Local Fractional Operator for the Solution of Quadratic Riccati Differential Equation with Constant Coefficients

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Abstract— **In this paper, we consider approximate solutions of fractional Riccati differential equations via the application of local fractional operator in the sense of Caputo derivative. The proposed semi-analytical technique is built on the basis of the standard Differential Transform Method (DTM). Some illustrative examples are given to demonstrate the effectiveness and robustness of the proposed technique; the approximate solutions are provided in the form of convergent series. This shows that the solution technique is very efficient, and reliable; as it does not require much computational work, even without given up accuracy.**

***Index Terms*— Fractional differential equations; Modified DTM; Riccati Differential Equation**

I. Introduction

The idea of Riccati Differential Equation (RDE) was introduced by the Italian: Count Jacopo Francesco Riccati (1676-1754), the detail of the basic theories associated with the RDEs are contained in Reid [1]. The applications of this form of differential equations are not limited to areas like random processes, diffusion problems, optimal control, stochastic realization theory, network analysis, financial mathematics, and robust stabilization [2,3].

In general, the RDE of integer order is expressed in the form of:

$$\frac{dy}{dt}=\sum_{i=0}^{2}p_i(t)\,y^i\,,\ \ t\geq 0,\ y(0)=y_0 \qquad (1.1)$$

where $p_i(t),\ i=0,1,2.$ are coefficient functions which may be constants or variables; hence, the notion of Riccati differential equation with constant or variable coefficients.

Equation (1.1) is a nonlinear differential equation whose approximate/analytical solutions and the likes can be obtained using some semi-analytical methods, say the decomposition method (ADM), Homotopy analysis method (HAM), Differential transformation method (DTM), Projected/Modified Differential transform method (MDTM), Forward-Euler method (FEM), Runge-Kutta method (R-KM), Perturbation Iteration Transform Method and so on [4-11].

Recently, Biazar and Eslami [12] proposed DTM for solving the quadratic RDE of the form (1.1) but with constant coefficients while Mukherjee and Roy [13] implemented the DTM for the solution of some RDEs with variable terms as coefficients.

In this work, we shall be considering the extension of (1.1) to fractional differential case, therefore, the fractional RDE (FRDE) of the form:

$$\frac{d^\alpha y}{dt^\alpha}=\sum_{i=0}^{2}p_i(t)\,y^i\,,\ \ t\geq 0,\quad \alpha\in(0,1] \qquad (1.2)$$

where $y(0)=y_0$ is an initial condition, and $p_i(t),\ i=0,1,2.$ are constant functions. Thus, a FRDE with constant coefficients.

Differential equations of fractional type (FDEs) act as generalizations of the classical differential equations of integer order [14].

Many researchers have considered obtaining approximate solutions of FDEs of the form in (1.2) and the likes via the application of semi-analytical methods [15-21].

In this present work, we will be considering the application of a local fractional differential operator (LFDO) based on MDTM for approximate solution of the FRDE in (1.2). This method involves less computational work, and requires less computational time.

II. Preliminaries and Notations on Fractional Calculus [14, 22, 23]

Here, a brief introduction of fractional calculus will be given as follows.

In fractional calculus, the power of the differential operator is considered a real or complex number. Hence, the following definitions:

Definition 1: Fractional derivative in gamma sense

Suppose $D=\dfrac{d(\cdot)}{dx}$ and J are differential and integration operators respectively, then:

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$$D^{\alpha}h(x)=\frac{d^{\alpha}h(x)}{dx^{\alpha}}=\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}. \quad (2.1)$$

We referred to (2.1) as a fractional derivative of $h(x)$, of order α, if $\alpha\in\mathbb{R}$.

Definition 2: Suppose $h(x)$ is defined for $x>0$, then:

$$(Jh)(x)=\int_0^x h(s)ds \quad (2.2)$$

and as such, an arbitrary extension of (2.2) yields:

$$(J^{\alpha}h)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}h(s)ds\ ,\ \alpha>0,\ t>0. \quad (2.3)$$

Equation (2.3) is regarded as Riemann-Liouville (R-L) fractional integration of order α.

Definition 3: R-L fractional derivative of ℓ is:

$$D^{\alpha}\ell \quad \frac{d^{\phi}\left(J^{\phi-\alpha}\ell\right.}{dx^{\phi}}. \quad (2.4)$$

Definition 4: Caputo fractional derivative of ℓ is:

$$D^{\alpha}\ell \quad J^{\phi-\alpha}\left(d^{\phi}\ell\right. \quad \mathbb{N}. \quad (2.5)$$

Note: The link between the R-L operator and the Caputo fractional differential operator is:

$$\left(J^{\alpha}D_t^{\alpha}\right)\ell \quad \ell \quad \ell \quad \sum_{k=0}^{n-1} \ell$$

$$,n-1<\alpha<n, n\in\mathbb{N}. \quad (2.6)$$

As such,

$$\ell \quad \ell \quad \sum_{k=0}^{n-1} \ell \quad . \quad (2.7)$$

Definition 5: The Mittag-Leffler (M-L) Function

The M-L function, $E_{\alpha}(z)$ is defined and denoted by the series representation as:

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)},\quad \alpha\geq 0,\ z\in\mathbb{C}. \quad (2.8)$$

Remark: For $\alpha=1$, $E_{\alpha}(z)$ in (2.8) becomes:

$$E_{\alpha=1}(z)=e^{z}. \quad (2.9)$$

III. Analysis of the DTM

The semi-analytical method (DTM) as noted by many researchers in literature, has proven to be simple and easier in the sense of application for both linear and nonlinear differential models because the DTM converts the problems under consideration to their equivalent forms in algebraic recursive relations, but this is not so when other semi-analytical techniques, say, ADM, VIM, HAM and so on are used) [24]. The DTM has been modified to handle models of nonlinear types and the likes [25-28].

A. The overview of the DTM

Let $v(x)$ be an analytic function at x_+ in a domain D, thus, the differential transform (DF) of $v(x)$ is defined and denoted by:

$$V(p)=\frac{1}{p!}\left[\frac{d^p v(x)}{dx^p}\right]_{x=x_+}. \quad (3.1)$$

And as such:

$$v(x)=\sum_{p=0}^{\infty}V(p)(x-x_+)^p \quad (3.2)$$

Equation (3.2) is the differential inverse transform (DIT) of $V(p)$, where $v(x)$ and $V(p)$ are the original and the transformed functions respectively.

B. The fundamentals of DTM: theorems and properties

Theorem 1: If $v(x)=\alpha v_a(x)\pm\beta v_b(x)$, then

$$V(p)=\alpha V_a(p)\pm\beta V_b(p).$$

Theorem 2 : If $v(x)=\frac{\alpha d^{\eta}v_+(x)}{dx^{\eta}}$, $\eta\in\mathbb{N}$, then

$$V(p)=\frac{\alpha(p+\eta)!}{p!}V_+(p+\eta).$$

Theorem 3: If $v(x)=\frac{\alpha dv_+(x)}{dx}$, then

$$V(p)=\frac{\alpha(p+1)!V_+(p+1)}{p!}=\alpha(p+1)V_+(p+1).$$

Theorem 4: If $v(x)=v_+^2(x)$, then

$$V(p)=\sum_{\eta=0}^{p}V_+(\eta)V_+(p-\eta).$$

Theorem 5: **(**PDTM of a fractional derivative**)**

If $f(x)=D_x^{\alpha}w(x)$, then

$$\Gamma\left(1+\frac{p}{q}\right)F(p)=\Gamma\left(1+\alpha+\frac{p}{q}\right)W(p+\alpha q).$$

Consequently, we have:

$$\Gamma\left(1+\alpha+\frac{p}{q}\right)W(p+\alpha q)=\Gamma\left(1+\frac{p}{q}\right)F(p). \quad (3.3)$$

Setting $\alpha q=1$ in (3.3) gives:

$$W(p+1)=\frac{\Gamma(1+\alpha p)}{\Gamma(1+\alpha(1+p))}F(p). \quad (3.4)$$

As such, for $w(x)$, α-analytic at $x_0=0$

$$w(x)=\sum_{\eta=0}^{\infty}W(\eta)x^{\frac{\eta}{q}}=\sum_{\eta=0}^{\infty}W(\eta)x^{\alpha\eta}. \quad (3.5)$$

C. Analysis of the Fractional DTM

Consider the nonlinear fractional differential equation (NLFDE):

$$D_x^{\alpha}w(x)+L_{\{x\}}w(x)+N_{\{x\}}w(x)=q_+(x),$$
$$w(x,0)=g_+(x),\ x>0 \quad (3.6)$$

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where $D_x^{\alpha} = \dfrac{d^{\alpha}}{dt^{\alpha}}$ is the fractional Caputo derivative of $w = w(x)$; whose projected differential transform is $W(p)$ while $L_{\{x\}}$ and $N_{\{x\}}$ are differential operators (with respect to x) of linear and nonlinear type respectively, and $q_+ = q_+(x)$ is the source term.

We rewrite (3.6) as:

$$\begin{cases} D_x^{\alpha} w(x) = -L_{\{x\}} w(x) - N_{\{x\}} w(x) q_+(x), \\ w(0) = g_+(x), \ \eta - 1 < \alpha < \eta, \ \eta \in \mathbb{N} \end{cases} \tag{3.7}$$

Applying the inverse fractional Caputo derivative, $D_x^{-\alpha}$ to both sides of (3.7) and with regard to (2.6) gives:

$$\begin{cases} w(x) = g(x) + D_x^{-\alpha}\left[-L_{[x]} w(x) - N_{[x]} w(x) + q(x)\right] \\ w(0) = g(x). \end{cases} \tag{3.8}$$

Thus, when $w(x)$ is expanded in terms of fractional power series, the inverse projected differential transform of $W(p)$ is given as follows:

$$\begin{cases} w(x) = \sum_{\eta=0}^{\infty} W(\eta) x^{\alpha\eta} = w(0) + \sum_{\eta=1}^{\infty} W(\eta) x^{\alpha\eta}, \\ w(x,0) = g_+(x). \end{cases} \tag{3.9}$$

IV. Illustrative Examples and Applications

In this subsection, the proposed method is applied with some illustrative examples for the solutions of fractional Riccati differential equations (FRDEs).

Problem 4.1: Consider the FRDE in (1.2) with $p_0(t) = 1, \ p_1(t) = 0, \ \& \ p_2(t) = -1$, thus, we have:

$$D_t^{\alpha} y(t) + y^2(t) - 1 = 0 \ , \& \ \ y(0) = 0. \tag{4.1}$$

Solution to problem 4.1:

We re-write (4.1) as:

$$D_t^{\alpha} y(t) = 1 - y^2(t) \ , \& \ \ y(0) = 0. \tag{4.2}$$

Taking the LFDT of (4.2) gives:

$$\begin{cases} LFDT\left[D_t^{\alpha} y(t) = 1 - y^2(t)\right], \\ LFDT\left[y(0) = 0\right]. \end{cases} \tag{4.3}$$

$\Rightarrow$

$$\begin{cases} \dfrac{\Gamma(1+\alpha(1+k))}{\Gamma(1+\alpha k)} Y(1+k) = \left(\delta(k) - \sum_{\eta=0}^{k} Y(\eta) Y(k-\eta)\right), \\ Y(0) = 0. \end{cases} \tag{4.4}$$

Thus, for $k=0$, $k=1$, $k=2$, $k=3$, $k=4$, $k=5\cdots$, we have respectively $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, \cdots$ as follows:

$$Y(1) = Y_1 = \frac{1}{\Gamma(1+\alpha)}\left\{1 - Y^2(0)\right\}, \tag{4.5}$$

$$Y(2) = Y_2 = \frac{-2\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left\{Y_0 Y_1\right\}, \tag{4.6}$$

$$Y(3) = Y_3 = \frac{-\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left\{2Y_0 Y_2 + Y_1 Y_1\right\}, \tag{4.7}$$

$$Y(4) = Y_4 = \frac{-2\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left\{Y_0 Y_3 + Y_1 Y_2\right\}, \tag{4.8}$$

$$Y(5) = Y_5 = \frac{-\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}\left\{2Y_0 Y_4 + 2Y_1 Y_3 + Y_2 Y_2\right\}, \tag{4.9}$$

$$Y(6) = Y_6 = \frac{-2\Gamma(1+5\alpha)}{\Gamma(1+6\alpha)}\left\{Y_0 Y_5 + Y_1 Y_4 + Y_2 Y_3\right\}. \tag{4.10}$$

Hence, using the initial condition, we obtain:

$$Y_0 = Y_2 = Y_4 = Y_6 = Y_8 = \cdots \qquad \mathbb{N},$$

$$Y_1 = \frac{1}{\Gamma(1+\alpha)}, \qquad Y_3 = \frac{-\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\right)^2,$$

$$Y_5 = \frac{2\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}\left\{\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\right)^3\right)\right\}, \cdots$$

$$\therefore \ y(t) = \sum_{h}^{\infty} Y_h t^{\alpha h}$$

$$= \frac{1}{\Gamma(1+\alpha)} t^{\alpha} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\right)^2 t^{3\alpha} + \frac{2\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\right)^3 t^{5\alpha} + \cdots \tag{4.11}$$

Remark (a): For $\alpha = 1$, we have $y(t) = t - \dfrac{t^3}{3} + \dfrac{2t^5}{15} - \dfrac{17t^7}{315} + \cdots$ which corresponds to the solution as contained in [4, 13] via a decomposed method.

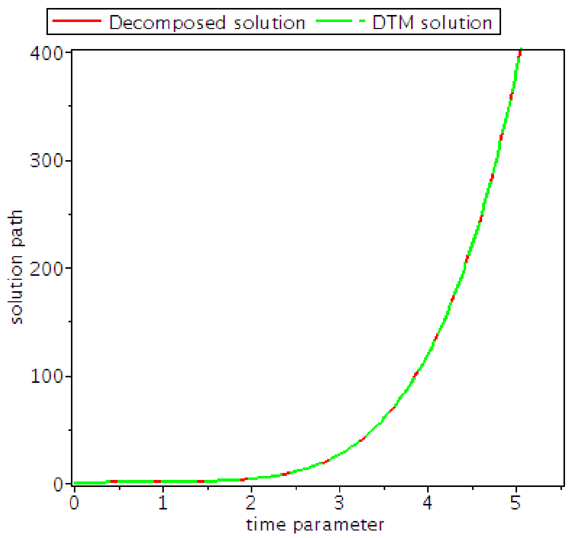

Fig.1: solution of 4.1 via decomposed and DT methods

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V. Concluding Remarks

We have considered in this work, the approximate solutions of fractional Riccati differential equations (FRDEs) via the application of the local fractional operator in the sense of Caputo derivative as a proposed semi-analytical method built on the basis of the classical DTM. To demonstrate the effectiveness and robustness of the present technique, we used some illustrative examples; the solutions are provided in the form of convergent series. This supports the efficiency, and reliability of the solution method; as it does not require much computational work, even without given up accuracy. We remarked that the solutions of RDEs at different (well-defined) values of fractional orders can easily be obtained. Thus, the method is recommended for the solutions linear and nonlinear time-fractional differential equations (TFDEs) with wider applications in other areas of applied sciences.

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