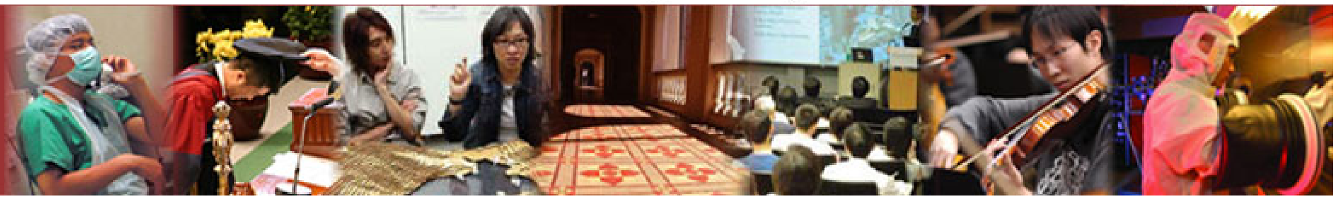

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$\mathcal{H}_\infty$ and $\mathcal{H}_2$ Norms of 2D Mixed Continuous-Discrete-Time Systems via Rationally-Dependent Complex Lyapunov Functions

Graziano Chesi* and Richard H. Middleton†

Abstract

This paper addresses the problem of determining the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of 2D mixed continuous-discrete-time systems. The first contribution is to propose a novel approach based on the use of complex Lyapunov functions with even rational parametric dependence, which searches for upper bounds on the sought norms via linear matrix inequalities (LMIs). The second contribution is to show that the upper bounds provided are nonconservative by using Lyapunov functions in the chosen class with sufficiently large degree. The third contribution is to provide conditions for establishing the tightness of the upper bounds. The fourth contribution is to show how the numerical complexity of the proposed approach can be significantly reduced by proposing a new necessary and sufficient LMI condition for establishing positive semidefiniteness of even Hermitian matrix polynomials. This result is also exploited to derive an improved necessary and sufficient LMI condition for establishing exponential stability of 2D mixed continuous-discrete-time systems. Some numerical examples illustrate the proposed approach. It is worth remarking that nonconservative LMI methods for determining the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of 2D mixed continuous-discrete-time systems have not been proposed yet in the literature.

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1 Introduction

2D mixed continuous-discrete-time systems is an important area of control systems. Such systems are characterized by both continuous-time and discrete-time dynamics, which mutually influence each other. The study of these systems has a long history, the reader is referred to [11, 20] for the introduction of basic models and fundamentals properties. 2D mixed continuous-discrete-time systems can be found in a number of applications, including repetitive processes [21], disturbance propagation in vehicle platoons [12], and irrigation channels [14, 16]. Fundamental problems in 2D systems include stability analysis, which has been considered in a number of works such as [1, 7, 13, 22] (see also [2, 3, 10, 15] for contributions to stability analysis of other classes of 2D systems). As in typical 1D systems, another fundamental problem in 2D systems is performance analysis.

This paper addresses the problem of determining the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of 2D mixed continuous-discrete-time systems. The first contribution is to propose a novel approach based on the use of complex Lyapunov functions with even rational parametric dependence, which searches for upper bounds on the sought norms via linear matrix inequalities (LMIs). The second contribution is to show that the upper bounds provided are nonconservative by using Lyapunov functions in the chosen class with sufficiently large degree. The third contribution is to provide conditions for establishing the tightness of the upper bounds. Such conditions are necessary and sufficient in the case of the $\mathcal{H}_\infty$ norm, and necessary in the case of the $\mathcal{H}_2$ norm. The fourth contribution of the paper is to show how the numerical complexity of the proposed approach can be significantly reduced by proposing a new necessary and sufficient LMI condition for establishing positive semidefiniteness of even Hermitian matrix polynomials. This result is also exploited to derive an improved necessary and sufficient LMI condition for establishing exponential stability of 2D mixed continuous-discrete-time systems. Some numerical examples illustrate the proposed approach and its advantages over existing methods. The LMI problems are solved with the toolbox SeDuMi [23] for Matlab on a standard computer (Windows 7, Intel Core 2, 3 GHz, 4 GB Ram).

The contribution of this paper with respect to the existing literature is to propose for the first time nonconservative LMI methods for determining the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of 2D mixed continuous-discrete-time systems. Indeed, the computation of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms has been investigated in [18, 19] via LMIs, however these conditions are only sufficient. Also, an LMI method based on the use of polynomial Lyapunov functions has been proposed in [7] for stability analysis and for the computation of the $\mathcal{H}_\infty$ norm. However, while this method is nonconservative for stability analysis, it can be conservative for the computation of the $\mathcal{H}_\infty$ norm (see Example 2 in this paper). Moreover, the numerical complexity of the LMI method in [7] is significantly larger than that of the novel approach proposed here (see Example 2 in this paper).

This paper extends the preliminary conference version [8] by adding Theorem 1 (complexity reduction), Theorem 3 (construction of upper bounds on the $\mathcal{H}_2$ norm), Theorem 5 (asymptotical nonconservatism of the upper bound on the $\mathcal{H}_2$ norm), Theorem 6 (tightness of the upper bound on the $\mathcal{H}_\infty$ norm), Theorem 7 (tightness of the upper bound on the $\mathcal{H}_2$ norm), and Corollary 1 (improved stability condition).

The paper is organized as follows. Section 2 provides the problem formulation. Section 3 investigates positive semidefiniteness of Hermitian matrix functions. Section 4 addresses the construction of the upper bounds on the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms. Section 5 analyzes the conservatism of these upper bounds. Section 6 provides the improved condition for exponential stability. Lastly, Section 7 concludes the paper with some final remarks.

2 Problem Formulation

Notation:

- $\mathbb{R}, \mathbb{C}$: real and complex number sets;
- j: imaginary unit;
- I: identity matrix (of size specified by the context);
- $\Re(\cdot)$, $\Im(\cdot)$: real and imaginary parts;
- $|\cdot|$: magnitude;
- $\|\cdot\|_2$: Euclidean norm;
- $adj(\cdot)$: adjoint;
- $\det(\cdot)$: determinant;
- $trace(\cdot)$: trace;
- $\bar{A}$: complex conjugate;

- A^T, A^H: transpose and complex conjugate transpose;
- $A \otimes B$: Kronecker product;
- $A \circ B$: Hadamard product;
- Hermitian matrix A: a complex square matrix satisfying $A^H = A$;
- $\star$: corresponding block in Hermitian matrices;
- $A > 0$, $A \geq 0$: Hermitian positive definite and Hermitian positive semidefinite matrix A;
- $\deg(\cdot)$: degree;
- $\|\cdot\|_{\mathcal{L}_2}$: $\mathcal{L}_2$ norm;
- $\|\cdot\|_{Z-\mathcal{H}_\infty}$, $\|\cdot\|_{Z-\mathcal{H}_2}$: Z $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms;
- $\|\cdot\|_{LZ-\mathcal{H}_\infty}$, $\|\cdot\|_{LZ-\mathcal{H}_2}$: Laplace-Z $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms.

Let us consider the 2D mixed continuous-discrete-time system described by

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}x_c(t,k) & = & A_{cc}x_c(t,k) + A_{cd}x_d(t,k) + B_c u(t,k) \\ x_d(t,k+1) & = & A_{dc}x_c(t,k) + A_{dd}x_d(t,k) + B_d u(t,k) \\ y(t,k) & = & C_c x_c(t,k) + C_d x_d(t,k) + Du(t,k) \end{array}\right. \tag{1}$$

where $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ are the continuous and discrete states, the scalars t and k are independent variables, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are the input and output, and $A_{cc} \in \mathbb{R}^{n_c \times n_c}$, $A_{cd} \in \mathbb{R}^{n_c \times n_d}$, $A_{dc} \in \mathbb{R}^{n_d \times n_c}$, $A_{dd} \in \mathbb{R}^{n_d \times n_d}$, $B_c \in \mathbb{R}^{n_c \times n_u}$, $B_d \in \mathbb{R}^{n_d \times n_u}$, $C_c \in \mathbb{R}^{n_y \times n_c}$, $C_d \in \mathbb{R}^{n_y \times n_d}$ and $D \in \mathbb{R}^{n_y \times n_u}$ are given matrices.

The system (1) is said to be exponentially stable if there exist $\beta > 0$ and $\gamma > 0$ such that

$$\left\| \begin{pmatrix} x_c(t,k) \\ x_d(t,k) \end{pmatrix} \right\|_2 \leq \beta \varrho e^{-\gamma \min\{t,k\}} \tag{2}$$

for all initial conditions $x_c(0,k) \in \mathbb{R}^{n_c}$ and $x_d(t,0) \in \mathbb{R}^{n_d}$ for all $t \geq 0$ and $k \geq 0$, where

$$\left\{\begin{array}{rcl} \varrho & = & \max\{\varrho_1, \varrho_2\} \\ \varrho_1 & = & \sup_{t \geq 0} \|x_d(t,0)\|_2 \\ \varrho_2 & = & \sup_{k \geq 0} \|x_c(0,k)\|_2 . \end{array}\right. \tag{3}$$

See [17, 25] for details on this definition and for alternative ones.

The $\mathcal{H}_\infty$ norm of (1) is known to be equal to its $\mathcal{L}_2$ gain, which is given by

$$\gamma_\infty = \sup_{u:\ \|u\|_{\mathcal{L}_2}\neq 0} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \tag{4}$$

where $\|\cdot\|_{\mathcal{L}_2}$ is the $\mathcal{L}_2$ norm defined as

$$\|u\|_{\mathcal{L}_2} = \sqrt{\sum_{k=0}^{\infty}\int_0^\infty \|u(t,k)\|_2^2 dt}. \tag{5}$$

The $\mathcal{H}_2$ norm of (1) is defined as

$$\gamma_2 = \sqrt{\sum_{l=1}^{n_u}\sum_{k=0}^{\infty}\int_0^\infty g^T(t,k,l)g(t,k,l)dt} \tag{6}$$

where $g(t,k,l)$ is the response of the system (1) due to an impulse applied at $k=0$ to the l-th channel, i.e., the solution of $y(t,k)$ for null initial conditions and $u(t,k)$ given by

$$u(t,k) = \begin{cases} \delta(t)b(l) & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where $\delta(t)$ is the Dirac function and $b(l)$ is the l-th canonical basis vector in $\mathbb{R}^{n_u}$.

Problems 1 and 2. The problems addressed in this paper consist of determining the $\mathcal{H}_\infty$ norm (Problem 1) and the $\mathcal{H}_2$ norm (Problem 2) of the system (1), i.e., γ_∞ and γ_2. $\square$

3 Semidefinite Hermitian Matrix Polynomials

In this section we introduce some preliminary results that will be exploited in the next sections for establishing whether a Hermitian matrix polynomial is positive semidefinite.

Let us start by introducing the following definition. For a complex matrix function $M : \mathbb{R} \to \mathbb{C}^{n_1 \times n_2}$, we say that $M(\omega)$ is *even* if

$$M(-\omega) = \overline{M(\omega)} \quad \forall \omega \in \mathbb{R} \tag{8}$$

and we say that $M(\omega)$ is *odd* if

$$M(-\omega) = -\overline{M(\omega)} \quad \forall \omega \in \mathbb{R}. \tag{9}$$

It follows that a complex matrix function $M : \mathbb{R} \to \mathbb{C}^{n_1 \times n_2}$ can be decomposed as

$$M(\omega) = M_{even}(\omega) + M_{odd}(\omega) \tag{10}$$

where the complex matrix functions $M_{even}, M_{odd} : \mathbb{R} \to \mathbb{C}^{n_1 \times n_2}$ are even and odd, respectively. In particular, one has

$$\left\{ \begin{array}{rcl} M_{even}(\omega) & = & \dfrac{M(\omega) + \overline{M(-\omega)}}{2} \\ M_{odd}(\omega) & = & \dfrac{M(\omega) - \overline{M(-\omega)}}{2}. \end{array} \right. \tag{11}$$

Let us define the set of Hermitian matrix polynomials

$$\begin{array}{c} \mathcal{P}(n) = \{M : \mathbb{R} \to \mathbb{C}^{n \times n},\ M(\omega) \text{ is a Hermitian} \\ \text{matrix polynomial}\}. \end{array} \tag{12}$$

Let us write $M \in \mathcal{P}(n)$ as

$$M(\omega) = M_R(\omega) + jM_I(\omega) \tag{13}$$

where $M_R, M_I : \mathbb{R} \to \mathbb{R}^{n \times n}$ are matrix polynomials with the property

$$\left\{ \begin{array}{rcl} M_R(\omega) & = & M_R(\omega)^T \\ M_I(\omega) & = & -M_I(\omega)^T. \end{array} \right. \tag{14}$$

Let us define $\Phi : \mathbb{C}^{n \times n} \to \mathbb{R}^{\varsigma \times \varsigma}$ as

$$\Phi(M(\omega)) = \left\{ \begin{array}{l} M(\omega) \text{ if } M(\omega) \text{ is real for all } \omega \in \mathbb{R} \\ \left(\begin{array}{cc} M_R(\omega) & M_I(\omega) \\ -M_I(\omega) & M_R(\omega) \end{array} \right) \text{ otherwise} \end{array} \right. \tag{15}$$

where

$$\varsigma = \left\{ \begin{array}{ll} n & \text{if } M(\omega) \text{ is real for all } \omega \in \mathbb{R} \\ 2n & \text{otherwise.} \end{array} \right. \tag{16}$$

From [4] and [7], one has that

$$M(\omega) \geq 0 \quad \forall \omega \in \mathbb{R} \tag{17}$$

if and only if

$$\Phi(M(\omega)) \text{ is SOS.} \tag{18}$$

The condition (18) amounts to solving an LMI feasibility test. Indeed, let d be the smallest integer such that $\deg(M) \leq 2d$, and let us express $\Phi(M(\omega))$ according to the square matrix representation (SMR) of matrix polynomials [4–6] as

$$\Phi(M(\omega)) = (b(\omega) \otimes I)^T \left(K + L(\alpha)\right) (b(\omega) \otimes I) \tag{19}$$

where $b(\omega) \in \mathbb{R}^{\sigma}$ is a vector whose entries are the monomials in ω of degree less than or equal to d with

$$\sigma = d + 1, \tag{20}$$

$K \in \mathbb{R}^{\sigma\varsigma \times \sigma\varsigma}$ is a symmetric matrix that satisfies

$$\Phi(M(\omega)) = (b(\omega) \otimes I)^T K \left(b(\omega) \otimes I\right), \tag{21}$$

$L : \mathbb{R}^{\tau} \in \mathbb{R}^{\sigma\varsigma \times \sigma\varsigma}$ is a linear parametrization of the subspace

$$\mathcal{L} = \left\{ \tilde{L} = \tilde{L}^T : \;\; (b(\omega) \otimes I)^T \tilde{L} \left(b(\omega) \otimes I\right) = 0 \;\forall \omega \in \mathbb{R} \right\} \tag{22}$$

and $\alpha \in \mathbb{R}^{\tau}$ is a free vector where the quantity τ is the dimension of $\mathcal{L}$ given by

$$\tau = \frac{\varsigma}{2} \left(\sigma \left(\sigma\varsigma + 1\right) - (\varsigma + 1)(2\sigma - 1)\right). \tag{23}$$

One has that (18) holds if and only if there exists $\alpha \in \mathbb{R}^{\tau}$ satisfying the LMI

$$K + L(\alpha) \geq 0. \tag{24}$$

The number of LMI scalar variables in (24) is given by the dimension of the vector α, i.e., τ.

In what follows we will propose a new necessary and sufficient LMI condition for establishing positive semidefiniteness of even Hermitian matrix polynomials, whose number of LMI scalar variables is significantly smaller than that of (24). Indeed, let us define the set

$$\mathcal{P}_{even}(n) = \left\{ M \in \mathcal{P}(n), \; M(\omega) \text{ is even} \right\}. \tag{25}$$

One can write $M \in \mathcal{P}_{even}(n)$ as

$$M(\omega) = \tilde{M}_R(\omega^2) + j\omega\tilde{M}_I(\omega^2) \tag{26}$$

where $\tilde{M}_R, \tilde{M}_I : \mathbb{R} \to \mathbb{R}^{n\times n}$ are matrix polynomials. It follows that

$$\Phi(M(\omega)) = \begin{cases} M(\omega) \text{ if } M(\omega) \text{ is real for all } \omega \in \mathbb{R} \\ \begin{pmatrix} \tilde{M}_R(\omega^2) & \omega\tilde{M}_I(\omega^2) \\ -\omega\tilde{M}_I(\omega^2) & \tilde{M}_R(\omega^2) \end{pmatrix} \text{ otherwise.} \end{cases} \tag{27}$$

Let us express $\Phi\left(M(\omega)\right)$ in (27) as

$$\Phi\left(M(\omega)\right) = \left(b(\omega)\otimes I\right)^T \left(K + L_{even}(\alpha_{even})\right)\left(b(\omega)\otimes I\right) \tag{28}$$

where $b(\omega)$ and K are as in (19), and $L_{even} : \mathbb{R}^{\tau_{even}} \to \mathbb{R}^{\sigma\varsigma\times\sigma\varsigma}$ is a linear parametrization of the subspace

$$\mathcal{L}_{even} = \mathcal{L} \cap \mathcal{E} \tag{29}$$

where $\mathcal{L}$ is as in (22) and $\mathcal{E}$ is defined as follows:

- if $M(\omega)$ is real for all $\omega \in \mathbb{R}$, then

$$\begin{array}{c} \mathcal{E} = \Big\{ E = E^T : \ E_{ik} = 0 \ \ \forall i,k : \\ b_i(\omega)b_k(\omega) \text{ is an odd power of } \omega \Big\} \end{array} \tag{30}$$

  where $E_{ik} \in \mathbb{R}^{n\times n}$, $i,k = 1,\ldots,\sigma$, partition $E \in \mathbb{R}^{\sigma\varsigma\times\sigma\varsigma}$ according to

$$E = \begin{pmatrix} E_{11} & \ldots & E_{1\sigma} \\ \vdots & \ddots & \vdots \\ \star & \ldots & E_{\sigma\sigma} \end{pmatrix}; \tag{31}$$

- otherwise,

$$\begin{array}{c} \mathcal{E} = \Big\{ E = E^T : \ E_{ikl} = 0 \ \ \forall i,k,l : \\ b_i(\omega)b_k(\omega) \text{ is an odd power of } \omega \text{ and } l = 1 \\ \text{or } b_i(\omega)b_k(\omega) \text{ is an even power of } \omega \text{ and } l = 2 \Big\} \end{array} \tag{32}$$

where $E_{ikl} \in \mathbb{R}^{n \times n}$, $i,k = 1,\dots,\sigma$ and $l = 1,2$, partition $E \in \mathbb{R}^{\sigma\varsigma \times \sigma\varsigma}$ according to

$$E = \begin{pmatrix} E_{111} & E_{112} & \dots & E_{1\sigma 1} & E_{1\sigma 2} \\ -E_{112} & E_{111} & \dots & -E_{1\sigma 2} & E_{1\sigma 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \dots & E_{\sigma\sigma 1} & E_{\sigma\sigma 2} \\ \star & \star & \dots & -E_{\sigma\sigma 2} & E_{\sigma\sigma 1} \end{pmatrix}. \tag{33}$$

The vector $\alpha_{even} \in \mathbb{R}^{\tau_{even}}$ is free, and τ_{even} is the dimension of $\mathcal{L}_{even}$.

The following result provides a necessary and sufficient LMI condition for establishing whether $M \in \mathcal{P}_{even}(n)$ is positive semidefinite over $\mathbb{R}$, whose number of LMI scalar variables is significantly reduced with respect to the LMI (24).

Theorem 1 *Let $M \in \mathcal{P}_{even}(n)$. Then,* (17)*,* (18) *and* (24) *are equivalent to the existence of $\alpha_{even} \in \mathbb{R}^{\tau_{even}}$ satisfying the LMI*

$$K + L_{even}(\alpha_{even}) \geq 0. \tag{34}$$

Proof. "$\Leftarrow$" Suppose that there exists α_{even} satisfying (34). Then, there exists α satisfying (24) because $L_{even}(\alpha_{even})$ parametrizes $\mathcal{L}_{even}$, which is a subset of the matrices parametrized by $L(\alpha)$, i.e., $\mathcal{L}$.

"$\Rightarrow$" Suppose that there exists α satisfying (24). From (29), there are two cases: $L(\alpha) \in \mathcal{L}_{even}$ or $L(\alpha) \notin \mathcal{L}_{even}$. In the case where $L(\alpha) \in \mathcal{L}_{even}$, it directly follows that there exists α_{even} such that

$$L_{even}(\alpha_{even}) = L(\alpha)$$

since $L_{even}(\alpha_{even})$ parametrizes $\mathcal{L}_{even}$. Hence, let us consider the case where $L(\alpha) \notin \mathcal{L}_{even}$. Let us observe that, if $\tilde{M}_I(\omega) \neq 0$ for some $\omega \in \mathbb{R}$, one can choose without loss of generality α such that $L(\alpha)$ has the structure of E in (33) due to the definition of $\Phi\left(M(\omega)\right)$. Let α_{even} be such that $L_{even}(\alpha_{even})$ is the projection of $L(\alpha)$ onto $\mathcal{L}_{even}$, i.e.

$$L(\alpha) = L_{even}(\alpha_{even}) + \tilde{L}$$

where $\tilde{L} \in \mathbb{R}^{\sigma\varsigma \times \sigma\varsigma}$ is a symmetric matrix such that

$$\tilde{L} \circ E = 0 \quad \forall E \in \mathcal{E}$$

and the operator "$\circ$" denotes the Hadamard product. Since the diagonal blocks of $\tilde{L}$ are null, and since the possible non-zero blocks of $\tilde{L}$ are null in $K + L(\alpha)$, it follows that

$$\nu_i \left(K + L(\alpha) \right) \leq \nu_i \left(K + L_{even}(\alpha_{even}) \right) \forall i = 1, \ldots, \sigma\varsigma$$

where $\nu_i(\cdot)$ denotes the i-th principal minor. This implies that (34) holds since $K + L(\alpha) \geq 0$. $\square$

Theorem 1 states that $M \in \mathcal{P}_{even}(n)$ is positive semidefinite over $\mathbb{R}$ if and only if $\Phi(M(\omega))$ is SOS, and this is equivalent to the feasibility of the LMI (34). The number of LMI scalar variables in this LMI is given by the dimension of the vector α_{even}, i.e., τ_{even}. With some calculations, one can show that, if $M(\omega)$ is real for all $\omega \in \mathbb{R}$,

$$\tau_{even} = \left\lfloor \frac{d^2}{4} \right\rfloor n^2, \tag{35}$$

otherwise

$$\tau_{even} = \frac{dn(dn+1)}{2}. \tag{36}$$

As shown by Tables 1 and 2, τ_{even} is significantly smaller than the number of LMI scalar variables in the LMI (24), i.e., τ.

Example 1. Let us consider $M \in \mathcal{P}_{even}(2)$ defined as

$$M(\omega) = \begin{pmatrix} \omega^2 + 1 & 1 + j2\omega \\ \star & \omega^2 + p \end{pmatrix}$$

where $p \in \mathbb{R}$ is a parameter. It follows from (27) that

$$\Phi\left(M(\omega)\right) = \begin{pmatrix} \omega^2 + 1 & 1 & 0 & 2\omega \\ \star & \omega^2 + p & -2\omega & 0 \\ \star & \star & \omega^2 + 1 & 1 \\ \star & \star & \star & \omega^2 + p \end{pmatrix}.$$

The matrix polynomial $\Phi\left(M(\omega)\right)$ can be written according to the SMR in (28) with

$$\left\{\begin{array}{l} b(\omega)=\begin{pmatrix}1\\ \omega\end{pmatrix},\ K=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\ \star & p & 0 & 0 & 0 & 0 & -1 & 0\\ \star & \star & 1 & 1 & 0 & -1 & 0 & 0\\ \star & \star & \star & p & 1 & 0 & 0 & 0\\ \star & \star & \star & \star & 1 & 0 & 0 & 0\\ \star & \star & \star & \star & \star & 1 & 0 & 0\\ \star & \star & \star & \star & \star & \star & 1 & 0\\ \star & \star & \star & \star & \star & \star & \star & 1 \end{pmatrix}\\ L_{even}(\alpha)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & \alpha_2\\ \star & 0 & 0 & 0 & 0 & 0 & \alpha_2 & \alpha_3\\ \star & \star & 0 & 0 & -\alpha_1 & -\alpha_2 & 0 & 0\\ \star & \star & \star & 0 & -\alpha_2 & -\alpha_3 & 0 & 0\\ \star & \star & \star & \star & 0 & 0 & 0 & 0\\ \star & \star & \star & \star & \star & 0 & 0 & 0\\ \star & \star & \star & \star & \star & \star & 0 & 0\\ \star & \star & \star & \star & \star & \star & \star & 0 \end{pmatrix}. \end{array}\right.$$

First, let us consider the case $p=1$. It follows that

$$\nexists \alpha_{even}:\ K+L_{even}(\alpha_{even})\geq 0,$$

which implies from Theorem 1 that $M(\omega)$ is not SOS and that $M(\omega)\ngeq 0$ for some $\omega\in\mathbb{R}$.

Second, let us consider the case $p=2$. It follows that

$$\alpha_{even}=\begin{pmatrix}-0.448\\ -0.224\\ 0.448\end{pmatrix} \quad\Rightarrow\quad K+L_{even}(\alpha_{even})\geq 0,$$

which implies from Theorem 1 that $M(\omega)$ is SOS and that $M(\omega)\geq 0$ for all $\omega\in\mathbb{R}$. $\square$

4 $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Norm Upper Bounds: Construction

In this section we address the construction of upper bounds on the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of the system (1), i.e., γ_∞ and γ_2 in (4) and (6). Let us start by introducing the following assumption, which is a necessary condition for

τ	$2d=2$	$2d=4$	$2d=6$	$2d=8$
$n=1$	0	1	3	6
$n=2$	1	6	15	28
$n=3$	3	15	36	66
$n=4$	6	28	66	120
$n=5$	10	45	105	190

(A)

τ_{even}	$2d=2$	$2d=4$	$2d=6$	$2d=8$
$n=1$	0	1	2	4
$n=2$	0	4	8	16
$n=3$	0	9	18	36
$n=4$	0	16	32	64
$n=5$	0	25	50	100

(B)

Table 1: Number of LMI scalar variables τ (A) and τ_{even} (B) for some values of n and $2d$ in the case where $M(\omega)$ is real for all $\omega \in \mathbb{R}$, $M \in \mathcal{P}_{even}(n)$.

exponential stability of (1).

Assumption 1. The matrix A_{cc} is Hurwitz (i.e., all its eigenvalues have negative real parts) and the matrix A_{dd} is Schur (i.e., all its eigenvalues have magnitude less than one). □

The fact that Assumption 1 is a necessary condition for exponential stability of (1) can be easily verified by considering $u(t,k)=0$. For $x_d(t,0)=0$, we obtain $x_c(t,0)=e^{A_{cc}t}x_c(0,0)$. For $x_c(0,k)=0$, we obtain $x_d(0,k)=A_{dd}^k x_d(0,0)$.

Let us denote by $U_L(s,k)$ and $Y_L(s,k)$ the Laplace transforms of $u(t,k)$ and $y(t,k)$, respectively, where $s\in\mathbb{C}$. Let us denote with $U_{LZ}(s,k)$ and $Y_{LZ}(s,k)$ the Z-transforms of $U_L(s,k)$ and $Y_L(s,k)$, respectively, where $z\in\mathbb{C}$. The Laplace-Z transfer function from $u(t,k)$ and $y(t,k)$ can be expressed as

$$F(s,z)=\frac{Y_{LZ}(s,z)}{U_{LZ}(s,z)} \tag{37}$$

and standard manipulations lead to

$$F(s,z)=F_3(s)\left(zI-F_1(s)\right)^{-1}F_2(s)+F_4(s) \tag{38}$$

τ	$2d=2$	$2d=4$	$2d=6$	$2d=8$
$n=1$	N/A	N/A	N/A	N/A
$n=2$	6	28	66	120
$n=3$	15	66	153	276
$n=4$	28	120	276	496
$n=5$	45	190	435	780

(A)

τ_{even}	$2d=2$	$2d=4$	$2d=6$	$2d=8$
$n=1$	N/A	N/A	N/A	N/A
$n=2$	3	10	21	36
$n=3$	6	21	45	78
$n=4$	10	36	78	136
$n=5$	15	55	120	210

(B)

Table 2: Number of LMI scalar variables τ (A) and τ_{even} (B) for some values of n and $2d$ in the case where $M(\omega)$ is not real for some $\omega \in \mathbb{R}$, $M \in \mathcal{P}_{even}(n)$.

where

$$\left\{ \begin{array}{rcl} F_1(s) & = & A_{dc}(sI - A_{cc})^{-1} A_{cd} + A_{dd} \\ F_2(s) & = & A_{dc}(sI - A_{cc})^{-1} B_c + B_d \\ F_3(s) & = & C_c(sI - A_{cc})^{-1} A_{cd} + C_d \\ F_4(s) & = & C_c(sI - A_{cc})^{-1} B_c + D. \end{array} \right. \tag{39}$$

We express $F_i(s)$, $i = 1, \ldots, 4$, as

$$F_i(s) = \frac{G_i(s)}{g(s)} \tag{40}$$

where $G_i(s)$, $i = 1, \ldots, 4$, are matrix polynomials of suitable size, and $g(s)$ is defined as

$$g(s) = \det(sI - A_{cc}) \tag{41}$$

(in the case where $G_1(s), \ldots, G_4(s), g(s)$ have common roots, one can redefine $G_1(s), \ldots, G_4(s)$, $g(s)$ eliminating such common roots in order to lower the degrees and, hence, the computational burden).

Let us consider firstly Problem 1, i.e., the determination of the $\mathcal{H}_\infty$ norm of (1). This norm can be calculated as

$$\gamma_\infty = \|F\|_{LZ-\mathcal{H}_\infty} \tag{42}$$

where $\|F\|_{LZ-\mathcal{H}_\infty}$ is the Laplace-Z $\mathcal{H}_\infty$ norm of $F(s,z)$ defined as

$$\|F\|_{LZ-\mathcal{H}_\infty} = \sup_{\substack{\omega\in\mathbb{R}\\ \theta\in[-\pi,\pi]}} \left\| F(j\omega, e^{j\theta}) \right\|_2 . \tag{43}$$

The $\mathcal{H}_\infty$ norm of (1) can also be rewritten as

$$\gamma_\infty = \sup_{\omega\in\mathbb{R}} \|F_\omega\|_{Z-\mathcal{H}_\infty} \tag{44}$$

where $F_\omega(z)$ is the Z transfer function

$$F_\omega(z) = F(j\omega, z) \tag{45}$$

and $\|F_\omega\|_{Z-\mathcal{H}_\infty}$ is the Z $\mathcal{H}_\infty$ norm of $F_\omega(z)$ defined as

$$\|F_\omega\|_{Z-\mathcal{H}_\infty} = \sup_{\theta\in[-\pi,\pi]} \left\| F_\omega(e^{j\theta}) \right\|_2 . \tag{46}$$

In order to construct upper bounds on γ_∞, we introduce the Lyapunov function candidate defined by

$$\left\{ \begin{array}{rcl} V_{RAT}(\omega) & = & \dfrac{V(\omega)}{v(\omega)} \\ V & \in & \mathcal{P}_{even}(n_d) \\ \deg(V) & \leq & 2d \end{array} \right. \tag{47}$$

where d is an integer and

$$v(\omega) = (1+\omega^2)^d. \tag{48}$$

Exploiting the KYP lemma for discrete-time systems, we define

$$Q(\omega) = \begin{pmatrix} q_1 & q_2 \\ \star & q_3 \end{pmatrix} \tag{49}$$

where

$$\left\{ \begin{array}{rcl} q_1 & = & \left|g(j\omega)\right|^2 V(\omega) - G_1(j\omega)V(\omega)G_1(j\omega)^H \\ & & -v(\omega)G_2(j\omega)G_2(j\omega)^H \\ q_2 & = & -G_1(j\omega)V(\omega)G_3(j\omega)^H - v(\omega)G_2(j\omega)G_4(j\omega)^H \\ q_3 & = & \xi v(\omega)\left|g(j\omega)\right|^2 I - G_3(j\omega)V(\omega)G_3(j\omega)^H \\ & & -v(\omega)G_4(j\omega)G_4(j\omega)^H \end{array} \right. \tag{50}$$

and $\xi \in \mathbb{R}$ is a variable. Since the matrices of (1) are real, one has

$$\forall \omega \in \mathbb{R} \left\{ \begin{array}{rcl} G_i(j\omega) & = & \overline{G_i(-j\omega)} \;\; \forall i = 1, \ldots, 4 \\ g(j\omega) & = & \overline{g(-j\omega)}. \end{array} \right. \tag{51}$$

This implies that

$$\left\{ \begin{array}{rcl} Q & \in & \mathcal{P}_{even}(n_q) \\ \deg(Q) & = & 2d + 2n_c \end{array} \right. \tag{52}$$

where

$$n_q = n_d + n_u. \tag{53}$$

The following result provides an LMI condition for establishing an upper bound on the $\mathcal{H}_\infty$ norm of (1).

Theorem 2 *Suppose that there exist $V \in \mathcal{P}_{even}(n_d)$ and $\xi, \varepsilon \in \mathbb{R}$ such that*

$$\left\{ \begin{array}{l} \Phi(Q(\omega) - \varepsilon v(\omega) \left| g(j\omega) \right|^2 I) \text{ is SOS} \\ \Phi(V(\omega) - \varepsilon v(\omega) I) \text{ is SOS} \\ \varepsilon > 0 \\ \deg(V) \leq 2d. \end{array} \right. \tag{54}$$

Then,

$$\sqrt{\xi} > \gamma_\infty. \tag{55}$$

Moreover, (54) *can be equivalently rewritten as a system of LMIs of the form* (34).

Proof. Suppose that there exist $V \in \mathcal{P}_{even}(n_d)$ and $\xi, \varepsilon \in \mathbb{R}$ such that (54) holds. From Theorem 1, the first constraint in (54) implies that

$$Q(\omega) - \varepsilon v(\omega) \left| g(j\omega) \right|^2 I \geq 0 \;\; \forall \omega \in \mathbb{R}.$$

Let us observe that

$$Q(\omega) = v(\omega) \left| g(j\omega) \right|^2 \tilde{Q}(\omega)$$

where

$$\tilde{Q}(\omega) = \left(\begin{array}{cc} \tilde{q}_1 & \tilde{q}_2 \\ \star & \tilde{q}_3 \end{array} \right)$$

and

$$\begin{cases} \tilde{q}_1 &= V_{RAT}(\omega) - F_1(j\omega)V_{RAT}(\omega)F_1(j\omega)^H \\ & \quad -F_2(j\omega)F_2(j\omega)^H \\ \tilde{q}_2 &= -F_1(j\omega)V_{RAT}(\omega)F_3(j\omega)^H - F_2(j\omega)F_4(j\omega)^H \\ \tilde{q}_3 &= \xi I - F_3(j\omega)V_{RAT}(\omega)F_3(j\omega)^H \\ & \quad -F_4(j\omega)F_4(j\omega)^H. \end{cases}$$

Since Assumption 1 implies that there exists $\varepsilon_1 > 0$ such that

$$|g(j\omega)| \geq \varepsilon_1 \quad \forall \omega \in \mathbb{R},$$

and since

$$v(\omega) \geq 1 \quad \forall \omega \in \mathbb{R},$$

one can write

$$\tilde{Q}(\omega) \geq \varepsilon I \quad \forall \omega \in \mathbb{R}.$$

Since $\varepsilon > 0$ due to the third constraint in (54), it follows that $\tilde{Q}(\omega) > 0$ for all $\omega \in \mathbb{R}$. Similarly, from the second constraint in (54), one has that $V_{RAT}(\omega) > 0$ for all $\omega \in \mathbb{R}$. By applying the Schur complement and exploiting the bounded real lemma, this implies that

$$\sqrt{\xi} > \|F_\omega\|_{Z-\mathcal{H}_\infty} \quad \forall \omega \in \mathbb{R},$$

see for instance [9] and references therein where a similar condition is obtained for systems not depending on uncertain parameters. From (44), this implies that (55) holds. Lastly, let us observe that (54) can be equivalently rewritten as a system of LMIs of the form (34) because, due to (48) and (52), $Q(\omega) - \varepsilon v(\omega) |g(j\omega)|^2 I$ and $V(\omega) - \varepsilon v(\omega) I$ are even Hermitian matrix polynomials. □

Theorem 2 provides a condition for establishing an upper bound on the $\mathcal{H}_\infty$ norm of (1), γ_∞. This condition is equivalent to an LMI feasibility test since $\Phi(Q(\omega) - \varepsilon v(\omega) |g(j\omega)|^2 I)$ and $\Phi(V(\omega) - \varepsilon v(\omega) I)$ are affine linear in the decision variables $V(\omega)$, ξ and ε. In particular, this LMI feasibility test can be built according to Theorem 1 by replacing $M(\omega)$ with $Q(\omega) - \varepsilon v(\omega) |g(j\omega)|^2 I$ and $V(\omega) - \varepsilon v(\omega) I$.

Let us observe that $V_{RAT}(\omega)$ defines a complex Lyapunov function candidate with rational dependence on ω of degree $2d$. It is possible to show that the conservatism of the condition provided by Theorem 2 is monotonically non-increasing with $2d$, i.e., (54) holds with $2d + 2$ if it holds with $2d$. Intuitively, this is due to the fact that any $V_{RAT}(\omega)$ of degree $2d$ can be expressed as a $V_{RAT}(\omega)$ of degree $2d+2$ (simply by multiplying numerator and denominator by $1+\omega^2$).

The number of LMI scalar variables in the condition provided by Theorem 2 is given by the number of free coefficients in $V(\omega)$, plus two (for the scalars ξ, ε), plus the length of the vectors α_{even} required to establish whether $\Phi(Q(\omega)-\varepsilon v(\omega)\left|g(j\omega)\right|^2 I)$ and $\Phi(V(\omega)-\varepsilon v(\omega)I)$ are SOS according to Theorem 1.

The best upper bound on the $\mathcal{H}_\infty$ norm of (1) provided by Theorem 2 for a chosen degree $2d$ of $V_{RAT}(\omega)$ is given by

$$\hat{\gamma}_\infty = \sqrt{\hat{\xi}} \tag{56}$$

where $\hat{\xi}$ is the solution of the semidefinite program (SDP)

$$\hat{\xi} = \inf_{\substack{V \in \mathcal{P}(n_d) \\ \xi, \varepsilon \in \mathbb{R}}} \xi \ \text{ s.t. } (54). \tag{57}$$

From Theorem 2 it follows that

$$\hat{\gamma}_\infty \geq \gamma_\infty. \tag{58}$$

The computation of this upper bound amounts to solving the optimization problem (57), which is an SDP since the cost function is linear and the constraints are LMIs. Table 3 shows the number of LMI scalar variables η_1 in (57) in some cases.

η_1	$2d=0$	$2d=2$	$2d=4$
$n=1$	5	13	26
$n=2$	25	56	100
$n=3$	85	158	256

Table 3: Number of LMI scalar variables η_1 in (57) for $n_c = n_d = n$ and $n_u = n_y = 1$ for some values of n and $2d$.

Example 2. Let us consider the problem of determining the $\mathcal{H}_\infty$ norm of (1) with

$$\begin{cases} A_{cc} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}, \ A_{cd} = \begin{pmatrix} 0.5 & 0.4 \\ -0.7 & 0 \end{pmatrix} \\ A_{dc} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \ A_{dd} = \begin{pmatrix} 0.4 & -0.5 \\ 0.3 & 0.6 \end{pmatrix} \\ B_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B_d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ C_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ C_d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

Hence, it turns out that $n_c = n_d = n_u = n_y = 2$. Let us observe that the matrices A_{cc} and A_{dd} are Hurwitz and Schur, respectively.

Let us compute the upper bound $\hat{\gamma}_\infty$ on the $\mathcal{H}_\infty$ norm γ_∞. We solve the SDP (57) by using $V_{RAT}(\omega)$ as in (47) with degree $2d = 2$. We find $\hat{\xi} = 155.276$ and, hence,

$$\hat{\gamma}_\infty = 12.461.$$

The found $V_{RAT}(\omega)$ is $\Re(V_{RAT}(\omega)) + j\Im(V_{RAT}(\omega))$ where

$$\begin{cases} \Re(V_{RAT}(\omega)) = \dfrac{\begin{pmatrix} 36.217 + 26.763\omega^2 & -6.015 - 6.358\omega^2 \\ \star & 18.449 + 29.654\omega^2 \end{pmatrix}}{1+\omega^2} \\ j\Im(V_{RAT}(\omega)) = \dfrac{\begin{pmatrix} 0 & j14.117\omega \\ \star & 0 \end{pmatrix}}{1+\omega^2}. \end{cases}$$

The number of LMI scalar variables is 89 and the computational time is less than 1 second.

We have also investigated our previous approach in [7] which uses complex Lyapunov function candidates with polynomial dependence. Interesting, by using the degree $2d = 2$ as before, one finds only the upper bound 55.228 (the number of LMI scalar variables is 289 and the computational time is less than 2 seconds). Also, we have tested this approach up to the degree $2d = 8$, but the upper bound 55.228 cannot be improved.

Lastly, we have tested the method in [18] for comparison, which provides the upper bound 55.228 through an SDP built with Lyapunov functions that do not depend on the frequency ω (the number of LMI scalar variables is 7 and the computational time is less than 1 second). □

Next, let us consider Problem 2, i.e., the determination of the $\mathcal{H}_2$ norm of (1). This norm can be calculated as

$$\gamma_2 = \|F\|_{LZ-\mathcal{H}_2} \tag{59}$$

where $\|F\|_{LZ-\mathcal{H}_2}$ is the Laplace-Z $\mathcal{H}_2$ norm of $F(s,z)$ defined as

$$\begin{array}{c} \|F\|_{LZ-\mathcal{H}_2} = \\ \frac{1}{2\pi}\sqrt{\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} trace\left(F(j\omega, e^{j\theta})^H F(j\omega, e^{j\theta})\right) d\theta d\omega.} \end{array} \tag{60}$$

The $\mathcal{H}_2$ norm of (1) can also be rewritten as

$$\gamma_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|F_\omega\|_{Z-\mathcal{H}_2}^2} \tag{61}$$

where $\|F_\omega\|_{Z-\mathcal{H}_2}$ is the Z $\mathcal{H}_2$ norm of $F_\omega(z)$ defined as

$$\|F_\omega\|_{Z-\mathcal{H}_2} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} trace\left(F(j\omega, e^{j\theta})^H F(j\omega, e^{j\theta})\right) d\theta.} \tag{62}$$

It hence follows that, in order for γ_2 to be finite, a necessary condition is

$$\lim_{\omega\to\infty} F(j\omega, e^{j\theta}) = 0. \tag{63}$$

The idea to construct upper bounds on γ_2 is to make use of the Lyapunov function candidate defined by

$$\left\{\begin{array}{rcl} W_{RAT}(\omega) & = & \dfrac{W(\omega)}{v(\omega)} \\ W & \in & \mathcal{P}_{even}(n_d) \\ \deg(W) & \leq & 2d-2 \end{array}\right. \tag{64}$$

where $v(\omega)$ is given by (48). Exploiting the controllability Gramian-based $\mathcal{H}_2$ norm characterization of discrete-time systems, we define $R \in \mathcal{P}_{even}(n_d)$ as

$$\begin{array}{rcl} R(\omega) & = & |g(j\omega)|^2\, W(\omega) - G_1(\omega)W(\omega)G_1(\omega)^H \\ & & -v(\omega)G_2(\omega)G_2(\omega)^H. \end{array} \tag{65}$$

The following result provides an LMI condition for establishing an upper bound on the $\mathcal{H}_2$ norm of (1).

Theorem 3 *Suppose that there exist $W \in \mathcal{P}_{even}(n_d)$ and $\varepsilon \in \mathbb{R}$ such that*

$$\begin{cases} \Phi(R(\omega) - \varepsilon v(\omega) \left| g(j\omega) \right|^2 I) \text{ is SOS} \\ \Phi(W(\omega) - \varepsilon v(\omega) I) \text{ is SOS} \\ \varepsilon > 0 \\ \deg(W) \leq 2d - 2. \end{cases} \tag{66}$$

Then,

$$\sqrt{\zeta} > \gamma_2 \tag{67}$$

where

$$\zeta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\omega) d\omega \tag{68}$$

and

$$\phi(\omega) = trace \left(F_3(\omega) W_{RAT}(\omega) F_3(\omega)^H + F_4(\omega) F_4(\omega)^H \right). \tag{69}$$

Moreover, (66) *can be equivalently rewritten as a system of LMIs of the form* (34).

Proof. Suppose that there exist $W \in \mathcal{P}_{even}(n_d)$ and $\varepsilon \in \mathbb{R}$ such that (66) holds. From Theorem 1, the first constraint in (66) implies that

$$R(\omega) - \varepsilon v(\omega) \left| g(j\omega) \right|^2 I \quad \forall \omega \in \mathbb{R}.$$

Let us observe that

$$R(\omega) = v(\omega) \left| g(j\omega) \right|^2 \tilde{R}(\omega)$$

where

$$\tilde{R}(\omega) = W_{RAT}(\omega) - F_1(\omega) W_{RAT}(\omega) F_1(\omega)^H - F_2(\omega) F_2(\omega)^H.$$

Since from the proof of Theorem 2 one has that $|g(j\omega)| \geq \varepsilon_1$ and $v(\omega) \geq 1$ for all $\omega \in \mathbb{R}$ for some $\varepsilon_1 > 0$, it follows that

$$\tilde{R}(\omega) \geq \varepsilon I \quad \forall \omega \in \mathbb{R}.$$

Since $\varepsilon > 0$ due to the third constraint in (66), it follows that $\tilde{R}(\omega) > 0$ for all $\omega \in \mathbb{R}$. Similarly, from the second constraint in (66), one has that $W_{RAT}(\omega) > 0$ for all $\omega \in \mathbb{R}$. This implies that (see, e.g., [24])

$$\phi(\omega) > \|F_\omega\|^2_{Z-\mathcal{H}_2} \quad \forall \omega \in \mathbb{R}.$$

Hence,

$$\sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\omega) d\omega} > \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|F_\omega\|^2_{Z-\mathcal{H}_2} d\omega},$$

that is, (67). Lastly, (66) can be equivalently rewritten as a system of LMIs of the form (34) because $R(\omega)-\varepsilon v(\omega)\left|g(j\omega)\right|^2 I$ and $W(\omega)-\varepsilon v(\omega)I$ are even Hermitian matrix polynomials. □

Theorem 3 provides a condition for establishing an upper bound on the $\mathcal{H}_2$ norm of (1), γ_2. This condition is equivalent to an LMI feasibility test that can be built according to Theorem 1 by replacing $M(\omega)$ with $R(\omega) - \varepsilon v(\omega)\left|g(j\omega)\right|^2 I$ and $W(\omega) - \varepsilon v(\omega)I$.

As in the case of Theorem 2, it is possible to show that the conservatism of the condition provided by Theorem 3 is monotonically non-increasing with $2d$, i.e., (54) holds with $2d+2$ if it holds with $2d$.

The number of LMI scalar variables in the condition provided by Theorem 3 is given by the number of free coefficients in $W(\omega)$, plus one (for the scalar ε), plus the length of the vectors α_{even} required to establish whether $\Phi(R(\omega) - \varepsilon v(\omega)\left|g(j\omega)\right|^2 I)$ and $\Phi(W(\omega) - \varepsilon v(\omega)I)$ are SOS according to Theorem 1.

The best upper bound on the $\mathcal{H}_2$ norm of (1) provided by Theorem 3 for a chosen degree $2d$ of $W_{RAT}(\omega)$ is given by

$$\hat{\gamma}_2 = \sqrt{\hat{\zeta}} \tag{70}$$

where $\hat{\zeta}$ is the solution of the SDP

$$\hat{\zeta} = \inf_{\substack{W\in\mathcal{P}(n_d)\\ \varepsilon\in\mathbb{R}}} \zeta \quad \text{s.t. (66).} \tag{71}$$

From Theorem 3 it follows that

$$\hat{\gamma}_2 \geq \gamma_2. \tag{72}$$

The computation of this upper bound amounts to solving the SDP (71). Table 4 shows the number of LMI scalar variables η_2 in (71) in some cases.

Example 3. Let us consider the problem of determining the $\mathcal{H}_2$ norm

η_2	$2d=0$	$2d=2$	$2d=4$	$2d=6$
$n=1$	N/A	1	3	6
$n=2$	N/A	13	31	57
$n=3$	N/A	51	99	165

Table 4: Number of LMI scalar variables η_2 in (71) for $n_c = n_d = n$ and $n_u = n_y = 1$ for some values of n and $2d$.

	$2d=2$	$2d=4$	$2d=6$	$2d=8$	$2d=10$
$\hat{\gamma}_2$	3.166	2.373	2.009	1.841	1.765
η_2	13	31	57	91	133

Table 5: Example 3: upper bound $\hat{\gamma}_2$ and number of LMI scalar variables η_2 for some values of d.

of (1) with

$$\left\{\begin{array}{c} A_{cc} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \ A_{cd} = \begin{pmatrix} 0.4 & -0.8 \\ 0.6 & 0.6 \end{pmatrix} \\ A_{dc} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ A_{dd} = \begin{pmatrix} 0.3 & -0.5 \\ 0.5 & 0.3 \end{pmatrix} \\ B_c = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \ B_d = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ C_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ C_d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{array}\right.$$

Hence, it turns out that $n_c = n_d = n_u = n_y = 2$. Let us observe that the matrices A_{cc} and A_{dd} are Hurwitz and Schur, respectively.

Let us compute the upper bound $\hat{\gamma}_2$. We solve the SDP (71) for different values of the degree $2d$ of $W_{RAT}(\omega)$. Table 5 shows the found upper bounds and the corresponding number of LMI scalar variables η_2 (the computational time is less than 2 seconds in all cases). Figure 1 shows the found $\phi(\omega)$.

Lastly, we have tested the method in [18] for comparison, which does not provide upper bounds for this example (the LMI condition is infeasible). □

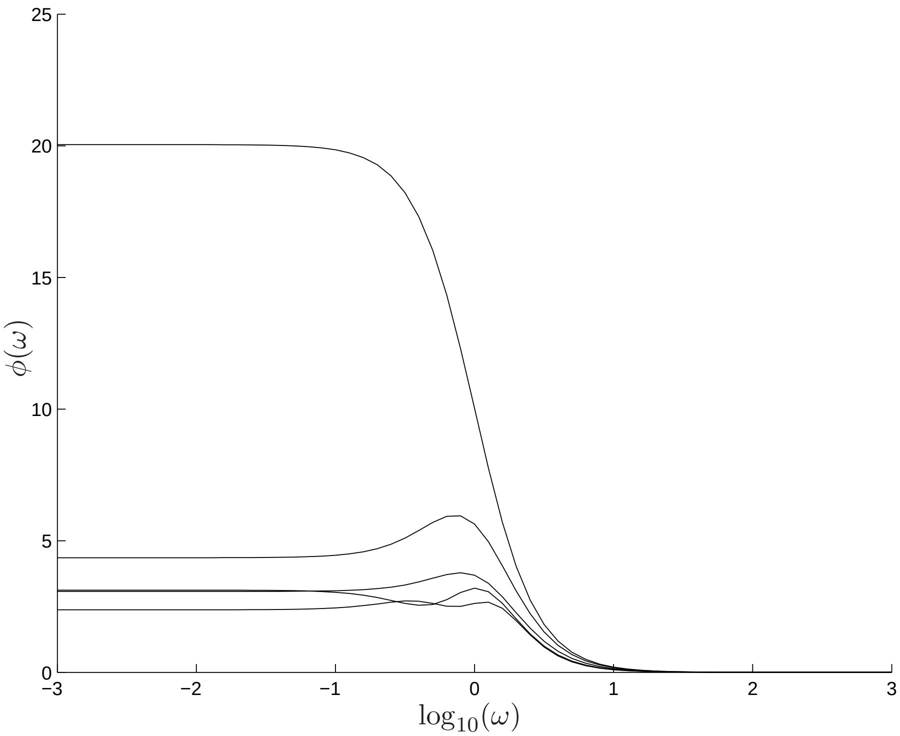

Figure 1: Example 3: function $\phi(\omega)$ corresponding to the upper bounds in Table 5.

5 $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Norm Upper Bounds: Convergence

The following result states an important property of the condition provided by Theorem 2, namely that this condition is nonconservative by using $V_{RAT}(\omega)$ of degree sufficiently large.

Theorem 4 *Let $\xi \in \mathbb{R}$ be such that*

$$\sqrt{\xi} > \gamma_\infty. \tag{73}$$

Then, there exists a sufficiently large integer d such that (54) *holds for some $V \in \mathcal{P}_{even}(n_d)$ and $\varepsilon \in \mathbb{R}$.*

Proof. Suppose that (73) holds. Then, there exists a Hermitian matrix function $\tilde{V}:\mathbb{R}\to\mathbb{C}^{n_d\times n_d}$ and a scalar $\tilde{\varepsilon}>0$ such that

$$\forall\omega\in\mathbb{R}\ \left\{\begin{array}{lcl}\tilde{Q}(\omega) & \geq & \tilde{\varepsilon}I\\ \tilde{V}(\omega) & \geq & \tilde{\varepsilon}I\end{array}\right.$$

where $\tilde{Q}(\omega)$ is defined as in the proof of Theorem 2 with $V_{RAT}(\omega)$ replaced by $\tilde{V}(\omega)$. The limit for ω that tends to infinity of $\tilde{V}(\omega)$ does exist, i.e.,

$$\lim_{\omega\to\infty}\tilde{V}(\omega)=\tilde{V}_\infty$$

for some symmetric matrix $\tilde{V}_\infty\in\mathbb{R}^{n_d\times n_d}$. Moreover, from (51) it follows that $\tilde{V}(\omega)$ can be assumed even without loss of generality.

Let us define

$$\tilde{V}_R(\omega)=\Re(\tilde{V}(\omega)).$$

Since $\tilde{V}(\omega)$ is an even Hermitian matrix function, it follows that $\tilde{V}_R(\omega)$ can be rewritten as

$$\tilde{V}_R(\omega)=\tilde{V}_1(\omega^2)$$

where $\tilde{V}_1:\mathbb{R}\to\mathbb{R}^{n_d\times n_d}$ is a symmetric matrix function. Let us define

$$\left\{\begin{array}{lcl}m_1(\psi) & = & \dfrac{\psi}{1-\psi}\\[2mm] m_2(\omega) & = & \dfrac{\omega^2}{1+\omega^2}\end{array}\right.$$

and

$$\tilde{V}_2(\psi)=\tilde{V}_1(m_1(\psi)).$$

It follows that $\tilde{V}_1(\omega^2)$ and $\tilde{V}_1(\psi)$ are the same function defined on different domains, i.e.,

$$\forall\omega\in\mathbb{R},\ \exists\psi=m_2(\omega)\in[0,1):\ \tilde{V}_1(\omega^2)=\tilde{V}_2(\psi).$$

Since $\tilde{V}_2(\psi)$ is continuous and the limit for ψ that tends to 1 of $\tilde{V}_2(\psi)$ does exist, in particular

$$\lim_{\psi\to1}\tilde{V}_2(\psi)=\tilde{V}_\infty,$$

it follows that $\tilde{V}_2(\psi)$ can be approximated arbitrarily well over $[0,1]$ by a symmetric matrix polynomial $\tilde{V}_3:\mathbb{R}\to\mathbb{R}^{n_d\times n_d}$. Hence, let us define

$$\tilde{V}_4(\omega)=\tilde{V}_3(m_2(\omega)).$$

It follows that $\tilde{V}_4(\omega)$ is a symmetric rational function that approximates arbitrarily well the continuous function $\tilde{V}_R(\omega)$. Moreover, since $\tilde{V}_4(\omega)$ is even, it follows that $\tilde{V}_4(\omega)$ has the form

$$\tilde{V}_4(\omega) = \frac{V_R(\omega)}{v(\omega)}$$

where $v(\omega)$ is as in (48) for a suitable integer d, and $V_R(\omega)$ is a symmetric matrix polynomial of degree $2d$ in the set $\mathcal{P}_{even}(n_d)$.

Next, let us define

$$\tilde{V}_I(\omega) = \Im(\tilde{V}(\omega)).$$

Since $\tilde{V}(\omega)$ is an even Hermitian matrix function, it follows that $\tilde{V}_I(\omega)$ can be rewritten as

$$\tilde{V}_I(\omega) = \omega\tilde{V}_5(\omega^2)$$

where $\tilde{V}_5 : \mathbb{R} \to \mathbb{R}^{n_d \times n_d}$ is a skew-symmetric matrix function. Similarly to $\tilde{V}_1(\omega^2)$, $\tilde{V}_5(m_1(\psi))$ can be approximated arbitrarily well by a skew-symmetric matrix polynomial $\tilde{V}_6 : \mathbb{R} \to \mathbb{R}^{n_d \times n_d}$ over $[0, 1]$, and hence

$$\tilde{V}_7(\omega) = j\omega\tilde{V}_6(m_2(\omega))$$

is a skew-symmetric rational function that approximates arbitrarily well $\tilde{V}_I(\omega)$. In particular,

$$\tilde{V}_7(\omega) = \frac{V_I(\omega)}{v(\omega)}$$

where $V_I(\omega)$ is a skew-symmetric matrix polynomial of degree $2d$, with $jV_I(\omega)$ in the set $\mathcal{P}_{even}(n_d)$.

Lastly, let us define $V_{RAT}(\omega)$ as in (47) with $V(\omega)$ given by

$$V(\omega) = V_R(\omega) + jV_I(\omega)$$

and let $\tilde{Q}(\omega)$ be defined as in the proof of Theorem 2 with such a $V_{RAT}(\omega)$. Due to the continuity of $\tilde{Q}(\omega)$ with $V_{RAT}(\omega)$, it follows that the degree $2d$ can be chosen such that $\tilde{Q}(\omega) \geq \varepsilon I$ and $V_{RAT}(\omega) \geq \varepsilon I$ for all $\omega \in \mathbb{R}$ for some $\varepsilon > 0$. This implies that $Q(\omega) \geq \varepsilon v(\omega) |g(j\omega)|^2 I$ and $V(\omega) \geq \varepsilon v(\omega) I$ for all $\omega \in \mathbb{R}$. Since $V \in \mathcal{P}_{even}(n_d)$, it follows from Theorem 1 that (54) holds. $\square$

Theorem 4 states that the condition provided by Theorem 2 is nonconservative by choosing an integer d sufficiently large, where d defines the degree of $V_{RAT}(\omega)$ given by $2d$. An analogous result holds for the condition provided by Theorem 3 as reported hereafter.

Theorem 5 *Let $\check{\zeta} \in \mathbb{R}$ be such that*

$$\sqrt{\check{\zeta}} > \gamma_2. \tag{74}$$

Then, there exists a sufficiently large integer d such that (66) *holds for some $W \in \mathcal{P}_{even}(n_d)$ and $\varepsilon \in \mathbb{R}$, and such that*

$$\sqrt{\check{\zeta}} > \sqrt{\zeta} > \gamma_2 \tag{75}$$

with ζ given by (68).

Proof. Suppose that (74) holds. Then, there exists a Hermitian matrix function $\tilde{W} : \mathbb{R} \to \mathbb{C}^{n_d \times n_d}$ and a scalar $\tilde{\varepsilon} > 0$ such that

$$\forall \omega \in \mathbb{R} \left\{ \begin{array}{rcl} \tilde{R}(\omega) & \geq & \tilde{\varepsilon} I \\ \tilde{W}(\omega) & \geq & \tilde{\varepsilon} I \end{array} \right.$$

and

$$\sqrt{\check{\zeta}} > \sqrt{\tilde{\zeta}}$$

where $\tilde{R}(\omega)$ is defined as in the proof of Theorem 3 with $W_{RAT}(\omega)$ replaced by $\tilde{W}(\omega)$, and

$$\tilde{\zeta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(\omega) d\omega$$

where

$$\tilde{\phi}(\omega) = trace\left(F_3(\omega)\tilde{W}(\omega)F_3(\omega)^H + F_4(\omega)F_4(\omega)^H \right).$$

The limit for ω that tends to infinity of $\tilde{W}(\omega)$ does exist, in particular

$$\lim_{\omega \to \infty} \tilde{W}(\omega) = 0.$$

Moreover, from (51) it follows that $\tilde{W}(\omega)$ can be assumed even without loss of generality. Hence, proceeding analogously to the proof of Theorem 4, it follows that there exists $\varepsilon > 0$ such that (66) holds for some $W \in \mathcal{P}_{even}(n_d)$ of degree $2d - 2$ for some d sufficiently large. $\square$

As a consequence of Theorems 4 and 5, one has that

$$\left\{ \begin{array}{rcl} \lim_{d\to\infty} \hat{\gamma}_\infty & = & \gamma_\infty \\ \lim_{d\to\infty} \hat{\gamma}_2 & = & \gamma_2. \end{array} \right. \tag{76}$$

Although the conditions provided by Theorems 2 and 3 are nonconservative by using $V_{RAT}(\omega)$ of degree sufficiently large according to Theorems 4 and 5, it is still unclear whether the upper bounds $\hat{\gamma}_\infty$ and $\hat{\gamma}_2$ found for a chosen value of d are tight. The following result provides a necessary and sufficient condition for answering this question in the case of the $\mathcal{H}_\infty$ norm.

Theorem 6 *Suppose $\hat{\gamma}_\infty < \infty$, and define*

$$\Omega = \left\{\omega \geq 0 : \ \det\left(\Phi\left(\hat{Q}(\omega)\right)\right) = 0\right\} \tag{77}$$

where $\hat{Q}(\omega)$ is $Q(\omega)$ evaluated for the optimal values of the decision variables in (57)*. Then,*

$$\hat{\gamma}_\infty = \gamma_\infty \tag{78}$$

if and only if there exists $\hat{\omega} \in \Omega \cup \{\infty\}$ such that

$$\|F_{\hat{\omega}}\|_{Z-\mathcal{H}_\infty} = \hat{\gamma}_\infty. \tag{79}$$

Proof. "$\Leftarrow$" If (79) holds, then $\hat{\gamma}_\infty \leq \gamma_\infty$ since γ_∞ is the supremum of $\|F_\omega\|_{Z-\mathcal{H}_\infty}$ for $\omega \in \mathbb{R}$, while Theorem 2 guarantees that $\hat{\gamma}_\infty \geq \gamma_\infty$. Hence, (78) holds.

"$\Rightarrow$" Suppose that (78) holds. If (79) holds with $\hat{\omega} = \infty$, then the proof is completed. Otherwise, from the definition of γ_∞ in (44), there exists $\hat{\omega} \in \mathbb{R}$ such that (79) holds. Without loss of generality, one can suppose that $\hat{\omega} \geq 0$ due to (51). Such a frequency belongs to Ω. In fact, if one supposes for contradiction that $\hat{\omega} \notin \Omega$, from (54) it would follow that

$$\Phi\left(\hat{Q}(\hat{\omega}) - \hat{\varepsilon} v(\hat{\omega}) \left|g(j\hat{\omega})\right|^2 I\right) > 0 \ \ \forall \omega \in \mathbb{R}$$

hence implying that there would exist $\tilde{\xi}$ such that (54) holds with $\xi = \tilde{\xi}$ and

$$\tilde{\xi} < \hat{\xi}$$

with $\hat{\xi}$ given by (57), which is impossible since $\hat{\xi}$ is the infimum of the admissible ξ in (54). $\square$

Theorem 6 provides a necessary and sufficient condition for establishing the tightness of the upper bound $\hat{\gamma}_\infty$ found for a chosen value of d. This condition consists of checking whether $\hat{\gamma}_\infty$ is achieved for any frequency in

$\Omega \cup \{\infty\}$. Let us observe that Ω is the set of real roots of the one variable polynomial $\det\left(\Phi\left(\tilde{Q}(\hat{\omega})\right)\right)$.

Example 2 (continued). Let us consider again Example 2 in Section 4 and the found upper bound $\hat{\gamma}_\infty = 12.461$. In order to establish whether this upper bound is tight, let us Theorem 6. We find that the set Ω in (77) is

$$\Omega = \{1.041\}.$$

Moreover, we find

$$\begin{cases} \hat{\omega} = 1.041 & \Rightarrow \quad \|F_{\hat{\omega}}\|_{Z-\mathcal{H}_\infty} = 12.461 \\ \hat{\omega} = \infty & \Rightarrow \quad \|F_{\hat{\omega}}\|_{Z-\mathcal{H}_\infty} = 2.779. \end{cases}$$

Hence, (79) holds with $\hat{\omega} = 1.041$ rad/s and, therefore, we conclude that the upper bound is tight, i.e.,

$$\hat{\gamma}_\infty = \gamma_\infty = 12.461.$$

In particular, $\left\|F_{\hat{\omega}}(e^{j\theta})\right\|_2 = 12.461$ for $\theta = 2.148$ rad. □

Lastly, we present the following result, which provides a necessary condition for establishing whether the upper bound $\hat{\gamma}_2$ found for a chosen value of d is tight.

Theorem 7 *Suppose that*

$$\hat{\gamma}_2 = \gamma_2. \tag{80}$$

Then,

$$f = 0 \tag{81}$$

where f is the nonnegative index

$$f = \int_{-\infty}^{\infty} \det\left(\frac{\hat{R}(\omega)}{v(\omega)\,|g(j\omega)|^2}\right) d\omega \tag{82}$$

and $\hat{R}(\omega)$ is $R(\omega)$ evaluated for the optimal values of the variables in (71)*.*

	$2d = 2$	$2d = 4$	$2d = 6$	$2d = 8$	$2d = 10$
f	1.665	0.455	0.010	0.026	0.008

Table 6: Example 3: index f corresponding to the upper bounds in Table 5.

Proof. Suppose that (80) holds. This implies that (81) holds because, if one supposes for contradiction that $f \neq 0$, it would follow that

$$\det\left(\frac{\hat{R}(\omega)}{v(\omega)\,|g(j\omega)|^2}\right) > 0 \quad \forall \omega \in \tilde{\Omega}$$

where $\tilde{\Omega}$ is a subset of $\mathbb{R}$ with nonzero measure since $\hat{R}(\omega) \geq 0$ and $v(\omega)\,|g(j\omega)|^2 > 0$ for all $\omega \in \mathbb{R}$. Consequently, there would exist $\tilde{W} : \mathbb{R} \to \mathbb{C}^{n_d \times n_d}$ and a scalar $\tilde{\varepsilon}$ such that (66)–(68) hold with $W(\omega) = \tilde{W}(\omega)$ and $\varepsilon = \tilde{\varepsilon}$, and also such that

$$\tilde{\zeta} < \hat{\zeta}$$

where $\hat{\zeta}$ is given by (71) and $\tilde{\zeta}$ is defined as in the proof of Theorem 5. This is impossible because one would have

$$\gamma_2 < \sqrt{\tilde{\zeta}} < \hat{\gamma}_2 = \gamma_2.$$

□

Theorem 7 provides a necessary condition for establishing the tightness of the upper bound $\hat{\gamma}_2$. This condition consists of checking whether the non-negative index f defined in (82) is zero. Let us observe that this index can be easily computed being the integral of a function of one scalar variable.

Example 3 (continued). Let us consider again Example 3 in Section 4 and the found upper bounds in Table 5. The tightness of these upper bounds can be investigated by using Theorem 7, in particular Table 6 shows the index f for each upper bound. □

6 An Improved Condition for Exponential Stability

Before concluding the paper, we show how the preliminary results presented in Section 3 can be used to obtain an improved necessary and sufficient LMI condition for establishing exponential stability of (1).

Corollary 1 *The system* (1) *is exponentially stable if and only if there exist* $V \in \mathcal{P}_{even}(n_d)$ *of degree* $2d$ *not greater than* 2μ *and* $\varepsilon \in \mathbb{R}$ *such that*

$$\begin{cases} \Phi\left(S(\omega) - \varepsilon\, |g(j\omega)|^2 I\right) \;\; \text{is SOS} \\ \Phi\left(V(\omega) - \varepsilon I\right) \;\; \text{is SOS} \\ \varepsilon > 0 \end{cases} \tag{83}$$

where

$$\mu = n_c n_d^2 \tag{84}$$

and

$$S(\omega) = |g(j\omega)|^2\, V(\omega) - G_1(j\omega)^H V(\omega) G_1(j\omega). \tag{85}$$

Moreover, (83) *can be equivalently rewritten as a system of LMIs of the form* (34).

Proof. First, from Theorem 3 in [7] one has that (1) is exponentially stable if and only if there exist a Hermitian matrix polynomial $V(\omega)$ of degree $2d$ not greater than 2μ and ε such that (83) holds. Since (51) holds, it follows that $V(\omega)$ can be assumed even without loss of generality, i.e., in the set $\mathcal{P}_{even}(n_d)$. Lastly, since $V(\omega)$ is even, it follows that also $S(\omega) - \varepsilon\, |g(j\omega)|^2 I$ and $V(\omega) - \varepsilon I$ are even, and, therefore, (83) can be equivalently rewritten as a system of LMIs of the form (34) due to Theorem 1. □

Corollary 1 provides a necessary and sufficient LMI condition for establishing exponential stability of (1) whose numerical complexity (specifically, the number of LMI scalar variables) is significantly reduced with respect to the original condition in Theorem 3 in [7]. Such a reduction is achieved, firstly, by restricting the Hermitian matrix polynomial $V(\omega)$ into the class $\mathcal{P}_{even}(n_d)$, and, secondly, by exploiting Theorem 1 to rewrite (83) as a system of LMIs.

Example 4. Let us consider (1) with

$$\begin{cases} A_{cc} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}, \ A_{cd} = \begin{pmatrix} 0.5 & 0.4 \\ -0.6 & 0.3 \end{pmatrix} \\ A_{dc} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \ A_{dd} = \begin{pmatrix} 0.4 & -0.5 \\ 0.3 & 0.6 \end{pmatrix}. \end{cases}$$

By searching for a complex Lyapunov function of degree 2 in the frequency ω, exponential stability can be proved through the LMI condition in [7] that, in such a case, has 84 LMI scalar variables. On the other hand, by using Corollary 1, exponential stability can be proved by searching for $V(\omega)$ of degree 2 and the number of LMI scalar variables in (83) is just 31. □

Example 5. Let us consider (1) with

$$\begin{cases} A_{cc} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{pmatrix}, \ A_{cd} = \begin{pmatrix} 0 & 0.5 & 0 \\ -0.5 & 1.5 & 1 \\ -0.5 & 0 & 1 \end{pmatrix} \\ A_{dc} = \begin{pmatrix} 0.2 & 0 & 0.4 \\ 0 & -0.3 & 0 \\ 0 & 0 & 0.3 \end{pmatrix}, \ A_{dd} = \begin{pmatrix} -0.4 & 0 & 0 \\ 0.2 & 0 & 0.3 \\ 0 & -0.4 & -0.2 \end{pmatrix}. \end{cases}$$

As in the previous example, exponential stability can be proved through the LMI condition in [7] by searching for a complex Lyapunov function of degree 2 in the frequency ω, and the number of LMI scalar variables is 318. On the other hand, by using Corollary 1, exponential stability can be proved by searching for $V(\omega)$ of degree 2 and the number of LMI scalar variables in (83) is just 99. □

7 Conclusion

We have proposed a novel approach for determining the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of 2D mixed continuous-discrete-time systems based on the use of complex Lyapunov functions with even rational parametric dependence. For any chosen degree of such Lyapunov functions, the proposed approach provides upper bounds on the sought norms via LMIs. We have shown that the provided upper bounds are nonconservative by using Lyapunov functions in the chosen class with degree sufficiently large. Also, we have provided conditions for establishing the tightness of the upper bounds found for any chosen degree of the Lyapunov functions in terms of simple numerical tests. Lastly, we have shown how the numerical complexity of the proposed approach can be significantly reduced by proposing a new necessary and sufficient LMI condition for establishing positive semidefiniteness of even Hermitian matrix polynomials. This result has been exploited also to derive an improved necessary and sufficient LMI condition for establishing exponential stability of 2D mixed continuous-discrete-time systems.

The numerical complexity of the proposed approach may be higher than that of existing LMI conditions for determining the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of 2D mixed continuous-discrete-time systems. This might be expected since the existing LMI conditions are conservative.

Several directions can be investigated in future works starting from the results proposed in this paper. One of these is the synthesis of feedback controllers minimizing the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of 2D mixed continuous-discrete-time systems. Such a synthesis is presently nonconvex since the LMIs become bilinear matrix inequalities (BMIs) when simultaneously looking for the Lyapunov function and controller. Nevertheless, one can expect that the proposed results will lead to less conservative approaches than the existing LMI conditions since the latter are conservative also for system analysis.

Other directions include the search for upper bounds on the degree of the Lyapunov functions in order to achieve estimates of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms within a pre-specified accuracy. Also, one could investigate the extension of the proposed results to more general models where the dimensions are not necessarily times.

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