ON COMPLETENESS OF PARTIAL METRIC SPACES, SYMMETRIC SPACES AND SOME FIXED POINT RESULTS

by

MAGGIE APHANE

submitted in fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in the subject

MATHEMATICS

at the

UNIVERSITY OF SOUTH AFRICA

Supervisor: PROF S.P. MOSHOKOA

DECEMBER 2016.

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Acknowledgements

The completion of this thesis is owed to my supervisor Professor Seithuti Moshokoa, who has been supportive and provided guidance as well as constructive criticisms. My sincere gratitude to his availability and readiness to help, his thorough scrutiny to my work throughout. Without him this work would have not been possible and successful. Thank you Professor S.P. Moshokoa for being an inspiring example to me.

My special gratitude also goes to my employer Tshwane University of Technology specifically the Department of Mathematics and Statistics in the Faculty of Science for their financial support, sabbatical leave and lecturer reliefs. To list few colleagues Dr Charles Nsukukasifani Msipha, Ms Minky Makgethwa Seromo and Dr Phumezile Kama for their friendship and support.

Thanks to the Department of Mathematical Sciences at the University of South Africa for the financial support specifically during research writing retreats. Also thanks goes to Dr Mack Matlabyana and Dr Oghenetega Ighedo. My sincere gratitude also goes to Dr Mokhwetha Mabula and Dr Esme Voges for proofreading my thesis.

There are no words to thank the Aphane's family: Jack Manaba, Masetene Mighty, Welheminah Magentlele, Mokgethwa Claudia, Ditebogo William, Boitumelo Mahlogonolo and Phetogo Lincoln for their patience and supporting career endeavors while I was preoccupied with my studies and attending conferences nationally and internationally. Finally, thanks to my mom, sisters, brother and inlaws for their unlimited support.

Abstract

The purpose of the thesis is to study completeness of abstract spaces. In particular, we study completeness in partial metric spaces, partial metric type spaces, dislocated metric spaces, dislocated metric type spaces and symmetric spaces that are generalizations of metric spaces. It is well known that complete metric spaces have a wide range of applications. For instance, the classical Banach contraction principle is phrased in the context of complete metric spaces. Analogously, the Banach's fixed point theorem and fixed point results for Lipschitzian maps are discussed in this context, namely in, partial metric spaces and metric type spaces. Finally, fixed point results are presented for symmetric spaces.

Keywords: Metric space, quasi metric space, dislocated metric space, partial metric space, TVS-cone metric space, TVS-partial cone metric space, dislocated cone metric space, convergent sequence, Cauchy sequence, 0-Cauchy sequence, convergence complete, Cauchy complete, 0-Cauchy complete, contraction constant, contraction map, Lipschitzian constant, Lipschitzian map, fixed point, metric type space, dislocated metric type space, partial metric type space, symmetric space.

Summary

The thesis deals with the completeness problem for spaces, this is a continuation to the study of completeness in metric spaces as presented in the literature. In this study, however, we look at more general spaces than metric spaces, for example the spaces considered are partial metric spaces and symmetric spaces. As applications, we also present some fixed point results in this context. In the literature [19] TVS-cone metric structures are introduced and are shown to be a generalization of metric spaces. More specifically, it is shown that topological properties of TVS-cone metric spaces arise easily from those of metric spaces see, [9] and [24]. So, we will also discuss the relationship on TVS-partial cone metric spaces and dislocated metric spaces.

The first part, with the main work starting at Chapter 2 of the thesis, begins by establishing that a TVS-partial cone metric space gives rise to a dislocated metric space. An interesting but unexpected result is that these spaces are not topologically equivalent. Later on, the quasimetrizability of partial metric spaces is discussed.

Chapter 3, focuses on partial metric type spaces as a generalization of metric type spaces. We define two types of Cauchy sequences, and consequently two types of completeness are discussed and some fixed point results are presented for these spaces. A fixed point result for TVS-partial cone metric space is also presented.

The final part, Chapter 4, deals with the study of completeness in symmetric spaces. In the literature two types of completeness for symmetric spaces are obtained. The one that deals with completeness using Cauchy sequences [17], [35] and the other one that deals with completeness without appealing to Cauchy sequences [32], [33]. We show that a symmetric space that is complete in the sense of [32] and [33] is actually Cauchy complete and present further properties of symmetric spaces, like products of complete symmetric spaces and some fixed point results for single-valued maps and multi-valued maps in the context of complete symmetric spaces.

Declaration

Student number: 32172125

I hereby declare that *On completeness of partial metric spaces, symmetric spaces and some fixed point results* is the result of my own work and that all sources I have used have been duly acknowledged and indicated in a complete list of references.

I further declare that I have not previously submitted this work, or part of it, for examination at any other higher education institution including Unisa for another qualification.

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(M. Aphane)

Chapter 1

Introduction

In this chapter we provide a background study of metric spaces [47] in Section 1.1, quasi metric spaces [26] in Section 1.2, partial metric spaces [28] in Section 1.3, TVS-cone metric spaces [19] in Section 1.4 and TVS-partial cone metric spaces [48] in Section 1.5. The work presented in this chapter shall be generalized in the subsequent chapters. Most of the results are taken from the literature and thus are well-known. Consequently, we shall omit most of the proofs of the results presented and provide references to guide the reader.

Our notation is fairly standard. For instance, we denote the set of real numbers by $\mathbb{R}$, the set of positive real numbers by $\mathbb{R}_0^+$, the set of positive integers by $\mathbb{N}$, and the set of rational numbers by $\mathbb{Q}$.

1.1 Metric spaces and some fixed point results

References used for metric spaces are [36] and [47].

Definition 1.1.1 Let X be a nonempty set. A map $d : X \times X \to [0, \infty)$ is said to be a **metric** on X if for all $x, y, z \in X$ the following conditions hold:

(i) $d(x, y) \geq 0$;

(ii) $d(x, y) = d(y, x)$;

(iii) $x = y$ iff $d(x, y) = 0$;

(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a **metric space**.

The reader should note that (ii) is called the symmetric property and (iv) is called the triangle inequality.

Example 1.1.1 [47, Example 1.2.2] Let $X = \mathbb{R}$, define a map $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $d(x, y) = |x - y|$ for all $x, y \in X$. A map d is called the **usual** or **standard metric** on $\mathbb{R}$. Thus (X, d) is a metric space.

Definition 1.1.2 Let (X, d) be a metric space. Then

(i) a sequence $\{x_n\}$ in (X, d) d-**converges** to a point $x \in X$ if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for each $n \geq N$.

(ii) a sequence $\{x_n\}$ in (X, d) is d-**Cauchy** if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for each $m, n \geq N$.

(iii) a metric space (X, d) is d-**Cauchy complete** if every d-Cauchy sequence $\{x_n\}$ d-converges to a point $x \in X$.

Remark 1.1.1 Let (X, d) be a metric space.

(i) If a sequence $\{x_n\}$ d-converges to a point $x \in X$ we shall also write $\lim_n d(x_n, x) = 0$ or simply $x_n \xrightarrow{d} x$.

(ii) If a sequence $\{x_n\}$ in (X, d) is d-Cauchy we shall also write $\lim_{n,m} d(x_n, x_m) = 0$.

Example 1.1.2 [47, Example 1.2.2] The metric space (X, d) of Example 1.1.1 is d-Cauchy complete.

Proposition 1.1.1 *Let (X, d) be a metric space, $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x \in X$. If $\lim_n d(x_n, x) = 0$ and $\lim_n d(y_n, y) = 0$, then $\lim_n d(x_n, y_n) = d(x, y)$.*

Proposition 1.1.2 *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. If a sequence $\{x_n\}$ is d-convergent then it is d-Cauchy.*

The converse of Proposition 1.1.2 is not necessarily true.

Example 1.1.3 [47, Example 1.4.14] Let $X = (0, \infty)$ be equipped with the usual metric d. A sequence $\{x_n = \frac{1}{n}, n \geq 1\}$ is d-Cauchy but does not d-converge to a point in X.

Proposition 1.1.3 *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. If $\lim_n d(x_n, x) = 0$ and $\lim_n d(x_n, y) = 0$, then $x = y$.*

Definition 1.1.3 Let (X, d) be a metric space. Define

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$$

for all $x \in X, \epsilon > 0$. The set $B_d(x, \epsilon)$ is called an **open ball** with center x and radius ϵ. The family $\{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$ is a **base** for the topology τ on X. We denote by $\tau_{(d)}$ the topology induced by d on X.

Definition 1.1.4 A topological space (X, τ) is **metrizable** if there exists a metric d on X such that $\tau_{(d)} = \tau$.

It should be observed that every metric space (X, d) is a T_2-space(**Hausdorff space**).

Definition 1.1.5 Let d_1 and d_2 be two metrics on a nonempty set X. Then d_1 and d_2 are **equivalent** when $\lim_n d_1(x_n, x) = 0$ if and only if $\lim_n d_2(x_n, x) = 0$ holds for a sequence $\{x_n\}$ in X and $x \in X$. If d_1 is equivalent to d_2 we will write $d_1 \simeq d_2$.

Next is an example of equivalent metrics on a set X.

Example 1.1.4 [47, Example 1.2.4] Let $X = \mathbb{R}$, $d_1(x, y) = |x - y|$ and $d_2(x, y) = \frac{|x-y|}{1+|x-y|}$ for all $x, y \in X$. Then $d_1 \simeq d_2$.

Remark 1.1.2 Let X be a nonempty set, d_1, d_2 and d_3 be metrics on X. If $d_1 \simeq d_2$ and $d_2 \simeq d_3$ then $d_1 \simeq d_3$.

Definition 1.1.6 Let (X, d) be a metric space and A be a nonempty subset of X. Then A is **bounded** if there exists an $N > 0$ such that $d(x, y) \leq N$ for all $x, y \in A$.

By reverting back to Example 1.1.4, we see that if (X, d_1) is a metric space, by defining $d_2 : X \times X \to [0, \infty)$ with $d_2(x, y) = \frac{d_1(x,y)}{1+d_1(x,y)}$, for all $x, y \in X$, we obtain a bounded metric space (X, d_2) associated with (X, d_1). Hence,

Proposition 1.1.4 *Every metric space* (X, d_1) *admits a bounded equivalent metric space* (X, d_2).

Definition 1.1.7 Let (X, d) be a metric space and $T : (X, d) \to (X, d)$ be a map. Then

(i) T is called a **Lipschitzian map** if there exists a constant λ, where $\lambda \geq 0$ such that $d(Tx, Ty) \leq \lambda d(x, y)$ for any $x, y \in X$. Furthermore, the number λ, where $\lambda \geq 0$ is called a **Lipschitzian constant**. The smallest constant λ will be denoted by $Lip(T)$.

(ii) T is called a **contraction map** if there exists a constant λ, where $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)$ for any $x, y \in X$. Furthermore, the number λ, where $0 \leq \lambda < 1$ is called a **contraction constant**.

(iii) a point $x \in X$ is said to be a **fixed point** of T if $Tx = x$.

It should be noted that from now on, both Lipschitzian maps and contraction maps considered in the entire thesis are self maps, that is for a map $T : (X, d) \to (X, d)$, we require that $TX \subseteq X$, unless stated otherwise.

Remark 1.1.3 Let $T : (X, d) \to (X, d)$ be a map between metric spaces. If T is a contraction map then it is a Lipschitzian map but the converse is not true.

Example 1.1.5 Let $X = \mathbb{R}$ be equipped with the usual metric d. Consider a map $T : (X, d) \to (X, d)$ defined by $Tx = 3x$ for all $x \in X$. Then T is a Lipschitzian map with a Lipschitzian constant $\lambda = 3$ but not a contraction.

Definition 1.1.8 Let (X, d_X) and (Y, d_Y) be metric spaces. A map $T : (X, d_X) \to (Y, d_Y)$ between metric spaces is **continuous** if a sequence $\{x_n\}$ d_X-converges to a point x in X implies that $\{T(x_n)\}$ d_Y-converges to a point $T(x)$ in Y.

Remark 1.1.4 It should be noted that a Lipschitzian map is continuous but a continuous map may neither be a contraction nor a Lipschitzian.

There are several results on fixed point theory in the literature; we recall only those that we shall use in the subsequent chapters.

Theorem 1.1.1 *Let (X, d) be a d-Cauchy complete metric space and $T : (X, d) \to (X, d)$ be a map. If T^n is a Lipschitzian map for all $n \geq 0$ and $\sum_{n=0}^{\infty} Lip(T^n) < \infty$, then T has a unique fixed point $x \in X$. Moreover, for any $x \in X$ the orbit $\{T^n x\}$ d-converges to x.*

Remark 1.1.5 Let (X, d) be a metric space and $T : (X, d) \to (X, d)$ be a map. Note the following:

(i) If T is a Lipschitzian map with a Lipschitzian constant $(\lambda \geq 0)$, then any iterate T^n is a Lipschitzian map with a Lipschitzian constant $(\lambda^n \geq 0)$.

(ii) A unique fixed point of T will also be a fixed point of any iterate T^n.

The converse of Remark 1.1.5 (ii) does not necessarily hold; see the example below.

Example 1.1.6 Let $X = \mathbb{R}$, endowed with the usual metric d and $T : (X, d) \to (X, d)$ be defined by $T(x) = 1 - x$ for all $x \in X$. T is a Lipschitzian map with the Lipschitzian constant $\lambda = 1$. T has a unique fixed point at $x_0 = \frac{1}{2}$. Furthermore, $T^2 x = x$ for all $x \in X$. So, $x_0 = \frac{1}{2}$ is a fixed point for T^2. Hence, $T^2 : (X, d) \to (X, d)$ does have a fixed point which is not unique. In this case $x_0^* = \frac{1}{4}$ is also a fixed point for $T^2 : (X, d) \to (X, d)$, but not a fixed point for $T : (X, d) \to (X, d)$.

We now present the well-known Banach's fixed point theorem which plays a fundamental role in many applications.

Theorem 1.1.2 *Let (X, d) be a d-Cauchy complete metric space and $T : (X, d) \to (X, d)$ be a map. If $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ and $0 \leq \lambda < 1$, then T has a unique fixed point.*

The d-Cauchy completeness of X in Theorem 1.1.2 cannot be removed. Indeed, a contraction on a metric space which is not d-Cauchy complete may fail to have a fixed point.

Example 1.1.7 Let $X = (1, \infty)$ and d be the usual metric on X. Then (X, d) is not a d-Cauchy complete metric space. Let $T : (X, d) \to (X, d)$ be a map defined by $T(x) = \frac{x+1}{2}$ for all $x \in X$. Then T is a contraction map without a fixed point.

1.2 Quasi metric spaces and their properties

We shall briefly present fundamental properties of quasi metric spaces. Details can be found in [13], [26] and [44].

Definition 1.2.1 Let X be a nonempty set. A map $q : X \times X \to [0, \infty)$ is a **quasi metric** on X if for all $x, y, z \in X$ the following conditions hold:

(i) $q(x, y) \geq 0$;

(ii) $x = y$ iff $q(x, y) = q(y, x) = 0$;

(iii) $q(x, z) \leq q(x, y) + q(y, z)$.

The pair (X, q) is called a **quasi metric space**.

Definition 1.2.2 Let (X, q) be a quasi metric space and $q^{-1} : X \times X \to [0, \infty)$ be defined by $q^{-1}(x, y) = q(y, x)$ for all $x, y \in X$. Then (X, q^{-1}) is also a quasi metric space. We call q^{-1} the **conjugate** of q on X. Now define $q^* : X \times X \to [0, \infty)$ by $q^*(x, y) = \max\{q(x, y), q(y, x)\}$. Then q^* is a **metric** on X, see, [44].

Example 1.2.1 [26, Example 1] Let $X = \mathbb{R}$ and $q : X \times X \to [0, \infty)$ be defined by $q(x, y) = \max\{x - y, 0\}$ for all $x, y \in X$. Then (X, q) is a quasi metric space. Define $q^{-1} : X \times X \to [0, \infty)$ by $q^{-1}(x, y) = \max\{y - x, 0\}$ for all $x, y \in X$. Then (X, q^{-1}) is also a quasi metric space. Furthermore, $q^* : X \times X \to [0, \infty)$ is given by $q^*(x, y) = |x - y|$ for all $x, y \in X$. Note that (X, q^*) is the **standard metric space**.

There are many notions related to convergence and completeness in quasi metric spaces [44]. In the thesis we shall focus on the following:

Definition 1.2.3 Let (X, q) be a quasi metric space. Then

(i) a sequence $\{x_n\}$ in (X, q) q-**converges** to a point $x \in X$ if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $q(x_n, x) < \epsilon$ for each $n \geq N$.

(ii) a sequence $\{x_n\}$ in (X, q) q^{-1}-**converges** to a point $x \in X$ if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $q^{-1}(x_n, x) < \epsilon$ for each $n \geq N$.

(iii) a sequence $\{x_n\}$ in (X, q) q^*-**converges** to a point $x \in X$ if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $q^*(x_n, x) < \epsilon$ for each $n \geq N$.

(iv) a sequence $\{x_n\}$ in (X, q) is q^*-**Cauchy** if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $q^*(x_n, x_m) < \epsilon$ for each $n, m \geq N$.

(v) a quasi metric space (X, q) is **Cauchy bicomplete** if every q^*-Cauchy sequence q^*-converges to a point $x \in X$.

Throughout the thesis we adhere to the following notation; given a quasi metric space (X, q) the function $q^* : X \times X \to [0, \infty)$ is defined by $q^*(x, y) = \max\{q(x, y), q(y, x)\}$ for all $x, y \in X$.

Remark 1.2.1 Note that every metric space is a quasi metric space but not every quasi metric space is a metric space. Therefore the class of quasi metric spaces is larger than the class of metric spaces.

Remark 1.2.2 Let (X, q) be a quasi metric space. Then

(i) if a sequence $\{x_n\}$ in (X, q) q-converges to a point $x \in X$ we shall write $\lim_n q(x_n, x) = 0$ or simply $x_n \xrightarrow{q} x$.

(ii) if a sequence $\{x_n\}$ in (X, q) q^{-1}-converges to a point $x \in X$ we shall write $\lim_n q^{-1}(x_n, x) = 0$ or simply $x_n \xrightarrow{q^{-1}} x$.

(iii) if a sequence $\{x_n\}$ in (X, q) is q-Cauchy we shall write $\lim_{n,m} q(x_n, x_m) = 0$.

(iv) $\lim_n q^*(x_n, x) = 0$ if and only if $\lim_n q(x_n, x) = 0$ and $\lim_n q(x, x_n) = 0$.

(v) $\lim_{n,m} q^*(x_n, x_m) = 0$ if and only if $\lim_{n,m} q(x_n, x_m) = 0$ and $\lim_{m,n} q(x_m, x_n) = 0$.

(vi) a quasi metric space (X, q) is Cauchy bicomplete if and only if the quasi metric space (X, q^{-1}) is Cauchy bicomplete.

(vii) a quasi metric space (X, q) is Cauchy bicomplete if and only if the metric space (X, q^*) is q^*-Cauchy complete.

An example of a Cauchy bicomplete quasi metric space is presented below.

Example 1.2.2 [26, Example 1] The quasi metric space (X, q) in Example 1.2.1 is Cauchy bicomplete.

Next is an example of a quasi metric space which is not Cauchy bicomplete.

Example 1.2.3 Let $X = (0, 1]$ and $q : X \times X \to [0, \infty)$ be defined by $q(x, y) = \max\{x - y, 0\}$ for all $x, y \in X$. The sequence $\{x_n = \frac{1}{n}, n \geq 1\}$ is q^*-Cauchy and does not converge to a point in X. Then (X, q) is not Cauchy bicomplete.

Definition 1.2.4 Let (X, q) be a quasi metric space. Define

$$B_q(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\}$$

for all $x \in X, \epsilon > 0$. The set $B_q(x, \epsilon)$ is called an **open ball** with the center x and radius ϵ. The family $\{B_q(x, \epsilon) : x \in X, \epsilon > 0\}$ is a **base** for the topology τ on X. We denote by $\tau_{(q)}$ the topology induced by q on X.

Given a quasi metric space (X, q) we see that X is endowed with two topologies namely, $\tau_{(q)}$ and $\tau_{(q^{-1})}$. Hence, $(X, \tau_{(q)}, \tau_{(q^{-1})})$ is a **bitopological space** [45].

Definition 1.2.5 A topological space (X, τ) is **quasi metrizable** if there exists a quasi metric q on X that induces the topology τ, such that $\tau = \tau_{(q)}$. In this case we say q is compatible with τ and that (X, τ) is a quasi metrizable topological space.

Remark 1.2.3 Observe that every quasi metric space (X, q) is a T_1-space [13].

1.3 Partial metric spaces and some Lipschitzian mappings

Some well-known results and notions on partial metric spaces due to Matthews are recalled; details can be found in [28] and [29]. We include some proofs for the sake of completeness.

Definition 1.3.1 Let X be a nonempty set. A map $p : X \times X \to [0, \infty)$ is a **partial metric** on X if for all $x, y, z \in X$, the following conditions hold:

(i) $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$;

(ii) $p(x, x) \leq p(x, y)$;

(iii) $p(x, y) = p(y, x)$;

(iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a **partial metric space**.

Note that if $p(x, y) = 0$, then from (i) and (ii) we obtain that $x = y$. But if $x = y$, then $p(x, y)$ is not necessarily zero, that is a partial metric space has nonzero self-distance property.

Example 1.3.1 [10, Example 1.3] Let $X = \mathbb{R}_0^+$. Define a map $p : X \times X \to [0, \infty)$ by $p(x, y) = \max\{x, y\}$, for all $x, y \in \mathbb{R}_0^+$. Then (X, p) is a partial metric space. We can see that $p(x, x) \neq 0$, for all $x \in \mathbb{R}_0^+$, hence, p is not a metric on X.

Example 1.3.2 [29, Example 3.2] Let $X = \{[a,b], a, b \in \mathbb{R}, a \leq b\}$ and define $p : X \times X \to [0, \infty)$ by $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$, for all $a,b,c,d \in X$. Then (X,p) is a partial metric space.

Remark 1.3.1 Let (X,p) be a partial metric space. Note that every metric space is a partial metric space but the converse is not true as reflected in Example 1.3.1. Therefore the class of partial metric spaces is larger than the class of metric spaces.

Definition 1.3.2 Let (X,p) be a partial metric space. Then

(i) a sequence $\{x_n\}$ in (X,p) p-**converges** to a point $x \in X$ if $\lim_n p(x_n, x) = p(x,x) = \lim_n p(x_n, x_n)$. This is equivalent to saying that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $p(x_n, x) < p(x,x) + \epsilon$ and $p(x_n, x_n) < p(x,x) + \epsilon$ for all $n \geq N$.

(ii) a sequence $\{x_n\}$ in (X,p) is p-**Cauchy** if the $\lim_{n,m} p(x_n, x_m)$ exists and is finite.

(iii) a partial metric space (X,p) is p-**Cauchy complete** if every p-Cauchy sequence $\{x_n\}$ p-converges to a point $x \in X$.

It is important to note that the limit need not be unique in partial metric spaces. Before providing an example of a p-Cauchy complete partial metric space, we present a definition and a remark.

Definition 1.3.3 [43] Let (X,p) be a partial metric space. Then

(i) a sequence $\{x_n\}$ in (X,p) is 0-**Cauchy** if $\lim_{n,m} p(x_n, x_m) = 0$.

(ii) a partial metric space (X,p) is 0-**Cauchy complete** if every 0-Cauchy sequence in X converges to a point $x \in X$ and $p(x,x) = 0$.

Remark 1.3.2 Let a partial metric space (X,p) be a metric space. Then

(i) a sequence $\{x_n\}$ is 0-Cauchy if and only if it is a p-Cauchy sequence in (X,p).

(ii) a metric space (X,p) is 0-Cauchy complete if and only if it is a p-Cauchy complete.

We present examples of p-Cauchy complete partial metric spaces.

Example 1.3.3 [10, Example 1.3] The partial metric space (X, p) in Example 1.3.1 is p-Cauchy complete.

Example 1.3.4 [10, Example 2.9] Let $X = [0, 1] \cup [2, 3]$ and define $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \begin{cases} \max\{x, y\} & \text{if} \quad (x, y) \cap [2, 3] \neq \emptyset \\ |x - y| & \text{if} \quad (x, y) \subset [0, 1], \end{cases}$$

for all $x, y \in X$. Then (X, p) is a p-Cauchy complete partial metric space.

The two examples provide a 0-Cauchy complete partial metric space that is not p-Cauchy complete. In particular, we show that Remark 1.3.2 does not necessarily hold in the partial metric space setting.

Example 1.3.5 [43, Page 3] Let $X = \mathbb{Q} \cap \mathbb{R}_0^+$, be endowed with a partial metric $p : X \times X \to [0, \infty)$ defined by $p(x, y) = \max\{x, y\}$, for all $x, y \in \mathbb{Q} \cap \mathbb{R}_0^+$. Then (X, p) is a 0-Cauchy complete partial metric space which is not a p-Cauchy complete partial metric space.

Example 1.3.6 [1, Example 2] Let $X = (1, \infty)$ be equipped with a partial metric $p : X \times X \to [0, \infty)$ defined by $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a 0-Cauchy complete but not a p-Cauchy complete partial metric space.

The importance of Definition 1.3.3 can be seen in [43]. It is shown in [43] that 0-Cauchy sequences cannot be replaced by p-Cauchy sequences.

Remark 1.3.3 Let (X, p) be a partial metric space. Then

(i) every 0-Cauchy sequence is a p-Cauchy sequence but not conversely [Example 1.3.5].

(ii) a nonzero constant sequence $\{x_n\}$ in (X, p) is a p-Cauchy sequence but not a 0-Cauchy sequence.

(iii) every p-Cauchy complete partial metric space is 0-Cauchy complete partial metric space but the converse does not necessarily hold [Example 1.3.6].

We are now ready to provide properties of sequences in partial metric spaces.

Proposition 1.3.1 [14] *Let (X,p) be a partial metric space, $\{x_n\}$ and $\{y_n\}$ be sequences in X. If $\{x_n\}$ p-converges to x and $\{y_n\}$ p-converges to y, then* $\lim_n p(x_n,y_n) = p(x,y)$.

Proof. Let (X,p) be a partial metric space, $\{x_n\}$ and $\{y_n\}$ be sequences in X. Suppose that $\{x_n\}$ p-converges to x and $\{y_n\}$ p-converges to y. Then $p(x,x) = \lim_n p(x,x_n) = \lim_n p(x_n,x_n)$ and $p(y,y) = \lim_n p(y,y_n) = \lim_n p(y_n,y_n)$. For $n \in \mathbb{N}$, we have $p(x_n,y_n) \le p(x_n,x) + p(x,y) + p(y,y_n) - p(x,x) - p(y,y)$ and

$$p(x,y) \le p(x,x_n) + p(x_n,y_n) + p(y_n,y) - p(y_n,y_n) - p(x_n,x_n).$$

Let $n \to \infty$. Then $\lim_n p(x_n,y_n) \le p(x,x)+p(x,y)+p(y,y)-p(x,x)-p(y,y) = p(x,y)$ and $p(x,y) \le p(x,x) + \lim_n p(x_n,y_n) + p(y,y) - p(x,x) - p(y,y) = \lim_n p(x_n,y_n)$. So, $\lim_n p(x_n,y_n) = p(x,y)$. □

Proposition 1.3.2 [14] *Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ p-converges to x, then $\{x_n\}$ is a p-Cauchy sequence.*

Proof. Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X such that $\{x_n\}$ p-converges to x. Then $p(x,x) = \lim_n p(x,x_n) = \lim_n p(x_n,x_n)$. For $n,m \in \mathbb{N}$, $p(x_n,x_m) \le p(x_n,x) + p(x,x_m) - p(x,x)$ and

$$p(x,x) \le p(x,x_n) + p(x_n,x_m) + p(x_m,x) - p(x_n,x_n) - p(x_m,x_m).$$

Let $n,m \to \infty$. Then $p(x,x) \le \lim_{n,m} p(x_n,x_m) \le p(x,x)$. So, $\lim_{n,m} p(x_n,x_m) = p(x,x)$, and then $\{x_n\}$ is a p-Cauchy sequence in (X,p). □

Proposition 1.3.3 *Let (X,p) be a partial metric space and $\{x_n\}$ be a 0-Cauchy sequence in X. If $\{x_n\}$ p-converges to x and p-converges to y, then $x = y$.*

Proof. Let $\{x_n\}$ be a 0-Cauchy sequence in a partial metric space (X,p). Suppose that $\{x_n\}$ p-converges to x and p-converges to y. Then $p(x,x) = p(y,y) = 0$. For each $\epsilon > 0$, find $N \in \mathbb{N}$ such that $p(x_n,x) < \epsilon + p(x,x), p(x_n,y) < \epsilon + p(y,y)$ and

$p(x_n, x_n) < \epsilon + p(x, x)$ for all $n \geq N$. Then

$$\begin{aligned} p(x,y) &\leq p(x,x_n) + p(x_n,y) - p(x_n,x_n). \\ &\leq 2\epsilon + 2p(x,x). \\ &= 2\epsilon. \end{aligned}$$

So, $p(x, y) - p(x, x) < 2\epsilon$ for all $n \geq N$ and since $p(x, y) \geq p(x, x)$ for all $x, y \in X$ and $\epsilon > 0$ is arbitrary, we conclude that $p(x, y) = p(x, x)$. Hence, $p(x, y) = p(x, x) = p(y, y)$. Therefore $x = y$. □

We present Definition 1.1.3 in the partial metric space setting.

Definition 1.3.4 Let (X, p) be a partial metric space. Define $B_p(x, \epsilon) = \{y \in X : p(x, y) < \epsilon + p(x, x), \epsilon > 0\}$ for all $x \in X$. The set $B_p(x, \epsilon)$ is called an **open ball** with center x and radius ϵ. The family $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$ is a **base** for the topology τ on X. We denote by $\tau_{(p)}$ the topology induced on X by p.

Remark 1.3.4 Observe that every partial metric space (X, p) is a T_0-space [28].

Lemma 1.3.1 [29] *Let (X, p) be a partial metric space. Then $q_p : X \times X \to [0, \infty)$ defined by*

$$q_p(x, y) = p(x, y) - p(x, x)$$

is a quasi metric on X for all $x, y \in X$. Furthermore, $\tau_{(p)} = \tau_{(q_p)}$.

Proof. We start by showing that $q_p(x, y) = p(x, y) - p(x, x)$ is a quasi metric for all $x, y \in X$. (i) $q_p(x, y) \geq 0$ for all $x, y \in X$, since $p(x, x) \leq p(x, y)$.

(ii) Suppose that $x = y$, then $q_p(x, x) = p(x, x) - p(x, x) = 0$. Conversely, suppose that $q_p(x, y) = 0$. Then $p(x, y) - p(x, x) = 0$, so, $p(x, y) = p(x, x)$ for all $x, y \in X$. We know that $q_p(x, y) = q_p(y, x)$ and $q_p(y, x) = p(y, x) - p(y, y)$ for all $x, y \in X$. Let $q_p(y, x) = 0, p(y, x) - p(y, y) = 0$. Then $p(y, x) = p(y, y)$ for all $x, y \in X$. Hence, $p(x, y) = p(x, x) = p(y, y)$. Therefore $x = y$.

(iii) We prove that $q_p(x,z) \leq q_p(x,y) + q_p(y,z)$.

$$\begin{aligned} q_p(x,z) &= p(x,z) - p(x,x) \\ &\leq p(x,y) + p(y,z) - p(x,x) - p(y,y) \\ &= p(x,y) - p(x,x) + p(y,z) - p(y,y) \\ &= q_p(x,y) + q_p(y,z). \end{aligned}$$

Therefore (X, q_p) is a quasi metric space. Next we show that $\tau_{(p)} = \tau_{(q_p)}$.

Suppose that $A \in \tau_{(q_p)}$. Then there exists $\epsilon > 0$ such that $B_{q_p}(x,\epsilon) \subset A$ for every $x \in A$. If $y \in B_{q_p}(x,\epsilon)$ and $q_p(x,y) < \epsilon$, it follows that $q_p(x,y) = p(x,y) - p(x,x) < \epsilon$. We know that $p(x,y) < \epsilon + p(x,x)$. Therefore $y \in B_p(x,\epsilon) \subset A$. Hence, $A \in \tau_{(p)}$. This implies that $\tau_{(q_p)} \subseteq \tau_{(p)}$.

Conversely, suppose that $A \in \tau_{(p)}$. Then there exists $\epsilon > 0$ such that $B_p(x,\epsilon) \subset A$ for every $x \in A$. If $y \in B_p(x,\epsilon)$ and $p(x,y) \leq \epsilon + p(x,x), p(x,y) - p(x,x) \leq \epsilon$. Then $q_p(x,y) = p(x,y) - p(x,x) \leq \epsilon$. Therefore $y \in B_{q_p}(x,\epsilon) \subset A$. Hence, $A \in \tau_{(q_p)}$. This implies that $\tau_{(p)} \subseteq \tau_{(q_p)}$. Therefore $\tau_{(p)} = \tau_{(q_p)}$. □

Remark 1.3.5 The reader should observe that from Definition 1.2.5, we see that a partial metric space (X, p), is quasi-metrizable.

Given a partial metric space (X, p), from now on, we shall denote a quasi metric space by (X, q_p) and a metric space obtained from (X, q_p) by (X, q_p^*), see, Definition 1.2.2.

Definition 1.3.5 [40] Let X be a nonempty set. A map $p : X \times X \to \mathbb{R}$ is a **dualistic partial metric** on X if for all $x, y, z \in X$, the following conditions hold:

(i) $x = y$ iff $p(x,x) = p(x,y) = p(y,y)$;

(ii) $p(x,x) \leq p(x,y)$;

(iii) $p(x,y) = p(y,x)$;

(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X, p) is called a **dualistic partial metric space**.

Theorem 1.3.1 [39] *A dualistic partial metric space* (X, p) *is* p*-Cauchy complete if and only if the metric space* (X, q_p^*) *is* q_p^**-Cauchy complete.*

Corollary 1.3.1 *A partial metric space* (X, p) *is* p*-Cauchy complete if and only if a metric space* (X, q_p^*) *is* q_p^**-Cauchy complete.*

Proposition 1.3.4 [16] *Let* (X, p) *be a partial metric space. Then* $d_p : X \times X \to [0, \infty)$ *defined by*

$$d_p(x, y) = \begin{cases} p(x, y) & \textit{whenever} \quad x \neq y \\ 0 & \textit{whenever} \quad x = y, \end{cases}$$

for all $x, y \in X$ *is a metric on* X*. Hence,* (X, d_p) *is a metric space.*

Proof. (i) Clearly $d_p(x, y) = 0$ if and only if $x = y$.

(ii) Since $p(x, y) = p(y, x)$ we get $d_p(x, y) = d_p(y, x)$ for all $x, y \in X$.

(iii) To show that $d_p(x, z) \leq d_p(x, y) + d_p(y, z)$ note that $d_p(x, z) \leq p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ and if $x \neq z$ and $x = y$, then

$$\begin{aligned} d_p(x, z) \leq p(x, z) &\leq p(x, y) + p(y, z) - p(y, y) \\ &= p(y, y) + p(y, z) - p(y, y) \\ &= p(y, z) \\ &= d_p(y, z). \end{aligned}$$

If $x \neq z$ and $y = z$, then

$$\begin{aligned} d_p(x, z) \leq p(x, z) &\leq p(x, y) + p(z, z) - p(z, z) \\ &= p(x, y) \\ &= d_p(x, y). \end{aligned}$$

If $x \neq y \neq z$, then $d_p(x, z) \leq d_p(x, y) + d_p(y, z)$. If $x = z$, then $d_p(x, z) = 0 \leq d_p(x, y) + d_p(y, z)$. Therefore (X, d_p) is a metric space. □

Theorem 1.3.2 [16] *Let (X,p) be a partial metric space. The metric space (X,d_p) is d_p-Cauchy complete if and only if the partial metric space (X,p) is 0-Cauchy complete.*

Proof. Suppose that (X,p) is 0-Cauchy complete, and $\{x_n\}$ be a d_p-Cauchy sequence in (X,d_p). Without loss of generality assume that $x_n \neq x_m$ for all $n \neq m$. Hence, $d_p(x_n,x_m) = p(x_n,x_m)$ for all $n,m \geq 1$ and $\lim_{n,m} d_p(x_n,x_m) = 0$. It follows that $\lim_{n,m} p(x_n,x_m) = 0$. Thus $\{x_n\}$ is a p-Cauchy sequence in (X,p). Since (X,p) is 0-Cauchy complete, $\lim_n p(x_n,x) = 0$ for some $x \in X$. Note that $x \neq x_n$ for all n. Therefore $\lim_n d_p(x_n,x) = 0$. So, (X,d_p) is a d_p-Cauchy complete metric space.

Conversely, suppose that (X,d_p) is d_p-Cauchy complete and $\{x_n\}$ be a 0-Cauchy sequence in (X,p). Without loss of generality assume that $x_n \neq x_m$ for all $n \neq m$. Then $p(x_n,x_m) = d_p(x_n,x_m)$ for all $n,m \geq 1$. So, $\lim_{n,m} d_p(x_n,x_m) = \lim_{n,m} p(x_n,x_m) = 0$. Hence, $\{x_n\}$ is a d_p-Cauchy sequence in (X,d_p). Since (X,d_p) is d_p-Cauchy complete, there exists $x \in X$ such that $\lim_n d_p(x_n,x) = 0$. Thus $\lim_n p(x_n,x) = 0$ and so, (X,p) is a 0-Cauchy complete partial metric space. □

Definition 1.3.6 Let (X,p) be a p-Cauchy complete partial metric space and $T : (X,p) \to (X,p)$ be a map. Then

(i) T is called a **Lipschitzian map** if there exists a constant λ, where $\lambda \geq 0$ such that $p(Tx,Ty) \leq \lambda p(x,y)$, for any $x,y \in X$. Furthermore, the constant λ, where $\lambda \geq 0$ is called a **Lipschitzian constant**. The smallest constant λ will be denoted by $Lip(T)$.

(ii) T is called a **contraction map** if there exists a constant λ, where $0 \leq \lambda < 1$ such that $p(Tx,Ty) \leq \lambda p(x,y)$, for any $x,y \in X$. Furthermore, the constant λ, where $0 \leq \lambda < 1$ is called the **contraction constant**.

(iii) a point x is said to be a **fixed point** of T if $Tx = x$.

Definition 1.3.7 Let (X,p_X) and (Y,p_Y) be partial metric spaces. A map $T : (X,p_X) \to (Y,p_Y)$ between partial metric spaces is **continuous** if a sequence $\{x_n\}$ p_X-converges to a point x in X implies that $\{T(x_n)\}$ p_Y-converges to a point $T(x)$

in Y.

Remark 1.3.6 Let $T : (X, p) \to (X, p)$ be a map between partial metric spaces.

(i) If T is a contraction map then it is a Lipschitzian map but the converse is not true.

(ii) Any contraction map and a Lipschitzian map is continuous, but a continuous map may neither be a contraction nor a Lipschitzian.

Theorem 1.3.3 [28] *Let (X, p) be a p-Cauchy complete partial metric space and $T : (X, p) \to (X, p)$ be a map. If $p(Tx, Ty) \leq \lambda p(x, y)$ for any $x, y \in X$ and $0 \leq \lambda < 1$, then T has a unique fixed point.*

Proof. Suppose that $T : X \to X$ is a contraction in a p-Cauchy complete partial metric space with partial metric $p : X \times X \to [0, \infty)$, and that $0 \leq \lambda < 1$ is such that for all $x, y \in X, p(T(y), T(x)) - p(T(x), T(x)) \leq \lambda(p(y, x) - p(x, x))$. Let $x \in X$ and $\{x_n\} \in X$ for all $n \in \mathbb{N}$ be such that $\{x_n\} = T^n(x)$ for all n.

We first show that $\{x_n\}$ is a p-Cauchy sequence. $T(x_{n+2}, x_{n+1}) - T(x_{n+1}, x_{n+1}) \leq \lambda(T(x_{n+1}, x_n) - T(x_n, x_n))$. Therefore for all $n \geq 0$. $T(x_{n+2}, x_{n+1}) - T(x_{n+1}, x_{n+1}) \leq \lambda^{n+1}(T(x_1, x_0) - T(x_0, x_0))$. For all $n, m \geq 0$.

$$T(x_{n+m+1}, x_n) - T(x_n, x_n) \leq T(x_{n+m+1}, x_{n+m}) - T(x_{n+m}, x_{n+m}) + T(x_{n+m}, x_n) - T(x_n, x_n).$$

$$\leq \lambda^{n+m}(T(x_1, x_0) - T(x_0, x_0) + T(x_{n+m}, x_n) - T(x_n, x_n)).$$

Therefore for all $n, m \geq 0$,

$$\begin{aligned} T(x_{n+m+1}, x_n) - T(x_n, x_n) &\leq (\lambda^{n+m} + ... + \lambda^n)(T(x_1, x_0) - T(x_0, x_0)) \\ &= \frac{\lambda^n(1 - \lambda^{m+1})}{1 - \lambda}(T(x_1, x_0) - T(x_0, x_0)) \\ &< \frac{\lambda^n}{(1 - \lambda)}(T(x_1, x_0) - T(x_0, x_0)). \end{aligned}$$

Thus, $\{x_n\}$ is a p-Cauchy sequence, since (X,p) is p-Cauchy complete and $\{x_n\}$ p-converges to $x^* \in X$.

We now show that x^* is a fixed point of T. Choose $\epsilon > 0$, then as $\{x_n\}$ p-converges to x^* we can find $m \geq 0$ such that, for all $n > m$. $p(x^*, x_n) - p(x_n, x_n) < \frac{\epsilon}{1+\lambda}$ and $p(x_n, x^*) - p(x^*, x^*) < \frac{\epsilon}{1+\lambda}$. Thus for all $n > m$.

$$\begin{aligned}
p(T(x^*), x^*) - p(x^*, x^*) &\leq p(T(x^*), x_{n+1}) - p(x_{n+1}, x_{n+1}) + p(x_{n+1}, x^*) - p(x^*, x^*) \\
&\leq \lambda(p(x^*, x_n) - p(x_n, x_n)) + p(x_{n+1}, x^*) - p(x^*, x^*) \\
&< \lambda(\frac{\epsilon}{1+\lambda}) + \frac{\epsilon}{1+\lambda} \\
&= \epsilon.
\end{aligned}$$

Thus, as ϵ is arbitrary, $p(T(x^*), x^*) = p(x^*, x^*)$ (1)

Similarly, for all $n > m$.

$$\begin{aligned}
p(T(x^*), x^*) - p(T(x^*), T(x^*)) &\leq p(T(x^*), x_{n+1}) - p(x_{n+1}, x_{n+1}) + p(x_{n+1}, x^*) \\
&\quad -p(T(x^*), T(x^*)) \\
&= (p(T(x^*), x_{n+1}) - p(T(x^*), T(x^*))) + p(x_{n+1}, x^*) \\
&\quad -p(x_{n+1}, x_{n+1}) \\
&\leq \lambda(p(x^*, x_n) - p(x^*, x^*)) + \frac{\epsilon}{1+\lambda} \\
&< \lambda(\frac{\epsilon}{1+\lambda}) + \frac{\epsilon}{1+\lambda} \\
&= \epsilon.
\end{aligned}$$

Thus, as ϵ is arbitrary, $p(T(x^*), x^*) = p(T(x^*), T(x^*))$. By property (i) of Definition 1.3.1 and (1) we have $x^* = T(x^*)$ and so, T has a fixed point. We now show that x^* is unique. Suppose that $y^* \in X$ and $y^* = T(y^*)$, then

$$\begin{aligned}
p(x^*, y^*) - p(y^*, y^*) &= p(T(x^*), T(y^*)) - p(T(y^*), T(y^*)) \\
&\leq \lambda p(x^*, y^*) - p(y^*, y^*).
\end{aligned}$$

Therefore $p(x^*, y^*) - p(y^*, y^*) = 0$ as $0 \leq \lambda < 1$. Similarly, $p(y^*, x^*) - p(x^*, x^*) = 0$. Therefore $x^* = y^*$. □

Theorem 1.3.4 [28] *Let (X,p) be a p-Cauchy complete partial metric space and $T:(X,p)\to(X,p)$ be a map. If T^n is a Lipschitzian map for all $n\geq 0$ and $\sum_{n=0}^{\infty} Lip(T^n)<\infty$, then T has a unique fixed point $x\in X$. Moreover, for any $x\in X$, the orbit $\{T^n x\}$ p-converges to x.*

1.4 Properties of TVS-cone metric spaces

In what follows we recall basic properties of TVS-cone metric spaces and refer the reader to [8], [12] and [19] for more details. In this section by (X,σ) we refer to (X,P,E,σ) where X is a nonempty set, E is a **normed topological vector space**, P is a normal cone in E with normal constant K and σ is a TVS-cone metric on X.

Definition 1.4.1 [15] A **topological vector space** (TVS) E is a vector space over a topological field $\mathbb{K}$ that is endowed with a topology such that the vector addition and scalar multiplication are continuous functions.

A subset P of a topological vector space (TVS) E is called a **cone** if

(i) P is closed, nonempty and $P\neq\{0\}$, here 0 is the zero vector in E;

(ii) $a,b\in\mathbb{R}, a,b\geq 0$ and $x,y\in P$, then $ax+by\in P$;

(iii) both $x\in P$ and $-x\in P$ then $x=0$.

Given a cone P in E a **partial ordering** $\preceq$ on E via P is defined by $x\preceq y$ if and only if $y-x\in P$ for $x,y\in E$. We write $x<y$ to indicate that $x\preceq y$ but $x\neq y$ while $x\prec\prec y$ will stand for $y-x\in int(P)$ where $int(P)$ denote the interior of P in the norm topology on E. The cone P in E is called **normal** if there exists a constant $K>0$ such that for all $a,b\in E, 0\preceq a\preceq b$ implies $||a||\leq K||b||$, where $||\cdot||:E\to[0,\infty)$ is a norm on E. We will always assume that a topological vector space E has a norm on it.

Example 1.4.1 Let $E=\mathbb{R}^2, X=\mathbb{R}$ and $P=\{(x,y):x\geq 0,y\geq 0\}$ be a subset of X. Equip $\mathbb{R}$ with the usual norm $||\cdot||$. Then P is a normal cone in E with constant

$K = 1$.

Definition 1.4.2 A TVS-**cone metric space** is an ordered pair (X, σ) where X is a nonempty set, E is a topological vector space, P be a normal cone in E and $\sigma : X \times X \to E$ is a map satisfying for all $x, y, z \in X$:

(i) $0 \preceq \sigma(x, y)$;

(ii) $\sigma(x, y) = 0$ iff $x = y$;

(iii) $\sigma(x, y) = \sigma(y, x)$;

(iv) $\sigma(x, z) \preceq \sigma(x, y) + \sigma(y, z)$.

Example 1.4.2 [19, Example 1] Let $E = \mathbb{R}^2$, where E is equipped with the usual norm $|| \cdot ||$, $P = \{(x, y) \in E, x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $\sigma : X \times X \to E$ be defined by $\sigma(x, y) = \{|y - x|, \alpha|y - x|\}$ for all $x, y \in X$, where $\alpha \geq 0$ is a constant. Then (X, σ) is a TVS-cone metric space.

Definition 1.4.3 Let (X, σ) be a TVS-cone metric space. Then

(i) a sequence $\{x_n\}$ in (X, σ) σ-**converges** to a point $x \in X$ if for each $c \in int(P)$ there exists an $N \in \mathbb{N}$ such that $\sigma(x_n, x) \prec\prec c$ for each $n \geq N$.

(ii) a sequence $\{x_n\}$ in (X, σ) is σ-**Cauchy** if there exists an $a \in P$ such that for every $\epsilon > 0$ there is a natural number N such that $m, n \geq N$, $||\sigma(x_n, x_m) - a|| < \epsilon$.

(iii) a TVS-cone metric space (X, σ) is σ-**Cauchy complete** if every σ-Cauchy sequence in X σ-converges to a point $x \in X$.

Remark 1.4.1 Let (X, σ) be a TVS-cone metric space and $\{x_n\}$ be a sequence in X. If a sequence $\{x_n\}$ is σ-convergent to a point $x \in X$ we shall also write $\lim_n \sigma(x_n, x) = 0$ or simply $x_n \xrightarrow{\sigma} x$.

We present examples of a σ-Cauchy complete TVS-cone metric space.

Example 1.4.3 [19, Example 1] The TVS-cone metric space (X, σ) in Example 1.4.2 is σ-Cauchy complete.

Example 1.4.4 [42, Example 9.2] Let $E = \mathbb{R}^2$, be equipped with the usual norm. $P = \{(x, y) \in E, x, y \geq 0\} \subset \mathbb{R}^2, X = P$ and $\sigma : X \times X \to E$ be defined by

$$\sigma(x,y) = \begin{cases} x+y & \text{if} \quad x \neq y \\ 0 \quad \text{if} & x = y \end{cases}$$

for all $x, y \in X$. Then (X, σ) is a σ-Cauchy complete TVS-cone metric space.

Definition 1.4.4 Let (X, σ) be a TVS-cone metric space. Define $B_\sigma(x, c) = \{y \in X : \sigma(x, y) \prec\prec c\}$ for all $x \in X, 0 \prec\prec c$. The set $B_\sigma(x, c)$ is called an **open ball** with center x and radius c. The family $\{B_\sigma(x, c) : x \in X, c \succ\succ 0\}$ is a **base** for the topology τ on X. We denote by $\tau_{(\sigma)}$ the topology induced by σ on X.

We now present some properties of sequences in a TVS-cone metric space.

Proposition 1.4.1 [19] *Let (X, σ) be a TVS-cone metric space, P be a normal cone in E with normal constant K, $\{x_n\}$ and $\{y_n\}$ be sequences in X. If $\lim_n \sigma(x_n, x) = 0$ and $\lim_n \sigma(y_n, y) = 0$, then $\lim_n \sigma(x_n, y_n) = \sigma(x, y)$.*

Proof. Let (X, σ) be a TVS-cone metric space, P be a normal cone in E with normal constant K, $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\lim_n \sigma(x_n, x) = 0$ and $\lim_n \sigma(y_n, y) = 0$. Then for every $\epsilon > 0$ choose $c \in E, c \succ\succ 0$ and $||c|| < \frac{\epsilon}{4K+2}$. Since $\lim_n \sigma(x_n, x) = 0$ and $\lim_n \sigma(y_n, y) = 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N, \sigma(x_n, x) \prec\prec c$ and $\sigma(y_n, y) \prec\prec c$. We have

$$\begin{aligned} \sigma(x_n, y_n) &\preceq \sigma(x_n, x) + \sigma(x, y) + \sigma(y, y_n) \\ &\preceq \sigma(x, y) + 2c. \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma(x, y) &\preceq \sigma(x, x_n) + \sigma(x_n, y_n) + \sigma(y_n, y) \\ &\preceq \sigma(x_n, y_n) + 2c. \end{aligned}$$

Hence, $0 \preceq \sigma(x, y) + 2c - \sigma(x_n, y_n) \preceq 4c$ and

$$\begin{aligned} ||\sigma(x_n, y_n) - \sigma(x, y)|| &\leq ||\sigma(x, y) + 2c - \sigma(x_n, y_n)|| + ||2c|| \\ &\leq (4K + 2)||c|| \\ &< \epsilon. \end{aligned}$$

Therefore $\lim_n \sigma(x_n, y_n) = \sigma(x, y)$. □

Proposition 1.4.2 [19] *Let (X, σ) be a TVS-cone metric space and $\{x_n\}$ be a sequence in X. If $\lim_n \sigma(x_n, x) = 0$, then $\{x_n\}$ is a σ-Cauchy sequence.*

Proof. Let (X, σ) be a TVS-cone metric space and $\lim_n \sigma(x_n, x) = 0$. Then for any $c \succ\succ 0, c \in E$ there is an $N \in \mathbb{N}$ such that for all $m, n \geq N, \sigma(x_n, x) \prec\prec \frac{c}{2}$ and $\sigma(x_m, x) \prec\prec \frac{c}{2}$. Hence,

$$\begin{aligned} \sigma(x_n, x_m) &\preceq \sigma(x_n, x) + \sigma(x, x_m) \\ &\prec\prec \frac{c}{2} + \frac{c}{2} \\ &\prec\prec c. \end{aligned}$$

Therefore $\{x_n\}$ is a σ-Cauchy sequence. □

Proposition 1.4.3 [19] *Let (X, σ) be a TVS-cone metric space, P be a normal cone in a normed topological space E with normal constant K and $\{x_n\}$ be a sequence in X. If $\lim_n \sigma(x_n, x) = 0$ and $\lim_n \sigma(x_n, y) = 0$, then $x = y$.*

Proof. Suppose that (X, σ) is a TVS-cone metric space, P be a normal cone in a normed topological space E with normal constant K, $\lim_n \sigma(x_n, x) = 0$ and $\lim_n \sigma(x_n, y) = 0$. Then for any $c \in E, c \succ\succ 0$, with $2K||c|| < \epsilon$, there is an $N \in \mathbb{N}$ such that for all $n \geq N, \sigma(x_n, x) \prec\prec c$ and $\sigma(x_n, y) \prec\prec c$. Hence,

$$\begin{aligned} \sigma(x, y) &\preceq \sigma(x_n, x) + \sigma(x_n, y) \\ &\preceq 2c. \end{aligned}$$

We have $||\sigma(x, y)|| \leq 2K||c||$. Since c is an arbitrary then $\sigma(x, y) = 0$. Therefore $x = y$. □

Lemma 1.4.1 [12] *Let (X, σ) be a TVS-cone metric space. Then $d : X \times X \to [0, \infty)$ defined by*

$$d(x, y) = \inf\{||u||, u \in P : \sigma(x, y) \preceq u, x, y \in X\},$$

is a metric on X.

Proof. (i) Clearly $d(x, y) \geq 0$ for all $x, y \in X$.

(ii) To prove that $d(x, y) = d(y, x)$ for all $x, y \in X$. We know that $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in X$. It follows that for all $x, y \in X$, $d(x, y) = d(y, x)$.

(iii) If $d(x, y) = \inf\{||u||, u \in P : \sigma(x, y) \preceq u, x, y \in X\} = 0$, then for each $n \in \mathbb{N}$ there exists $u_n \in P$, $\sigma(x, y) \preceq u_n$ such that $||u_n|| < \frac{1}{n}$. Since $\sigma(x, y) \preceq u_n$ and $u_n \to 0$ as $n \to \infty$, by Proposition 1.4.3 we have $0 \succeq \sigma(x, y)$ which implies that $\sigma(x, y) \in P \cap (-P)$. Hence, $\sigma(x, y) = 0$ and $x = y$.

(iv) Now we prove that $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. In fact, let

$$d(x, z) = \inf\{||u_1|| : \sigma(x, z) \preceq u_1, u_1 \in P\},$$

$$d(x, y) = \inf\{||u_2|| : \sigma(x, y) \preceq u_2, u_2 \in P\},$$

$$d(y, z) = \inf\{||u_3|| : \sigma(y, z) \preceq u_3, u_3 \in P\}.$$

Since $u_2, u_3 \in P$, we have $\sigma(x, y) \preceq u_2, \sigma(y, z) \preceq u_3$. Now

$$\begin{aligned} \sigma(x, z) &\preceq \sigma(x, y) + \sigma(y, z) \\ &\preceq u_2 + u_3. \end{aligned}$$

So, $\{u_2 + u_3 \in P : \sigma(x, y) \preceq u_2, \sigma(y, z) \preceq u_3\} \subset \{u_1 \in P : \sigma(x, z) \preceq u_1\}$ which implies that $\inf\{||u_1|| : \sigma(x, z) \preceq u_1\} \leq \inf\{||u_2 + u_3|| : \sigma(x, y) \preceq u_2, \sigma(y, z) \preceq u_3\}$. Note that $\inf\{||u_2 + u_3|| : \sigma(x, y) \preceq u_2, \sigma(y, z) \preceq u_3\} \leq \inf\{||u_2|| + ||u_3|| : \sigma(x, y) \preceq u_2, \sigma(y, z) \preceq u_3\} = \inf\{||u_2|| : \sigma(x, y) \preceq u_2, u_2 \in P\} + \inf\{||u_3|| : \sigma(y, z) \preceq u_3, u_3 \in P\}$. Thus, $\inf\{||u_1|| : \sigma(x, z) \preceq u_1, u_1 \in P\} \leq \inf\{||u_2|| : \sigma(x, y) \preceq u_2, u_2 \in P\} + \inf\{||u_3|| : \sigma(y, z) \preceq u_3, u_3 \in P\}$. That is

$$d(x, z) \leq d(x, y + d(y, z).$$

By $(i), (ii), (iii)$ and (iv), d is a metric on X. □

Next, the notion of equivalent TVS-cone metrics (Definition 1.1.5) is presented.

Definition 1.4.5 Let $E, \bar{E}$ be normed topological vector spaces, $\sigma_1 : X \times X \to E$ and $\sigma_2 : X \times X \to \bar{E}$ be two TVS-cone metrics on a nonempty set X. Then σ_1 and σ_2 are said to be **equivalent** if a sequence $\{x_n\}$ σ_1-converges to x if and only if $\{x_n\}$ σ_2-converges to x for a sequence $\{x_n\}$ in X and $x \in X$. If σ_1 is equivalent to σ_2 we will write $\sigma_1 \simeq \sigma_2$.

Remark 1.4.2 Let X be a nonempty set, σ_1, σ_2 and σ_3 be TVS-cone metrics on X. If $\sigma_1 \simeq \sigma_2$ and $\sigma_2 \simeq \sigma_3$, then $\sigma_1 \simeq \sigma_3$.

Theorem 1.4.1 [8] *Let (X, σ) be a TVS-cone metric space, P be a normal cone with normal constant K and (X, d) be a metric space, where $d : X \times X \to [0, \infty)$ is defined by*

$$d(x, y) = \inf\{||u||, u \in P : \sigma(x, y) \preceq u, x, y \in X\}$$

on X. Then for every TVS-cone metric $\sigma : X \times X \to E$ there exists a metric $d : X \times X \to [0, \infty)$ such that $d \simeq \sigma$ on X.

Proof. By Lemma 1.4.1, d is a metric on X. Suppose that $\{x_n\}$ d-converges to a point x in (X, d). Then $\lim_n d(x_n, x) = 0$. For each $c \succ\succ 0$ there exists $\epsilon > 0$ such that $K||c|| < \epsilon$, where K is a normal constant. Since $d(x_n, x) = \inf\{||u_{nm}|| : \sigma(x_n, x) \preceq u_{nm}\}$ for all $n, m \in N$, then there exists u_{nm} such that

$$||u_{nm}|| < d(x_n, x) + \frac{1}{m}, \quad \sigma(x_n, x) \preceq u_{nm}.$$

Let $v_n = u_{nn}$. Then $||v_n|| < d(x_n, x) + \frac{1}{n}, (n \to \infty)$ and $\sigma(x_n, x) \preceq v_n$. Now if x_n d-converges to a point x in (X, d) then $\lim_n d(x_n, x) = 0$ and v_n d-converges to 0. Therefore for all $c \succ\succ 0$ there exists an $N \in \mathbb{N}$ such that $v_n \prec\prec c$ for all $n \geq N$. This implies that $\sigma(x_n, x) \prec\prec c$ for all $n \geq N$. That is, $\lim_n \sigma(x_n, x) = 0$.

Conversely, for every real $\epsilon > 0$ choose $c \in E$ with $c \succ\succ 0$ and $||c|| < \epsilon$. Then there exists an $N \in \mathbb{N}$ such that $\sigma(x_n, x) \prec\prec c$ for all $n \geq N$. This means that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x) \leq ||c|| < \epsilon$ for all $n \geq N$. Therefore $\lim_n d(x_n, x) = 0$ as $n \to \infty$, so, $\{x_n\}$ d-converges to x in (X, d). $\square$

An example to support Theorem 1.4.1 is presented below. For more examples the reader should consult [8].

Example 1.4.5 [8, Example 2.3] Let $E = \mathbb{R}^2$, be equipped with the usual norm, $X = \mathbb{R}, P = \{(x, y) \in E, x, y \geq 0\} \subset \mathbb{R}^2, a \in P, a \neq 0$ with $||a|| = 1$ and $\sigma : X \times X \to E$ be defined by

$$\sigma(x, y) = \begin{cases} a & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

for all $x, y \in X$. Then σ is a TVS-cone metric on X and its equivalent metric $d : X \times X \to [0, \infty)$ is defined by

$$d(x, y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

for all $x, y \in X$. Note that d is a discrete metric on X. Hence, $\tau_{(\sigma)}$ is a discrete topology on X.

We discuss the relationship between σ-Cauchy sequence in a TVS-cone metric space (X, σ) and d-Cauchy sequence in a metric space (X, d).

Lemma 1.4.2 [12] *Let (X, σ) be a TVS-cone metric space and (X, d) be a metric space where $d : X \times X \to [0, \infty)$ is defined by*

$$d(x, y) = \inf\{||u||, u \in P : \sigma(x, y) \preceq u, x, y \in X\}$$

on X. Then $\{x_n\}$ is d-Cauchy in (X, d) if and only if it is σ-Cauchy sequence in (X, σ).

Proof. Suppose that $\{x_n\}$ is a d-Cauchy sequence in a metric space (X, d). For any $c \succ\succ 0$ there exists $\epsilon > 0$ such that $c + \mathcal{B}(0, \epsilon) \subset P$. Note that $\{x_n\}$ is a d-Cauchy sequence, there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \leq \frac{\epsilon}{4}$ for $m, n > N$, that is

$$d(x_n, x_m) = \inf\{||u|| : \sigma(x_n, x_m) \preceq u, u \in P\} \leq \frac{\epsilon}{4}$$

for all $n, m \geq N$. Hence, there exists $v \in P, ||v|| \leq \frac{\epsilon}{2}$ such that $\sigma(x_n, x_m) \preceq v$. Note that $c - v \in int(P)$, thus $\sigma(x_n, x_m) \preceq u \prec\prec c$ for $m, n > N$, which implies $\{x_n\}$ is a σ-Cauchy sequence of (X, σ).

Conversely, suppose that $\{x_n\}$ is a σ-Cauchy sequence in a TVS-cone metric space (X, σ). Given $c \succ\succ 0$ and a positive number $\delta > 0$, there is $K \geq 1$ such that $||\frac{c}{K}|| < \delta$. Noting that $\frac{c}{K} \succ\succ 0$ and $\{x_n\}$ be a σ-Cauchy sequence in (X, σ), there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $\sigma(x_n, x_m) \prec\prec \frac{c}{K}$. Hence,

$$d(x_n, x_m) = \inf\{||u|| : \sigma(x_n, x_m) \preceq u\} \leq \frac{||c||}{K} < \delta$$

for all $m, n \geq N$, which implies that $\{x_n\}$ is a d-Cauchy sequence in (X, d). □

By virtue of Lemma 1.4.2 and Theorem 1.4.1 we immediately have the next theorem.

Theorem 1.4.2 [12] *Let (X, σ) be a TVS-cone metric space and (X, d) be a metric space, where $d : X \times X \to [0, \infty)$ is defined by*

$$d(x, y) = \inf\{||u||, u \in P : \sigma(x, y) \preceq u, x, y \in X\}$$

on X. Then (X, d) is d-Cauchy complete if and only if (X, σ) is σ-Cauchy complete.

Definition 1.4.6 Let (X, σ) be a TVS-cone metric space and $T : (X, \sigma) \to (X, \sigma)$ be a map. Then

(i) T is called a **Lipschitzian map** if there exists a constant λ, where $\lambda \geq 0$ such that $\sigma(Tx, Ty) \preceq \lambda\sigma(x, y)$ for any $x, y \in X$. Moreover, the constant λ, where $\lambda \geq 0$ is called a **Lipschitzian constant**. The smallest constant λ will be denoted by $Lip(T)$.

(ii) T is called a **contraction map** if there exists a constant λ, where $0 \leq \lambda < 1$ such that $\sigma(Tx, Ty) \preceq \lambda\sigma(x, y)$ for any $x, y \in X$. Moreover, the constant λ, where $0 \leq \lambda < 1$ is called a **contraction constant**.

(iii) a point $x \in X$ is said to be a **fixed point** of T if $Tx = x$.

Definition 1.4.7 Let (X, σ_X) and (Y, σ_Y) be TVS-cone metric spaces. A map $T : (X, \sigma_X) \to (Y, \sigma_Y)$ between TVS-cone metric spaces is **continuous** if a sequence $\{x_n\}$ σ_X-converges to a point x in X implies that $\{T(x_n)\}$ σ_Y-converges to a point $T(x)$ in Y.

Theorem 1.4.3 [19] *Let (X, σ) be a σ-Cauchy complete TVS-cone metric space, P be a normal cone in E with normal constant K. Suppose that a map $T : (X, \sigma) \to (X, \sigma)$ satisfies the contraction condition $\sigma(Tx, Ty) \preceq \lambda\sigma(x, y)$ for some $\lambda \in (0, 1)$ and $x, y \in X$. Then T has a unique fixed point in X. And for any $x \in X$ iterative sequence* $\lim_n \sigma(T^n x, x) = 0$.

Proof. Let (X, σ) be a σ-Cauchy complete TVS-cone metric space, P be a normal cone in E with normal constant K. Choose $x_0 \in X$, and $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$ Then we have

$$\begin{aligned}
\sigma(x_{n+1}, x_n) &= \sigma(Tx_n, Tx_{n-1}) \\
&\preceq \lambda\sigma(x_n, x_{n-1}) \\
&\preceq \lambda^2\sigma(x_{n-1}, x_{n-2}) \\
&\preceq ... \\
&\preceq \lambda^n\sigma(x_1, x_0).
\end{aligned}$$

So, for $n > m$,

$$\begin{aligned}
\sigma(x_n, x_m) &\preceq \sigma(x_n, x_{n-1}) + \sigma(x_{n-1}, x_{n-2}) + ... + \sigma(x_{m+1}, x_m) \\
&\preceq (\lambda^{n-1} + \lambda^{n-2} + ... + \lambda^m)\sigma(x_1, x_0) \\
&\preceq \frac{\lambda^m}{1-\lambda}\sigma(x_1, x_0).
\end{aligned}$$

Then we get $||\sigma(x_n, x_m)|| \leq \frac{\lambda^m}{1-\lambda}K||\sigma(x_1, x_0)||$. This implies that $\lim_{n,m} \sigma(x_n, x_m) = 0$. Hence, $\{x_n\}$ is a σ-Cauchy sequence. Since X is σ-Cauchy complete there exists $x^* \in X$ such that $\lim_n \sigma(x_n, x^*) = 0$. Hence,

$$\begin{aligned}
\sigma(Tx^*, x^*) &\preceq \sigma(Tx_n, Tx^*) + \sigma(Tx_n, x^*) \\
&\preceq \lambda\sigma(x_n, x^*) + \sigma(x_{n+1}, x^*).
\end{aligned}$$

So,

$$||\sigma(Tx^*, x^*)|| \leq K(\lambda||\sigma(x_n, x^*)|| + ||\sigma(x_{n+1}, x^*)||)$$

and

$$K(\lambda||\sigma(x_n, x^*)|| + ||\sigma(x_{n+1}, x^*)||) \to 0$$

as $n \to \infty$. Hence, $||\sigma(Tx^*, x^*)|| = 0$. This implies that $Tx^* = x^*$. So, x^* is a fixed point of T. Now if y^* is another fixed point of T, then $\sigma(x^*, y^*) = \sigma(Tx^*, Ty^*) \preceq \lambda\sigma(x^*, y^*)$. Hence, $||\sigma(x^*, y^*)|| = 0$ implies that $\sigma(x^*, y^*) = 0$ for all $x^*, y^* \in X$. By condition (ii) of Definition 1.4.2 it follows that $x^* = y^*$. Therefore the fixed point of T is unique. □

Example 1.4.6 [19, Page 8] Let $E = \mathbb{R}^2$, be equipped with the usual norm, $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ and $X = \{(x, 0) \in \mathbb{R}^2 | 0 \leq x \leq 1\} \cup \{(0, x) \in \mathbb{R}^2 | 0 \leq x \leq 1\}$. Define $\sigma : X \times X \to E$ by

$$\sigma((x, 0), (y, 0)) = (\frac{4}{3}|x - y|, |x - y|),$$

$$\sigma((0, x), (0, y)) = (|x - y|, \frac{2}{3}|x - y|),$$

$$\sigma((x, 0), (0, y)) = \sigma((0, y), (x, 0)) = (\frac{4}{3}x + y, x + \frac{2}{3}y)$$

for all $x, y \in X$. Then (X, σ) is a σ-Cauchy complete TVS-cone metric space. Define a mapping $T : (X, \sigma) \to (X, \sigma)$ by $T(x, 0) = (0, x)$ and $T(0, x) = (\frac{1}{2}x, 0)$. Then T satisfies the contractive condition $\sigma((Tx_1, Tx_2), (Ty_1, Ty_2)) \preceq \lambda\sigma((x_1, x_2), (y_1, y_2))$, for all $(x_1, x_2), (y_1, y_2) \in X$, with $\lambda = \frac{3}{4} \in [0, 1)$. Furthermore, T has a unique fixed point $(0, 0) \in X$ and it is not contractive.

Theorem 1.4.4 [12] *Let (X, σ) be a σ-Cauchy complete TVS-cone metric space, $T : (X, \sigma) \to (X, \sigma)$ be a map and $d : X \times X \to [0, \infty)$ be a metric defined by*

$$d(x, y) = \inf\{||u|| : \sigma(x, y) \preceq u, u \in P, x, y \in X\}$$

on X. If $\sigma(Tx, Ty) \preceq \lambda\sigma(x, y)$ for some $\lambda \in (0, 1)$ and $x, y \in X$, then $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ and $\lambda \in (0, 1)$.

Proof. Suppose that $T : (X, \sigma) \to (X, \sigma)$ is a contraction. Let $v \in P, \sigma(x, y) \preceq v$ and $||u_n|| \leq v$ for $x, y \in X, n \geq 1$. Then

$$\sigma(Tx, Ty) \preceq \lambda\sigma(x, y) \preceq \lambda v$$

for all $x, y \in X, \lambda \in (0,1)$. Since $\{\lambda v : \sigma(x,y) \preceq v, v \in P\} \subset \{u : \sigma(Tx,Ty) \preceq u, u \in P\}$. Then

$$\begin{aligned} d(Tx,Ty) &= \inf\{||u|| : \sigma(Tx,Ty) \preceq u, u \in P\} \\ &\leq \inf\{||\lambda v|| : \sigma(x,y) \preceq v, v \in P\} \\ &= \lambda \inf\{||v|| : \sigma(x,y) \preceq v, v \in P\} \\ &= \lambda d(x,y). \end{aligned}$$

Therefore $d(Tx,Ty) \leq \lambda d(x,y)$ and $T : (X,d) \to (X,d)$ is a contraction. □

Theorem 1.4.5 [12] *Let (X,σ) be a σ-Cauchy complete TVS-cone metric space, $T : (X,\sigma) \to (X,\sigma)$ be a map and $d : X \times X \to [0,\infty)$ be a metric defined by*

$$d(x,y) = \inf\{||u|| : \sigma(x,y) \preceq u, u \in P, x, y \in X\}$$

on X. If $T : (X,\sigma) \to (X,\sigma)$ has a fixed point, then $T : (X,d) \to (X,d)$ has a fixed point.

Proof. Let (X,σ) be a σ-Cauchy complete TVS-cone metric space, and for $T : (X,\sigma) \to (X,\sigma)$ we have $\sigma(Tx,Ty) \preceq \lambda\sigma(x,y)$, where $\lambda \in (0,1)$. Then $T : (X,\sigma) \to (X,\sigma)$ has a fixed point. Let x be a fixed point for T. Then it follows that $T : (X,d) \to (X,d)$ has a fixed point. □

1.5 TVS-partial cone metric spaces and their properties

Some well know fundamental results and notions of TVS-partial cone metric spaces are presented in this section. We refer the reader to [11] and [48]. Note that by (X,σ_p) we refer to (X,P,E,σ_p) where X is a nonempty set, E is a **normed topological vector space**, P is a normal cone in E with normal constant K and σ_p is a TVS-partial cone metric on X. We begin with:

Definition 1.5.1 [48] Let X be a nonempty set. A map $\sigma_p : X \times X \to E$ is called TVS-**partial cone metric** on X if for all $x, y, z \in X$, the following conditions hold:

(i) $x = y$ iff $\sigma_p(x,y) = \sigma_p(x,x) = \sigma_p(y,y)$;

(ii) $\sigma_p(x,y) = \sigma_p(y,x)$;

(iii) $\sigma_p(x,x) \preceq \sigma_p(x,y)$;

(iv) $\sigma_p(x,z) \preceq \sigma_p(x,y) + \sigma_p(y,z) - \sigma_p(y,y)$.

The pair (X, σ_p) is called a TVS-**partial cone metric space**.

A TVS-cone metric space is necessarily a TVS-partial cone metric space, but the converse does not necessarily hold, see the upcoming example:

Example 1.5.1 [20, Example 2] Let $E = C^1_{\mathbb{R}}[0,1]$ with the norm $||u|| = ||u_\infty|| + ||u'||_\infty$ and $X = P = \{u \in E : u(t) \geq 0, t \in [0,1]\}$. Define a map $\sigma_p : X \times X \to E$ by

$$\sigma_p(x,y) = \begin{cases} x & \text{if} \quad x = y \\ x + y & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Then σ_p is a TVS-partial cone metric, but not a TVS-cone metric since $\sigma_p(x,x) \neq 0$, for some $x \in X$ with $x \neq 0$.

Remark 1.5.1 Let (X, σ_p) be a TVS-partial cone metric space. Then the class of TVS-partial cone metric spaces is larger than the class of TVS-cone metric spaces.

Definition 1.5.2 Let (X, σ_p) be a TVS-partial cone metric space. Then

(i) a sequence $\{x_n\}$ in (X, σ_p) σ_p-**converges** to a point $x \in X$ if for each $c \in int(P)$ there exists an $N \in \mathbb{N}$ such that $\sigma_p(x_n, x) \prec\prec \sigma_p(x,x) + c$ and $\sigma_p(x_n, x_n) \prec\prec \sigma_p(x,x) + c$ for each $n \geq N$.

(ii) a sequence $\{x_n\}$ in (X, σ_p) is σ_p-**Cauchy** if there exists an $a \in P$ such that for every $\epsilon > 0$ there is a natural number N such that $m, n \geq N$, $||\sigma_p(x_n, x_m) - a|| \leq \epsilon$.

(iii) a TVS-partial cone metric space (X, σ_p) is σ_p-**Cauchy complete** if every σ_p-Cauchy sequence in X σ_p-converges to a point $x \in X$.

Remark 1.5.2 In [48] a sequence $\{x_n\}$ in (X, σ_p) is defined to σ_p-converge to $x \in X$ if $\lim_n \sigma_p(x_n, x) = \sigma_p(x, x)$. It is also mentioned that $\sigma_p(x_n, x) \to \sigma_p(x, x)$ implies that $\sigma_p(x_n, x_n) \to \sigma_p(x, x)$. This is not true in general see the following example.

Example 1.5.2 Let $X = [0, \infty), E = X, P = \{x \in X : x \geq 0\}$, and define $\sigma_p : X \times X \to P$ by $\sigma_p(x, y) = \max\{x, y\}$ for all $x, y \in X$. A sequence $\{x_n = \frac{1}{n+1}, n \geq 1\}$ in (X, σ_p) σ_p-converges to $x = 2$, per definition in [48](Remark 1.5.2). Clearly $\lim_n \sigma_p(x_n, x) = \sigma_p(x, x)$ and $\lim_n \sigma_p(x_n, x_n) \neq \sigma_p(x, x)$. So, $\{x_n\}$ does not σ_p-converge to $x = 2$, using Definition 1.5.2. Note that according to Definition 1.5.2 (i) we have $\lim_n \sigma_p(x_n, x_n) = \sigma_p(x, x) = \lim_n \sigma_p(x_n, x)$, where $x = 0$. Therefore $\{x_n\}$ σ_p-converges to 0.

Remark 1.5.3 The reader should note that with Definition 1.5.2 (i) a sequence $\{x_n\}$ in a TVS-partial cone metric space (X, σ_p) can σ_p-converge to at most one point.

Next is an example of a σ_p-Cauchy complete TVS-partial cone metric space.

Example 1.5.3 [48, Example 1] Let $E = \mathbb{R}^2$, be equipped with the usual norm, $X = \mathbb{R}_0^+, P = \{(x, y) \in E, x, y \geq 0\}$ and $\sigma_p : X \times X \to E$ be defined by $\sigma_p(x, y) = (\max\{x, y\}, \alpha \max\{x, y\})$ for all $x, y \in X$, where $\alpha \geq 0$ is a constant. Then (X, σ_p) is a σ_p-Cauchy complete TVS-partial cone metric space.

Definition 1.5.3 Let (X, σ_p) be a TVS-partial cone metric space. Then

(i) a sequence $\{x_n\}$ in (X, σ_p) is 0-**Cauchy** if $\lim_{n,m} \sigma_p(x_n, x_m) = 0$.

(ii) a TVS-partial cone metric space (X, σ_p) is 0-**Cauchy complete** if every 0-Cauchy sequence in X σ_p-converges to a point $x \in X$ and $\sigma_p(x, x) = 0$.

Remark 1.5.4 Let (X, σ_p) be a TVS-partial cone metric space. If (X, σ_p) is σ_p-Cauchy complete then this implies that it is 0-Cauchy complete, but not conversely.

Next is an example of 0-Cauchy complete TVS-partial cone metric space which is not a σ_p-Cauchy complete TVS-partial cone metric space.

Example 1.5.4 Let $E = \mathbb{R}^2$, be equipped with the usual norm, $X = \mathbb{Q} \cap [0, \infty)$, $P = \{(x, y) \in E, x, y \geq 0\}$. Then P is normal with normal constant $K = 1$. Let $\sigma_p : X \times X \to E$ be defined by $\sigma_p(x, y) = (\max\{x, y\}, \alpha \max\{x, y\})$ for all $x, y \in X$ where $\alpha \geq 0$. Then (X, σ_p) is 0-Cauchy complete TVS-partial cone metric space which is not σ_p-Cauchy complete.

Remark 1.5.5 It worth noting that if a TVS-partial cone metric space (X, σ_p) is a TVS-cone metric space, then a sequence $\{x_n\}$ is 0-Cauchy if and only if it is σ_p-Cauchy in (X, σ_p). Furthermore, a TVS-cone metric space (X, σ_p) is 0-Cauchy complete if and only if it is σ-Cauchy complete.

The proof of the following two results use σ_p-convergence as in [11] (Remark 1.5.2). Note that since σ_p-convergence as in Definition 1.5.2 implies σ_p-convergence as in [11] (Remark 1.5.2). Therefore the following two results also hold when Definition 1.5.2 (i) is used.

Proposition 1.5.1 [11] *Let (X, σ_p) be a TVS-partial cone metric space, where P is a normal cone in E with normal constant K, $\{x_n\}$ and $\{y_n\}$ be sequences in X. If $\{x_n\}$ σ_p-converges to x and $\{y_n\}$ σ_p-converges to y, then* $\lim_n \sigma_p(x_n, y_n) = \sigma_p(x, y)$.

Proof. Let (X, σ_p) be a TVS-partial cone metric space, P be a normal cone in E with normal constant K, $\{x_n\}$ and $\{y_n\}$ be sequences in X. Suppose that $\{x_n\}$ σ_p-converges to x and $\{y_n\}$ σ_p-converges to y. Then for every $\epsilon > 0$ choose $c \in E, c \succ\succ 0$ and $||c|| < \frac{\epsilon}{4K+2}$. Since $\{x_n\}$ σ_p-converges to x and $\{y_n\}$ σ_p-converges to y then there exists an $N \in \mathbb{N}$ such that for all $n \geq N, \sigma_p(x_n, x) \prec\prec c + \sigma_p(x, x)$ and $\sigma_p(y_n, y) \prec\prec c + \sigma_p(y, y)$. We have for all $n \geq N$,

$$\begin{aligned}\sigma_p(x_n, y_n) &\preceq \sigma_p(x_n, x) + \sigma_p(x, y) + \sigma_p(y, y_n) - \sigma_p(x, x) - \sigma_p(y, y) \\ &\preceq \sigma_p(x, y) + 2c.\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma_p(x,y) &\preceq \sigma_p(x,x_n)+\sigma_p(x_n,y_n)+\sigma_p(y_n,y)-\sigma_p(x_n,x_n)-\sigma_p(y_n,y_n)\\ &\preceq \sigma_p(x_n,y_n)+2c.\end{aligned}$$

Hence, for all $n>N$, $0\preceq\sigma_p(x,y)+2c-\sigma_p(x_n,y_n)\preceq 4c$ and so, for $n>N$,

$$\begin{aligned}||\sigma_p(x_n,y_n)-\sigma_p(x,y)|| &\leq ||\sigma_p(x,y)+2c-\sigma_p(x_n,y_n)||+||2c||\\ &\leq (4K+2)||c||\\ &< \epsilon.\end{aligned}$$

Therefore $\lim_n \sigma_p(x_n,y_n)=\sigma_p(x,y)$. □

Proposition 1.5.2 [11] *Let (X,σ_p) be a TVS-partial cone metric space, P be a normal cone in E with normal constant K and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ σ_p-converges to x, then $\{x_n\}$ is a σ_p-Cauchy sequence.*

Proof. Let (X,σ_p) be a TVS-partial cone metric space, P be a normal cone in E with normal constant K and $\{x_n\}$ be a sequence in X that σ_p-converges to a point $x\in X$. Then for any $\epsilon>0$ choose $c\succ\succ 0, c\in E$ with $K||c||<\epsilon$, there is an $N\in\mathbb{N}$ such that for all $m,n>N, \sigma_p(x_n,x)\prec\prec\frac{c}{2}+\sigma_p(x,x)$ and $\sigma_p(x_m,x)\prec\prec\frac{c}{2}+\sigma_p(x,x)$. Then for any $n,m>N$

$$\begin{aligned}\sigma_p(x_n,x_m) &\preceq \sigma_p(x_n,x)+\sigma_p(x,x_m)-\sigma_p(x,x)\\ &\preceq \frac{c}{2}+\frac{c}{2}+\sigma_p(x,x)\\ &\prec c+\sigma_p(x,x).\end{aligned}$$

So, $||\sigma_p(x_n,x_m)-\sigma_p(x,x)||\leq K||c||<\epsilon$. Therefore $\{x_n\}$ is a σ_p-Cauchy sequence. The proof is complete. □

Proposition 1.5.3 *Let (X,σ_p) be a TVS-partial cone metric space, P be a normal cone in E with normal constant K, $\{x_n\}$ be a 0-Cauchy sequence in X and $x,y\in X$. If $\{x_n\}$ σ_p-converges to x and σ_p-converges to y, then $x=y$.*

Proof. Suppose that (X, σ_p) is a TVS-partial cone metric space, P be a normal cone in E with normal constant K and $\{x_n\}$ be a 0-Cauchy sequence in X. Suppose that $\{x_n\}$ σ_p-converges to x and to y. Then $\sigma_p(x,x) = \sigma_p(y,y) = 0$. For any $c \in E, c \succ\succ 0$, such that, $2K||c|| < \epsilon$ there is an $N \in \mathbb{N}$ such that for all $n \geq N, \sigma_p(x_n, x) \prec\prec c + \sigma_p(x,x), \sigma_p(x_n, y) \prec\prec c + \sigma_p(y,y)$ and $\sigma_p(x_n, x_n) \prec\prec c + \sigma_p(x,x)$. So, for $n \geq N$

$$
\begin{aligned}
\sigma_p(x,y) &\preceq \sigma_p(x,x_n) + \sigma_p(x_n,y) - \sigma_p(x_n,x_n) \\
&\preceq c + \sigma_p(x,x) + c + \sigma_p(x,x) - \sigma_p(x_n,x_n) \\
&\preceq 2c + 2\sigma_p(x,x) \\
&= 2c \\
||\sigma_p(x,y)|| &< 2K||c|| \\
&< \epsilon.
\end{aligned}
$$

So, $\sigma_p(x,y) = 0$. This implies that $\sigma_p(x,y) = \sigma_p(x,x) = \sigma_p(y,y)$. Therefore $x = y$. □

Definition 1.5.4 Let (X, σ_p) be a TVS-partial cone metric space. Define $B_{\sigma_p}(x,c) = \{y \in X : c \prec\prec \sigma_p(x,x) - \sigma_p(x,y)\}$ for all $x \in X, c \succ\succ 0$. The set $B_{\sigma_p}(x,c)$ is called an **open ball** with the center x and the radius c. The family, $\{B_{\sigma_p}(x,c) : x \in X, c \succ\succ 0\}$ is a **base** for the topology $\tau_{(\sigma_p)}$ on X. We denote by $\tau_{(\sigma_p)}$ the topology on X induced by σ_p.

Definition 1.5.5 Let (X, σ_p) be a TVS-partial cone metric space and $T : (X, \sigma_p) \to (X, \sigma_p)$ be a map. Then

(i) T is called a **Lipschitzian map** if there exists a constant λ, where $\lambda \geq 0$ such that $\sigma_p(Tx, Ty) \preceq \lambda\sigma_p(x,y)$, for any $x, y \in X$. Furthermore, the constant λ, where $\lambda \geq 0$ is called a **Lipschitzian constant**. The smallest constant λ will be denoted by $Lip(T)$.

(ii) T is called a **contraction map** if there exists a constant λ, where $0 \leq \lambda < 1$ such that $\sigma_p(Tx, Ty) \preceq \lambda\sigma_p(x,y)$, for any $x, y \in X$. Furthermore, the constant λ, where $0 \leq \lambda < 1$ is called a **contraction constant**.

(iii) a point $x \in X$ is said to be a **fixed point** of T if $Tx = x$.

Definition 1.5.6 Let (X, σ_{p_X}) and (Y, σ_{p_Y}) be TVS-partial cone metric spaces. A map $T : (X, \sigma_{p_X}) \to (Y, \sigma_{p_Y})$ between TVS-partial cone metric spaces is **continuous** if a sequence $\{x_n\}$ σ_{p_X}-converges to a point x in X implies that $\{T(x_n)\}$ σ_{p_Y}-converges to a point $T(x)$ in Y.

Theorem 1.5.1 [48] *Let (X, σ_p) be a σ_p-Cauchy complete TVS-partial cone metric space, P be a normal cone in E with normal constant K. Suppose that the map $T : (X, \sigma_p) \to (X, \sigma_p)$ satisfies the contractive condition $\sigma_p(Tx, Ty) \preceq \lambda\sigma_p(x, y)$ for some $\lambda \in (0, 1)$ and $x, y \in X$. Then T has a unique fixed point in X. And for any $x \in X$ iterative sequence $\{T^n x\}$ σ_p-converges to the fixed point.*

Proof. Let (X, σ_p) be a σ_p-Cauchy complete TVS-partial cone metric space, P be a normal cone in E with normal constant K. Choose $x_0 \in X$, and $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$ Then we have for $m > n$,

$$\begin{aligned}
\sigma_p(x_m, x_n) &\preceq \sigma_p(x_m, x_{m-1}) + \sigma_p(x_{m-1}, x_{m-2}) + ... + \sigma_p(x_{n+2}, x_{n+1}) + \\
&\quad \sigma_p(x_{n+1}, x_n) - \sum_{i=1}^{m-n-1} \sigma_p(x_{m-i}, x_{m-i}) \\
&\preceq (\lambda^{m-1} + \lambda^{m-2} + ... + \lambda^n)\sigma_p(x_1, x_0) \\
&= \lambda^n \frac{1 - \lambda^{m-n}}{1 - \lambda}\sigma_p(x_1, x_0) \\
&\preceq \lambda^n \frac{1}{1 - \lambda}\sigma_p(x_1, x_0)
\end{aligned}$$

we get $||\sigma_p(x_m, x_n)|| \leq \lambda^n K \frac{1}{1-\lambda} ||\sigma_p(x_1, x_0)||$. Thus $\{T^n x\}$ is a σ_p-Cauchy sequence in (X, σ_p) such that $\lim_{n,m} \sigma_p(T^n x_0, T^m x_0) = 0$. As (X, σ_p) is σ_p-Cauchy complete there exists $x_0 \in X$ such that $\{T^n x_0\}$ σ_p-converges to x^* and

$$\sigma_p(x^*, x^*) = \lim_n \sigma_p(x_n, x^*) = \lim_n \sigma_p(x_n, x_n) = 0.$$

Now for any $n \in N$, we have that $\sigma_p(Tx^*, x^*) \preceq \sigma_p(Tx^*, T^{n+1}x_0) + \sigma_p(T^{n+1}x_0, x^*) -$

$\sigma_p(T^{n+1}x_0, T^{n+1}x_0) \preceq \lambda\sigma_p(x^*, T^n x_0) + \sigma_p(T^{n+1}x_0, x^*)$.

$$||\sigma_p(Tx^*, x^*)|| \leq K\lambda||\sigma_p(x^*, T^n x_0)|| + ||\sigma_p(T^{n+1}x_0, x^*)|| \to 0.$$

Hence, $\sigma_p(Tx^*, x^*) = 0$. But since $\sigma_p(Tx^*, Tx^*) \preceq \lambda\sigma_p(x^*, x^*) = 0$. We have that $\sigma_p(Tx^*, Tx^*) = \sigma_p(Tx^*, x^*) = \sigma_p(x^*, x^*) = 0$ which implies that $Tx^* = x^*$. Now if y^* is another fixed point of T, then $\sigma_p(x^*, y^*) = \sigma_p(Tx^*, Ty^*) \preceq \lambda\sigma_p(x^*, y^*)$. Since $\lambda < 1$ we have $\sigma_p(x^*, y^*) = \sigma_p(x^*, x^*) = \sigma_p(y^*, y^*)$. Hence, $x^* = y^*$, thus the fixed point of T is unique. □

Chapter 2

Relationship between TVS-partial cone metric spaces, dislocated metric spaces and metric spaces

In the literature, it has been established that TVS-cone metric spaces and metric spaces are equivalent see, [8] and [12]. Also, it is shown that every partial metric space gives rise to a metric space [16]. In this chapter we discuss the relationship between TVS-partial cone metric spaces, dislocated metric spaces and metric spaces.

In particular, we show that a TVS-partial cone metric space does not gives rise to a partial metric space, unlike in the case where a TVS-cone metric space give rise to an equivalent metric space as seen in Chapter 1 and [8], [12]. In fact, TVS-partial cone metric space gives rise to a dislocated metric space but the two are not equivalent.

The chapter shall unfold as follows: Section 2.1 present some relationship on TVS-partial cone metric spaces, dislocated metric spaces and partial metric spaces. In Section 2.2 we show that every TVS-partial cone metric space gives rise to a TVS-quasi cone metric space. The relations between TVS-partial cone metric spaces and TVS-cone metric spaces are discussed in Section 2.3. Fixed point results on TVS-partial cone metric spaces will be discussed in Chapter 3.

It is important to note that in this chapter by (X, σ_p) we refer to (X, E, P, σ_p) where X is a nonempty set, E is a **normed topological vector space**, P is a normal cone in E with normal constant K and $\sigma_p : X \times X \to E$ may be a TVS-partial cone metric or a dislocated cone metric on X, or any well known mapping. Each context will be made explicit in order to avoid a possible confusion.

2.1 More properties of TVS-partial cone metric spaces

Definition 2.1.1 [25] Let X be a nonempty set. A map $p : X \times X \to [0, \infty)$ is a **dislocated metric** on X if for all $x, y, z \in X$ the following conditions hold:

(i) $p(x, y) = 0 \Rightarrow x = y$;

(ii) $p(x, y) = p(y, x)$;

(iii) $p(x, z) \leq p(x, y) + p(y, z)$.

The pair (X, p) is then called a **dislocated metric space**.

Note that every partial metric space is a dislocated metric space but the converse is not true.

Example 2.1.1 [2, Example 1] Let $X = \{0, 1\}$ and $p : X \times X \to [0, \infty)$ be defined by

$$p(x, y) = \begin{cases} 2 & \text{if} \quad x = y = 0 \\ 1 & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Then (X, p) is a dislocated metric space, but not a partial metric space, since $p(0, 0) \not\leq p(0, 1)$.

Definition 2.1.2 Let (X, p) be a dislocated metric space. Then

(i) a sequence $\{x_n\}$ in (X, p) is p-**convergence** to a point $x \in X$ if $\lim_n p(x_n, x) = p(x, x)$. This is equivalent to saying that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $p(x_n, x) < p(x, x) + \epsilon$.

(ii) a sequence $\{x_n\}$ in (X, p) is p-**Cauchy** if the $\lim_{n,m} \sigma_p(x_n, x_m)$ exists and is finite.

(iii) a dislocated metric space (X, p) is p-**Cauchy complete** if every p-Cauchy sequence p-convergence to a point $x \in X$.

(iv) a sequence $\{x_n\}$ is 0-**Cauchy** if the $\lim_{n,m} p(x_n, x_m) = 0$.

(v) a dislocated metric space (X, p) is 0-**Cauchy complete** if every 0-Cauchy sequence p-convergence to a point $x \in X$ and $p(x, x) = 0$.

The reader should note that convergence of dislocated metric space in this thesis is due to Amini-Harandi in [2].

Definition 2.1.3 Let X be a nonempty set and E be a normed topological vector space with a normal cone P. A map $\sigma_p : X \times X \to E$ is called a **dislocated cone metric** such that for all $x, y, z \in X$:

(i) $\sigma_p(x, y) = 0 \Rightarrow x = y$;

(ii) $\sigma_p(x, y) = \sigma_p(y, x)$;

(iii) $\sigma_p(x, z) \preceq \sigma_p(x, y) + \sigma_p(y, z)$.

The pair (X, σ_p) is called a **dislocated cone metric space**.

We now show that every dislocated cone metric σ_p on X gives rise to a dislocated metric p on X.

Theorem 2.1.1 *Let (X, σ_p) be a dislocated cone metric space. Then $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = \inf\{||u|| : \sigma_p(x, y) \preceq u, u \in P, x, y \in X\}$$

is a dislocated metric on X.

Proof. (i) Suppose that $p(x, y) = 0$, that is, $\inf\{||u|| : \sigma_p(x, y) \preceq u, u \in P, x, y \in X\} = 0$, then for an arbitrary $n \in N$, there exists $u_n \in P, u_n \succeq \sigma_p(x, y)$ such that $||u_n|| < \frac{1}{n}$. Since $u_n \succeq \sigma_p(x, y)$ and $u_n \to 0$ as $n \to \infty$, we have $\sigma_p(x, y) \preceq 0$ which implies that $\sigma_p(x,y) \in P \cap (-P)$. Thus $\sigma_p(x,y) = 0$. This means $\sigma_p(x,x) = \sigma_p(x,y) = \sigma_p(y,y)$. Therefore $x = y$.

(ii) To prove that $p(x,y) = p(y,x)$ for all $x, y \in X$. We note that $\sigma_p(x,y) = \sigma_p(y,x)$ for all $x, y \in X$. It follows that for all $x, y \in X$, $p(x,y) = p(y,x)$.

(iii) Now we prove that $p(x,z) \leq p(x,y) + p(y,z)$ for all $x, y, z \in X$. First, we note that $\sigma_p(x,z) \preceq \sigma_p(x,y) + \sigma_p(y,z) - \sigma_p(y,y) \preceq \sigma_p(x,y) + \sigma_p(y,z)$. We follow the same process as in the proof of Lemma 1.4.1 (iv). Let

$$p(x,z) = \inf\{||u_1|| : \sigma_p(x,z) \preceq u_1, u_1 \in P\},$$

$$p(x,y) = \inf\{||u_2|| : \sigma_p(x,y) \preceq u_2, u_2 \in P\},$$

$$p(y,z) = \inf\{||u_3|| : \sigma_p(y,z) \preceq u_3, u_3 \in P\}.$$

Since $u_2, u_3 \in P, \sigma_p(x,y) \preceq u_2, \sigma_p(y,z) \preceq u_3$.

Then

$$\begin{aligned} \sigma_p(x,z) &\preceq \sigma_p(x,y) + \sigma_p(y,z) \\ &\preceq u_2 + u_3. \end{aligned}$$

So, $\{u_2 + u_3 \in P : \sigma_p(x,y) \preceq u_2, \sigma_p(y,z) \preceq u_3\} \subset \{u_1 \in P : \sigma_p(x,z) \preceq u_1\}$ which implies that $\inf\{||u_1|| : \sigma_p(x,z) \preceq u_1\} \leq \inf\{||u_2 + u_3|| : \sigma_p(x,y) \preceq u_2, \sigma_p(y,z) \preceq u_3\}$. Note that:

$\inf\{||u_2 + u_3|| : \sigma_p(x,y) \preceq u_2, \sigma_p(y,z) \preceq u_3\} \leq \inf\{||u_2|| + ||u_3|| : \sigma_p(x,y) \preceq u_2, \sigma_p(y,z) \preceq u_3\} = \inf\{||u_2|| : \sigma_p(x,y) \preceq u_2, u_2 \in P\} + \inf\{||u_3|| : \sigma_p(y,z) \preceq u_3, u_3 \in P\}$. Thus, $\inf\{||u_1|| : \sigma_p(x,z) \preceq u_1, u_1 \in P\} \leq \inf\{||u_2|| : \sigma_p(x,y) \preceq u_2, u_2 \in P\} + \inf\{||u_3|| : \sigma_p(y,z) \preceq u_3, u_3 \in P\}$. That is

$$p(x,z) \leq p(x,y) + p(y,z).$$

By $(i), (ii), (iii)$ and (iv) we conclude that p is a dislocated metric on X. □

Note that every metric space (X,d) can be regarded as a TVS-cone metric space.

Example 2.1.2 Let $E = C^1_{\mathbb{R}}[0,1]$ with $||u|| = ||u||_\infty + ||u'||_\infty$ and $X = P = \{u \in X : u(t) \geq 0$ for all $t \in [0,1]\}$. Define $\sigma_p : X \times X \to E$ by $\sigma_p(x,y) = x$ if $x = y$ and $\sigma_p(x,y) = x+y$, otherwise. It is easy to check that σ_p is a TVS-partial cone metric on X. Define $p : X \times X \to [0,\infty)$ by $p(x,y) = \inf\{||u|| : \sigma_p(x,y) \preceq u, u \in P, x, y \in X\}$, then $p(x,y) = ||\sigma_p(x,y)||$, for all $x, y \in X$. Note that for $x, y \in P$ such that

$$\text{supp} \quad x \quad \cap \quad \text{supp} \quad y = \emptyset, ||x|| = ||y||$$

we have $||x+y|| = ||x|| = ||y||$ which implies that $p(x,y) = p(x,x) = p(y,y)$ but, $x \neq y$. Hence, (X,p) is not a partial metric space but a dislocated metric space.

Example 2.1.3 Consider the TVS-partial cone metric space as in Example 2.1.2. For $p(x,y) = \inf\{||u|| : \sigma_p(x,y) \preceq u, u \in P, x, y \in X\}$, we get $p(x,y) = ||\sigma_p(x,y)||$. Let $\{x_n\}$ be a normalized sequence in P and $\{x\}$ be a normalized vector in P such that

$$\text{supp} \quad \{x_n\} \quad \cap \quad \text{supp} \quad \{x\} = \emptyset.$$

We have $x_n \neq x$. So, $\sigma_p(x_n, x) = x_n + x$ and $\sigma_p(x,x) = x$. Now $||\sigma_p(x_n,x)|| = 1$ and $||\sigma_p(x,x)|| = 1$. So, $p(x_n,x) \to p(x,x)$ but $\sigma_p(x_n,x)$ does not converge to $\sigma_p(x,x)$.

Proposition 2.1.1 *Let (X,σ_p) be a TVS-partial cone metric space and (X,p) be a dislocated metric space. Then $\{x_n\}$ is a 0-Cauchy sequence in (X,σ_p) implies that it is a 0-Cauchy sequence in (X,p).*

Proof. Let $\{x_n\}$ be a 0-Cauchy sequence in (X,σ_p). For each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for each $0 \prec\prec c$, we have $\sigma_p(x_n,x_m) \prec\prec \epsilon\frac{c}{||c||}$ for $m, n \geq N$. Thus $p(x_n,x_m) = \inf\{||u|| : \sigma_p(x_n,x_m) \preceq u, u \in P\} < \epsilon||\frac{c}{||c||}|| = \epsilon$. This shows that $\{x_n\}$ is a 0-Cauchy sequence in (X,p). □

2.2 TVS-partial cone metric spaces and TVS-quasi cone metric spaces

We begin with a definition.

Definition 2.2.1 [46] Let X be a nonempty set and P be a normal cone in E. Suppose that the mapping $\sigma_q : X \times X \to E$ satisfies the following conditions for all $x, y, z \in X$;

(i) $0 \preceq \sigma_q(x, y)$;

(ii) $x = y$ iff $\sigma_q(x, y) = 0 = \sigma_q(y, x)$;

(iii) $\sigma_q(x, z) \preceq \sigma_q(x, y) + \sigma_q(y, z)$.

Then σ_q is called the TVS-**quasi cone metric** on X, and (X, σ_q) is called TVS-**quasi cone metric space**.

Note that by (X, σ_q) we refer to (X, E, P, σ_q), where X is a nonempty set, E is a normed topological vector space with a norm, P is a normal cone in E with normal constant K and σ_q is a TVS-quasi cone metric on X.

Let (X, σ_q) be a TVS-quasi cone metric space. Define $\sigma_q^* : X \times X \to E$ by $\sigma_q^*(x, y) = \sigma_q(x, y) + \sigma_q^{-1}(x, y), x, y \in X$. Then (X, σ_q^*) is a TVS-**cone metric space**.

Definition 2.2.2 Let (X, σ_q) be a TVS-quasi cone metric space. Then

(i) a sequence $\{x_n\}$ in (X, σ_q) σ_q-**converges** to a point x if for $0 \prec\prec c$, there exists $N \in \mathbb{N}$ such that $\sigma_q(x_n, x) \prec\prec c$ for all $n \geq N$.

(ii) a sequence $\{x_n\}$ in (X, σ_q) σ_q^{-1}-**converges** to a point x if for $0 \prec\prec c$, there exists $N \in \mathbb{N}$ such that $\sigma_q^{-1}(x_n, x) \prec\prec c$ for all $n \geq N$.

(iii) a sequence $\{x_n\}$ in (X, σ_q) σ_q^*-**converges** to a point x if for $0 \prec\prec c$, there exists $N \in \mathbb{N}$ such that $\sigma_q^*(x_n, x) \prec\prec c$ for all $n \geq N$.

(iv) a sequence $\{x_n\}$ in (X, σ_q) is σ_q^*-**Cauchy** if there exists an $a \in P$ such that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $||\sigma_q^*(x_n, x_m) - a|| < \epsilon$ for each $n, m \geq N$.

(v) a TVS-quasi cone metric space (X, σ_q) is **Cauchy bicomplete** if every σ_q^*-Cauchy sequence σ_q^*-converges to a point $x \in X$.

Remark 2.2.1 Let (X, σ_q) be a TVS-quasi cone metric space. Then

(i) if a sequence $\{x_n\}$ σ_q-converges to a point x we shall write $\lim_n \sigma_q(x_n, x) = 0$ or simply $x_n \xrightarrow{\sigma_q} x$.

(ii) if a sequence $\{x_n\}$ σ_q^{-1}-converges to a point x we shall write $\lim_n \sigma_q^{-1}(x_n, x) = 0$ or simply $x_n \xrightarrow{\sigma_q^{-1}} x$.

(iii) $\lim_n \sigma_q^*(x_n, x) = 0$ if and only if $\lim_n \sigma_q(x_n, x) = 0$ and $\lim_n \sigma_q^{-1}(x_n, x) = 0$.

Definition 2.2.3 Let (X, σ_q) be a TVS-quasi cone metric space. Define $B_{\sigma_q}(x, c) = \{y \in X : c \prec\prec \sigma_q(x, x) - \sigma_q(x, y)\}$ for all $x \in X, c \succ\succ 0$. The set $B_{\sigma_q}(x, c)$ is called an **open ball** with the center x and the radius c. The family, $\{B_{\sigma_q}(x, c) : x \in X, c \succ\succ 0\}$ is a **base** for the topology $\tau_{(\sigma_q)}$ on X. We denote by $\tau_{(\sigma_q)}$ the topology on X induced by σ_q.

We now relate TVS-partial cone metric space and TVS-quasi cone metric space.

Lemma 2.2.1 *Let (X, σ_p) be a TVS-partial cone metric space. Then $\sigma_{q_p} : X \times X \to [0, \infty)$ defined by*

$$\sigma_{q_p}(x, y) = \sigma_p(x, y) - \sigma_p(x, x)$$

is a TVS-quasi cone metric for all $x, y \in X$. Furthermore, $\tau_{(\sigma_p)} = \tau_{(\sigma_{q_p})}$.

Proof. (i) $\sigma_{q_p}(x, y) \succeq 0$ for all $x, y \in X$, since $\sigma_p(x, x) \preceq \sigma_p(x, y)$.

(ii) Suppose that $x = y$. Then $\sigma_{q_p}(x, y) = \sigma_{q_p}(x, x) = \sigma_p(x, x) - \sigma_p(x, x) = 0$. Conversely, suppose that $\sigma_{q_p}(x, y) = 0$ and $\sigma_{q_p}(y, x) = 0$. Then $\sigma_p(x, y) - \sigma_p(x, x) = 0 \Rightarrow \sigma_p(x, y) = \sigma_p(x, x)$ and $\sigma_p(y, x) - \sigma_p(y, y) = 0 \Rightarrow \sigma_p(y, x) = \sigma_p(y, y)$. Therefore $\sigma_p(x, y) = \sigma_p(y, y) = \sigma_p(x, x)$. Thus $x = y$ by (i) of Definition 1.5.1.

(iii) We prove that $\sigma_{q_p}(x, z) \preceq \sigma_{q_p}(x, y) + \sigma_{q_p}(y, z)$.

$$\begin{aligned}
\sigma_{q_p}(x, z) &= \sigma_p(x, z) - \sigma_p(x, x) \\
&\prec \sigma_p(x, y) + \sigma_p(y, z) - \sigma_p(x, x) - \sigma_p(y, y) \\
&= \sigma_p(x, y) - \sigma_p(x, x) + \sigma_p(y, z) - \sigma_p(y, y) \\
&= \sigma_{q_p}(x, y) + \sigma_{q_p}(y, z).
\end{aligned}$$

Therefore (X, σ_{q_p}) is a TVS-quasi cone metric space.

We show that $\tau_{(\sigma_p)} = \tau_{(\sigma_{q_p})}$.

Suppose that $A \in \tau_{(\sigma_{q_p})}$. Then there exists $0 \prec\prec c$ such that $B_{\sigma_{q_p}}(x,c) \subset A$ for all $x \in A$. If $y \in B_{\sigma_{q_p}}(x,c)$, then $\sigma_{q_p}(x,y) \prec c$, it follows that $\sigma_{q_p}(x,y) = \sigma_p(x,y) - \sigma_p(x,x) \prec c$. We know that $0 \prec c - \sigma_p(x,y) + \sigma_p(x,x)$. Therefore $y \in B_{\sigma_p}(x,c)$. Hence, $A \in \tau_{(\sigma_p)}$. This implies that $\tau_{(\sigma_{q_p})} \subseteq \tau_{(\sigma_p)}$.

Conversely, suppose that $A \in \tau_{(\sigma_p)}$. Then there exists $0 \prec\prec c$ such that $B_{\sigma_p}(x,c) \subset A$ for all $x \in A$. If $y \in B_{\sigma_p}(x,c)$, then $0 \preceq c - \sigma_p(x,y) + \sigma_p(x,x)$. So, $\sigma_{q_p}(x,y) = \sigma_p(x,y) - \sigma_p(x,x) \prec c$. Therefore $y \in B_{\sigma_{q_p}}(x,y) \subset A$. Hence, $A \in \tau_{(\sigma_{q_p})}$. This implies that $\tau_{(\sigma_p)} \subseteq \tau_{(\sigma_{q_p})}$. In conclusion $\tau_{(\sigma_p)} = \tau_{(\sigma_{q_p})}$. □

Note that if σ_{q_p} is a TVS-quasi cone metric on X, then $\sigma_{q_p^*} : X \times X \to [0, \infty)$ defined by $\sigma_{q_p^*}(x,y) = \sigma_{q_p}(x,y) + \sigma_{q_p}(y,x)$, for all $x, y \in X$, is a TVS-**cone metric** on X.

In the sequel consider (X, σ_p) to be a TVS-partial cone metric space and (X, σ_{q_p}) be a TVS-quasi cone metric space, where $\sigma_{q_p} : X \times X \to [0, \infty)$ is defined by $\sigma_{q_p}(x,y) = \sigma_p(x,y) - \sigma_p(x,x)$ for all $x, y \in X$ [Lemma 2.2.1].

Theorem 2.2.1 *A TVS-partial cone metric space (X, σ_p) is σ_p-Cauchy complete if and only if a TVS-cone metric space $(X, \sigma_{q_p^*})$ is $\sigma_{q_p^*}$-Cauchy complete.*

Proof. Suppose that $(X, \sigma_{q_p^*})$ is $\sigma_{q_p^*}$-Cauchy complete. Let $\{x_n\}$ be a σ_p-Cauchy sequence in (X, σ_p), then $\{x_n\}$ is a $\sigma_{q_p^*}$-Cauchy sequence in $(X, \sigma_{q_p^*})$. Hence, there exists $x \in X$, such that $x_n \xrightarrow{\tau(\sigma_{q_p^*})} x$. Thus $x_n \xrightarrow{\tau(q_p)} x$, so, $\sigma_{q_p}(x_n, x) = \sigma_p(x_n, x) - \sigma_p(x,x) \to 0$, and $\sigma_{q_p}(x, x_n) = \sigma_p(x, x_n) - \sigma_p(x_n, x_n) \to 0$. For $0 \prec\prec c, N \in \mathbb{N}, \sigma_{q_p}(x_n, x) \prec c \Rightarrow \sigma_p(x_n, x) - \sigma_p(x,x) \prec c, n \geq N$. This implies that $\sigma_p(x_n, x) \prec c + \sigma_p(x,x)$ and $\sigma_p(x_n, x_n) \prec c + \sigma_p(x,x)$. Thus $\lim_n \sigma_p(x_n, x) = \sigma_p(x,x) = \lim_n \sigma_p(x_n, x_n)$.

Conversely, suppose that (X, σ_p) is σ_p-Cauchy complete. Let $\{x_n\}$ be a $\sigma_{q_p^*}$-Cauchy sequence in $(X, \sigma_{q_p^*})$. So, $\{x_n\}$ is a σ_p-Cauchy sequence in (X, σ_p). Find a point $x \in X$, such that $x_n \xrightarrow{\tau(\sigma_p)} x$. That is, $\lim_n \sigma_p(x_n, x) = \sigma_p(x,x) = \lim_n \sigma_p(x_n, x_n)$. We shall show that there exists $N \in \mathbb{N}$, and $0 \prec\prec c$ such that $\sigma_{q_p^*}(x_n, x) \prec\prec c$, for all $n > N$ and $||c|| < \epsilon$. Note that if $x_n \xrightarrow{\tau(\sigma_p)} x$, then there exists $N \in \mathbb{N}$, such that $0 \prec c - \sigma_p(x_n, x) + \sigma_p(x,x)$ for all $n \geq N$.

Now

$$\begin{aligned}\sigma_{q_p}(x_n, x) &= \sigma_p(x_n, x) - \sigma_p(x_n, x_n)\\ &\preceq c + \sigma_p(x, x) - \sigma_p(x_n, x_n)\\ &\preceq 2c, \quad n \geq N.\end{aligned}$$

Also,

$$\begin{aligned}\sigma_{q_p}(x, x_n) &= \sigma_p(x, x_n) - \sigma_p(x, x)\\ &\preceq c, \quad n \geq N.\end{aligned}$$

This shows that $x_n \xrightarrow{(\sigma_{q_p})} x$, and $x_n \xrightarrow{(\sigma_{q^{-1}p})} x$. Therefore $x_n \xrightarrow{(\sigma_{q_p^*})} x$. □

2.3 TVS-partial cone metric spaces and TVS-cone metric spaces

In this section we start by showing that every TVS-partial cone metric space gives rise to a TVS-cone metric space. The following result extends Proposition 1.3.4 the proof is similar.

Theorem 2.3.1 *Let (X, σ_p) be a TVS-partial cone metric space and $\sigma_{d_p} : X \times X \to E$ be defined by*

$$\sigma_{d_p}(x, y) = \begin{cases} 0 & \textit{whenever} \quad x = y \\ \sigma_p(x, y) & \textit{otherwise} \end{cases}$$

for all $x, y \in X$. Then σ_{d_p} is a TVS-cone metric on X.

Proof. (i) Clearly $\sigma_{d_p}(x, y) = 0$ if and only if $x = y$.

(ii) Since $\sigma_p(x, y) = \sigma_p(y, x)$ we get $\sigma_{d_p}(x, y) = \sigma_{d_p}(y, x)$ for all $x, y \in X$.

(iii) To show that $\sigma_{d_p}(x, z) \preceq \sigma_{d_p}(x, y) + \sigma_{d_p}(y, z)$ note that $\sigma_{d_p}(x, z) \preceq \sigma_p(x, z) \preceq \sigma_p(x, y) + \sigma_p(y, z) - \sigma_p(y, y)$.

If $x \neq z$ and $x = y$, then

$$\begin{aligned}\sigma_{d_p}(x,z) \preceq \sigma_p(x,z) &\preceq \sigma_p(x,y)+\sigma_p(y,z)-\sigma_p(y,y)\\ &= \sigma_p(y,y)+\sigma_p(y,z)-\sigma_p(y,y)\\ &= \sigma_p(y,z)\\ &= \sigma_{d_p}(y,z).\end{aligned}$$

If $x \neq z$ and $y = z$, then

$$\begin{aligned}\sigma_{d_p}(x,z) \preceq \sigma_p(x,z) &\preceq \sigma_p(x,y)+\sigma_p(y,y)-\sigma_p(y,y)\\ &= \sigma_p(x,y)\\ &= \sigma_{d_p}(x,y).\end{aligned}$$

If $x \neq z$ and $y \neq z$, then $\sigma_{d_p}(x,z) \preceq \sigma_{d_p}(x,y) + \sigma_{d_p}(y,z)$.

If $x = z$, then $\sigma_{d_p}(x,z) = 0 \preceq \sigma_{d_p}(x,y) + \sigma_{d_p}(y,z)$. Therefore (X, σ_{d_p}) is a TVS-cone metric space. □

In the following results let (X, σ_{d_p}) be a TVS-cone metric space as in Theorem 2.3.1 and (X, σ_p) be a TVS-partial cone metric space. The following result extends Theorem 1.3.2 the proof is similar.

Theorem 2.3.2 *Let (X, σ_p) be a TVS-partial cone metric space, P be a normal cone in E and (X, σ_{d_p}) be a TVS-cone metric space. Then (X, σ_{d_p}) is σ_{d_p}-Cauchy complete if and only if (X, σ_p) is 0-Cauchy complete.*

Proof. Suppose that TVS-partial cone metric space (X, σ_p) is 0-Cauchy complete, P be a normal cone in E with normal constant K and $\{x_n\}$ be a σ_{d_p}-Cauchy sequence in (X, σ_{d_p}). Without loss of generality assume that $x_n \neq x_m$ for all $n \neq m$. Then $\{x_n\}$ is a 0-Cauchy sequence in (X, σ_p). Since (X, σ_p) is 0-Cauchy complete, $\lim_n \sigma_p(x_n, x) = 0$ for some $x \in X$. Note that $x \neq x_n$ for all n. Therefore $\lim_n \sigma_{d_p}(x_n, x) = 0$. So, (X, σ_{d_p}) is σ_{d_p}-Cauchy complete.

Conversely, suppose that (X, σ_{d_p}) is σ_{d_p}-Cauchy complete, P be a normal cone in E with normal constant K and $\{x_n\}$ be a 0-Cauchy sequence in (X, σ_p). Without loss of generality assume that $x_n \neq x_m$ for all $n \neq m$. So, $\sigma_p(x_n, x_m) = \sigma_{d_p}(x_n, x_m) \to 0$, where 0 is the neutral element in E. Hence, $\{x_n\}$ is a σ_{d_p}-Cauchy sequence in (X, σ_{d_p}). Since (X, σ_{d_p}) is σ_{d_p}-Cauchy complete then there exists $x \in X$ such that $\lim_n \sigma_{d_p}(x_n, x) = 0$. Thus $\lim_n \sigma_p(x_n, x) = 0$ and so, (X, σ_p) is 0-Cauchy complete. □

Chapter 3

Partial metric type structures and Lipschitzian mappings

The notion of metric type structure was originally developed and studied by Khamsi [22] in 2010 as a generalization of metric spaces. Inspired and motivated by this notion we introduce partial metric type structure as a generalization of partial metric spaces and metric type structures and present some fixed point results of Lipschitzian mappings in this setting.

The chapter is aligned as follows: Preliminaries and some basic notions are recalled in Section 3.1. In Section 3.2 we present some fundamental properties of partial metric type structures and further show that every partial metric type space gives rise to a metric type space. Lipschitzian mappings and some fixed point results on metric type structures and partial metric type structures are discussed in Section 3.3. Most results in this chapter can be found in the papers [5] and [6].

3.1 Metric type structures and Lipschitzian mappings

We start by showing that TVS-cone metric space gives rise to a metric type structure.

Theorem 3.1.1 [22] *Let (X, σ) be a TVS-cone metric space over a Banach space E with a normal cone P and normal constant K. The map $D : X \times X \to [0, \infty)$ defined by*

$$D(x, y) = ||\sigma(x, y)||$$

for all $x, z, y_i \in X, i = 1, 2, ..., n$ satisfies the following properties:

(i) $D(x, y) = 0$ *iff* $x = y$;

(ii) $D(x, y) = D(y, x)$;

(iii) $D(x, z) \leq K[D(x, y_1) + D(y_1, y_2) + ... + D(y_n, z)]$.

Proof. (i) Let $x = y$. Then this implies that $D(x, y) = D(x, x) = ||\sigma(x, x)|| = ||0|| = 0$. Hence, $D(x, y) = 0$. Conversely, suppose that $D(x, y) = 0$. Then this implies that $||\sigma(x, y)|| = 0 \Rightarrow \sigma(x, y) = 0$. Therefore $x = y$.

(ii) To prove that $D(x, y) = D(y, x)$ for all $x, y \in X$, we know that $||\sigma(x, y)|| = ||\sigma(y, x)||$ for all $x, y \in X$. It follows that $D(x, y) = D(y, x)$ for all $x, y \in X$.

(iii) Let $x, z, y_1, y_2, ..., y_n$ be any points in X. By the inequality we get

$$\sigma(x, z) \preceq \sigma(x, y_1) + \sigma(y_1, y_2) + ... + \sigma(y_n, z).$$

Since P is normal in E with normal constant K we get

$$||\sigma(x, z)|| \leq K[||\sigma(x, y_1) + \sigma(y_1, y_2) + ... + \sigma(y_n, z)||],$$

which implies that

$$||\sigma(x, z)|| \leq K[||\sigma(x, y_1)|| + ||\sigma(y_1, y_2)|| + ... + ||\sigma(y_n, z)||].$$

It follows that, $D(x, z) \leq K[D(x, y_1) + D(y_1, y_2) + ... + D(y_n, z)]$. □

Definition 3.1.1 Let X be a nonempty set and $D : X \times X \to [0, \infty)$ be a function for all $x, y_i, z \in X, i = 1, 2, ..., n$, for some $K > 1$ such that:

(i) $D(x, y) = 0$ iff $x = y$;

(ii) $D(x, y) = D(y, x)$;

(iii) $D(x,z) \leq K[D(x,y_1) + D(y_1,y_2) + ... + D(y_n,z)]$.

The pair (X,D) is called **metric type space**.

Note that metric type spaces are not Hausdorff.

Remark 3.1.1 It is important to note that, every metric space is a metric type space, but not conversely.

Definition 3.1.2 Let (X,D) be a metric type space. Then

(i) a sequence $\{x_n\}$ in (X,D) D-**converges** to a point $x \in X$ if $\lim_n D(x_n,x) = 0$.

(ii) a sequence $\{x_n\}$ in (X,D) is D-**Cauchy** if $\lim_{n,m} D(x_n,x_m) = 0$.

(iii) a metric type space (X,D) is D-**Cauchy complete** if every D-Cauchy sequence in X D-converges to a point $x \in X$.

Remark 3.1.2 Note that if a sequence $\{x_n\}$ in a metric type space (X,D) is D-convergent to a point $x \in X$, we shall write $\lim_n D(x_n,x) = 0$ or simply $x_n \xrightarrow{D} x$.

We shall present some properties of sequences in metric type structures. The next proposition is our own contribution.

Proposition 3.1.1 *Let (X,D) be a metric type space and $\{x_n\}$ be a sequence in X. If $\lim_n D(x_n,x) = 0$ and $\lim_n D(x_n,y) = 0$, then $x = y$.*

Proof. Let (X,D) be a metric type space and $\{x_n\}$ be a sequence in X. Suppose that $\lim_n D(x_n,x) = 0$ and $\lim_n D(x_n,y) = 0$. Then

$$\begin{aligned}
D(x,y) &\leq K[D(x,x_n) + D(x_n,y)] \\
\lim_n D(x,y) &\leq K \lim_n [D(x,x_n) + D(x_n,y)] \\
&= 0 \\
D(x,y) &= 0.
\end{aligned}$$

So, $x = y$. □

Definition 3.1.3 Let (X, D) be a metric type space and $T : (X, D) \to (X, D)$ be a map. Then

(i) T is called a **Lipschitzian map** if there exists a constant λ, where $\lambda \geq 0$ such that $D(Tx, Ty) \leq \lambda D(x, y)$, for any $x, y \in X$. Moreover, the constant λ, where $\lambda \geq 0$ is called a **Lipschitzian constant**. A smallest constant λ will be denoted by $Lip(T)$.

(ii) T is called a **contraction map** if there exists a contraction constant λ, where $0 \leq \lambda < 1$ such that $D(Tx, Ty) \leq \lambda D(x, y)$, for any $x, y \in X$. Moreover, the constant λ, where $0 \leq \lambda < 1$ is called a **contraction constant**.

(iii) a point $x \in X$ is said to be a **fixed point** of T if $Tx = x$.

Definition 3.1.4 Let (X, D_X) and (Y, D_Y) be metric type spaces. A map $T : (X, D_X) \to (Y, D_Y)$ between metric type spaces is **continuous** if a sequence $\{x_n\}$ D_X-converges to a point x in X implies that $\{T(x_n)\}$ D_Y-converges to a point $T(x)$ in Y.

Remark 3.1.3 Let $T : (X, D) \to (X, D)$ be a map between metric type spaces.

(i) If T is a contraction map then it is a Lipschitzian map but the converse is not true.

(ii) Any contraction map and a Lipschitzian map is continuous but a continuous map may neither be a contraction nor a Lipschitzian.

Theorem 3.1.2 [22] *Let (X, D) be a D-Cauchy complete metric type space and $T : (X, D) \to (X, D)$ be a map. If T^n is a Lipschitzian map for all $n \geq 0$ and $\sum_{n=0}^{\infty} Lip(T^n) < \infty$, then T has a unique fixed point $\omega \in X$. Moreover, for any $x \in X$, the orbit* $\lim_n D(T^n x, \omega) = 0$.

Proof. Let $x \in X$ and $m, n \geq 0$. Then

$$\begin{aligned} D(T^{n+m}x, T^n x) &= D(T^n(T^m x), T^n(x)) \\ &\leq Lip(T^n) D(T^m x, x). \end{aligned}$$

Note that

$$D(T^m x, x) \leq K[D(x, Tx) + D(Tx, T^2x) + ... + D(T^{m-1}x, T^m x)].$$

Then

$$\begin{aligned} D(T^{n+m}x, T^n x) &\leq K[Lip(T^n)\sum_{i=0}^{m-1} D(T^{i+1}x, T^i x)] \\ &\leq K[Lip(T^n)[\sum_{i=0}^{m-1} Lip(T^i)]D(x, Tx)]. \end{aligned}$$

Since $\sum_{n=0}^{\infty} Lip(T^n)$ is convergent then $\sum_{n=0}^{\infty} Lip(T^n) = 0$. Thus $\lim_{n,m} D(T^{n+m}x, T^n x) = 0$. Therefore $\{T^n x\}$ is a D-Cauchy sequence. Since X is D-Cauchy complete then $\lim_n D(T^n x, \omega) = 0$.

$$\begin{aligned} D(T^{n-1}x, \omega(x)) &\leq K[D(T^{n-1}x, T^n x) + D(T^n x, \omega(x))] \\ &\leq K[D(T^{n-1}(x, Tx)) + D(T^n x, \omega(x))] \\ &\leq K[Lip(T^{n-1})D(x, Tx) + D(T^n x, \omega(x))]. \end{aligned}$$

Now

$$\begin{aligned} D(\omega(x), T\omega(x)) &\leq K[D(T\omega(x), T^n x) + D(T^n x, \omega(x))] \\ &\leq K[D(\omega(x), T^n x) + D(T(T^{n-1}x, T\omega(x)))] \\ &\leq K[D(\omega(x), T^n x) + Lip(T)D(T^{n-1}x, \omega(x))]. \end{aligned}$$

Furthermore,

$D(\omega(x), T\omega(x)) \leq K[D(\omega(x), T^n x) + Lip(T)[KD(T^{n-1}x, T^n x) + KD(T^n x, \omega(x))]]$

$D(\omega(x), T\omega(x)) \leq K[D(\omega(x), T^n x) + Lip(T)[KLip(T^{n-1})D(x, Tx) + KD(T^n x, \omega(x))]]$

$D(\omega(x), T\omega(x)) \leq K(1 + KLip(T))D(\omega(x), T^n x) + Lip(T)KLip(T^{n-1})D(x, Tx)$. So,

$D(\omega(x), T\omega(x)) \leq K(1 + KLip(T))D(\omega(x), T^n x) + Lip(T)KLip(T^{n-1})D(x, Tx)$.

$$D(\omega(x), T\omega(x)) \leq K(1 + KLip(T))D(\omega(x), \omega(x)) + Lip(T)KLip(T^{n-1})D(x, Tx).$$

Let $n \to \infty$. Then we get $D(\omega(x), T\omega(x)) = 0$. Therefore $T\omega(x) = \omega(x)$.

Next we show that T has at most one fixed point. Let ω_1 and ω_2 be two fixed points of T. Then

$$D(\omega_1, \omega_2) = D(T^n\omega_1, T^n\omega_2) \leq Lip(T^n)D(\omega_1, \omega_2)$$

for any $n \geq 1$. Since $\lim_n Lip(T^n) = 0$, we get $D(\omega_1, \omega_2) = 0$. Therefore $\omega_1 = \omega_2$. That is $\omega(x) = \omega(y)$ for all $x, y \in X$. □

Regarding Definition 3.1.1 it worthwhile to mention that we require

$$\sum_{n=0}^{\infty} Lip(T^n) < \infty$$

such that property (iii) is satisfied. In particular, a more natural condition is presented as follows:

$$(iii)' D(x, z) \leq K(D(x, y) + D(y, z))$$

for all $x, y, z \in X$ and some constant $K \geq 1$.

We present an example which fulfills $(iii)'$.

Example 3.1.1 [22, Example 3.2] Let X be the set of the Lebesque measurable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^2 dx < \infty$. Define $D : X \times X \to [0, \infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx$$

for all $x \in X$. Then

(i) $D(f, g) = 0$ iff $f = g$;

(ii) $D(f, g) = D(g, f)$ for all $f, g \in X$;

(iii) $D(f, g) \leq 2(D(f, h) + D(h, g))$ for all $f, g, h \in X$.

It worth noting that Example 3.1.1 is also a TVS-cone metric space with $K > 1$.

Theorem 3.1.3 [22] *Let (X, σ) be a TVS-cone metric space over the Banach space E with a normal cone P and normal constant K. Let $T : (X, \sigma) \to (X, \sigma)$ be a contraction with constant $\lambda < 1$. Then T has a unique fixed point ω, and any orbit σ-converges to ω.*

Proof. Let (X, σ) be a TVS-cone metric space over the Banach space E with a cone P which is normal with normal constant K. Consider $D : X \times X \to [0, \infty)$ defined by $D(x, y) = ||\sigma(x, y)||$ for all $x, y \in X$. Let $T : (X, \sigma) \to (X, \sigma)$ be a contraction with constant $\lambda < 1$. Then

$$\sigma(Tx, Ty) \preceq \lambda\sigma(x, y), \quad \text{so,} \quad \sigma(T^n x, T^n y) \preceq \lambda^n \sigma(x, y).$$

Hence,

$$\begin{aligned} ||\sigma(T^n x, T^n y)|| &\leq ||K\lambda^n \sigma(x, y)|| \\ &\leq K\lambda^n ||\sigma(x, y)||. \end{aligned}$$

Thus, $\sigma(T^n x, T^n y) \preceq K\lambda^n \sigma(x, y)$ for any $x, y \in X$ and $n \geq 0$. Hence, $Lip(T^n) \leq K\lambda^n$, for any $n \geq 0$. Therefore $\sum_{n\geq 0} Lip(T^n)$ is convergent, which implies that T has a unique fixed point ω, and any orbit σ-converges to ω. □

3.2 Dislocated metric type structures

We shall show that every TVS-partial cone metric space gives rise to a dislocated metric type structure.

Theorem 3.2.1 *Let (X, σ_p) be a TVS-partial cone metric space over a Banach space E with a normal cone P and normal constant K. A map $D_p : X \times X \to [0, \infty)$ defined by*

$$D_p = ||\sigma_p(x, y)||$$

for all $x, y \in X$ satisfies the following properties:

(i) $D_p(x, y) = 0 \Rightarrow x = y$;

(ii) $D_p(x, y) = D_p(y, x)$;

(iii) $D_p(x, z) \leq K[D_p(x, y_1) + D_p(y_1, y_2) + ... + D_p(y_n, z)]$.

Proof. Let (X, σ_p) be a TVS-partial cone metric space over a Banach space E with a normal cone P and normal constant K. Define a map $D_p : X \times X \to [0, \infty)$ by $D_p(x, y) = ||\sigma_p(x, y)||$ for all $x, y \in X$.

(i) Let $D_p(x,y) = 0$. This implies that $||\sigma_p(x,y)|| = 0, \sigma_p(x,y) = 0$. Then $\sigma_p(x,x) = \sigma_p(y,y) = 0$. So, $\sigma_p(y,x) = \sigma_p(x,x) = \sigma_p(y,y)$. Therefore $x = y$.

(ii) To prove that $D_p(x,y) = D_p(y,x)$ for all $x,y \in X$. We know that $||\sigma_p(x,y)|| = ||\sigma_p(y,x)||$ for all $x,y \in X$. It follows that $D_p(x,y) = D_p(y,x)$ for all $x,y \in X$.

(iii) Let $x, y_1, ..., y_n, z \in X$. Then by the inequality, $D_p(x,z) = ||\sigma_p(x,z)|| \leq ||\sigma_p(x,y_1) + \sigma_p(y_1,y_2) + ... + \sigma_p(y_n,z)||$. Since P is normal in E with normal constant K we get $||\sigma_p(x,z)|| \leq K[||\sigma_p(x,y_1) + \sigma_p(y_1,y_2) + ... + \sigma_p(y_n,z)||]$, which implies that $||\sigma_p(x,z)|| \leq K[||\sigma_p(x,y_1)|| + ||\sigma_p(y_1,y_2)|| + ... + ||\sigma_p(y_n,z)||]$. It follows that

$$D_p(x,z) \leq K[D_p(x,y_1) + D_p(y_1,y_2) + ... + D_p(y_n,z)].$$

This complete our proof. □

Definition 3.2.1 Let X be a nonempty set and $D_p : X \times X \to [0,\infty)$ be a function for all $x, y_i, z \in X, i = 1,2,...,n$ for some $K > 1$ such that:

(i) $D_p(x,y) = 0 \Rightarrow x = y$;

(ii) $D_p(x,y) = D_p(y,x)$;

(iii) $D_p(x,z) \leq K[D_p(x,y_1) + D_p(y_1,y_2) + ... + D_p(y_n,z)]$.

The pair (X, D_p) is called **dislocated metric type space**.

Remark 3.2.1 We would like to point out the following:

(i) Every metric space is a dislocated metric type space, but not conversely.

(ii) Every metric type space is a dislocated metric type space, but the converse does not necessarily hold.

3.2.1 TVS-partial cone metric spaces and partial metric type spaces

Definition 3.2.2 Let X be a nonempty set and $D_p : X \times X \to [0,\infty)$ be a function for all $x, y, z \in X$ such that:

(i) $x = y$ iff $D_p(x,x) = D_p(x,y) = D_p(y,y)$;

(ii) $D_p(x,y) \geq 0$;

(iii) $D_p(x,y) = D_p(y,x)$;

(iv) $D_p(x,x) \leq D_p(x,y)$;

(v) $D_p(x,z) \leq K[D_p(x,y_1) + D_p(y_1,y_2) + ... + D_p(y_n,z)] - \sum_{i=1}^{n} D_p(y_i,y_i)$, for some constant $K > 1$ and $i = 1,2,...,n$.

The pair (X, D_p) is called **partial metric type space**.

Remark 3.2.2 Every partial metric type space is a dislocated metric type space, but not conversely.

It should be observed that if a sequence converges to a point then its self distance converges to the self distance of that point. In particular, we define convergence and Cauchy completeness (Definition 3.1.2) in partial metric type space settings.

Definition 3.2.3 Let (X, D_p) be a partial metric type space. Then

(i) a sequence $\{x_n\}$ in (X, D_p) D_p-**converges** to a point $x \in X$ if $\lim_n D_p(x_n, x) = D_p(x,x) = \lim_n D_p(x_n, x_n)$.

(ii) a sequence $\{x_n\}$ in (X, D_p) is D_p-**Cauchy** if $\lim_{n,m} D_p(x_n, x_m)$ is finite and exists.

(iii) a partial metric type space (X, D_p) is D_p-**Cauchy complete** if every D_p-Cauchy sequence in X D_p-converges to a point $x \in X$.

Definition 3.2.4 Let (X, D_p) be a partial metric type space. Then

(i) a sequence $\{x_n\}$ is 0-**Cauchy** if $\lim_{n,m} D_p(x_n, x_m) = 0$.

(ii) a partial metric type space (X, D_p) is 0-**Cauchy complete** if every 0-Cauchy sequence in X converges to a point $x \in X$ and $D_p(x,x) = 0$.

It worth noting that if (X, D) is a metric type space, then a sequence $\{x_n\}$ is 0-Cauchy if and only if it is a D-Cauchy sequence in (X, D). Furthermore, a metric type space (X, D) is 0-Cauchy complete if and only if it is D-Cauchy complete.

Remark 3.2.3 Let (X, D_p) be a partial metric type space. Observe that a 0-Cauchy sequence is a D_p-Cauchy sequence but the converse is not necessarily true.

The example below shows that the implications of Remark 3.2.3 are not reversible.

Example 3.2.1 Let $X = \{a, b\}$ and $D_p : X \times X \to [0, \infty)$ be defined by

$$D_p(x, y) = \begin{cases} 1 & \text{if} \quad x = y \\ 2 & \text{otherwise} \end{cases}$$

for all $x, y \in X$. We note that (X, D_p) is a partial metric type space. Consider a sequence $\{x_n = a, n \geq 1\}$. Then $\{x_n\}$ is a D_p-Cauchy sequence but not a 0-Cauchy sequence.

Remark 3.2.4 The reader should observe that a D_p-Cauchy complete partial metric type space is a 0-Cauchy complete partial metric type space but the converse is not necessarily true.

Proposition 3.2.1 *Let (X, D_p) be a 0-Cauchy complete partial metric type space and $\{x_n\}$ be a 0-Cauchy sequence in X. If $\{x_n\}$ D_p-converges to a point x and $\{x_n\}$ D_p-converges to a point y, then $x = y$.*

Proof. Let (X, D_p) be a 0-Cauchy complete partial metric type space and $\{x_n\}$ be a 0-Cauchy sequence in X. Suppose that $\{x_n\}$ D_p-converges to a point x in X and $\{x_n\}$ D_p-converges to a point y in X. Then

$$\begin{aligned} D_p(x, y) &\leq K[D_p(x, x_n) + D_p(x_n, y)] - D_p(x_n, x_n) \\ \lim_n D_p(x, y) &\leq K \lim_n [D_p(x, x_n) + D_p(x_n, y)] - \lim_n D_p(x_n, x_n) \\ &\leq KD_p(x, x) + KD_p(y, y) - D_p(x, x) \\ D_p(x, y) &\leq (K - 1)D_p(x, x) + KD_p(y, y). \end{aligned}$$

Note that $D_p(x, x) = D_p(y, y) = 0$, since $\{x_n\}$ is a 0-Cauchy sequence and (X, D_p) is a 0-Cauchy complete partial metric type space. So, $D_p(x, y) = 0$ then this implies that $x = y$. □

Theorem 3.2.2 *Let (X, D_p) be a partial metric type space. Define $d_{D_p} : X \times X \to [0, \infty)$ by*

$$d_{D_p}(x, y) = \begin{cases} 0 & \textit{whenever} \quad x = y \\ D_p(x, y) & \textit{whenever} \quad x \neq y \end{cases}$$

for all $x, y \in X$. Then d_{D_p} is a metric type distance function, hence, (X, d_{D_p}) is a metric type space.

Proof. (i) Clearly, $d_{D_p}(x, y) = 0$ if and only if $x = y$.

(ii) Since $D_p(x, y) = D_p(y, x)$ we get $d_{D_p}(x, y) = d_{D_p}(y, x)$ for all $x, y \in X$.

(iii) To show that $d_{D_p}(x, z) \leq K[d_{D_p}(x, y_1) + d_{D_p}(y_1, y_2) + ... + d_{D_p}(y_n, z)]$ for all $x, y_i, z \in X$. Note that if $x = z$, then $d_{D_p}(x, z) = 0$. So, without loss of generality, we assume that $x \neq z$ and $y_i \neq y_j$ for all $i \neq j = 1, 2, ..., n$. Then $d_{D_p}(x, z) = D_p(x, z)$ and

$$D_p(x, z) \leq K[D_p(x, y_1) + D_p(y_1, y_2) + ... + D_p(y_n, z)] - \sum_{i=1}^{n-1} D_p(y_i, y_i).$$

Furthermore, we assume that $x \neq y_1$ and $z = y_n$. Indeed, if $x = y_1$, and $z = y_n$, then the inequality holds. So, this implies that

$$d_{D_p}(x, z) \leq K[d_{D_p}(x, y_1) + d_{D_p}(y_1, y_2) + ... + d_{D_p}(y_n, z)].$$

Therefore (X, d_{D_p}) is a metric type space. □

Let (X, D_p) be a partial metric type space and $d_{D_p} : X \times X \to [0, \infty)$ be a metric type distance function defined by

$$d_{D_p}(x, y) = \begin{cases} 0 & \text{whenever} \quad x = y \\ D_p(x, y) & \text{whenever} \quad x \neq y \end{cases}$$

for all $x, y \in X$ [Theorem 3.2.2].

Lemma 3.2.1 *Let (X, D_p) be a partial metric type space and $d_{D_p} : X \times X \to [0, \infty)$ be a metric type distance function. If $\{x_n\}$ is a d_{D_p}-Cauchy sequence in (X, d_{D_p}) then it is either a 0-Cauchy sequence or eventually constant sequence in (X, D_p).*

Proof. Let (X, D_p) be a partial metric type space. If $\{x_n\}$ is a d_{D_p}-Cauchy sequence in (X, d_{D_p}), then $\lim_{n,m} d_{D_p}(x_n, x_m) = 0$. There are two cases to consider. Case (i) $x_n = x_m$ for all $m, n \geq 1$ in this case $\{x_n\}$ is an eventually constant sequence in (X, D_p). Case (ii) $x_n \neq x_m$ for some $m, n \geq 1$ and $\lim_{n,m} d_{D_p}(x_n, x_m) = 0$. Clearly, in this case we have $\lim_{n,m} D_p(x_n, x_m) = 0$. This shows that the sequence is a 0-Cauchy sequence in (X, D_p). □

Lemma 3.2.2 *Let (X, D_p) be a partial metric type space and $d_{D_p} : X \times X \to [0, \infty)$ be a metric type distance function. If $\{x_n\}$ is a 0-Cauchy sequence in (X, D_p) then it is a d_{D_p}-Cauchy sequence in (X, d_{D_p}).*

Remark 3.2.5 Let (X, D_p) be a partial metric type space. It is important note that the converse of Lemma 3.2.2 does not necessarily hold.

Hence,

Theorem 3.2.3 *Let (X, D_p) be a partial metric type space and $d_{D_p} : X \times X \to [0, \infty)$ be a metric type distance function. Then (X, D_p) is 0-Cauchy complete if and only if (X, d_{D_p}) is d_{D_p}-Cauchy complete.*

Proof. Let (X, D_p) be a 0-Cauchy complete partial metric type space and $\{x_n\}$ be a d_{D_p}-Cauchy sequence in (X, d_{D_p}). By Lemma 3.2.1, $\{x_n\}$ is a 0-Cauchy sequence in (X, D_p). Thus there exists $x \in X$ such that $\{x_n\}$ D_p-converges to x in the partial metric type space (X, D_p). Hence, (X, d_{D_p}) is d_{D_p}-Cauchy complete.

Conversely, assume that (X, d_{D_p}) is d_{D_p}-Cauchy complete, and let $\{x_n\}$ be a 0-Cauchy sequence in (X, D_p). Without loss of generality we assume that $x_n \neq x_m$ for all $n, m \geq 1$. By Lemma 3.2.1, $\{x_n\}$ is a d_{D_p}-Cauchy sequence in (X, d_{D_p}). Thus, there exists $x \in X$, such that $\{x_n\}$ d_{D_p}-converges to x in the metric type space (X, d_{D_p}). Now $D_p(x_n, x) = d_{D_p}(x_n, x)$ D_p-converges to 0 as n tends to ∞. This shows that (X, D_p) is 0-Cauchy complete. □

Remark 3.2.6 Every TVS-cone metric space (X, σ) gives rise to a metric type structure (X, D) [Theorem 3.1.1]. In a similar vein, every TVS-partial cone metric space (X, σ_p) gives rise to a dislocated metric type structure (X, D_p) [Theorem 3.2.1]. Moreover, for every partial metric type space (X, D_p) there exists a metric type space (X, d_{D_p}) [Theorem 3.2.2].

3.3 Lipschitzian mappings and some fixed point results in partial metric type structures

We begin with a definition.

Definition 3.3.1 Let (X, D_p) be a partial metric type space and $T : (X, D_p) \to (X, D_p)$ be a map. Then

(i) T is called a **Lipschitzian map** if there exists a constant λ, where $\lambda \geq 0$ such that $D_p(Tx, Ty) \leq \lambda D_p(x, y)$, for any $x, y \in X$. Moreover, the constant λ, where $\lambda \geq 0$ is called a **Lipschitzian constant**. A smallest constant λ will be denoted by $Lip(T)$.

(ii) T is called a **contraction map** if there exists a contraction constant λ, where $0 \leq \lambda < 1$ such that $D_p(Tx, Ty) \leq \lambda D_p(x, y)$, for any $x, y \in X$. Moreover, the constant λ, where $0 \leq \lambda < 1$ is called a **contraction constant**.

(iii) a point $x \in X$ is said to be a **fixed point** of T if $Tx = x$.

Definition 3.3.2 Let (X, D_{p_X}) and (Y, D_{p_Y}) be partial metric type spaces. A map $T : (X, D_{p_X}) \to (Y, D_{p_Y})$ between partial metric type spaces is **continuous** if a sequence $\{x_n\}$ D_{p_X}-converges to a point x in X implies that $\{T(x_n)\}$ D_{p_Y}-converges to a point $T(x)$ in Y.

Theorem 3.3.1 *Let (X, D_p) be a 0-Cauchy complete partial metric type space and $T : (X, D_p) \to (X, D_p)$ be a map. If T^n is a Lipschitzian map for all $n \geq 0$ and $\sum_{n=0}^{\infty} Lip(T^n) < \infty$, then T has a unique fixed point $\omega \in X$. Moreover, for any $x \in X$, the orbit $\{T^n x\}$ D_p-converges to ω.*

Proof. Let (X, D_p) be a 0-Cauchy complete partial metric type space. By Theorem 3.2.2, (X, d_{D_p}), is a metric type space. Then (X, d_{D_p}) is d_{D_p}-Cauchy complete by Theorem 3.2.3. Clearly, $T : (X, d_{D_p}) \to (X, d_{D_p})$ satisfies the properties in Theorem 3.1.2. Hence, T has a unique fixed point $\omega \in X$. Moreover, for any $x \in X$, the orbit $\{T^n x\}$ d_{D_p}-converges to ω. □

Theorem 3.3.2 *Let (X, D_p) be a D_p-Cauchy complete partial metric type space and $T : (X, D_p) \to (X, D_p)$ be a map. If T^n is a Lipschitzian map for all $n \geq 0$ and $\sum_{n=0}^{\infty} Lip(T^n) < \infty$, then T has a unique fixed point $\omega \in X$. Moreover, for any $x \in X$, the orbit $\{T^n x\}$ D_p-converges to ω.*

Proof. Let (X, D_p) be a D_p-Cauchy complete partial metric type space. Then (X, D_p) is 0-Cauchy complete. The proof is complete by Theorem 3.3.1. □

Regarding Definition 3.2.2 it worthwhile to mention that we require

$$\sum_{n=0}^{\infty} Lip(T^n) < \infty$$

such that property (v) is satisfied. In particular, a more natural condition is presented as follows:

$$(v)' D_p(x, z) \leq K(D_p(x, y) + D_p(y, z)) - D_p(y, y)$$

for all $x, y, z \in X$ and some constant $K \geq 1$.

Now we present an example which fulfills $(v)'$.

Example 3.3.1 Let $X = \mathbb{N}$ and $D_p : X \times X \to [0, \infty)$ be defined by $D_p(x, y) = [\max\{x, y\}]^p + |x - y|^p$ for all $x, y \in X$ and $p \geq 1$. Then

(i) $x = y$ iff $D_p(x, x) = D_p(x, y) = D_p(y, y)$;

(ii) $D_p(x, y) \geq 0$;

(iii) $D_p(x, y) = D_p(y, x)$;

(iv) $D_p(x, x) \leq D_p(x, y)$;

$(v)'$ $D_p(x, z) \leq 2^p[D_p(x, y) + D_p(y, z)] - D_p(y, y)$.

Therefore (X, D_p) is a partial metric type space. Note that $K = 2^p, p \geq 1$.

Theorem 3.3.3 *Let (X, σ_p) be a TVS-partial cone metric space and (X, D_p) be the dislocated metric type space. The function $d_{D_p} : X \times X \to [0, \infty)$ defined by*

$$d_{D_p}(x, y) = \begin{cases} 0 \quad whenever \quad x = y \\ D_p(x, y) \quad whenever \quad x \neq y \end{cases}$$

for all $x, y \in X$ is a metric type on X.

Theorem 3.3.4 *Let (X, σ_p) be a TVS-partial cone metric space. If (X, σ_p) is 0-Cauchy complete, then (X, d_{D_p}) is d_{D_p}-Cauchy complete.*

Theorem 3.3.5 *Let (X, σ_p) be a 0-Cauchy complete TVS-partial cone metric space over the Banach space E with a normal cone P and normal constant K. Let $T : (X, \sigma_p) \to (X, \sigma_p)$ be a contraction with constant $\lambda < 1$. Then T has a unique fixed point ω, and any orbit σ_p-converges to ω.*

Proof. Let (X, σ_p) be a 0-Cauchy complete TVS-partial cone metric space over the Banach space E with a cone P which is normal with normal constant K. Consider $D_p : X \times X \to [0, \infty)$ defined by $D_p(x, y) = ||\sigma_p(x, y)||$ for all $x, y \in X$. Then D_p is a dislocated metric type on X and (X, d_{D_p}) is a d_{D_p}-Cauchy complete metric type space [Theorem 3.3.3]. Let $T : (X, \sigma_p) \to (X, \sigma_p)$ be a contraction with constant $\lambda < 1$. Then $\sigma_p(Tx, Ty) \preceq \lambda \sigma_p(x, y)$ so, $\sigma_p(T^n x, T^n y) \preceq \lambda^n \sigma_p(x, y)$. Without loss of generality, assume that $x \neq y$. So,

$$\begin{aligned} ||\sigma_p(T^n x, T^n y)|| &\leq ||K \lambda^n \sigma_p(x, y)|| \\ &\leq K \lambda^n ||\sigma_p(x, y)||. \end{aligned}$$

Actually, $d_{D_p}(T^n x, T^n y) \leq K \lambda^n d_{D_p}(x, y)$ for any $x, y \in X$ and $n \geq 0$. So, by Theorem 3.1.2 T has a unique fixed point ω, and any orbit σ_p-converges to ω. □

Since a σ_p-Cauchy complete TVS-partial cone metric space is 0-Cauchy complete:

Corollary 3.3.1 *Let (X, σ_p) be a σ_p-Cauchy complete TVS-partial cone metric space over the Banach space E with a normal cone P and normal constant K. Let $T : (X, \sigma_p) \to (X, \sigma_p)$ be a contraction with constant $\lambda < 1$. Then T has a unique fixed point ω, and any orbit σ_p-converges to ω.*

Chapter 4

Completeness in symmetric spaces and some fixed point results

In the literature the notion of completeness for metric spaces is discussed in terms of Cauchy sequences. Unlike in the classical case (metric spaces), in symmetric spaces not every convergent sequence is a Cauchy sequence. Motivated by the notion of absolute closure [41], the author in [31] defined a new notion of completeness for symmetric spaces. It should be observed that this new notion is equivalent to completeness when restricted to the class of metric spaces. A similar study was done for quasi pseudo metric spaces [31] and for symmetric spaces [32].

The reader should also note that using the classical notion of (Cauchy) completeness for symmetric spaces, analogous fixed point results are presented in the literature see for example [17] for single valued maps; [30], [35], [37], [38] and [49] for multivalued maps. We note that the paper [33] presents a fixed point theory result of a single valued maps as presented in [17], without appealing to Cauchy sequences.

The notion of a complete metric space and symmetric space is very important and so, is the completion of such structures; recently a completion for the dislocated metric spaces is presented in [25].

The purpose of this chapter is to revisit the notion of convergence completeness in symmetric spaces and discuss some properties of symmetric spaces. Furthermore, some fixed point results are discussed in this setting. We recall some well known results in Section 4.1. Completeness in symmetric spaces are discussed in Section 4.2. Products of symmetric spaces are presented in Section 4.3. In Section 4.4 we present symmetric spaces and some fixed point results in this setting. Some work presented in this chapter is from the paper [4].

4.1 Some properties on symmetric spaces

Definition 4.1.1 [7] A **symmetric space** (X, s) is a nonempty set X together with a real-valued function $s : X \times X \to [0, \infty)$ such that for all $x, y \in X$, the following conditions are satisfied:

(i) $s(x, y) = 0$ if and only if $x = y$;

(ii) $s(x, y) = s(y, x)$.

As in [32] and [33] we denote the class of all symmetric spaces by $\mathcal{S}$ and the class of all metric spaces by $\mathcal{M}$.

Example 4.1.1 [32, Example 1.1] Let $X = \mathbb{R}$ and $s : X \times X \to [0, \infty)$ be defined by $s(x, y) = (x - y)^2$ for all $x, y \in X$. Then (X, s) is a symmetric space but not a metric space. Choose $x = 1, y = 2$ and $z = 3$, then $s(x, z) > s(x, y) + s(y, z)$. Hence, triangle inequality does not hold.

Definition 4.1.2 [32] Let (X, s) be a symmetric space. Then

(i) a sequence $\{x_n\}$ in (X, s) is **convergent** to a point x if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $s(x_n, x) < \epsilon$ whenever $n \geq N$. The point $x \in X$ will be called a **limit** of a sequence $\{x_n\}$.

(ii) a sequence $\{x_n\}$ of points in (X, s) is called **Cauchy sequence** if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $s(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

(iii) a symmetric space (X, s) is said to be **Cauchy complete** if every Cauchy sequence in X converges to a point in X.

(iv) a point x in X is a **cluster point** of a sequence $\{x_n\}$ in X if there are positive integers $n_1 < n_2 < ...$ such that $\lim_k s(x_{n_k}, x) = 0$, for $k \geq 1$.

Remark 4.1.1 Note the following facts:

(i) if a sequence $\{x_n\}$ converges to a point x in (X, s) we shall write $\lim_n s(x_n, x) = 0$ or simply $x_n \xrightarrow{s} x$.

(ii) if a sequence $\{x_n\}$ is Cauchy in (X, s) we shall write $\lim_{n,m} s(x_n, x_m) = 0$.

The class of all symmetric spaces where convergent sequences have unique limits is denoted by $\mathcal{U}$, as in [32] and [33].

The next examples illustrates the following:

(i) A symmetric space in which the limits of a sequence may not be unique.

(ii) A symmetric space in which sequences that converge will have unique limits but the space is not necessarily a metric space.

(iii) The class of $\mathcal{U}$, is larger than the class of $\mathcal{M}$.

Example 4.1.2 [32, Example 1.2] Let $X = \mathbb{N}$. Define $s : X \times X \to [0, \infty)$ by $s(x, y) = (x - y)^2$, for all $x, y \in X$. Then (X, s) belong to $\mathcal{S}$. We construct $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s). Let $\bar{X} = \mathbb{N} \cup \{\alpha, \beta\}, \alpha \neq \beta$ both do not belong to $\mathbb{N}$. Define $\bar{s} : \bar{X} \times \bar{X} \to [0, \infty)$ by

$$\bar{s}(x, y) = \begin{cases} (x - y)^2 & \text{if} \quad x, y \in X \\ 0 & \text{if} \quad x = y \\ \frac{1}{x} & \text{if} \quad x \in X, y = \alpha \\ \frac{1}{y} & \text{if} \quad y \in X, x = \alpha \\ \frac{1}{x} & \text{if} \quad y = \beta, x \in X \\ \frac{1}{y} & \text{if} \quad y \in X, x = \beta \\ 1 & \text{if} \quad y = \alpha, x = \beta \\ 1 & \text{if} \quad x = \alpha, y = \beta \end{cases}$$

for all $x, y \in \bar{X}$. Now let $\{x_n = n, n \geq 1\}$ in X. Then $(\bar{X}, \bar{s})$ is a symmetric space and the sequence $\{x_n\}$ converges to both α and β with respect to $\bar{s}$.

Example 4.1.3 [32, Example 1.3] Let $X = \mathbb{N}$ be equipped with the discrete metric s. Then (X, s) belong to $\mathcal{S}$. We construct $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s). Let $\bar{X} = \mathbb{N} \cup \{\alpha\}, \alpha \notin \mathbb{N}$. Define $\bar{s} : \bar{X} \times \bar{X} \to [0, \infty)$ by

$$\bar{s}(x,y) = \begin{cases} s(x,y) & \text{if} \quad x, y \in X \\ 0 \quad \text{if} & x = y \\ \frac{1}{x} \quad \text{if} & x \in X, y = \alpha \\ \frac{1}{y} \quad \text{if} & x = \alpha, y \in X \end{cases}$$

for all $x, y \in \bar{X}$. We easily see that $(\bar{X}, \bar{s}) \in \mathcal{S}$. A sequence in $(\bar{X}, \bar{s})$ that converges with respect to $\bar{s}$ has a unique limit. As we now show this, let $\{x_n\}$ be a sequence in $\bar{X}$ that converges to x and y in $\bar{X}$ with respect to $\bar{s}$. Without loss of generality, assume that $\{x_n\}$ belongs to $X, x \in X$ and $y = \alpha$. Now $\lim_n s(x_n, x) = 0$. It follows that $\{x_n\}$ is eventually constant. Let $x_{n_k} = x, k \geq 1$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Also we have $\bar{s}(x_{n_k}, \alpha) \to 0$. Thus $\bar{s}(x, \alpha) = 0$, hence, $x = \alpha$. However, we clearly observe that $\bar{s}$ is not a metric.

Definition 4.1.3 [32] Let A be a subset of $(X, s) \in \mathcal{S}$, we shall say that A is $\mathcal{S}$-**closed** if for a sequence $\{x_n\}$ in A and $x \in X$ then $\lim_n s(x_n, x) = 0$, imply that $x \in A$.

Definition 4.1.4 [32] Let $(X, s) \in \mathcal{S}$ and $(\bar{X}, \bar{s}) \in \mathcal{S}$. Then we write $(X, s) \subset (\bar{X}, \bar{s})$ to mean that $X \subseteq \bar{X}$ and $\bar{s}|_{X \times X} = s$.

In the sequel, as in [32] and [33], we shall say $(\bar{X}, \bar{s})$ **contains** (X, s) to mean that $(X, s) \subset (\bar{X}, \bar{s})$. We shall call the $(\bar{X}, \bar{s})$ an extension of (X, s).

More specifically, if $(X, s) \subset (\bar{X}, \bar{s})$ and $(\bar{X}, \bar{s}) \in \mathcal{U}$, then $(X, s) \in \mathcal{U}$. However, it is possible to have $(X, s) \in \mathcal{U}$ with $(X, s) \subset (\bar{X}, \bar{s})$ but $(\bar{X}, \bar{s})$ is not necessarily in $\mathcal{U}$. We shall make it explicit when we require both (X, s) and its extension $(\bar{X}, \bar{s})$ to belong to the same class.

4.2 Completeness in symmetric spaces

Definition 4.2.1 [32] Let (X, s) be a symmetric space. Then

(i) (X, s) is **$\mathcal{S}$-convergence complete** if for every $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s) a sequence $\{x_n\}$ in X that converges to $\bar{x} \in \bar{X}$ with respect to $\bar{s}$ also converges to some point $x \in X$ with respect to s.

(ii) (X, s) is **weakly $\mathcal{S}$-convergence complete** if for every $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s) a sequence $\{x_n\}$ in X that converges to $\bar{x} \in \bar{X}$ with respect to $\bar{s}$ has a subsequence $\{x_{n_k}\}$ that converges to some point $x \in X$ with respect to s.

Definition 4.2.2 [32] Let $\{x_n\}$ be a sequence in $(X, s) \in \mathcal{S}$. We say that $\{x_n\}$ has **property $\mathcal{F}$** if for every finite subset F of X there is $N \in \mathbb{N}$ such that $n \geq N$ implies that $x_n \notin F$.

The proof of the following result is a slight modification of what appears in [33], regarding the construction of $(\bar{X}, \bar{s})$.

Proposition 4.2.1 [32] *A non-convergent sequence in* $(X, s) \in \mathcal{S}$ *converges in* $(\bar{X}, \bar{s}) \in \mathcal{S}$ *that contains* (X, s) *if and only if it has property* $\mathcal{F}$.

Proof. Let $\{x_n\}$ be a non-convergent sequence in $(X, s) \in \mathcal{S}$. Suppose that $\lim_n \bar{s}(x_n, \bar{x})$ $= 0$, where $\bar{x}$ belongs to $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s). Let $\mathcal{F}$ be a finite subset of X and fix $a \in \mathcal{F}$. Then a cannot appear as x_n for infinitely many indices n, otherwise there would be a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = a$ for all k. Now $\lim_k \bar{s}(x_{n_k}, \bar{x}) = 0$ implies that $\bar{s}(a, \bar{x}) = 0$, hence $\bar{x} = a$. It follows that $\{x_n\}$ converges in X, a contradiction. Conversely, let $\{x_n\}$ be a non-convergent sequence in X with property $\mathcal{F}$. Let $p = \{x_n\}$ and $\bar{X} \cup \{p\}$. Define a function $\bar{s} : \bar{X} \times \bar{X} \to [0, \infty)$ by

$$\begin{aligned}
\bar{s}(x,y) &= s(x,y) \quad \text{if} \quad x,y \in X; \\
\bar{s}(x,p) &= 1 \quad \text{if} \quad x \notin \{x_n : n \in N\}; \\
\bar{s}(p,y) &= 1 \quad \text{if} \quad y \notin \{x_n : n \in N\}; \\
\bar{s}(p,p) &= 0; \\
\bar{s}(x,p) &= \frac{1}{N} \quad \text{if} \quad x \neq x_n, \quad \text{for all} \quad n \geq N \quad \text{and} \quad N \\
&\quad \text{is the smallest integer with this property;} \\
\bar{s}(p,y) &= \frac{1}{N} \quad \text{if} \quad y \neq x_n, \quad \text{for all} \quad n \geq N \quad \text{and} \quad N \\
&\quad \text{is the smallest integer with this property.}
\end{aligned}$$

It easily follows that $(\bar{X}, \bar{s}) \in \mathcal{S}$ and contains (X, s). We will show that $\lim_n \bar{s}(x_n, p) = 0$. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \epsilon$. Fix N, then $\{x_1, x_2, ..., x_N\}$ is finite. Now there is $K \in \mathbb{N}$ such that $k > N$ and $k \geq K$ implies that $x_k \notin \{x_1, x_2, ..., x_N\}$, so, for $n \geq K$ we have $\bar{s}(x_n, p) = \frac{1}{k} < \frac{1}{N} < \epsilon$. □

Lemma 4.2.1 *Let (X, s) be a symmetric space, $\{x_n\}$ be a Cauchy sequence in (X, s) which is not eventually constant. Then $\{x_n\}$ has property $\mathcal{F}$.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in a symmetric space (X, s) which is not eventually constant. Assume that $\{x_n\}$ does not have property $\mathcal{F}$. That is, there exists a finite subset F of X such that $x_n \in F$ for all $n \in \mathbb{N}$. Let $\epsilon = \min\{s(x, y) : x, y \in F, x \neq y\}$. Find $N \in \mathbb{N}$ such that $s(x_n, x_m) < \epsilon$ for all $m, n \geq N$. Hence, $s(x_n, x_N) = 0$, for $n \geq N$. So, $x_n = x_N$ for every $n \geq N$. This shows that $\{x_n\}$ is eventually constant. A contradiction. It follows that $\{x_n\}$ has property $\mathcal{F}$. □

The proof of the next result is a slight modification of the result in [4].

Theorem 4.2.1 [4] *Let (X, s) be $\mathcal{S}$-convergence complete, then (X, s) is Cauchy complete.*

Proof. Suppose that (X, s) is $\mathcal{S}$-convergence complete but not Cauchy complete. Let $\{x_n\}$ be a Cauchy sequence in X. If $\{x_n\}$ is eventually constant, then there exists $x \in X$, such that $\lim_n s(x_n, x) = 0$. A contradiction. On the other hand assume that $\{x_n\}$ is not eventually constant. Then by Lemma 4.2.1, $\{x_n\}$ has property $\mathcal{F}$. By Proposition 4.2.1, $\{x_n\}$ converges to $\bar{x}$ with $\bar{x} \notin X$ and $(\bar{X}, \bar{s})$ contains (X, s). Note that (X, s) is $\mathcal{S}$-convergence complete. Hence, $\lim_n s(x_n, x) = 0$ for some $x \in X$, which is also a contradiction. Therefore (X, s) is Cauchy complete. □

Theorem 4.2.2 [4] *Let (X, s) be weakly $\mathcal{S}$-convergence complete. Then every Cauchy sequence in (X, s) has a cluster point.*

Proof. Let (X, s) be a weakly $\mathcal{S}$-convergence complete symmetric space and $\{x_n\}$ be a Cauchy sequence in X. If $\{x_n\}$ is eventually constant, our proof is complete. Assume that $\{x_n\}$ does not cluster in X. We may assume that $\{x_n\}$ is not eventually constant, then by Lemma 4.2.1, $\{x_n\}$ has property $\mathcal{F}$. Thus $\{x_n\}$ converges to $p \in (\bar{X}, \bar{s})$, where $p \notin X$ and $(\bar{X}, \bar{s})$ contains (X, s). Hence, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $x \in X$ such that $\lim_k s(x_{n_k}, x) = 0$. It follows that x is a cluster point of $\{x_n\}$, this leads to a contradiction. Hence, our proof is complete. □

In the following result we require both (X, s) and its extension $(\bar{X}, \bar{s})$ to belong to the class $\mathcal{U}$. We use the same construction as in Theorem 4.2.1 and Theorem 4.2.2 and the fact that a subsequence in the extension $(\bar{X}, \bar{s})$ can have at most one limit point.

Corollary 4.2.1 *Let (X, s) be weakly $\mathcal{U}$-convergence complete. Then (X, s) is Cauchy complete.*

Example 4.2.1 [32, Example 2.1] Let (X, s) be defined as in Example 4.1.3. We know that (X, s) is a complete metric space. Let $\{x_n = n, n \geq 1\}$ and $(\bar{X}, \bar{s})$ be defined as in Example 4.1.3. Then we have $\lim_n \bar{s}(x_n, \alpha) = 0$. Thus $\{x_n\}$ converges in $(\bar{X}, \bar{s})$ but not in (X, s). Therefore a complete metric space in the classical case is not necessarily $\mathcal{S}$-convergence complete.

It was observed in [32] and [33] that a sequence in a symmetric space (X, s) that converges in $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s) is not necessarily a Cauchy sequence. In fact, Example 4.1.3 is an example of a Cauchy complete metric space which is not $\mathcal{S}$-convergence complete, nor it is $\mathcal{U}$-convergence complete.

The next result discuss $\mathcal{S}$-convergence complete on subspaces of symmetric space (X, s).

Lemma 4.2.2 [32] *Let $(X, s) \in \mathcal{S}$ and A be a subspace of X. If $(\acute{A}, \acute{s}) \in \mathcal{S}$ contains A there exists $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s) and $(\acute{A}, \acute{s})$.*

Proof. Suppose that A is a subspace of $(X, s) \in \mathcal{S}$ and $(\acute{A}, \acute{s}) \in \mathcal{S}$ contains A. Without loss of generality, we may assume that $\acute{A} = A \cup \{\acute{a}\}$ and $\acute{a} \notin X$. Now let $\bar{X} = X \cup \{\acute{a}\}$. Define a function $\bar{s} : \bar{X} \times \bar{X} \to [0, \infty)$ by

$$\bar{s}(x, y) = \begin{cases} s(x, y) \quad \text{if} \quad x, y \in X \\ 0 \quad \text{if} \quad x = y = \acute{a} \\ \inf_{a \in A}\{\acute{s}(\acute{a}, a) + s(a, y)\} \quad \text{if} \quad x = \acute{a}, y \in X \end{cases}$$

for all $x, y \in \bar{X}$. Then $(\bar{X}, \bar{s}) \in \mathcal{S}$ and contains (X, s) as well as $(\acute{A}, \acute{s})$. □

As for metric spaces, $\mathcal{S}$-convergence complete spaces are closed hereditarily and closed under countable products.

Theorem 4.2.3 [32] *Let (X, s) be a symmetric space. If A is an $\mathcal{S}$-closed subset of $\mathcal{S}$-convergence complete (X, s), then A is $\mathcal{S}$-convergence complete.*

Proof. Let A be an $\mathcal{S}$-closed subset of $\mathcal{S}$-convergence complete (X, s). To show that A is $\mathcal{S}$-convergence complete, consider a sequence $\{a_n\}$ in A such that for $(\acute{A}, \acute{s}) \in \mathcal{S}$ that contains (A, s_A) we have $\lim_n \acute{s}(a_n, \acute{a}) = 0$, for some $\acute{a} \in (\acute{A}, \acute{s})$. We shall show that $\{a_n\}$ converges in A. Let $\bar{X} = X \cup \{\acute{a}\}$. By Lemma 4.2.2 we obtain $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s) and $(\acute{A}, \acute{s})$. Now $\lim_n \bar{s}(a_n, \acute{a}) = 0$, where $\bar{s}$ is constructed as in the proof of Lemma 4.2.2. It follows that $\lim_n s(a_n, x) = 0$ for some $x \in X$ since X is $\mathcal{S}$-convergence complete. Now A is sequentially closed in X, so, we get $x \in A$. □

The converse holds for the class of symmetric spaces where limits of sequences are unique. In fact,

Theorem 4.2.4 [32] *Let $(X, s) \in \mathcal{U}$. Then $\mathcal{U}$-convergence complete subspace A of X is sequentially closed.*

Hence, the following corollary follows.

Corollary 4.2.2 [32] *Let A be an $\mathcal{S}$-closed subset of a weakly $\mathcal{S}$-convergence complete $(X, s) \in \mathcal{S}$. Then A is weakly $\mathcal{S}$-convergence complete.*

The proof of the next result follows from [32] and Corollary 4.2.1 and it is therefore omitted.

Proposition 4.2.2 [4] *Let A be an $\mathcal{S}$-closed subset of an $\mathcal{S}$-convergence complete $(X, s) \in \mathcal{S}$. Then A is Cauchy complete.*

The result below concerns an $\mathcal{S}$-closed subset of a Cauchy complete space.

Proposition 4.2.3 [4] *Let A be an $\mathcal{S}$-closed subset of a Cauchy complete $(X, s) \in \mathcal{S}$. Then A is Cauchy complete.*

Definition 4.2.3 Two symmetrics s and $\tilde{s}$ on the same nonempty set X are said to be **equivalent** if for every sequence $\{x_n\}$ in X and a point x in X, then we have $\lim_n s(x_n, x) = 0$ if and only if $\lim_n \tilde{s}(x_n, x) = 0$.

Definition 4.2.4 [35] Let (X, s) be a symmetric space and A be a nonempty subset of X. Then A is **bounded** if there exists an $N > 0$ such that $s(x, y) \leq N$ for all $x, y \in A$. If A is bounded we define the **diameter** of A as $\delta(A) < \infty$, where $\delta(A) = \sup\{s(x, y) : x, y \in A\}$.

For every $(X, s) \in \mathcal{S}$, there exists $(X, \tilde{s}) \in \mathcal{S}$ such that:

(i) $(X, \tilde{s})$ is bounded.

(ii) $\tilde{s}$ is equivalent to s.

To see this, given $(X, s) \in \mathcal{S}$ define $\tilde{s}(x, y) = \min\{1, s(x, y)\}$, then $(X, \tilde{s})$ has the desired properties. We shall call $(X, \tilde{s})$ a **bounded equivalent** of (X, s).

Lemma 4.2.3 [4] *Let $(X, s) \in \mathcal{S}$ and $(X, \tilde{s})$ be its bounded equivalent. If $(X, s) \subset (\bar{X}, \bar{s})$, then there exists $(X^*, s^*) \in \mathcal{S}$ such that (X^*, s^*) is equivalent to $(\bar{X}, \bar{s})$ and contains $(X, \tilde{s})$.*

Proof. Let $(X, s) \in \mathcal{S}$ and $(X, \tilde{s})$ be its bounded equivalent. Suppose that $(X, s) \subset (\bar{X}, \bar{s})$ holds. We construct $(X^*, s^*) \in \mathcal{S}$ such that $(X, \tilde{s}) \subset (X^*, s^*)$. Let $X^* = \bar{X}$ and define $s^* : X^* \times X^* \to [0, \infty)$ by

$$s^*(x, y) = \begin{cases} \tilde{s}(x, y) \quad \text{if} \quad x, y \in X \\ 0 \quad \text{if} \quad x = y \\ \min\{1, \bar{s}(x, y)\} \quad \text{otherwise} \end{cases}$$

for all $x, y \in X^*$. Then easily $(X^*, s^*) \in \mathcal{S}$ and it is equivalent to $(X^*, \bar{s})$. Further, we have $(X, \tilde{s}) \subset (X^*, s^*)$. □

Lemma 4.2.4 [4] *Let $(X, s) \in \mathcal{S}$ and $(X, \tilde{s})$ be its bounded equivalent. If $(X, \tilde{s}) \subset (\bar{X}, \bar{s})$, then there exists $(X^*, s^*) \in \mathcal{S}$ such that (X^*, s^*) is equivalent to $(\bar{X}, \bar{s})$ and contains (X, s).*

Proof. A similar construction as in Lemma 4.2.3 will do. □

Using Lemma 4.2.3 and Lemma 4.2.4 the following two theorems follows.

Theorem 4.2.5 [4] *A symmetric space (X, s) is $\mathcal{S}$-convergence complete if and only if $(X, \tilde{s})$ is $\mathcal{S}$-convergence complete.*

Theorem 4.2.6 [4] *A symmetric space (X, s) is weakly $\mathcal{S}$-convergence complete if and only if $(X, \tilde{s})$ is weakly $\mathcal{S}$-convergence complete.*

Proposition 4.2.4 [4] *Let (X, s) be $\mathcal{S}$-convergence complete. Then $(X, \tilde{s})$ is weakly $\mathcal{S}$-convergence complete.*

Proof. Let (X, s) be $\mathcal{S}$-convergence complete and $(X, \tilde{s})$ be its bounded equivalent. Since $\mathcal{S}$-convergence complete implies weakly $\mathcal{S}$-convergence complete, it follows that (X, s) is weakly $\mathcal{S}$-convergence complete. By Theorem 4.2.6, $(X, \tilde{s})$ is weakly $\mathcal{S}$-convergence complete. □

In fact, the following results hold.

Proposition 4.2.5 [4] *A symmetric space (X, s) is Cauchy complete if and only if $(X, \tilde{s})$ is Cauchy complete.*

Proposition 4.2.6 [4] *Let (X, s) be $\mathcal{S}$-convergence complete. Then $(X, \tilde{s})$ is Cauchy complete.*

4.3 Products of symmetric spaces

In this section we present products of symmetric spaces. Note that Lemma 4.2.2 is crucial for the proof of the following results on product of $\mathcal{S}$-convergence complete symmetric spaces.

Theorem 4.3.1 [34] *Let $\{(X_i, s_i) : i = 1, 2, 3, ...\}$ be a collection of symmetric spaces and suppose that $X = \Pi_i^\infty X_i$ is $\mathcal{S}$-convergence complete with X equipped with $s : X \times X \to [0, \infty)$ defined by $s(x, y) = \Sigma_{i=1}^\infty \frac{\min\{1, s_i(x_i, y_i)\}}{2^i}$ for all $x, y \in X$. Then (X_i, s_i) is $\mathcal{S}$-convergence complete for each $i = 1, 2, 3, ...$*

Proof. Let $X = \Pi_i^\infty X_i$ and $s : X \times X \to [0, \infty)$ be defined by $s(x, y) = \Sigma_{i=1}^\infty \frac{\min\{1, s_i(x_i, y_i)\}}{2^i}$ for all $x, y \in X$. Then s is a symmetric on X. Suppose that $(\bar{X}, \bar{s}) \in \mathcal{S}$ contains (X, s) and that a sequence $\{x_n = (x_n^1, x_n^2, ...)\}$ in X converges in $(\bar{X}, \bar{s})$ to some $\alpha \in \bar{X}$ with respect to $\bar{s}$. Let $\tilde{X}_i = X_i \cup \{\alpha\}$ for each $i = 1, 2, ...$ and define a function $\tilde{s}_i : \tilde{X}_i \times \tilde{X}_i \to [0, \infty)$ by

$$\tilde{s}_i(x, y) = \begin{cases} 0 \quad \text{if} \quad x = y = \alpha \\ s_i(x, y) \quad \text{if} \quad x, y \in X_i \\ \inf_{a=(a^1, a^2) \in X}\{\bar{s}(\alpha, a) + s_i(a^i, y)\} \quad \text{if} \quad y \in X_i \end{cases}$$

for all $x, y \in \tilde{X}_i$. Then for each $i = 1, 2, ...,$ $(\tilde{X}_i, \tilde{s}_i) \in \mathcal{S}$ and contains (X_i, s_i). Now

$$\tilde{s}_i(\alpha, x_n^i) \leq \bar{s}(\alpha, a) + s_i(a^i, x_n^i)$$

for all $a \in X$. In particular, let $a = \{x_n\}$. Then

$$\tilde{s}_i(\alpha, x_n^i) \leq \bar{s}(\alpha, x_n) + s_i(x_n^i, x_n^i)$$

and this gives $\lim_n \tilde{s}_i(\alpha, x_n^i) = 0$. So, for each $i = 1, 2, ...$ a sequence $\{x_n^i\}$ in (X_i, s_i) converges to α. Hence, by assumption, $\{x_n^i\}$ converges to some point y_i in (X_i, s_i). Let $y = (y_1, y_2, ...)$. Then $y \in X$ and by the above construction $\lim_n s(x_n, y) = 0$. Conversely, let $X = \prod X_i$. The product symmetric is a function $s : X \times X \to [0, \infty)$ defined by

$$s(x, y) = \Sigma_{i=1}^{\infty} \frac{\min\{1, s_i(x_i, y_i)\}}{2^i}$$

where $x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in X$. Suppose that for all $i = 1, 2, ...$ a sequence $\{x_n^i\}$ in (X_i, s_i) converges to α_i in $(\bar{X}_i, \bar{s}_i) \in \mathcal{S}$ that contains (X_i, s_i). We show that $\{x_n^i\}$ converges in (X_i, s_i) for each $i = 1, 2, ...$ Without loss of generality, assume that $\bar{X}_i = X_i \cup \{\alpha_i\}$ for all $i = 1, 2, ...$ Let $\bar{X} = \prod \bar{X}_i$ and define a function $\bar{s} : \bar{X} \times \bar{X} \to [0, \infty)$ by

$$\bar{s}(x, y) = \Sigma_{i=1}^{\infty} \frac{\min\{1, \bar{s}_i(x_i, y_i)\}}{2^i}$$

for all $x, y \in \bar{X}$. Then $(\bar{X}, \bar{s}) \in \mathcal{S}$ and contains (X, s). Now consider a sequence $\{x_n = (x_n^1, x_n^2, ...)\}$ in X. It can be checked that $\lim_n \bar{s}(x_n, \alpha) = 0$, where $\alpha = (\alpha_1, \alpha_2, ...) \in \bar{X}$. Therefore there exists some point $\beta \in X$ such that $\lim_n s(x_n, \beta)$. Let $\beta = (\beta_1, \beta_2, ...)$. Then it follows that $\lim_n s_i(x_n^i, \beta_i) = 0$. □

Theorem 4.3.2 [4] *Let $\{(X_i, s_i) : i = 1, 2, 3, ...\}$ be a collection of symmetric spaces and suppose that $X = \Pi_i^{\infty} X_i$ is weakly $\mathcal{S}$-convergence complete with X equipped with $s : X \times X \to [0, \infty)$ defined by $s(x, y) = \Sigma_{i=1}^{\infty} \frac{\min\{1, s_i(x_i, y_i)\}}{2^i}$ for all $x, y \in X$. Then (X_i, s_i) is weakly $\mathcal{S}$-convergence complete for each $i = 1, 2, 3, ...$*

Proof. Suppose that (X, s) is weakly $\mathcal{S}$-convergence complete with the symmetric $s : X \times X \to [0, \infty)$ defined by $s(x, y) = \Sigma_{i=1}^{\infty} \frac{\min\{1, s_i(x_i, y_i)\}}{2^i}$ for all $x, y \in X$. We want to show that (X_i, s_i) is weakly $\mathcal{S}$-convergence complete for $i = 1, 2, ...$ For a sequence $\{x_n\}$ in (X, s), note that $\{x_n = (x_n^1, x_n^2, ...)\}$. Fix $i \in \{1, 2, ..., \}$ arbitrarily and assume that a sequence $\{x_n^i\}$ in (X_i, s_i) converges to α in $(\bar{X}_i, \bar{s}_i)$, with respect to $\bar{s}_i$, where $(\bar{X}_i, \bar{s}_i)$ contains (X_i, s_i). Without loss of generality we put $\bar{X}_i = X_i \cup \{\alpha\}$, and $\bar{X}_j = X_j$, for $i \neq j$. Next we let $\bar{X} = \Pi_j \bar{X}_j$. Define $\bar{s} : \bar{X} \times \bar{X} \to [0, \infty)$ by

$$\bar{s}(\bar{x}, \bar{y}) = \begin{cases} \Sigma_{j=1}^{\infty} \frac{\min\{1, s_j(x_j, y_j)\}}{2^j} \quad \text{if} \quad \bar{x}, \bar{y} \in X, \\ 0 \quad \text{if} \quad \bar{x} = \bar{y}, \\ \bar{s}_i(x^i, \alpha) \quad \text{if} \quad \bar{x} \in X \quad \text{and} \\ \bar{y} \in \bar{X}, \text{that is,} \quad \bar{y} = (y^1, y^2, ...), \quad \text{with} \quad y^i = \alpha, \\ \bar{s}_i(\alpha, y^i) \quad \text{if} \quad \bar{x} \in \bar{X}, \quad \text{that is,} \\ \quad \bar{x} = (x^1, x^2, ...) \quad \text{with} \quad x^i = \alpha \quad \text{and} \quad \bar{y} \in X \end{cases}$$

for all $x, y \in \bar{X}$. Clearly, $(X, s) \subset (\bar{X}, \bar{s}) \in \mathcal{S}$. Define $\varphi_j : \Pi_j \bar{X}_j \to \bar{X}_j$ by $\varphi_j(x = (x^1, x^2, ...,)) = x^j$ for $(x^1, x^2, ...) \in \Pi_j \bar{X}_j$ and $x^j \in X_j$ for all $j = 1, 2, ...,$ otherwise we put $\varphi_j(x) = \alpha$, if $j = i$, and $x^i = \alpha$. For each $j \in \{1, 2, ...\}$ and a sequence $\{x_n^j : n \geq 1\}$ in X_j, define $\{x_n\}$ in X by $x_n = \varphi^{-1}(x_n^j), n \geq 1$. In particular, for a fixed $i \in \{1, 2, ...\}$ and $\{x_n^i\}$ we find a sequence $\{x_n\}$ in X such that $x_n = \varphi^{-1}(x_n^i), n \geq 1$. Now we have $\lim_n \bar{s}(x_n, \alpha) = \lim_n \bar{s}_i(x_n^i, \alpha) = 0$. Since (X, s) is weakly $\mathcal{S}$-convergence complete, we can find $x \in X$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_k s(x_{n_k}, x) = 0$. In particular, let $x = (x^1, x^2, ...) \in X$ and $x_{n_k} = (x_{n_k}^1, x_{n_k}^2, ...)$, where $\{x_{n_k}^i\}$ is a subsequence of $\{x_n^i\}$. Then $\lim_k s_i(x_{n_k}^i, x^i) = 0$, it follows that (X_i, s_i) is weakly $\mathcal{S}$-convergence complete. Since the fixed i was arbitrary, the proof is complete. □

Our next result characterizes weakly $\mathcal{S}$-convergence completeness, albeit in a special case, namely, in the product of a finite sequence of symmetric spaces.

Theorem 4.3.3 [4] *Let $\{(X_i, s_i) : i = 1, 2, 3, ..., n\}$ be a finite collection of symmetric spaces. Then $X = \Pi_i^n X_i$ is weakly $\mathcal{S}$-convergence complete if and only if (X_i, s_i) is weakly $\mathcal{S}$-convergence complete for each $i = 1, 2, 3, ..., n$, where X is equipped with the function $s : X \times X \to [0, \infty)$ defined by*

$$s(x, y) = \max_{1 \leq i \leq n} \{s_i(x_i, y_i)\}$$

for all $x, y \in X$.

Proof. Let $\{(X_i, s_i) : i = 1, 2, 3, ..., n\}$ be a collection of finite symmetric spaces and suppose that $X = \Pi_i^n X_i$ with the symmetric $s : X \times X \to [0, \infty)$ defined by $s(x, y) = \max\{s_i(x_i, y_i) : i = 1, 2, ..., n.\}$ for all $x, y \in X$, is weakly $\mathcal{S}$-convergence complete. Emulating the proof of Theorem 4.3.2 we can show that (X_i, s_i) is weakly $\mathcal{S}$-convergence complete for each $i = 1, 2, 3, ..., n$. Conversely, suppose that (X_i, s_i) is weakly $\mathcal{S}$-convergence complete for all $i = 1, 2, ..., n$. We want to show that (X, s) is weakly $\mathcal{S}$-convergence complete. We shall provide proof for $i = 1, 2$ and by induction our proof shall be complete. We use a technique in [34], see pages 101-102 to construct an extension of (X, s). Consider an arbitrary sequence $\{x_n = (x_n^1, x_n^2)\}$ in (X, s) that converges to α in $(\bar{X}, \bar{s}) \in \mathcal{S}$ that contains (X, s) with respect to $\bar{s}$. Let $\tilde{X}_i = X_i \cup \{\alpha\}$ for each $i = 1, 2$ and the function $\tilde{s}_i : \tilde{X}_i \times \tilde{X}_i \to [0, \infty)$ for $i = 1, 2$ be defined by

$$\tilde{s}_i(x, y) = \begin{cases} 0 \quad \text{if} \quad x = y = \alpha \\ s_i(x, y) \quad \text{if} \quad x, y \in X_i \\ \inf_{a=(a^1,a^2)\in X}\{\bar{s}(\alpha, a) + s_i(a^i, y)\} \quad \text{if} \quad y \in X_i \end{cases}$$

for all $x, y \in \tilde{X}_i$. Then $(\tilde{X}_i, \tilde{s}_i) \in \mathcal{S}$ and contains (X_i, s_i) for each $i = 1, 2$. In particular,

$$\tilde{s}_i(\alpha, x_n^i) \leq \bar{s}(\alpha, a) + s_i(a^i, x_n^i)$$

for all $a \in X$. Let $a = \{x_n\}$, then

$$\tilde{s}_i(\alpha, x_n^i) \leq \bar{s}(\alpha, x_n) + s_i(x_n^i, x_n^i).$$

This gives $\lim_n \tilde{s}_i(x_n^i, \alpha) = 0$ for $i = 1, 2$. Since (X_i, s_i) is weakly $\mathcal{S}$-convergence complete we find a sequence of positive integers $n_1 < n_2 < ...$ such that $\{x_{n_k}^1\}$ is a subsequence of $\{x_n^1\}$ in X_1 and $\lim_k s_1(x_{n_k}^1, x_1) = 0$ for some $x_1 \in X_1$. Also, (X_2, s_2) is weakly $\mathcal{S}$-convergence complete, from the sequence of positive integers $n_1 < n_2 < ...$ we can find a sequence of positive integers $n_{k_1} < n_{k_2} < ...$ such that $\{x_{n_{k_j}}^2\}$ is a subsequence of $\{x_n^2\}$ and $\lim_j s_2(x_{n_{k_j}}^2, x_2) = 0$ for some $x_2 \in X_2$. Note that $\{x_{n_{k_j}}^1\}$ is a subsequence of $\{x_n^1\}$ and $\lim_j s_1(x_{n_{k_j}}^1, x_1) = 0$. Now let $x = (x_1, x_2) \in X$. Then $\{x_{n_{k_j}}\} = (x_{n_{k_j}}^1, x_{n_{k_j}}^2)$ is a subsequence of $\{x_n\}$ and $\lim_j s(x_{n_{k_j}}, x) = 0$. This shows that (X, s) is weakly $\mathcal{S}$-convergence complete. □

4.4 Symmetric spaces and fixed point results

Definition 4.4.1 Let (X, s) be a symmetric space. A mapping $T : (X, s) \to (X, s)$ is called a **contraction mapping** on X if there exists a constant $\lambda \in (0, 1)$ such that $s(T(x), T(y)) \leq \lambda s(x, y)$ for all $x, y \in X$.

We shall say that a map $T : (X, s_X) \to (Y, s_Y)$ between symmetric spaces is **continuous**, if for every sequence $\{x_n\}$ in X, $\lim_n s_X(x_n, x) = 0$ implies that $\lim_n s_Y(T(x_n), T(x)) = 0$.

Definition 4.4.2 [33] Let (X, s) be a symmetric space and $T : (X, s) \to (X, s)$ be a self map. Then a point $x \in X$ such that $T(x) = x$ is called a **fixed point** of T.

Let $(X, s) \in \mathcal{S}$ and denote by $\mathcal{C}$ the family of all non-empty and $\mathcal{S}$-closed subsets in X.

Definition 4.4.3 [17] Let (X, s) be a symmetric space, $\{x_n\}, \{y_n\}, x$ and $y \in X$. Then

$(W.3)$ $\lim_n s(x_n, x) = 0$ and $\lim_n s(x_n, y) = 0$ imply that $x = y$.

$(W.4)$ $\lim_n s(x_n, x) = 0$ and $\lim_n s(x_n, y_n) = 0$ imply that $\lim_n s(y_n, x) = 0$.

Definition 4.4.4 [21] Two mappings $T : (X, s) \to (X, s)$ and $g : (X, s) \to (X, s)$ of a symmetric space (X, s) are said to be **commuting** if $T(g(x)) = g(T(x))$ for all $x \in X$.

It is important to note that the uniqueness condition denoted by $(W.3)$ was previously denoted as (X, s) belongs to the class $\mathcal{U}$.

Theorem 4.4.1 [17] *Let (X, s) be a bounded and Cauchy complete symmetric space satisfying $(W.3)$ and $T : (X, s) \to (X, s)$ be a continuous map. Then T has a fixed point if and only if there exists a $\lambda \in (0, 1)$ and a continuous function $g : (X, s) \to (X, s)$ which commutes with T and satisfies*

$$g(X) \subset T(X) \quad \textit{and} \quad s(gx, gy) \leq \lambda s(Tx, Ty) \quad \textit{for all} \quad x, y \in X \qquad (1)$$

Therefore T and g has a unique common fixed point if (1) *holds.*

Proof. Suppose that $T(a) = a$ for some $a \in X$. Let $g(x) = a$ for all $x \in X$. Then $g(T(x)) = a$ and $T(g(x)) = T(a) = a$ for $x \in X$, so, $g(T(x)) = T(g(x))$ for all $x \in X$ and g commute with T. Moreover, $g(x) = a = T(a)$ for all $x \in X$, so, that $g(X) \subset T(X)$. Finally, for any $\lambda \in (0, 1)$ we have for all $x, y \in X$:

$$s(gx, gy) = s(a, a) = 0 \leq \lambda s(Tx, Ty).$$

Thus (1) holds. Suppose that there exists λ and g, so, that (1) holds. Let $M = \sup\{s(x, y) : x, y \in X\}$ and $x_0 \in X$. Choose x_1 such that $g(x_0) = T(x_1)$. In general choose x_n such that $T(x_n) = g(x_{n-1})$. We show that $s(T(x_n), T(x_{n+m})) \leq \lambda^n M$. Now

$$\begin{aligned} s(T(x_n), T(x_{n+m})) &= s(g(x_{n-1}), g(x_{n+m-1})) \\ &\leq \lambda^1 s(T(x_{n-1}), T(x_{n+m-1})) \\ &\leq ... \leq \lambda^n s(T(x_0), T(x_m)) \\ &< \lambda^n M. \end{aligned}$$

Clearly, $\{T(x_n)\}$ is a Cauchy sequence and the Cauchy completeness of (X, s) gives an $x \in X$ with $\lim_n s(T(x_n), x) = 0$. Note that g is continuous implies $\lim_n s(g(T(x_n)), T(x))$ $= 0$. Now $T(x_n) = g(x_{n-1})$ such that $\lim_n s(g((x_n), x)) = 0$. T is continuous gives $\lim_n s(T(g(x_n)), T(x)) = 0$. Since

$$Tg = gT, \quad \text{and} \quad \lim_n s(T(g(x_n)), T(x)) = \lim_n s(T(g(x_n)), g(x)) = 0.$$

By $(W.3)$ we obtain $T(x) = g(x)$. Also $T(g(x)) = g(T(x))$. Thus $T(T(x)) = T(g(x)) = g(T(x)) = g(g(x))$ and $s(g(x), g(g(x))) \leq \lambda s(T(x), T(g(x))) = \lambda s(g(x), g(g(x)))$ implies that $g(x) = g(g(x))$. Hence, $g(x) = g(g(x)) = T(g(x))$, so, $g(x)$ is a common fixed point of T and g. If $x = T(x) = g(x)$ and $y = T(y) = g(y)$, then (1) gives $s(x, y) = s(g(x), g(y)) \leq \lambda s(T(x), T(y)) = \lambda s(x, y)$ or $x = y$. □

Theorem 4.4.2 [33] *Let $(X, s) \in \mathcal{U}$ be bounded and $\mathcal{S}$-convergence complete and $T : (X, s) \to (X, s)$ be a continuous map. Then T has a fixed point if and only* *if there exists a* $\lambda \in (0,1)$ *and a continuous function* $g : (X,s) \to (X,s)$ *which commutes with* T *and satisfies*

$$g(X) \subset T(X) \quad \textit{and} \quad s(gx,gy) \leq \lambda s(Tx,Ty) \quad \textit{for all} \quad x,y \in X \tag{1}$$

Therefore T *and* g *has a unique common fixed point if* (1) *holds.*

Proof. Suppose that $T(a) = a$ for some $a \in X$. Let $g(x) = a$ for all $x \in X$. Then $g(T(x)) = a$ and $T(g(x)) = T(a) = a$ for $x \in X$, so, $g(T(x)) = T(g(x))$ for all $x \in X$ and g commute with T. Moreover, $g(x) = a = T(a)$ for all $x \in X$, so, that $g(X) \subset T(X)$. Finally, for any $\lambda \in (0,1)$ we have for all $x,y \in X$:

$$s(gx,gy) = s(a,a) = 0 \leq \lambda s(Tx,Ty).$$

Thus (1) holds.

Conversely, suppose that there exists $\lambda \in (0,1)$ and a continuous function $g : (X,s) \to (X,s)$, so, that (1) holds. Now since (X,s) is bounded, put $M = \sup\{s(x,y) : x,y \in X\}$. Let $x_0 \in X$. Choose x_1 such that $g(x_0) = T(x_1)$. In general choose x_n such that $T(x_n) = g(x_{n-1})$. It follows that $s(T(x_n),T(x_{n+m})) = s(g(x_{n-1}),g(x_{n+m-1}))$ and $s(g(x_{n-1}),g(x_{n+m-1})) \leq \lambda s(T(x_{n-1}),T(x_{n+m-1}))$. Similarly, we have

$$s(T(x_{n-1}),T(x_{n+m-1})) \leq \lambda^2 s(T(x_{n-2}),T(x_{n+m-2})) \leq ...,$$

hence, $s(T(x_n),T(x_{n+m})) \leq \lambda^n s(T(x_0),T(x_m)) \leq \lambda^n M$. We shall now show that the sequence $\{T(x_n)\}$ converges to a point in a symmetric space $(\bar{X},\bar{s})$ that contains (X,s). Note that $\lim_{m,n} s(T(x_m),T(x_n)) = 0$. If $\{T(x_n)\}$ is eventually constant, then we are done. Assume, that the sequence $\{T(x_n)\}$ is not eventually constant. Then by Lemma 4.2.1, $\{T(x_n)\}$ has property $\mathcal{F}$, hence, the sequence $\{T(x_n)\}$ converges to a point in $(\bar{X},\bar{s}) \in \mathcal{S}$ and $(\bar{X},\bar{s})$ contain (X,s).

Since (X,s) is $\mathcal{S}$-convergence complete, there exists $a \in X$ such that $\lim_n s(T((x_n),a))$ $= 0$. By continuity of g we get $\lim_n s(g(T(x_n)),g(a)) = 0$. Now $T(x_n) = g(x_{n-1})$, so, that $\lim_n s(g(x_n),a) = 0$ and continuity of T gives $\lim_n s(T(g(x_n)),T(a)) = 0$. Since $Tg = gT$, we have $\lim_n s(T(g(x_n)),T(a)) = 0 = \lim_n s(T(g(x_n)),g(a))$, hence,

$T(a) = g(a)$, as $(X, s) \in \mathcal{U}$. Also, $T(g(a)) = g(T(a))$. Thus $T(T(a)) = T(g(a)) = g(T(a)) = g(g(a))$ and

$$s(g(a), g(g(a))) \leq \lambda s(T(a), T(g(a))) = \lambda s(g(a), g(g(a)))$$

implies that $g(a) = g(g(a))$, hence, $g(a) = g(g(a)) = T(g(a))$, so, that $g(a)$ is a common fixed point of T and g. So, if $a = T(a) = g(a)$ and $b = T(b) = g(b)$, then we get $s(a, b) = s(g(a), g(b)) \leq \lambda s(T(a), T(b)) = \lambda s(a, b)$ which implies $s(a, b) \leq 0$, since $(1-\lambda) > 0$. Therefore $s(a, b) = 0$ thus, $a = b$. □

We provide an alternative prove of Theorem 4.4.2. Restrict $\mathcal{S}$ to $\mathcal{U}$.

Theorem 4.4.3 *Let $(X, s) \in \mathcal{U}$ be bounded and $\mathcal{S}$-convergence complete and $T : (X, s) \to (X, s)$ be a continuous map. Then T has a fixed point if and only if there exists a $\lambda \in (0, 1)$ and a continuous function $g : (X, s) \to (X, s)$ which commutes with T and satisfies*

$$g(X) \subset T(X) \quad \textit{and} \quad s(gx, gy) \leq \lambda s(Tx, Ty) \quad \textit{for all} \quad x, y \in X \tag{1}$$

Therefore T and g has a unique common fixed point if (1) holds.

Proof. Let $(X, s) \in \mathcal{U}$ be bounded and $\mathcal{S}$-convergence complete. Then (X, s) is a bounded Cauchy complete symmetric space by Theorem 4.2.1. The proof is complete by Theorem 4.4.1. □

Next is a generalization of the Hausdorff distance to the setting of symmetric spaces.

Definition 4.4.5 [35] Let A and B be two non empty elements of $\mathcal{C}$ in a bounded symmetric space (X, s). We define their **Hausdorff distance** $s_H(A, B)$ by

$$s_H(A, B) = \max\{\sup_{a\in A} s(a, B), \sup_{b\in B} s(A, b)\}.$$

Lemma 4.4.1 [35] *Let (X, s) be a bounded symmetric space. Suppose that $A, B \in (\mathcal{C}, s_H)$ and $\lambda > 1$. For each $a \in A$, there exists $b \in B$ such that $s(a, b) \leq \lambda s_H(A, B)$.*

Theorem 4.4.4 [35] *Let (X, s) be bounded and Cauchy complete symmetric space satisfying $(W.4)$. Suppose that $T : (X, s) \to (\mathcal{C}, s_H)$ is a multivalued map satisfying*

$$s_H(Tx, Ty) \leq ks(x, y), \quad k \in (0, 1) \quad \text{for all} \quad x, y \in X. \tag{1}$$

Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $x_1 \in X$ and $\lambda \in (k, 1)$. Since Tx_1 is nonempty, there exists $x_2 \in Tx_1$ such that $s(x_1, x_2) > 0$ (if not, then x_1 is a fixed point of T). In view of (1), we have:

$$s(x_2, Tx_2) \leq s_H(Tx_1, Tx_2) \leq ks(x_1, x_2) < \lambda s(x_1, x_2)$$

using $s(x_2, Tx_2) = \inf\{s(x_2, b) : b \in Tx_2\}$, it follows that there exists $x_3 \in Tx_2$ such that

$$s(x_2, x_3) < \lambda s(x_1, x_2).$$

Similarly, there exists $x_4 \in Tx_3$ such that $s(x_3, x_4) < \lambda s(x_2, x_3)$.

Continuing in this fashion, there exists a sequence $\{x_n\}$ in X satisfying $x_{n+1} \in Tx_n$ and

$$s(x_n, x_{n+1}) < \lambda s(x_{n-1}, x_n).$$

We claim that $\{x_n\}$ is a Cauchy sequence. Indeed, we have

$$\begin{aligned} s(x_n, x_{n+m}) &< \lambda s(x_{n-1}, x_{n+m-1}) \\ &< \lambda^2 s(x_{n-2}, x_{n+m-2}) \\ &< \ldots < \lambda^{n-1} s(x_1, x_{m+1}) \\ &< \lambda^{n-1} \delta_s(X). \end{aligned}$$

So, $\{x_n\}$ is a Cauchy sequence. Hence, $\lim_n s(u, x_n) = 0$ for some $u \in X$. Now we are able to show that $u \in Tu$. Let $k\epsilon < 1$. From Lemma 4.4.1, for each $n \in \{1, 2, \ldots\}$ there exists $y_n \in Tu$ such that

$$s(x_{n+1}, y_n) \leq \epsilon s_H(Tx_n, Tu) \leq \epsilon ks(x_n, u), n = 1, 2, \ldots$$

which implies that $\lim_n s(x_{n+1}, y_n) = 0$. In view of $(W.4)$, we have $\lim_n s(y_n, u) = 0$ therefore $u \in \bar{T}u = Tu$. □

We conclude this section with the following result and we restrict $\mathcal{S}$ to $\mathcal{U}$. Furthermore, let $(X, s) \in \mathcal{U}$. We shall require the $(W.4)$ property.

Theorem 4.4.5 [4] *Let $(X, s) \in \mathcal{S}$ be bounded and $\mathcal{S}$-convergence complete symmetric space. Suppose that $T : (X, s) \to (\mathcal{C}, s_H)$ satisfies*

$$s_H(Tx, Ty) \le \lambda s(x, y), \quad \lambda \in (0, 1) \quad \textit{for all} \quad x, y \in X. \tag{2}$$

Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $(X, s) \in \mathcal{S}$ be bounded and $\mathcal{S}$-convergence complete symmetric space. Then (X, s) is a bounded Cauchy complete symmetric space by Theorem 4.2.1. The proof is complete by Theorem 4.4.4. □

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