p,p',v)\\&\quad\ll 2^{4u}r^4(r,\mathbf{d}^2)^{1/2}(r,\mathbf{t}^2)^{1/2}\Delta^{-2}R^4(\pi\pi')^5(1+\Delta R^{-2})^{-1},\end{aligned} \tag{4.7}$$

where we have used the decomposition $q=2^u\pi\pi'r$ with $(2pp',r)=1$. On applying the estimate (4.7), we get

$$\mathcal{Y}_1^{(1)}\ll P^{-2}R^4\sum_{|v|\le P}\frac{1}{1+|v|}\sum_{\mathbf{d},\mathbf{t}}\sum_{\Delta\,|\,(pp')^2}\sum_{2^u\pi\pi'r\le P}\frac{\Delta\pi\pi'(r,\mathbf{d}^2)^{1/2}(r,\mathbf{t}^2)^{1/2}}{1+\Delta R^{-2}}.$$

Recalling the condition (2.8), we obtain

$$\begin{aligned}\sum_{r\le\frac{P}{2^u\pi\pi'}}(r,\mathbf{d}^2)^{1/2}(r,\mathbf{t}^2)^{1/2}&\le\Bigg(\sum_{r\le\frac{P}{2^u\pi\pi'}}(r,\mathbf{d}^2)\Bigg)^{1/2}\Bigg(\sum_{r\le\frac{P}{2^u\pi\pi'}}(r,\mathbf{t}^2)\Bigg)^{1/2}\\&\ll P^{\varepsilon}\Bigg(\sum_{r\le\frac{P}{2^u\pi\pi'}}(r,d_1^2d_2^2d_3^2)\Bigg)^{1/2}\Bigg(\sum_{r\le\frac{P}{2^u\pi\pi'}}(r,t_1^2t_2^2t_3^2)\Bigg)^{1/2}\\&\ll\frac{P^{1+\varepsilon}}{2^u\pi\pi'}.\end{aligned}$$

Then one can get

$$\sum_{2^u\pi\pi' r\le P} \pi\pi'(r,\mathbf{d}^2)^{1/2}(r,\mathbf{t}^2)^{1/2} \ll P^{1+\varepsilon}.$$

It is easy to see

$$\sum_{\Delta\,|\,(pp')^2} \frac{\Delta}{1+\Delta R^{-2}} \ll R^2.$$

We can now conclude that

$$\mathcal{Y}_1^{(1)} \ll (D_1D_2D_3)^2 P^{\varepsilon-1}R^6.$$

Combining (4.7) and

$$\mathcal{C}\Big(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},\frac{pp'}{\Delta}\Big) \ll \frac{q}{P}\max\Big\{\frac{|n_j|}{d_j pp'/\Delta},\frac{|l_i|}{t_i}\Big\}^{-1},$$

one can deduce that

$$\mathcal{Y}_1^{(2)} \ll P^{-1}R^4 \sum_{|v|\le P}\frac{1}{1+|v|}\sum_{\mathbf{d},\mathbf{t}}\sum_{\Delta\,|\,(pp')^2}\sum_{2^u\pi\pi' r\le P}\sum_{\substack{(\mathbf{n},\mathbf{l})\ne\mathbf{0}\in\mathbb{Z}^6\\ |n_j|\le d_jP^\varepsilon pp'/\Delta\\ |l_i|\le t_iP^\varepsilon\\ \eta=0}}\frac{\Delta}{2^u r(1+\Delta R^{-2})}$$
$$\times (r,\mathbf{d}^2)^{1/2}(r,\mathbf{t}^2)^{1/2}\max\Big\{\frac{|n_j|}{d_j pp'/\Delta},\frac{|l_i|}{t_i}\Big\}^{-1}.$$

Note that

$$\sum_{2^u\pi\pi' r\le P}\frac{1}{2^u r}(r,\mathbf{d}^2)^{1/2}(r,\mathbf{t}^2)^{1/2} \ll P^\varepsilon,$$

we have

$$\mathcal{Y}_1^{(2)} \ll P^{-1+\varepsilon}R^4\sum_{\Delta\,|\,(pp')^2}\frac{\Delta}{1+\Delta R^{-2}}\sum_{|v|\le P}\frac{1}{1+|v|}$$
$$\times\sum_{\mathbf{d},\mathbf{t}}\sum_{\substack{(\mathbf{n},\mathbf{l})\ne\mathbf{0}\in\mathbb{Z}^6\\ |n_j|\le d_jP^\varepsilon pp'/\Delta\\ |l_i|\le t_iP^\varepsilon\\ \eta=0}}\max\Big\{\frac{|n_j|}{d_j pp'/\Delta},\frac{|l_i|}{t_i}\Big\}^{-1}.$$

Then one can conclude, by appealing to Lemma 12, that

$$\mathcal{Y}_1^{(2)} \ll P^{-1+\varepsilon}R^4\sum_{\Delta\,|\,(pp')^2}\frac{\Delta}{1+\Delta R^{-2}}(D_1D_2D_3)^2\Big(1+\frac{pp'}{\Delta}\Big)$$
$$\ll (D_1D_2D_3)^2P^{\varepsilon-1}R^6.$$

For $\mathcal{Y}_1^{(3)}$ we shall apply

$$W(\Delta,q;\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},p,p',v) \ll 2^{4u}r^3(r,\eta)(r,\mathbf{d})(r,\mathbf{t})R^4(\pi\pi')^4$$

by Lemma 11. Note that $\mathcal{N}_{q,\mathbf{d}}(pp'/\Delta)=\mathcal{N}_{r,\mathbf{d}}(pp'/\Delta)$ for $q=2^u\pi\pi' r$, and

$$\mathcal{C}\Big(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},\frac{pp'}{\Delta}\Big) \ll \frac{q^2}{P^2}\max\Big\{\frac{n_j^2}{d_j^2(pp'/\Delta)^2},\frac{l_i^2}{t_i^2}\Big\}^{-1}.$$

Thus, one has

$$\mathcal{Y}_1^{(3)} \ll P^{-2}R^4 \sum_{\Delta \mid (pp')^2} \Delta \sum_{|v|\le P} \frac{1}{1+|v|} \sum_{2^u\pi\pi' r\le P} \frac{1}{r} \times \sum_{\mathbf{d},\mathbf{t}} \sum_{\substack{(\mathbf{n},\mathbf{l})\neq \mathbf{0}\in\mathbb{Z}^6\\ \mathbf{n}\in\mathcal{N}_{r,\mathbf{d}}(\Delta/(pp'))\\ \mathbf{l}\in\mathcal{N}_{r,\mathbf{t}}\\ \eta\neq 0}} (r,\eta)(r,\mathbf{d})(r,\mathbf{t}) \max\left\{\frac{n_j^2}{d_j^2(pp'/\Delta)^2}, \frac{l_i^2}{t_i^2}\right\}^{-1}.$$

By Lemma 13, we have

$$\mathcal{Y}_1^{(3)} \ll P^{-2+\varepsilon}R^4 \sum_{\Delta \mid (pp')^2} \Delta (D_1D_2D_3)^4\left(1+\frac{pp'}{\Delta}\right)^3 \ll (D_1D_2D_3)^4 P^{\varepsilon-2}R^{10}.$$

The assertion is established. □

Lemma 15. *Let*

$$\mathcal{F} = \sum_{0<|l|\le R^2Q^2P^{\varepsilon-2}} \left(\frac{l}{pp'}\right) \times \sum_{\substack{q_1,q_2\\ (q_1q_2,pp')=1\\ |l|(pp')^{-1}P^{2-\varepsilon}<q_1q_2}} (q_1q_2)^{-3}\left(\frac{q_1q_2}{pp'}\right) e_{pp'}(-\overline{q_1q_2}lN)e\left(\frac{ln}{q_1q_2pp'}\right) \times \sum_{\substack{a_1(q_1)^*,\,a_2(q_2)^*\\ pp'(a_1q_2+a_2q_1)\equiv l \pmod{q_1q_2}}} S_{\mathbf{d}}(q_1,-a_1)S_{\mathbf{t}}(q_2,-a_2).$$

Then we have

$$\mathcal{F} \ll P^{-4+\varepsilon}Q^3R^2 + P^{-4+\varepsilon}Q^{14/5}R^{24/5}.$$

Proof. We modify the argument of Heath-Brown and Tolev [9]. Let

$$q_j = g_jb_j, \quad 1\le j\le 2,$$

where

$$g_1 = \prod_{\substack{p^k \parallel q_1\\ p\nmid 2d_1d_2d_3(q_1,q_2)}} p^k \quad\text{and}\quad g_2 = \prod_{\substack{p^k \parallel q_2\\ p\nmid 2t_1t_2t_3(q_1,q_2)}} p^k.$$

Then

$$(g_1,g_2) = (g_1g_2, b_1b_2) = (g_1, 2d_1d_2d_3) = (g_2, 2t_1t_2t_3) = 1.$$

Let $a_j = \alpha_jb_j + \beta_jg_j$ for $j=1,2$. Then

$$pp'(a_1q_2+a_2q_1) \equiv l \pmod{q_1q_2}$$

is equivalent to the three conditions

$$pp'\alpha_1b_1b_2g_2 \equiv l \pmod{g_1},$$
$$pp'\alpha_2b_1b_2g_1 \equiv l \pmod{g_2},$$
$$pp'g_1g_2(\beta_1b_2+\beta_2b_1) \equiv l \pmod{b_1b_2}.$$

Hence by Lemma 1 (i), the inner multiple sum over a_1, a_2 in $\mathscr{F}$ is equal to

$$\left(\frac{-pp'b_1b_2g_2l}{g_1}\right)\left(\frac{-pp'b_1b_2g_1l}{g_2}\right)S^3(g_1,1)S^3(g_2,1)\Xi,$$

where

$$\Xi = \Xi(pp'g_1g_2) = \sum_{\substack{\beta_1(b_1)^*,\,\beta_2(b_2)^* \\ pp'g_1g_2(\beta_1b_2+\beta_2b_1)\equiv l \pmod{b_1b_2}}} S_{\mathbf{d}}(b_1,-\beta_1)S_{\mathbf{t}}(b_2,-\beta_2).$$

Let $b_j = B_j\Delta$ for $j = 1,2$ with $\Delta = (b_1,b_2)$. Then $\Xi = 0$ if $\Delta \nmid l$, and for $l = \Delta v$ we have

$$\Xi(pp'g_1g_2) = \sum_{\substack{\beta_1(B_1\Delta)^*,\,\beta_2(B_2\Delta)^* \\ pp'g_1g_2(\beta_1B_2+\beta_2B_1)\equiv v \pmod{B_1B_2\Delta}}} S_{\mathbf{d}}(B_1\Delta,\beta_1)S_{\mathbf{t}}(B_2\Delta,\beta_2).$$

In order to change variables, we introduce the conditions

$$(4.8)\qquad \begin{cases} (g_1,g_2) = (g_1g_2, B_1B_2\Delta) = (B_1,B_2) = 1, \\ (g_1, 2d_1d_2d_3) = (g_2, 2t_1t_2t_3) = 1, \\ \rho_{\mathbf{d}}(B_1\Delta) = \rho_{\mathbf{d}}(\Delta), \quad \rho_{\mathbf{t}}(B_2\Delta) = \rho_{\mathbf{t}}(\Delta), \\ (B_1B_2\Delta g_1g_2, pp') = 1, \quad |v|(pp')^{-1}P^{2-\varepsilon} < B_1B_2\Delta g_1g_2, \end{cases}$$

where for $\mathbf{k} = (k_1,k_2,k_3)$ we use the notation

$$\rho_{\mathbf{k}}(m) = \prod_{\substack{p\,|\,m \\ p\,\nmid\,2k_1k_2k_3}} p.$$

By changing variables we arrive at

$$\mathscr{F} = \sum_{\substack{v,\Delta \\ 0<|v\Delta|\le R^2Q^2P^{\varepsilon-2}}} \left(\frac{v\Delta}{pp'}\right) \sum_{B_1,B_2} (B_1B_2)^{-3}\Delta^{-6}\left(\frac{B_1B_2}{pp'}\right)$$
$$\times \sum_{\substack{g_1,g_2 \\ (4.8)}} (g_1g_2)^{-3}\left(\frac{g_1g_2}{pp'}\right) e_{pp'}(-\overline{g_1g_2B_1B_2\Delta}vN)\, e\left(\frac{vn}{g_1g_2B_1B_2\Delta pp'}\right)$$
$$\times \left(\frac{-pp'B_1B_2g_2\Delta v}{g_1}\right)\left(\frac{-pp'B_1B_2g_1\Delta v}{g_2}\right)S^3(g_1,1)S^3(g_2,1)\Xi(pp'g_1g_2).$$

Let

$$\xi(g_1,g_2) = e\left(\frac{vn}{g_1g_2B_1B_2\Delta pp'}\right)\left(\frac{g_1g_2}{pp'}\right)\left(\frac{-pp'g_2}{g_1}\right)\left(\frac{-pp'g_1}{g_2}\right)\frac{S^3(g_1,1)S^3(g_2,1)}{g_1^3g_2^3}.$$

Note that $\Xi(\lambda)$ depends on $\lambda \pmod{B_1B_2\Delta}$ only. Hence

$$\mathscr{F} = \sum_{\substack{v,\Delta \\ (v\Delta,pp')=1 \\ 0<|v\Delta|\le R^2Q^2P^{\varepsilon-2}}} \left(\frac{v}{pp'}\right) \sum_{B_1,B_2} (B_1B_2)^{-3}\Delta^{-6} \sum_{\lambda \pmod{8B_1B_2\Delta}} \Xi(\lambda)\left(\frac{B_1B_2\Delta}{\lambda}\right)$$
$$\times \sum_{\substack{g_1,g_2 \\ (4.8) \\ pp'g_1g_2\equiv\lambda \pmod{8B_1B_2\Delta}}} e_{pp'}(-\overline{g_1g_2B_1B_2\Delta}vN)\left(\frac{v}{g_1}\right)\left(\frac{v}{g_2}\right)\xi(g_1,g_2).$$

Moreover, in view of (4.8) and the condition

$$pp'g_1g_2 \equiv \lambda \ (\mathrm{mod}\ 8B_1B_2\Delta),$$

the summation $\sum_{\lambda \ (\mathrm{mod}\ 8B_1B_2\Delta)}$ can be replaced by $\sum_{\lambda \ (\mathrm{mod}\ 8B_1B_2\Delta)^*}$.

From Lemma 1 (ii), (iii), (iv), and (2.8),

$$\Xi(\lambda) \ll \Delta^4(B_1B_2)^{3/2}(\Delta B_1, \mathbf{d}^2)^{1/2}(\Delta B_2, \mathbf{t}^2)^{1/2} \ll \Delta^5(B_1B_2)^2P^{\varepsilon}.$$

By the dyadic argument, we have

$$\mathcal{F} \ll \log^2 P \sup_{\substack{G_1\le G_1'\le 2G_1\\ G_2\le G_2'\le 2G_2}} \mathcal{F}(G_1,G_2), \tag{4.9}$$

where

$$\mathcal{F}(G_1,G_2) = \sum_{\substack{v,\Delta\\ (v\Delta,pp')=1\\ 0<|v\Delta|\le R^2Q^2P^{\varepsilon-2}}} \sum_{B_1,B_2} (B_1B_2)^{-1}\Delta^{-1} \sum_{\lambda \ (\mathrm{mod}\ 8B_1B_2\Delta)^*}$$
$$\times \Bigg| \sum_{\substack{G_1\le g_1\le G_1', G_2\le g_2\le G_2'\\ (4.8)\\ pp'g_1g_2\equiv\lambda \ (\mathrm{mod}\ 8B_1B_2\Delta)}} e_{pp'}(-\overline{g_1g_2B_1B_2\Delta}vN)\left(\frac{v}{g_1}\right)\left(\frac{v}{g_2}\right)\xi(g_1,g_2)\Bigg|.$$

Without loss of generality, we assume $G_1 \le G_2$. In view of (4.8), the multiple summations over B_1, B_2 are naturally restricted by

$$\begin{cases} \rho_{\mathbf{d}}(B_1\Delta) = \rho_{\mathbf{d}}(\Delta), \quad \rho_{\mathbf{t}}(B_2\Delta) = \rho_{\mathbf{t}}(\Delta),\\ (B_1B_2, pp') = 1,\\ |v|R^{-2}P^{2-\varepsilon} \le 16B_1B_2\Delta G_1G_2. \end{cases} \tag{4.10}$$

In view of the congruence condition $pp'g_1g_2 \equiv \lambda \ (\mathrm{mod}\ 8B_1B_2\Delta)$, for fixed p, p', λ and g_1, we have

$$\xi(g_1,g_2) = e\left(\frac{vn}{g_1g_2B_1B_2\Delta pp'}\right)g_1^{-3/2}g_2^{-3/2}\epsilon,$$

where $|\epsilon| = 1$ and ϵ is independent of g_2. Partial summation gives

$$\mathcal{F}(G_1,G_2) \ll \sup_{\substack{G_2\le G,G'\le G_2'\\ G_1\le g_1\le G_1'}} G_1^{-1/2}G_2^{-3/2}P^{\varepsilon} \sum_{\substack{v,\Delta\\ (v\Delta,pp')=1\\ 0<|v\Delta|\le R^2Q^2P^{\varepsilon-2}}} \Delta^{-1} \sum_{\substack{B_1,B_2\\ (4.10)}} (B_1B_2)^{-1}$$
$$\times \sum_{\lambda \ (\mathrm{mod}\ 8B_1B_2\Delta)^*} \Bigg| \sum_{\substack{G\le g_2\le G'\\ (4.11)\\ pp'g_1g_2\equiv\lambda \ (\mathrm{mod}\ 8B_1B_2\Delta)}} e_{pp'}(-\overline{g_1g_2B_1B_2\Delta}vN)\left(\frac{v}{g_2}\right)\Bigg|,$$

where the condition (4.11) comprises

$$\begin{cases} (g_1,g_2) = (g_2, B_1B_2\Delta) = 1,\\ (g_2, 2t_1t_2t_3) = 1,\\ (g_2,pp') = 1, \quad |v|(pp')^{-1}P^{2-\varepsilon} < B_1B_2\Delta g_1g_2. \end{cases} \tag{4.11}$$

As in [9], we use two different estimates according to $G_2 \geq H$ or otherwise, where H will be chosen later. The summation over g_2 is of the type

$$\mathcal{M} = \sum_{\substack{K \leq g \leq K' \\ (g,u)=1 \\ g \equiv \lambda' \pmod{8B_1B_2\Delta}}} e_{pp'}(\overline{gm_1}m_2)\left(\frac{v}{g}\right), \tag{4.12}$$

where $G \leq K, K' \leq G'$ and $(m_1m_2, pp') = (2B_1B_2\Delta, \lambda') = 1$. We shall prove

$$\mathcal{M} \ll P^{\varepsilon} G_2 (B_1B_2\Delta pp')^{-1} + |v|R^2P^{\varepsilon}. \tag{4.13}$$

With (4.13), we can get

$$\begin{aligned}\mathcal{F}(G_1, G_2) &\ll G_1^{-1/2}G_2^{-3/2}P^{\varepsilon} \sum_{\substack{v,\Delta \\ 0<|v\Delta| \leq R^2Q^2P^{\varepsilon-2}}} \sum_{\substack{B_1,B_2 \\ (4.10)}} (G_2(B_1B_2\Delta pp')^{-1} + |v|R^2) \\ &\ll G_1^{-1/2}G_2^{-3/2}P^{\varepsilon} \sum_{\substack{v,\Delta \\ 0<|v\Delta| \leq R^2Q^2P^{\varepsilon-2}}} (|v|^{-1}P^{-2}G_1G_2^2 + |v|R^2) \sum_{\substack{B_1,B_2 \\ (4.10)}} 1.\end{aligned}$$

Note that

$$\sum_{\substack{B_1,B_2 \\ (4.10)}} 1 \leq \sum_{\substack{B_1 \leq P \\ p \mid B_1 \Rightarrow p \mid 2d_1d_2d_3\Delta}} \sum_{\substack{B_2 \leq P \\ p \mid B_2 \Rightarrow p \mid 2t_1t_2t_3\Delta}} 1 \ll P^{\varepsilon}. \tag{4.14}$$

We now conclude

$$\begin{aligned}\mathcal{F}(G_1, G_2) &\ll P^{-4+\varepsilon}R^2Q^2G_1^{1/2}G_2^{1/2} + G_1^{-1/2}G_2^{-3/2}R^6Q^4P^{-4+\varepsilon} \\ &\ll P^{-4+\varepsilon}R^2Q^3 + R^6Q^4P^{-4+\varepsilon}H^{-3/2}.\end{aligned}$$

To prove (4.13), we first remove the restriction $(g, u) = 1$, getting

$$\mathcal{M} = \sum_{w|u} \mu(w)\left(\frac{v}{w}\right) \sum_{\substack{Kw^{-1} \leq g \leq K'w^{-1} \\ gw \equiv \lambda' \pmod{8B_1B_2\Delta}}} e_{pp'}(\overline{gwm_1}m_2)\left(\frac{v}{g}\right).$$

We divide the inner summation into $O(G_2w^{-1}(8B_1B_2\Delta pp'|v|)^{-1})$ complete sums and at most one incomplete sum

$$\mathcal{M} \ll \sum_{w \mid u} (G_2w^{-1}(8B_1B_2\Delta pp'|v|)^{-1}|\mathcal{M}_0| + O(|v|R^2)), \tag{4.15}$$

where

$$\mathcal{M}_0 = \sum_{\substack{g \pmod{8B_1B_2\Delta pp'|v|} \\ gw \equiv \lambda' \pmod{8B_1B_2\Delta}}} e_{pp'}(\overline{gwm_1}m_2)\left(\frac{v}{g}\right)$$

and the error $O(|v|R^2)$ is the contribution from the incomplete sum. Recalling the condition $(B_1B_2\Delta v, pp') = 1$, we change variables, by $g = s8B_1B_2\Delta|v| + kpp'$, to deduce that

$$\mathcal{M}_0 = \sum_{\substack{k \pmod{8B_1B_2\Delta|v|} \\ kpp'w \equiv \lambda' \pmod{8B_1B_2\Delta}}} \left(\frac{v}{kpp'}\right) \sum_{s \pmod{pp'}^*} e_{pp'}(\overline{s8B_1B_2\Delta|v|wm_1}m_2).$$

The inner sum is a Ramanujan sum, so $\mathcal{M}_0 \ll |v|$. Now (4.13) follows from (4.15).

When $G_2 < H$, we apply the trivial bound to get

$$\mathcal{F}(G_1, G_2) \ll \sup_{G,G'} G_1^{-1/2} G_2^{-3/2} \sum_{\substack{v,\Delta \\ 0<|v\Delta|\le R^2Q^2P^{\varepsilon-2}}} \Delta^{-1} \times \sum_{\substack{B_1,B_2 \\ (4.10)}} (|v|R^{-2}P^{2-\varepsilon}(\Delta G_1 G_2)^{-1})^{-1} G_2.$$

Recalling (4.14), we arrive at

$$\mathcal{F}(G_1, G_2) \ll P^{-4+\varepsilon} R^4 Q^2 G_1^{1/2} G_2^{1/2} \ll P^{-4+\varepsilon} R^4 Q^2 H. \tag{4.16}$$

We choose

$$H = R^{4/5} Q^{4/5},$$

by equating $R^6 Q^4 P^{-4} H^{-3/2} = P^{-4} R^4 Q^2 H$, to conclude finally

$$\mathcal{F}(G_1, G_2) \ll P^{-4+\varepsilon} Q^3 R^2 + P^{-4+\varepsilon} Q^{14/5} R^{24/5}. \tag{4.17}$$

The proof is completed by putting (4.17) into (4.9). □

5. Invoking the square sieve

By the dyadic argument, our task is to prove

$$\sum_{k\le P} \Bigg| \sum_{D_i'\le d_i<2D_i'\ (1\le i\le 3)} \beta(\mathbf{d})(\Omega_{\mathbf{d}}(N-k^2) - \mathcal{M}_{\mathbf{d},Q}(N-k^2)) \Bigg| \ll P^{2-\varepsilon}. \tag{5.1}$$

Suppose that $\kappa \in (0, 1/2)$. Set $m = 4\lceil \kappa^{-1} \rceil$, and denote by $\mathcal{R}_j$ the interval $[2^{j-1}P^\kappa, 2^j P^\kappa)$ for $1 \le j \le m$. Let $\pi(\mathcal{R}) = \prod_{p\in\mathcal{R}} p$. We have the partitions

$$[D_i', 2D_i') = \bigsqcup_{J\subseteq\{1,\dots,m\}} \mathcal{D}_J,$$

where $\mathcal{D}_J = \{D_i' \le d_i < 2D_i' : (d_i, \pi(\mathcal{R}_j)) \ne 1 \iff j \in J\}$. Hence (5.1) can be deduced from

$$\sum_{k\le P} \Bigg| \sum_{d_i\in\mathcal{D}_{J_i}\ (1\le i\le 3)} \beta(\mathbf{d})(\Omega_{\mathbf{d}}(N-k^2) - \mathcal{M}_{\mathbf{d},Q}(N-k^2)) \Bigg| \ll P^{2-\varepsilon}.$$

By Cauchy's inequality, it suffices to prove

$$\sum_{k\le P} \Bigg| \sum_{d_i\in\mathcal{D}_{J_i}\ (1\le i\le 3)} \beta(\mathbf{d})(\Omega_{\mathbf{d}}(N-k^2) - \mathcal{M}_{\mathbf{d},Q}(N-k^2)) \Bigg|^2 \ll P^{3-\varepsilon}. \tag{5.2}$$

Note that $D_i' \le D_i \le P^{1/2} < P$. So if $\mathcal{D}_J$ is non-empty, then $|J| \le m/4$. Therefore for any triple $\mathcal{D}_{J_1}, \mathcal{D}_{J_2}, \mathcal{D}_{J_3}$ satisfying $\mathcal{D}_{J_i} \ne \emptyset$ $(1 \le i \le 3)$, there exists $1 \le j \le m$ such that $j \notin J_1 \cup J_2 \cup J_3$. In other words, we have $(p, d_i) = 1$ for all $p \in \mathcal{R}_j$ and $d_i \in \mathcal{D}_{J_i}$.

In view of the square sieve of Heath-Brown [8] and the above preparations, Proposition 4 can be reduced to the estimate

$$\mathcal{E}_0 = \sum_{1\le n<N}\Big|\sum_{\mathbf{d}}\beta(\mathbf{d})(\Omega_{\mathbf{d}}(n)-\mathcal{M}_{\mathbf{d},Q}(n))\Big|^2\bigg(\frac{\log R}{R}\sum_{\substack{p\nmid N\\ R\le p<2R}}\Big(\frac{N-n}{p}\Big)\bigg)^2 \tag{5.3}$$
$$\ll P^{4+\varepsilon}R^{-1}Q^{-1}+(D_1D_2D_3)^2P^{2+\varepsilon}R^3+P^{\varepsilon}Q^{14/5}R^{29/5},$$

where $\beta(\mathbf{d})$ is supported on $d_i\in\mathcal{D}_i\subseteq[D_i,2D_i)$, $1\le i\le 3$, satisfying (2.6), (2.7), (2.8), and

$$(d_1d_2d_3,p)=1\quad\text{for all } d_i\in\mathcal{D}_i,\ R\le p<2R,$$

and R is restricted by

$$P^{1+\varepsilon}Q^{-1}<R<P^{2-\varepsilon}Q^{-2}. \tag{5.4}$$

Although our choices will be of type $Q^{1-\varepsilon}$ and $R=P^{\varepsilon}$, there is really some space for the parameters Q and R so that $\mathcal{E}_0\ll P^{3-\varepsilon}$ and (5.4) hold simultaneously.

6. Proof of Proposition 4

We start to estimate $\mathcal{E}_0$ which is bounded by

$$\mathcal{E}_0\ll\frac{\log^2R}{R^2}|\mathcal{E}_1|+\frac{\log R}{R}\mathcal{E}_2, \tag{6.1}$$

where

$$\mathcal{E}_1=\sum_{\mathcal{R}}\Big(\frac{N-n}{pp'}\Big)\sum_{1\le n<N}\Big|\sum_{\mathbf{d}}\beta(\mathbf{d})\Big(\Omega_{\mathbf{d}}(n)-\mathcal{M}_{\mathbf{d},Q}(n)\Big)\Big|^2$$

and

$$\mathcal{E}_2=\sum_{1\le n<N}\Big|\sum_{\mathbf{d}}\beta(\mathbf{d})\Big(\Omega_{\mathbf{d}}(n)-\mathcal{M}_{\mathbf{d},Q}(n)\Big)\Big|^2.$$

Throughout, we use $\sum_{\mathcal{R}}$ to indicate that the summation is taken over prime numbers p,p' with $R\le p\ne p'<2R$ and $(pp',N)=1$.

We expand the square to obtain

$$\mathcal{E}_2=\mathcal{E}_2^{(1)}-2\mathcal{E}_2^{(2)}+\mathcal{E}_2^{(3)}, \tag{6.2}$$

where

$$\mathcal{E}_2^{(i)}=\sum_{\mathbf{d}}\sum_{\mathbf{t}}\beta(\mathbf{d})\beta(\mathbf{t})J_2^{(i)} \tag{6.3}$$

with

$$J_2^{(1)}=\int_0^1 f_{\mathbf{d}}(\alpha)f_{\mathbf{t}}(-\alpha)d\alpha,$$
$$J_2^{(2)}=\int_0^1 f_{\mathbf{d}}(\alpha)\mathcal{W}_{\mathbf{t},Q}(-\alpha)d\alpha,$$
$$J_2^{(3)}=\int_0^1 \mathcal{W}_{\mathbf{d},Q}(\alpha)\mathcal{W}_{\mathbf{t},Q}(-\alpha)d\alpha.$$

Let

$$\mathcal{T}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l}) = \sum_{a(q)^*} \int_{\mathcal{B}(q,a)} S_{\mathbf{d}}(q,a,\mathbf{n}) S_{\mathbf{t}}(q,-a,-\mathbf{l}) I_{\mathbf{d}}\Big(\beta N, -\frac{P}{q}\mathbf{n}\Big) I_{\mathbf{t}}\Big(-\beta N, \frac{P}{q}\mathbf{l}\Big)\, d\beta,$$

where

$$\mathcal{B}(q,a) = \Big[-\frac{1}{q(q+q')}, \frac{1}{q(q+q'')}\Big]$$

with q' and q'' satisfying

$$P < q+q', q+q'' \le q+P, \quad aq' \equiv 1 \ (\mathrm{mod}\ q), \quad aq'' \equiv -1 \ (\mathrm{mod}\ q).$$

Set

$$\mathcal{T}_2(q,\mathbf{d},\mathbf{t}) = \mathcal{T}_2(q,\mathbf{d},\mathbf{t},\mathbf{0},\mathbf{0}).$$

By Lemma 6, we have

$$\begin{aligned} J_2^{(3)} &= \sum_{q\le P} \sum_{a(q)^*} \int_{\beta\in\mathcal{B}(q,a)} \mathcal{W}_{\mathbf{d},Q}\Big(\frac{a}{q}+\beta\Big) \mathcal{W}_{\mathbf{t},Q}\Big(-\frac{a}{q}-\beta\Big)\, d\beta \\ &= \frac{P^6}{d_1d_2d_3t_1t_2t_3} \sum_{q\le Q} \frac{1}{q^6} \mathcal{T}_2(q,\mathbf{d},\mathbf{t}) + O(P^{-A}). \end{aligned}$$

So

$$\mathcal{E}_2^{(3)} = \mathcal{E}_2'(Q) + O(P^{-A}), \tag{6.4}$$

where

$$\mathcal{E}_2'(K) = P^6 \sum_{\mathbf{d}} \sum_{\mathbf{t}} \frac{\beta(\mathbf{d})\beta(\mathbf{t})}{d_1d_2d_3t_1t_2t_3} \sum_{q\le K} \frac{1}{q^6} \mathcal{T}_2(q,\mathbf{d},\mathbf{t}). \tag{6.5}$$

According to Lemma 5, one has

$$J_2^{(1)} = \frac{P^6}{d_1d_2d_3t_1t_2t_3} \sum_{q\le P} \frac{1}{q^6} \sum_{\mathbf{n}\in\mathcal{N}_{\mathbf{d},q}} \sum_{\mathbf{l}\in\mathcal{N}_{\mathbf{t},q}} \mathcal{T}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l}) + O(P^{-A}). \tag{6.6}$$

Combining Lemma 5 and Lemma 6, we derive

$$J_2^{(2)} = \frac{P^6}{d_1d_2d_3t_1t_2t_3} \sum_{q\le Q} \frac{1}{q^6} \sum_{\mathbf{n}\in\mathcal{N}_{\mathbf{d},q}} \mathcal{T}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{0}) + O(P^{-A}). \tag{6.7}$$

We exchange the summation over a and the integration by the standard technique to get

$$\begin{aligned} \mathcal{T}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l}) = \int_{|\beta|\le\frac{1}{qP}} \sum_{|v|\le P} \sigma(v;\beta,q) \sum_{a(q)^*} e_q(\overline{a}v) S_{\mathbf{d}}(q,a,\mathbf{n}) S_{\mathbf{t}}(q,-a,-\mathbf{l}) \\ \times I_{\mathbf{d}}\Big(\beta N, -\frac{P}{q}\mathbf{n}\Big) I_{\mathbf{t}}\Big(-\beta N, \frac{P}{q}\mathbf{l}\Big)\, d\beta, \end{aligned} \tag{6.8}$$

where the function σ satisfies $\sigma(v;\beta,q) \ll 1/(1+|v|)$. For this technique, one may refer to [9, (98) and (99)], for example. One can also refer to Estermann [5, proof of Lemma 13].

Recalling (4.1), we have

$$(6.9)\qquad \mathcal{T}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l}) \ll \sum_{|v|\le P}\frac{1}{1+|v|}\int_{|\beta|\le\frac{1}{qP}}|\theta(q;\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,1,1)| \times\left|I_{\mathbf{d}}\Big(\beta N,-\frac{P}{q}\mathbf{n}\Big)I_{\mathbf{t}}\Big(-\beta N,\frac{P}{q}\mathbf{l}\Big)\right|d\beta \le \sum_{|v|\le P}\frac{1}{1+|v|}|\theta(q;\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,1,1)|\mathcal{J}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l}),$$

where

$$\mathcal{J}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l}) = \int_{-\infty}^{\infty}\left|I_{\mathbf{d}}\Big(\beta N,-\frac{P}{q}\mathbf{n}\Big)I_{\mathbf{t}}\Big(-\beta N,\frac{P}{q}\mathbf{l}\Big)\right|d\beta.$$

Let

$$\mathcal{X}_2 = P^6\sum_{|v|\le P}\frac{1}{1+|v|}\sum_{\mathbf{d},\mathbf{t}}\frac{|\beta(\mathbf{d})\beta(\mathbf{t})|}{d_1d_2d_3t_1t_2t_3} \times\sum_{q\le P}\frac{1}{q^6}\sum_{\substack{\mathbf{n}\in\mathcal{N}_{\mathbf{d},q},\mathbf{l}\in\mathcal{N}_{\mathbf{t},q}\\ (\mathbf{n},\mathbf{l})\ne\mathbf{0}\in\mathbb{Z}^6}}|\theta(q;\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},v,1,1)|\mathcal{J}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l}).$$

By (6.3), (6.5), (6.6) and (6.9), we get

$$(6.10)\qquad \mathcal{E}_2^{(1)} = \mathcal{E}_2'(P) + O(\mathcal{X}_2) + O(P^{-A}).$$

Similarly from (6.3), (6.5), (6.7) and (6.9) we deduce that

$$(6.11)\qquad \mathcal{E}_2^{(2)} = \mathcal{E}_2'(Q) + O(\mathcal{X}_2) + O(P^{-A}).$$

One has, by Lemma 1,

$$\sum_{a(q)^*}\int_{\beta\in\mathcal{B}(q,a)}S_{\mathbf{d}}(q,a)S_{\mathbf{t}}(q,-a)|I(\beta N)|^6d\beta \ll P^{-2}q^4(q,\mathbf{d}^2)^{1/2}(q,\mathbf{t}^2)^{1/2}.$$

This yields

$$(6.12)\qquad \mathcal{E}_2'(P)-\mathcal{E}_2'(Q) \ll P^4\sum_{\mathbf{d}}\sum_{\mathbf{t}}\frac{|\beta(\mathbf{d})\beta(\mathbf{t})|}{d_1d_2d_3t_1t_2t_3}\sum_{Q<q\le P}\frac{1}{q^2}(q,\mathbf{d}^2)^{1/2}(q,\mathbf{t}^2)^{1/2} \ll Q^{-1}P^{4+\varepsilon}.$$

We conclude from (6.2), (6.4), (6.10), (6.11) and (6.12) that

$$\mathcal{E}_2 \ll Q^{-1}P^{4+\varepsilon} + \mathcal{X}_2.$$

From Lemma 4 (i),

$$\mathcal{J}_2(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l}) \ll P^{-2+\varepsilon}\min\left\{\frac{q^2}{P^2}\left(\sum_{j=1}^{3}\Big(\frac{|n_j|}{d_j}+\frac{|l_i|}{t_i}\Big)\right)^{-2},\frac{q}{P}\left(\sum_{j=1}^{3}\Big(\frac{|n_j|}{d_j}+\frac{|l_i|}{t_i}\Big)\right)^{-1}\right\}.$$

Applying Lemma 14, we see that

$$\mathcal{X}_2 \ll P^{4+\varepsilon}(D_1D_2D_3)^{-2}\mathcal{Y}_2 \ll P^{3+\varepsilon} + (D_1D_2D_3)^2P^{2+\varepsilon}.$$

Hence our final estimate for $\mathcal{E}_2$ is

$$(6.13)\qquad \mathcal{E}_2 \ll P^{4+\varepsilon}Q^{-1} + (D_1D_2D_3)P^{2+\varepsilon}.$$

Now we consider $\mathcal{E}_1$. Following the argument in [9, Section 3.4.1], we arrive at

$$\mathcal{E}_1 = \mathcal{E}_1^{(1)} - 2\mathcal{E}_1^{(2)} + \mathcal{E}_1^{(3)}, \tag{6.14}$$

where

$$\mathcal{E}_1^{(i)} = \sum_{\mathcal{R}} \frac{\gamma(pp')}{pp'} \sum_{\mathbf{d}} \sum_{\mathbf{t}} \beta(\mathbf{d})\beta(\mathbf{t}) \sum_{s(pp')^*} \left(\frac{s}{pp'}\right) e_{pp'}(-sN) J_1^{(i)} \tag{6.15}$$

with

$$J_1^{(1)} = \int_0^1 f_{\mathbf{d}}\left(\alpha + \frac{s}{pp'}\right) f_{\mathbf{t}}(-\alpha)\,d\alpha,$$
$$J_1^{(2)} = \int_0^1 f_{\mathbf{d}}\left(\alpha + \frac{s}{pp'}\right) \mathcal{W}_{\mathbf{t},Q}(-\alpha)\,d\alpha,$$
$$J_1^{(3)} = \sum_{n\in\mathbb{Z}} \mathcal{M}_{\mathbf{d},Q}(n)\mathcal{M}_{\mathbf{t},Q}(n) e_{pp'}(sn).$$

We decompose the integral

$$\begin{aligned} J_1^{(1)} &= \sum_{q\le P} \sum_{a(q)^*} \int_{\beta\in\mathcal{B}(q,a)} f_{\mathbf{d}}\left(\beta + \frac{a}{q} + \frac{s}{pp'}\right) f_{\mathbf{t}}\left(-\beta - \frac{a}{q}\right) d\beta \\ &= \sum_{q\le P} \sum_{\Delta} \sum_{\substack{a(q)^* \\ (app'+sq,qpp')=\Delta}} \int_{\mathcal{B}(q,a)} f_{\mathbf{d}}\left(\beta + \frac{(app'+sq)\Delta^{-1}}{qpp'\Delta^{-1}}\right) f_{\mathbf{t}}\left(-\beta - \frac{a}{q}\right) d\beta. \end{aligned}$$

Note that $(app'+sq, qpp') \mid (pp')^2$, we deduce by Lemma 5

$$\begin{aligned} J_1^{(1)} = &\sum_{\Delta\mid(pp')^2} \sum_{q\le P} \frac{P^6\Delta^3}{q^6 d_1d_2d_3t_1t_2t_3(pp')^3} \\ &\times \sum_{\mathbf{n}\in\mathcal{N}_{\mathbf{d},q}(pp'/\Delta)} \sum_{\mathbf{l}\in\mathcal{N}_{\mathbf{t},q}} \sum_{\substack{a(q)^* \\ (app'+sq,qpp')=\Delta}} \int_{\mathcal{B}(q,a)} S_{\mathbf{d}}\left(\frac{qpp'}{\Delta}, \frac{app'+sq}{\Delta}, \mathbf{n}\right) \\ &\times S_{\mathbf{t}}(q,-a,-\mathbf{l}) I_{\mathbf{d}}\left(\beta N, -\frac{P\Delta}{qpp'}\mathbf{n}\right) I_{\mathbf{t}}\left(-\beta N, \frac{P}{q}\mathbf{l}\right) d\beta + O(P^{-A}). \end{aligned}$$

Let

$$\begin{aligned} \mathcal{T}_1(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},p,p',\Delta) = &\sum_{s(pp')^*} \left(\frac{s}{pp'}\right) e_{pp'}(-sN) \sum_{a(q)^*} \int_{\mathcal{B}(q,a)} S_{\mathbf{d}}\left(\frac{qpp'}{\Delta}, \frac{app'+sq}{\Delta}, \mathbf{n}\right) \\ &\times S_{\mathbf{t}}(q,-a,-\mathbf{l}) I_{\mathbf{d}}\left(\beta N, -\frac{P\Delta}{qpp'}\mathbf{n}\right) I_{\mathbf{t}}\left(-\beta N, \frac{P}{q}\mathbf{l}\right) d\beta, \end{aligned}$$

and let

$$\mathcal{T}_1(q,\mathbf{d},\mathbf{t},p,p',\Delta) = \mathcal{T}_1(q,\mathbf{d},\mathbf{t},\mathbf{0},\mathbf{0},p,p',\Delta).$$

From above we obtain

$$\begin{aligned} \mathcal{E}_1^{(1)} = P^6 &\sum_{\mathcal{R}} \frac{\gamma(pp')}{(pp')^4} \sum_{\mathbf{d},\mathbf{t}} \frac{\beta(\mathbf{d})\beta(\mathbf{t})}{d_1d_2d_3t_1t_2t_3} \\ &\times \sum_{\Delta\mid(pp')^2} \sum_{q\le P} \frac{\Delta^3}{q^6} \sum_{\mathbf{n}\in\mathcal{N}_{\mathbf{d},q}(pp'/\Delta)} \sum_{\mathbf{l}\in\mathcal{N}_{\mathbf{t},q}} \mathcal{T}_1(q,\mathbf{d},\mathbf{t},\mathbf{n},\mathbf{l},p,p',\Delta). \end{aligned} \tag{6.16}$$

By interchanging the summation over a and the integration, we see that $\mathcal{T}_1(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', \Delta)$ is equal to

$$\int_{|\beta|\le \frac{1}{qP}} \sum_{|v|\le P} \sigma(v; q, \beta) \sum_{\substack{a(q)^* \\ (app'+sq, qpp')=\Delta}} e_q(\overline{a}v) S_{\mathbf{d}}\Big(\frac{qpp'}{\Delta}, \frac{app'+sq}{\Delta}, \mathbf{n}\Big) S_{\mathbf{t}}(q, -a, -\mathbf{l}) \times I_{\mathbf{d}}\Big(\beta N, -\frac{P\Delta}{qpp'}\mathbf{n}\Big) I_{\mathbf{t}}\Big(-\beta N, \frac{P}{q}\mathbf{l}\Big) d\beta.$$

Recalling the definition of $W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', v)$, we have

$$\mathcal{T}_1(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', \Delta) \ll \sum_{|v|\le P} \frac{1}{1+|v|} |W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', v)| \mathcal{J}_1\Big(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, \frac{pp'}{\Delta}\Big), \tag{6.17}$$

where

$$\mathcal{J}_1(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, H) = \int_{-\infty}^{\infty} \Big| I_{\mathbf{d}}\Big(\beta N, -\frac{P}{qH}\mathbf{n}\Big) I_{\mathbf{t}}\Big(-\beta N, \frac{P}{q}\mathbf{l}\Big)\Big| d\beta.$$

To discuss the contribution from $(\mathbf{n}, \mathbf{l}) = \mathbf{0} \in \mathbb{Z}^6$, we define

$$\begin{aligned}\mathcal{T}_1'(q, \mathbf{d}, \mathbf{t}, p, p', \Delta) &= \sum_{s(pp')^*} \Big(\frac{s}{pp'}\Big) e_{pp'}(-sN) \times \sum_{a(q)^*} \int_{|\beta|\le \frac{1}{2qP}} S_{\mathbf{d}}\Big(\frac{qpp'}{\Delta}, \frac{app'+sq}{\Delta}\Big) S_{\mathbf{t}}(q, -a) |I(\beta)|^6 d\beta \\ &= W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{0}, \mathbf{0}, p, p', 0) \int_{|\beta|\le \frac{1}{2qP}} |I(\beta)|^6 d\beta.\end{aligned}$$

We deduce from Lemma 4 (ii) that

$$\mathcal{T}_1'(q, \mathbf{d}, \mathbf{t}, p, p', \Delta) - \mathcal{T}_1(q, \mathbf{d}, \mathbf{t}, p, p', \Delta) \ll P^{-4} q^2 \sum_{|v|\le P} \frac{1}{1+|v|} |W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{0}, \mathbf{0}, p, p', v)| \tag{6.18}$$

and

$$\mathcal{T}_1'(q, \mathbf{d}, \mathbf{t}, p, p', \Delta) - \sigma_0 P^{-2} W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{0}, \mathbf{0}, p, p', 0) \ll P^{-4} q^2 |W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{0}, \mathbf{0}, p, p', 0)|. \tag{6.19}$$

Let

$$\mathcal{E}_1'(K) = \sigma_0 P^4 \sum_{\mathcal{R}} \frac{\gamma(pp')}{(pp')^4} \sum_{\mathbf{d},\mathbf{t}} \frac{\beta(\mathbf{d})\beta(\mathbf{t})}{d_1 d_2 d_3 t_1 t_2 t_3} \sum_{\Delta \mid (pp')^2} \sum_{q\le K} \frac{\Delta^3}{q^6} W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{0}, \mathbf{0}, p, p', 0),$$

and define

$$\mathcal{X}_1 = P^4 R^{-7} \sum_{\mathcal{R}} \sum_{|v|\le P} \frac{1}{1+|v|} \sum_{\mathbf{d},\mathbf{t}} \frac{|\beta(\mathbf{d})\beta(\mathbf{t})|}{d_1 d_2 d_3 t_1 t_2 t_3} \sum_{\Delta \mid (pp')^2} \sum_{q\le P} \frac{\Delta^3}{q^6} \times \sum_{\mathbf{n}\in \mathcal{N}_{\mathbf{d},q}(pp'/\Delta)} \sum_{\mathbf{l}\in \mathcal{N}_{\mathbf{t},q}} |W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, p, p', v)| \mathcal{C}\Big(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, \frac{pp'}{\Delta}\Big).$$

Here the function $\mathcal{C}(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, pp'/\Delta)$ is defined in (4.6). By Lemma 4 (i), if $(\mathbf{n}, \mathbf{l}) \neq \mathbf{0} \in \mathbb{Z}^6$, then

$$\mathcal{I}_1\left(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, \frac{pp'}{\Delta}\right) \ll P^{-2+\varepsilon}\mathcal{C}\left(q, \mathbf{d}, \mathbf{t}, \mathbf{n}, \mathbf{l}, \frac{pp'}{\Delta}\right). \tag{6.20}$$

We conclude from (6.16), (6.17), (6.18), (6.19) and (6.20) that

$$\mathcal{E}_1^{(1)} = \mathcal{E}_1'(P) + O(\mathcal{X}_1) + O(P^{-A}). \tag{6.21}$$

Similarly we can also obtain

$$\mathcal{E}_1^{(2)} = \mathcal{E}_1'(Q) + O(\mathcal{X}_1) + O(P^{-A}). \tag{6.22}$$

We shall prove

$$\mathcal{E}_1^{(3)} = \mathcal{E}_1' + O(P^{\varepsilon}Q^3R^3) + O(P^{\varepsilon}Q^{14/5}R^{29/5}), \tag{6.23}$$

where

$$\begin{aligned}\mathcal{E}_1' = \sigma_0 P^4 \sum_{\mathcal{R}} \frac{\gamma(pp')}{(pp')^4} \sum_{\mathbf{d},\mathbf{t}} \frac{\beta(\mathbf{d})\beta(\mathbf{t})}{d_1d_2d_3t_1t_2t_3} \\ \times \sum_{\Delta \mid (pp')^2} \Delta^3 \sum_{q \le \min\{Q, Q\Delta(pp')^{-1}\}} \frac{1}{q^6} W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{0}, \mathbf{0}, p, p', 0).\end{aligned}$$

Note that $\min\{Q, Q\Delta(pp')^{-1}\} \le Q < P$, we introduce

$$\begin{aligned}\mathcal{X}_3 = P^4R^{-7} \sum_{\mathcal{R}} \sum_{\mathbf{d},\mathbf{t}} \frac{|\beta(\mathbf{d})\beta(\mathbf{t})|}{d_1d_2d_3t_1t_2t_3} \\ \times \sum_{\Delta \mid (pp')^2} \sum_{\min\{Q, Q\Delta(pp')^{-1}\} \le q \le P} \frac{\Delta^3}{q^6} |W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{0}, \mathbf{0}, p, p', 0)|.\end{aligned}$$

Thus by (6.14), (6.21), (6.22) and (6.23) we arrive at

$$\mathcal{E}_1 \ll O(\mathcal{X}_1) + O(\mathcal{X}_3) + O(P^{\varepsilon}Q^3R^3) + O(P^{\varepsilon}Q^{14/5}R^{29/5}).$$

By Lemma 14, we have

$$\mathcal{X}_1 \ll P^{4+\varepsilon}R^{-7}(D_1D_2D_3)^{-2} \sum_{\mathcal{R}} \mathcal{Y}_1 \ll P^{3+\varepsilon}R + (D_1D_2D_3)^2P^{2+\varepsilon}R^5.$$

From Lemma 11,

$$\begin{aligned}&W(\Delta, q; \mathbf{d}, \mathbf{t}, \mathbf{0}, \mathbf{0}, p, p', v) \\ &\qquad \ll 2^{4u}r^4(r, \mathbf{d}^2)^{1/2}(r, \mathbf{t}^2)^{1/2}\Delta^{-2}R^4(\pi\pi')^5(1 + \Delta R^{-2})^{-1}.\end{aligned} \tag{6.24}$$

Substituting (6.24) into the definition of $\mathcal{X}_3$, we can deduce that

$$\begin{aligned}\mathcal{X}_3 &\ll P^4R^{-3} \sum_{\mathcal{R}} \sum_{\mathbf{d},\mathbf{t}} \frac{|\beta(\mathbf{d})\beta(\mathbf{t})|}{d_1d_2d_3t_1t_2t_3} \\ &\qquad \times \sum_{\Delta \mid (pp')^2} \sum_{\min\{Q, Q\Delta(pp')^{-1}\} \le 2^u\pi\pi'r} \frac{\Delta(r, \mathbf{d}^2)^{1/2}(r, \mathbf{t}^2)^{1/2}}{2^{2u}\pi\pi'r^2(1 + \Delta R^{-2})} \\ &\ll P^{4+\varepsilon}R^{-3} \sum_{\mathcal{R}} \sum_{\Delta \mid (pp')^2} \Delta \min\{Q, Q\Delta(pp')^{-1}\}^{-1}(1 + \Delta R^{-2})^{-1} \\ &\ll P^{4+\varepsilon}RQ^{-1}.\end{aligned}$$

Thus for $\mathcal{E}_1$ we finally obtain

$$\mathcal{E}_1 \ll P^{4+\varepsilon} R Q^{-1} + (D_1 D_2 D_3)^2 P^{2+\varepsilon} R^5 + P^{\varepsilon} Q^{14/5} R^{29/5}. \tag{6.25}$$

Combining (6.13), (6.25) and the lines around (5.3) and (6.1), we established Proposition 4.

We are left to establish (6.23). Recalling the definition of $J_1^{(3)}$, one has

$$J_1^{(3)} = \frac{P^2}{d_1 d_2 d_3 t_1 t_2 t_3} \sum_{q_1, q_2 \le Q} (q_1 q_2)^{-3} \sum_{\substack{a_1 (q_1)^* \\ a_2 (q_2)^*}} S_{\mathbf{d}}(q_1, -a_1) S_{\mathbf{t}}(q_2, -a_2)$$
$$\times \sum_{n \in Z} H^2\left(\frac{n}{N}\right) e\left(n\left(\frac{a_1}{q_1} + \frac{a_2}{q_2} + \frac{s}{pp'}\right)\right).$$

By Lemma 7, we get

$$\mathcal{E}_1^{(3)} = \mathcal{E}_1'' + E + O(P^{-A}), \tag{6.26}$$

where

$$\mathcal{E}_1'' = \sigma_0 P^4 \sum_{\mathcal{R}} \frac{\gamma(pp')}{pp'} \sum_{\mathbf{d}} \sum_{\mathbf{t}} \frac{\beta(\mathbf{d})\beta(\mathbf{t})}{d_1 d_2 d_3 t_1 t_2 t_3} \sum_{s (pp')^*} \left(\frac{s}{pp'}\right) e_{pp'}(-sN)$$
$$\times \sum_{q_1, q_2 \le Q} (q_1 q_2)^{-3} \sum_{\substack{a_1 (q_1)^*, a_2 (q_2)^* \\ \frac{a_1}{q_1} + \frac{a_2}{q_2} + \frac{s}{pp'} \in \mathbb{Z}}} S_{\mathbf{d}}(q_1, -a_1) S_{\mathbf{t}}(q_2, -a_2)$$

and

$$E = \sum_{\mathcal{R}} \frac{\gamma(pp')}{pp'} \sum_{\mathbf{d}, \mathbf{t}} \frac{P^2 \beta(\mathbf{d})\beta(\mathbf{t})}{d_1 d_2 d_3 t_1 t_2 t_3} \sum_{n \in \mathbb{Z}} H^2\left(\frac{n}{N}\right) \sum_{s (pp')^*} \left(\frac{s}{pp'}\right) e_{pp'}(-sN) \sum_{q_1, q_2 \le Q} \frac{1}{q_1^3 q_2^3}$$
$$\times \sum_{\substack{a_1 (q_1)^*, a_2 (q_2)^* \\ 0 < \| \frac{a_1}{q_1} + \frac{a_2}{q_2} + \frac{s}{pp'} \| \le P^{\varepsilon - 2}}} S_{\mathbf{d}}(q_1, -a_1) S_{\mathbf{t}}(q_2, -a_2) e\left(n\left(\frac{a_1}{q_1} + \frac{a_2}{q_2} + \frac{s}{pp'}\right)\right).$$

Notice that

$$\frac{a_1}{q_1} + \frac{a_2 pp' + s q_2}{q_2 pp'} = \frac{a_1}{q_1} + \frac{a_2}{q_2} + \frac{s}{pp'} \in \mathbb{Z}$$

is equivalent to

$$q_1 = q_2 pp' \Delta^{-1} \quad \text{and} \quad -a_1 \equiv (a_2 pp' + s q_2)\Delta^{-1} \pmod{q_1},$$

where $\Delta = (a_2 pp' + s q_2, q_2 pp')$. Hence

$$\mathcal{E}_1'' = \mathcal{E}_1'. \tag{6.27}$$

Now we handle E. By Lemma 1 (vi),

$$E \ll P^4 R \sup_{n, p, p'} \sum_{\mathbf{d}} \sum_{\mathbf{t}} \frac{|\beta(\mathbf{d})\beta(\mathbf{t})|}{d_1 d_2 d_3 t_1 t_2 t_3} |\mathcal{F}|, \tag{6.28}$$

where

$$\mathcal{F} := \sum_{s(pp')^*} \left(\frac{s}{pp'}\right) e_{pp'}(-sN) \sum_{q_1,q_2 \le Q} (q_1q_2)^{-3} \times \sum_{\substack{a_1(q_1)^*, a_2(q_2)^* \\ 0<\|\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{s}{pp'}\| \le P^{\varepsilon-2}}} S_{\mathbf{d}}(q_1,-a_1) S_{\mathbf{t}}(q_2,-a_2) e\left(n\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{s}{pp'}\right)\right).$$

Note that

$$\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{s}{pp'} = \frac{pp'(a_1q_2+a_2q_1)+sq_1q_2}{q_1q_2pp'}.$$

We divide the summations according to $pp'(a_1q_2+a_2q_1)+sq_1q_2 \pmod{q_1q_2pp'}$,

$$\mathcal{F} = \sum_{0<|l|\le R^2Q^2P^{\varepsilon-2}} \sum_{s(pp')^*} \left(\frac{s}{pp'}\right) e_{pp'}(-sN) \sum_{\substack{q_1,q_2 \le Q \\ |l|(pp')^{-1}P^{2-\varepsilon} < q_1q_2}} (q_1q_2)^{-3} \times \sum_{\substack{a_1(q_1)^*, a_2(q_2)^* \\ pp'(a_1q_2+a_2q_1)+sq_1q_2 \equiv l \pmod{q_1q_2pp'}}} S_{\mathbf{d}}(q_1,-a_1) S_{\mathbf{t}}(q_2,-a_2) \times e\left(n\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{s}{pp'}\right)\right).$$

Here the restriction $0<|l|\le R^2Q^2P^{\varepsilon-2}$ and $|l|(pp')^{-1}P^{2-\varepsilon} < q_1q_2$ come from

$$0 < \left\|\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{s}{pp'}\right\| \le P^{\varepsilon-2}.$$

The congruence $pp'(a_1q_2+a_2q_1)+sq_1q_2 \equiv l \pmod{q_1q_2pp'}$ implies

$$e\left(n\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{s}{pp'}\right)\right) = e\left(\frac{ln}{q_1q_2pp'}\right).$$

Thus

$$\mathcal{F} = \sum_{0<|l|\le R^2Q^2P^{\varepsilon-2}} \sum_{s(pp')^*} \left(\frac{s}{pp'}\right) e_{pp'}(-sN) \sum_{\substack{q_1,q_2 \le Q \\ |l|(pp')^{-1}P^{2-\varepsilon} < q_1q_2}} (q_1q_2)^{-3} \times e\left(\frac{ln}{q_1q_2pp'}\right) \sum_{\substack{a_1(q_1)^*, a_2(q_2)^* \\ pp'(a_1q_2+a_2q_1)+sq_1q_2 \equiv l \pmod{q_1q_2pp'}}} S_{\mathbf{d}}(q_1,-a_1) S_{\mathbf{t}}(q_2,-a_2).$$

Recall the assumption (5.4), we easily know $(l, pp') = 1$. This implies $(q_1q_2, pp') = 1$. So

$$pp'(a_1q_2+a_2q_1)+sq_1q_2 \equiv l \pmod{q_1q_2pp'}$$

is equivalent to

$$pp'(a_1q_2+a_2q_1) \equiv l \pmod{q_1q_2} \quad \text{and} \quad sq_1q_2 \equiv l \pmod{pp'}.$$

Thus

$$\mathcal{F} = \sum_{0<|l|\le R^2Q^2P^{\varepsilon-2}} \sum_{\substack{q_1,q_2\le Q \\ |l|(pp')^{-1}P^{2-\varepsilon}<q_1q_2}} (q_1q_2)^{-3}\left(\frac{q_1q_2l}{pp'}\right)e_{pp'}(-\overline{q_1q_2}lN)$$
$$\times e\left(\frac{ln}{q_1q_2pp'}\right) \sum_{\substack{a_1(q_1)^*,\,a_2(q_2)^* \\ pp'(a_1q_2+a_2q_1)\equiv l \pmod{q_1q_2}}} S_{\mathbf{d}}(q_1,-a_1)S_{\mathbf{t}}(q_2,-a_2).$$

Now (6.23) follows from Lemma 15, (6.26), (6.27) and (6.28). The proof of Proposition 4 is complete.

7. Proof of Proposition 3

Let us define

$$\widetilde{\mathcal{H}}(D_1,D_2,D_3) = \sum_{\substack{d_i\le D_i\ (1\le i\le 3) \\ (d_i,d_j)\le P^{\varepsilon}\ (1\le i<j\le 3)}} \beta(\mathbf{d})\left(\mathcal{L}_{\mathbf{d}}(N) - \frac{\mathcal{N}_0(N)\Sigma_0(\mathbf{d},N)}{d_1d_2d_3}\right),$$

where $\beta(\mathbf{d}) = \beta(d_1,d_2,d_3)$ is a real function satisfying (2.6) and (2.7). We have the following result.

Proposition 5. *Suppose that $D_1D_2D_3 < P^{1/2-\varepsilon}$. We have*

$$\widetilde{\mathcal{H}}(D_1,D_2,D_3) \ll P^2(\log P)^{-A}.$$

We first explain that Proposition 3 can be deduced from Proposition 5. In order to prove

$$\mathcal{H}(D) \ll P^2(\log P)^{-A} \quad \text{for } D \le P^{1/2-5\varepsilon},$$

we divide the underlying summation into two parts. Note that

$$\mathcal{H}(D) = \sum_{\substack{d_1,d_2,d_3 \\ [d_1,d_2,d_3]\le D}} \beta(\mathbf{d})\left(\mathcal{L}_{\mathbf{d}}(N) - \frac{\mathcal{N}_0(N)\Sigma_0(\mathbf{d},N)}{d_1d_2d_3}\right)$$
$$= \left(\sum_{\substack{d_1,d_2,d_3 \\ \max_{i<j}(d_i,d_j)\le P^{\varepsilon} \\ [d_1,d_2,d_3]\le D}} + \sum_{\substack{d_1,d_2,d_3 \\ \max_{i<j}(d_i,d_j)> P^{\varepsilon} \\ [d_1,d_2,d_3]\le D}}\right)\beta(\mathbf{d})\left(\mathcal{L}_{\mathbf{d}}(N) - \frac{\mathcal{N}_0(N)\Sigma_0(\mathbf{d},N)}{d_1d_2d_3}\right)$$
$$=: \mathcal{H}_1(D) + \mathcal{H}_2(D).$$

Since $\max_{i<j}(d_i,d_j) \le P^{\varepsilon}$ and $[d_1,d_2,d_3] \le D$ together imply $d_1d_2d_3 \le P^{1/2-2\varepsilon}$, by the dyadic argument and Proposition 5, we can conclude

$$\mathcal{H}_1(D) \ll P^2(\log P)^{-A}.$$

Now we consider $\mathcal{H}_2(D)$ which is bounded by $\mathcal{H}'P^{\varepsilon/9}$, where

$$\mathcal{H}' = \sum_{\substack{d_j\le D\ (1\le j\le 3) \\ \max_{i<j}(d_i,d_j)\ge P^{\varepsilon}}} \left|\mathcal{L}_{\mathbf{d}}(N) - \frac{\mathcal{N}_0(N)\Sigma_0(\mathbf{d},N)}{d_1d_2d_3}\right|.$$

It suffices to verify $\mathscr{H}' \ll P^{2-\varepsilon/5}$. By symmetry, the assertion is a consequence of

$$\mathscr{H}_1' := \sum_{\substack{d_j \le D\ (1\le j\le 3)\\ (d_1,d_2)\ge P^\varepsilon}} \left| \frac{\mathscr{N}_0(N)\Sigma_0(\mathbf{d},N)}{d_1 d_2 d_3} \right| \ll P^{2-\varepsilon/5} \tag{7.1}$$

and

$$\mathscr{H}_2' := \sum_{\substack{d_j \le D\ (1\le j\le 3)\\ (d_1,d_2)\ge P^\varepsilon}} \mathscr{L}_{\mathbf{d}}(N) \ll P^{2-\varepsilon/5}. \tag{7.2}$$

It has been pointed out in [9] that

$$\Sigma_0(\mathbf{d},N) \ll \tau^2(d_1^2)\tau^2(d_2^2)\tau^2(d_3^2)\log\log P \quad \text{and} \quad \mathscr{N}_0(N) \asymp \frac{P^2}{\log P}.$$

Hence

$$\mathscr{H}_1' \ll P^{2+\frac{\varepsilon}{10}} \sum_{\substack{d_j \le D\ (1\le j\le 3)\\ (d_1,d_2)\ge P^\varepsilon}} \frac{1}{d_1 d_2 d_3} \ll P^{2+\frac{\varepsilon}{5}} \sum_{\substack{d_1\le D,\, d_2\le D\\ (d_1,d_2)\ge P^\varepsilon}} \frac{1}{d_1 d_2}. \tag{7.3}$$

Note that $\omega(\mathbf{x}) \ll 1$, we have

$$\mathscr{H}_2' \ll \sum_{\substack{d_1\le P,\, d_2\le P\\ (d_1,d_2)\ge P^\varepsilon}} \sum_{\substack{x_1,x_2\\ d_1\mid x_1,\, d_2\mid x_2}} \sum_{\substack{q,x_3,d_3\\ q^2+x_1^2+x_2^2+x_3^2=N\\ d_3\mid x_3}} 1.$$

For fixed x_1, x_2, the inner sum is bounded by $P^{\varepsilon/10}$. Hence

$$\mathscr{H}_2' \ll P^{\varepsilon/10} \sum_{\substack{d_1\le P,\, d_2\le P\\ (d_1,d_2)\ge P^\varepsilon}} \sum_{\substack{x_1,x_2\\ d_1\mid x_1,\, d_2\mid x_2}} 1 \ll P^{\varepsilon/10} \sum_{\substack{d_1\le P,\, d_2\le P\\ (d_1,d_2)\ge P^\varepsilon}} \frac{P^2}{d_1 d_2}. \tag{7.4}$$

As an exercise, we have

$$\sum_{\substack{d_1\le P,\, d_2\le P\\ (d_1,d_2)\ge P^\varepsilon}} \frac{1}{d_1 d_2} \le \sum_{\delta\ge P^\varepsilon} \sum_{\substack{d_1'\le D/\delta\\ d_2'\le D/\delta}} \frac{1}{d_1' d_2' \delta^2} \ll P^{\varepsilon/10} \sum_{\delta\ge P^\varepsilon} \frac{1}{\delta^2} \ll P^{-\frac{9}{10}\varepsilon}.$$

Now the estimates (7.1) and (7.2) follow from (7.3) and (7.4), respectively.

The remaining of this section is to show Proposition 5 by invoking Proposition 4. The proof follows the argument of Heath-Brown and Tolev closely. Let

$$\mathscr{H}_1 = \sum_{\substack{d_i\le D_i\ (1\le i\le 3)\\ (d_i,d_j)\le P^\varepsilon\ (1\le i<j\le 3)}} \beta(\mathbf{d})\mathscr{L}_{\mathbf{d}}(N).$$

By applying Proposition 4, we see that

$$\mathscr{H}_1 = \sum_{k\in\mathscr{A}_j} \omega(k) \sum_{\substack{d_i\le D_i\ (1\le i\le 3)\\ (d_i,d_j)\le P^\varepsilon\ (1\le i<j\le 3)}} \beta(\mathbf{d})\Omega_{\mathbf{d}}(N-k^2) = \mathscr{H}_2 + O(P^{2-\varepsilon}),$$

where

$$\mathcal{H}_2 = \sum_{k \in \mathcal{A}_j} \omega(k) \sum_{\substack{d_i \le D_i \ (1 \le i \le 3) \\ (d_i, d_j) \le P^{\varepsilon} \ (1 \le i < j \le 3)}} \beta(\mathbf{d}) \mathcal{M}_{\mathbf{d}, Q}(N - k^2).$$

Recalling the definition of $\mathcal{M}_{\mathbf{d}, Q}(n)$, we know

$$\mathcal{H}_2 = P \sum_{d_i \le D_i \ (1 \le i \le 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q \le Q} q^{-3} \sum_{a(q)^*} S_{\mathbf{d}}(q, a) \times \sum_{k \in \mathcal{A}_j} \omega(k) H\left(1 - \frac{k^2}{N}\right) e\left(\frac{a}{q}(k^2 - N)\right),$$

where $\widetilde{\beta}(\mathbf{d})$ is $\beta(\mathbf{d})$ if $(d_i, d_j) \le P^{\varepsilon}$ $(1 \le i < j \le 3)$, and zero otherwise. Now partial summation gives

$$\mathcal{H}_2 = -P \int \mathcal{B}(x) \left(\frac{d}{dx} \omega(x) H\left(1 - \frac{x^2}{N}\right)\right) dx, \tag{7.5}$$

where

$$\mathcal{B}(x) = \sum_{d_i \le D_i \ (1 \le i \le 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q \le Q} q^{-3} \sum_{a(q)^*} S_{\mathbf{d}}(q, a) \mathcal{Z}(x)$$

and

$$\mathcal{Z}(x) = \sum_{\substack{k \le x \\ k \in \mathcal{A}_j}} e\left(\frac{a}{q}(k^2 - N)\right).$$

Let us write

$$\mathcal{Z}_0(x) = \sum_{\substack{k \le x \\ k \in \mathcal{A}_j \\ (k,q)=1}} e\left(\frac{a}{q}(k^2 - N)\right).$$

Then we have

$$\mathcal{Z}(x) = \mathcal{Z}_0(x) + \sum_{\substack{k \le x \\ k \in \mathcal{A}_j \\ (k,q)>1}} e\left(\frac{a}{q}(k^2 - N)\right),$$

and thereby

$$\mathcal{B}(x) = \mathcal{B}_0(x) + \mathbf{E}(x),$$

where

$$\mathcal{B}_0(x) = \sum_{d_i \le D_i \ (1 \le i \le 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q \le Q} q^{-3} \sum_{a(q)^*} S_{\mathbf{d}}(q, a) \mathcal{Z}_0(x)$$

and

$$\mathbf{E}(x) = \sum_{d_i \le D_i \ (1 \le i \le 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q \le Q} q^{-3} \sum_{a(q)^*} S_{\mathbf{d}}(q, a) \sum_{\substack{k \le x \\ k \in \mathcal{A}_j \\ (k,q)>1}} e\left(\frac{a}{q}(k^2 - N)\right).$$

If $(k,q) > 1$ and $k \in \mathcal{A}_j$, then $(k,q) \geq P^{1/16}$. So we have

$$\mathbf{E}(x) = \sum_{h \geq P^{1/16}} \sum_{d_i \leq D_i \ (1 \leq i \leq 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q \leq Q/h} \sum_{\substack{k \leq x/h \\ kh \in \mathcal{A}_j \\ (k,q)=1}} q^{-3} h^{-3} \times \sum_{a(hq)^*} S_{\mathbf{d}}(hq,a) e\left(\frac{a}{hq}(k^2h^2 - N)\right).$$

For the inner sum above, we have the bound

$$\sum_{a(q)^*} S_{\mathbf{d}}(q,a) e\left(-\frac{a}{q} n\right) \ll (n,q)(q,d_1^2)^{1/2}(q,d_2^2)^{1/2}(q,d_3^2)^{1/2} q^2.$$

Then we get

$$\mathbf{E}(x) \ll P^{\varepsilon} \sum_{h \geq P^{1/16}} \sum_{q \leq Q/h} q^{-1} h^{-1} \sum_{k \leq x/h} (N - k^2 h^2, qh).$$

We finally find that

$$\mathbf{E}(x) \ll P^{1-1/16+\varepsilon}.$$

For $(m,q) = 1$, we introduce the notation

$$\Delta_j(x;q,m) = \sum_{\substack{k \leq x \\ k \in \mathcal{A}_j \\ k \equiv m \ (\mathrm{mod}\, q)}} 1 - \frac{1}{\phi(q)} \Lambda_j(x;q),$$

where

$$\Lambda_j(x;q) = \sum_{\substack{k \leq x \\ k \in \mathcal{A}_j \\ (k,q)=1}} 1.$$

Then we have

$$\mathcal{Z}_0(x) = \frac{1}{\phi(q)} e\left(\frac{-aN}{q}\right) T(q,a) \Lambda_j(x;q) + \sum_{m(q)^*} e\left(\frac{a(m^2 - N)}{q}\right) \Delta_j(x;q,m).$$

Therefore

$$\mathcal{B}_0(x) = \widetilde{\mathcal{B}}_0(x) + \mathcal{C}(x),$$

where

$$\widetilde{\mathcal{B}}_0(x) = \sum_{d_i \leq D_i \ (1 \leq i \leq 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q \leq Q} h_{\mathbf{d}}(q) \Lambda_j(x;q)$$

and

$$\mathcal{C}(x) = \sum_{d_i \leq D_i \ (1 \leq i \leq 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q \leq Q} q^{-3} \sum_{a(q)^*} S_{\mathbf{d}}(q,a) \sum_{m(q)^*} e\left(\frac{a(m^2 - N)}{q}\right) \Delta_j(x;q,m).$$

Define

$$L = \sum_{q \leq Q} \sum_{m(q)^*} \Delta_j(x;q,m)^2$$

and

$$M = \sum_{q \le Q} \sum_{m(q)^*} \Gamma(q,m)^2,$$

where

$$\Gamma(q,m) = q^{-3} \sum_{d_i \le D_i \ (1 \le i \le 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{a(q)^*} S_{\mathbf{d}}(q,a) e\left(\frac{a}{q}(m^2 - N)\right).$$

It has been proved in [9] that $M \ll (\log P)^C$ for some absolute constant $C > 0$ (see [9, (274)]). The Generalized Barban–Davenport–Halberstam Theorem states that

$$L \ll P^2 (\log P)^{-A}.$$

Observing that

$$\mathcal{C}(x) = \sum_{q \le Q} \sum_{m(q)^*} \Delta_j(x; q, m) \Gamma(q,m),$$

one can conclude by Cauchy's inequality

$$\mathcal{C}(x) \ll P(\log P)^{-A}.$$

Let

$$A(x) = \sum_{\substack{k \le x \\ k \in \mathcal{A}_j}} 1.$$

Note that $\Lambda_j(x; q) = A(x) + O(P^{1-1/16+\varepsilon})$ and

$$\sum_{q \le Q} |h_{\mathbf{d}}(q)| \ll \tau^2(d_1) \tau^2(d_2) \tau^2(d_3) \log\log P.$$

Then we have

$$\widetilde{\mathcal{B}}_0(x) = \mathcal{B}_0 A(x) + O(P^{1-1/16+\varepsilon}),$$

where

$$\mathcal{B}_0 = \sum_{d_i \le D_i \ (1 \le i \le 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q \le Q} h_{\mathbf{d}}(q).$$

It is not hard to see that

$$\mathcal{B}_0 = \mathcal{B}_1 + O(P^{-1/2+\varepsilon}),$$

where

$$\mathcal{B}_1 = \sum_{d_i \le D_i \ (1 \le i \le 3)} \frac{\widetilde{\beta}(\mathbf{d})}{d_1 d_2 d_3} \sum_{q=1}^{\infty} h_{\mathbf{d}}(q).$$

Therefore

$$\mathcal{B}(x) = \mathcal{B}_1 A(x) + O(P(\log P)^{-A}). \tag{7.6}$$

By the Prime Number Theorem, we have

$$A(x) = \int_2^x \frac{C_j(t)}{\log t} dt + O(P(\log P)^{-A}).$$

Now combining (7.5) and (7.6), we arrive at

$$\mathcal{H}_2 = P \mathcal{B}_1 \int \omega(x) H\left(1 - \frac{x^2}{N}\right) \frac{C_j(x)}{\log x} dx + O(P^2 (\log P)^{-A}).$$

The proof of Proposition 5 is completed and therefore Proposition 3 is also established.

8. The review of the three-dimensional sieve

In this section, we recall the Diamond–Halberstam–Richert sieves. One may refer to [7] for the details. We shall focus on the case that the sieve dimension $\kappa = 3$. Let σ_κ be the continuous solution of the differential delay problem

$$u^\kappa \sigma_\kappa(u) = (2e^\gamma)^\kappa (\Gamma(\kappa+1))^{-1}, \quad 0 < u \le 2,$$
$$(u^\kappa \sigma_\kappa(u))' = -\kappa u^{-\kappa-1} \sigma_\kappa(u-2), \quad u > 2,$$

where γ is Euler's constant and Γ is Euler's gamma function.

Let $F_\kappa(u)$ and $f_\kappa(u)$ be the continuous solutions of the simultaneous differential delay system

$$\tag{8.1} \begin{cases} F_\kappa(u) = \dfrac{1}{\sigma_\kappa(u)}, & 0 < u \le \alpha_\kappa, \\ f_\kappa(u) = 0, & 0 \le u \le \beta_\kappa, \\ (u^\kappa F_\kappa(u))' = \kappa u^{\kappa-1} f_\kappa(u-1), & u > \alpha_\kappa, \\ (u^\kappa f_\kappa(u))' = \kappa u^{\kappa-1} F_\kappa(u-1), & u > \beta_\kappa, \end{cases}$$

where α_κ and β_κ are real numbers such that

$$3 < \beta_\kappa + 1 < \alpha_\kappa.$$

We note that

(i) $F_\kappa(u)$ decreases monotonically toward 1 as $u \to \infty$,

(ii) $f_\kappa(u)$ increases monotonically toward 1 as $u \to \infty$.

Suppose that $\{a_n\}$ is a (finite) sequence of non-negative real numbers. Then we introduce

$$\mathcal{A}(d) = \sum_{n \equiv 0 \pmod{d}} a_n.$$

It is expected that $\frac{\Omega(t)}{t} X$ is a good approximation to $\mathcal{A}(t)$, where $\Omega(t)$ is a multiplicative function satisfying

$$\tag{8.2} 0 \le \Omega(p) < \min\{p, c\}$$

for some constant c, and

$$\tag{8.3} \prod_{w_1 \le p < w} \left(1 - \frac{\Omega(p)}{p}\right)^{-1} \le \left(\frac{\log w}{\log w_1}\right)^3 \left(1 + \frac{c_1}{\log w_1}\right), \quad 2 \le w_1 < w,$$

for some constant $c_1 > 0$. Suppose that there exists a constant $c_2 \ge 2$ such that

$$\tag{8.4} \sum_{t \le D} \mu^2(t)\tau^2(t) \left| \mathcal{A}(t) - \frac{\Omega(t)}{t} X \right| \le c_2 \frac{X}{(\log X)^4}.$$

Let $\Pi(z) = \prod_{p \le z} p$ and define

$$V(x) = \prod_{p \le x} \left(1 - \frac{\Omega(p)}{p}\right).$$

Then one has

$$\sum_{(n,\Pi(z))=1} a_n \le XV(z)\left(F_\kappa(s) + O\left(\frac{(\log\log X)^2}{(\log X)^{1/8}}\right)\right) \tag{8.5}$$

and

$$\sum_{(n,\Pi(z))=1} a_n \ge XV(z)\left(f_\kappa(s) + O\left(\frac{(\log\log X)^2}{(\log X)^{1/8}}\right)\right), \tag{8.6}$$

where $s = \log D/\log z$.

By (8.1), for $s > \beta_\kappa$ we have

$$s^\kappa f_\kappa(s) = \int_{\beta_\kappa}^{s} \kappa u^{\kappa-1} F_\kappa(u-1)\,du > F_\kappa(s)\int_{\beta_\kappa}^{s} \kappa u^{\kappa-1}\,du = F_\kappa(s)(s^\kappa - \beta_\kappa^\kappa).$$

Therefore one has

$$\frac{f_\kappa(s)}{F_\kappa(s)} > 1 - \left(\frac{\beta_\kappa}{s}\right)^\kappa. \tag{8.7}$$

9. Applications of the switching principle

Let

$$\mathcal{A}(t) = \mathcal{A}(t,N) := \mathcal{A}^{(j)}(t,N) = \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ q\in\mathcal{A}_j \\ t \mid x_1x_2x_3}} \omega(q)\omega(\mathbf{x}).$$

Note that the function

$$g(t) = \mu(t) \sum_{\substack{\mathbf{d} \\ [d_1,d_2,d_3]=t \\ d_j \mid x_j \ (1\le j\le 3)}} \mu(d_1)\mu(d_2)\mu(d_3)$$

is a multiplicative function of t. If t is square-free and $t \mid x_1x_2x_3$, then $g(t)=1$, and $g(t)=0$ otherwise. Hence for t square-free, we have

$$\mathcal{A}(t) = \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ q\in\mathcal{A}_j \\ t \mid x_1x_2x_3}} \omega(q)\omega(\mathbf{x})\mu(t) \sum_{\substack{\mathbf{d} \\ [d_1,d_2,d_3]=t \\ d_j \mid x_j \ (1\le j\le 3)}} \mu(d_1)\mu(d_2)\mu(d_3).$$

By interchanging the summations, we get

$$\mathcal{A}(t) = \mu(t) \sum_{\substack{\mathbf{d} \\ [d_1,d_2,d_3]=t}} \mu(d_1)\mu(d_2)\mu(d_3)\mathcal{L}_{\mathbf{d}}(N), \tag{9.1}$$

where $\mathcal{L}_{\mathbf{d}}(N)$ is defined in (2.4). The expected main term for $\mathcal{A}(t)$ is

$$\mathcal{M}_t = \mu(t) \sum_{\substack{\mathbf{d} \\ [d_1,d_2,d_3]=t}} \mu(d_1)\mu(d_2)\mu(d_3)\frac{\Sigma_0(\mathbf{d},N)\mathcal{N}_0(N)}{d_1d_2d_3}, \tag{9.2}$$

where $\Sigma_0(\mathbf{d},N)$ and $\mathcal{N}_0(N)$ are given in (2.2) and (2.5) respectively.

Let $X = \Sigma_0(\mathbf{e}, N)\mathcal{N}_0(N)$ with $\mathbf{e} = (1,1,1)$. We point out that for $N \equiv 4 \pmod{24}$ (cf. [9, (311)]),

$$1 \ll \Sigma_0(\mathbf{e}, N) \ll \log\log N.$$

Therefore we can define

$$\Omega(t) = t\mu(t) \sum_{\substack{\mathbf{d} \\ [d_1,d_2,d_3]=t}} \mu(d_1)\mu(d_2)\mu(d_3)\frac{\Sigma_0(\mathbf{d}, N)}{d_1d_2d_3\Sigma_0(\mathbf{e}, N)}.$$

Then we see that

$$\mathcal{M}_t = \frac{\Omega(t)}{t}X. \tag{9.3}$$

For $p > 2$, we define

$$h_0(p) = \begin{cases} \frac{1}{p} & \text{if } p \mid N, \\ \frac{-1}{p(p-1)}\left(1 + \left(\frac{-N}{p}\right)\right) & \text{if } p \nmid N, \end{cases}$$

$$h_1(p) = \begin{cases} \frac{-1}{p}\left(\frac{-1}{p}\right) & \text{if } p \mid N, \\ \frac{1}{p-1}\left(\left(\frac{-N}{p}\right) + \frac{1}{p}\left(\frac{-1}{p}\right)\right) & \text{if } p \nmid N, \end{cases}$$

$$h_2(p) = \begin{cases} \left(\frac{-1}{p}\right) & \text{if } p \mid N, \\ \frac{-1}{p-1}\left(\left(\frac{-1}{p}\right) + \left(\frac{N}{p}\right)\right) & \text{if } p \nmid N, \end{cases}$$

$$h_3(p) = \begin{cases} -1 & \text{if } p \mid N, \\ \frac{1}{p-1}\left(p\left(\frac{N}{p}\right) + 1\right) & \text{if } p \nmid N. \end{cases}$$

The function $\Omega(t)$ is multiplicative with

$$\Omega(p) = \frac{3(1 + h_1(p))}{1 + h_0(p)} - \frac{3(1 + h_2(p))}{p(1 + h_0(p))} + \frac{1 + h_3(p)}{p^2(1 + h_0(p))}$$

for $p > 2$ and $\Omega(2) = 0$. One can easily show that

$$0 \le \Omega(p) < \min\{p, 8\}$$

and

$$\Omega(p) = 3 + O\left(\frac{1}{p}\right).$$

Hence (8.2) and (8.3) are established. Now we turn to (8.4). We have

$$\begin{aligned} E(D) &:= \sum_{t \le D} \mu^2(t)\tau^2(t)\left|\mathcal{A}(t) - \frac{\Omega(t)}{t}X\right| \\ &= \sum_{t \le D} \mu^2(t)\tau^2(t)\xi(t, N)\left(\mathcal{A}(t) - \frac{\Omega(t)}{t}X\right), \end{aligned}$$

where

$$\xi(t, N) = \begin{cases} |\mathcal{A}(t) - \frac{\Omega(t)}{t}X|(\mathcal{A}(t) - \frac{\Omega(t)}{t}X)^{-1} & \text{if } \mathcal{A}(t) - \frac{\Omega(t)}{t}X \text{ is non-zero}, \\ 0 & \text{otherwise}. \end{cases}$$

Recalling (9.1), (9.2) and (9.3), we have

$$E(D) = \sum_{\substack{d_1,d_2,d_3 \\ [d_1,d_2,d_3]\le D}} \beta(d_1,d_2,d_3)\left(\mathcal{L}_{\mathbf{d}}(N) - \frac{\Sigma_0(\mathbf{d},N)\mathcal{N}_0(N)}{d_1d_2d_3}\right), \tag{9.4}$$

where

$$\beta(d_1,d_2,d_3) = \mu(d_1)\mu(d_2)\mu(d_3)\mu([d_1,d_2,d_3])\tau^2([d_1,d_2,d_3])\xi([d_1,d_2,d_3],N).$$

Obviously,

$$\beta(d_1,d_2,d_3) \ll \prod_{j=1}^{3}\tau^2(d_j).$$

Invoking Proposition 3, we see that (8.4) holds true with $D = P^{1/2-\varepsilon}$. Therefore the inequalities (8.5) and (8.6) hold for the sequence

$$a_n := a_n^{(j)} = \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ q\in\mathcal{A}_j \\ x_1x_2x_3=n}} \omega(q)\omega(\mathbf{x}).$$

Lemma 16. *Let $c_1(t)=1$ or $c_1(t)=0$ according to $t\ge 1$ or $t<1$. We define $c_j(t)$ inductively by*

$$c_j(t) = \int_j^{\max(j,t)} \frac{c_{j-1}(x-1)}{x-1}\,dx.$$

Then for $1\le j\le 15$, one has

$$\frac{N}{\log N} \ll \mathcal{N}_0^j(N) \ll \frac{N}{\log N}$$

and

$$\mathcal{N}_0^j(N) = \left(c_j(16) + O\left(\frac{1}{\log N}\right)\right)\mathcal{N}_0^1(N).$$

Proof. In view of [2, (2.16)], we have

$$C_j(x) = c_j(16) + O\left(\frac{1}{\log N}\right) \quad \text{for } \frac{P}{2} < x < P.$$

Hence

$$\mathcal{N}_0^j(N) - c_j(16)\mathcal{N}_0^1(N) \ll \frac{N}{\log^2 N}.$$

We get the desired results by observing that

$$\frac{N}{\log N} \ll \mathcal{N}_0^1(N) \ll \frac{N}{\log N}. \qquad \square$$

Remark. We record some numerical values:

$$c_2(16) > 2.70805, \quad c_3(16) > 2.912112, \quad c_4(16) > 1.663428, \quad c_5(16) > 0.563668.$$

Proof of Theorem 1.1. Our objective is to prove

$$R(N) := \sum_{\substack{p^2+x_1^2+x_2^2+x_3^2=N \\ x_1,x_2,x_3\in\bigcup_{j=1}^{5}\mathcal{A}_j}} \omega(p)\omega(\mathbf{x}) > 0.$$

Observe that

$$\sum_{\substack{p\in\mathcal{A}_1 \\ x_1,x_2,x_3\in\bigcup_{j=1}^{5}\mathcal{A}_j}} = \sum_{\substack{p\in\mathcal{A}_1 \\ x_1,x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} - \sum_{\substack{p\in\mathcal{A}_1 \\ x_1\in\bigcup_{j=6}^{15}\mathcal{A}_j \\ x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} - \sum_{\substack{p\in\mathcal{A}_1 \\ x_1\in\bigcup_{j=1}^{5}\mathcal{A}_j \\ x_2\in\bigcup_{j=6}^{15}\mathcal{A}_j \\ x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} - \sum_{\substack{p\in\mathcal{A}_1 \\ x_1,x_2\in\bigcup_{j=1}^{5}\mathcal{A}_j \\ x_3\in\bigcup_{j=6}^{15}\mathcal{A}_j}}$$

$$\geq \sum_{\substack{p\in\mathcal{A}_1 \\ x_1,x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} - \sum_{\substack{q\in\bigcup_{j=1}^{15}\mathcal{A}_j \\ x_1\in\bigcup_{j=6}^{15}\mathcal{A}_j \\ x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} - \sum_{\substack{q\in\bigcup_{j=1}^{15}\mathcal{A}_j \\ x_2\in\bigcup_{j=6}^{15}\mathcal{A}_j \\ x_1,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} - \sum_{\substack{q\in\bigcup_{j=1}^{15}\mathcal{A}_j \\ x_3\in\bigcup_{j=6}^{15}\mathcal{A}_j \\ x_1,x_2\in\bigcup_{j=1}^{15}\mathcal{A}_j}}.$$

By switching the roles of q and x_j, we obtain

$$R(N) \geq \sum_{\substack{p^2+x_1^2+x_2^2+x_3^2=N \\ x_1,x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} \omega(p)\omega(\mathbf{x}) - 3 \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ q\in\bigcup_{j=6}^{15}\mathcal{A}_j \\ x_1,x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} \omega(q)\omega(\mathbf{x}).$$

In view of the three-dimensional sieve, we have

$$\sum_{\substack{p^2+x_1^2+x_2^2+x_3^2=N \\ x_1,x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} \omega(p)\omega(\mathbf{x}) = \sum_{(n,\Pi(z))=1} \sum_{\substack{p^2+x_1^2+x_2^2+x_3^2=N \\ x_1x_2x_3=n}} \omega(p)\omega(\mathbf{x})$$

$$\geq (f_3(8-\varepsilon)-\varepsilon)\Sigma_0(\mathbf{e},N)\mathcal{N}_0^1(N)V(z),$$

where $z = P^{1/16}$. Similarly,

$$\sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ q\in\bigcup_{j=6}^{15}\mathcal{A}_j \\ x_1,x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} \omega(q)\omega(\mathbf{x}) = \sum_{j=6}^{15} \sum_{(n,\Pi(z))=1} \sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ q\in\mathcal{A}_j \\ x_1x_2x_3=n}} \omega(q)\omega(\mathbf{x})$$

$$\leq (F_3(8-\varepsilon)+\varepsilon)\Sigma_0(\mathbf{e},N)\sum_{j=6}^{15}\mathcal{N}_0^j(N)V(z).$$

By Lemma 16, we obtain

$$\sum_{\substack{q^2+x_1^2+x_2^2+x_3^2=N \\ q\in\bigcup_{j=6}^{15}\mathcal{A}_j \\ x_1,x_2,x_3\in\bigcup_{j=1}^{15}\mathcal{A}_j}} \omega(q)\omega(\mathbf{x}) \leq \left(F_3(8-\varepsilon)\sum_{j=6}^{15}c_j(16)+\varepsilon\right)\Sigma_0(\mathbf{e},N)\mathcal{N}_0^1(N)V(z).$$

Therefore

$$R(N) \geq \left(f_3\left(\frac{16}{2}\right) - 3F_3\left(\frac{16}{2}\right) \sum_{j=6}^{15} c_j(16) - \varepsilon \right) \Sigma_0(\mathbf{e}, N) \mathcal{N}_0^1(N) V(z)$$
$$= \left(\frac{f_3(8)}{3F_3(8)} - \sum_{j=6}^{15} c_j(16) - \varepsilon \right) 3F_3(8) \Sigma_0(\mathbf{e}, N) \mathcal{N}_0^1(N) V(z)$$
$$= (C_0 - \varepsilon) 3F_3(8) \Sigma_0(\mathbf{e}, N) \mathcal{N}_0^1(N) V(z),$$

where

$$C_0 = \frac{f_3(8)}{3F_3(8)} - \sum_{j=6}^{15} c_j(16).$$

By (8.7), we have

$$\frac{f_3(8)}{F_3(8)} > 1 - \left(\frac{\beta_3}{8}\right)^3,$$

where $\beta_3 \leq 6.640859$. Thus

$$C_0 > \frac{1}{3}\left(1 - \left(\frac{\beta_3}{8}\right)^3\right) - \sum_{j=6}^{15} c_j(16).$$

Now numerical computations reveal that $C_0 > 0.003$. Thus Theorem 1.1 is established. □

One can do numerical computations in the following way. We have

$$C_0 > \frac{1}{3}\left(1 - \left(\frac{\beta_3}{8}\right)^3\right) + \sum_{j=1}^{5} c_j(16) - \sum_{j=1}^{15} c_j(16).$$

Brüdern and Kawada pointed out that as a consequence of linear sieve, one has (see [2, (6.35)])

$$\sum_{j=1}^{15} c_j(16) \leq 16e^{-\gamma} F_1(16) \leq 16e^{-\gamma}(1 + 10^{-9}).$$

Therefore,

$$C_0 > \frac{1}{3}\left(1 - \left(\frac{\beta_3}{8}\right)^3\right) + \sum_{j=1}^{5} c_j(16) - 16e^{-\gamma}(1 + 10^{-9}).$$

Then we actually need the numerical values of $c_j(16)$ for $1 \leq j \leq 5$.

Proof of Theorem 1.2. The proof is as same as Theorem 1.1 except that we consider

$$R'(N) := \sum_{\substack{q^2 + x_1^2 + x_2^2 + x_3^2 = N \\ q, x_1, x_2, x_3 \in \bigcup_{j=1}^4 \mathcal{A}_j}} \omega(q)\omega(\mathbf{x}).$$

In view of the switching principle and the three-dimensional sieve, the lower bound for $R'(N)$ is

$$\begin{aligned} R'(N) &\geq \left(f_3(8) \sum_{j=1}^{4} c_j(16) - 3F_3(8) \sum_{j=5}^{15} c_j(16) - \varepsilon \right) \Sigma_0(\mathbf{e}, N) \mathcal{N}_0^1(N) V(z) \\ &= \left(\frac{f_3(8)}{3F_3(8)} \sum_{j=1}^{4} c_j(16) - \sum_{j=5}^{15} c_j(16) - \varepsilon \right) 3F_3(8) \Sigma_0(\mathbf{e}, N) \mathcal{N}_0^1(N) V(z) \\ &= (C_0' - \varepsilon) 3F_3(8) \Sigma_0(\mathbf{e}, N) \mathcal{N}_0^1(N) V(z), \end{aligned}$$

where

$$C_0' = \frac{f_3(8)}{3F_3(8)} \sum_{j=1}^{4} c_j(16) - \sum_{j=5}^{15} c_j(16).$$

Similarly, we have

$$C_0' > \frac{1}{3}\left(1 - \left(\frac{\beta_3}{8}\right)^3\right) \sum_{j=1}^{4} c_j(16) + \sum_{j=1}^{4} c_j(16) - 16e^{-\gamma}(1 + 10^{-9}).$$

Again a numerical computation reveals that $C_0' > 0$. We point out that we gain P_4 instead of P_5 due to the constant $\sum_{j=1}^{4} c_j(16)$ in place of 1. This completes the proof of Theorem 1.2. □

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