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ON THE CONVERGENCE OF MODIFIED THREE-STEP ITERATION PROCESS FOR GENERALIZED CONTRACTIVE-LIKE OPERATORS

(COMMUNICATED BY MARTIN HERMANN)

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Abstract. In this paper, we introduce a new Jungck-three step iterative scheme and call it modified three-step iteration process. A strong convergence theorem is proved using this iterative process for the class of generalized contractive-like operators introduced by Olatinwo [14] and Bosede [3] respectively, in a Banach space. The results obtained in this paper improve and generalize among others, the results of Bosede [3], Olatinwo and Imoru [13], Shaini and Singh [16], Jungck [6] and Berinde [2].

1. Introduction and Preliminaries

One of the remarkable generalization of Banach contraction mapping principle is the Jungck contraction principle proved by Jungck [6] in 1976. The author [6] proved the theorem by replacing the identity map with a continuous map.

Theorem 1 [6]. Let f be a continuous mapping of a complete metric space (X, d) into itself and let $g : X \to X$ be a map that satisfy the following conditions:
(a) $g(X) \subseteq f(X)$
(b) g commute with f
(c) $d(gx, gy) \leq kd(fx, fy)$ for all $x, y \in X$ and for some $0 \leq k < 1$. Then f and g have a unique common fixed point provided f and g commute.

Recently, several authors have studied the Jungck-multistep iterative schemes to approximate the coincidence points and common fixed points of the Jungck-type operators in Banach spaces (for details see [12], [13], [14] and [18]).

In this paper, a modified three-step iterative is introduced and a strong convergence theorem is proved for the class of generalized contractive-like operators in a Banach space. The iteration process is defined as follows.

Let E be a Banach space and Y an arbitrary set. Let $S, T : Y \to E$ be two nonselfmappings such that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of E. Then

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for $x_0 \in Y$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} Sx_{n+1} &= (1-a_n^1-b_n^1-c_n^1)Sx_n + a_n^1 Ty_n + b_n^1 Tz_n + c_n^1 Tx_n \\ Sy_n &= (1-b_n-c_n)Sx_n + b_n Tz_n + c_n Tx_n \\ Sz_n &= (1-a_n)Sx_n + a_n Tx_n, \qquad n \geq 0 \end{aligned} \tag{1.1}$$

for every $x, y, z \in E$, where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^1\}, \{b_n^1\}, \{c_n^1\}$ are appropriate sequences in [0,1). If $a_n = c_n = b_n^1 = c_n^1 = 0$, then (1.1) reduces to the Jungck-Ishikawa (two-step) iterative process [13].

$$\begin{aligned} Sx_{n+1} &= (1-a_n^1)Sx_n + a_n^1 Ty_n \\ Sy_n &= (1-b_n)Sx_n + b_n Tx_n, \qquad n \geq 0, \end{aligned} \tag{1.2}$$

where $\{a_n^1\}$ and $\{b_n\}$ are appropriate sequences in [0,1).
Also, if $a_n = b_n = c_n = b_n^1 = c_n^1 = 0$, then (1.1) reduces to the Jungck-Mann iterative process [18].

$$Sx_{n+1} = (1-a_n^1)Sx_n + a_n^1 Tx_n, \qquad n \geq 0, \tag{1.3}$$

where $\{a_n^1\}$ is an appropriate sequence in [0,1).
If $a_n^1 = 1$ and $Y = E$, (1.3), reduces to the Jungck iteration process [6].

$$Sx_{n+1} = Tx_n, \qquad n \geq 0. \tag{1.4}$$

If $S = id$ (identity operator), Y = E, then (1.1), (1.2), (1.3), (1.4) reduces to the iterative processes introduced by Shaini and Singh [16], Ishikawa [5], Mann [8] and Picard iterations respectively.

Olatinwo [14] introduced the following class of generalized contractive-like operators to obtain some stability results for the Jungck-Noor iterative process in an arbitrary Banach space.

Definition 1.1 [14]. For $S, T : X \to X$ with $T(Y) \subset S(Y)$, where $S(Y)$ is a complete subspace of X, there exist a real number $\delta \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and for every $x, y \in Y$, then

$$\|Tx - Ty\| \leq \delta\|Sx - Sy\| + \varphi(\|Sx - Tx\|). \tag{1.5}$$

Recently, Bosede [3] introduced a new class of generalized contractive-like operators independent of (1.5) and obtained a strong convergence results for the Jungck-Ishikawa and Jungck-Mann iteration processes for this class of operators in a Banach space.

$$\|Tx - Ty\| \leq e^{L\|Sx-Tx\|}(\delta\|Sx - Sy\| + 2\delta\|Sx - Tx\|), \tag{1.6}$$

where $\delta \in [0, 1)$ and e^x denotes the exponential function of $x \in Y$.

Definition 1.2 [1]. A point $x \in X$ is called a coincidence point of self maps S, T if there exists a point q (called a point of coincidence) in X such that $p = Sq = Tq$. Self-maps S and T are said to be weakly compatible if they commute at their coincidence pointe, that is, if $Sx = Tx$ for some $x \in X$, then $STx = TSx$.

The purpose of this paper is to establish strong convergence results for modified three-step iterative process in a Banach space using contractive conditions (1.5) and (1.6) respectively. Our results improve and generalize, among others, the results of Bosede [3], Olatinwo and Imoru [14], Shaini and Singh [16], Jungck [6] and Berinde [2].

Lemma 1.3 [12]: *Let* $\{\theta_n\}_{n=0}^{\infty}$ *be sequence of nonnegative numbers satisfying*

$$\theta_{n+1} \leq (1-\lambda_n)\theta_n, \qquad n \geq 0,$$

where $\lambda_n \in [0,1)$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$. *Then* $\lim_{n\to\infty} \theta_n = 0$.

2. Main Result

Theorem 2.1. *Let* E *be a Banach space and* $S, T : Y \to E$ *for an arbitrary set* Y *such that* $\|Tx - Ty\| \leq \delta\|Sx - Sy\| + \varphi(\|Sx - Tx\|)$ *holds and* $T(Y) \subset S(Y)$. *Assume that* S *and* T *have a coincidence point* q *such that* $Tq = Sq = p$. *For any* $x_0 \in Y$, *the modified three step iterative process*
(1.1) $\{Sx_n\}$ *converges to* p, *where* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^1\}$, $\{b_n^1\}$, $\{c_n^1\}$ *are real sequences in [0,1) such that* $b_n + c_n$ *and* $a_n^1 + b_n^1 + c_n^1$ *are in [0,1) for all* $n \geq 0$ *satisfying one of the following conditions:*
$(i) \sum_{n=0}^{\infty} a_n^1 = \infty, \quad (ii) \sum_{n=0}^{\infty} b_n^1 = \infty, \quad (iii) \sum_{n=0}^{\infty} c_n^1 = \infty.$
Further, if $Y = E$ *and* S *and* T *are weakly compatible (i.e* S, T *commute at* p*), then* p *is the unique common fixed point of* S, T.

Proof:
We now use contractive condition (1.5) to establish that the common fixed point of S and T is unique.

$$\begin{aligned}
\|Sx_{n+1} - p\| &= \|(1 - a_n^1 - b_n^1 - c_n^1)Sx_n + a_n^1 Ty_n + b_n^1 Tz_n + c_n^1 Tx_n \\
&\quad -(1 - a_n^1 - b_n^1 - c_n^1 + a_n^1 + b_n^1 + c_n^1)p\| \\
&= \|(1 - a_n^1 - b_n^1 - c_n^1)(Sx_n - p) + a_n^1(Ty_n - p) \\
&\quad + b_n^1(Tz_n - p) + c_n^1(Tx_n - p)\| \\
&\leq (1 - a_n^1 - b_n^1 - c_n^1)\|Sx_n - p\| + a_n^1\|Ty_n - p\| \\
&\quad + b_n^1\|Tz_n - p\| + c_n^1\|Tx_n - p\|. \qquad (2.1)
\end{aligned}$$

$$\|Ty_n - p\| = \|Ty_n - Tq\| \qquad \|Tz_n - p\| = \|Tz_n - Tq\|$$

For $x = q$, $y = y_n$ in (1.5), we have

$$\begin{aligned}
\|Tq - Ty_n\| &\leq \delta\|Sq - Sy_n\| + \varphi(\|Sq - Tq\|) \\
&= \delta\|Sy_n - p\|. \qquad (2.2)
\end{aligned}$$

Similarly,

$$\|Tq - Tz_n\| \leq \delta\|Sz_n - p\|. \qquad (2.3)$$

and $\|Tq - Tx_n\| \leq \delta\|Sx_n - p\|$ (2.4)

Using (2.2), (2.3) and (2.4) in (2.1), we have

$$\begin{aligned}
\|Sx_{n+1} - p\| &\leq (1 - a_n^1 - b_n^1 - c_n^1)\|Sx_n - p\| + \delta a_n^1\|Sy_n - p\| \\
&\quad + \delta b_n^1\|Sz_n - p\| + \delta c_n^1\|Sx_n - p\|. \qquad (2.5)
\end{aligned}$$

$$\begin{aligned}
\|Sy_n - p\| &= \|(1 - b_n - c_n)Sx_n + b_n Tz_n + c_n Tx_n \\
&\quad -(1 - b_n - c_n + b_n + c_n)p\| \\
&\leq (1 - b_n - c_n)\|Sx_n - p\| + b_n\|Tz_n - p\| + c_n\|Tx_n - p\| \\
&\leq (1 - b_n - c_n)\|Sx_n - p\| + \delta b_n\|Sz_n - Sq\| \\
&\quad + \delta c_n\|Sx_n - Sq\| \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}\|Sz_n - p\| &\le (1-a_n)\|Sx_n - p\| + a_n\|Tx_n - p\| \\ &\le (1-a_n)\|Sx_n - p\| + \delta a_n\|Sx_n - p\| \\ &= (1-a_n+\delta a_n)\|Sx_n - p\|. \qquad (2.7)\end{aligned}$$

Substituting (2.7) in (2.6), we have

$$\begin{aligned}\|Sy_n - p\| &\le (1-b_n-c_n)\|Sx_n - p\| + \delta b_n(1-a_n+\delta a_n)\|Sx_n - p\| \\ &\quad +\delta c_n\|Sx_n - p\| \\ &= (1-b_n-c_n+\delta b_n-\delta a_n b_n+\delta^2 a_n b_n+\delta c_n)\|Sx_n - p\|. \qquad (2.8)\end{aligned}$$

Substituting (2.7) and (2.8) in (2.5), we have

$$\begin{aligned}\|Sx_{n+1} - p\| &\le (1-a_n^1-b_n^1-c_n^1)\|Sx_n - p\| \\ &\quad +\delta a_n^1(1-b_n-c_n+\delta b_n-\delta a_n b_n+\delta^2 a_n b_n+\delta c_n) \\ &\quad +\delta b_n^1(1-a_n+\delta a_n)\|Sx_n - p\| + \delta c_n^1\|Sx_n - p\| \qquad (2.9) \\ &= (1-a_n^1+\delta a_n^1-b_n^1+\delta b_n^1-c_n^1+\delta c_n^1-\delta a_n^1 b_n \\ &\quad +\delta^2 a_n^1 b_n-\delta^2 a_n^1 a_n b_n+\delta^3 a_n^1 a_n b_n-\delta a_n c_n \\ &\quad +\delta^2 a_n c_n-\delta a_n b_n^1+\delta^2 a_n b_n^1)\|Sx_n - p\| \\ &= [1-a_n^1(1-\delta)-b_n^1(1-\delta)-c_n^1(1-\delta) \\ &\quad -a_n^1 b_n\delta(1-\delta)-a_n^1 a_n b_n\delta^2(1-\delta)-a_n^1 c_n\delta(1-\delta) \\ &\quad -a_n b_n^1\delta(1-\delta)]\|Sx_n - p\| \qquad (2.10) \\ &\le [1-a_n^1(1-\delta)]\|Sx_n - p\|, \qquad (2.11)\end{aligned}$$

for $n = 0, 1, 2, ...$

It follows from the given conditions and Lemma 1.3 that

$$\lim_{n\to\infty}[1-(1-\delta)a_n^1] = 0.$$

Thus by (2.11), it follows that

$$\lim_{n\to\infty}\|Sx_{n+1} - p\| = 0.$$

Therefore, $\{Sx_{n+1}\}_{n=0}^{\infty}$ converges strongly to p.

Next, we show that p is unique. Suppose there exist another point of coincidence p^*, then there is a $q^* \in E$ such that $Tq^* = Sq^* = p^*$. Hence, from (1.5), we have $\|p - p^*\| = \|Tq - Tq^*\| \le \delta\|Sq - Sq^*\| + \varphi(\|Sq - Tq\|) = \delta\|p - p^*\|$.

Since S, T are weakly compatible, then $TSq = STq$ and so $Tp = Sp$. Hence p is a coincidence point of S, T and since the coincidence point is unique, then $p = p^*$ and hence $Sp = Tp = p$ and therefore p is the unique common fixed point of S, T. This completes the proof.

Corollary 2.2. *Let E be a Banach space and $S, T : Y \to E$ for an arbitrary set Y such that $\|Tx - Ty\| \le \delta\|Sx - Sy\| + \varphi(\|Sx - Tx\|)$ holds and $T(Y) \subset S(Y)$. Assume that S and T have a coincidence point q such that $Tq = Sq = p$. For any $x_0 \in Y$, the Jungck-Noor iteration process [14] $\{Sx_n\}$ converges to p, where $\{a_n^1\}, \{b_n\}$ and $\{a_n\}$ are real sequences in [0,1) such that $\sum a_n^1 = \infty$. Further, if $Y = E$ and S and T are weakly compatible (i.e S, T commute at p) then p is the unique common fixed point of S, T.*

Corollary 2.3. *Let E be a Banach space and $S,T : Y \to E$ for an arbitrary set Y such that $\|Tx-Ty\| \leq \delta\|Sx-Sy\| + \varphi(\|Sx-Tx\|)$ holds and $T(Y) \subset S(Y)$. Assume that S and T have a coincidence point q such that $Tq = Sq = p$. For any $x_0 \in Y$, the Jungck-Ishikawa iteration process (1.2) $\{Sx_n\}$ converges to p, where $\{a_n^1\}, \{b_n\}$ are real sequences in [0,1) such that $\sum a_n^1 = \infty$. Further, if $Y = E$ and S and T are weakly compatible (i.e S,T commute at p) then p is the unique common fixed point of S,T.*

Corollary 2.4. *Let E be a Banach space and $S,T : Y \to E$ for an arbitrary set Y such that $\|Tx-Ty\| \leq \delta\|Sx-Sy\| + \varphi(\|Sx-Tx\|)$ holds and $T(Y) \subseteq S(Y)$. Assume that S and T have a coincidence point q such that $Tq = Sq = p$. For any $x_0 \in Y$, the Jungck-Mann iteration process (1.3) $\{Sx_n\}$ converges to p, where $\{a_n^1\}$ is real sequence in [0,1) such that $\sum a_n^1 = \infty$. Further, if $Y = E$ and S and T are weakly compatible (i.e S,T commute at p) then p is the unique common fixed point of S,T.*

Remark 2.5. (i) Our Theorem 2.1 is a generalization and extension of Theorem 3.1 of Olatinwo and Imoru [13] in the sense that the Jungck-Ishikawa iterative process used in [13] is a special case of the modified three-step iterative scheme (1.1) in Theorem 2.1. Also, with $\varphi(t) = 2\delta t$ [13], the generalized contractive-like operator (1.5) used in Theorem 2.1 reduces to the generalized Zamfirescu operator used in [13].

(ii) Our Theorem 2.1 extends and generalizes Theorem 2.1 of Shaini and Singh [16] in the sense that when $S = id$ (identity) in modified three-step iteration (1.1), we have the three-step iteration introduced by Shaini and Singh [16]. Also, with $S = id$ (identity) and $\varphi(t) = 2\delta t$, inequality (1.5) reduces to the Zamfirescu operator used in [12]

(iii) Berinde's Theorem ([2], Theorem 2) follows as a Corollary from Corollary 2.3 with $\varphi(t) = 2\delta t$ and $S = id$ (identity operator).

Theorem 2.6. *Let E be a Banach space and $S,T : Y \to E$ for an arbitrary set Y such that $\|Tx - Ty\| \leq e^{L\|Sx-Tx\|}(\delta\|Sx - Sy\| + 2\delta\|Sx - Tx\|)$ holds and $T(Y) \subseteq S(Y)$. Assume that S and T have a coincidence point q such that $Tq = Sq = p$. For any $x_0 \in Y$, the modified three step iterative process (1.1) $\{Sx_n\}$ converges to p, where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^1\}, \{b_n^1\}, \{c_n^1\}$ are real sequences in [0,1) such that $b_n + c_n$ and $a_n^1 + b_n^1 + c_n^1$ are in [0,1) for all $n \geq 0$ satisfying one of the following conditions:*
(i) $\sum_{n=0}^{\infty} a_n^1 = \infty$, *(ii)* $\sum_{n=0}^{\infty} b_n^1 = \infty$, *(iii)* $\sum_{n=0}^{\infty} c_n^1 = \infty$.
Further, if $Y = E$ and S and T are weakly compatible (i.e S,T commute at p), then p is the unique common fixed point of S,T.

Proof:
We now use contractive condition (1.6) to establish that the common fixed point of S and T is unique.

$$\begin{aligned} \|Sx_{n+1}-p\| &= \|(1-a_n^1-b_n^1-c_n^1)Sx_n+a_n^1Ty_n+b_n^1Tz_n+c_n^1Tx_n \\ &\quad -(1-a_n^1-b_n^1-c_n^1+a_n^1+b_n^1+c_n^1)p\| \\ &= \|(1-a_n^1-b_n^1-c_n^1)(Sx_n-p)+a_n^1(Ty_n-p) \\ &\quad +b_n^1(Tz_n-p)+c_n^1(Tx_n-p)\| \\ &\leq (1-a_n^1-b_n^1-c_n^1)\|Sx_n-p\|+a_n^1\|Ty_n-p\| \\ &\quad +b_n^1\|Tz_n-p\|+c_n^1\|Tx_n-p\|. \end{aligned} \tag{2.12}$$

$$\|Ty_n-p\|=\|Ty_n-Tq\| \qquad \|Tz_n-p\|=\|Tz_n-Tq\|$$

For $x=q$, $y=y_n$ in (1.6), we have

$$\begin{aligned} \|Tq-Ty_n\| &\leq e^{L\|Sq-Tq\|}(\delta\|Sq-Sy_n\|+2\delta\|Sq-Tq\|) \\ &= e^{L\|p-p\|}(\delta\|p-Sy_n\|+2\delta\|p-p\|) \\ &= \delta\|Sy_n-p\|. \end{aligned} \tag{2.13}$$

Similarly,

$$\|Tq-Tz_n\| \leq \delta\|Sz_n-p\|. \tag{2.14}$$

and $\|Tq-Tx_n\| \leq \delta\|Sx_n-p\|$ (2.15)

Using (2.13), (2.14) and (2.15) in (2.12), we have

$$\begin{aligned} \|Sx_{n+1}-p\| &\leq (1-a_n^1-b_n^1-c_n^1)\|Sx_n-p\|+\delta a_n^1\|Sy_n-p\| \\ &\quad +\delta b_n^1\|Sz_n-p\|+\delta c_n^1\|Sx_n-p\|. \end{aligned} \tag{2.16}$$

$$\begin{aligned} \|Sy_n-p\| &= \|(1-b_n-c_n)Sx_n+b_nTz_n+c_nTx_n \\ &\quad -(1-b_n-c_n+b_n+c_n)p\| \\ &\leq (1-b_n-c_n)\|Sx_n-p\|+b_n\|Tz_n-p\|+c_n\|Tx_n-p\| \\ &\leq (1-b_n-c_n)\|Sx_n-p\|+\delta b_n\|Sz_n-Sq\| \\ &\quad +\delta c_n\|Sx_n-Sq\| \end{aligned} \tag{2.17}$$

$$\begin{aligned} \|Sz_n-p\| &\leq (1-a_n)\|Sx_n-p\|+a_n\|Tx_n-p\| \\ &\leq (1-a_n)\|Sx_n-p\|+\delta a_n\|Sx_n-p\| \\ &= (1-a_n+\delta a_n)\|Sx_n-p\| \end{aligned} \tag{2.18}$$

Substituting (2.18) in (2.17), we have

$$\begin{aligned} \|Sy_n-p\| &\leq (1-b_n-c_n)\|Sx_n-p\|+\delta b_n(1-a_n+\delta a_n)\|Sx_n-p\| \\ &\quad +\delta c_n\|Sx_n-p\| \end{aligned} \tag{2.19}$$

Substituting (2.18) and (2.19) in (2.16), we have

$$\begin{aligned} \|Sx_{n+1}-p\| &\leq (1-a_n^1-b_n^1-c_n^1)\|Sx_n-p\| \\ &\quad +\delta a_n^1(1-b_n-c_n+\delta b_n-\delta a_nb_n+\delta^2a_nb_n+\delta c_n) \\ &\quad +\delta b_n^1(1-a_n+\delta a_n)\|Sx_n-p\|+\delta c_n^1\|Sx_n-p\| \qquad (2.20) \\ &= (1-a_n^1+\delta a_n^1-b_n^1+\delta b_n^1-c_n^1+\delta c_n^1-\delta a_n^1b_n \\ &\quad +\delta^2a_n^1b_n-\delta^2a_n^1a_nb_n+\delta^3a_n^1a_nb_n-\delta a_nc_n \\ &\quad +\delta^2a_nc_n-\delta a_nb_n^1+\delta^2a_nb_n^1)\|Sx_n-p\| \\ &= [1-a_n^1(1-\delta)-b_n^1(1-\delta)-c_n^1(1-\delta) \\ &\quad -a_n^1b_n\delta(1-\delta)-a_n^1a_nb_n\delta^2(1-\delta)-a_n^1c_n\delta(1-\delta) \\ &\quad -a_nb_n^1\delta(1-\delta)]\|Sx_n-p\| \qquad (2.21) \\ &\leq [1-a_n^1(1-\delta)]\|Sx_n-p\|, \qquad (2.22) \end{aligned}$$

for $n=0,1,2,...$

It follows from the given conditions and Lemma 1.3 that

$$\lim_{n\to\infty}[1-(1-\delta)a_n^1]=0.$$

Thus by (2.22), it follows that

$$\lim_{n\to\infty}\|Sx_{n+1}-p\|=0.$$

Therefore, $\{Sx_{n+1}\}_{n=0}^{\infty}$ converges strongly to p.

Next, we show that p is unique. Suppose there exist another point of coincidence p^*, then there is a $q^*\in E$ such that $Tq^*=Sq^*=p^*$. Hence, from (1.6), we have $\|p-p^*\|=\|Tq-Tq^*\|\leq e^{L\|Sq-Tq\|}(\delta\|Sq-Sq^*\|+2\delta\|Sq-Tq\|)=\delta\|p-p^*\|$.

Since S,T are weakly compatible, then $TSq=STq$ and so $Tp=Sp$. Hence p is a coincidence point of S,T and since the coincidence point is unique, then $p=p^*$ and hence $Sp=Tp=p$ and therefore p is the unique common fixed point of S,T. This completes the proof.

Remark 2.7. Our Theorem 2.6 generalizes and extends Theorems 3.1 and 3.2 of Bosede [3] in the sense that the concept of weak compatibility was employed and injectivity of the map S was not assumed. Also the Jungck-Ishikawa (1.2) and Jungck-Mann (1.3) iterative processes are special cases of the modified three-step iterative process (1.1) considered in this work.

Example 2.8. Let $Y=\Big([0,2],|.|\Big)$. Define T and S by

$$Tx=\begin{cases}\frac{1}{2}, & \text{if } \quad x\in(0,1]\\ 0, & \text{if } \quad x\in\{0\}\cup(1,2]\end{cases} \quad \text{and } Sx=\begin{cases}0, & \text{if} \quad \text{x=0}\\ x+1, & \text{if} \quad x\in(0,1]\\ x-1, & \text{if} \quad x\in(1,2]\end{cases}$$

$\|Tx-Ty\|\leq\delta\|Sx-Sy\|+\varphi(\|Sx-Tx\|)$, where $\delta=\frac{1}{2}$ and $\varphi(t)=2\delta t$.
$T(Y)=\{0\}\cup\{\frac{1}{2}\}$ and $S(Y)=[0,2]$. Then $T(Y)\subseteq S(Y)$. It is easy to see that $S(0)=T(0)=0$ and $ST(0)=S(0)=0$, $TS(0)=T(0)=0$.
Hence the common fixed point of S and T is 0.

Running a MATLAB 7.10.0 script, with $a_n=\frac{2}{3}$, $b_n=c_n=\frac{1}{2n+4}$, $a_n^1=b_n^1=c_n^1=\frac{1}{4}$ for all $n>0$ and $x_0=1$ we have the following results:

$$\begin{aligned}
Sx_1 &= 0.750000000000000\\
Sx_2 &= 0.187500000000000\\
Sx_3 &= 0.046875000000000\\
Sx_4 &= 0.011718750000000\\
Sx_5 &= 0.002929687500000\\
Sx_6 &= 0.000732421875000\\
Sx_7 &= 0.000183105468750\\
Sx_8 &= 0.000045776367188\\
Sx_9 &= 0.000011444091797\\
Sx_{10} &= 0.000002861022949
\end{aligned}$$

We notice that $\{Sx_n\}$ in (1.1) converges to 0 which is the common fixed point of S and T.

Example 2.9 ([20]). Let $(X, d) = ([0, 10], |.|)$. Define S and T by

$$Sx = \begin{cases} 3 & \text{if} \quad x \in (0, 2] \\ \\ 0 & \text{if} \quad x \in \{0\} \cup (2, 10] \end{cases} \quad \text{and } Tx = \begin{cases} 0 & \text{if} \quad \text{x=0} \\ \\ x + 8 & \text{if} \quad x \in (0, 2] \\ \\ x - 2 & \text{if} \quad x \in (2, 10] \end{cases}$$

Then
$Sx = Tx$ iff $x = 0$,
$ST(0) = T(0) = 0$, $TS(0) = S(0) = 0$.
Therefore S and T are weakly compatible.

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