Mathematical Symbolisation: Challenges and Instructional Strategies for Limpopo Province Secondary School learners

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LIST OF ACRONYMS

ACME	Advisory Committee on Mathematics Education
ANA	Annual National Assessments
APOS	Activity-Process-Object-Schema
CAPS	Curriculum Assessment Policy Statement
DBE	Department of Basic Education
DoE	Department of Education
FET	Further Education and Training
IESS	International Encyclopaedia of Social Sciences
MMR	Mixed Methods Research
NCS	National Curriculum Statement
PISA	Programme for International Student Assessment
SACMEQ	Southern Africa Consortium for Monitoring Education Quality
TIMSS-RS	Trends in International Mathematics and Science Study- Repeat Survey

LIST OF TEXTBOXES

DECLARATION

I, the undersigned, hereby declare that the work in this thesis titled "***Mathematical Symbolisation: Challenges and Instructional Strategies for Limpopo Province Secondary School learners***" is my own original work and that I have not previously submitted it in its entirety or in part to any other university for purposes of obtaining a degree. I also declare that that all the sources I quoted were acknowledged by means of complete references.

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DEDICATION

I dedicate this thesis work to my family, friends and colleagues. Special gratitude goes to my loving parents Mr and Mrs Mutodi whose words of encouragement and motivation continued ringing in my ears during the four years of my study. My brothers Peter, Johnson, Tinashe, Rodgers, Odious, Ishmael and Elson missed me but never abandoned me and are very special.

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ABSTRACT

This study reports on an investigation into the manner in which mathematical symbols influence learners' understanding of mathematical concepts. The study was conducted in Greater Sekhukhune and Capricorn districts of Limpopo Province, South Africa. Multistage sampling (for the district), simple random sampling (for the schools), purposive sampling (for the teachers) and stratified random sampling with proportional allocation (for the learners) were used. The study was conducted in six schools randomly selected from rural, semi-urban and urban settings. A sample of 565 FET learners and 15 FET band mathematics teachers participated in the study. This study is guided by four interrelated constructivist theories: symbol sense, algebraic insight, APOS and procept theories. The research instruments for the study consist of questionnaires and interviews. A mixed method approach that was predominantly qualitative was employed. An analysis of learners' difficulties with mathematical symbols produced three (3) clusters. The main cluster consists of 236 (41.6%) learners who indicate that they experience severe challenges with mathematical symbols compared to 108 (19.1%) learners who indicated that they could confidently handle and manipulate mathematical symbols with understanding. Six (6) categories of challenges with mathematical symbols emerged from learners' encounters with mathematical symbols: reading mathematical text and symbols, prior knowledge, time allocated for mathematical classes and activities, lack of symbol sense and problem contexts and pedagogical approaches to mathematical symbolisation. Two sets of theme classes related to learners' difficulties with mathematical symbols and instructional strategies emerged. Learners lack symbol sense for mathematical concepts and algebraic insight for problem solving. Learners stick to procedurally driven symbols at the expense of conceptual and contextual understanding. From a pedagogical perspective teachers indicated that they face the following difficulties when teaching: the challenge of introducing unfamiliar notation in a new topic; reading, writing and verbalising symbols; signifier and signified connections; and teaching both symbolisation and conceptual understanding simultaneously. The study recommends teachers to use strategies such as informed choice of subject matter and a pedagogical approach in which concepts are understood before they are symbolised.

CHAPTER 1: INTRODUCTION

This chapter introduces the study. The background of the study together with an explanation of the context and focus of the study are discussed. The chapter also discusses the problem statement and research questions. The purpose, significance of the study and its limitations and delimitations are also discussed. Research assumptions are described and the researcher’s position is clarified. The chapter concludes by defining the key terms of the study and a description of the organisation as well as the contents of study chapters.

1.1 Introduction

The challenges and difficulties associated with the teaching and learning of mathematics are multidimensional. One of the obstacles envisaged in this study is the use of mathematical symbols. Research on learners’ understanding of mathematical symbols at secondary level reveals that the conciseness and abstract nature of symbols can be a barrier to learning (Adu & Olaoye, 2014). Symbols form the foundation of mathematical communication. However, the increase in symbol load due to unfamiliarity and increased density may cause learners to lose confidence and develop negative conceptions about mathematics (Bardinia & Pierce, 2015). Many mathematical symbols and notations are figured routinely by learners as they learn mathematics in classroom contexts. Mathematical symbols obscure learners from understanding mathematical concepts and sometimes lead to misunderstandings (Buhari, 2012).

The main distinguishing feature of mathematics is the property of having an extensive symbol system. Mathematics is abstract and “pure” and its subject matter is cognitive (Hegel, 2010). Abstract symbols reside within a complex system of rules and internal relationships that make it possible to both communicate and generate powerful mathematical ideas (Drouhard & Teppo, 2004). Knowledge of mathematics symbols is important for understanding mathematical concepts. Learners need to acquire the ability to use mathematical symbols and representational forms in ways that represent their use across the mathematical communities (Jao, 2012). The use of mathematical symbols is to represent relations, patterns, expressions, formulas, diagrams, drawings and to support thinking. Mathematical symbols provide shorthand for representing mathematical processes and concepts. Learners experience difficulties when using symbols, and to gain that confidence, they need to understand their meanings. From my experience as a high school mathematics teacher, I discovered that learners experience difficulties in using symbols to understand mathematical concepts. This intrigued me to investigate further into these challenges and instructional strategies that mathematics teachers can use to mitigate the effects of symbolic obstacles.

Mathematics derives much of its power from the use of symbols but their conciseness and abstractness can be a barrier to learning (Arcavi, 2005). Mathematical symbols give meaning to the subject, but present pedagogical strains to mathematics education especially in Algebra (Szydlik, 2015). Mathematical symbols make mathematics a highly specialised and technical language that is difficult to decode (Dale & Tanner, 2012). This specialisation presents problems to learners when interpreting and conceptualising mathematical texts, particularly word problems (Jan & Rodrigues, 2012). Mathematical language coupled with its symbolic syntactic structure, presents challenges to learners whose first language differs from the medium of instruction (Garegae, 2011). Bell (2003) also asserts that mathematics vocabulary, special syntactic structures, mathematical inference and discourse patterns in written text compound the difficulties learners experience when learning mathematics.

The use of mathematical symbols presents multifaceted problems but the researcher suspects that one factor, though not fully investigated, is a barrier caused by the transition in the use of symbols between senior and FET band in secondary school mathematics. The problem is heightened by variation in symbol use between mathematics and other science subjects. The issue of reading, recognising and understanding symbols underpins all mathematics topics. A study conducted by Hiebert (2013) reveals that the use of mathematical symbols is one of the reasons why learners experience difficulties. Learners who expressed dislike for mathematics pointed out at symbolisation as one of major reasons for their distaste of the subject (Peter & Olaoye, 2013). Chirume (2012) viewed learning mathematics as a complex process and highlighted the challenges of mathematical symbolisation as the first hurdle that learners must overcome in order to succeed in the subject. Mathematical symbols together with a variety of representations provide tools for conveying mathematical knowledge. However, as noted by Koedinger, Alibali & Nathan (2008) learners have trouble in understanding mathematical concepts and processes due to symbols and representations that are not part of their formal reality.

Research has shown that learners prefer a symbolic strategy even when a different representation would be more helpful; although learners may attempt to use more than one strategy, they often regress to using the symbolic representation (Senk & Thompson, 2006; Huntley & Davis, 2008; Moreno, Ozogul & Reisslein, 2011). Mathematical symbols are essential ingredients of mathematical language that constitute the components of mathematical language that enable teachers and learners to engage in discourse about abstract mathematics concepts (Berger, 2004). Symbols also serve as tools through which mathematical thoughts and ideas are communicated (Chae, 2005). They provide shorthand for representing complex word-names, abstract mathematical processes and concepts. They provide a means of manipulating mathematical concepts and processes in accordance with specific rules in a condensed form (K'Odhiambo & Gunga, 2011).

Most mathematical activities eventually lead to mathematical ideas that are eventually represented as symbolic objects (Altun & Yilmaz, 2011). Whitebread, Basilio, Kuvalja and Verma (2012) emphasised that the growth of modern scientific disciplines depends on mathematics and their evolution is measured by their growing reliance on symbols. It is therefore reasonable to infer that learners' difficulties with understanding mathematical concepts have their origins in the problem of symbolisation. For many learners, mathematics is seen as a 'foreign language' (Adoniou & Yi, 2014:3). Unfamiliar symbols and representations of mathematical concepts present barriers to understanding (Naidoo, 2016). There is scant literature and knowledge of how the symbolic language of mathematics obscures learners from understanding mathematical concepts (Maguire, 2012). This gap requires an understanding of how learners interact with and perceive the symbolic and abstract nature of mathematics.

Mathematical symbols are a crucial component of the subject. They facilitate the representation of mathematical operations to the external environment (De Cruz & De Smedt, 2013). They provide an external representation of abstract mathematical objects. Bellotti (2011) maintained that symbols allow mathematical objects to exist independent of their concrete representations. In this view, mathematical symbols do not only express mathematical concepts but they constitute mathematical concepts themselves (Lolli, Panza & Venturi, 2014). Mathematical symbols can also be viewed in two ways: as epistemic actions, which enable complex concepts to be represented physically and as a notational system that frees up cognitive resources to offload abstract ideas into the environment (Coolidge & Overmann, 2012). Freeguard (2014) also submitted that symbols build an intimate relationship between mathematical concepts and mathematical cognition. Despite all these advantages, the consensus among researchers is that the use of mathematical symbols continues to be an obstacle that cannot be soon eliminated from mathematics classrooms (Schleppegrell, 2010; Cobb, Yackel & McClain, 2012). Traditional teaching has not particularly encouraged the development of sense of symbols, nor has it developed habits of mind for inducing the interplay between representations.

Mathematical symbols serve as means of perceiving, recognising, and creating meaning out of patterns and configurations drawn from real-life experiences or communication (Radford, 2008). This is where the strengths of symbols lie; they enable us to solve problems without making reference to concrete objects. Mazur (2014) concurs with this assertion, arguing that mathematical symbols have a definite purpose, that is, to unpack complex information in order to facilitate understanding. Presmeg (2006) and Sfard (2008) also made similar sentiments, arguing that mathematical symbols provide a language to record mathematical ideas and processes. Another essential point proposed by Gray and Tall (1992) is that symbols are treated as objects in mathematics, and mathematicians manipulate them as if they are the objects signified. O’Halloran (2005) brought another dimension of symbolism as an information dense language. According to this view, symbolism can be regarded as a language with specialised strategies for organising meaning. Hammill (2010) also argued that because of mathematical symbolism, operations, relations, and existential meaning can be operated on to solve mathematical problems without resorting to their concrete world. Nunes, Bryant and Watson (2007), however contend that learning mathematics through symbolisation is a complex exercise due to the detachment of algebra from the original meaning of a problem.

The use of mathematical symbols also allows the essence of mathematical thought to be recorded and passed on from one generation to another (Firth, 2011). Symbols enable mathematical thinking to be recorded in a compressible way (Gray and Tall, 2007). Without proper knowledge and understanding of symbols, it would be very difficult for learners to express mathematical procedures or relations (Moschkovich, 2008). The use of symbols and the process of symbolisation pave way for a symbolic logic and the discourse of modern mathematics (Sarukkai, 2005). Mathematics register is dominated by symbols, hence it is imperative that learners understand and use them fluently. Lee (2010) urged that the most important thing about written statements in symbolic form is the meaning that the symbol invokes in the mind of the learner. Thomas and Hong (2001) concurred with Gray and Tall (2007) that some symbols invoke action or processes while others are perceived as objects or concepts.

The efficacy of mathematical symbols is variously interpreted in literature (Pyke, 2003). Symbols can be used as names or labels for mathematical objects ideas and processes. They also play the role of signifiers and as a form of shorthand during classroom communication or instruction. Symbols also provide entities that are used to present and simplify the solution process during problem solving. Barwell (2007) and Karam (2014) concurred that symbols are used to reveal structure of mathematical objects as well as displaying their relationships. Mathematical symbols can be utilised as the semiotic resource through which mathematical solution processes can be presented (O’Halloran, 2005). Meaney (2005) asserts that the high symbolic density of mathematical language allows great flexibility in the way symbols are used. In order to deal with this complexity, learners should possess specific skills of drawing meanings. Meaney (2005) and O’Halloran (2005) shared common views pertaining to the challenges of mathematical symbolisation. They argued that symbolisation is not taught as a way of developing mathematical language. As a result, learners struggle to master it. The teaching of the symbolic component of mathematics text is often neglected and not planned for and teachers take a naïve approach that language-reading skills are transferable through reading mathematics. Shepherd (2005) reported that English reading skills are taught in confined ways that cannot be transferred to content areas such as mathematics.

With these mixed interpretations and functions, it is not a surprise that the symbolic language of mathematics brings a lot of misunderstandings and present difficulties for learners (Stacey & MacGregor, 1997). Chae (2005) concurred with Hiebert's (1997) explanation that the challenges of using symbols as learning tools are attributed to the fact that meaning does not reside in symbols, but something one makes from signs. There is also a consensus view among researchers that mathematical meaning is not attached to symbols automatically and that without meaning, symbols cannot be used effectively (Redish & Gupta, 2009; Chirume, 2012). A mathematical symbol's potential to effect meaning and convey an idea depends on how the interpreter reads the symbol, the so-called symbol-object relation (Mingers & Willcocks, 2014). The interpretation of new mathematical ideas creates new symbols. In mathematics, new symbols are created through interpretation and communication with old symbols (Steinbring, 2006). Symbols themselves have the potential of generating new meanings and challenging old ones (Preucel & Bauer 2001).

Nicol, Oesterle, Liljedahl and Allan (2014) highlighted that the symbolic language makes mathematics more powerful and applicable by removing subjective elements that can be found in vernacular. However, the powerful and yet de-contextualized language presents difficulties for novice mathematics learners. Mathematical symbolism exerts cognitive demands on learners to the extent of treating symbolic representations as mathematical objects or operations (De Cruz, 2006). Furthermore, Limjap (2009) observed that modifying a learner' informal interpretations of certain symbols and replacing them with formal symbols present further cognitive burdens on learners.

Experts in mathematics such as teachers are able to manipulate and to understand mathematical concepts through its symbolic representations, while learners experience challenges in this endeavour. Mathematics deals with relationships between numbers, categories, geometric forms and variables. These relationships are linked and expressed symbolically. Since the relationships are abstract, they are only accessed through language and a unique symbol system (Zeljić, 2015). Mathematical symbols are interpreted linguistically. This contradicts the commonly held view that mathematics is a language free area (Wilburne & Napoli, 2008). Tsanwani (2009) strongly argued that learning mathematics depend on learners' language competences. Henry, Baltes and Nistor (2014) observed that low language proficiency and mathematical underachievement are highly correlated.

Kirschner, Sweller and Clark (2006) noted that if learners fail to solve a mathematics problem successfully; the teacher might think that the learners need more time to practice or understand. However, Kenney (2008) argues that allowing more time and practice while learners are in a confused state may aggravate the confusion with understanding mathematical symbols. When introducing a new topic teachers often fail to teach three essential things: new symbols $(\pi, \theta, \Sigma, and\ \infty),$ new words (parallel, tangent, and normal), phrases (sum of, product of), mathematical terms (function, domain, and derivative) and new grammar (expressing equations in a logical and consistent manner).

According to Sloutsky, Kaminski and Heckler (2005) learners bring to the classroom and sometimes stick to misinterpretations and misconceptions of some symbols as a result of their previous encounters in earlier mathematics classes. Learners over rely on the syntax of natural language (English) to understand and make sense of the language of mathematics (Chirume, 2012). Firouzian (2014) also describes another common difficulty, called "manipulation focus," in which learners select their strategies and procedures to problem solving based on the given symbols and pay little attention to the meanings of the symbols. Teaching by simply pointing out that the rules are not the same is not guarantee that they will understand the symbolic notations. Lack of fluency with the symbolic language of mathematics negatively affects learners' problem solving skills (Peter & Olaoye, 2013). Consequently, this causes learners to look for alternative ways of solving mathematical problems without paying attention to the meanings of symbols.

Mathematical language derives some of its meanings from natural language and kinaesthetic actions such as counting, dividing and measuring (Christie & Maton, 2011).

However, learners lack the skills to transfer such actions into symbolic forms. The grammar of mathematical symbolism is specially organised. Symbolism allows relations between mathematical objects to be rearranged and simplified in a logical manner. The grammatical strategies found in mathematical symbolism are the opposite of what is found in scientific language. Mathematical symbolism works through deep embedding of configurations of mathematical concepts and processes (O'Halloran, 2011). It preserves mathematical objects and the processes such that they can be reconfigured to solve problems, according to pre-established results, laws and axioms. Mathematical symbolism has a range of grammatical strategies which make the preservation, rearrangement and simplification of mathematical processes and participant configurations possible, such as generalised participants, use of spatial notation (for example, division and powers) and brackets, ellipsis of processes and rules of order which stipulate the sequence in which mathematical processes unfold. The sequence of unfolding processes in mathematical statements is not linear, but it is predetermined in specific ways by mathematical rules.

Mathematical symbolism is a carefully designed tool that aids logical reasoning (Sapire & Reed, 2011). It does this by encoding of mathematical concepts and processes in a format that facilitates their rearrangement. It is this rearrangement that brings about understanding. However, it can act as a cognitive barrier to understanding mathematical concepts (Heeffer, 2013). There are on-going debates pertaining to when and how to introduce symbolism within the school curriculum. If it is introduced too early, (Heeffer, 2014) argued that learners may lack the maturity to understand and reason symbolically. However, (Zvawanda, 2014) had a contrary view, he argued that if symbols are introduced too late, some mathematical methods and concepts cannot be taught as they rely on symbolism.

1.2 Background of the study

The history of mathematics education in South African secondary schools is characterised by changes in curriculum. The Curriculum Assessment Policy Statement (CAPS) curriculum is the fourth wave of curriculum reforms in the post-apartheid South Africa. A number of curriculum reforms have been designed to suit both international and national shifts and developments in mathematics education, theory and practice. Classroom based and content-based research has played an insignificant role in the direction or form taken by the curriculum over time (Guomundsdottir, 2015). None of these curriculum shifts has emphasised on the need to address why learners continue to struggle with the transition from arithmetic to algebra.

The Trends in International Mathematics and Science Study Repeat Survey (TIMSS-RS) of the world wide trends in scholastic performance in Mathematics and Science revealed that South African learners' performance in mathematics is poor (Mullis, Martin, Foy & Arora, 2012). South African learners perform poorly in tests that measure knowledge of basic mathematical skills (Spaull, 2013). Further evidence of South African learners' underperformance in mathematics were recorded in summative national and international assessments such as the Programme for International Student Assessment (PISA), Southern Africa Consortium for Monitoring Education Quality (SACMEQ) and national assessments such as the Annual National Assessments (ANA), and the National Senior Certificate (NSC) examinations. Sepeng and Madzorera (2014) also contributed to the debate by revealing that South African learners struggled to deal with problems related to mathematical symbols and communication. Moreover, the Annual National Assessment (ANA) revealed that, "the overall performance of learners was very low with average scores of 30%" (DBE, 2011, p. 2). In addition, poor performance in higher grades 9-12 is linked to poor performance in algebra (Mashazi, 2014).

Bernstein (2013) reported that the high failure rate in Mathematics at secondary school level in South Africa remains unacceptably high. The matric pass rate is far below the national expected standard (DoE, 2015). Reddy & van Rensburg (2011) analysed the mathematical performance of the South African schooling population and concluded that the national average mathematics performance score for different grade levels across the schooling system is similar and stable, ranging from 30% to 40% across all the grade levels. This raises the question of whether improved schooling makes any difference in performance (Reddy & van Rensburg, 2011). Good matriculation results, especially in Mathematics and Science determine whether a learner will be accepted in the sought-after technological and scientific fields of study at tertiary institutions. These fields of study are largely out of reach for many black learners. The lack of adequate basic academic skills and competencies to transition from secondary to tertiary education coupled with a lack of adequate support systems further prevent many potential mathematics and science graduates from completing their studies. This gives learners limited opportunities to study Mathematics and Science further and secure employment. This is so because many learners from rural and township secondary schools fail to achieve university entry of which a pass in Mathematics is one of such requirements (Moloi & Strauss, 2005).

A study of South African secondary school learners conducted by Spangenberg (2012) revealed that many learners lack basic knowledge and skills for problem solving. Mogari, (2014) made similar sentiments arguing that there are deficiencies in knowledge of basic mathematical concepts. Teaching of basic mathematical concepts is superficial and promotes rote memorisation of mathematical concepts. Senoamali (2016) blamed most mathematics teachers for teaching to the test and this practice impacts negatively on the learners' conceptual understanding. The quality of performance reflected in the 2014 Annual National Assessments (ANA) demonstrates learners' lack of conceptual understanding (DBE, 2014).

Makgato (2007) and Pooran (2011) investigated the problem of mathematics underachievement in South African secondary schools. Their findings include poor social background, lack of support materials, and poor quality of teaching and language of instruction. Mathematics teaching and learning in South African secondary schools is susceptible to poor instruction, teachers continue to present in a way that strongly encourages reticence, conformation to rules and use of sophisticated language (Maree & De Boer, 2003). There is little emphasis on conceptual understanding. Moyer (2001) reiterated that teachers do not emphasise the utilisation of mathematical symbols to construct concepts.

Mwakapenda (2008) noted that the approaches to mathematics teaching and learning in South Africa have little emphasis on conceptual understanding. Concepts are not adequately connected with symbols together with their meanings. Mulwa (2015) also revealed that learners' performance is highly correlated to their understanding of mathematical concepts and symbols. Furthermore, Bardini and Pierce (2015) highlighted the importance of paying attention to potential barriers to learning because of heightened complexity in the use of symbols. Mathematical language uses symbols and notations that are not common in ordinary English and the various languages across South Africa. Maree et al, (2006) argue that learners from impoverished backgrounds lack informal mathematics knowledge which is a prerequisite for developing strategies for solving formal mathematical problems. Many learners have difficulty with the new and more intense ways in which symbols are used at secondary school level. This leads to a decrease in positive affect, which in turn discourages enrolment in mathematics related fields.

Despite all these efforts by researchers to get to the root causes of poor performance in mathematics at secondary school level, few attempts have been made to research and assess learners' challenges in the different mathematics curricula. No attempt has been made to look into the specific challenges that teachers and learners face when implementing the curriculum. The high failure rate in secondary school mathematics and cognitive gaps in the conception of mathematical concepts are attributed to learners' failure to acquire the language system of mathematics that is dominated by unfamiliar and confusing symbols (Nunes & Bryant, 2015). Mathematics presents many unique challenges during teaching and learning. The most noticeable barrier to communication is that mathematics is heavily laden with symbolism (Sheikh & Randa, 2013).

When learners are introduced to a new mathematical concept for the first time, the new symbols involved overwhelm them and concentrate on symbols instead of the meanings. (Arcavi, 1994) argued that a strong symbol sense ought to be developed. However, Steinbring (2012) warns that there is a danger of acquiring meaning by considering concrete materials as other forms of representation. In order to acquire meanings for symbols, Brown, McNeil and Glenberg (2009), recommend that teachers should engage learners in ways that promote the connection of abstract symbols and their concrete representations. However, the potential for these connections to create understanding is complicated by the fact that concrete materials themselves are representations of mathematical relationships and quantities. Thus, the usefulness of concrete materials as referents for symbols depends both on their embodiments of mathematical relationships and on their connections to written symbols.

Mathematical knowledge is normally conveyed and imparted in classrooms in the form of symbols. Mathematics classes rarely use discourse and talk as modes of instruction (Walshaw and Anthony, 2008). Mathematics teachers seldom engage learners in mathematical discourse. Teachers tend to direct and dominate classroom activities instead of engaging learners through discourse. The National Council of Teachers of Mathematics (NCTM, 2000) encouraged teachers to use classroom discourse in math classes, to support both learners' ability to reason mathematically and their ability to communicate that reasoning. Mathematics teachers are expected to emphasise and inculcate knowledge of how to use the unique symbols of mathematics. Buchanan (2007) reveals that learners often struggle with reading, verbalising and writing in mathematics. These skills are important in the mathematics classroom. One of the new goals for learning in the Curriculum Assessment Policy Statement (CAPS) requires learners to develop the power to use mathematical signs, symbols, concepts and terms of mathematics (DB E, 2010). This is best accomplished if instruction allows learners the opportunity to read, write, and discuss ideas in which the use of the language and symbols of mathematics becomes natural.

Meiers, Reid, McKenzie and Mellor (2013) note that learners devote little time working with mathematical text. Learners need to develop special skills of reading, verbalising and writing mathematics. Learners lack strategies for articulating word symbols that guide thought and allow for the attachment of mathematical meaning (McIntosh, Jarrett & Peixotto, 2000). Woolley (2011) viewed reading as part of thinking that involves interpreting symbols, decoding meanings of symbols, and extracting ideas from symbols. Learners should be able to handle mathematical ideas through the manipulating abstract symbols and notation. These efficient, but abstract, symbols and notation present a special concern to the mathematics teacher. The ability to decode mathematical symbols and to associate meaning with them is a special prerequisite to mathematics learning. Learners see mathematics an intimidating subject which is difficult to understand, difficult to master while teachers find it difficult to teach. Learners find mathematics as a completely different language to learn. Meanings of mathematical symbols are not static (Pimm, 2002). In some cases, they represent operations (Usiskin, 2015) while in other situations they constitute concepts (Stahl, 2007). Furthermore, operations performed on symbols and using symbols are interchangeable and require different operations in different contexts (Sherman and Bisanz, 2009).

According to Chae (2005), mathematical symbols serve two fundamental functions based on two types of connections. Symbols are used as communication tools to convey mathematical ideas (concept or objects) or actions (processes). Symbols have also a private function in which symbols are used to organise and manipulate ideas based on the connection within the symbol system. Similarly, Gray and Tall (1994) regard symbols as pivots between processes and concepts in the notion of "procept". According to the "procept" view of mathematical symbols, they provide a link between the image (of a symbol) and the interiorised operations for carrying out mathematical processes.

Anthony and Walshaw (2009) advocate for classroom practices that encourage learners to demonstrate multiple ways of presenting and representing mathematical concepts, promoting mathematical discourse, language and symbolic proficiency. The challenges of teaching and learning mathematics involve difficulties that are inherent in the nature of the subject, particularly the symbolic, abstract and visual nature of mathematics (Adler & Pillay, 2007). Given these perceived challenges, why should teachers continue to teach mathematics to learners who have not acquired the language and symbol system of the subject? It is against this background that the researcher decided to obtain an in-depth understanding of the challenges posed by mathematical symbolisation. The study aims to explore, find, and suggest possible instructional strategies to mitigate the aforementioned problems.

1.3 Context of the Study

Mahn and Steiner (2013) argue that learners' mathematical production and thinking modes depend on the social and cultural contexts in which they develop. Presmeg (2007) concedes that mathematics, long considered value- and culture-free, is indeed a cultural product, and hence that the role of culture-with all its complexities and contestations is an important aspect of mathematics education. Thus, learning mathematics in a particular social and cultural context is some kind of enculturation (van Schaik & Burkart, 2011).

Hence, it is therefore important to include and discuss the context and sites in which this study was undertaken.

The study was conducted in Greater Sekhukhune and Capricorn districts of Limpopo Province, South Africa, where the researcher observed that learners had problems in understanding mathematical concepts due to, among other factors the symbolic language of mathematics text. Limpopo province is mainly rural and participants for the study were drawn from rural, semi-urban and urban backgrounds. Learners from semi-urban schools either commute from surrounding rural villages or live in the service centres. Mathematics performance is poor in Limpopo Province schools, especially in schools that are based in former homelands and townships (Mouton, Louw & Strydom, 2013). A study by Sinyosi (2015) also highlighted some socio-cultural factors that hinder learners from learning mathematics. Most schools in Limpopo Province are located in impoverished areas where learning resources are limited and scarce. On average, learners in the province perform significantly lower than the national average in National examinations (Reddy et al, 2012).

The matric results of 2015 indicated that Limpopo Province had the worst performance in mathematics with 32.4% of the learners achieving a mark of 40% and above (Gavin, 2016). Rammala (2009) posited that learners' poor performance in mathematics could be linked to multiple factors such as: poverty, lack of resources and infrastructure of schools, low teacher qualification, and poor learning cultures in schools. Language proficiency was also identified as a contributory factor. From a socio-cultural point of view, Weeks (2012) argued that creating an ideal learning environment is necessary to allow a dynamic interaction between teachers and learners. The quality of tasks selected by teachers should provide learners with opportunities to create their own knowledge during interaction with peers (Moreeng & Du Toit, 2013). However, this cannot be said of learners in rural settings. They need the teacher to guide them to unpack meanings of mathematical symbols and understand concepts.

1.4 The missing phenomenon

The key to comprehending mathematical concepts lies in understanding and interpreting symbols and the role they play in conceptual development (Limjap, 2009). It is essential for learners to understand the role and meaning of symbols and be able to appreciate their usefulness in problem solving. Symbols are the backbone of mathematical language and are vital tools that make it a universal science (Jamison, 2000). Learners concentrate on the procedures of manipulating symbols in problem solving instead of understanding their meanings (Koedinger & Nathan, 2004). Learners sometimes mistakenly treat symbols as mathematical ideas yet they are representatives of the intangible or abstract ideas. This research investigates learners' understanding of mathematical concepts through symbolisation. More importantly, the research expands into the process of mathematical abstraction by looking at the ways in which symbols facilitate or obscure learners' understanding of mathematical concepts, problem solving and solution processes.

Reading mathematics text requires learners to master the distinct and special-purpose symbolic language of the subject (Selden & Shepherd, 2013). The findings of this research could possibly provide teachers with insights into learners' difficulties with multiple representations of mathematical processes and concepts. The knowledge of these difficulties enables teachers to provide learners with multiple ways of representing mathematical ideas in a manner that facilitates understanding. By identifying learners' difficulties in connecting mathematical concepts and their meanings, teachers can anticipate the problems and learning gaps that learners are likely to encounter and suggest remedies for such difficulties. Preventing learners from obtaining partial and surface understanding help them to achieve a robust understanding of the mathematical concept or process and its symbols in breadth and depth.

1.5 Problem Statement

In an ideal mathematics learning situation, learners are expected to be competent in representing mathematical situations and recognising structure and meaning in symbolic expressions (Moschkovich, 2008). Learning mathematics with understanding involves acquiring the knowledge of concepts and mastering the skills of encoding symbol meanings. Learning mathematics requires learners to be efficient and fluent in using symbols, and to manipulate symbols effectively to discover and make new mathematical concepts (Tarasenkova, 2013). However, this is not the case in most South African mathematics classrooms. Many learners find mathematics overwhelming because it is highly symbolic, contains unfamiliar notations and conventions (Chinn, 2016). Even more, the symbolic formulation is dense with meaning, and learners are often disinclined to unpack meanings. As a consequence, learners resort to meaningless ‘symbol pushing’, which obscures further mathematics learning (Thompson, Cheepurupalli, Hardin, Lienert and Selden, 2010). Many learners experience mathematics as “rules without reason or marks without meaning” (Mueller, Yankelewitz & Maher, 2010). Learners do not make connections among and between concepts and symbolic expressions. De Lima & Tall (2008) also reported that learners mentally use symbols and manipulate them according to rules without grasping their meanings. Learners do not reason about an overall goal or the concepts involved in a problem, but instead they look for an implied procedure inherent in the symbols.

The researcher observed that most learners have challenges in understanding mathematical concepts due to syntactic features of the subject. The researcher speculated that learners’ failure to conceptualise mathematical concepts could be linked to unfamiliar symbols which are confusing and sometimes contradictory. As learners interact with symbols they have to endow them with meaning, understand the context in which they are used as well as recognising concepts, models and actions associated with the symbols. A similar claim was made by Lockhart (2009) who cited mathematical symbolism as an obstacle to mathematical learning and teaching. Mathematical symbols obscure learners from understanding mathematical concepts and processes as well as limiting their problem-solving endeavours (Heeffer, 2012). Thus, learners struggle to understand mathematical concepts especially algebra due to lack of knowledge of algebraic symbols. This problem emanates from the fact that symbols assume dual roles: they represent mathematical processes and concepts (Tall, 2008). Symbolic language remains a challenge for South African learners such that teachers continuously pursue effective instructional strategies to curb this problem.

Mathematics is more than just numbers; it involves symbols, terminology and syntax which complicate concepts for most learners. Thus, the problems addressed in this study relate to the nature of challenges that learners experience with symbolic representations.

The following questions summarise the problem statement for this study:

a) What challenges do secondary school learners encounter when interpreting and using mathematical symbols to understand mathematical concepts and problem solving procedures?
b) What instructional strategies can mathematics teachers use to mitigate the effects of symbolic obstacles?

The study focuses on the problems relating to how learners interpret and use the language, symbols, and syntax of mathematics when reading mathematics text, during problem-solving and algebraic reasoning. Communication in mathematics is strongly correlated to a learner's problem solving and reasoning abilities (Neria & Amit, 2004). As a result, it is importance for teachers to be aware of these difficulties. Misconceptions about the use of the symbols and syntax of mathematics force some learners to develop informal techniques for understanding and solving problems (Reynders, 2014).

As a consequence of learners' symbolic illiteracy, mathematics has become one of the most unpopular subjects in South African secondary schools (Spaull, 2013). Learners do not perform well in the subject (Mogari, Coetzee & Maritz, 2009). The spectrum of causes associated with this poor performance includes among other things, deficits in learning mathematical concepts (Carnoy & Chisholm, 2008). The other causes of poor performance were cited by Ramohapi, Maimane and Rankhumise (2015) as: learners' attitudes towards mathematics; the use of English as a medium of instruction; teachers' lack of content knowledge and pedagogy; learning resources and support from parents.

1.6 Purpose Statement

The purpose of this study is to obtain insights into learners' difficulties with mathematical symbolism. It also examines how teachers teach symbolism and recommends instructional strategies and practices to address learners' shortcomings. The study sought to obtain in-depth understanding of how secondary learners perceive mathematical concepts focusing on how they interpret mathematical symbols. The study further enquires on how symbolism influences learners' problem-solving approaches or reading mathematics text. The key attributes that teachers should attend to include the symbol sense that learners exhibit during problem solving. In particular, the study intends to inform mathematics teachers on how symbols can help learners to construct meanings of mathematical concepts. It can be argued that a better conceptual understanding of symbolism by teachers will prepare them for possible difficulties that learners will confront in the classroom.

The study also sought to sensitise teachers on the need to select instructional activities that support the development of algebra as a sense-making activity. Kieran (2004) emphasised that the transition from arithmetic to algebra requires teachers to focus learners' attention to how they build meanings for algebraic concepts and processes. There is need for teachers to guide learners to see algebraic symbols as tools for thinking rather than as bags of tricks. Algebraic symbols should not be viewed as procedural tools but as representational aids. According to Sfard (2000), algebraic symbols do not speak for themselves or have meanings inherent in themselves. They depend on what learners are prepared to notice and able to perceive. In other words, meaningfulness is derived from the ability to see abstract ideas beneath the symbols.

1.7 Rationale of the study

This study is important due to the fundamental educational necessity of understanding the challenges faced by learners, and to provide clear and coherent instructional symbol usage to facilitate meaningful learning and teaching of mathematical concepts in general. Confusion and misconceptions resulting from the improper or inconsistent use of symbols are detrimental to a learner's attempt to define the content presented in any given learning environment. Rubenstein and Thomson (2001) stressed that learners who cannot develop fluency with the use of mathematical symbols are prone to slow growth in their mathematical development. Radford (2008) also stressed the importance of investigating the way learners interpret mathematical symbols and how teachers present such symbols to learners when they attempt to endow them with meaning when learners encounter them for the first time.

This study is also crucial since it sought to establish the extent and manner of use of mathematical symbols at secondary school level and to establish the perceived level of learner confusion as a result of the use of such symbols. It is anticipated that such determination will influence the manner in which teachers choose to present such symbols in future classes. Findings from the inquiry of this nature can also contribute to the body of knowledge regarding the best instructional practices for teaching mathematics and to the field of mathematics by answering the aforementioned research questions.

This study is significant to both mathematics teachers and learners. For the teachers the research serves to inform and potentially modify their pedagogical practices to reduce potential learner confusion due to mathematical symbolisation. For learners the study seeks to reduce or diminish their level of mathematical confusion due to the use of mathematical symbolisation and potentially lead to an increase in conceptual understanding and achievement in examinations. Finally, since the study is exploratory in nature the results may serve as a foundation for further investigation and inquiry.

1.8 Research Questions

Algebra is a branch of mathematics that uses symbols or letters to represent variables, values or numbers. Mathematical symbols are an integral part of Algebra used to express operations, relationships and to solve problems. Learners need to master the symbolic language of mathematics because symbols are the standard nomenclature used in mathematical discourse, reasoning and problem solving. Bakker, Doorman and Drijvers (2003) maintain that there is no Algebra without the use of mathematical symbols. The intertwinement of symbols, representation and meaning presents additional problems for mathematics education. Mathematicians, teachers and instructional designers regard symbols as carriers of meaning (Stacey, Chick & Kendal, 2006). Learners, however, lack the necessary mathematical background to interpret symbolic representations. Teachers should therefore explain symbolic representations to learners and demonstrate how to use them in problem solving.

The present study specifically focused on FET learners who encounter problems with mathematical symbols; what they mean and how to use them in problem solving. At FET level, more complicated and sophisticated symbols are introduced to lay a foundation for advanced mathematical concepts. The research presumes that learners' experiences with mathematical symbolism occur in lessons. Experiences consist of participating during classroom engagement, reading mathematics text, doing hands-on activities, observing how the teacher and other learners use symbols. Thus, the study is based on learners' attempt to grasp mathematical concepts through symbolisation during classroom activities or extracting meanings from textbooks.

The following research questions guided this study:

a) What challenges do secondary school learners encounter when interpreting and using mathematical symbols to understand mathematical concepts and problem solving procedures?
b) What instructional strategies can mathematics teachers use to mitigate the effects of symbolic obstacles?

Sub- problems

a) How do learners connect symbols and their meanings?
b) How do learners use conventional mathematical symbols in problem solving?
c) In what ways are learners' problem-solving goals and activities influenced by mathematical symbols?
d) How do teachers connect learners' informal and formal conceptions of mathematical symbolism?

The first research question investigates the challenges secondary school learners encounter when interpreting mathematical symbols during problem solving or decoding meanings from textbooks. The expectation is that if learners are competent, fluent and capable of communicating using mathematical symbols and notation, their performance in mathematics shows improvement (Blanton & Kaput, 2005). Learners acquire notations and symbols for mathematical concepts and processes during engagement in mathematical activities in the classroom setting and as they read mathematics textbooks. However, if teachers simply cue up procedures for learners to perform the appropriate calculations, understanding will be jeopardised. In some cases, teachers interpret problems for their learners; this deprives learners the opportunity to learn autonomously.

Mathematics lessons are characterised by classrooms discourse that involves decoding information, compressing long mathematical sentences, representing, and analysing data. These processes utilise and exploit the spatial features of mathematical symbolisms. The problems of failing to use and interpret symbols hinder conceptual understanding in most mathematics classes. Chow (2011) noted that, if learners are thoroughly taught the meanings of the symbols, and know how to use them, the compact form makes it easier to recognise critical relationships. The correct interpretation of these conventions reveals the power of mathematical symbolism. Chirume (2012) pointed out the weaknesses and problems of mathematical symbolisation are centred on using, reading and interpreting mathematical symbols. A number of researchers explained how the use of mathematical symbols influences conceptual understanding and mastery in mathematics:

- Garegae (2011) argues that mathematical symbols and language are seldom used at home so individual study with a textbook is a challenge. Learners studying alone do not know how to read and endow meanings to symbols they encounter during reading.
- Chirume (2012) reveals that learners struggle to read mathematical symbols with comprehension due to their prior encounters with those symbols on previous grades or classes.
- According to Tall (2009) symbols, have dual functions: they play the role of objects or concepts of mathematics or as ideas and processes that they represent.

The second research question seeks to investigate the possible instructional strategies that teachers can utilise in order to curb the effects of symbolic obstacles. One central argument raised by Bruner and Haste (2010) is that learners attach personal and informal meanings to abstract symbols. The transition from informal symbols and ways of thinking to formal school mathematics presents teachers and learners with pedagogical and learning problems. Carruthers and Worthington (2006) further highlighted this problem. They argued that the symbol system is not fully understood. For example, meanings letters of alphabet and numerals have no specific meaning, but convey information when they are combined in systematic ways. It is therefore important for learners to not only make sense of individual symbols but also need to understand them when used within a system.

Studies on the development of symbol writing indicate that learners bring forth strategies, which teachers can support to enhance their understanding. For instance, Machaba and

Lenyai (2014) suggest teachers should ensure that they make connections between learners' informal knowledge and the abstract system of mathematical symbolism. Hand and Taylor (2008) argued that the gap in knowledge and symbol use between learners' informal approaches and formal procedures is a critical cause of learners' failure to understand mathematics. Fisher (2010) echoed the same sentiments, arguing that such connections are imperative since they prepare learners to be critical mathematical thinkers rather than mindless manipulators of mathematical procedures. However, determining ways of fostering these connections is a challenge for teachers but failure to do so magnifies learners' difficulties with mathematics symbolism. Novak and Cañas (2008) observe that even though teachers make efforts to illustrate the symbols and operations with pictures and other concrete objects, it has been observed that learners continue to struggle to establish crucial links. Whilst researchers emphasise and encourages learners to use their own marks, teachers find this highly challenging as majority of learners rely on textbooks as sources of knowledge (Botes & Mji, 2010). The use of manipulatives is a vital way to engage various senses when learning mathematical concepts. Bruins (2014) maintains that instruction-involving manipulatives helps to engage as many senses as possible. Such an approach helps to simplify the abstract to be more concrete and understandable to the learner.

Sub-research questions seek to investigate the challenges learners encounter as they link mathematical symbols and problem solving procedures. The aim is to investigate learners' experiences in making connections, if ever they are able to do so, for example, how concepts and skills from one strand of mathematics are related to those from another (Fogarty & Pete, 2009). As learners make such connections, they begin to realise that mathematical concepts are not learnt in isolation, but knowledge from one area of mathematics a prerequisite to understand another. Establishing relationships among symbols, procedures and concepts also helps deepen learners' mathematical understanding (Mwakapenda, 2008).

1.9 Hypothesis

Tests of hypotheses were conducted to test the effects of moderating variables of the study units. Participants for the study were drawn from different genders, different age groups, different grade levels, different physical locations as well as a variety of language backgrounds. These variables can influence the findings of this study, hence and thus produces an interaction effect. It is therefore essential to investigate their influences on learners' responses.

The following hypotheses were envisaged in this study:

H_0: There are no gender differences in learners' difficulties with mathematical symbolisation.

H_1: There are gender differences in learners' difficulties with mathematical symbolisation.

H_0: There is no grade, age, language, residential area differences with regard to learners' difficulties with mathematical symbolisation.

H_1: There are grades, age, language, residential area differences with regard to learners' difficulties with mathematical symbolisation.

1.10 Definitions of Terms

Mathematical Symbols

Cobb (2000) defined symbols:

> *"...any situation in which a concrete entity such as a mark on paper, an icon on a computer screen, or an arrangement of physical materials is interpreted as standing for or signifying something else"* (p. 17).

However, the above definition is wide as it applies to both mathematical symbols and contemporary symbols. So in order to define symbols in a mathematical context Cobb's (2000) definition was modified to:

> *".... a concrete entity that stands for or signifies a mathematical idea or object or concept or process".*

Teachers should bear in mind that an entity like $sin\theta$ is not a symbol at all for a learner is seeing it for the first time. However, $sin\theta$ is a symbol for a learner who knows its meaning.

In the context of this study, the term *symbol* also refers to mathematical entities such as letters (a,b,c,h,m) numbers $(\pi:3,e)$, arithmetic signs $(=,+,-,x,/)$, parentheses ( ), square root signs ($\sqrt{\ }$) and all other symbolic inscriptions found in mathematics textbooks. Symbols are a form of *representation;* hence, it is important to define the term representation.

Furthermore, as Langer (2009) explains,

> *"...symbols are not proxy for their objects, but are vehicles for the conception of objects. To conceive a thing or a situation is not the same thing as to 'react toward it' overtly, or to be aware of its presence. In talking about things we have conceptions of them, not the things themselves; and it is the conceptions, not the things, that symbols directly 'mean'."* (p: 60-61).

Representation

Goldin and Kaput (1996) defined representation as:

> *"...a configuration of some kind, that, as a whole or part by part, corresponds to, is referentially associated with, stands for, symbolises, interacts in a special manner with, or otherwise symbolises something else"* (p. 398*).*

A representation can be also viewed as the mediator that links the mathematical concept and its rea-life object. Objects inscribed in textbooks such as formulae, tables, graphs, numerals and equations are all mathematical representations used to represent real life ideas and relationships. A representation is a form of symbolisms that plays a crucial role in teaching and learning mathematics. Without representation, mathematics would be totally abstract and inaccessible (Bolden, Barmby & Harries, 2013).

There are two categories of mathematical representations: external representations (notation systems) and internal representations (mental structures). External representations are physical objects such as symbols, equations, algebraic expressions, graphs, or diagrams that teachers write or draw as a way of illustrating a mathematical idea to their learners. On the other hand, internal representations are mental constructs of mathematical ideas developed through interaction with external representations (Goldin & Shteingold, 2001). This study focuses mainly on external representations that learners can read in textbooks, write in their books and verbalise as they read and communicate mathematical concepts and processes. However, there is a thin line between the two, as external representations build internal representations, which are mental constructs that help us to remember concepts.

Multiple representations are the different ways of symbolising and describing the same mathematical entity (Cobb, Yackel & McClain, 2012). They are used to represent the same concept or process in different ways.

Sign and Symbol

There is a difference between a mathematical sign and a symbol. It is important to clarify the difference between the two. A sign is what is often mistakenly perceived of a symbol. Cassirer (1944) describes the difference between signs and symbols in this way. "Signals and symbols belong to two different universes of discourse: a signal is a part of the physical world of being; a symbol is a part of the human world of meaning. Signals are "operators"; symbols are "designators." Signals, even when understood and used as such, have nevertheless a sort of physical or substantial being; symbols have only a functional value." (p. 32). A sign is the perceptible aspect of a symbol (Jolley, 2014). It is a written mark, or a sound. A symbol is a sign or a mark together with its meaning.

According to Sebeok (2001), a symbol is a combination of a sign together with its meaning or sense. A symbol can be perceived as something that stands or suggests an idea or object or process due to relationship, association, convention, or accidental but not intentional resemblance. Mazur (2014) argues that the above definition does not quite fit the collective experience of its use. He extended it to include some cultural and non-arbitrary, something representative of an object or concept that it does not resemble in sound or look and something that gives no preconception of the thing it resembles.

S**yntax** refers to the ways in which words are arranged according to the rules of a given language (Webster & Fisher, 2003);

Notation is system that uses symbols to record mathematical concepts (Webster & Fisher, 2003).

The **symbolic structure** refers to a situation in which a learner is attending to a group of symbols that are being used together in a representation instead of focusing on a single symbol (Holloway, Battista, Vogel & Ansari, 2013).

Syntactic structure refers to symbolic structure of a mathematical concept or process together with the relations, rules, and formal grammar that accompany it Goldin and Kaput (1996).

'**Symbol load'** refers to learners' experience of the changes in symbols, frequency of symbol use, and the various meanings of symbols that they need to deal with as they progress in mathematics. Bardinia and Piercea (2015) highlighted that the increase in symbol load due to unfamiliarity and increased density may cause learners to lose confidence and subsequently choose a study path that minimises their need for mathematics.

Symbol density refers to the 'the number of symbols' in a mathematical text.

Symbol familiarity

Pimm (2002) provides a framework for explaining how familiarity with symbolism develops. He identified three attributes of a mathematical symbol as: **materiality**, which refers to what the symbol looks like, and **syntax**, which deals with how the symbol is combined with other symbols, and **meaning**.

The word **"understanding"** is widely used in this study. It can mean many things. In the teaching and learning domain, it refers to the acquisition and retention of mathematical ideas. For this study, the definition is derived from the work of Dewey (1910) and Piaget (1978).

For Dewey (1910), understanding means

> *"….to grasp a meaning, to understand, to identify a thing in a situation in which it is important"* (p. 118).

Thus, a learner shows understanding of a mathematical concept if he is able to able to give its meaning and express it using appropriate symbols.

According to Piaget (1978) understanding means being able to explain how things work or does not work. Understanding cannot be separated from the realm of reason. A learner is considered to have shown understanding of a mathematical concept or process if s/he can provide a correct mathematical conception and communicate ideas consistent with what is accepted by the mathematical community. According to Sfard (1994), understanding can be conceived of as grasped meaning. It is a mediation process between the individual mind and the universally experienced. It involves building links between symbols and certain mind-dependent realities.

Mathematical symbolisation is the replacement of a mathematical object or process by a symbol. There are different kinds of such replacement. For example, one can replace `height' by h, a number by `n', a particular number 'nine' as 9, the idea of `variable in an equation' by x, the concept of relation or mapping by f (as in function), a derivative by $f'(x)$ and so on. In most cases such replacement or naming is conventional and arbitrary. The process of symbolisation should not and does not modify or distort that which it stands for. This character has often been interpreted as the 'strength' of symbolisation in logic and mathematics (Sarukkai, 2008). Every mathematical concept or process requires certain symbols to code knowledge. However, symbols do not have meaning in themselves. The meanings have to be constructed by the learner using suitable reference contexts. Meanings of mathematical symbols are actively constructed by the learner or teacher as interrelationships between sign symbol systems and reference contexts (Steinbring, 2008).

The next terms are related to the theoretical framework(s) used in this study.
The phrase, **symbol sense**, refers to a list of attributes and competencies about the use of symbols. It involves the learner's ability to appreciate the power of symbols, to have a feel of when the use of symbols is appropriate or inappropriate and an ability to handle and understand of symbols in different contexts (Pope & Sharma, 2001). Symbol sense also emphasises on the development of skills for using symbols and understanding of the situation. A common assumption made by many researchers is that a learner with symbol sense is less likely to encounter difficulties in understanding mathematical concepts or processes due to symbol barrier.

A mathematical concept is a general idea behind an equation, problem or formula in mathematics. A math concept is the 'why' or 'big idea' of mathematics. A learner who understands mathematical concepts can operate at higher levels of advanced learning that involves abstract thinking and dominated by symbols. Understanding mathematical concepts replaces learning by rote memorisation of procedures and answers to problems.

According to Cruz and De Smedt (2013), a learner who understands a concept is able to re-identify entities with fair reliability under a wide variety of conditions. Understanding a math concept, means being able to think about and process mathematical facts abstractly.

Conceptual understanding refers to the learner's ability to comprehend key ideas and to draw inferences about those ideas. It also involves being able to strategically use them to solve problems and to learn new concepts and avoid common misunderstandings.

Mathematical context*:* The term 'context' means several things when used in an educational setting. Fraser and Greenhalgh (2001) viewed context as the learning environment or situation in which learning takes place while Van Den Heuvel-Panhuizen (2005) described it as a characteristic of a task presented to learners. These characteristics include words and pictures that help learners to understand the task, or concerning the situation or event in which the task is situated. In this study context refers to the situation in which some symbols are used.

Algebra is branch of mathematics in which arithmetic relations are generalised and explored by using letter symbols to represent numbers, variable quantities, or other mathematical entities. Algebra can be viewed as a human activity that deals with the construction of tools and knowledge that can be used for solving recognisable problems (Drijvers, 2011). On the other hand, algebra can be viewed as a brain activity that deals with the abstract world of mathematical object (Hansen & Gray, 2010).

A "**procept"** is word derived from the work of Gray and Tall (1994) which refers to a combination of: a process (for example addition*)* which produces a mathematical object (sum) and a symbol(s) which is/are used to represent either process or object.

A multiple meaning mathematical symbol refers to a mathematical symbol, which can represent more than one mathematical entity, or a symbol for which multiple instructional definitions exist (Phillips, 2008). Some symbols have different meanings in different contexts. Multiple meanings of letter symbols are a source of difficulties in algebra. Note, however, that this is also, what makes algebra a powerful language and thinking tool. Multiple meanings can create obstacles in mathematical conversations because learners often use colloquial meanings while the teacher (or other learners) may use mathematical meanings (Moschkovich, 2007).

1.11 Significance of the Study

This study contributes to the understanding of challenges that learners and teachers encounter in the learning and teaching of mathematical concepts through symbolisation. It also explores how learners perceive and think about mathematical symbols and how such processes are affected by how they interpret mathematical symbols. The aim was to identify and describe the challenges that secondary school learners encounter when interpreting and using mathematical symbols to understand mathematical concepts and problem solving procedures. Specifically, the researcher sought to obtain insights into learners' perceptions about working on and communicating with mathematical symbols during mathematical engagements in different settings as well as using textbooks. Furthermore, the study suggests instructional strategies that mathematics teachers can use to mitigate the effects of symbolic obstacles.

1.12 Limitations of the study

Researches, both qualitative and quantitative have limitations and delimitations. The limitations of the study are those characteristics of design or methodology that set parameters on the application or interpretation of the results of the study; that is, the constraints on generalizability and utility of findings that are the result of the devices of design or method that establish internal and external validity. Limitations refer to the scope of the study (Simon & Goes, 2013). Creswell (2002) defines limitations as potential weaknesses in a study that the researcher has control over. These constraints affect the generalizability and utility of findings that are the result of the ways in the design of the study was chosen and/or the method used to establish internal and external validity.

In this study, the researcher combined both probabilistic and non-probabilistic sampling procedures. Thus, the outcomes of this research cannot be generalised to all the FET learners in Limpopo province, but can only be used as a guide for further study. A longitudinal study could have been conducted over an extended period to obtain topic-specific difficulties with mathematical symbols; however, this was not possible due the limited time allocated for research activities in the selected schools. The study only included learners drawn from the FET phase in the selected districts of Limpopo province. The study was also restricted to learners enrolled in the FET phase. Limpopo province is mainly rural; hence, participants were drawn mainly from rural settings.

1.13 Delimitations of the study

Delimitations refer to the boundaries set by the researcher in order to control the range of a study (Sharma, 2014). In this instance, the delimitations in social research refer to the various boundaries used in the study such as the participants, instruments used, and the geographical placement. The delimitations are characteristics of the study that can be controlled by the researcher such as limiting the scope and defining the boundaries of the study (Simon & Goes, 2013). This study was delimited to questioning learners enrolled in grade10-12 and teachers teaching mathematics at this level. Furthermore, the area of mathematical symbolisation is broad and can be studied from different perspectives. This study has been narrowed to explore and gain insights into learners and teachers' perceived mathematical symbolisation challenges. The study is specifically intended to provide information that may be used to change the complexion of mathematics instruction especially in South African rural secondary schools. The results of this study can be generalised to other South African provinces with same characteristics especially rural settings. However, the results may not be generalised to urban and white dominated schools.

1.14 Assumptions of the Study

According to Creswell and Plano Clark (2007), most research studies are grounded in a variety of assumptions and all designs are confined by sundry limitations. According to Leedy and Ormrod (2010):

> *"…assumptions are so basic that, without them, the research problem itself could not exist"* (p. 62).

A number of assumptions peculiar to this study and to studies of this nature were identified. This study utilised a survey research design, which rely mainly on questionnaires and interviews for data collection. One assumption that was made in this study was that the information supplied by participants was accurate and truthful. The researcher also assumed that the questions in both instruments were sufficiently valid, reliable and addressed the issues under investigation based on the pilot survey findings. The inclusion criteria of the sample were appropriate and therefore, assure that the participants have all experienced the same or similar phenomenon of the study. Prospective participants for the study were deemed suitable since they had enough exposure and experience with the symbolic language of mathematics. A mixed methods research approach (MMR) was utilised based on the assumption that the use of both quantitative and qualitative approaches provides a better understanding of research problems than either approach alone.

1.15 Overview of thesis chapters

This thesis is divided into six chapters.

Chapter 1

This chapter introduces the study. It begins by presenting a synopsis of the background and motivation for the study as well as highlighting some of the problems faced by learners in learning mathematics through symbolisation. Some of the learners' challenges were identified and highlighted from the researcher's experiences as a mathematics teacher. The research questions and hypotheses were also stated and briefly discussed.

Chapter 2

This chapter reviews the literature on the issues and challenges currently experienced in mathematics education due to mathematical symbolisation. Key aspects and themes were outlined in relation to how they influence learners' understanding of mathematical concepts. The chapter also discusses, in detail, the theoretical perspectives that underpin this study, namely, Arcavi's (1994) symbol sense, Pierce and Stacey's (2001) framework for algebraic insight, Dubinsky and McDonald's (2002) APOS theory and Tall's (2004)

Procept Theory. These frameworks provide explanations for associating mathematical symbols and their meanings. Symbol sense and Algebraic insight are problem-solving frameworks while APOS and Procept are frameworks of conceptual growth. The four frameworks allow researchers to evaluate learners' understanding of mathematical symbols and observe the way learners learn. Furthermore, they help teachers to cover a wide spectrum of representations in the classroom that would help learners build symbolic fluency.

In Chapter 3

In this chapter, the methods used to collect data in this study are outlined. The main theoretical influences on the methodology of the study as well as the processes of data collection and analysis are discussed. The chapter highlights issues related to data collection methods, research approach, ethical issues, trustworthiness and generalisability in research. This study proposes a mixture of qualitative and quantitative researches. The collection of data report is a hybrid consisting of questionnaire and focus group interviews.

Chapter 4

This chapter presents and analyses data associated with learners' challenges with mathematical symbols and teachers' instructional strategies to alleviate the difficulties. The organisation of the report is a hybrid form consisting of descriptive and statistical reports. Responses from questionnaires and interviews were analysed and categorised into themes, which are eventually used to report the findings.

Chapter 5

This chapter discusses the findings in relation to the research questions, the literature reviewed and the conceptual frameworks that guide the study. Lessons emerging from the study are discussed in relation to the two domains of interest in this study: mathematical symbolisation challenges and teachers' instructional practices.

Chapter 6

This chapter summarises the main conclusions concerning learners' challenges with mathematical symbols and teachers' instructional strategies, arrived at in this study. This chapter also sets out limitations of the study, implications of findings, directions for further research and concluding remarks.

1.16 Summary

This chapter introduces the study on the challenges experienced by learners due to mathematical symbolisation. The focus of the study is to gain insights into learners' difficulties with mathematical symbolisation and sensitise teachers so that they can prescribe appropriate intervention strategies. The chapter also outlined the background, the problem statement, the motivation for the study. Pertinent research questions and the general and specific objectives were also addressed. A brief outline of the chapters of the study was also provided. This chapter provided a summary of what the study intends to investigate. The next chapter reviews the literature and the conceptual framework related to the study.

CHAPTER 2: LITERATURE REVIEW

The review of literature in this study is organised thematically. The discussion is organised around themes and theoretical concepts related to challenges and instructional strategies for teaching mathematical symbolisation. This structure is preferable to the chronological organisation because it enables the researcher to define the theories and constructs that are important to the study (Levy & Ellis, 2006). The sequence of themes moves from broad to specific in a funnel approach where the discrete sub-concepts and themes are funneled from higher-level concepts to the specific cases upon which this research is based.

The chapter provides an overview of current and previous research on mathematical symbolisation. It connects and correlates the current study to findings of previous relevant research and expert opinion on symbolism. It provides a justification for the need to review literature concerning the symbolisation challenges experienced by learners and the instructional practices on the use of mathematical symbols. The chapter also discusses and connects a number of frameworks that guide the study. The purpose of reviewing literature is to survey previous studies on knowledge regarding the challenges of mathematical symbolisation and link it with current trends and classroom practices. The review looks at the nature of mathematical symbolism, the role of symbolism and learners' difficulties with symbolism. It also provides detailed insights into the reasons why learners have trouble with symbols when learning mathematical concepts and during problem solving. The reviewing literature was done to guide the selection and identification of key data collection requirements for the research to be conducted, and it forms part of the emergent research design process (Giles, King & de Lacey, 2013).

The discussion of literature is divided into sections and each section revolves around a theme. In the first section, the discussion involves literature about the use of symbolic representations in mathematics. It discusses literature related to: (a) the processes of mathematical symbolisation in mathematics education, (b) the challenges and difficulties experienced by learners in learning mathematics concepts through symbolisation (c) instructional strategies for teaching mathematics through symbolisation (d) connections among symbols and concepts. The second section discusses the pedagogical strategies recommended by various researchers for teaching and learning mathematical concepts and symbols for understanding. The third section discusses the conceptual framework and theories that support the teaching and learning of mathematical concepts through mathematical symbolisation. Four (4) conceptual frameworks: Symbol sense symbol sense (Arcavi, 1994), Algebraic Insight framework (Pierce & Stacey, 2001), APOS theory (Dubinsky & McDonald's, 2002) and Procept Theory (Gray & Tall, 1994) were condensed into a quadrilateral frame of theories and serve as lens for focusing and guiding this study.

2.1 Mathematical Symbolisation

Santos and Thomas (2011) define symbolisation as a process that involves forming a correspondence between a mathematical concept and its meaning. Chandler (2007) conceives a symbolic representation as an externally written or spoken symbol that stands for something other than itself. According to Godino, Godino, and Batanero (2003) symbolisation refers to the relationship between the represented and the representing worlds. Symbolic representations such as formal equations and line graphs eliminate extraneous surface details, are arbitrarily related to their referents, and represent the underlying structure of the referent more efficiently. Thus, they allow greater flexibility and generalizability to multiple contexts, but may appear as meaningless symbols to learners who lack conceptual understanding (Nathan, 2012).

Symbolisation is also viewed as a process involving assigning meanings and defining relationships between mathematical objects and their external representations (Thomas, 2003). Symbols are used by teachers and other experts in mathematics to code problem-solving situations and context into symbolic forms. These forms allow the problem to be solved without reverting to the original real-life problem situation. Symbolic forms or representations take various forms such as graphs, symbols, language and organisational schemes that describe the concept. According to Kollár (2014), symbolisation is engrained in a learner's ability to interact with the external environment. Symbolisation produce mental structures, which when acted upon by the mind produce mental or cognitive structures (Fiorini, Gärdenfors & Abel, 2014). Thus, meanings of mathematical concepts evolve through the association of mental operations with mathematical symbols (Kvasnička, 2008).

Mathematical symbols serve several roles such as illustrating and describing the structure of mathematical concepts, manipulation routines such as addition, subtraction, division and multiplication (Steinbring, 2006). Mathematical symbols allow teachers to express mathematical concepts compactly and help learners to make reflections about mathematics. Mathematical symbols allow thought and solution processes to be expressed permanently (Rubenstein & Thompson, 2001). Ganesalingam (2013) describes mathematical symbols as characters of written mathematic statements that are important for the construction of mathematical knowledge. Written mathematics differs from other disciplines with the property of having vast amounts of symbols. Farrugia (2013) also singles out the symbolic feature of mathematics as the subject's most apparent and distinctive feature. The symbolic language of mathematics often presents learners with challenges as they try to write, read and verbalise these symbols.

Delice and Aydin (2006) found that learners conceive symbols as objects with some meaning rather than thinking of process-object duality. At high school level, it has been observed that, the processes of manipulating symbols meaningfully with correct procedures and notation varies from learner to learner (Fyfe et al, 2014). Learners have difficulties in expressing their thoughts using appropriate mathematical symbols. When learners memorise mathematical expressions, they conceive symbols as objects with some meaning rather than thinking of process-object duality. According to Santos and Thomas (2001), learners' inability to see a mathematical concept from two perspectives, the symbolic and its description form seem to limit learners during problem solving.

Symbols are special features of mathematical representations. Harel and Kaput (2002) describe symbols as strings of characters used to represent a mathematical process or object. The symbols are the mathematical marks that do not constitute ordinary language, and are manipulated according to certain well-defined rules. Even though symbols have specific mathematical meanings, learners often have their own constructed meanings that are shaped by socio-cultural factors, experiences, knowledge and cognitive abilities. Learners understand mathematics if they are actively engaged in the construction of new knowledge from past experiences. Hence, learners' past encounters with some of the mathematical symbols and concepts influence their understanding of new symbols and concepts (Luna & Fuscablo, 2002).

2.1.1 Mathematical symbols and symbol systems

There is a need to clarify the distinction between symbols, symbol systems and symbol products. A symbol is any entity or object, whether material or abstract that stands for another object (DeLoache, 2004). Langer (2009) defined a symbol as an "an instrument of thought", that enables us to think about something and to form a concept in the absence of that object itself. They are, according to Vygotsky (1978), "tools for the mind." Symbols create those possibilities of thought that are uniquely human. Pierce (2006) asserts that symbols have a triad-meaning, which suggests that meaning arise from a relationship among three things: the object or referent, the person (interpreter) and the sign. The sign presents the object in the mind of the interpreter. Meaning thus depends on the mental image or thought of the person in relation to the sign and the object the sign represents. The most distinctive feature of Peirce (2006) account is best thought of as the understanding that we have of the sign/object relation. The importance of the interpretant for Peirce (2006) is that signification is not a simple dyadic relationship between sign and object: a sign signifies only what is being interpreted. This makes the interpretant central to the content of the sign, in that, the meaning of a sign is manifest in the interpretation that it generates in sign users.

Systems of symbols are human inventions and thus are cultural tools that have to be taught. Mathematical symbols are human-made tools that improve our ability to control and adapt to the environment. Each system makes specific cognitive demands on the learner, who has to understand the systems of representation and relations that are being represented. Learners can behave as if they understand how the symbols work while they do not understand them completely: they can learn routines for symbol manipulation that remain disconnected from meaning. Learners acquire informal knowledge in their everyday lives, which can be used to give meaning to mathematical symbols learned in the classroom.

Mathematical symbols do not necessarily need to have any logical or natural connection to the things they represent (Wolfram, 1999). Symbolic systems provide the structuring matrices of human consciousness. Symbols for mathematical concepts assume various forms such as diagrams, pictures, variables, tables and numbers. A symbol system is a combination of symbols that are arranged in as specific manner according to some rules (Pollatsek & Treiman, 2015). Symbol systems sustain entire realms of thought in pure abstraction (mathematics comes to mind). In so doing they create additional ranges of human consciousness that simply would not exist in their absence. Moreno-Armella, Hegedus and Kaput (2008) add that symbols are meaningful if they correspond to known fields of reference. The field of reference gives meaning to symbols and rules for combining them. Symbols are entities that the mathematics community created in order to communicate mathematical knowledge with other experts in the subject. Symbols are part of mathematical language with unique meanings that others in the field can understand, interpret, appreciate, criticise or transform.

Another way of comprehending mathematical symbols is to consider the context in the symbol is being used and topics being studied (Szydlik, 2015). As reported by Ongstad (2007), meanings are also derived from convention, that is, meaning of particular symbols were decided and agreed upon by mathematicians and scientists. Symbol systems are those cognitive "tools" that, often in written form, allow us to record and communicate ideas without the immediate presence and participation of actual things in the environment. Symbols allow us to entertain ideas because they serve to evoke those ideas.

One area of mathematics that requires learners to be fluent and competent with symbols is problem- solving. Problem- solving is a critical mathematics skill that requires learners to convert a problem from a symbolic representation to an alternative form. Many South African secondary school mathematics learners lack this skill and problem-solving continues to be a serious challenge for them especially in financial mathematics and applications of derivatives (Brijlall & Ndlovu, 2013). To solve problems in mathematics learners, need to be competent in the three senses: number, symbol and function. If learners do not recognise a symbol or misinterpret the vocabulary of a symbol, their performance may suffer (Powell, 2011). A study conducted by Shavelson, Webb and Lehman (2000) indicates that learners' understanding of mathematics content depends on how learners decode and symbolically represent information to themselves (aptitude). Consequently; learners' understanding of a mathematical concept depends on their interpretation of symbols used in instruction. Learners' understanding and interpretation of mathematical concepts depend on their preferred mode of representation.

2.1.2 Meaning of mathematical symbols

The term 'symbol' refers to different things in the branches of mathematics. In mathematics and other scientific fields, it refers to a mark that is mapped to some referent object or point (Deacon, 2011). It can be combined with other marks according to specific rules. In this way, a symbol is conceived as a code that represents a mathematical concept. In the context of this study, a mathematical symbol contains two ideas: that of the signifier and that of the signified. Developing meanings of symbols is a compound process of conjectures, analyses, and descriptions of the sense, in this case, the concept that the symbol might represent. Studying the development of the meaning of symbols has strong implications for the study of understanding.

Harel, Fuller and Rabin (2008) suggested that meanings of mathematical concepts are best learned by paying attention to the context in which they are used. They noted that learners manipulate symbols without a meaningful basis that is grounded in the context of the symbols. This behaviour of operating on symbols as if they possess a life of their own, rather than treating them as representations of entities in a coherent reality, is referred to as the non-referential symbolic way of thinking (Harel et al, 2008). Sapire (2011) observed that when reading symbols, words, and letters do not make or carry meaning until the reader associate them with real life contexts. Thus mathematical symbols are brought to existence through associations and ideas that learners and teachers bring into mind during the teaching and learning process. As recommended by Phillips (2008) mathematics teachers need to keep this in mind before they attempt to introduce mathematical symbols in general.

Mathematical symbols are important for representing mathematical ideas and problem solving procedures but the learner interacting with them has to endow them with meaning, has to keep the context in mind, and recognise models and actions associated with the symbol. It is also worth mentioning that symbols are not the main goal of learning mathematics, they are not static (Langer, 2009). In some cases, they represent processes of mathematics, and in other instances, they represent mathematical objects (Gray & Tall, 1991). Symbols at times are construed as ‘objects’ which can be used without having to root them back in any model or context. Thus, there are dangers of being misled if teachers look at learners’ symbolic representations and manipulations and assuming that these reveal what they know about mathematics. Based on the findings of Naidoo (2009) it can be argued that many learners are proficient in using the rules for manipulating symbols without having a strong sense of what the symbols represent.

According to Amit and Neria (2004), the meanings of mathematical symbols are derived from four main sources: algebraic structure (letter-symbol form), other mathematical representations, and problem context and real-life applications. A number of researchers have attempted to distinguish between the meanings attached to features of symbolic inscriptions. Skemp (1987) describes two levels of structure related to features of symbolic inscriptions: surface structures and deep structures. Surface structures involve the written symbols, whereas the more difficult deep structures of language are those that involve the conceptual meanings of the symbolic inscriptions. In a similar manner, Yerushalmy (2005) differentiates between two levels of meaning learners attach to symbolic inscriptions. At the lower level is syntactic manipulation in which learners operate with basic algebraic rules such as order of operations. These are constructed from common mathematics instructions such as expanding brackets, collecting like terms, reducing to lowest terms and taking out the common factors. The other set of meanings for mathematical symbols is derived from semantic interpretations of higher cognitive properties of algebraic expressions such as number of zeros of a polynomial, degree of a polynomial, remainder, parameters, or constraints.

Perceptual symbolism is another source of difficulty for learners (Ottmar, Landy & Goldstone, 2012). Perceptual symbols are symbols that arise from performing a mathematical action. In an action such as counting, the symbols used$(1; 2; 3; 4 \ldots.)$ are thinkable concepts, such as number. A symbol such $\int \sin x \, dx$ represents both a process (integration) to be executed and the resulting thinkable concept (integral).Tall (2008) refers to such an amalgamation of symbols, processes, and concepts as “procept”.

2.1.3 Learning mathematics through symbolisation

Kenney (2008) viewed mathematical symbols as the objects of mathematical language that facilitate communication between teachers and learners. The function of symbols in the teaching and learning process is well documented in literature. However, their impact on conceptual understanding and learner achievement remain largely unexplored. Bergen (2002) and Azzarello and Edwards (2005) acknowledge that linking mathematical concepts and operations or processes to mathematical symbols is a complex intellectual activity. This is because symbols lack a one-one correspondence with their meanings or references. The semiotic structure of mathematical concepts and processes causes conceptual difficulties for learners due to the multiple ways in which symbols are used. Symbols perform multiple functions such as naming, labelling, signifying, communicating, simplifying, representing, revealing structure, and displaying relationships (Moschkovich, 2015). Symbolisms play a crucial role in teaching and learning mathematics. They allow communication of mathematical ideas to the learners in a coherent and consistent way and provide a common language that the members of teaching-learning community use to express their thoughts, to share their ideas with the others, and to reflect collectively upon a mathematical notion being investigated (Bayazıt & Aksoy, 2010). Because of the multiplicity of interpretations and meanings of mathematical symbols, it is not surprising that the symbolic language of mathematics confuses learners (Kailikole, 2009).

Expert mathematicians or mathematics teacher are able to manipulate mathematical representations, whereas learners struggle. As learners are schooled they learn the symbols, they learn the meaning of the symbol and the use of the symbol. These meanings and uses are established in relation to the other symbols in the system. The whole gestalt of meanings has to be negotiated, revisited from time to time, and adjusted as necessary. Once the symbolic connection has been established between the symbol and its object, then we are able to set the objects aside and operate only with the symbols. This defines abstract thought.

In classroom contexts where learners' experiences differ, new symbol are variously interpreted. During mathematics classes, learners try to assimilate new symbols to their existing schemas, which may bring clusters of templates where it may fit, evoking meaning within available schemas derived from individual prior experiences. The meaning constituted by the symbol is adopted when learners discuss its meaning among themselves, or with the teacher, through negotiation. Thus, the negotiation of meaning between the teacher and learners is essential, as the teacher directs to learners to understand the symbol, together with its meaning. Sfard (2000) recommends that conversational feedback play a central role in discursive and experiential background for the introduction of the symbol.

Mathematical symbols paved the way for the translation of human activities into symbolic models. Symbols are needed to deal with quantity, shape and change. This is how mathematics was born. Mathematics is a symbolic version of nature built on basic intuitions. When learners are dealing with a certain symbol for the first time, the reference field can be very narrow. However, as they progress with learning, they become more proficient with its use, and the corresponding reference field begins to widen. Various researchers have stressed that the symbolic formulation of relations between variables raise specific problems for novice learners (Azzarello, 2006; Radford, 2008). Although particular difficulties experienced by learners have been widely reported and documented by the aforementioned works, Radford (2011) argues that more research is still needed since learners continue to struggle to endow symbols with meaning.

The history of mathematics evolved through a series of attempts to represent the mathematical concepts symbolically. Despite concerted efforts to produce clear and concise symbolically representative systems, most attempts have resulted in imperfect representations. Such imperfect systems ended up with too few symbols, too many symbols, unclear symbols, or symbols which carry multiple meanings. For example, the ancient Babylonians failed to create a symbol to represent the quantity zero. This omission led to confusion and uncertainty regarding the precise quantity embodied by various symbolic representations (Cajori, 1993). Similarly, the Romans never developed a symbol for zero and introduced an additional element of confusion by allowing the symbols $(I = 1), (V = 5), (X = 10), (L = 50), (C = 100), (D = 500)\ and\ (M = 1000)$ to embody multiple meanings as both letters of the alphabet and numerals (Keppie, 2002).

Attempts to develop symbolic representations in the various cultures such Chinese, Sumerian, Greek, Phoenician, and Cadmea cultures led to further communicative complications and confusions. According to Sun (2006), most mathematical symbols have multiple meanings, inconsistent and ambiguous. For example, the ancient Sumerians had six different symbols, used interchangeably to represent the modern day letters O and U (Waddell, 2004). Thus the impact of incomplete or overabundant and multi-meaning symbolic systems and the detriments of employing unclear symbols are important and certainly worthy of studying. These detrimental effects of symbolic representations infiltrate classroom discourse, influence instructional practices and affect learning outcomes. The confusion associated with the use of multiple meaning symbols has detrimental effects on learners' conceptions and understanding of mathematical concepts. There is limited research on instructional use of multiple meanings and abstract nature of math symbols as well as their impact on learners' comprehension of mathematical concepts.

The development of mathematical symbols is a result of conventions by the mathematics community, comprised of mathematicians, teachers and theorists. Conventions are agreed by the mathematics community and lead to the use of certain symbols to represent mathematical properties, operations, or concepts, thereby endowing such symbols with meanings beyond the symbols themselves. Many researchers have conducted studies on the impact of symbols on mathematics education (Rubenstein & Thompson, 2001; Adams, 2003; Steinbring, 2006). Another group of scholars (Shaftel, Belton-Kocher, Glasnapp & Poggio, 2006) investigated the instructional use of multiple meaning of mathematical words, but very little has been explored on perceived learner confusion resulting from the use of such symbols in trying to understand mathematical concepts as well as instructional strategies to foster understanding.

2.1.4 Multiple meanings of symbols

The multiple meaning nature of a mathematical symbol is another feature of mathematical symbols that confuses most learners (Chirume, 2012). The cognitive objects to which the symbols and words refer are constructs that reflect different webs of meaning that, for each individual, might be said to be part of their personal system of algebra (Drouhard & Teppo, 2004). Mathematics, as a scientific field requires learners to think and organise their thinking in terms of symbols, concepts and abstract ideas. Garrison and Mora (1999) describe mathematics as a subject in which ideas, words and relationships are compressed into a single symbol. For instance, a set of parentheses () has at least five different meanings depending on the context and situation under consideration. Such multiple meanings have the potential of introducing confusion and disorientation for mathematics learners as they attempt to remember all the applications of the same symbol and the appropriate circumstance in which to use each one.

Parentheses are used as grouping symbols in order to facilitate the order of operations when simplifying mathematical expressions. They are also used to indicate multiplication between two terms. Another common use of parentheses is to indicate a point on a graph such as $(3,5)$. Parentheses are also commonly used in function notation $f(x)$ to define relationships between variables. This particular representation possesses the greatest potential for learner confusion in that, at first glance, two terms separated by parentheses appears to be representing multiplication, since $2(3) = 6$ or $h(2) = 2h$. King (2002) observes that many novice algebra learners not only struggle with the concept of functions but also mistake function notation $f(x)$ as a multiplication indicator fx. Finally, parentheses can be used to indicate a range of numbers on a number line such as in $(3, 5)$. This particular symbolic presentation is designed to convey the meaning that one wishes to consider all of the real numbers which are greater than three and less than five $(3 < x < 5)$. It is particularly problematic since it takes on the exact form used to indicate a point on a graph.

Working fluently the language of mathematics requires learners to develop a strong symbol sense (Essien, 2011). Symbol sense involves having an ability to create symbolic relationships that represent written information; experiencing different roles played by symbols; and appreciating the power of symbols as tools for displaying and explaining relationships expressed using natural language (Arcavi, 2005). However, it is not easy for learners to connect natural language and symbolic representations, particularly in the context of word problems. Mathematics is a language in itself, composed of natural language and a symbolic system of mathematical signs, graphs, and diagrams (Hammill, 2010).

Mathematical language is heavily dependent on the symbolic language that includes syntax and organisation of symbols and the natural language of instruction (Moschkovich, 2007). On the other hand, mathematical notation enables ideas and concepts to be expressed unambiguously and to enable and encourage a corresponding way of thinking. Mathematical symbols are essential for coding, constructing and communicating mathematical knowledge. However, they do not carry mathematical meaning and conceptual ideas themselves. Instead, meaning is negotiated through interaction with the symbol and its reference.

Schleppegrell (2007) explains that an interplay between symbolic and natural language is clearly present when solving word problems where learners are required to decode not only the language of the question and the overlaying context, but must also have knowledge of and be able to represent words with the appropriate mathematical symbols needed to effectively solve the problem. Recent developments in mathematics have shown that many learners encounter difficulties when making connections between words and mathematical symbols in word problems (Reynders, 2014; Sepeng and Madzorera, 2014). Some of the suggested reasons for added difficulties for learners on word problems include a lack of built-in contextual clues found in literary narratives (Fernandes, Anhalt & Civil, 2009), unfamiliar cultural contexts and interpretations (Solano-Flores & Trumbull, 2003), reading comprehension issues (Schleppegrell, 2007), and the artificial contexts of word problems (Wiest, 2001).

Many countries, including South Africa adopted the Arabic system of numeration, thereby making symbols universal in mathematics. However, this symbol universality across languages is heavily criticised for encouraging teachers to move too quickly to the symbolic expressions before the conceptual foundation has been built (Sloutsky, Kaminski & Heckler, 2005). It encourages learners to acquire the skills for manipulating symbols without a proper conceptual foundation. Consequently, this limits their progress into higher mathematics, since they lack the basis for conceptual foundation for advanced mathematics. Learners should access the language of mathematics through multiple semiotic systems that fulfil different functions: (a) natural language introduces, contextualizes, and describes a mathematical problem; (b) symbolism is used for finding the solution of the problem; and (c) visual images deal with visualizing the problem graphically or diagrammatically (de Oliveira & Cheng, 2011). All of these systems may involve vocabulary, sentence structures, contexts, and representations that are new or unfamiliar to learners.

Clement (2004) noted that learners often find it relatively easy to represent mathematical concepts in a variety of modes such as manipulatives, pictures, diagrams, spoken languages. However, the same cannot be said about the written form that is dominated by symbols. It is this symbolic nature of mathematics that scares them. Previous studies on mathematical symbolism have demonstrated a series of misconceptions learners have when using mathematical symbols. For example, Knuth, Stephens, McNeil and Alibali (2006) outlined learners' misconceptions with the equal sign. Primary school learners often misinterpret the equal sign ($=$) as an operational instead of a relational symbol. Learners often view symbols as labels for objects (Christou, Vosniadou & Vamvakoussi, 2007). Many learners mention the use of symbols as the origin of their difficulties, saying that they understood mathematics algebraic symbols were introduced (Christou and Vosniadou, 2005).

Another difficulty that learners experience when using symbols is the use of symbols is known as 'lack of closure' error (Herscovics & Linchevski, 1994). This error is committed when a learner does not accept symbolic expressions as final answers. For example, when simplifying: $\frac{2}{5}(5x+10) = 2x+4$, learners may proceed further to solve for x, $2x = 4$, $x = 2$. Christou et al. (2007) suggested that learners view mathematics as an empirical subject, where mathematical calculations must always lead to numerical answers only. When learners are introduced to a new topic, they face the difficult task of assigning meanings to new symbols and assigning new meanings to old symbols, which they learned in the previous topics. A study by Chow (2011) reveals that learners' misconceptions, errors, and cognitive dissonance with the use of symbols originate from the inappropriate transfer of prior knowledge from previous encounters.

An interplay between symbolic and natural language is clearly present when solving mathematical word problems where learners must be able to translate not only the language of the question and the overlaying context, but should also have knowledge of and be able to represent words with the appropriate mathematical symbols needed to effectively represent the situation and answer the question. For some learners, mathematics presents a “third language” which is heavily symbolic and too specific (Reynders, 2014). Some of the suggested reasons for added difficulties for learners on word problems include a lack of built-in contextual clues found in literary narratives (Kenney & de Oliveira, 2012), unfamiliar cultural contexts and interpretations (Wilburne, Marinak & Strickland, 2011), reading comprehension issues (Schleppegrell, 2007), and the artificial contexts of word problems (Wiest, 2001).

2.1.5 The influence of symbols in algebraic thinking

If learners are unable to see abstract ideas beneath the symbols, they develop an impoverished understanding of algebraic concepts (MacGregor & Stacey, 1997). As learners progress into secondary and tertiary scientific fields, symbols play an indispensable role in representing mathematical concepts. The transition from arithmetic thinking to algebraic thinking requires learners to make sense of the symbolic notation. Brijlall and Ndlovu (2013) lamented of the cognitive gap between learners’ arithmetic and algebraic thinking. They noted that learners lack skills to operate with or on the unknown as they move to algebraic thinking. They reported that learners are not able to view literal symbols as generalized numbers and unable to operate with the symbols themselves. If learners are not given sufficient time to develop this type of meaning, many will struggle to progress from arithmetic thinking to algebraic thinking. As a result, when learners fail to construct meaning for the new symbolism and they resort to performing meaningless manipulations of symbols without understanding their meanings.

2.1.6 Mathematical symbols and signs

It is important to provide a clarification of what mathematical symbols and signs are. Jao (2012) described symbols as abstractions entities that represent of mathematical ideas, concepts, or processes. This study adopts Foucault (1966) and Presmeg (2006)'s definitions that regard a symbol as a signifier that represents, signifies, or replaces a mathematical idea, concept, or process. Rouse (2000) also defined a mathematical symbol as a character that is used to indicate a mathematical relation or operation. Combining the two definitions, Redish and Gupta (2009) concluded that mathematical symbols have a definite initial purpose: to methodically unpack complex information in order to facilitate understanding. Steinbring (2006) described mathematical signs as means of communicating abstract mathematical ideas (oral function), of indicating (deictic function) and of writing (symbolic function). Mitchelmore and White (2008) referred to mathematical symbols as shorthand marks that are used to represent mathematical concepts, ideas and processes. Hiebert (1988) defined symbols as entities that represent mathematical ideas or processes. Researchers in mathematics education have concurred that the development of mathematical notation is closely connected with the overall development of the concepts and methods of mathematics (Cajori, 2010).

The connection between the meaning of a concept and its mathematical symbol is not always obvious. Various notions of the meaning of symbols have been studied in mathematics education. Sowa (2010) identified mathematics as one area that lacks a one-to-one correspondence between mathematical symbols and the words/concepts they represent. In order to understand mathematical symbols and their meanings there are two things to help us; the context in which we are working, or the particular topics being studied, and convention, where mathematicians and scientists have decided that particular symbols will have particular meanings. Tall (2004) hinted that mathematics is powerful because of its symbolism. He noted two contrasting effects that written symbols have for learners as a two-edged sword: they can help them cope or they can overwhelm them. Thus, mathematical symbols, interpreted as either processes or objects, symbols allow a duality of thought. According to Tall (2004) this view is a perceptual *divide*: only those who come to think flexibly about processes and objects become successful in mathematics. Gray and Tall (1991) define a "*procept* "as a combination of a process and a concept in which a mathematical process and object/ product is represented by the same symbol. Thus according to this view the symbol for a procept can evoke either process or concept. For example the sight of the symbol, $f'(x)$ or $\frac{dy}{dx}$ invokes the process of differentiation and a derivative at the same time.

2.1.7 The nature of mathematics

By tracing the history and developments in mathematics, one gets the impression that the essence of modern mathematics is symbolic mathematics. Mathematics is the construction of knowledge that deals with qualitative and quantitative relationships of space and time (Mdaka, 2006). Thus, mathematics is a language that has its own symbols, syntax, grammar, and a variety of representations. It also relies on an intensive use of different types of symbols to represent variables, signs for numbers, diagrams, formulas, and algorithms. The dominant entities that dominate mathematics are numbers and algebra. These involve processes that are eventually symbolised into both process and concepts. However, the dual use of a symbol as either process or concept causes great difficulty for many learners. Tall (1992) asserts that symbols on their own cannot provide a complete environment for mathematical thinking. They are more powerful if they do so in a flexible proceptual way. The power is further enhanced if there are alternative representations available that increase the flexibility of thinking.

Mathematics can be viewed as a human cultural activity that deals with patterns, problem-solving, and logical thinking in an attempt to understand the world and make use of that understanding (Adler, 2006). This understanding is expressed, developed and contested through language, symbols, and social interaction. Mathematical literacy provides powerful numeric, spatial, temporal, symbolic, communicative, and other conceptual tools, skills, knowledge, attitudes and values to analyze; make and justify critical decisions; and take transformative action in society. Reynders (2014) observed that one of the problems for mathematics learners is related to syntax, the sentence structure and semantic components of language in the mathematics classes. The lack of one-to-one correspondence between mathematical symbols and the concepts they represent was singled out as one feature that present problems to learners.

2.1.8 Differentiating symbol systems

Researchers proposed two dimensions that can be used to differentiate symbolic systems: resemblance and notationality (Blumson, 2014). Resemblance refers to the extent to which symbols resemble their referents. Symbols that resemble or look like their referents are called iconic symbols or replica models, for example, geometrical shapes such as a rectangle to represent a rectangular field. The advantage of using iconic models is the models' correspondence with the reality of appearance. In other words, the model user can tell exactly what the proposed object will look like. Schematic models are more abstract than physical models. While they do have some visual correspondence with reality, they look much less like the physical reality they represent.

Graphs and charts are schematic models that provide pictorial representations of mathematical relationships. Symbols that do not represent their referents are referred to as analogues. Various researchers classify mathematical symbols and systems differently. Sowell (1974) classify symbols as concrete, pictorial and abstract while Shavelson, Webb, and Lehman (1986) classifies symbols as representational (realistic depictions), conventional (symbols stand for ideas or events in a particular culture), connotative (symbols results from the distortions of conventional symbols) and qualitative (symbols represent some idea or feeling). However, this classification was heavily criticised by Goodman (1968) and Salomon (1979) who argued that resemblance is not a satisfactory way of defining symbol systems. They argued that resemblance is ambiguous since symbols can represent their references in multiple ways. They further argued that symbol systems can be notational, non-notational or somewhere between these two extremes.

Shavelson, Webb and Lehman (1986) provided an exhaustive distinction between notational and non-notational systems. In notational systems, the symbols are discrete and discontinuous and there is a one–to-one correspondence between symbols and their referents. In non-notational systems, symbols are not disjointed but are continuous and each element does not correspond to one and only one referent. For example, pictures are non-notational because each element could represent many things, for example, a line can represent length, depth and the picture could lead to many interpretations. However, notationality was criticised for being too abstract to help define taxonomies of symbol systems for particular knowledge domains. Harkin and Rising (1974) classified mathematical symbols into five categories: ambiguous symbols, synonymous symbols, archaic symbols, inappropriate symbols and contradictory symbols.

Ambiguous Symbols

This class of symbols consists of symbols whose meanings are not clear when the symbol is used in isolation. Context clues are necessary for clarification. A dash $(-)$ is an example of an ambiguous mathematical symbol that carries three distinct meanings. It carries meaning if it is part of a chain of symbols that represent a mathematical concept or process. For example, it can denote the binary operation of subtraction in $5-3$, in another context it is used to indicate a negative integer , $-4 < -3$, and it can be used as an additive inverse (opposite) of a number, for example, $-(-2) = 2$. It can also represent a range as $10 - 20$ in grouped data.

Sajka (2003) observe that, one of the learners' difficulties in understanding the concept of function stems from its dual nature. In fact, Dede and Soybas (2011) note that a function can be understood in two essentially different ways: (i) structurally, as an object; and (ii) operationally, as a process. In the first instance, the function is a set of ordered pairs, and in the operational way, it is a computational process or well defined method for getting from one system to another. These two ways of understanding functions, although apparently ruling out one another, however, should complement each other and constitute a coherent unit. For example, the function $f(x) = 2x + 3$ has two meanings. The first meaning is "how to calculate the value of the function for particular arguments (evoking the process), secondly it encapsulates the whole concept of function for any given argument (thus presenting the object). Therefore, the function $f(x)$ represents both a process and a concept. In addition, in the context of functions, when we write y, sometimes we are referring to a certain value of the function; at other times, we are referring to the ordinate of a certain point in the coordinates system, and yet in other times we are referring to an argument. The interpretation depends on the context, which can confuse a learner. This notation of function is ambiguous and presents some difficulties among learners. For Sajka (2003), the causes of learners' symbols difficulties also depend on the contexts in which the symbols are worked in mathematics classes, and on the teachers' limited choices of mathematical tasks. For some learners, the concept of a function is often linked to the concept of formula, and sometimes learners connect the concept of function to the graphing process, where a formula is necessary to draw it.

Synonymous Symbols

Synonymous symbols are multiple representations or a group of symbols that are associated with the same concept. For example, a linear function may be expressed in different ways or notations. These different notations and symbols invoke different conceptions of the concept. For example, the line with gradient of three and passing through the point $(0, 1)$ can be expressed in different ways: $y = 3x + 1$; $f(x) = 3x + 1$ and $f: x \to 3x + 1$.

Archaic Symbols

The language of mathematics is archaic. The notation used to describe mathematical objects and processes is confusing. The names that are assigned to the symbols and concepts are poor. Names are important. They drive our thoughts. However, when names become disconnected to the things they represent, they become a source of confusion (Lockhart, 2009). It is easy to forget if the symbols are separate from the references. For example, the sine of angle $A\hat{B}C$ in a triangle ABC drawn on the chalkboard is easier to conceptualise than $sin\ \theta$.

Inappropriate Symbols

Inappropriate mathematical symbols refer to symbols that encourage misconceptions due to the learner's level of intellectual attainment (Post, 1988). For example, a learner may think that letters of the alphabet represent objects or numbers, $h =$ height, $b = 2$, since b is the second letter of the alphabet. Learners may also simplify the expression $2m + 5 = 7m$ in two different mathematical contexts. These contexts are expanding brackets containing unknown and simplifying expressions by collecting like terms. Appropriate use of symbols should begin early in the primary grades; however, in the search by human intelligence or coherence in our world, misconceptions play an important transitional role. The world of the learner is particularly full of relativism. A learner's cognitive growth depends on his/her ability to establish the gross essence of concepts on an intellectual as well as a perceptual and an emotional level. The entire situation can be viewed as a structure of ideas in which he seeks those connections that seem most pervasive.

Contradictory Symbols

Contradictory mathematical symbols are symbols that have different meanings in forms or different topics. They are designated as inconsistent symbols (Lankham, Nachtergaele and Schilling, 2007). Many symbols mean different things in different contexts or topics. For example, the use of parentheses is a frequent source of confusion. For example, $2(3) = 6$ (product) but $f(x) \neq fx$ (function). To solve this confusion, learners must pay attention to the context in which mathematical symbols are used (Reys, Lindquist, Lambdin and Smith, 2014). When learners fail to give meaning to a symbol by drawing upon the context in which it occurs, they often give up on developing understanding of the symbols. Instead, they simply look for clues as to what algorithm the symbol suggests.

2.1.9 The Role of Symbols

Cockcroft (1982) viewed mathematical symbolism as both the strength and weakness of mathematical communication. Grey and Tall (1994) took this fundamental paradox a stage further; and regard mathematical symbolism as a major source of both success and distress in mathematics learning. Mathematics is taught symbolically because symbolic representations are the most effective way of recording mathematics and transferring mathematical knowledge from one generation to another (Anthony & Walshaw, 2010). Symbols are valuable in showing what one cannot say. They express inexpressible concepts, abstract ideas, and particularly complex significations that are difficult to articulate (Burbidge, 2013). Symbols are way of representing and expressing mathematical thoughts, knowledge, and communicate in discourse. Learners' ability to use symbols expands their cognitive and communicative power. Symbols are a means of taking the present into the future, the past into the present (Bevan, 2016). Symbol enables the present generation of mathematicians to learn from the proceeding generations. Because symbols are such an important source of learning and knowledge, it is important for learners to become symbol-minded (DeLoache, 2004). Symbols play a crucial role in advanced mathematical thinking by providing flexibility and reducing cognitive load.

They have a dual nature since they signify both processes and objects of mathematics (Güçler, 2014). It is important to note that the key to understanding mathematics lies with the interpretation and distinguishing between the concept and processes, and not really, in the nature or amount of symbols and the role they play.

However, to understand mathematical concepts learners must appreciate the role and meaning of symbols and to appreciate their usefulness. Symbols are useful as substitutes for abstract ideas. Arcavi (1994) and Pimm (1995) concur that at times learners work manipulate symbols correctly and efficiently without paying much attention to their referents. This practice has its roots in symbol pushing. Symbol pushing involves concentrating on the symbols rather than interpreting the symbols as representing concepts (Hersh, 2013). Crooks and Alibali (2014) reported that mathematical thinking is conceptual thinking and not procedural thinking. Symbols can be transformed or replaced while the meaning remains the same. Understanding mathematical symbols by "symbol pushing" is not real understanding. Teachers should strongly discourage this style of learning since it is unproductive in the long run and lead to erroneous conclusions such as $\sqrt[n]{x^3 + y^2} = \sqrt[n]{x^3} + \sqrt[n]{y^2}$ as a result of over-generalising the rules such as, $\sqrt{9x^4y^2} = \sqrt{9} \times \sqrt{x^4} \times \sqrt{y^2} = 3x^2y$. It is important to note that the key to comprehending mathematics lies with the interpretation of the concept and not really in the nature or amount of symbols and the role they play. Symbols do not have meanings of their own; this has to be produced by the learner by means of establishing mediation to suitable reference contexts.

Another key argument raised by researchers is that learners have a tendency to wait for the teacher to interpret symbols for them and to show them how they are used in problem-solving (Bakker, Doorman & Drijvers, 2003 Advocates of constructivist philosophy argue that human mind does not hold abstract notions; rather it possesses symbolism that contains distilled meaning of mathematical concepts (Gray et al, 1999). Constructivists argue that it is not ideal for learners to understand concepts and symbols by being simply told what to know. Symbols and syntactic rules of mathematics do not have meaning for learners until they are interpreted by the individual (Lee & Hollebrands, 2008). Learners have a tendency to bring their own interpretations of symbols to the classroom, based on their previous encounters symbols in past math classes (Saraiva & Teixeira, 2009). Learners depend on the rules and syntax of English to interpret mathematical language in order to unpack meanings of mathematical concepts (Cook, 2013).

Rubenstein and Thompson (2001) listed the different roles for mathematical symbols. Some of the roles include naming concepts, stating relationships between concepts, indicating mathematical operations and processes, abbreviate words, and indicate grouping. However, they failed to highlight the multiple roles that symbols play within a single mathematical statement. Other researchers (Ursini & Trigueros, 2004; Bardini, Radford & Sabena, 2005) posit that letter symbols can be used in algebra as generalized numbers, parameters, unknown numbers, and variables. For example, in representing the equation of a line as $y = mx + c$, the learners must differentiate the letters y and x as variables and the letters m and c as parameters that define the gradient and intercept of a line. It is therefore imperative for learners to be able to appreciate the different roles played by letters, operators, and other notational devices in order to communicate fluently in mathematics.

Various attempts have been made to define and describe symbol sense. For example, Fey (1990) described symbol sense as:

> *"…an informal skill required to deal effectively with symbolic expressions and algebraic operations"* (p. 80).

Arcavi (1994) defines it as:

> *"…a quick or accurate appreciation, understanding, or instinct regarding symbols that is involved at all stages of mathematical problem solving"* (p. 31*).*

Kinzel (2001) described symbol sense as a sense of "algebratizing" a situation: creating algebraic expressions that accurately represent relevant quantities within a situation. Equally important is the fact that such representational awareness should be accompanied by the skill to manipulate and interpret these expressions. In this regard, the combination of awareness and skill seems to imply a sense of symbols and their role in a mathematical activity. If learners are to be competent and fluent users of symbols they should have notational options available and be able to judge when such options are appropriate or not.

In order to work fluently with mathematical symbols learners, need to develop a strong symbol sense (Rubenstein, 2009; Chirume, 2012). None of these researchers has attempted to provide an exhaustive definition of symbol sense, arguing that doing so is difficult since symbol sense is closely related to other senses such as number or function senses. Instead, they listed features of what it means for a learner to have symbol sense. The list of these characterisations include among others: knowing when to use symbols during problem solving and when to abandon them for better tools; understanding the need to continuously reflecting on meanings of symbols and compare with one’s own expectations and intuitions; and having an appreciation of the communicability and power of symbols to display and prove relationships. Arcavi (1994) noted that learners do not see mathematical symbols as tools for understanding, communicating, and making connections, even after several years of study. He views the development of symbol sense as an important component of meaning making in mathematics. Symbol sense makes provision for learners to read and the meaning of a problem and checks the reasonableness of the solution process. Pierce and Stacey (2001) expanded the symbol sense framework and emphasise the need for learners to distinguish between meanings of letters as symbols and operators.

2.1.10 The importance of symbols in mathematics

Mathematical symbols and signs are mainly viewed as “instruments” for coding and describing mathematical knowledge, for communicating mathematical knowledge as well as for operating with mathematical knowledge and generalizing it (Steinbring, 2006). Mathematics requires certain sign or symbol systems in order to keep a record of and code the knowledge. Mathematics is primarily made up of two basic entities: numbers and symbols. Symbols are found in simple mathematics, algebra, geometry, calculus and statistics. Symbols are essentially representative of a value and without mathematical symbols, one cannot perform mathematics operations and procedures.

Lester (2007) points out that symbols support understanding and provide a universally accepted way of showing certain mathematical functions and patterns. In order to understand a mathematical concept, as opposed to rote memorization of rules without reason, Skemp (1971), posits that it is crucial and good instructional practice for teachers to link abstract mathematical symbolism with representations from the everyday world whenever this is possible. The fundamental need in mathematics at all levels of learning is the ability to represent the relationship between a sign and the number or value it refers. Certain ideas and concepts can be clearly illustrated only by the creation and use of symbols. Measuring the relationship between numbers and representing the relationship symbolically not only serves to simplify the process but also gains a better understanding of the concept than a wordy description of the same. This is where the issue of languages comes in.

2.1.11 Algebraic Reasoning and Symbolisation

According to Blanton and Kaput (2005), algebraic reasoning involves generalising mathematical ideas from a set of instances, establishing those generalisations through the discourse of argumentation, and expresses them in formal ways using appropriate symbols. Zorn (2002) refereed to this kind of meta-knowledge as symbol sense. Drijvers (2011) viewed algebraic reasoning as the literacy that operates in the background without our conscious awareness during problem solving. Algebraic reasoning can be construed as the learner's ability to model a situation using appropriate functional relationships and symbols. It involves formalising experiences and ideas into a symbol system (Lapp, Ermete, Brackett & Powell, 2013). It bridges the cognitive gap between arithmetic in primary school grades and abstract algebraic topics such as functions, calculus and other topics in secondary grades.

The use of formal symbolic representations, such as equations, gives learners to access abstract concepts. It provides a foundation for the development of abstract mathematical understanding. Algebraic reasoning provides tools for mathematicians to explore the structure of mathematics and supports mathematical thinking. Koedinger, Alibali & Nathan (2008) advised that teachers should focus on developing learners' algebraic reason prior to formal symbolic representation and manipulation. Algebraic reasoning is a facet of symbol sense. Algebra requires learners to decode the symbolic language of algebra (Bednarz, Kieran & Lee, 2012). The main aim of learning algebra is to develop symbol sense. This is because learners' ability to recognise and generalise mathematical situations depends on their competence in using symbols. Symbol sense and algebraic reasoning provide learners with the ability to represent and draw inferences about algebraic relations and functions.

2.1.12 Switching Representations

Mathematical ideas and modelling are usually represented in the form of numeric, geometric, graphical, algebraic, pictorial, and concrete representations. Based on the findings of Flanders (2014), it can be argued that learners have problems of switching from one representation to another (triangulation), recognising the connections between representations, and using the different representations appropriately and as needed to solve problems. Learning the various forms of representation helps learners to understand mathematical concepts and relationships; communicate their thinking, arguments, and understandings; recognise connections among related mathematical concepts; and use mathematics to model and interpret mathematical, physical, and social phenomena. When learners are able to represent concepts in various ways, they develop flexibility in their thinking about those concepts. They are not inclined to perceive any single representation as "the math"; rather, they understand that it as one of representations that help them to understand a concept.

2.2 Challenges of teaching mathematical symbolisation

Rubenstein and Thompson (2001) identified the challenges to mathematical symbolisation as: (a) the same symbol may have different meanings, (b) multiple symbols may represent the same concept, (c) symbols that are used as specific variables in specific contexts, and (d) the family to which a function belongs is embedded in its symbolization. Koedinger, Alibali and Nathan (2008) cited the use of symbolism in mathematics is as the main reason for the lack of understanding and difficulties in learning mathematics. Learners who express hatred for and aversion to mathematics cited its reliance on symbolism as the main reason for their distaste. There is a strong emphasis placed on symbolic or abstract representations of problems (Bryant, 2011). When learning new concepts, learners are quickly rushed into using those symbolic representations, before they ever understand what the symbols represent. Therefore, mathematics becomes an overwhelming mental exercise in the memorization and manipulation of symbols.

Steinbring (2006) indicates that attempts to expound mathematics concepts without using symbolism yield nothing. There are ongoing debates on the question of when and how to introduce symbolism within the mathematics curriculum. Heeffer (2013) argued that if it is introduced prematurely learners might lack the maturity to understand and reason symbolically. On the other hand, if it is delayed, some mathematical concepts cannot be taught as they rely heavily on symbolism. Current understanding of symbolism provides a picture that they pose threats as well as opportunities for the mathematics curriculum. Teachers should take cognisant of the fact that symbolism does not act in a completely abstract way. An insight in how perceptual processes direct learners' understanding of symbolism prepares teachers for possible mistakes and difficulties in classroom practice. Historical epistemology and cognitive psychology drawn from recent findings singled out symbolism as a conceptual barrier in understanding mathematical concepts (Heeffer, 2013). The following section discusses some of the challenges of mathematical symbolisation identified in literature.

2.2.1 Lack of correspondence between symbols and referents

Written mathematical symbols play an important role in the teaching and learning of mathematics, but learners often experience challenges in constructing mathematical meanings of symbols (Yetkin, 2003). One such challenge identified in literature is that learners do not make connections between symbols and their meanings or referents (Adams, Thangata & King, 2005; Hammill, 2010). Studies by Heath (2010) have also proven that symbols are effective when learners understand the connection between the symbol and the mathematical concept. Heath (2010) further argued that it is more important for learners to understand what the symbol means than its name. Marshall (2006) urged mathematics teachers to help learners to understand symbols and avoid rote instruction. He recognised that when learners work with symbols, they must know what they mean and where they come from.

Learners should be able to make use of mathematical concepts using symbols in many settings. Learners derive meaning for the symbols from either connecting with other forms of representations such as graphs, concrete objects, pictures and spoken language or establishing connections within the symbol systems (Hiebert & Carpenter, 1992). However, there is a drawback in using these representations to facilitate learning written symbols; they have limited potential to create understanding of written symbols, since they are representations themselves. McNeil, Uttal, Jarvin and Sternberg (2009) report that learners experience difficulties in understanding the meaning of a written symbols if the referents do not well represent the mathematical meaning or if the connection between the referent and the written symbol is not appropriate. Pimm (1995) advises:

> *"…through working with symbols we gain experience of the thing substituted for. However, we also lose sight of the fact that what we have is a symbol and not the real thing we originally desired"* (p.109).

Pimm (1995) emphasizes the importance of keeping track of symbol meaning during teaching. Similarly, van Oers (2000) considered symbols and meanings to be "*inextricably linked*" (p. 148), and considers reflection on the relationship between symbols and their referents to be a critical part of constructing meaning. Van Oers (2000) also argued that it is not enough for learners to be able to use symbols correctly; but they must also understand their meanings in order to determine their relevance in a particular situation.

Azer, Guerrero and Walsh (2013) also stress the importance of reflection on connections. They suggest that teachers should be explicit about what is being represented by symbols and should encourage learners to continually reflect on symbol meanings. Sajka (2003) studied learners' misunderstanding of the symbols used in functional notation and identification of their possible sources. He posits three kinds of sources: the intrinsic ambiguities of the mathematical notation; the restricted contexts in which some symbols occur in teaching, and a limited choice of mathematical tasks at school; learners' idiosyncratic interpretation of school mathematical tasks.

2.2.2 Dual nature of mathematical symbols

The use of symbols can be described in two ways: as processes and as objects. This dual nature describes a symbol serving both as an indicator of a particular operation (process) and an object upon which to be operated, can be an additional difficultly for learners trying to interpret and work with mathematical symbols (Kenney, 2008). For example, a symbolic expression such as $f(x) = 3x + 1$ can be interpreted as a rule for a procedure, or as an object that can be manipulated (Kinzel, 1999). Saraiva (2009) concluded that learners face many difficulties when they attempt to understand it and when they need to indicate the chain of symbols that are connected with it. Rojano (2002) also reminds mathematics teachers to be cautious of the change in meaning of mathematical symbols during the transition from arithmetic to algebra. The transition phase presents obstacles in the subject's evolution towards the acquisition of algebraic language and reasoning. The differences in meaning of some symbols present difficulties for learners in algebra, challenging the old idea that algebra could be conceived, for teaching purposes, as "an extension of arithmetic" (p.145).

From a procept standpoint of mathematical logic, the following main groups of mathematical symbols can be noted: symbols designating objects ($\frac{dy}{dx}$), symbols designating mathematical operations or processes ($\int f(x)dx$), and symbols designating relations ($f^{-1}(x)$. A fourth group borders on these three main groups of mathematical symbols: auxiliary symbols that establish the sequence in which symbols are combined. For instance, parentheses, which indicate the order in which operations are performed, provide an adequate idea of such symbols. Researchers unanimously agree that recalling or recognising symbols is not complex (Quinnell & Carter, 2012). However, learners struggle with the semantics and meanings of symbols or the concepts that they represent (Hourihan, 2009). Quinnell and Carter (2012) further noted that the syntax of symbols further brings additional complexities for learners. They also presented compelling evidence that learners struggle with decoding and verbalising mathematical symbols relevant for their grade level.

2.2.3 Attributing personal meaning to mathematical symbols

Another source of learners' difficulties with symbolism cited by Howard (2008) is that learners apply personal meanings to symbols. According to Kenney (2007), mathematical notations can only become representations if someone endows and constructs an interpretation for them. For someone who has not developed meaning for them, they are regarded as potential representations. Learners' interpretations always differ based on their prior experiences they bring to the classroom. Knowledge of mathematical symbols is also based on learners' experiences when they met the symbol for the first time. As Schleppegrell (2007) pointed out, learners have informal ideas about symbols and their uses in mathematics. Learners' prior experiences often hinder their understanding of mathematical language and notation. For example, Kinzel (1999) found that when told to use the letter h for height in a word problem, some learners assigned the value 8 to h because it is the eighth letter of the alphabet. Van Oers (2000) explains that such interpretations are promoted in daily life with puzzles and games that involve using $a = 1, b = 2$, and so on. Anthony and Walshaw (2009) suggest that teachers need to guide learners to identify and use the conventions of mathematical language.

According to Kilpatrick, Swafford and Findell (2001), many learners who have trouble with mathematics bring to school informal conceptions of mathematical understanding. Consequently, they encounter difficulties in connecting this prior knowledge base to formal procedures, language, and symbolic notation system of school mathematics. Teachers should therefore pay attention to the informal ideas that learners bring to the learning situation. Teachers should strive to close this gap between informal and formal mathematical conceptions.

There is growing literature on mathematical symbols that support that learners' inability to comprehend mathematical symbols hampers their aspirations to pursue mathematics related careers (Holtman et al, 2008). The findings on a research conducted by Kalloo and Mohan (2011) confirm that many learners were able to do mathematics up to introduction of algebra. The ability to manipulate symbols according to rules is an important skill of mathematics. If learners lack this skill, they find it difficult to understand the concepts. Symbols allow complicated concepts and procedures to be eventually compressed and represented symbolically in a way that can hardly be conveyed in words.

The progression through to secondary school marks a growing collection of new and advanced notation and symbols. Sepeng and Madzorera (2014) noted that the abundance of symbols carries the potential to confuse and disorient learners who are attempting to understand and comprehend mathematical concepts. A study conducted by Heath (2010) reveals that a learner who cannot establish the meaning of signs and symbols struggle with mathematical concepts. Thus, from a teaching perspective, Naik, Banerjee and Subramanian (2004) support the view that before introducing new mathematical symbols it is important to consider meanings of symbols, context and the topic under study.

2.2.4 The uniqueness and complexity of mathematical language

Mathematical language is dominated by symbols and unique notation that can only be interpreted by mathematically literate people (Baber, 2011). Algebra is one branch of mathematics where this language is mostly dominant. Researchers have noted that the confusion between mathematics symbols and their meanings is the root cause of difficulties experienced by learners in understanding mathematical concepts (Saraiva & Teixeira, 2009; Chirume, 2012). The sight of the symbols often produces disturbance to cognition. According to Biro et al (2005) mathematics is a language that has its own vocabulary, symbols and tools that are used in specific circumstances.

Mathematics language is unique and complex. The use of symbols and abstract notations adds uniqueness and complexity to the mathematical register. Quinnell and Carter (2012) adds that learners are able to think mathematically in the absence of symbols; however, communicating using written mathematical ideas cannot be achieved without the use of mathematical symbols. Mathematics language problems are evident when learners have difficulties in using mathematical symbols, expressing mathematical concepts to others, and listening to mathematics explanations. Learners also struggle with reading or writing word problems and writing and expressing math “sentences”, (Garnett,1998). Proficiency in mathematical language provides the link between the concrete and the abstract mathematical representations. At advanced levels of mathematics learning, language aids mathematical thinking, manipulating concepts and ideas without relying on concrete materials. Teaching approaches based on lecture, demonstration and worksheets should be used with caution since they limit learners' language development and conceptual growth.

2.2.5 The symbolic nature of mathematical concepts

All mathematical activities are eventually expressed in terms of symbols and symbolic expressions (Corry, 2015). The many diverse activities of mathematicians have symbolic inscriptions as their common features. Modern disciplines that depend upon mathematics could be measured by their growing reliance on symbols. It is reasonable to conjecture that much of the difficulties experienced by learners in mathematics, and the lack of popularity of the subject in higher education could be linked to the problem of symbolisation. Behind the formal symbols of mathematics, lies a wealth of experience that provides meaning for those symbols. Scott-Wilson (2014) noted that rushing learners into the world of symbols impoverishes the background experiences and lead to trouble in advanced mathematics. They recommended that it is essential to provide learners with time to talk about their activities and developing their own informal records using concrete manipulatives before introducing the formal symbols of adult mathematicians.

There are two approaches in which learners acquire the meaning of mathematical symbols: nominalism and conceptualism. The distinction between nominalism and conceptualism is most evident in the way proponents of each account for the meaning of mathematical symbols. The nominalist argues that the meaning of mathematical symbols is derived from the context in which the symbols are used. Rotman (2000) argues that on one hand, symbols can be construed as means to think about mathematical relations and objects, and on the other, they are the products of such thinking since new mathematical signs are generated. If a learner is asked to calculate the area of a triangle, for example, the meaning of the symbols "A" "h" and "b " would be derived from the area formulas in which these symbols appear. There is no need to argue that the symbols refer to postulated cognitive entities.

From the conceptualist point of view, the meaning of a mathematical symbol cannot be totally specified by describing the behaviour of those who use the symbol. When a learner is asked to calculate the area of a rhombus, for example, the meaning of the symbols "A," "h" and "b" is derived, not just from the area formulas that the learner manipulates, but also from the mathematical ideas to which these symbols refer. Conceptualists view mathematical symbols as cognitive constructs (Gärdenfors, 1997). For the conceptualists, the concept is more important than the symbols used to construct it.

2.2.6 Mathematical symbols and contexts

Mathematical symbols mean different things in different contexts (Haylock and Cockburn, 2008). Similarly, learners hold various conceptions of symbols, letters and signs in different settings. Effective learning of mathematics requires learners to acquire conceptual understanding about the use of the symbols and the context in which they are used. Sapire (2011) posited that when learners memorise rules for moving symbols around on paper they may be learning something other than mathematics. Moreover, using symbols without understanding their meanings is detrimental to learners' relational understanding of Mathematics. Wilson and Peterson (2006) pointed out that teaching abstract idea without paying attention to meaning deter conceptual understanding. They suggested that if teachers intend to enhance learners' understanding of mathematical concepts then they should engage them with a deeper understanding of the use of symbols and their meanings in different contexts.

According to Sullivan (2011), to foster symbolic literacy, teachers should be aware of how they approach the symbols of mathematics. Phillips (2008) maintained that mathematical symbols themselves bear neither meaning nor any purpose until someone endows such meaning or purpose through relational conveyance. In mathematics classrooms, teachers are the agents of the endowment. Teachers tend to depend on their education, experience, and textbook information to assign meaning to symbols, but research has shown that the assignment of such meaning requires deeper thought and analysis. Mathematical symbols do not have meaning until they are meditated by the epistemological nature of the subject into reference contexts (Steinbring, 2005). It is therefore important for teachers to keep this in mind before they attempt to introduce new mathematical symbols.

Teachers should provide clear and coherent instructional symbol usage to facilitate meaningful learner understanding and comprehension of mathematics in general. Phillips (2008) argued that the ability to use symbols enables learners to imagine, select, and create and to define the situations to which they respond. Any ambiguity or confusion resulting from the improper or inconsistent use of such symbols would be detrimental to learners' attempt to conceptualise mathematical concept.

2.2.7 Learners' prior knowledge of Algebra

Stacey and MacGregor (1997) provide evidence that learners have misconceptions about the use of mathematical symbols. Prior research points to the many difficulties learners have with the formal and abstract concepts in linear algebra. A study conducted by Sin (2006) reveals that learners have misconceptions about the use of symbols. This negatively affects their understanding of mathematical concepts. In his study, Ali (2011) argues that the problems encountered by the learners in understanding mathematical concepts originate from their lack of prior knowledge and could be a result of teaching they experience in learning mathematics prior to secondary schooling level.

Nalube (2014) suggested that primary school teachers need to encourage learners to develop skills for observing patterns and relationships. The next step is to model the situation, first in words, and later moving towards standard notational representations. As learners make sense of simple relationships and practice verbalising those relationships, they gain experience with the concept of abstraction from the earliest grades, which prepares them for the increasingly rigorous use of symbolic notation in later grades. Learners are often asked to perform actions in questions like simplify, evaluate and solve rather than actually using algebraic concepts and symbols to represent and solve real or relevant situations (Egodawatte, 2011). Learners lack exposure to the process of algebraic thinking and reasoning, the rules for manipulating and interpreting symbolic expressions have little meaning and are simply rules to memorise, or "rules without reasons." Instead, as suggested by the NCTM (2000) standards, learners need exposure to the process of modelling real-life contexts, beginning with a situation, representing and generalizing the mathematical relationships with symbols, and using equations to model the situation.

Nunes, Bryant and Watson (2009) recommended that in order to understand algebraic symbolisation, learners should have knowledge of operations and be fluent with the notation. The symbols and their meanings are successfully understood when learners know what is being expressed and have time to become fluent at using the notation. Learners lack prior experience of recognising the different roles of letters as: unknowns, variables, constants and parameters. These meanings are not always distinct in algebra and do not relate unambiguously to arithmetical understandings. Mapping symbols to meanings is not learnt in a one-off experience but it is a process. Welder (2006) asserts that prior to learning algebra; learners must have an understanding of numbers, ratios, proportions, and the order of operations, equality, algebraic symbolism, algebraic equations and functions. Barsalou (1999) also mentioned that the introduction to algebra marks a cognitive milestone for learners. Learners begin to explore the more abstract concepts of numeric relationships, representations and symbolism. Prior to algebra, learners must have essential prerequisite knowledge.

2.2.8 Mathematical language is compact and precise

Mathematical language consists of strings of formal symbols that can be processed according to some grammatical rules, and, conversely, generation of new strings according to the grammatical rules (Gärdenfors, 1997). The language of mathematics is unique and complex (Moschkovich, 2010). Mathematical language is used by mathematicians to communicate mathematical ideas among themselves. This language consists of a substrate of some natural language (English) using technical terms and grammatical conventions that are peculiar to mathematical discourse supplemented by a highly specialized symbolic notation for mathematical formulas.

A notable feature of mathematical register is the use of symbols. Symbols communicate complicated mathematical concepts clearly and efficiently. Their uniformity enables people to share mathematical and scientific knowledge (Krippendorff, 2012). Whilst it is possible for learners to think mathematically in the absence of symbols, the written communication of mathematical ideas cannot be achieved concisely without the use of mathematical symbols (Quinnell & Carter, 2012). Further, it is possible to suggest that the fear and dislike of mathematics can be attributed to learners' inability to decode fully the symbols inherent in this area of mathematics. Written text can be defined as symbols or signs that convey mathematical meaning (Siegel, 2006). These symbols can take on many forms, such as letters, numbers, mathematical signs. These symbols have specific meanings. Meanings, however, are not arbitrary. Once the meanings of the symbols have been established and acknowledged, learners need to be able to understand these combinations of symbol strings in mathematical concepts and procedures.

Mathematics text is best described as compact, dense and precise (Österholm & Bergqvist, 2013). This means that a lot of information can be represented by a few symbols. The English text can be understood despite spelling mistakes and wrong word usage, comparable errors in the use of mathematical symbols can have a significant influence on the meaning. However, minor changes in the use of symbols can cause major changes in the meaning of a mathematical statement. Teachers usually hold the assumption that mathematical symbols and notations are figured routinely by learners as they learn mathematics in the school contexts. However, on the bases of the evidence currently available in most classes many learners are struggling to understand the meaning of those mathematical symbols and notations, and sometimes lead them to misunderstandings (Buhari, 2012).

2.2.9 The dynamic natures of mathematics register

Another noticeable challenge of mathematical notation and symbol system is that it is constantly evolving (De Cruz & De Smedt, 2010). Mathematical notation evolves constantly as people continue to invent new ways of approaching and expressing ideas. There is abundant evidence that supports the view that mathematicians continually invent new notations to present innovative concepts and ideas together with new symbols (Kaput, Noss & Hooleys, 2002). Mathematical ideas can exist independently of the notation that represents them. However, the connection between meaning and notation is subtle, and part of the power of mathematics to describe and analyse derives from its ability to represent and manipulate ideas in symbolic form.

Modern mathematical symbols are a product of centuries of refinement (Schliemann & Carraher, 2002). Mazur (2014) also investigates the subconscious and psychological effects that mathematical symbols have on mathematical thoughts, moods, meaning, communication, and comprehension. He considers how symbols influence conceptual understanding through similarity, association, identity, resemblance, and repeated imagery, how they lead to new ideas by subconscious associations, how they make connections between experience and the unknown, and how they contribute to the communication of basic mathematics.

2.2.10 Communicating mathematically

The issues around communicating mathematically include what it means to be able to communicate mathematically, why it is important and what are the implications for classroom practice. The term communicating mathematically is being used in this thesis to mean using mathematical language and representations to formulate and express mathematical ideas in written, oral and diagrammatic form in a way that is acceptable to the wider mathematical community. Communicating mathematically involves more than having the ability to apply mathematical conventions and linguistic formulations appropriately. It includes knowing mathematics in depth and breadth (that is, internalization) and thinking mathematically (Khisty & Morales, 2004). Communicating mathematically comprises a particular type of discourse and register (Schleppegrell, 2007). Depending on the context, the meanings that emerge in discourse are multiple, changing, situated, and determined socially and culturally (Adjei, 2013). Communicating mathematically and doing mathematics are inseparable. Both involve acting, as well as using tools, symbols and objects. Gwengo (2013) argues that the ability to communicate mathematically enables learners to contribute effectively in the negotiation of mathematical meaning and better understanding of the mathematical concepts.

Communication in mathematics can be referred to as the ability to represent mathematical ideas in multiple ways and to make connections among different representations (Clement, 2004). NCTM (2000) noted that the rules for interpreting and manipulating mathematical symbols are not always in agreement with the way relationships are expressed through the English language. Tanner (2003) describes mathematical language is a collection of symbols, letters, or words with arbitrary meanings that are governed by rules and used to communicate concepts. Language can be thought of as a system of communication that uses symbols to convey deep meaning. Symbols can be words, images, body language and sounds. Language is symbolic in that the symbols used have a deeper "symbolic and semantic" meaning beyond their literal meaning. It consists of words or symbols that represent objects without being those objects. This can cause difficulties for learners.

According to Braiden (2011), the processes of language and mathematics diverge above the level of symbolic processing. Competence in one does not correlate with competence in the other. This divergence is partly due to differences in syntax. The syntax of language and syntax of mathematics both evolve from the ability to process symbols. Both need to be taught and learned. Good writing, reading and grammatical skills do not in and of themselves translate into good arithmetic computation and problem solving skills. However, poor language skills do correlate with poor mathematical skills, suggesting that both require a basic level of competence in symbol processing, that is, deriving meaning from symbols. Being able to think mathematically is reflected by the ability to read & comprehend mathematical symbolism in much the same way one reads words in English.

With regard to reading Daroczy, Wolska, Meurers and Nuerk (2015) argue that mathematics is an abstract and cognitive process that requires a working knowledge of the interaction of numerous discrete skills. Mathematical symbols tend to be more precise than language. Multiple interpretations and ambiguity are not generally considered as part of mathematics register or computation until it is used as a tool in such fields as statistical inference. There is danger of pre-maturely focusing on symbols. Symbols are abstract and have no meanings. The symbols that learners read and write must have meaning to them. Starting with the abstract nature of symbolism will almost assuredly lead to failure. Mathematical symbols become meaningful if teaching begins with concrete and semi-concrete examples that can be attached to meaningful verbal comprehension (Fite, 2002).

2.2.11 Informal and formal mathematics controversy

A critical analysis of the results of a study conducted by O'Toole (2006) provides confirmatory evidence that learners who encounter difficulties with mathematical symbols bring to school a strong foundation of informal mathematics understanding. They encounter challenges when trying to connect this knowledge to the more formal symbolic notation of school mathematics. Many learners struggle to understand the new world of written mathematical symbols onto the known world of quantities, actions as well as the peculiar mathematics language. Learners' confusion with the conventions of written mathematical symbols is normally sustained at the primary school level by the practice of workbooks filled with problems to be solved (Fuchs & Fuchs, 2007). This kind of instruction encourages learners to act as problem solvers rather than as demonstrators of mathematics knowledge.

Chirume (2012) acknowledged that learners see written mathematical symbols as an unfamiliar foreign language causing considerable difficulties for their understanding of Mathematics. Carruthers and Worthington (2005) pointed at the gap between learners' self-invented strategies and school-taught, formal mathematical symbols as a likely cause of learners' difficulties with school mathematics. Worthington and Carruthers (2003) also made the same sentiments, arguing that making connections between formal mathematical symbols and the learner's own informal mathematics is imperative. Doig, McCrae and Rowe (2003) propose that meaningful mathematics learning occurs when learners associate some personal experience negotiated through social experience with others symbols. The consensus view amongst researchers seems to be that the clash between learners' self- invented strategies and formal mathematical symbols is one cause of the conflict.

From a Vygotskian perspective, symbols or graphical representations close the gap between 'enactive, perception-bound thinking and abstract symbolic thinking' (van Oers, 1997, p.237). A study conducted by Deloache (1991) reveals that learners are able to represent a mathematical concept in two different ways. This flexibility of meaning and object allows learners to understand that written mathematical symbols stand for something other than themselves. Deloache (1998) points out that the symbol system is not fully transparent. For example, letters of the alphabet and numerals have no inherent content or meaning, but convey information in systematic ways. Learners not only have to make sense of individual symbols or in isolation but need to understand their role within a system whether for example, letters within a written word, marks that denote parts of a drawing or a mathematical symbol within a written calculation. Understanding abstract symbols in written language or mathematics begins long before learners enter school: they have a 'pre-history' that Vygotsky (1978) believed originates in both gesture and the alternative meanings that learners assign to objects within their play.

Lee and Ginsburg (2007) outline three features of learner's written symbolism in mathematics, namely, understanding of written symbolism generally lags behind learners' informal arithmetic, learners interpret written symbolism in terms of what they already know and good teaching attempts to foster connections between the learners' informal knowledge and the abstract and arbitrary system of symbolism. From these three features, one gets the impression that learners possess considerable informal mathematics by the time they start formal learning.

There has been what Munn (2001), describes as 'a considerable gap in our knowledge of how learners develop the ability to use number symbols and the development of learners' use and understanding of written numerals' (p. 35). Supporting learners' early writing and reading is problematic for some mathematics teachers and it appears that introducing abstract symbolism of mathematics is more so. Primary school teachers emphasise on the concrete approach to teaching mathematics. Thus, most of the work is left for secondary school teacher to introduce the bulk of mathematical symbols. Carruthers and Worthington (2006) observe, even though teachers illustrate the symbols and operations with pictures and objects, many learners still have trouble with establishing important links. Determining ways to foster these connections has been a challenge for teachers but Hughes (1986) observed that failure to do this is likely to be where many learners' difficulties lie.

2.2.12 The Abstract and Virtual reality of mathematical concepts

According to Decon (2011), abstraction is a characteristic feature of the symbolic representation of mathematical concepts. This is an essential feature of mathematics, and again is one part that makes mathematics incomprehensible to learners. Abstraction in mathematics is the process of extracting the underlying essence of a mathematical concept, removing any dependence on real life objects with which it might originally have been connected, and generalizing it so that it has wider applications or matching among other abstract descriptions of equivalent phenomena (Saitta & Zucker, 2013). Mason (2004) perceives abstraction as a spiral process. It is an on-going process in mathematics. Unlike most other subjects, mathematics is a quest for abstract principles, without any necessary connection to concrete facts. Many mathematical topics and concepts exhibit a progression from the concrete to the abstract. At the lowest level, one begins by manipulating concrete objects (for example, a sequence of numbers). After a while, one "gets a sense of" those objects and begins to be able to articulate rules and properties that describe those objects (for example, certain terms in the sequence are divisible by a number). Although most learners easily pick up elementary knowledge through the use of concrete objects, they should be encouraged to use symbols and other mathematical notation to represent their understanding. Reading mathematics requires learners to develop skills at the symbol processing level (Große, 2014). Symbol processing involves the ability to derive meaning from symbols, whether they are words, letters, numbers or equations. If a learner lacks the ability to process symbols, then he/she cannot read nor do mathematics.

Abstraction in mathematics is based on the assumption that mathematics is self-contained, that is, is an abstract mathematical object takes its meaning only from the system within which it is defined (Duval, 2006). Having rules, symbols and properties to work with instead of the real objects themselves is one level of abstraction. A limitation in coping with abstraction presents the greatest barrier to handle mathematical procedures and concepts. The disadvantage of abstraction is that highly abstract concepts are difficult to learn. Mitchelmore and White (2004) propounds that a certain degree of mathematical maturity and experience may be needed for conceptual assimilation of abstractions. He further proposed that learners must be encouraged to move from concrete examples to abstract thinking.

Tillema (2007) viewed mathematics as a science focusing on symbols in a sense. He noted that the comprehension of the symbols used in mathematics is particularly important for understanding the universal and abstract language of mathematics. Arcavi (1994) introduces the notion of symbol sense as a "…desired goal for mathematics education" (p.32). Pope and Sharma (2001) expanded the symbol sense notion to incorporate the ability to appreciate the power of symbols, to know when the use of symbols is appropriate and ability to manipulate and make sense of symbols in a range of contexts. Symbol sense actually develops skills of using of symbols and understanding the situation where they are useful and where they are not. The assumption of symbol sense is based on the premise that a learner with symbol sense is less likely to experience difficulties in understanding abstract concepts. Symbol sense actually develops skills on the use of symbols and understanding of the situation. According to Santos and Thomas (2001), mathematicians seek precision and unique definitions, but cognitively they seem to use symbols ambiguously to represent either processes to do mathematics or concepts to think about. He argues that mathematicians and other experts in mathematics have a sense of symbols that enables them to handle symbols in a flexible and imaginative manner.

In mathematics, unlike other science subjects, objects do not have a tangible existence and are not directly accessible to perception. The only way to access them is via symbolic representations (Fagnant, 2005). In contrast to other school subjects, the "objects" dealt with in mathematics are symbols that do not refer to specific objects or events in the real world. The representation and processing of knowledge in mathematics is abstract and requires more abstraction in the domain of mathematics than in other subjects in the school curriculum. Mathematics belongs to what Sfard (2000) calls “virtual reality” as opposed to actual reality (p. 39). Actual reality communication may be perceptually mediated by the objects that are being discussed, whereas in the virtual reality discourse perceptual mediation is scarce and is only possible with the help of what is understood as symbolic substitutes of objects under consideration. Symbols are therefore an integral part of mathematical reasoning.

Cobb (2000) advocates the idea according to which “the ways that symbols are used and the meanings they come to have are mutually constitutive and emerge together” (p. 18). When teaching symbolisation, teachers should not concentrate on symbols and their meanings but rather on the activity of symbolising and meaning making (Yackel, 2000). Fagnant (2005) summarises learners’ difficulties at the symbolisation stage: learners are not always capable of producing a correct number sentence when they are confronted with a problem, even if they have solved it correctly. In other words, learners experience difficulties in making connections between their informal approaches to problem solving and their use of mathematical symbolism.

Drawing from Reynders (2014), learners' difficulties in learning written symbols, concepts, and procedures in mathematics has been a source of concern for many researchers. Standard written symbols in school mathematics textbooks play an important role in the learning of mathematics, but learners may experience difficulties in constructing mathematical meanings of symbols. Learners tend to derive meaning for the symbols from either connecting with other forms of representations (for example physical objects, pictures and spoken language) or establishing connections within the symbol systems (Yetkin, 2003). Meanings of numerical and operational symbols are constructed by connecting with concrete materials, everyday experiences or language (McNeil, Uttal, Jarvin & Sternberg, 2009). An understanding of a mathematical concept might therefore involve facts about that concept, pictures, symbols or procedures learners might draw on in order to explore the concept, and how we have felt in the past working with that concept. In order to improve learners' understanding of mathematical concepts, teachers need to link together these separate representations to create a more complex understanding about that concept (Barmby, Harries, Higgins & Suggate, 2007).

2.3 Instructional Strategies

One of the challenges of mathematics teaching is to create instructional sequences in which learners generate, refine, and extend their intuitive and informal thinking to more sophisticated and formal ways of reasoning (Rasmussen & Blumenfeldg, 2007). The design of such learning sequences requires teachers to carefully analyse learners' existing or informal knowledge that can be leveraged for the development of formal or conventional mathematics. An important aspect of mathematics learning suggested by Quinnell and Carter (2012) is the need to give learners opportunities to read, write and verbalise symbols and explanations to aid learning. Learners have a tendency to undervalue, and often avoid entirely, expressing their mathematical thoughts verbally (Duval, 2006). Learners often struggle to sound out symbols. Asking learners to read mathematical expressions and problems aloud is one way to identify misconceptions (Rubenstein & Thompson, 2001).

2.3.1 Precision with mathematical symbols

Teachers should approach mathematical symbolism with caution. Mathematical symbols need to be written very carefully taking into account the size, position, and order (Rubenstein & Thompson, 2001). Links and connections need to be made among symbolic, written, graphic, and oral language. Rubenstein and Thompson (2001) suggested that learners should draw examples and counter examples of statements such as, or write symbolic statements that apply to certain diagrams, or practice by reading and writing statements containing symbols. Bossé and Faulconer (2008) recommend that the development of a learner's power to be fluent in mathematics involves learning the signs, symbols and terms of mathematics. This is best accomplished in problem situations in which learners have an opportunity to read, write, and discuss ideas in which the use of the language of mathematics becomes natural. As learners communicate their ideas, they learn to clarify, refine, and consolidate their thinking.

Communication in mathematics can be referred to as the ability to represent mathematical ideas in multiple ways and to make connections among different representations (Clement, 2004). The NCTM (2000) notes that the rules for interpreting and manipulating mathematical symbols are not always in agreement with the way relationships are expressed in English language. Mathematical language is a collection of symbols, letters, or words with arbitrary meanings that are governed by rules and used to communicate concepts. It consists of words or symbols that represent objects without being those objects. This can cause difficulties for learners.

According to Matejko and Ansari (2016) the processes of language and mathematics diverge above the level of symbolic processing. Competence in one does not correlate with competence in the other. This divergence is partly due to differences in syntax. The syntax of language and syntax of mathematics both evolve from the ability to process symbols. Both need to be taught and learned. Good writing, reading and grammatical skills do not in and of themselves translate into good arithmetic computation and problem solving skills. However, poor language skills do correlate with poor mathematical skills, suggesting that both require a basic level of competence in symbol processing, that is, deriving meaning from symbols. Being able to think mathematically is reflected by the ability to read and comprehend mathematical symbolism in much the same way one reads words in English.

2.3.2 Classroom Discourse

Another important aspect of learning mathematics is to equip learners with the skills to communicate what they know, or think. One of the recommended ways is to encourage communication from all learners is through classroom discussion or small group work (Ololube, 2015). There has always been the notion that learners learn best when they actually have to teach or explain a concept to their peers (Kihlstrom, 2011). This means being able to verbalise what they know. Therefore, teachers need to encourage their learners to verbalise their own knowledge so that they can learn more efficiently. Learners on the listening end also benefit from hearing explanations from their classmates. When learners listen to each other, they often benefit from hearing concepts being explained from different points and in ways that might be closer to their ways of thinking. When learners listen effectively, they generate questions to further their thinking.

The process of attaching appropriate meanings to mathematical symbols may be undermined by teaching that is heavily weighted in favour of instrumental learning (Goldstone, 2012). Such a learning environment encourages a process-oriented view of mathematics where the object of study is not cognitively engaged, and hence pseudo-conceptions are more likely to occur. Once these pseudo-conceptions are in place they can be very resistant to change and may act as cognitive obstacles when a learner is encouraged to perceive a mathematical object, such as an equation, via its properties.

2.3.3 Timeous introduction of symbolism

Reacting to the difficulties demonstrated by learners, several researchers (Radford, 2006; Drews, 2007; Mduli, 2014; Boorman, 2015) recommend early teaching of problems in order to give a variety of meanings to mathematical symbolism. Why have these recommendations not always been followed? For Berliner and Calfee (2013), a persistent idea in educational thinking is that knowledge should first be acquired, and that applications for reasoning and problem solving should be delayed. However, the creation of links between problems and mathematical symbols is a complex process that cannot be reduced to a simple translation.

Doig, McCrae and Rowe (2003) recommended that learners should understand symbols by making connections within the system. Mathematics teachers should be mindful of these difficulties and provide learners with opportunities to make connections within symbol system. It has been well documented that it is key to support learners so that they form links between their own informal mathematics and the abstract symbolism of school-based mathematics (Worthington & Carruthers, 2003). Learners' difficulties in learning written symbols can be reduced by creating learning environments that help learners build connections between their formal and informal mathematical knowledge and by using appropriate representations relevant to the given problem context.

With regard to reading Daroczy, Wolska, Meurers and Nuerk (2015) argue that mathematics is an abstract and cognitive process that requires a working knowledge of the interaction of numerous discrete skills. Mathematical symbols tend to be more precise than language. Multiple interpretations and ambiguity are not generally considered as part of mathematics register or computation until it is used as a tool in such fields as statistical inference. There is danger of pre-maturely focusing on symbols. Symbols are abstract and have no meanings. The symbols that learners read and write must have meaning to them. Starting with the abstract nature of symbolism will almost assuredly lead to failure. Symbols become meaningful if teaching begins with concrete and semi-concrete examples that can be attached to meaningful verbal comprehension.

One way to help learners with potentially confusing symbolism is to provide a historical insight into the development of those symbols. For example, a story about the development of Leibniz notation might help learners understand the integral notation. Another way to alleviate confusion is to explicitly point out to learners that symbols often have different meanings in different contexts, and that alternate symbolism often exists with the same meaning. Unpacking complex symbolism piece by piece can also enhance learners' understanding. This includes breaking the expression into smaller reference units that are easy to understand. By habitually unpacking symbolic statements' meanings, learners can more readily attach meaning to symbols and extract meaning from symbolic expressions. Mathematics teachers will find that a new culture emerges in their classrooms when they are conscientiously and consistently sensitive to learners' meaningful use of symbols. Learners will make connections between mathematics concepts and the symbolism used to represent these concepts. As a by-product, learners will develop symbol sense and will become better symbolic reasoners.

2.3.4 Connecting manipulatives and written mathematical symbols

The manipulation of concrete objects is not, in itself, enough to give learners the opportunity to understand abstract, symbolic representations of mathematical ideas (Blair, Blair & Schwartz, 2012). It is critically important that learners understand these symbolic representations as they advance through school (Uttal, O'Doherty, Newland, Hand & DeLoache, 2009). Manipulating concrete objects in order to understand mathematical concepts is certainly important, particularly in the early stages of learning, but learners must be able to connect concrete and symbolic representations. Thus, the essential duty for mathematics teachers is to help learners to understand, and to manipulate, symbolic representations.

Learners need repeated experiences and a wide variety of concrete materials to make these connections strong and stable. Teachers often compound difficulties at this stage of learning by asking learners to match pictured groups with number sentences before they acquire sufficient experience of relating varieties of physical representations with the various ways of stringing mathematics symbols together, and the different ways we refer to these things in words. The fact that concrete materials can be moved, held, and physically grouped and separated makes them much more vivid teaching tools than pictorial representations.

Because pictures are semi-abstract symbols, if introduced too early, they may confuse the delicate connections being formed between existing concepts and the new language of mathematics. Similarly, Marshall and Paul (2008) note that structured concrete materials are beneficial at the conceptual development stage for mathematics topics at all grade levels. Concrete objects provide a way around the opaqueness of written mathematical symbols. Evidence from research indicates that learners who use concrete materials actually develop more precise and more comprehensive mental representations often show more motivation and on-task behaviour, better understanding of mathematical ideas, and are able to apply these to real-life situations (Hiebert & Grouws, 2007).

According to DeLoache (2004) the concept of dual representation can shed light on this fundamental problem. The central tenet of this concept is that all symbolic objects have a dual nature; they are simultaneously objects in their own right and representations of something else. To use a symbolic object effectively, one must focus more on what the symbol is intended to represent and less on its physical properties. Symbols may be difficult to teach to learners who have not yet grasped the concepts that they represent (Ball, Thames & Phelps, 2008). At the same time, the concepts may be difficult to teach to learners who have not yet mastered the symbols. This scenario presents teachers with a dilemma of how to sequence concepts and symbols during teaching.

Hiebert (1988) proposes a theory that may help to explain learners' “overly mechanical behaviour” of learning. The theory is based on how learners develop competence in dealing with the written symbol systems of mathematics. Hiebert (1988) suggests a series of cognitive processes whose cumulative effect yield competence with written mathematical symbols. He identified five major types of processes:(1) connecting individual symbols with referents; (2) developing symbol manipulation procedures; (3) elaborating procedures for symbols; (4) routinizing the procedures for manipulating symbols; and (5) using the symbols and rules as referents for building more abstract symbol systems.

Connecting symbols with referents: In school mathematics, written marks in textbooks represent quantities or operations (processes) on quantities. To connect written symbols with appropriate referents, learners must be familiar with the relevant quantities and actions on the quantities, and they must be familiar with the written characters that will be used to stand for the quantities and actions. Then they must create a correspondence between the written characters and the quantities or actions to which they refer. Familiarity with quantities that can be used as referents is part of many learners' informal knowledge. Learners often engage in activities with materials and ideas to find how many, how much and when. These everyday experiences generate knowledge of quantities and actions on quantities that can provide the initial referents for written mathematical symbols (Nunes, Bryant & Watson, 2007). Learners competence with written symbols develops as construct connections between individual symbols and familiar referents. Meanings for individual symbols are created as connections are established between the written marks on paper and the quantities or actions that they represent (Pape & Tchoshanov, 2001). The process involves building bridges between symbols and referents and crossing over them mentally many times.

The significance of the connections between numeric symbols and quantities is that they provide mental paths from the symbol to the referent. Learners can recall the mental image of related quantities and reason directly about the quantity to solve the problem if it is presented to learners in the form of written symbols (as in ordinary classroom lessons). The advantage is that the quantities serve as "conceptual entities" (Greeno, 1983), as cognitive objects that the problem solving procedures take as arguments. For learners who are new to the domain, such conceptual entities are likely to support the problem solving process.

Developing symbol manipulation procedures: The second cognitive process required to continue the development of competence with symbols is directed towards the development of symbol procedures. The procedures are formulated by manipulating the referents of the individual symbols, observing the result, and then paralleling the action on referents with an action on symbols.

Routinizing symbol procedures*:* The symbol system is used more efficiently if the procedures are well practiced. When procedures are practiced so often, they can be executed automatically, with little conscious thought, and then the user achieves maximal efficiency.

Building more abstract symbol systems*:* Symbol systems themselves develop by building on one another (Goldin & Kaput, 1987). Learners' competence with symbols continues to develop as more abstract systems are encountered, and the ways in which they build on earlier familiar systems are recognised. One way in which later systems can build on earlier ones is through the transfer of meaning directly from the early symbols and rules to the later system. A second way is through the recognition of a correspondence between two different symbol systems. Learners can transfer meaning from a familiar symbol system to a new, more abstract system if they have established meanings for the familiar symbols (the first two processes have been thoroughly engaged), and if they recognise a mapping between the systems so that the familiar symbols and rules can serve as referents for the new system.

2.3.5 Strategies for teaching mathematical symbolisation

Teachers should be aware of the difficulties that symbolism creates for learners. Symbolism is a form of mathematical language that is compact, abstract, specific, and formal. Mathematical symbolism is largely limited to the mathematics classroom. Therefore, opportunities to use that language should be regular, rich, meaningful, and rewarding. According to Bruner (1960) learning should proceed from concrete to abstract. Mathematical symbolism and mathematical understanding are intertwined, but meaning must generally precedes symbolisation (Rubenstein & Thompson, 2001). Teachers should engage learners in contexts, problems, and activities that move them from familiar to newer mathematical ideas; this stage is called the *enactive stage.* The products from these activities may then be expressed in tables or pictures, the *iconic stage*. Ultimately, learning is expressed in common oral English with mathematical vocabulary and, in written English with mathematical symbols; this stage is called the *symbolic stage.*

Mathematics teachers need to verbalise everything they write and be precise and fluent in mathematical language. It is very important for all learners to use as many senses as possible when learning new mathematics concepts. They need to read a new mathematics problem, write it, listen to it, tactically explore it through manipulatives, and when possible move their body and/or manipulative through space.

The poor performance of South African learners in mathematics can be traced to the methods used to teach mathematics at the primary school level (Siyepu, 2013). The focus is on specific problems and does not building on the theoretical foundations necessary for understanding general mathematics at higher level (Wilson, 2006). These foundations can only be built with a mathematics program that teaches concepts and skills, and problem-solving (Daro, 2006). The reform movement in mathematics education can be traced to the mid-1980's and was a response to the failure of traditional teaching methods, the impact of technology on curriculum and the emergence of new approaches to the scientific study of how mathematics is learned (Battista, 1999). Learners must be able to read, write and discuss mathematics, use demonstrations, drawings and real-world objects, and participate in formal mathematical and logical arguments. Meaningful mathematics learning is a product of purposeful engagement and interaction that builds on prior experience (Romberg, 2000).

Sabean and Bavaria (2005) compiled a list of the most significant principles related to mathematics teaching and learning. The list includes expectations that teachers know what learners need to learn based on what they know. Teachers ask questions focusing on developing conceptual understanding, experiences and prior knowledge provide the basis for learning mathematics with understanding, learners provide written justification for problem solving strategies, problem based activities focus on concepts and skills, and that the mathematics curriculum emphasizes conceptual understanding.

2.3.6 Teaching reading in mathematics

Of all the content-area texts that secondary school learners read, mathematics is arguably the most difficult (Barton, Heidema & Jordan, 2002). Learners face challenges when reading mathematics text. Mathematics is a language that requires the use of vocabulary and symbols to translate problems from word form to algebraic form. Adams (2003) characterised mathematics as a language of words, numerals, and symbols that are at times interrelated and interdependent and at other times disjointed and autonomous. Adams (2003) states that weakness in learners' mathematics ability is often due in part to the obstacles they face in focusing on these symbols as they attempt to read the language of mathematics.

Textbooks are commonly written in a concise manner using symbols and diagrams. The conceptual density of mathematics text is one of the major challenges. Metsisto (2005) maintains that mathematics texts contain more concepts per line, sentence, and paragraph than any other kind of text. In addition, reading mathematics requires special reading skill, skills that learners may not have used in other content areas. For example, in addition to comprehending text passages, learners must be able decode and comprehend scores of scientific and mathematical signs, symbols, and graphics. Learners also need to read and interpret information presented in unfamiliar ways not only from left to right, but also right to left (number lines), top to bottom (tables), and even diagonally (graphs). Further, learners must learn how to read text that is organized differently from that in other core subjects. For example, reading limits of functions present challenges for some learners: $\lim_{h\to 0}\ (2x+3h)$ can be read as the limit of $2x+3h$ as h tends to zero or the limit as h tends to zero of $2x+3h$.

Given these challenges, it is no wonder why one should ask the question: "how can teachers help learners become more successful at reading and learning mathematics texts"? In response to this, Burton, Heidema and Jordan (2002) suggest that teachers can incorporate reading as part of instruction to help learners activate prior content knowledge, master vocabulary, and make sense of unfamiliar text styles. Vacca and Vacca (2005) also contended that a learner's prior knowledge is the single most important resource in learning mathematics text. Each learner actively draws on prior knowledge and experience to make sense of new information. The more knowledge of symbols and skills that learners bring to a text, the better they will learn from and remember what they read. Activating learners' prior knowledge prepares them to make logical connections, draw conclusions, and assimilate new ideas.

The ability to read, write, and verbalise mathematical terms is often overlooked during instruction. These skills are necessary for learners to be able to understand and communicate during mathematical discourse. One strategy that can be of great assistance in learning to speak, read, and write the language of mathematics is diagramming. Rubenstein and Thompson (2000) suggest that diagramming is a tool that learners can use to make connections between different mathematical vocabularies. From reading, to writing, to verbalising, learners throughout history have struggled with mathematics. Moreover, teachers should remember that there is no one list of strategies that is all-inclusive. The possibilities are endless. The main challenge is that learners who do not know how to read, write, or verbalise mathematical terms and ideas have an even harder time trying to learn how to do the actual mathematics.

The Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) in America proposed the need for learners to learn to communicate mathematically. They proposed that:

"... The development of a learner's power to use mathematics involves learning the signs, symbols, and terms of mathematics. This is best accomplished in problem solving situations in which learners have an opportunity to read, write, and discuss ideas in which the use of the language of mathematics becomes natural. As learners communicate their ideas, they learn to clarify, refine, and consolidate their thinking" (p. 6).

However, Callan (2004) later noticed that it is rare to find a mathematics classroom in which reading experiences are thoroughly integrated into mathematics instruction. Borasi and Siegel (2005) then proposed that learners could use transactional reading strategies to learn from any kind of mathematical texts. These strategies engage readers in active meaning-making in the sense that interpretations are constructed through reflective thought motivated by ambiguity. Later, Duke and Pearson (2008) argue that it is not only what learners read, but also how they read that could make a difference in their learning.

2.3.7 Scaffolding

Proponents of the constructive theory argue that learning occurs when individual is prompted to move past current levels of performance and develop new abilities (Ertmer and Newby, 2013). Thus, the provision of external support from the instructor, peers, experts, artifacts or tools is essential for learners to construct knowledge. The guidance that the teacher extends to the learners is termed scaffolding Hammond and Gibbons (2005). It is assumed that through scaffolding, learners can become independent learners. Scaffolding techniques such as clarifying doubts, inviting responses, focusing on task, reinforcing important facts and evaluating learners' works can be used by teachers to enhance understanding. The teacher initially provides extensive instructional support, or scaffolding, necessary to help learners build their own understanding of new concepts or skills. Scaffolding is a term in the world of education that exists in modern constructivist theory of learning. In learning, scaffolding takes a very important role in the development of learner learning. Each time the learners reach a certain developmental stage in learning which is characterized by the fulfilment of indicators in certain aspects, the learners will require scaffolding. Bassiri (2012) suggests that scaffolding is the concept of learning with assistance (assisted learning).

According to Vygotsky (1986), the functions of higher mental, including memory and the ability to direct attention to specific goals and the ability to think in symbols, is a behaviour that requires assistance, especially in the form of media. Scaffolding is derived from the view that learning mathematics needs a multiway interaction, teacher-learner, learner-learner, learner-teaching materials so that learners-based on experience-can develop mathematical knowledge and strategies to respond to mathematical problem given. Allowing learners to work out mathematical problems using symbols initially and then discussing the reasoning may also be an effective way to scaffold mathematical understanding. Hammond and Gibbons (2009) views scaffolding as a form of support in which learners take increased responsibility for their learning. Vygotsky (1986) coined "the zone of proximal development" to describe the gap between what a learner can do independently and what they can do with help. Teachers need to provide high levels of support when necessary while ensuring that learners are challenged enough to make progress.

2.4 Challenges related to learning mathematical symbols

2.4.1 Difficulties of learning written mathematical symbols

Learning mathematics with understanding is the vision of school mathematics recommended by the National Council of Teachers of School Mathematics (2000). Learners struggle with a very narrow form of mathematical language, namely formal symbolism. The special written symbolism of mathematics is the hardest form of language for learners to learn. In order to design and develop learning environments that promote understanding efficiently, teachers need to be aware of learners' difficulties in learning mathematics.

Standard written symbols play an important role in learning of mathematics, but learners may experience difficulties in constructing mathematical meanings for symbols. Learners derive meaning for the symbols from either connecting with other forms of representations (e.g. physical objects, pictures and spoken language) or establishing connections within the symbol systems (Yetkin, 2003). Meanings of numerical and operational symbols such as $2, -4, 3/4, 2.4,$ and $\pm$ are constructed by connecting with concrete materials, everyday experiences or language. For example, the symbol $" +$

" takes meaning if it is connected with the joining idea in situations like "I have four marbles. My mother gave me five more marbles. How many marbles do I have altogether?" Although these representations facilitate learning written symbols, the potential for them to create understanding of written symbols is limited, since they are representations themselves. Learners might have difficulty in understanding the meaning of a written symbol if the referents do not well represent the mathematical meaning or if the connection between the referent and the written symbol is not appropriate (Yetkin, 2003). For example, geometric regions are the models most commonly used to represent fractions. These models represent the part-whole interpretation of rational numbers. However, the symbol $\frac{a}{b}$ also refers to a relationship between two quantities in terms of the ratio interpretation of rational numbers. Similarly, $\frac{a}{b}$ may be used to refer to division operation. For this reason, teachers need to use other types of representations such as sets of discrete objects and the number line to promote conceptual understanding of the symbol $\frac{a}{b}$.

One of the reasons advanced for the difficulty in understanding symbols comes from the fact that in their standard form, written symbols might take on different meanings in different settings. For instance, in solving the equation $2x+3=4$, x is an unknown that does not vary, whereas it varies depending on y in the equation $2x+3=y$ (Janvier, Girardon & Moorland, 1993). In order to understand mathematical symbols, learners need to learn multiple meanings of the symbols depending on the given problem context. Therefore, they should be provided with a variety of appropriate materials that represent the written mathematical symbols, and they should also be aware of the meaning of mathematical symbols in different problem contexts. Furthermore, concepts are learned best when they are encountered in a variety of contexts and expressed in a variety of ways, for that ensures that there are more opportunities for them to become imbedded in a student's knowledge system (Bransford, Brown and Cocking, 1999).

Learners also build understanding for written symbols by making connections within the system. For example, a numeral such as 3254 can express the number of the units of any power of ten. In other words, it represents three thousand, two hundred, fifty-four units as well as three hundred twenty-five tens; thirty-two hundred; and three thousand. Although these patterns are evident for adults, learners might not easily construct these relationships by themselves (Whitebread, 2012). Therefore, teachers should be aware of these difficulties and provide learners with opportunities to recognise the patterns and make connections within symbol system. Developing understanding in mathematics is an important but difficult goal. Being aware of learner difficulties and the sources of the difficulties, and designing instructions to diminish them, are important steps in achieving this goal.

Because mathematics is so often conveyed in symbols, oral and written communication about mathematical ideas is not always recognised as an important part of mathematics education. Learners do not necessarily talk about mathematics naturally; teachers need to help them learn how to do so (NCTM, 2000). As learners progress through the grades, the mathematics about which they communicate should become more complex and abstract. Learners' repertoire of tools and ways of communicating, as well as the mathematical reasoning that supports their communication, should become increasingly sophisticated. To this regard, Hattie and Donoghue (2016) encourages teachers to establish classroom cultures that foster learning for learners to develop ability of effective communication that promotes deeper learning, but this condition alone is not sufficient to make learning with deeper understanding take place. Learners whose primary language is not English may need some additional support in order to benefit from communication-rich mathematics classes, but they can participate fully if classroom activities are appropriately structured (Ferreira, 2011).

Wilder (2013) discusses how human beings possess “symbolic initiative” that enables them to “assign symbols to stand for objects or ideas, set up relationships between them, and operate with them on a conceptual level” (p. 5). He credits much of mathematics achievement to this uniquely human capacity. Human beings possesses what is called symbolic initiative; that is, they assign symbols to stand for objects or ideas, set up relationships between them, and operate with them as though they were physical objects.

2.4.2 Verbalisation challenges

Verbalisation challenges involve translating mathematical symbols into spoken language. Verbalisation refers to the surface structures used to transmit ideas (K'Odhiambo & Gunga, 2010). Thompson and Rubenstein (2000) posit that if a learner does not know how to read mathematics aloud; it is difficult to register the mathematics. Reading is a link to understanding. Some symbols require multiple words to pronounce, and others are verbalised in multiple ways. At times, the verbalisation of a symbol changes depending on the context. Learners may need to be reintroduced to verbalisations of familiar symbols when they are doing more advanced work (Maharaj, 2008). Therefore, learners must not only recognise the symbols, but they must also learn to associate them with particular concepts, procedures and the words used to express those concepts.

Verbalising mathematics is a skill that learners must develop. Learners need to routinely participate in dialogue and discussion on mathematics related topics and also to discover methods in mathematics. Studies, conducted by Siegel and Fonzi (1995), reveal lack of verbal exchange between learners and their peers, and also with their teachers within the classroom, and instead portrayed classrooms as a standard input/output situation. Teachers should learn to give up part of the educational reigns of their classroom and allow the learners to become more than just passive receivers of the materials at which they need to become skilled in. Engaging learners in talking about mathematical concepts is one of the ways to engage in formative assessment. An additional benefit is that learners may themselves realise what they do not understand. This allows them to adjust their own reasoning, and over time it may improve their metacognitive abilities. Teaching through discussion supports robust learning by boosting memory, deeper reasoning, development of language and social skills (Coe, Aloisi, Higgins & Major, 2014).

Another aspect of verbalising mathematics is the use of correct terminology and vocabulary. If learners do not speak the language of mathematics, how do they understand the mathematics? Mercer and Sams (2006) feel the need for learners and teachers to converse using terms that are functional, not only for communication but for reasoning. Part of understanding mathematics is being able to use its vocabulary correctly in daily conversation. Teachers need to be aware that learning vocabulary is not just learning definitions. “Just giving learners vocabulary lists with definitions, or asking them to look up definitions, is not enough for them to develop the conceptual meaning behind the words or to read and use the vocabulary accurately” (Kenney, Hancewicz, Heuer, Metsisto & Tuttle, 2005:26). Teachers and learners should correctly use the vocabulary daily in their classroom interactions. This inclusion will help make the vocabulary a natural part of learners’ spoken language and will aid in their understanding. “Mathematics is a foreign language for many learners; it is learned at school and is not spoken at home. Mathematics is not a ‘first’ language; that is, it does not originate as a spoken language” (Kenney, Hancewicz, Heuer, Metsisto & Tuttle, 2005:6). Learners need to recognise that for them to learn the material; they have to become participants, not observers, of their education process. They must be active learners.

Mathematics is often conveyed in symbols, the oral and written communication about mathematical ideas is not always recognised as an important part of mathematics education. Learners do not necessarily talk about mathematics naturally; hence teachers need to help them learn how to do so (O’Connell and Croskey, 2007). As learners progress through the grades, the mathematics that they communicate becomes more complex and abstract. Learners' repertoire of tools and ways of communicating, as well as the mathematical reasoning that supports their communication, should become increasingly sophisticated. Support for learners is vital. Eisenchlas, Schalley and Guillemin (2013) recommend that learners whose primary language is not English may need some additional support in order to benefit from communication-rich mathematics classes, but they can participate fully if classroom activities are appropriately structured.

The language policy in South Africa stipulates that English language is the medium of instruction at the secondary school level (Mncwango, 2012). But, mathematics is conceived everywhere in the world has a subject with internationally accepted terminologies and a symbol system that has condensed meaning (Wasike, 2006). These symbols and terminologies are not familiar and sometimes have contradicting meanings with ordinary English especially in the area of statistics.

Learners need to learn mathematical symbols and ideas so that they can communicate with others mathematically. As learners strive to express and expand their mathematical understanding through the communication of their ideas, they learn to clarify, refine, and consolidate their thinking (NCTM 2000). Mathematics is a communication system that can be used to describe and communicate life experiences, yet Mulwa (2014) further discerns that communication about mathematics requires genuine negotiation and sharing of meaning. The meanings are conveyed through symbols. Learners' literature involving mathematics provides a common, natural context for the sharing of mathematics. Mathematical discourse not only promotes learners' oral language skills, but it also advances learners' abilities to think and communicate mathematically (Moyer, 2000).

Communication is an essential part of mathematics through which ideas become objects of reflection, refinement, discussion, and amendment. The communication process also helps build meaning and permanence for ideas and makes them public. When learners are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing. Listening to others' explanations gives learners opportunities to develop their own understanding. Conversations in which mathematical ideas are explored from multiple perspectives help the participants sharpen their thinking and make connections. Learners who are involved in discussions in which they justify solution especially in the face of disagreement will gain better mathematical understanding as they work to convince their peers about differing points of view (Smith, Silver & Stein, 2005). Such an activity helps learners to develop a language for expressing mathematical ideas and an appreciation of the need for precision in that language.

Learners who have opportunities, encouragement, and support for speaking, writing, reading, and listening in mathematics classes reap dual benefits: they communicate to learn mathematics, and they learn to communicate mathematically (Falk-Ross & Evans, 2014). There is little research on how mathematics teachers and learners acquire verbalisation. Research on learners' handling of the verbal and symbolic elements of mathematics language often focused on learners' comprehension and response to mathematical texts, rather than learners' own generated verbal utterances.

2.4.3 Reading challenges

Reading challenges refers to difficulties learners' encounter when reading mathematical concepts in textbooks. Mathematics is a language that can neither be read nor understood without initiation (Simonson & Gouvea, 2003). The issue of reading, recognising and understanding symbols underpins all mathematics topics (Bardinia & Pierce, 2015). Reading is a skill that goes beyond pronouncing and attaching meaning to symbols. Reading in mathematics entails more than a mechanical or manipulative approach to numbers. Reading a mathematics text requires an understanding of symbols in order to master two basic processes: classification and the study of relationships. Therefore, any approach to improving reading skills in mathematics must focus primarily on comprehension, on understanding abstract ideas in order to improve learners' understanding of concepts.

Reading a mathematical text requires a reading protocol, which is a set of strategies that a reader must use in order to benefit fully from reading the text. Reading a mathematics text requires cross references, reflecting, scanning, pausing revisiting and re-reading. In mathematical writing, mathematicians appear to prize conciseness and precision of meaning (Shepherd, Selden & Selden, 2009). Most mathematics textbooks used in South African secondary schools contain a text exposition of concepts and processes, definitions of key terms and vocabulary, theorems related to the concepts and less formal mathematical assertions, graphical representations, figures, tables, worked examples, and exercises at the end of a sub-unit or concept and a summative exercise or topic at the end of the topic.

Mathematics textbooks contain many confusing symbols that function as ideographs rather than letters. An ideograph is a graphic symbol that represents an idea or concept. Some ideograms are understandable only by familiarity with prior convention; others convey meaning through pictorial resemblance to a physical object, and thus may also be referred to as pictograms. The meaning of such complexes cannot be "spelled or sounded out" while learners read. Reading mathematics text requires analysis and the generation of meaning from a symbol system and involves two types of comprehension: literal, including word meanings, sentence meanings, and getting the main idea; and inferential, including drawing conclusions, making judgments, and using symbolic language (Randi, Newman & Grigorenko, 2010). Reading mathematics can be challenging. Some mathematical words have more than one meaning, depending on the branch of mathematics, for example, “inverse” in arithmetic and in functions. In arithmetic the inverse of 2 is $\frac{1}{2}$ (fraction or reciprocal) while in functions the inverse of the function $f(x) = 3x + 1$ is not $\frac{1}{3x+1}$ but $f^{-1}(x) = \frac{x-1}{3}$.

Some adjectives used in mathematics can substantially change the meaning of some words, such as “value of $3x + 1$” or “absolute value of $3x + 1$”. Learners must comprehend the words, symbols, signs and sentences they are reading in order to understand the concept. Zambo and Cleland (2005) argue that reading activities such as relating the symbols to personal experience, and concentration-type games have a place in mathematics instruction when vocabulary development is an objective.

According to Tall and Gray (2001) many learners have difficulty moving beyond simple arithmetic to understanding the symbolic nature of algebra and variables. Anthony and Walshaw (2009) posit that providing learners at any age with opportunities to converse, read, and write about mathematics enhances the development of concepts. When concept development is the desired goal, verbal interaction among peers is a tremendous facilitator (Dennen, 2004). However, few learners get the chance to verbalise mathematical understandings and symbols. Pillay and Adler (2015) indicate that school mathematics learning is dominated by teacher presentations and independent silent work. Group discussions are no longer a common feature of modern classrooms. It is important that both teachers and learners acknowledge that errors and misinterpretations are a natural and valuable, part of the learning process. The ability to share one's ideas and justify them to others helps develop a solid understanding of those ideas.

The National Council of Teachers of Mathematics (2000) contends that learners who have opportunities, encouragement, and support for writing, reading, and listening in mathematics classes reap dual benefits. They communicate their ideas to others and they learn to communicate mathematically. Barton and Heidema (2002) further pointed out that:

“……*They learn to use language to focus on and work through problems, to communicate ideas coherently and clearly, to organize ideas and structure arguments, to* *extend their thinking and knowledge to encompass other perspectives and experiences, to understand their own problem-solving and thinking process as well as those of others, and to develop flexibility in representing and interpreting ideas* (p. 4).

For the learner, reading the mathematics textbook or handouts or extended response problems presents built-in challenges. The vocabulary of mathematics can be confusing, with some words meaning one thing in one mathematical context and another in everyday settings. Symbols can look alike, and different symbols can represent the same operation. Graphs vary in format, even when representing the same data.

The ability to read mathematics is an extremely important and necessary skill for learners to master. Learners who can read and comprehend mathematical text and language are better able to understand and succeed in mathematics (Buchanan, 2007). Weinberg and Wiesner (2011) explored the potential for mathematics instruction using reading strategies based on the transactional reading theory. They explained what makes reading mathematics text a more complicated endeavour than reading other types of text as well as what skills are needed to be able to understand it successfully. The key to successful reading of technical mathematics texts lies in the learner's ability to decode mathematical symbols and the special and unique language used in such texts.

The ability to read and understand mathematical text also benefits learners in their daily school work, examinations, and even college entrance tests or other types of assessments. Teaches need to teach learners the skill of how to read and understand mathematics as a language to learners. One way to do this is by treating mathematics as a second language that needs to be taught, learned, practiced and understood. Mastering the skills to read and comprehend mathematical text is not a natural skill, but instead a skill that must be practised and learned. Adams (2003) outlined some of the skills that learners lack when reading mathematics text. He argued that reading is often excluded or given little attention in mathematics classes. Reading mathematics is a multidimensional task because the reader is challenged to acquire comprehension and mathematical understanding with fluency and proficiency through the reading of numerals and symbols, in addition to words. Many learners have weakness in their mathematics ability due in part to the obstacles they face in focusing on these symbols as they attempt to read the mathematical language.

One of the tools for helping learners to succeed at reading the mathematical text is to teach them how to read the text and then constantly practice this skill (Buchannan, 2007). Teachers need to teach the skill of how to read and understand mathematics as a language to learners. One way to do this is by treating mathematics as a second language that needs to be taught, learned, practiced and understood. Mastering how to read and comprehend mathematical text is not a natural skill, but instead a skill that must be learned. Learning to read mathematical text and write mathematical ideas in written expressions seem to have a symbiotic relationship with each other. If a learner can do one skill, it makes the other skill easier, and vice versa (Rosa & Orey, 2010). Reading mathematics is different from reading a novel because mathematical writing is very different from fiction and even most types of nonfiction. Mathematical writing is concise and dense. New concepts build logically upon previously introduced concepts. Specialized vocabulary, abundant symbols, and detailed diagrams challenge the reader.

Shepherd, Selden and Selden (2009) summarised learners' difficulties in reading mathematics textbooks as: (1) learners bring insufficient prior knowledge as a result of underdeveloped concept images; (2) learners struggle with the syntax and precision of mathematical definitions, examples, and lack exposition in mathematical writing; and (3) grounding the abstractness of mathematical ideas in concrete objects or actions while reading.

2.4.4 Writing Challenges

Writing challenges refers to inability to produce appropriate symbols for a given mathematical situation. With regard to learners' own writing Phillips (2008) suggested that writing sentences helps learners write correct symbolic expressions. However, many learners struggle to effectively communicate mathematical ideas in writing. Most learners believe that this ability is not important. Mathematical writing, however, has its own particular style. The focus of good mathematics writing is on clarity and precision.

By habitually unpacking symbolic statements' meanings, learners can more readily attach meaning to symbols and extract meaning from symbolic expressions. Mathematics teachers will find that a new culture emerges in their classrooms when they are conscientiously and consistently sensitive to learners' meaningful use of symbols.

Learners will make connections between mathematics concepts and the symbolism used to represent these concepts. As a by-product, learners will develop symbol sense and will become better symbolic reasoners. Many learners' progress in mathematics is hampered by math symbols. To battle symbol confusion; learners should familiarise themselves with symbols in advance and perhaps even write out in words what they mean. For example, they can write out that an exponent means, "Multiply a number by itself." That way, learners will be able to understand and quickly interpret symbols on mathematics tests and will not allow the language of math to confuse them.

2.4.5 Multiple Representations of mathematical concepts

Kirsh (2010) defined multiple representations as external mathematical embodiments of ideas and concepts that provide the same information in more than one form. Mathematical concepts or processes may be represented in a number of different ways. These include verbal, symbolic (numerical or algebraic), pictorial/diagrammatical (geometrical), as a table of values (spreadsheet), graphical or as a physical model. They are used to understand, to develop, and to communicate different mathematical features of the same object or operation, as well as connections between different processes.

Teachers should use multiple representations of mathematical ideas and concepts when teaching mathematics and encourage learners to use multiple representations to help solve mathematical problems. Research focusing on the use of multiple representations in teaching and learning reveals that learners learn more readily under this regime and gain deeper mathematical understanding (Kaput & Goldin, 2002). Hegedus and Kaput (2007) found convincing evidence that learners using dynamically-linked representations gained in understanding by seeing how a change in one representation produced changes in the others. Studies by Chittleborough and Treagust (2008) provided a great deal of evidence to support the argument that learners working with multiple representations gain a deeper understanding of the mathematical concepts involved.

Hoong, Kin and Pien (2015) reveal that learners learn through several modes of representations. Similarly, Kaput (1989); Skemp (1987); Hiebert and Carpenter (1992), illustrate that multiple representations of concepts can be utilized to help learners to develop deeper, and more flexible understanding. Bal (2015) argues that representations are inherent in mathematics; they provide multiple concretisations of a concept; they could be used to mitigate certain difficulties; and they are intended to make mathematics more attractive and interesting. Dreher, Kuntze and Lerman (2015) listed some potential benefits of multiple representations: (a) they provide multiple concretisations of a concept, (b) selective emphasis and de-emphasis different aspects of complex concepts, and (c) facilitate cognitive linking of representations.

Goldin and Shteingold (2001) classify representations as external and internal. An internal representation consists of mental images corresponding to internal formulations constructed out of reality (signified). External representations refer to external symbolic entities such as symbols, schema and diagrams that are used to represent a certain mathematical reality. External representations are the means by which mathematical ideas could be communicated and they are presented as physical objects, pictures, spoken language, or written symbols. External representations such as pictures, diagrams, and physical models are grounded in familiar experiences, connect with learners' prior knowledge, and have an identifiable perceptual correspondence with their referents (Fyfe, McNeil, Son & Goldstone, 2014). However, they may contain extraneous perceptual details that distract learners from relevant information or inhibit transfer of knowledge to novel situations (Sloutsky & Heckler, 2008).

External representations act as stimuli on the senses and include charts, tables, graphs, diagrams, models, computer graphics, and formal symbol systems. They are often regarded as embodiments of ideas or concepts. External representations are the symbols (signifiers) while internal representations are called the signified. Mason (2002) presented the idea that teaching schemes are a spiral movement. As they pass through the spiral, learners have mental transformations from using manipulable external representations to gain meaning of internal representations to symbolic representations. Symbolic representations such as formal equations and line graphs eliminate extraneous surface details, are more arbitrarily related to their referents, and represent the underlying structure of the referent more efficiently (Chu, 2015). Thus, they allow greater flexibility and generalizability to multiple contexts, but may appear as meaningless symbols to learners who lack understanding of the symbols (Nathan, 2012).

Stylianou (2010) further elaborate on the two forms of representation as: External representations are the representations we can easily communicate to other people; they are the marks on the chalkboard, paper, the drawings, the geometry sketches, and the equations. Internal representations are the images we create in our minds for mathematical objects and processes. Goldin and Shteingold (2001) expand the discussion on the types of representation arguing that: external systems of representation range from the conventional symbol systems of mathematics (such as base-ten numeration, formal algebraic notation, the real number line, or Cartesian coordinate representation) to structured learning environments (for example, those involving concrete manipulative materials or computer-based micro worlds). Internal systems, in contrast, include learners' personal symbolisation constructs and assignments of meaning to mathematical notations, as well as their natural language, their visual imagery and spatial representation, their problem-solving strategies and heuristics, and (very important) their affect in relation to mathematics. (p. 2).

In trying to relate internal and external representations in mathematics, Goldin and Shteingold (2001) propose two important terms in their discussion: homonymy and synonymy. The first phenomenon in mathematics is found when one representation has two different meanings. That is, from an external representation there are two different internal representations. The second term refers to when one mental object is denoted in many representations: from two different external representations there is one internal representation. According to Goldin and Shteingold (2001) homonymy, as well as synonymy cannot be avoided in mathematics. Learners show certain preferences for certain external representations. Hart (1991) studied learners' preferred representations and observed that they vary depending on the problem. Her findings are complementary to Arcavi (1994)'s attributes of symbol sense:

1. Learners seek alternate representations when they are not successful at finding solutions using symbols.
2. Learners' choice of representation depends on the complexity of the symbolic information provided.
3. Some learners do not prefer certain representations because they do not recognise them as viable choices.

4. Learners lack confidence in using certain representations.
5. Learners who are not conversant with graphs do not choose to use the graphical representation.

2.4.6 Abstraction in mathematics

Abstraction in mathematics means extracting the underlying essence of a mathematical concept (Hollihan & Baaske, 2015). Their meanings are defined within the world of mathematics, and they exist quite apart from any external reference. It removes any dependence on real world objects with which the concept might have been connected to (Joan, 2015). One of the features that make mathematics difficult is that it deals with abstract concepts that are represented by abstract symbols. Mathematics concepts are modelled at the abstract level using only numbers, notation and mathematical symbols. Mathematical cognition only takes place after converting mathematical symbols into appropriate inner codes (De Cruz & De Smedt, 2013). At the elementary level, these symbols may not be adequately explained and thus learners fail to perform mathematical operations when the abstractions are more complex.

The abstract nature of mathematical symbols and concepts is one of the reasons why mathematics is so difficult. Abstraction is one of the underlying powers of mathematics (Wilson, 2006). Most of the strands of mathematics begin with the study of real world problems, before the underlying rules and concepts are identified and defined as abstract structures. Abstraction and mathematical symbolisation are ongoing processes in mathematics and the historical development of many mathematical topics exhibits a progression from the concrete to the abstract. For example, physical manipulatives act as teaching aids that can help learners to understanding mathematical concepts. They are not, in and of themselves, mathematics, but are teaching tools to help get to the heart of mathematics.

2.5 THEORETICAL FRAMEWORK

This study is guided by four interrelated constructivist theories. In the constructivist perspective, the learner must be actively involved in the construction of one's own knowledge rather than passively receiving knowledge. The teacher's responsibility is to arrange situations and contexts within which the learner constructs appropriate knowledge. According to the constructivist theory of learning learners are viewed as active mathematical thinkers, who try to construct meaning and make sense for themselves of what they are doing, based on their personal experience (Shuard, 1986). Understanding the nature of learners' challenges with mathematical symbols is complex, and there is a need for organisational structures such as frameworks to examine the nature of learners' reasoning about symbols and understand what this entails about conceptual understanding. This theoretical analysis aims to suggest a framework that teachers and learners can use to construct meanings for mathematical symbols that aid understanding of mathematical concepts. This study is guided by a combination of symbol sense (Arcavi, 1994), Algebraic Insight framework (Pierce & Stacey, 2001), APOS theory (Dubinsky & McDonald's, 2002) and Procept Theory (Gray & Tall, 1994). These frameworks are interrelated and all shed light into the aspects of symbol sense that are challenging for learners as they reason and use symbols in mathematical activities and problem solving. These frameworks are described in detail below.

2.5.1 Symbol Sense Framework

The proponents of the symbol sense framework are Fey (1990) and Arcavi (1994). Symbol sense is considered as the heart of algebraic competency (Arcavi, 1994). It is difficult to define symbol sense because it interacts with other senses like number sense, function sense, and graphical sense in problem-solving situations. Arcavi (1994) made a remarkable attempt to characterise symbol sense through a rich variety of examples and illustrations of mathematical behaviours (Zehavi, 2002). Kinzel (2001) describes symbol sense as the combination of notational awareness of expressions and the skill to manipulate and interpret these expressions. Boero (2001) uses the terms "transformation and anticipation" to analyse behaviours in algebraic problem solving. He refers to the continuous tension between "foreseeing and applying" as a dialectic relationship. Zorn (2002) viewed symbol sense as the ability to extract mathematical meaning and structure from symbols, to encode meaning efficiently by symbols, and to manipulate symbols effectively to discover new mathematical meaning and structure. In order to be proficient, mathematics learners must acquire an understanding of letters, variables and objects (Arcavi, 2005).

Arcavi (2005) argues that having 'symbol sense' is central to mathematics learning and good teaching aims to achieve 'symbol sense'. Symbol sense is an essential prerequisite for advanced mathematics and science and is the primary purpose of algebra (Sullivan, 2013). Some keys themes for teaching symbol sense were suggested by Fey (1990) and Arcarvi (1994). Arcavi (1994) modified the list proposed by Fey (1990) and considers that the symbol sense must include, among others, an understanding of and an aesthetic feel for the power of symbols, which brings the idea of visual salience (Kirshner & Awtry (2004); an ability to manipulate and to "read" symbolic expressions as two complimentary aspects of solving algebraic problems. Arcavi (1994) further asset that knowing the algebraic manipulations to solve problems it is not enough, instead it is necessary to understand the meaning of the symbols. He identified four key behaviours: reading instead of manipulation of the symbols; reading and manipulation; reading as the goal for manipulation, reading for reasonableness.

Goldin (2002) explains that communication in mathematics is viable if symbolic systems are understood and relations between systems could be used to enhance symbolic understanding. Holmqvist et al, (2011) define symbol sense as a complex and multifaceted "feel" for symbols. Zehavi (2002), like Arcavi (2005) conceded that it is difficult to define symbol sense because it interacts with other senses like number, function and graphical in problem-solving situations. In an attempt to define symbol sense, Hawkins and Allen (1991) described it as an accurate choice of symbols to represent a mathematical situation or concept. Pope and Sharma (2001) provided a comprehensive definition in which they defined symbol sense as the ability to appreciate the power of symbols, to know when the use of symbols is appropriate, and to manipulate and make sense of symbols in a range of contexts. Thus, there is no concise definition of symbol sense but descriptions of behaviours that illustrate whether a learner has symbol sense or not.

Arcavi (1994) characterised symbol sense as an:

a) understanding of situation and stage where symbols can be and should be used in order to display relationships;
b) ability to abandon certain symbols in favour of other approaches in order to make progress in solving a problem;
c) ability to carry out mathematical processes and to “read” symbolic expression as complementary aspects of solving algebraic problems;
d) awareness that one can initiate symbolic relationships that express the verbal or graphical information needed to make progress in solving a problem; or
e) ability to select a possible symbolic representation of a problem.

Learning algebra requires learners to have symbol sense (Naidoo, 2009). Algebra involves much more than mastering basic skills; it also involves choosing sensible strategies to tackle problems, maintaining an overview of the solution process, creating a model, taking a global view of expressions, wisely choosing subsequent steps, distinguishing between relevant and less relevant characteristics and interpreting results in meaningful ways. Symbol sense is regarded as a type of meta-knowledge in algebra. Symbol sense involves the flexible algebraic expertise or algebraic literacy that often operates in the background without our conscious awareness. Based on insight into the underlying concepts, it directs the implementation of the basic routines. It plays a role in planning, coordinating and interpreting basic operations and consists of three interrelated skills:

i. The strategic skills and heuristics to approach a problem; the capacity to maintain an overview of this process, to make effective choices within the approach, or if a strategy falls short, to seek another approach.

ii. The ability to view expressions and formulas globally, to understand the meaning of symbols in the context and to formulate expressions in another way. Process-object duality plays a role in that skill.

iii. The capacity for algebraic reasoning. This often involves qualitative reflections on terms and factors in expressions, symmetry considerations or reasoning with particular or extreme cases.

Arcavi (1994) states that many learners fail to see symbols as tools for understanding, communicating, and making connections, even after several years of study. He sees development of symbol sense as a necessary component of sense-making in mathematics. He argues that having 'symbol sense' is a fundamental requirement for the study of mathematics especially algebra.

Bergsten (2000) describe symbol sense as an appreciation for the power of symbolic thinking, an understanding of when and why to apply it, and a feel for mathematical structure. Adams, Pegg and Case (2015) compared symbol sense with number sense and found it to be a higher level of mathematical literacy. Wu (2009) explains that communication in mathematics is viable if symbolic systems are understood and relations between systems could be used to enhance symbolic understanding. Arzarello, Ferrara, Robutti and Sabena (2009) urged learners to acquire skills in manipulating various symbols in order to solve a mathematical problem or to prove a formula. Research has revealed how learners interpret and make use of mathematical symbols, a facet of the work on *symbol sense.* Arcavi (1994) described it as "making friends with symbols" (p. 25), including an understanding and feel for symbols, how to use and read them. While solving a mathematical problem, the learner is required to analyse, identify and recognise the relevance of critical areas of a mathematical representation. Kenney (2008) adopted a symbol sense framework constructed using the work of Pierce and Stacey (2001, 2002) and Arcavi (1994, 2005), to investigate learners' reasoning with mathematical symbols at different problem solving stages. She identified the following components of symbol sense:

1. Friendliness with symbols

This includes understanding of and an aesthetic feel for the power of symbols, how and when symbols can and should be used in order to display relationships, generalizations and proofs that otherwise are hidden and invisible. Arcavi (1994) found that most learners lack substantial background in algebra and do not resort to symbols as tools to enable them to investigate it in a general way.

In some cases, invoking mathematics symbols may be costly in terms of the amount of work and time required to execute the mathematics task compared to other approaches. Thus, researchers claim that learners who know how to perform algebraic manipulations, but do not consider the possible relevance of symbols to reveal the structure of a problem that has aroused their curiosity, did not fully develop their symbol sense (Drijvers, 2003; Arcavi, 2005). Having symbol sense includes the relevant invocation of algebra; that is, to have symbols readily available as possible sense making tools. A further indication of lack of symbol sense is also noticed when, in the process of solving mathematical problems algebraically, learners are usually unable to recognise and express solutions in symbolic forms or having symbolic as final answers. Even when symbols are used, and the solution they yield is recognised, it would be desirable that learners appreciate the "power of symbols": Only with the use of symbols, a conjecture or an argument can be conclusively accepted or dismissed.

Arcavi (2005) further posits that symbol sense should include, beyond the relevant invocation of symbols and their proper use, the appreciation of the elegance, conciseness, the communicability and the power of symbols to display and prove relationships in a way that arithmetic cannot. Thus symbol sense requires learners to invoke symbols when they are appropriate and it requires them to abandon symbols when they are likely "to drown" in complicated technical manipulations. The ability to discard the almost unavoidable initial temptation to proceed mostly symbolically, in favour of the search for another approach, requires a healthy blend of "control" with symbol sense. Control refers to “a category of behaviour which deals with the way individuals use the information potentially at their disposal (Schoenfeld, 2014. It focuses on major decisions about what to do in a problem, decisions that in and of themselves may “make or break” an attempt to solve a problem. To sum up, this component or theme of symbol sense implies that learners should cultivate a culture of trying alternative ways to represent the problem, in the belief that more elegant and straightforward approaches may exist and should be considered.

2. *Manipulating and 'reading through 'symbolic in solving algebraic problems*

One of the strengths of symbols is that they enable us to detach from, and even "forget", their referents in order to produce results efficiently. On the one hand, the detachment of meaning coupled with a global 'gestalt' view of symbolic expressions is needed for the manipulations to be relatively quick and efficient (Drijvers, 2011). On the other hand, the reading of and through the symbolic expressions towards meaning adds layers of connections and reasonableness to the results. An observation made by Chirume (2012) on learners performing tasks involving symbols indicates automatic manipulation of symbols without understanding their meanings. Another strategy used by learners involves the use of the *a-priori* inspection of the symbols with the anticipation of gaining a feel for the problem and its meaning, and its *a-posterior* checking to contrast meaning-making with symbolic manipulations are instances of symbol sense (Hurlburt, 2009).

For example when solving the equation, $\frac{5x+2}{10x+4}=2$ learners should try to 'read' meaning into the symbols. One might notice that, whatever x, since the numerator is half the denominator; this equation cannot have a solution. Tall (1996) claim that this *a-priori* inspection of the symbols with the expectancy of gaining a feel for the problem and its meaning is another instance of symbol sense. This also corresponds to algebraic expectation of the Algebraic insight framework.

3. Initiating symbolic relationships

This refers to the ability to successfully initiate mathematical symbolic relationships that express verbal or graphical information needed to make progress in a problem. This scheme shows a higher cognitive level of symbol sense than the ones discussed above. It suggests that, given the symbols, learners with symbol sense should be able to "read" meaning from the symbols themselves. It proposes that symbol sense also includes: firstly, an appreciation that an ad hoc symbolic expression can be created for a desired purpose and that one can engineer it; secondly, and more specifically, the realization that an expression, with certain characteristics is what is needed; finally, symbol sense should include the ability to engineer that expression successfully.

4. The ability to select symbolic representation for a problem

A learner who has symbol sense should be able to assign a symbol for a certain variable, situation, idea or process and have the courage to recognise and have dissatisfaction with that choice to search for better ones. This re-conceptualisation emerged from regarding equivalent symbolic expressions as possible sources of new meanings.

5. Reflecting on the meanings of symbols during problem-solving

This involves checking for symbol meanings during the implementation of a procedure, the solution to a problem, or, during the inspection of a result. When learners translate a situation into symbols, the first step is to choose what and how to represent. The choices that learners make crucially affect their solution process as well as the results. In this regard, a learner with a developed symbol sense makes the appropriate choice by taking into account the goal of the problem. The choice of symbols may not only obscure part of the situation, but it may also impede the whole solution process.

6. Symbols have different roles and meanings in different context

This component of symbol sense involves the realisation that symbols play different roles in different contexts such as, variables or parameters (Gutiérrez, Leder and Boero, 2016). Thus, learners should develop an intuitive feel for those different contexts. In this case a learner is expected to appreciate the desirable components of symbol sense which consists of the "in-situ" and operative recognition of the different (and yet similar) roles which symbols can play in high school algebra. This entails that the learner with symbol sense should be able to sort out the multiplicity of the meanings of symbols depending on the context. In addition, the ability to handle different mathematical objects and processes involved (Tarasenkova, 2013). In order to understand mathematical symbols, Yetkin (2003) recommends that learners should be exposed to multiple meanings of the symbols in different problem contexts.

Reflections on symbol sense

A number of researchers attempted to review the symbol sense framework. Arcavi (2005) further characterises (5) and (6) as showing a higher cognitive levels of symbol sense than (1) and (2). Kinzel (2001) describes symbol sense as the combination of notational awareness of expressions and the skill to manipulate and interpret symbolic expressions.

Pierce and Stacey (2002) adopt Arcavi's work in suggesting a practical research framework called algebraic insight as a subset of symbol sense, and their focus was mainly on algebraic expectations.

Zorn (2002) takes a broader view of advanced symbol sense to mean "...the general ability to extract mathematical meaning from and recognize structure in symbolic expressions, to encode meaning efficiently in symbols, and to manipulate symbols effectively to discover new mathematical meaning and structure" (p. 4). Zehavi (2004) coined the term advanced symbol sense to refer to problem-solving behaviours that involve masterful insight and judgment of the problem and its solution. A further reflection by Naidoo (2009) on the attributes of symbol sense revealed that the six components of 'symbol sense' are interrelated and closely linked. In other words, if a learner has one component then she/he will probably display other components. However, lacking one component might result in not having any of the components. In other words, if a learner shows 'friendliness with symbols' then the learner is likely able to manipulate and read symbolic expressions.

Inculcating symbol sense

There are ongoing debates on whether symbol sense is taught or is just acquired naturally; the so-called nature or nurture controversy. The debate is centred on the following questions: Is symbol sense something that only mathematically able people develop by themselves, or can most people develop it at least partially? Can symbol sense be taught? Arcavi (2005) proposes that: symbol sense can be nurtured, and one necessary condition for symbol sense to develop is to provide supportive instructional practices. Bokhove and Drijvers (2010) describe symbol sense as an intertwinement between procedural skills and conceptual understanding as complementary aspects of algebraic expertise. Good teaching aims to address both procedural skills and symbol sense in algebra, as they are intimately related: understanding of concepts makes basic skills understandable, and basic skills can enforce conceptual understanding (Arcavi, 2005). Teachers should discourage from jumping to symbols, but to make sense of the problem, to draw a table, a graph or a picture, to encourage them to describe what they see and to reason about it.

2.5.2 Algebraic Insight Framework

The proponents of the symbol sense framework are Pierce and Stacey (2001). The Algebraic Insight framework is embedded in the Symbol Sense framework. Algebraic Insight is the subset of symbol sense that enables the partnership of the thinking involved at all stages of mathematical problem solving including formulating the problem and interpreting the solution. The theory helps in the formulation of mathematical solutions to problems (Pierce & Stacey, 2001). The framework breaks algebraic insight into two components: ability to link representations (symbolic, numeric, graphical); and algebraic expectation, the cognitive skill required to monitor symbolic work. Pierce and Stacey (2001) describe algebraic insight as the algebraic knowledge and understanding which allows a learner to correctly monitor algebraic expressions during problem solving.

According to Pierce and Stacey (2004), the algebraic insight, has two aspects: "algebraic expectation" and ability to link representations. The term algebraic expectation refers to the thinking process that takes place when an experienced mathematician figures out the result they expect to obtain as the outcome of some algebraic process. Pierce and Stacey (2004), divide the algebraic expectation into three elements: a) recognition of conventions and basic properties, which common instances are the knowledge of the meaning of the symbols, the order and the properties of operations; (b) identification of structure, which common instances are the identification of objects and of strategic groups of components and recognition of simple factors ; c) identification of key features, related to the identification of the form and the dominant term, as well as the union of the form with the type of solution.

Algebraic expectation focuses on the application of Algebraic Insight *within* the symbolic representation of a mathematical problem. For example, an estimate of the product of 5000 and 4200 will be in millions. Algebraic expectation may involve expecting the product $(2-x+x^2)$ (x^4-x^3+2x-9) to be a polynomial of degree seven or the expansion of $(1+2x)^5$ has 6 terms one of which one is a constant. It is important to note that Algebraic expectation does not produce an approximate solution but rather noticing conventions, symbols, structure and key features of an expression that determine features which may be expected in the solution. Algebraic is characterised by the following features:

Ability to recognise of conventions and basic properties

Learners must recognise the conventional meaning of symbols used in algebra. This involves both operators and ‘letters’. While the operators $+, -$, and x should be familiar from arithmetic the convention in pen and paper algebra of implicit multiplication, where xy means x times y, is a source of confusion. Letters are used in a number of ways in algebra. For example, a standard quadratic function is commonly expressed as $y = ax^2 + bx + c$. This requires a learner to recognise that the letters a, b and c are parameters while x and y are variables, two different meanings for letters in the same algebraic sentence. Thus, a learner with algebraic expectation has knowledge of meanings of symbols, order of operations and properties of operations.

Ability to identify structure

Recognising structure of an algebraic expression can mean seeing at a glance, a learner can realise that $3x-1$ is a common factor, in the expression $(3x-1)^2 - 5x(3x-1)$ but looking at $(3x+1)^2 - 5x(3x-1)$ and noting that the bracketed objects differ.

Ability to identify key features

When solving equations, identification of key features may lead to expectation about the type of solution, number of solutions, type of solution, whether a point is maxima and minima, domain and range. Identifying the correct form of equation helps the learner to apply associated knowledge required to solve the problem. For example, $2 + e^x = 0$ is a linear equation in e^x while $e^{2x} + 5e^x = -6$ is a quadratic equation. A learner with a good algebraic insight can realise that the first equation has one unique solution while the second equation has at most three distinct solutions.

Ability to link representations

The ability to link representations involves the learners’ ability to move cognitively between symbolic (algebraic) representations and graphical or numeric representations. Such linking is also concerned with expectations, but expectations across representations.

Algebraic Insight is shown when a learner has expectations about graphs and tables that are linked to features of the symbolic representation of the problem. For example, upon recognising the function $f(x) = 3x + 1$, the learner can tell that the graph of this function is a straight line of gradient 3 and a y −intercept of 1.

Furthermore, the learner should be able to tell that the orientation of the graph stretches from the bottom left corner to upper right corner of the Cartesian plane. Pierce and Stacey (2001) describe algebraic insight as the algebraic knowledge that enables a learner to correctly use conventional mathematical symbols. It involves knowledge of linking multiple representations. A mathematical idea can be represented symbolically, graphically, numerically, or in other ways. Having algebraic insight involves being able to anticipate what the graphical or numerical representation looks like given a symbolic representation, or vice versa.

Pierce and Stacey (2001) recommended learners to recognise the meanings of both letter and operator symbols in order to inform their understanding of transitions between symbols and graphs or tables. Recognising and understanding of the structure of mathematical concept are features of problem solving (Pierce & Stacey, 2001; Rubenstein & Thompson, 2001; Neria & Amit, 2004; Kieran, 2007a). Arcavi (1994) considers the ability to identify symbols to reveal the structure of a problem as an important part of symbol sense, and Pierce and Stacey (2001) stress that, a structural view of expressions will inform algebraic expectation. Thus, the two theories blend well.

Ability to link of Algebraic and graphical representations

The ability to link algebraic and graphical representations of a mathematical concept involves associating algebraic form to the shape and key features such as orientation, intercepts and asymptotes. Linking of shape to form is shown when a learner looks at a function like $f(x) = 2\sin(x - 30^0)$ recognises that this is the graph of the sine function in which the modulus has been doubled and translated by 30° to the right. In general, identifying form provides enough information about a graph to be able to draw the basic shape ‘in the air’ with a hand wave.

Ability to link symbolic and numeric representations

The ability to link symbolic and numeric representations is shown when a learner links number patterns to formula. For example a learner's ability to link symbolic and numeric representations is shown when a learner can represent the pattern: $2, 4;\ 6;\ 8$ .... as $T_n = 2n$. Algebraic insight framework is a framework for reflecting symbol sense at the solving stage. The framework addresses ways of planning, assessing, and reflecting on learners' understanding when working with mathematical symbols to solve mathematical problems. Blending this framework and expanding it to include aspects of symbol sense at all levels of problem solving assists in the task of identifying learners' progress in developing activity-effect relationships (Simon et al, 2004). In analysing learners' competency with mathematical symbols the researcher can look for signs of recognition of conventions, understanding of the meaning of symbols and order of operations. The researcher can also look for instances of learners' verbalising or indicating connections that they are making between what is being done on paper and what is needed to meet their goal. This framework provides observable aspects on which the researcher can focus when interviewing and working with learners in the study.

There are limitations to the Pierce and Stacey's Algebraic insight framework as a lens for describing learners' reasoning about symbols since it is designed to apply only to elements of symbol sense at the stage of solving an already formulated problem. It does not describe the activity in other stages of problem solving, such as formulating the problem and interpreting the solution. This is the problem-solving stage where learners seem to have challenges (Evans and Swan, 2014). Kenney (2008) expanded the Algebraic Insight framework by incorporating features for identifying learners' uses and understanding of symbolic structures in the other stages of problem solving. However, her frameworks were criticised for lacking the back-and-forth movement between representations that is typical of learners' reasoning about symbols. Although her framework was useful in identifying and categorising some aspects of symbol sense, it was criticised for not providing a lens for examining some of the challenges in learners' reasoning about symbols that she found in her study. The current research seems to need a framework that can incorporate these two frameworks, hence the need to envisage other frameworks.

2.5.3 Action–Process– Object–Schema (APOS) Theory

APOS theory is grounded in the philosophical beliefs of constructivism and focuses on the mental constructions made by learners as they attempt to make sense of mathematical concepts. The proponents of this theory are Dubinsky and McDonald (2001). APOS is an attempt to understand the mechanism of reflective abstraction, introduced by Piaget (1968) to describe the development of logical thinking in learners. It is resolutely grounded in the tenets of constructivism, contending that learning is not passively received but rather constructed by an active participant.

APOS is an acronym that stands for the types of mental structures (Action, Process, Object, and Schema) which learners build in their attempt to understand mathematical concepts (Brown, De Vries, Dubinsky, Mathews & Thomas, 1996; Dubinsky & McDonald, 2001). Arnon et al. (2014) state that "…APOS is a theory which explains how learners learn mathematical concepts" (p.1). According to the APOS-theory the learner constructs a mathematical concept so that an action performed to an object is interiorized to a process which then encapsulates to an object (Hähkiöniemi, 2006). APOS theory is a useful theoretical framework for studying and explaining learners' conceptual development. It is closely related to Piaget's (1968) notions of reflective abstraction; it claims that mathematical knowledge develops as learners perform actions that become interiorized to form a process or a concept, which eventually leads learners to a higher level of awareness or object understanding of a mathematical concept. Finally, the learner organises these mental images to make a schema that enables him to conceptualise a mathematical situation.

APOS theory claims that mathematical objects are constructed by reflective abstraction in the sequence A-P-O-S, beginning with **A**ctions that are perceived as external, interiorised into internal **P**rocesses, encapsulated as mental **O**bjects developing within a coherent mathematical **S**chema. APOS theory views mathematical knowledge as an individual's tendency to deal with perceived mathematical problem situations by constructing mental actions, processes, and objects and organizing them in schemas to make sense of the situations and solve the problems (Dubinsky and McDonald, 2001). Mathematical knowledge in this theory is modelled through those constructions by making inferences from learners' activity with specific mathematical tasks. APOS theory proposes that a learner should possess certain mental structures to make sense of a given mathematical concept. It is therefore recommended that before teaching a concept, the teacher should design suitable learning activities to support the construction of these mental structures.

APOS is a cognitive theory (Arnon et al, 2013). Objects in this framework are considered as mental objects that individuals construct in order to learn about mathematical objects, as defined by the mathematics community. The theory proposes that mathematical knowledge is constructed by making mental *actions*, *processes*, and *objects* and organising them in *schemas* to make sense of the situations and solve problems. APOS theory is a tool that objectively explains learner difficulties with a broad range of mathematical concepts and to suggest ways that learners can learn these concepts. It can inform teachers on the pedagogical strategies that lead to marked improvement in learner learning of complex or abstract mathematical concepts and learners' use of these concepts to prove theorems, provide examples, and solve problems. There seems to be considerably widespread agreement that mathematical ideas begin with human activity and move from there to abstract (Dubinsky, 1991).

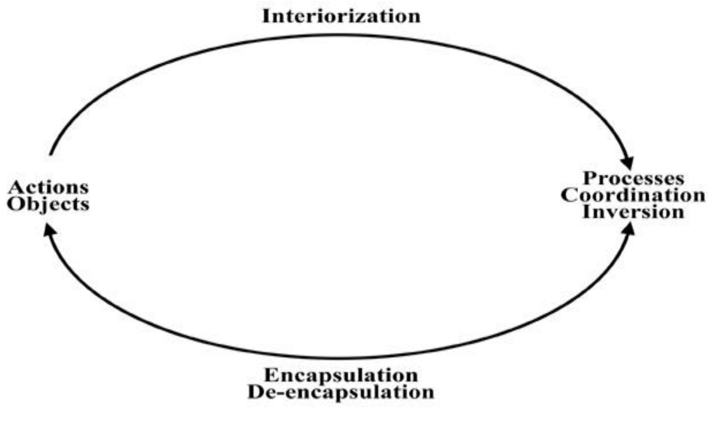


Figure 2-1: Construction of mathematical knowledge

APOS claims that for one to understand a mathematical concept, one must begin by invoking previously constructed mental or physical objects in the learner's mind to form actions. Actions would then be interiorised to form processes that are then encapsulated to form objects. These objects could be de-encapsulated back to the processes from which they are formed, which would be finally organised in schemas. Jojo (2014) stated a learner who has developed a schema for a concept has developed a process or object conception of the concept, that is, the learner can understand the concept as a process or as an object.

APOS theory claims that the formation of a mathematical concept involves transforming existing objects into new objects. An action is any transformation of objects according to an explicit algorithm in order to obtain other objects, and is seen as being at least somewhat externally driven. As an action is repeated and the individual reflects upon it, it may be interiorized into a mental process. An important characteristic of a process is that the individual is able to describe, or reflect upon, the steps of the transformation wholly in her/his mind without actually performing those steps. Additionally, once a mental process exists, it is possible for an individual to think of it in reverse and possibly construct a new process (a reversal of the original process) (Font et al., 2008).

When a learner becomes aware of the process as a totality and is able to transform it by some action, we say that the process has been encapsulated as an object. When necessary, an individual may de-encapsulate an object back to its underlying process. In other situations, the individual may think of the transformation in terms of actions. A schema for a certain mathematical concept is an individual's collection of actions, processes, objects and other schemas linked consciously or unconsciously in a coherent framework in the individual's mind. The research method or investigative approach of this framework consists of three-step cycles. The first step is a theoretical analysis of the actions, processes, objects, and schemas that a learner may construct in order to learn a given/specific mathematical concept.

According to Berger (2005), the use of a symbol to refer to an object prior to 'full' understanding resonates with how a learner makes a new mathematical object meaningful to herself. In practice, the learner starts communicating with peers, with teachers or the potential readers using the signs of the new mathematical object before she has full comprehension of the mathematical sign. This communication with signs gives initial access to the new object. According to Vygotsky (1986), the central role in concept formation is a functional use of the word, or any other sign, selecting distinctive features and analysing and synthesizing them. He also argued that the learner does not spontaneously develop concepts independent of their meaning in the social world. The meaning of a concept is 'imposed' upon the learner and this meaning is not assimilated in a ready-made form.

A learner is expected to construct a concept whose use and meaning is compatible with its use in mathematics and is accepted by the mathematics community. To do this, a learner needs to use the mathematical symbols in communicating with more socialised others (including the use of textbooks which embody the knowledge of more learned others). In this way, concept construction becomes socially regulated. Vygotsky (1978) regarded all higher human mental functions as products of mediated activity. The role of the mediator is played by psychological tools, such as words, graphs, algebraic symbols, or a physical tool. Vygotsky (1978), views action mediated by symbols as the fundamental mechanism which links the external social world to internal human mental processes and he argues that it is by mastering semiotically mediated processes and categories in social interaction that human consciousness is formed in the individual (Berger, 2005).

The constructs of APOS theory

Mulqueeny (2012) summarised the four constructs of APOS theory of conceptual understanding as follows:

The action construct

An action is a physical or mental manipulation that transforms objects. Learners develop an action construct of a mathematical concept if they have an external perception of the mathematical concept. This means an individual can only carry out symbolic manipulations via specific external cues and detailed step by step procedure. A learner whose knowledge of algebra is limited to an action conception reacts to external cues of mathematical symbols by giving precise details on what steps to take. Learners who have an action conception of symbols see algebraic expressions as commands to follow a certain procedure. In order to alleviate learners' misconceptions at the action level teachers need to address the symbols. Working blindly with symbols that are not understood leads to incorrect solution processes. Learners tend to invent their own procedures to deal with or avoid symbols they do not fully understand. For example, an expression such as $\sin 2\theta$ consists of three distinct pieces, each of which needs attention. This cognitively obscures the learner and challenges the teacher in terms of finding a convincing explanation that can be understood by learners. The symbol "sin" does not offer any intuitive notion of an action while "2″ means doubling the angle (θ). The whole expression can be mistakenly as $2\,sin\,\theta$. Learners struggle to see how this new information can "fit" into their existing cognition.

The **process construct**

A process is an action that takes place entirely in the mind. Exteriorisation occurs when the individual reflects upon the action that he or she is performing. A learner who is at the process level of understanding can "reflect on, describe, or even reverse the steps" of a previously learnt concept without actually performing those steps. A learner who has acquired the processes level can view the function $\sin 2\theta$ as a sine function in which the angle has been doubled or $\sin(\theta+\theta)$. If a learner has moved to this next level of understanding, they should be able to apply the identity: $\sin(A+B)=\sin A\cos B+\sin B\cos A$ to get $\sin(\theta+\theta)=\sin\theta\cos\theta+\sin\theta\cos\theta=2\sin\theta\cos\theta$. A learner with a process conception is able to see that the expression stands for compound angle, which in this case is a double angle. Teachers therefore need to focus their attention to this kind of learners' use of symbols as this has a potential to support the development of symbol sense (Bills, 2001) and scaffold the learner to a process level of understanding.

The Object construct

A process is encapsulated into a cognitive object; the learner is able to reflect on many different representations of the concept. Dubinsky (1991) speculated that encapsulation is difficult to see and researchers can only infer that this level of understanding has been achieved from statements made by a learner. Asiala (1996) described this phenomenon as the ability of a learner to "reflect upon operations applied to a particular process and become aware of the process as a totality. A learner with an object construct realise that the $sine$ of a doubled angle is not the same as twice the $sine$ of the angle ($\sin 2\theta \neq 2sin\theta$). Sfard (1991) describes this ability as structural thinking. Seeing a mathematical concept as an entity enables the learner to "recognise the idea at a glance and manipulate it as a whole, without going into detail" (Sfard, 1991, p. 4). At this developmental stage, thinking is detailed and dynamic. The learner is able to move freely from object to process. Once this is achieved, the concept is said to be at the object level. Thus, the learner should be able to see that $\sin 2\theta = \sin(\theta + \theta) = 2sin\theta \cos\theta$, without invoking the identity $\sin(A+B) = \sin A \cos B + \sin B \cos A$.

The **schema construct**

A schema is a collection of cognitive objects and internal processes for manipulating these objects (Brijlall & Ndlovu, 2009). According to Dubinsky (1991), a schema helps learners to:

> *"... understand, deal with, organise, or make sense out of a perceived problem situation"* (p.102).

Skemp (1981) considers a schema as a conceptual structure stored in memory. He argues that a schema integrates existing knowledge and, even more than a concept, greatly reduces cognitive load. Skemp argues that inappropriate early schemas will make the assimilation of later ideas much more difficult, perhaps impossible. A learner who has developed a schema for the double angle identity should realise that $\sin 2\theta = 2sin\theta \cos\theta$ without reverting to the double angle identity, $\sin(A+B) = \sin A \cos B + \sin B \cos A$.

Sfard (1991) pointed out that concepts can be conceived in two fundamentally different ways: as processes (operationally) or objects (structurally). In APOS, theory action and process can be regarded as operational conceptions, while object and schema are structural conceptions. Sfard (1995) used the term "reification" to characterise the "act of turning computational operations (processes) into permanent object-like entities" (objects). The development of mathematics often proceeds by taking processes as operators and then turning them into objects. Examples of processes as operators are counting, calculating using a formula (for example, using the n^{th} term (Tn) of a sequence to generate successive terms) and differentiating; while examples of resulting objects are numbers, algebraic expressions (for example, the n^{th} (general) term of a sequence) and the first derivative ($\frac{dy}{dx}$) or $f'(x)$ of a function . Therefore, reification, which refers to a transition from an operational to a structural mode of thinking, is a basic phenomenon in the formation of a mathematical concept since it brings the concept "... into existence and thereby deepens our understanding" (Sfard & Linchevski 1994, p.54). Both operational (procedural) and structural thinking are important in mathematics since both contribute to the hierarchical structure of algebra, which is used to represent mathematical concepts symbolically.

Tall (2004) introduces the idea of three worlds of mathematics, the embodied, symbolic and formal. The worlds describe a hierarchy of qualitatively different ways of thinking that individuals develop as new conceptions are compressed into concepts that are more thinkable. The embodied world, containing embodied objects, is where we think about the things around us in the physical world, and it "includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuo-spatial imagery" (Tall, 2004, p. 30). The symbolic world is the world of procepts, where actions, processes and their corresponding objects are realized and symbolised. According to Tall, Thomas, Davis, Gray and Simpson (2000) the formal world of thinking comprises defined objects, presented in terms of their properties, with new properties deduced from objects by formal proof.

APOS theory is similar to the concept image that Tall and Vinner (1981) introduce in "Concept image and concept definition in mathematics" with particular reference to limits and continuity. The development of a schema occurs during a process called reflective abstraction (Arnon, Cottrill & Dubinsky, 2013). Reflective abstraction is a concept introduced by Piaget (1978) to describe the construction of logico-mathematical structures by an individual during the course of cognitive development. Piaget (1978) made two important observations. Firstly, that reflective abstraction has no absolute beginning, but is present at the very earliest ages in the coordination of sensori-motor structures (Piaget & Beth, 1966, pp. 203-208). Secondly, that it continues up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction (Piaget, 1985). This process utilizes two mechanisms: projection unto a higher level of abstraction and reflection aimed at reconstruction and reorganisation into larger systems. The process of reflective abstraction is the means by which concepts can evolve from actions to processes to objects and finally into schemes. These processes are termed exteriorisation, encapsulation and schematization, respectively.

Interiorisation

Transformation of an action is the process by which a physical series of actions can be performed in the mind without the need to be prompted or having to perform every learner step. For example, $6 + 8 = 14$, can be done mentally without counting pebbles. Once achieved, it can be said that a given action has been interiorized into a process. For an action to be interiorized into a process it must be repeated and the learner reflects upon it. When the learner is able to describe, or reflect upon, the steps of the transformation wholly in her/his mind using abstract symbols without actually performing those steps, we conclude that the actions have been interiorized into a process.

Encapsulation

When a learner becomes aware of the process as a totality and can apply actions to it, the process is encapsulated and an object is constructed. Thus, a mathematical process is encapsulated when the given mathematical concept exists abstractly without the need to perform any specific actions or steps. At this stage, the concept gains invariant properties. Once this is achieved, the concept can be transformed and new actions can be learned using the encapsulated mathematics process, now said to be at the object level. For example, a learner knows how to find the derivative $(\frac{dy}{dx})$ or $f'(x)$ of a function$f(x)$, use it to find turning points, to determine concavity, points of inflection and n^{th} derivatives.

Schematisation

Schematisation is the process by which multiple objects, processes, and actions, form a coherent body, called a schema, where concepts can be manipulated and related to one another. Schematisation implies the possibility of thinking of a schema as a whole, to act on it or make transformations on it and study its properties. It also involves the possibility to dissect, break down, examine its parts, and reassemble it as a whole. García, Llinares and Sánchez-Matamoros (2011) characterized the derivative schema is in terms of the learners' ability to explicitly transfer the relationship between a function and its first derivative to the derivative function and the second derivative.

Designing and implementing instruction according to APOS

The design of instruction based on APOS is based on the assumption that learning is a non-linear process. APOS theorists claim learners gain partial knowledge and repeatedly return to this knowledge in an attempt to organise their knowledge structures. The learner first develops partial understanding, repeatedly returns to the same idea, and periodically summarizes and tries to pull the ideas together. APOS theory assumes moreover that learning is fundamentally dependent on cognitive conflicts whose overcoming requires a "re-equilibration" of previously developed mental constructions (Piaget, 1985). Cognitive conflicts may arise when the learner's ideas contrast ideas of others. Therefore, in a classroom based on APOS theory, learners are usually organized into groups where they can work cooperatively and are encouraged to reflect on procedures that they perform. This is intended to drive the learners into an environment where their mental constructions can disequilibrate, or start to contradict each other in the learner's mind. The effort to overcome those contradictions may lead to the formation of new mental constructions. According to Dubinsky (2010), APOS theory's application to teaching and learning is based on two assumptions mathematical knowledge and learning.

Implications of the assumptions

One of the implications of the assumptions made above is that a learner must possess the appropriate mental structures to make sense of a given mathematical concept and its symbolic structure. Maharaj (2013) also studied learners' mental structures for understanding the limit process and found that many learners lack mental structures at the process, object and schema levels. The mental structures refer to the likely actions, processes, objects and schema required to learn the concept. The theory requires teaching and learning to be structured in such a way that before a given mathematical concept is taught or learnt the likely mental structures needed to support understanding should be detected, and then suitable learning activities should be designed to support the construction of those mental structures. Thus, the assumptions imply the selection of teaching strategies that help learners to build appropriate mental structures, and guiding them to apply these structures to construct new understanding of mathematical concepts.

Instructional approaches suitable for APOS theory requires teachers to start with a breakdown of the topic or concept into simpler concepts which are combined to give the overall picture. The teacher should anticipate a set of mental constructions that learners might form as they begin to explore the concepts. This provides an initial theoretical perspective used to guide instruction. The theory proposes that teachers begin instruction by giving explicit directions, enabling learners to carry out routine procedures. Repeating these actions, coupled with teacher-guided questioning and cueing strategies that encourage reflection provides a framework for the development of an action conception of the concept. At this level, teachers will in fact giving learners tools to think with. When leaners no longer need external cues to manipulate mathematical symbols, they begin to realise that symbolic notation is related to the concept, interiorize these actions to form processes which in turn form concept images.

APOS and mathematical representation

Representation is an essential tool for expressing mathematical concepts and thoughts when learning mathematics. Representations and symbol systems are fundamental to mathematics as a discipline since mathematics is "inherently representational in its intentions and methods"(Kaput, 1989, p. 169). Panasuk (2010) views representation as an attribute of mathematical concepts, which are defined by three variables: (i) the situation that makes the concept useful and meaningful, (ii) the operation that can be used to deal with the situation, (iii) and the set of symbolic, linguistic and graphic representation that can be used to represent situations and procedures. Hiebert and Carpenter (1992) propose a framework for understanding based on the constructivist perspective that sheds light on how mathematics understanding occurs. Representations are essential elements for supporting learners' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others (Clement, 2004). Hiebert and Carpenter (1992) make a distinction between the external and internal representation of mathematical ideas, pointing out that, to think and communicate mathematical ideas, learners need to represent them in some way. External representation refers to observable symbols, figures and tables, models, and images (Adu-Gyamfi & Bossé, 2014). Communication requires that the representations be external, taking the form of spoken language, written symbols, drawings or concrete objects. Internal representation refers to the mental images constructed by a learner.

Learners can use external representation to produce an internal representation of mathematical concepts. When the various changes in the internal representation of a mathematical concept and the functional relationships among these changes can be developed, we can say that this concept has been learned (Kaput, 1987). Goldin (2001) identifies five different forms of internal representation systems: (a) verbal/syntactic, (b) imagistic, (c) formal notational, (d) strategic and heuristic, and (e) affective. According to Goldin (2001) the study of learners' conception and understanding of a concept should focus on studying learner's internal representations. This is done by interpreting learners' interaction with, discourse about, or production of external representations. A concept is learned when a variety of appropriate internal representations have been developed with functioning relationships among them.

External and internal systems of representation and their interaction are essential to mathematics teaching and learning (Goldin & Shteingold, 2001). Internal representations are usually associated with mental images individuals create in their minds. Bruner (1966) proposed to distinguish three different modes of mental representations: the sensory-motor (physical action upon objects), the iconic (creating mental images) and the symbolic (mathematical language and symbols). Panasuk (2010) posits that internal representation is an attribute of high-order human cognitive processes; it involves abstraction to represent the entity of the object of communication in symbols. Pape and Tchoshanov (2001) described mathematical representation as an internal abstraction of mathematical ideas or cognitive schemata that the learner constructs to establish internal mental network or representational system Hiebert and Carpenter (1992). Thus, one can assert that internal representation and abstraction are closely related mental constructs.

External representations are associated with the knowledge and structure of the environment, physical symbols, objects, or dimensions as well as external rules, constraints, or relations embedded in physical configurations (Khosla, Sethi and Damiani, 2013). Goldin and Shteingold (2001) suggested that an external representation "is typically a sign or a configuration or signs, characters, or objects "and that external representation can symbolise something other than itself" (p. 3). Most of the external representations in mathematics (for example, signs of operations, symbols or composition of signs and symbols used to represent certain relationships) are conventional; they are objectively determined, defined and accepted. In distinguishing internal representations and external representations, Kaput (1999) used the term "fusion" to emphasize the actions surrounded by the experience of internalising the external representation. Through classroom discourse and various experiences, teachers facilitate interaction between external representations and the learners' internal representation systems and assist the learners in the process of building into their internal mental structure the images of the external representations (Goldin and Shteingold, 2001, p.2). For instance, to introduce the notion of multiplication, the teacher gives certain meanings and interpretations to the multiplication symbol (x) as an external representation (external abstraction) that replaces repeated addition symbols (for example, $4+4+4=4\times3$).

Because of interaction of "learners' personal symbolisation constructs" with the external representation (Goldin & Shteingold, 2001, p. 2), multiplication sign, learners build into their mental structure the image of the operation of multiplication that becomes their internal representation. Goldin and Shteingold (2001) stress that learners' internal representations are affected by their visual imagery, natural language, problem solving abilities and their attitude toward mathematics. Mathematical relationships, principles, and ideas can be expressed in multiple representations including visual representations (i.e. diagrams, pictures, or graphs), verbal representations (written and spoken language) and symbolic representations (numbers, letters). Each type of representation articulates different meanings of mathematical concepts.

According to Goldin (1998) representation systems are proposed to develop through three stages, so that first, new signs are taken to symbolise aspects of a previously established system of representation. Then the structure of the new representation system develops in the old system and finally the new system becomes autonomous. Therefore, a mathematical concept can be represented in multiple ways. Different forms of representation can be used to express or build the same concept, and each representation has advantages that make it superior to other representations. In discussing these advantages, Tall (2004) felt that graphical representations provide qualitative and comprehensive insight, quantitative results, and symbols provide a powerful capacity for manipulation. APOS and representation theories allow researchers to examine the same phenomenon from two different but complementary viewpoints. In APOS theory, by using actions or processes of representation to describe the theory, reflection on actions can produce meaningful viewpoints or properties, causing the actions to become internalized as processes. By integrating representation theory, the researcher can clarify the role of these actions by emphasizing the necessity of distinct viewpoints or properties. APOS theory can be used to describe the relationship between two objects in the same schema, or the relationships among objects, processes, or actions with different representations. For example, symbolic representations of a cubic function are $y = x^3$ or $f'(x)$. The symbols of its derived functions are $\frac{dy}{dx} = 3x^2$ or $f^{'}(x) = 3\ x^2$.

2.5.4 The Procept Theory

Another theoretical framework adopted in this study is Procept Theory. The proponents of this theory are Gray and Tall (1994). Procept refers to the dual nature of mathematical symbols both as a process (such as addition) and as a concept (the sum) (Tall, 1992). The notion of procept helps to explain the dual nature of mathematical symbols. The procept theory enables us to think about different kinds of encapsulation in different contexts and to see how learners face cognitive difficulties related to symbolism (Tall, 1995). It includes different symbols and different processes that give rise to the same mental object in the mind of learner. This phenomenon of the duality and ambiguity of mathematical notation perceived as procedure and concept has been proposed by Gray and Tall (1991) as an explanation of an underlying cause of learners' success or lack of success in mathematics. This theory postulates a duality between a process and a concept in mathematics. One way in which this duality becomes apparent is that a single symbol is often used to represent both a process (such as the addition of two numbers $(12 + 7)$ and the sum of that process (the sum of 19), which is the object.

Gray and Tall (1994) described this dual nature of symbols as a "procept". In an attempt to define procept, they introduced the term elementary procept. It consists of an amalgamation of three components: a *process* that produces a mathematical *object* (or concept) and a *symbol* that represents either the process or the object (Gray & Tall, 1994). The processes often begin as step-by-step procedures that are slowly routinized into processes that can be thought of as a whole without needing to carry them out. Symbols allow the mind to pivot between the procedure and process on one hand and the mental concept on the other. A procept conceives symbols flexibly both as processes to do and concepts to think about. This flexibility allows more powerful mental manipulations and reflections to build new theories.

The Procept theory suggests that there is a non-linear progressive and recursive relationship between signifier (symbol) and signified (object) in constructing and communicating a mathematical object. A symbol that evokes a process or product is called a *procept*. Such a symbol stands dually for both a *pro*cess and a *concept.* It gives great flexibility in mathematics. This flexibility makes matters particularly difficult for the learner. Learners who implicitly sense the flexible power of symbolism succeed in understanding mathematical concepts, while those who do not, are likely to fail. In a sense, if a symbol is used as a signifier to refer to a signified, that is, procept, a successful learner should be able to *s*ee process acting on an input to produce an output as concept. Moreover, later on, the learner can perform actions/transformations on the signified they already perceived. The symbol D_x in $D_x\left[2x^2+\frac{1}{2}x^4-3\right]$ represents both a process of differentiating a function and derivative of the function.

According to Gray and Tall (2001) the concept acquisition can start by an action performed on an object, but also by making a perception of an object. Gray and Tall (2001) call this kind of perceived objects embodied objects. The embodied objects are mental constructs of perceived reality, and through reflection and discourse they can become more abstract constructs, which do not anymore refer to specific objects in the real world (Gray & Tall 2001). Hence learner's conception can start to develop from perceptual or from symbolic representations, and it is important to connect these representations. Table 2.1 below summarises some of the symbolic expressions or phrases that represent both mathematical processes and objects.

Table 2-1: Procept theory- processes and objects

Expression	Process	Concept/Object
$5 + 3$	Addition	sum
$5\ x\ 3$	Multiplication	product
$3/4$	Division	Fraction/ ratio
$+4$	Adding four	Positive number
$3 + 5x$	adding 3 to the product of 5 and x	Algebraic expression
π	approximating π	Infinite fraction

The flexible use of a symbolism as either process or concept causes conceptual difficulties for learners. In the minds of successful mathematicians, a symbol evokes either process or concept, whichever is appropriate, and this is done so subconsciously that we may be unaware that it is happening. In algebra, learners who view symbols as procedures to be carried out are less likely to understand the meaning of mathematical concepts (Oksuz, 2007).

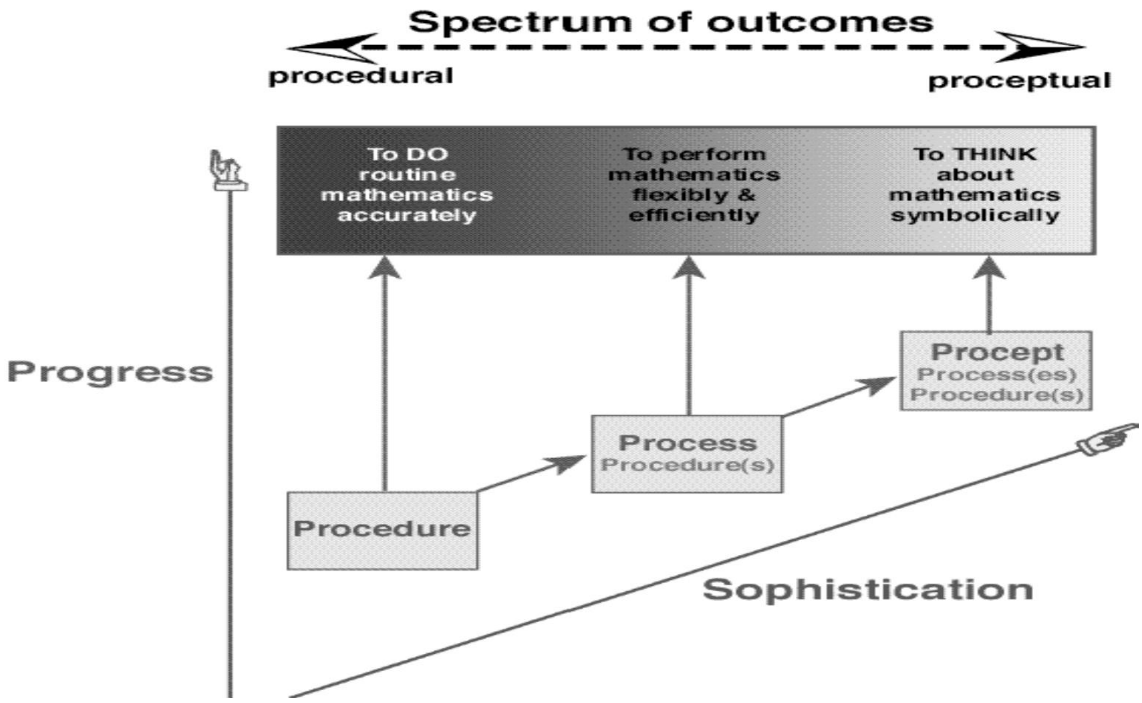


Figure 2-2: Procept theory (*Adapted from Tall, 1994*)

An action-based learning process begins by making some actions on the objects. At first, a sequence of actions, a procedure, is performed by using a step-by-step algorithm. After several repetitions, the procedure is automatized, and a learner is able to see it as an entity so that he/she can consider it without referring to the single steps. Then the process is encapsulated as a mental object. This stage is similar to the APOS theory (Dubinsky & McDonald, 2001) and Sfard's (1991) reification theory that describes the cognitive development of processes into objects.

Sfard (1991) pointed out that mathematics concepts could be conceived in two fundamental ways: structurally and operationally which respectively results in "objects" and "process." She distinguished those two conceptions in the following way: There is a deep ontological gap between operational and structural conceptions. Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing, a static structure, existing somewhere in space and time. It also means being able to reorganize the idea "at a glance" and to manipulate it as a whole, without going into details. In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential, and detailed (p. 4). In another article, Sfard and Linchevski (1994) maintained that learners need to switch from process to object in order to understand concepts. They specified three stages in the transition: interiorisation, condensation, and reification. Therefore, Sfard's theory about understanding concepts is startlingly consistent with Chi's (2002, 2005). The transition of process to object is also consistent with Piaget's theory of "reflective abstraction" (Simon, Heinz & Kinzel, 2004) which has two phases: "*a projection phase in which the actions at one level become the objects of reflection at the next and a reflection phase in which a reorganisation takes place*" (p. 313).

2.6 Justification for combining frameworks

The procept notion has strong links with APOS theory, but there are significant differences. Procept and APOS theories that seek to explain how learners learn new mathematics content. They are all frameworks of conceptual growth. The implication of the two theories is that learners play an active role in their own learning and action is required on their part to develop a deep level of mathematical understanding. Learners who do not see an object as more than a procedure may well be good at performing computations and succeed in the short term but in the long term they may lack the flexibility that will give greater success. Precise definitions of mathematical concepts that are given in class presentations focus on the object at the expense of the inner process. This prevents a larger number of learners, who do not sense the flexible power of symbolism from succeeding in mathematics. Despite the fact that Dubinsky's APOS theory refers to learners' mental views and Tall (2008)'s worlds are about mathematical thinking, the theories seem to blend naturally together. Such a framework allows the researcher to evaluate learners' conceptual understanding of mathematical symbols and observe the way learners learn. Furthermore, it was designed to help teachers and instructors to cover a spectrum of representations in the classroom in such a way that teaching based on it would help learners build symbolic knowledge.

On the other hand, symbol sense and algebraic insight frameworks also blend well since algebraic insight is embedded in symbol sense. Algebraic Insight is the component of symbol sense that helps in solving algebraically formulated mathematical problems. The first five attributes of symbol sense apply to the 'solve' section of the Algebraic Insight model (Pierce and Stacy, 2001). Algebraic insight is a specific symbol sense needed at the solving stage. Algebraic expectation focuses on the application of algebraic insight within the symbolic representation of mathematics while ability to link representations deals with the learners' ability to move cognitively between symbolic (algebraic) representations and graphical or numeric representations.

Algebraic insight framework addresses ways to plan, assess, and reflect on learners' understanding when solving mathematical problems (Pierce & Stacey, 2001). Incorporating this framework and aspects of symbol sense at all levels of problem solving assists in the task of identifying learners' progress in developing activity-effect relationships. In analysing learners' execution of the activity, the researcher can look for signs of recognition of conventions and properties to identify some of the aspects of symbol sense, including learners' understanding of the meaning of symbols and of order of operations.

Procept and APOS frameworks are cognitive oriented frameworks that provide useful tools for modelling learners' conceptual growth and explain the way learners learn new concepts. APOS is applicable as a tool to questions such as: "What pedagogical strategies can help learners in the mental construction of a particular concept?" A new mathematical concept is best learned if it involves an action conception of the concept, a process conception of the concept. A learner with an object conception can think about, name and manipulate an object without necessarily focusing on how it is formed. On the other hand, a learner with a process conception can think about problem-solving procedures and solution processes with little emphasis on what the object is. For this kind of learner, the process is more important than the product.

The four theoretical frameworks have representation as a common feature. Kaput (2000) describes a representation as some kind of relationship between a symbol and its referent. According to Goldin (1998) representation systems are proposed to develop through three stages, so that first, new signs are taken to symbolize aspects of a previously established system of representation. Then the structure of the new representation system develops in the old system and finally the new system becomes autonomous. Thus, in order to interact with concept, solve a problem, to act on an object, or experience a process, it must be cognitively represented in some way to facilitate meaning-making. Each of these theoretical positions makes an important contribution to the understanding of mathematical symbolisation and its contribution to mathematics teaching and learning. The composite conceptual framework is shown in Figure 2-3 below:

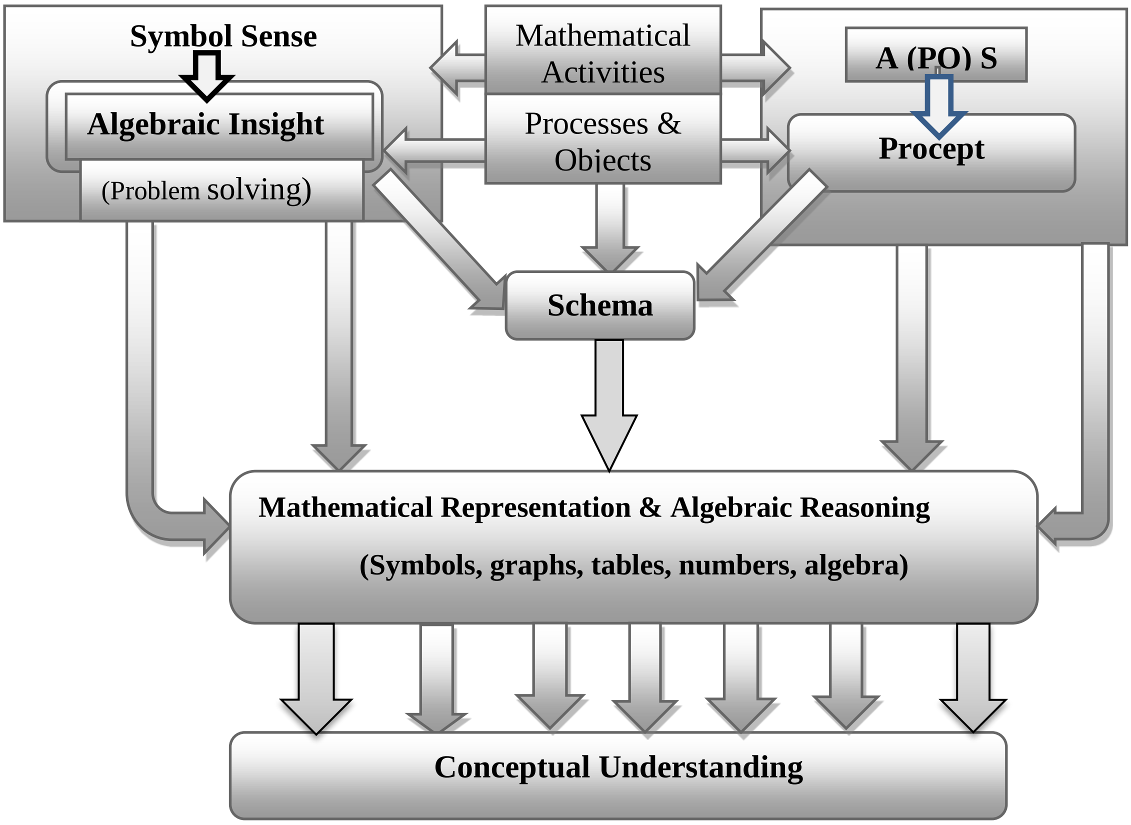


Figure 2-3: Theoretical Framework: Quadrilateral Frame of Theory

2.7 Summary

This chapter has discussed literature on past work that has been conducted to examine the nature of challenges that learners experience in trying to understand various mathematical concepts through their symbols. The review reveals significant extant literature on the specifics of the topic of investigation for this research. Literature on learners' experiences with mathematical symbolism appeared abundant relative to studies on learners' specific learning experiences and difficulties with mathematical symbolism. Some studies focused on mathematical symbolism itself to study learners' difficulties in manipulating symbols as mathematical objects and modifying their interpretations of symbols (Stacey & Macgregor, 1997). Some investigated how meaning for symbols could be developed

(Kieran, 1981) and some studied how mathematical symbols are used to delegate some mathematical operations to the external environment (De Cruz & De Smedt, 2013). Other studies investigated how learners draw meaning of symbols from inside of the symbol systems (Hiebert & Carpenter, 1992).

Current researches focus on symbolisation challenges specific to certain topics such as translating word problems to algebraic statements (Silver, 2013; Reynders, 2014), functions (Chirume, 2012), derivatives (Zweng, 2012). This study contributes to this debate by looking at the symbolisation challenges experienced by secondary school learners in the South African FET band when interpreting mathematical concepts and problem solving. Furthermore, the study investigates into the instructional strategies teachers can use to mitigate the effects of symbolic obstacles. Four (4) conceptual frameworks were condensed into a quadrilateral frame of theories that serve as lens for focusing and guiding this study. The next chapter discusses the methods that were used to conduct this study.

CHAPTER 3: RESEARCH METHODOLOGY

This chapter discusses the research methodology and design, including sampling, population, establishing rigour during and after data collection, ethical considerations and data analysis. The chapter explains how the research was conducted. A number of measures were taken to ensure that quality data is collected. Ethical considerations and trustworthiness are also discussed.

3.1 Research questions

The selection of the methodology for collecting and analysing data was guided by the following research questions:

a) What challenges do secondary school learners encounter when interpreting and using mathematical symbols to understand mathematical concepts and problem solving procedures?
b) What instructional strategies can mathematics teachers use to mitigate the effects of symbolic obstacles?

3.2 Research Methodology

Methodology encompasses concepts such as research paradigms, theoretical models and quantitative or qualitative techniques. Burns and Grove (2003) describe methodology as the means or methods of conducting research, which includes the design, setting, sample, methodological imitations, and the data collection and analysis techniques in a study. According to Holloway (2005), methodology means a framework of theories and principles on which methods and procedures are based. In this study, methodology describes how the research was conducted, what data was collected and how it was analysed.

A mixed methods approach was utilised in this study. Mixed methods research refers to quantitative and qualitative procedures of collecting and analysing data in the study (Creswell, 2013). Creswell and Plano-Clark (2007) define mixed methods as a methodology that involves the collection and analysis of qualitative and quantitative data in a single study or series of studies. The main reason for mixing the two research approaches is to obtain better understanding of research problems that either approach cannot achieve alone. The study focused on exploring and describing the experiences of learners as they struggle with the symbolic barrier to understanding mathematical concepts therefore the research approach was dominantly qualitative.

3.2.1 Research paradigm

This study is guided by a constructivist paradigm. Creswell and Plano-Clark (2007) defined a paradigm as a worldview. A paradigm is an interpretative framework, which is guided by a set of beliefs and feelings about the world and how it should be understood and studied (Lincoln & Guba, 2000). Constructivism as a paradigm posits that learning is an active, constructive process. The learner is an information constructor. The goals of constructivist research are understanding and structuring, as opposed to prediction. This study explored and described the experiences of FET band learners as they integrate the symbolism in mathematical concepts. The conception of mathematical symbols is constructed through the APOS, Symbol sense, and Procept and Algebraic Insight theories. Different types of data have to be used to construct a complete picture of mathematical symbols.

3.2.2 Qualitative Approach

The dominant research approach for this study is qualitative, since the natural setting is the direct source of the data (Fraenkel & Wallen, 2003). For this study, data was collected from the participants in their natural setting without controlling any aspect of the research situation. Qualitative methodology is interactive and interpretive. In the interaction between the researcher and participants, the researcher discovers the participant’s world and interprets it (De Vos, 2002). This study intended to find out challenges and difficulties learners encounter when dealing with mathematical symbols to develop concepts in the teaching and learning process. The first research question for this study was best answered through a qualitative paradigm. This design allows an in-depth understanding of learners’ challenges about the use of symbols in algebra and in exploring the factors that affect them in learning algebra. In this study, a qualitative method explored and described the challenges teachers and learners encounter when dealing with mathematical symbols, learners' interpretation of mathematical symbols and instructional strategies to reduce symbolic obstacles.

3.2.3 Quantitative Approach

Quantitative approach measures and analyses the causal relationships between variables. In order to eliminate the weaknesses and limitations of qualitative and quantitative approaches, Laxman (2015) suggests combining them in a mixed methods design. The main weakness of the quantitative paradigm is that the researcher is inseparable from the object of observation (Kura & Sulaiman, 2012). On the other hand, the qualitative research does not generate predictive models that generalise to larger populations. The quantitative paradigm tests and validates existing theories generalising research findings (Johnson & Onwuegbuzie, 2004). Thus, the strengths of both paradigms were combined to offset their mutual limitations.

3.3 Research Design

Research design is the overall plan for obtaining answers to the research questions (Polit & Beck, 2004). It is a plan of action that links the philosophical assumptions to specific methods (Creswell, 2013). The research design for this study is in two levels: the logic of the research and at another level, the research design reflects on the purpose of the inquiry, which in this case, is both exploratory and descriptive.

Exploratory research examines a theoretical idea. The researcher has an idea and seeks to understand more about it. This study was informed by the researcher's observation of learners' use and manipulation of mathematical symbols without understanding their meanings or concepts they represent. The exploratory research lays the groundwork for future studies on the idea. What is being observed might also be explained by a currently existing theory. Exploratory research identifies the boundaries of the environment in which the problems, opportunities or situations of interest are likely to reside and to elicit the salient factors or variables that might be found there and be of relevance to the research.

On one hand, a descriptive research design provides an accurate and valid representation of the variables that are pertinent and relevant to the research question (van Wyk, 2012).

Methods, on the other hand, refer to specific techniques that are used for data collection and analysis (Creswell, 2003). Kumar (2010) viewed it as a blueprint of how a research study is conducted. It operationalises variables so that they can be measured from a sample and analysis of the data therefrom. This procedure must be carefully adapted by the researcher to answer questions validly, objectively, accurately and economically. Thus, the research design minimizes the chances of drawing incorrect causal inferences from data.

3.3.1 Descriptive Research Design

A descriptive research design was used for the quantitative data collected using the questionnaire survey. Quantitative research designs emphasise objective measurements of data (Babbie, 2010). The study described the status of learners' understanding of mathematical symbols and their use in conceptual understanding. The dependent or criterion variable is a phenomenon that one is attempting to explain or predict. In this study, the phenomena of interest cover the difficulties that learners and teachers experience due to mathematical symbolisation. Since this study is non-experimental, there are no independent variables that can be manipulated to explain or predict the dependent variable. However, extraneous variables such as demographics of participants need to be controlled in order to obtain meaningful results. Hence, variables such as grade, gender, social economic status, age, home language, geographical location of participants and ethnicity were considered to see the extent to which they influence learners' understanding of mathematical symbols.

3.3.2 Phenomenological research Design

A phenomenological research study attempts to understand people's perceptions, perspectives and understandings of a phenomenon (McConnell, Chapman & Francis, 2009). The aim of phenomenological study is to obtain descriptions of experiences from learners who experience problems with mathematical symbols. The aim of the research is to describe the phenomenon of learners' symbol sense as accurately as possible. Similarly, Sterley (2014) believes that phenomenologists seek to understanding phenomena from the perspectives of the participants. From these descriptions, themes, typologies emerge. It involves interpreting the original descriptions of symbols using reflective analysis and interpretation of the participants' accounts. Primary methods of data collection are audio-recorded-conversations.

A phenomenological methodology was also utilised in this study. Interviews were designed to build a description of the participant's experiences with symbols. The fundamental assumption made is that the important reality is what people perceive it to be (Alibakhshi, 2015). This perception builds a description of a learner's conception of mathematical symbols that build mathematical concepts. Thus, the phenomenological interview is a technique ideally suited for data collection in this study.

Intuiting

This process involves thinking through the data in order to obtain a comprehensive and accurate interpretation of what participants mean in a particular description (Leech & Onwuegbuzie, 2007). In order to achieve this, the researcher remains open to the meanings and issues raised by participants in terms of the difficulties they experience with mathematical symbolisation. Intuition leads to a common understanding about the phenomenon that is being studied. It also requires that the researcher creatively analyses the data until such a common understanding emerges. The researcher must be totally immersed in the study of the phenomenon.

Analysing

Analysing involves listening to, comparing and contrasting descriptions of learners' conceptions of mathematical symbols in to identify the essence of the phenomenon under investigation. Analysis seeks to make sense of the essential meanings of the phenomenon. Common themes emerge as the researcher works with the descriptive data.

Bracketing

Bracketing is a qualitative research technique that suspends assumptions and presuppositions about any knowledge of learners' difficulties with symbolisation and teachers' approaches to symbolisation to limit interference with the information given by the participants (Tufford & Newton, 2010). Bracketing improves rigour and reduces bias in research. In this exploration, the researcher suspends his assumptions and preconceptions especially during data analysis. As recommended by Castellan (2010), the researcher remained neutral with respect to belief or disbelief in the existence of the phenomenon. The researcher first identified learners' preconceptions about mathematical symbolisation. Researcher also had to suspend all prior knowledge about learners' challenges, to allow the trustworthy "truth" to emerge.

Describing

This is the final step in which the researcher describes distinct, critical elements of the phenomenon. The researcher avoided premature to description of the phenomenon, a common methodological error in this type of research (Vilakati, 2009). In this study, phenomenological describing involved classifying all critical elements common to learners' challenges in understanding mathematical symbols.

'Memoing' was also used in this study. This is recording what the visual, auditory impressions and thoughts of the researcher in the course of collecting and reflecting on the process Groenewald (2004). The researcher complied field notes of what participants were raising during the data-collection process and reflected on the data analysis. As recommended by Ejimabo (2015) the researcher kept updated memos and later correlates them with the data.

In view of the issues discussed above, phenomenology was considered the best method and approach to address the qualitative part of the study.

3.3.3 Reflective analysis

Reflexivity is an aspect of a phenomenological research in which researcher assumes the roles of a researcher and the participant at the same time (Finlay, 2012). Researchers continuously reflect on their own preconceived values, participants' perception of the researcher and reflecting on how it will influence the data collected. In this study, the researcher maintained as self-monitoring stance in order to prevent bias and increase objectivity of the study. As recommended by Holloway and Wheeler (2002) the researcher continuously reflected on his own feelings, actions and conflicts during the research so that they do not affect the credibility of the study.

3.3.4 Mixed Method Approach

Rich and Brown (2014) defined mixed methods as 'research in which the researcher collects, analyses, mixes, and draws inferences from both quantitative and qualitative data in a single study. Creswell et al (2006:5) define it as "…. a methodology, it involves philosophical assumptions that guide the direction of the collection and analysis of data and the mixture of qualitative and quantitative approaches in the research process". The researcher selected this approach on the basis that the combined use of quantitative and qualitative approaches provides a better understanding of research problems than either approach alone. Integrating methodological approaches strengthens the research design, as the strength of one approach offsets the weakness of the other (Creswell & Plano-Clark, 2011). The other practical benefit of using a mixed method research is derived from Baran and Jones (2016) who reveal that it encourages interdisciplinary collaboration and use of multiple paradigms in a research.

Although there are on-going debates about whether MMR is a research design or methodology, this study takes a middle ground. MMR is a research design with philosophical assumptions as well as quantitative and qualitative methods. Wilson (2016) describes mixed methods as a research methodology in which data is collected, analysed, and inferences drawn from both quantitative and qualitative data in a study. Qualitative and quantitative designs, methods, data collection and analysis techniques were utilised to provide data that was later mixed to provide a big picture of the findings of this study. The choice of a mixed method approach was derived from the nature of research questions and the kind of instruments used to solicit the data.

The first research question for this study seeks to explore the challenges that learners encounter when interpreting and using mathematical symbols to understand mathematical concepts and problem solving procedures. The second research question is based on instructional strategies that mathematics teachers can use to reduce the effects of mathematical symbolisation obstacles. To address these research questions a survey questionnaire consisting of closed and open-ended questions was used. Quantitative data analysis methods were used to summarise data in the form of descriptive statistics. Open-ended questions were analysed by drawing a list of broad categories that were later qualitatively researched using focus group interviews. Thus, the study utilised qualitative research to gain access to participants' views about symbolisation while quantitative research allow researcher to make statistical inferences about the phenomenon.

3.3.5 Mixed method designs

There are many mixed methods designs in literature, each emphasising different dimensions. However, all of them share two common basic dimensions: timing of the integration and purpose of integration (Guest, Namey & Mitchell, 2013). Timing of integration refers to the stage at which qualitative and quantitative data sets are used. The purpose of integrating both methods is to overcome weaknesses in using one method with the strengths of another.

Morse (1991) describes simultaneous and sequential mixed designs: In simultaneous triangulation, qualitative and quantitative methods are used simultaneously but there is limited interaction between the two sources of data during the data collection stage, and the findings (at the data interpretation stage) complement one another. Triangulation combines methodologies in the study of the same phenomenon to decrease the bias inherent in using a particular method (Morse, 1991). In the sequential design, one form of data, either the qualitative or quantitative, is collected before the other. When the results of one approach are necessary for planning the next method, sequential triangulation is utilized. Quantitative data can support qualitative research components by explaining the emerging phenomenon and the reverse is true for qualitative data illuminating quantitative components by development of the conceptual model.

The design for this study is a sequential mixed design. Data were collected in two phases. First, data were collected using a questionnaire consisting of closed and open-ended questions. Quantitative statistical methods were used to analyse the closed questions to determine which findings to explore further and augment in the next phase. The researcher reviewed and analysed the survey results and tailored the subsequent in-depth interview instrument to follow-up on significant responses. Participants were purposively selected based on the issues they raised in the open-ended questions. Predictor importance values were utilised to inform and select questionnaire items that needed further investigation using focus group interviews. Secondly, questionnaire number codes were used to select in-depth interview participants. The subsequent in-depth, semi-structured interview schedule consisted of questions intended to explore particularly interesting survey responses. Figure 3-1 below shows the detailed summary of the sequential exploratory design used in this study.

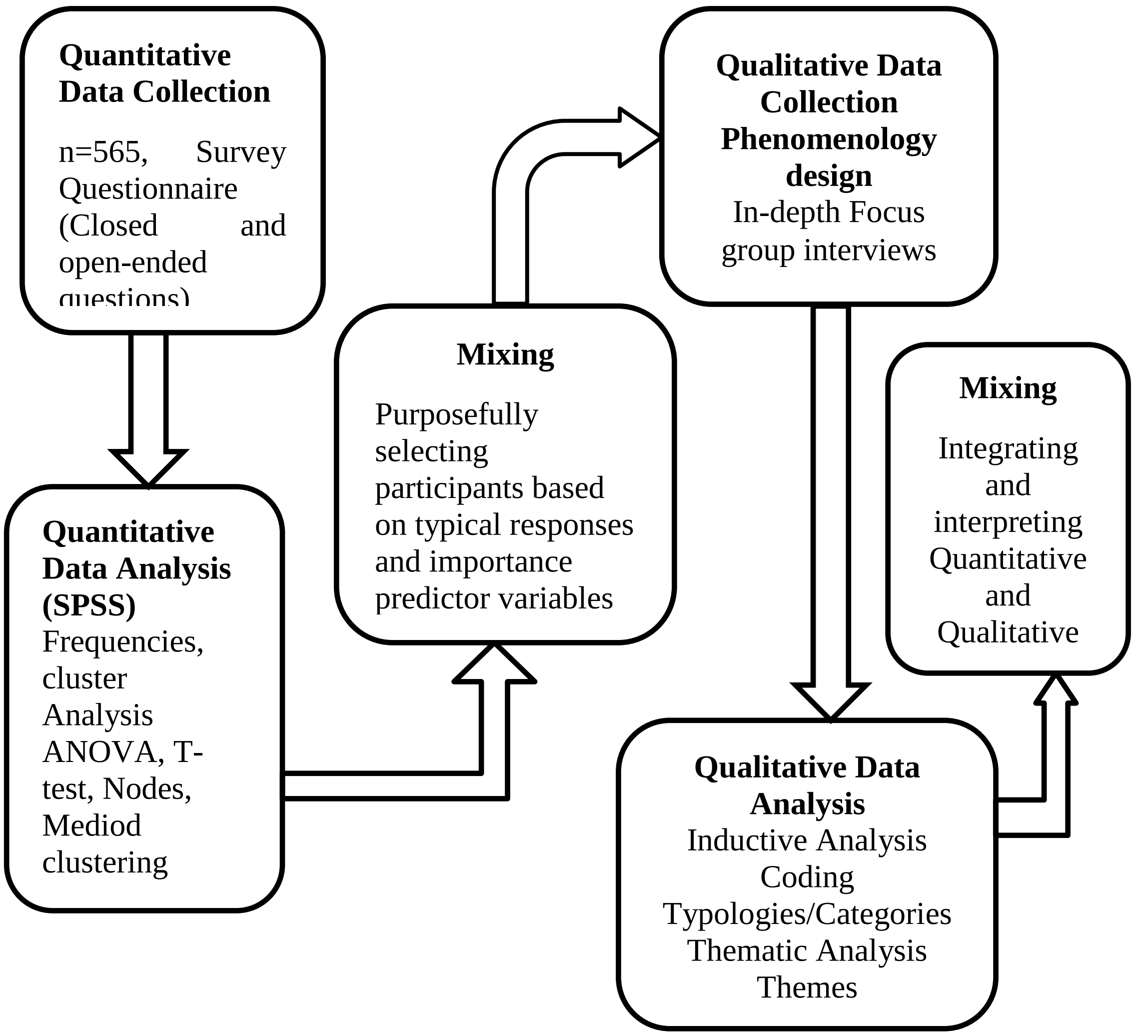


Figure 3-1: Sequential explanatory design (*Adapted from Creswell & Garrett, 2008*)

3.4 POPULATION AND SAMPLING

Polit and Beck (2014) define a population as the entire aggregation of units that meet a designated set of criteria. A population is also defined as all the individuals who have certain characteristics and are of interest to a researcher (Teddlie & Yu, 2007). Two types of population in research are: target population and accessible population. The target population is the total of cases that the researcher would like to make generalisations about (Polit & Beck, 2004). In this study, the target population consists of learners enrolled in Grades 10 to 12 in Greater Sekhukhune and Capricorn Districts in Limpopo Province. The reason for involving learners in theses grade levels was that they had adequate exposure to a variety of mathematical symbols inscribed in their textbooks. The research also targeted 18 mathematics teachers as valuable sources of data regarding the challenges of mathematical symbols since they are likely to observe these as they engage learners during the teaching and learning process. The population from which the researcher draws their conclusions is the accessible population. This population is a subset of the target population and is also known as the study population. In this study, the accessible population consists of 800 Grade 10-12 learners and 15 mathematics teachers who participated in the study.

3.4.1 Eligibility criteria

Eligibility criteria specify the characteristics of prospective participants that make them to be considered for inclusion in the study (Shamseer, Galipeau, Turner & Moher, 2013). These characteristics must be shared by all participants. The researcher enrolled participants with similar characteristics to ensure that the results will be due to what is under study and not extraneous factors. In this way, the eligibility criteria helped the researcher to achieve accurate and meaningful results. A well-defined eligibility criterion makes research protocol safe, ethical and scientifically valid (Humphreys, Harris & Weingardt, 2008). For eligibility to this study, participants had to:

- be Grade 10-12 learners enrolled in public secondary and high schools in Limpopo Province, South Africa
- have enough exposure to a variety of mathematics textbooks and are able to read, write and verbalise mathematical symbols.
- Secondary school mathematics teachers.

3.4.2 Sampling method

This study adopted Kemper, Stringfield and Teddies' (2003) guidelines for choosing a sample. The sample was selected such that it could furnish sufficient data on the phenomena being studied. Conducting a mixed method research requires the researcher to satisfy the requirements of the qualitative and quantitative samples. Teddlie and Yu (2007) referred to these requirements as representativeness and saturation. Sampling in quantitative research aims to achieve representativeness, that is, a sample has to be so large enough so that it reflects the true characteristics of the population. In this study, a sample of $N = 565$ learners selected from the FET phase was deemed large enough to represent the population.

3.4.3 Multistage random sample

A multistage random sampling design was used for this study. Multistage sampling refers to survey designs in which the population units are hierarchically arranged and the sample is selected in stages corresponding to the levels of the hierarchy (Uthayakumaran & Venkatasubramanian, 2015). A multistage random sample is obtained by taking a series of simple random samples in stages. Multi-stage sampling represents is a form of cluster sampling in which large clusters are subdivided into small, more targeted groupings for the purposes of surveying (Rao, 2011).

At each stage, only units selected at the previous stage are considered. In this study, the first-stage units were districts, the second-stage units were circuits while the third stage units were the schools, and the fourth stage involves selecting learners and teachers who participate in the study. Multi-stage sampling does not require a complete list of members in the target population, which greatly reduces sample preparation cost. The list of members is required only for those clusters used in the final stage. The main disadvantage of multi-stage sampling is the same as for cluster sampling: lower accuracy due to higher sampling error. A large sample size (565 learners) was therefore selected from the population in order to reduce sampling error. A large sample size decreases the potential for deviations from the actual population (Lenth, 2001). A stratification protocol was implemented by selecting 32 learners from three grade levels per school and selecting three schools from each of the geographical locations of the participants: rural, semi-urban and urban schools.

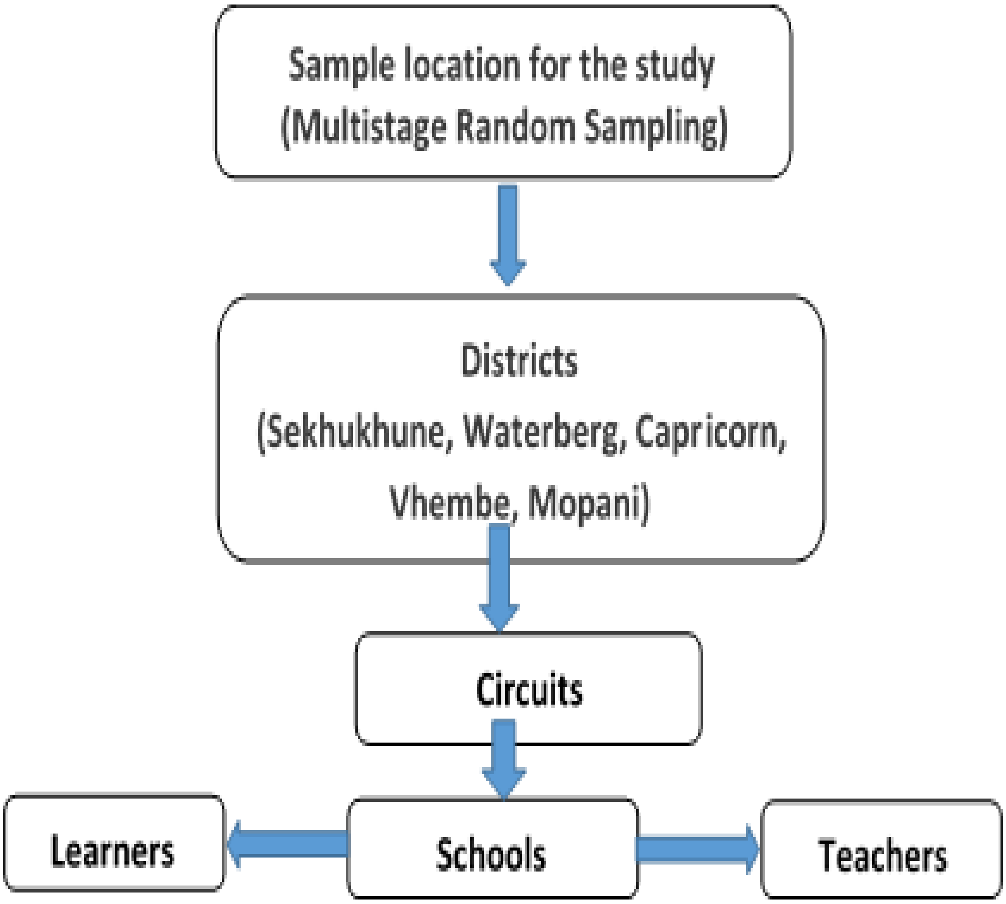


Figure 3-2: The sampling process

3.4.4 The Study Sample

The sample is a subset of a population selected to participate in a research study (Dul & Hak, 2008). For the sample, three schools from three circuits were randomly selected from the chosen districts to participate in the study (Banerjee & Chaudhury, 2010). A random sample of 96 learners consisting of 32 learners per grade level per school was selected from a population of FET band learners at the selected schools. At the end of data collection, 565 out of 800 questionnaires were successfully completed. This gives a response rate of 70.63%. Teacher participants were purposefully selected; they were all teaching Grades 10-12. These mathematics teachers were assumed to have adequate knowledge of the difficulties learners experience with mathematical symbolism. In a phenomenological study, “the phenomenon dictates the method, not vice-versa, including even, the type of participants” (Hycner, 1999:156). Purposive sampling is virtually synonymous with qualitative research. It is sometimes referred to as expert sampling since the researcher is looking for individuals who have particular expertise. Maxwell

(2008) also defines purposive sampling as one in which particular settings, persons or events are deliberately selected for the important information they can provide that cannot be obtained from other choices. In this study teachers who were teaching learners in the FET band were purposively selected as the researcher assume that they have experienced or observed learners struggling to understand mathematical concepts due to lack of symbol sense.

3.5 Data Collection

3.5.1 Research Instruments

In this study, questionnaires and focus group interviews were utilised because they supplement each other and their combination boosts the validity and dependability of the data. In the main study, quantitative data were obtained through closed-ended questionnaires and the qualitative data through open-ended questionnaires and focus interviews. Creswell (2011) hinted that a survey design provides a quantitative description of a sample that can be in turn generalised to the population from which it was drawn. The researcher found it useful to use a questionnaire since it was not possible to observe the phenomenon directly. The researcher is not a high school teacher and this requires a longitudinal study that can produce results after a long period of engaging learners. Thus, the data gathered through questionnaires allow the researcher to reconstruct learners' experience and perceptions of the phenomena (Alshenqeeti, 2014).

The items of the questionnaires were derived from research objectives and research questions. The questionnaire for this study consists of a mixture of closed-ended and open-ended. Closed-ended questionnaires are more convenient because of their ease of analysis (Seliger & Shohamy, 1989) while open questions can lead to a greater level of discovery (Gillham, 2000), because participants can express what they want to say (Zohrabi, 2013). Therefore, it is better that a questionnaire includes both closed-ended and open-ended questions to complement each other.

A group-administered questionnaire was issued to participants all at one time and place. Bee and Murdoch-Eaton (2016) recommended group-administered questionnaire because the return rate is high, the researcher is present to explain any unclear questions and knows the conditions under which the questionnaires were filled. The cover letter is an integral part of the questionnaire (De Vos, 1998), it informs the participants about the nature of the study and the value of their participation.

3.5.2 Questionnaire for Learners

The questionnaire for learners consisted of closed and open- ended questions. It addressed issues related to the research objectives. It consists of a covering letter and three sub-sections. Section A focused on participants' demographic data. Section B consisted of closed questions that explored learners' experiences, challenges and obstacles, encountered when using mathematical symbols. A 5-point Likert scale (1 = strongly disagree, 2 = disagree, 3 = neutral, 4 = agree, 5 = strongly agree) was used. The scale enabled respondents to report their experiences (Subedi, 2016). The last section consists of open-ended questions that solicited information relating to the teaching and learning approaches that are utilised in classrooms. Reja, Manfreda, Hlebec and Vehovar (2003) reasoned that open-ended questionnaire items work to elicit responses that individuals give spontaneously, avoiding the bias that may result from suggesting responses to individuals.

As highlighted by Stacey (2013) open-ended questions are used where the issue is complex, relevant dimensions are not known, or where a process is being explored. Harvey (2011) also recommended the use of a 'mixed' questionnaire is a best approach, arguing that researchers should avoid a restrictive questionnaire or even one that is too open and difficult to analyse. Bird (2009) also noted that open-ended items are used by participants to elaborate on the reasons underlying their answers to the closed-form items. Open-ended items in this questionnaire required learners to write their responses that were used to compile a list of questions for focus group interviews.

3.5.3 Administration of Questionnaires

The researcher personally administered the questionnaire to the participants at their schools. This has a fast response, as the researcher can get the questionnaires completed and collected quickly as compared to the postal method, where participants might postpone responding or questionnaire are delayed in transit (Sekaran & Bougie, 2013). The meanings of the questions were clarified to ensure that the participants were answering the questions in the sense that the researcher intended. The researcher also had the opportunity to introduce the research topic and motivate participants to offer frank responses. The researcher also explained the importance of the research and its significance to them. Self-administering the questionnaire also ensured better response rates because of the personal persuasion when researcher is present (Beukenhorst & Kerssemakers, 2012). However, the researcher was very careful to avoid introducing bias when explaining some of the questions to participants, especially in rural and semi-urban schools where learners had language problems.

3.5.4 Questionnaire for Mathematics teachers

The researcher prepared a perception questionnaire for teachers. Perception questionnaire asked questions concerning the feelings, thoughts, knowledge and opinions of participants (Mackay, 2004). The questionnaire for teachers was designed to obtain information about teachers' strategies for teaching mathematical concepts through symbolisation. The questionnaire for teachers focused on thoughts and perceptions related to mathematics education, classroom practical experiences with mathematical symbolisation. It also attached a covering letter on the nature and value of the research. Section A focused on participants' demographic data. Section B was made up of open-ended questions that explore teachers' experiences, challenges and obstacles, encountered with regard to the use of mathematical symbols when teaching mathematical concepts. The last section solicited information about the teaching and learning approaches that are utilised in classrooms. Only open-ended questions were used in this section.

3.5.5 Focus Group Interviews

In order to seek clarification to learners' responses to open-ended questions and to overcome difficulties in interpreting learners' mental processes, the researcher conducted focus group interviews that contained carefully constructed items and questions to identify learners' experiences, views, reflections, and symbol sense. Participants for focus group interviews were purposefully selected based on their responses to open-ended questions. Focus group interview is a type of in-depth discussion accomplished in a group, whose meetings present characteristics defined with respect to the proposal, size, composition, and interview procedures. The focus group research method generates ideas for investigation for generating additional or information for a study (Gill, Stewart, Treasure & Chadwick, 2008). Focus group interviews were most suited for this study since the objective was to further explore and understand learners' experiences of mathematical symbolisation based on their responses to closed ended questions.

The researcher allowed respondents the time and scope to express their opinions about mathematical symbolisation. The interviewer could explain questions that the respondent did not understand. Interviews also allowed the researcher to probe deeply into the problem to uncover new clues, to open up new dimensions of a problem, or to secure vivid, accurate and detailed accounts that are based on the personal experience of the participant (Zhou, Perera, Udeaja & Paul, 2012).

An interview guide was prepared ahead of time with questions and tasks to present to the participants (see Appendix B). At times, the interviewer allowed participants to guide the interview to a certain extent, as long as conversation remained within the realms of the study (Kenney, 2008). Different questions were used with different participants, depending on the details of responses and on the types of follow-up questions needed for a particular response. However, care was made to ensure that the discussions resonate around the targeted areas of study.

The researcher first established rapport with the respondents. Dundon and Ryan (2008) reported that if the participants do not trust the researcher, they will not describe their true feelings, thoughts, and intentions. Complete rapport is built over time as people get to know and trust one another. The researcher used a digital recorder to capture data because it has the advantage of preserving the entire verbal part of the interview for later analysis.

According to Harris and Brown (2010) structured questionnaires and structured interviews are often used in mixed method studies to generate confirmatory results despite differences in methods of data collection, analysis and interpretation. Questionnaires and interviews have different and complementary strengths and weaknesses (Lai & Waltman, 2008). Kendall (2008) asserts that while questionnaires can provide evidence of patterns amongst large populations, qualitative interview data provide more in-depth insights on participant attitudes, thoughts, and actions. Robinson

(2011) suggested that participants actually respond differently to questionnaire and interview prompts. Face-to-face interviews tend to trigger strong affective responses while questionnaires permit a wide range of cognitively dispassionate responses. Thus this research utilised the two approaches so that the weaknesses of one method are offset by the other method.

Qualitative research addresses the sample size issue by saturating the information. O'Reilly and Parker (2012) described saturation as point at which all the range of ideas and opinions about a phenomenon have been exhausted. Data collection went on until no more new information was generated. Focus group interviews went on until no new information or themes emerged from learners' narrations of their experiences or difficulties with mathematical symbolisation were generated.

3.6 Data Analysis

The data collected in this study was analysed using Statistical Package for Social Sciences (SPSS) version 23. A mixed analysis strategy was used to analyse the data. The rationale for conducting the mixed analysis was to ensure that results from one analysis type (qualitative) are interpreted to enhance or expand, findings derived from the other strand (quantitative). Analysing data in a mixed research study requires the researcher to integrate quantitative and qualitative results in a coherent and meaningful manner to produce reliable inferences (Powell et al, 2008).

The researcher adopted Creswell and Plano-Clark (2007)'s procedure which involves analysing the quantitative data using descriptive statistics and the qualitative data using thematic analysis. In this study a sequential explanatory analysis of quantitative and qualitative analyses was conducted with the aid of cluster analysis using Statistical Package for Social Sciences (SPSS Version 23). Quantitative data analysis involved descriptive statistics (frequency tables, clusters, Silhouette measures) and inferential statistics (T- and ANOVA tests, correlations and tests of hypothesis). Qualitative data analysis utilised cluster nodes generated from cluster analysis as well as interview data from both teachers and learners to create typologies or categories of mathematical symbolisation challenges and pedagogical strategies. Interview transcripts of participants 'words were content-analysed and themes emerge. Thematic analysis was conducted to identify themes and patterns of meaning across the dataset in relation to research questions. The process involves searching for themes among categories, reviewing themes, defining and naming themes, and validating the themes.

3.7 Pilot study

The researcher conducted a pilot study survey to ensure that quality is maintained throughout the study. A pilot study examining tools and processes in a research, drawing attention to problems before the main study begins (Secomb, 2011). Pilot studies examine study methods and data collection processes prior to a study (Leon, Davis & Kraemer, 2011). The researcher consulted peers and experts in Mathematics education to provide information on the appropriateness of intended instruments in order to validate the research processes before a major study begins.

It is important to clarify the pilot as it is used in this study. The term pilot study has two different meanings. On one hand it refers to the feasibility studies that are "small scale versions, or trial runs, done in preparation for the major study" (Polit & Bungler, 2004: 46). On the other hand, a pilot study also pre-tests research instrument (Sarandakos, 2012). Bless and Higson-Smith (2000) defined pilot study as a

"... small study conducted prior to a larger piece of research to determine whether the methodology, sampling, the instruments and analysis are adequate and appropriate" (p.155).

This mini-research exposes deficiencies of the measuring instruments or the procedure to be followed in the actual project. Pilot surveys are more common in quantitative studies, since adjustment after the beginning of fieldwork is less possible than in qualitative work (Shanyinde, Pickering & Weatherall, 2011).

The pilot survey was conducted at three selected secondary schools (urban, semi-urban and rural) which were omitted in the main survey. This was done to guard against contamination. Contamination arises when data from the pilot study are included in the main study (Collins, 2010).

The value of first piloting in this study was essentially to prevent waste of time, energy and money. In this study the pilot study was conducted based on Welman and Kruger (1999) recommendations that specifically aim to:

a) Detect possible flaws in measurement procedures such as clarifying instructions, time limits, and wording. The feedback from learners and teachers was helpful in restructuring some of the questions. This study utilised self-designed questionnaires, therefore piloting was necessary to adjust unclear and ambiguous questions.
b) Identify the non-verbal behaviour of participants in the study. This may give important information about any embarrassment or discomfort that can be experienced by participants due to the content or wording of items in the questionnaire.
c) Identify any sensitive issues that might reduce the response rate, obtain advance warning about potential weaknesses of the project, indicating where research protocols might be violated compromising the quality of the findings.
d) Identify and rectify practical problems of the research procedure, indicate whether proposed methods or instruments are inappropriate or too complicated.

3.7.1 Research Context and Setting

The study context is significant in qualitative research. The social context of the study is viewed as a crucial and integral element of analysis. According to Savikko, Routasalo, Tilvis and Pitkälä (2010) research context refers to the environment and conditions in which the study was conducted as well as the culture of the participants and location. The participants in this study were Grade 10-12 learners and mathematics teachers teaching Grade 10-12. The research was conducted in two districts in Limpopo province: Greater Sekhukhune and Capricorn. Greater Sekhukhune is a rural district, where most of the learners come from low social economic and poor backgrounds. A study conducted by Fabi (2013) revealed that the state of mathematics teaching and learning in Greater Sekhukhune District is below national standard. Some of the challenges highlighted include teachers lack the capacity to perform their mandate as instructed by the department. District and circuits offices are dysfunctional due to lack of subject advisors, planning monitoring. Greater Sekhukhune has 33 circuits and 25 (83.3%) are underperforming (Hindle, 2010).

Capricorn is rural, semi-urban and urban. The dominant language is Sepedi. Schools in this district are not well resourced. Most Limpopo schools are rural and these are characterised by high levels of poverty and unemployment. On average learners in Limpopo province, perform significantly below the national average in national matriculation examinations (Howie, 2006). This is because of poor teacher competence in content subjects and English language. Many teachers fail to provide appropriate mediation for learners to develop adequate cognitive functions in their subjects (Department of Education, 2014). Ramokgopa's (2013) findings show that current teachers in these schools do not perform at the grade level they are teaching. Teachers do not have the necessary subject content knowledge to enable them to teach the subjects in the grades they have been assigned to teach. Learners' performance has been a cause for concern. The performance of the province in international studies (TIMMS, 2012) has shown that learners generally perform below the expected grade levels in Literacy and Numeracy in Grade 3 and Languages in Grade 6 (Spaull, 2013).

3.7.2 Validity

Validity refers to the meaningfulness of research components (Drost, 2011). It is the amount of systematic or built-in error in measurement (Rao, 2007) and is established by a panel of experts and a field test. In this study, the questionnaire was pre-tested to enhance its face and content validity. According to Polit and Beck (2008), face validity is how far the instrument appears measures the appropriate construct. Face validity is a subjective and weak judgment on the operationalisation of a construct (Drost, 2011). In Content validity the analyst judges whether the measures fully represent the domain (Bollen, 2015). Content validity is a qualitative means of ensuring that the questionnaire has the meaning of a concept as defined by the experts in the same field.

To ensure validity in this study, the questionnaire was assessed by four mathematics education experts. The criteria for questionnaire evaluation were provided. The criteria consist of technical soundness, item clarity and relevance of the items. The researcher incorporated suggestions from the experts. A statistician did not make any amendments, suggesting that the descriptive analyses were mainly correct. Respondents were asked if they experienced difficulties in respect of being able to or willing to respond to the questionnaire. A checklist adopted from McMurray, Pace and Scott (2004) was used to monitor potential difficulties that can arise from the wording of the questions.

3.7.3 Trustworthiness

Trustworthiness is an aspect of the validity of the study (Loh, 2013). According to Anney (2014) trustworthiness refers the degree to which data is believable. It also refers to a set of criteria that can be used to judge the quality of qualitative inquiries. Schwandt (2001) also viewed trustworthiness as "that quality of an investigation and its findings that makes it noteworthy to audiences" (p.258). In order to improve the trustworthiness of the data collected the following criteria were used: credibility, transferability, dependability and conformability, and are constructed parallel to the analogous quantitative criteria of internal and external validity, reliability and neutrality (Denzin & Lincoln, 2000).

3.7.4 Credibility

Credibility measures how well the data and data analysis are believable and trustworthy (Davis & Buskist, 2008). Credibility is the careful attention by the researcher to establish trustworthiness. It measures the extent to which research findings reflect reality (Krippendorff, 2004). Credibility pays attention to assurances that respondents' views fit the inquirer's reconstruction, representation and interpretation (Schwandt, 2001). The validity of qualitative research is relative to the researcher and not necessarily to others due to the multiple realities. The reader must judge the extent of its credibility based on how they understand the study. From a rationalist's perspective there is no universal reality, instead, each individual constructs a personal reality (Smith & Ragan, 2005). Therefore, understanding is co-created and objective truth does not exist. In this study, the researcher included member checks into the findings to validate data, interpretations and conclusions using feedback from the participants. Furthermore, the researcher used persistent observation and triangulation to provide the assurance that what the researcher reports is a true reflection of the collected data and is consistent with the participants' views**.**

3.7.5 Triangulation

Triangulation validates data by cross referencing with two or more sources (Johnson, Onwuegbuzie & Turner, 2007). It refers to the application of several research methodologies such as multiple cases, multiple investigators, and multiple theoretical perspectives to verify that the validity criteria are met (Schwandt, 2001). The main objective of triangulation is to examine a conclusion from more than one vantage point. In this study, the researcher collected data and utilised multiple methods to analyse the evidence collected. The evidence for triangulation in this study collected includes observation notes, interviews, and questionnaire responses. However, it is debatable whether triangulation adequately verifies findings. Many viewpoints result in the argument that the worth of triangulation is the provision of broader insights. Thus triangulation is used to evaluate the findings of this study. Data obtained from qualitative explorative analysis and quantitative descriptive analyses were combined together and give meaning to the overall outcomes of the study.

3.7.6 Member Checks

The process of member checking obtains feedback from the participants about findings. It asks whether the researcher accurately described and interpreted the participants' experiences according to them by sharing the interview transcripts, analytical thoughts, and drafts of the final report. This ensures that the researcher has represented the ideas of the participants accurately (Lietz & Zayas, 2010). The researcher also allowed participants to see what was written about them.

3.7.7 Transferability

In qualitative research, transferability refers to the degree to which the findings can be applied and transferred to another group or to other context with similar conditions (Green & Thorogood, 2013). The reader is provided with rich, detailed information ("thick description") about the context that has been investigated. Transferability enables extrapolation of the findings across individual cases (Ary, Jacobs, Sorensen & Walker, 2013). The findings of this study can be used to understand learners from other schools, districts or provinces that have the same background as those participated in this study.

3.7.8 Dependability

According to Marshall and Rossman (2014) dependability refers to the degree to which research findings can be replicated in a similar context. Dependability emphasises the need for the researcher to account for the ever-changing context within which research occurs. Dependability ensures that the study process was logical, traceable, and well-documented (Shenton, 2004). It emphasises the importance of the researcher accounting for or describing the changing contexts and circumstances that are fundamental to guarantee consistency of the research outcome. Due to the evolving nature of the study, consistency is viewed as the extent to which variation can be explained or tracked (Ary, Jacobs, Sorensen & Walker, 2013). Triangulation was the strategy utilised to investigate dependability in this study.

3.7.9 Confirmability

Confirmability refers to the extent to which experts and researchers can corroborate findings (Ary, Jacobs, Sorensen & Walker, 2013; Lipscomb, 2012). Confirmability establishes that the evidence and interpretations of the study are not fabricated by the researcher. Strategies of confirmability included triangulation, audit trail, and member checks. Bitzer and Botha (2011) also recommended that auditing should be done to establish conformability. Here the researcher makes the provision of a methodological self-critical account of how the research was conducted. In order to make auditing possible by other researchers, all collected data was archived in a retrievable form, in case the findings are challenged and it becomes necessary to check the original data.

3.7.10 Audit Trail

An audit trail describes the research steps taken through the study to the development and reporting of findings (Bolar, 2015). The records of what was done in study are safely kept. Koch (2006) suggests that a study's trustworthiness may be established if a reader is able to audit the events, influences and actions of the researcher.

An audit trail ensures dependability and confirmability. In this study, the researcher maintained a journal of field observations and field notes. Documents such as write ups, observations note, and transcribed interviews are organised and filed as the audit trail.

The audit trail enables an independent auditor to examine the researcher's findings in order to attest to the dependability of the employed procedures (Ary, Jacobs, Sorensen & Walker, 2013).

3.7.11 Reliability

Phelan and Wren (2006) defined reliability as the degree to which a research instrument produces stable and consistent results. However, according to Streiner and Norman (2007) reliability refers to two things. On one hand, the researcher should get similar results if they repeated their questionnaires soon afterwards with the same participants. The "repeatability" of the questionnaire would be high. This is called test-retest reliability. It refers to questionnaire item consistency. If all the questions relate to the same phenomena, all the responses are expected to be fairly consistent.

Reliability was established using a pilot test. Data collected from pilot test was analysed using SPSS for correlation matrix and Cronbach's alpha coefficients (α). The Cronbach's alpha coefficient measures the internal consistency of a scale. It is the extent to which all the items in a questionnaire measure the same construct. Reliability coefficient (alpha) ranges from 0 to 1, with 0 representing an instrument with many errors and 1 representing total absence of errors. A reliability coefficient (alpha) of 0.70 or higher is considered acceptable reliability (Tavakol & Dennick, 2011). The Cronbach's alpha coefficient for this study (closed questions) of learners' questionnaire is shown in table 3-1 below.

Table 3-1: Cronbach's alpha coefficient

SECTION	Number of Items	Cronbach's Alpha Coefficient
B	26	0.716

The alpha coefficient of 0.716, suggests that the items have a high internal consistency.

3.8 Ethical Considerations

Liamputtong (2006) defined research ethics as a system of moral values that ensure that research procedures obey professional, legal and sociological obligations to participants. The researcher sought consent from participants before engaging them. Participants were informed about what participation in the research would involve, and what the possible risks were before they agree to take part. The researcher was guided by and complied with the Constitution of the Republic of South Africa, 1996 (Act No 108 of 1996) and potential participants were provided with information about the study. It was written at the appropriate reading age of potential participants. Finally, the researcher requested all the participants to sign consent forms before completing the questionnaire. Participants were assured that they could withdraw their consent and discontinue their participation at any time without penalty.

3.8.1 Rights of the institutions involved

Research Ethics Committee of the Institute of Science and Technology Education (ISTE-UNISA) reviewed the research proposal. The committee approved the proposal and granted permission to proceed with the study.

3.8.2 Respect for the rights of participants

The participants consented to participate in the study. Participants acknowledged that they had adequate information about the research, could comprehend the information and could discontinue from the research at any point. The nature of the study and its purpose were clearly explained. The researcher assured participants that their involvement in the study was voluntary. Failure of participants to comply with the research process or withdrawal from the study would not result in any consequences. The researcher's contact details were made available to respondents in case they needed to contact him regarding the study and their participation.

The researcher also committed to maintaining anonymity and confidentiality. The respondents were assured that anonymity and confidentiality would be maintained throughout the study. Participants were asked not write their names or any other personal details on the questionnaire to ensure anonymity (Cottrell & McKenzie, 2011). Confidentiality was maintained throughout the study since participants' identities were not linked to the information they provided. Number codes (for example, 023, for participant number 23) were used during data capture and data management. The responses were not discussed outside the research process.

PILOT SURVEY RESULTS

The purpose of a pilot study was to assess the feasibility of the research instruments. Theban et al. (2010) indicated that the goal of a pilot study is to assess the feasibility of the proposed study "so as to avoid potentially disastrous consequences of embarking on a large study, which could potentially 'drown' the whole research effort" (p. 1). The pilot study was mainly for testing the feasibility of the study, recruitment of participants, research tools and data analysis. The pilot study was necessary and useful in providing the groundwork for the study. However, this data might be irrelevant if there are problems with the methods. On the other hand, if a pilot study does not lead to modification of materials or procedures then the data might be suitable for incorporation into the main study (Kannan & Gowri, 2015). The presentation of the pilot study results was restricted to summary and descriptive statistics of the data as recommended by Arain, Campbell, Cooper and Lancaster (2010). Data presentation was mainly summary and descriptive statistics because the sample size was too small to detect differences and to make inferences. In addition, estimates of sample size, which are determined based on pilot data, may lead to insignificant statistical inferences. Thus, caution was undertaken when determining sample size for the main study.

Table 3-2: Demographic variables

Variable	Category	Frequency (f)	Percentage (%)
Gender	Female	73	66.4
	Male	37	33.6
Age (years)	11-15	13	11.8
	16-20	96	87.3
	21 Years and above	1	0.9
Home Language	Sepedi	108	98.2
	Sesotho	1	0.9
	Other languages	1	0.9
Grade	Grade 10	36	32.7
	Grade 11	36	32.7
	Grade 12	38	34.6
Residential Area	Urban	34	30.9
	Semi-Urban	36	32.7
	Rural	16	14.5
	Deep Rural	21	19.1
	Other	2	2.8
Household Size	Alone	1	0.9
	Family of two	6	5.5
	Family of three	12	10.9
	Family of four	35	31.8
	Above five	56	50.9

Participants

The sample for the pilot survey consists of 73(66.4%) females and 37(33.6% males. The sample was made up of 36(32.7%) Grade 10 learners, 36(32.7%) Grade 11 learners and 38(34.6%) Grade 12 learners. Ninety-six (87.3%) of the participants were in the 16-20-year-old category. The majority of participants were and Sepedi speakers (98. 2%).The researcher drew an equal number of learners from Grade 10 and 11 cohorts and 2 extra learners from Grade 12. The majority 37(33.6%) of the participants were drawn mainly from rural schools. There was one extreme age group (21 years and above) with one learner who had dropped out of school and decided to continue.

Table 3-3 : Frequencies of Responses

Key:

1 = Strongly Disagree 2 = Disagree

3 = Neutral 4 = Agree

5 = Strongly Agree

Questionnaire Item		Strongly Disagree 1	Disagree 2	Neutral 3	Agree 4	Strongly Agree 5
		Frequencies				
C1	Mathematical symbols affect my understanding of mathematics concepts.	34	46	5	13	12
C2	I understand the symbols and formulae in the current textbooks	5	66	20	14	5
C3	I am able to express word problems compactly using appropriate symbols.	29	33	8	31	9
C4	When I fail to cope with some symbol, I seek help instead of taking them as they are.	4	21	10	55	20
C5	I am able to handle expressions and equations using appropriate symbols.	6	52	22	19	11
C6	I struggle to assign meanings to the symbols and this negatively affects my conceptualisation.	6	27	15	48	14
C7	Unfamiliar mathematical symbols in a concept/topic often mark the point where I fail to understand the topic.	9	9	19	49	24
C8	I am able to learn how to use all symbols and language used in the textbooks.	11	31	13	42	13
C9	Navigating through the symbols and their meanings is easy to do.	19	47	12	20	12
C10	Mathematical symbols strongly affect my understanding of Algebra and related topics.	9	12	22	53	14
C11	Sometimes my own meanings of mathematical symbols often contradicts with the actual meaning and this often hampers my progress in problem solving	9	16	22	42	21
C12	My interpretation and use of mathematical symbols affect my competence in mathematics.	6	33	8	42	21
C13	The symbols in a formula sometimes contradict with my thinking.	11	13	15	54	17
C14	Linking concepts and appropriate symbols is easy.	11	53	15	14	17
C15	I am flexible to move from one formula to another in relation to the demands of task using appropriate symbols.	11	30	16	38	15
C16	The teaching and learning methods used by my current teacher enhance my understanding of the use of the various mathematical symbols	15	32	15	32	16
C17	Mathematics teachers who taught me in lower grades attempted to foster the connection between symbols and their meanings.	40	24	21	7	18
C18	I get my mathematics tasks done quickly with clear understanding of the symbols and features used in the task.	40	32	8	20	10
C19	Discovering new symbols and features with their meanings is easy.	14	45	17	21	13
C20	Mathematical symbols and formula strings are satisfying to use	13	46	22	12	17
C21	The symbols in a mathematical problem have a significant influence on my attempt to solve a problem	7	19	24	46 *	14
C22	The symbols in a mathematical problem influence my goals, activities and organisation of results when solving a mathematical problem.	7	21	18	40	24
C23	I am able to switch representations from geometric situations to algebraic and algebraic situations to geometric.	7	50	14	25	14
C24	I am able to define the meaning of symbols introduced to solve problems, including specifying units and distinguishing among the	22	31	10	33	14

	three main uses of variables(unknowns, placeholders, parameters)					
C25	I am able to read expressions, formulae in different ways.	7	18	17	44	24
C26	I read the question several times to gain the meaning of the problem together with the symbols before solving it.	6	18	14	35	37 *

3.8.3 Discussion of results

C1: Mathematical symbols affect learners' understanding of mathematical concepts

Eighty (72%) of the participants indicated that mathematical symbols present obstacles that prevent them from understanding mathematical concepts. Only 25(23%) learners indicated that they understood the symbols used in mathematics textbooks. Five (4.5%) learners indicated that they could cope with mathematics symbols depending on the topic under discussion. Further probing into the issue indicated that most learners familiarise themselves with symbols used in a particular topic and associate the symbols with the concept. These findings are consistent with Worthington and Carruthers (2003) who observed that learners find it difficult to understand symbol systems and this obscure them understanding mathematical concepts. Yetkin (2003) also noted that learners had trouble in constructing mathematical meanings of standard written symbols. Learners struggle to understand written symbols by making connections within the symbol system.

C2: Symbols and formulae in the current textbooks

Participants indicted that they do not understand the symbols and formulae in their current mathematics textbooks. Seventy-one (64.5%) of the participants confirmed that they have trouble in understanding the symbols and formulae when reading mathematics textbooks. Learners confirmed that they encounter difficulties in transferring and connecting knowledge from the abstract aspects of mathematics with reality. Understanding what symbols represent in the physical world is important to how well and how easily a learner will remember a concept. Holding and inspecting a rectangle, is much more meaningful to a learner than simply being told what that the rectangle is. A similar study conducted by Murray (2009) revealed that many learners find mathematics difficult because they have trouble learning mathematics formulas and understanding symbols in mathematics formulas. So before learners can understand a new mathematics topic or concept and its formulas they need to learn meanings of the symbols and concepts they represent. Only 20 (18.2%) indicated that they understand the symbols and formulae in the current textbooks and can use the textbook as a learning resource.

C4: Learners use symbols even without understanding their meanings

The majority 75(68.2%)) of respondents indicated that they seek help from teachers when they fail to cope with unfamiliar mathematical symbols. Twenty-five (22.7%) were opposed to the idea of consulting teachers but memorising the procedures together with their symbol strings. Ten (9%) learners indicated that they resort to meaningless "symbols pushing", which is, using the symbols without understanding their meanings. Findings from this study are consistent with the findings of Chan and Yeung (2000) who indicated that math symbols have very specific meanings. She recommended that if one is not certain about the meaning of a math symbol s/he look it up, or ask someone to explain it instead of just taking as it is. Thompson, Cheepurupalli, Hardin, Lienert and Selden (2010) further revealed that symbol pushing is counterproductive in the end.

C10: Manipulating expressions and equations using appropriate symbols

The results, as seen in Table 3-3 indicate that 58(52.7%) participants struggle to manipulate expressions and equations using appropriate symbols. Only 30(27.3%) confirmed that they can use symbols to represent information compactly. Twenty-two (20%) participants were undecided. There are several possible explanations for this result. For example, participants may fail to understand the question and settle for "Neutral". This was further investigated in the interviews.

C12: Mathematical symbols affect conceptualisation of concepts

Sixty-two (56.4%) participants indicate that their major challenge is to assign meanings to math symbols and this negatively affects their conceptualisation while 33 (30%) strongly opposed the claim. Fifteen (13.6%) participants indicated they are neither challenged by mathematics symbols nor their conceptualisation affected by symbols. Mathematical process (such as counting) can be symbolised, then the symbol is treated as a mathematical concept and itself manipulated as a mental object (Tall, 1994). Thus for some learners the symbol can be thought of *either* as a process, *or* as a concept. This dual nature of a symbol is a cause of confusion for some learners.

C16: Teaching methods to enhance understanding of mathematical concepts

There were mixed reactions to this item. Forty-seven (42.7%) participants acknowledged that the teaching and learning methods used by their teachers do not enhance their understanding of various mathematical symbols. Forty-eight (43.6%) confirmed that the teaching and learning methods used by teachers enhance their understanding of various mathematical symbols. Fifteen (13.6%) participants were not sure whether the teaching methods enhance their understanding of mathematical symbols. Yetkin (2003) observed that learners' challenges with written symbols, concepts and procedures can be reduced by creating learning environments that help learners to connect their formal and informal mathematical knowledge; using appropriate representations depending on the given problem context; and helping them connect procedural and conceptual knowledge.

C17: Prior knowledge and conceptions of concepts, symbols and meanings

Sixty-four (58.2%) participants acknowledged that mathematics teachers who have taught them in the lower grades made little attempts to foster the connection between symbols and their meanings. However, this is not the case with 25(22.7%) who confirmed that their teachers attempted to foster connections between symbols and referents. These findings are consistent with those of Yetkin (2003) who found that learners experience difficulties in connecting symbols and their references. Teachers need to design instruction that helps learners construct overarching ideas. The symbolic representation of mathematics concepts is abstract and more difficult to learn than concrete representations or drawings. The same observation was also made by Garrison and Mora (1999) who revealed that the ability to manipulate symbols without the proper conceptual foundation limits progress into higher mathematics, since conceptual understanding is the basis for advanced mathematics. The same observations were also made by Gurganus (2010) who noted that preceding experiences from lower grades affects learners' proficiency with mathematical symbols. If concepts and their symbols were not well explained in the early years, mathematics learning in later years is affected.

C18: Mathematical Symbols are a threat to problem–solving progress

Seventy-two (65.5%) participants disagreed with the statement and acknowledged that they take too long to go through their tasks due limited understanding of the symbols and features used in the task. Thirty (27.2%) participants conformed that they are able to do mathematics tasks quickly with clear understanding of the symbols and features used in the tasks. These findings are consistent with the findings of Reynders (2014) who observed that the written expression of symbols such as numbers, letters and unfamiliar notations are a threat to learners' progress. Gurganus (2010) also observed that these problems are evident in learners who experience difficulties in differentiating numbers or symbols that are close in form, copying shapes or symbols, following directions with algorithms or graphs, recognizing patterns or sequences, and understanding oral directions or drills.

C19: Discovering new and their meanings is a challenge

Fifty-eight (53.6%) of participants indicated that discovering new symbols and their meanings is one of the huddles when attempting a new topic. However, 30(26.4%) of the participants claimed that they do not encounter difficulties in learning new symbols together with their meanings. Ali (2011) found similar observations and relates this to language problems. These problems emerge when learners cannot use mathematical symbols to express mathematical concepts. Rubenstein and Thompson (2001) made similar observations and concluded that the symbolic language of mathematics is a cause of great confusion for learners. A similar study conducted by Bakker, Doorman and Drijvers (2003) revealed that mathematics teachers are able to work with and to "see" the mathematics through its symbolic representations, whereas learners often struggle in this endeavour; they may need to be told what to see and how to reason with mathematical symbols. Thus, learners cannot discover new mathematical symbols and their meanings without the teacher's help.

C23: Switching representations is a challenge

The results, as indicated in Table 3-3 above show that 57 (51.8%) of the participants struggle to switch representations while 39 (35.5%) acknowledged that they can switch representations from geometric situations to algebraic and algebraic situations to geometric. One result is that learners cannot realise that a mathematical concept may be represented in a number of different ways. These include verbal, symbolic (numerical or algebraic), pictorial/ diagrammatical (geometrical), as a table of values (spreadsheet), graphical or as a physical model. The ability to switch representations is a measure of a learner's symbol sense. This is achieved if learners are able to identify the mathematical aspects of a problem, choose between representations, simplify the problem and represent it mathematically, using appropriate variables, symbols, diagrams and models, then select appropriate mathematical information, methods and tools to use.

3.8.4 Inferential Statistics

Table 3-4: Grade and Difficulties cross tabulations

Grade			Level of Difficulty			Total
			Mild difficulty	Moderate Difficulty	Severe Difficulty	
Grade 10	Gender	Male	4(36%)	6(55%)	1(9%)	11
		Female	6(24%)	18(72%)	1(4%)	25
	Total		10(27.8%)	24(66.7%)	2(4.5%)	36
Grade 11	Gender	Male	2(20%)	8(80%)	0	10
		Female	6(21.4%)	20(71.4%)	2(7.2%)	28
	Total		8(21.1%)	28(73.4%)	2(5.5%)	38
Grade 12	Gender	Male	3(18.7%)	13(81.3%)	0	16
		Female	5(25%)	15(75%)	0	20
	Total		8(22.2%)	28(77.8%)	0	36
Total	Gender	Male	9(24.3%)	27(72.9%)	1(2.7%)	37
		Female	17(23.3%)	53(72.6%)	3(4.1%)	73
	Total		26(23.6%)	80(72.7%)	4(3.7%)	110

Learners' difficulties with mathematics symbols were coded according to the mean responses per questionnaire item for each participant. Classification codes were used to classify learners' level difficulties: 1= no difficulties; 2 = mild difficulties 3 = moderate difficulties and 4 = severe difficulties. This analysis was carried out for each grade as well as according to gender. The summary of these results is shown in Table 3.4 above. The results show that participants experience mild to severe difficulties with mathematics symbols.

Moderate difficulties were experienced across all the grade levels. Female learners experience more difficulties than their male counterparts do. Severe difficulties were experienced in Grade 10 and 11 while no learner in Grade 12 reported challenges with mathematical symbols. In summary, of all the participants, 26(23.6%) learners indicated that they experience mild difficulties, 80(72.7%) experience moderate difficulties and

4(3.7%) experience severe difficulties. However, these findings are preliminary; some tests of hypotheses may shed more light on the differences on difficulties noted so far.

3.9 Descriptive Statistics

Table 3-5: Summary measures

Gender	Mean	N	Std. Deviation
Male	2. 4603	37	.55073
Female	2.5205	73	.50303
Total	2.5455	110	.51822

The means for males and females are almost the same suggesting that that learners experience the same difficulties when dealing with mathematics symbols. The standard deviations for the different gender groups were almost the same as the standard deviation for the whole group suggesting that there is little variability in terms of challenges experienced by learners when working with mathematical symbols. However, this is a preliminary finding; a hypothesis test for the difference of two gender means will be conducted to ascertain this claim.

Table 3-6: T-test

	Test Value = 0					
					95% Confidence Interval of the Difference	
	t	do	Sig. (2-tailed)	Mean Difference	Lower	Upper
Gender	36.762	109	.000	1.6636	1.574	1.753

The following postulated hypotheses were designed to test if gender has a significant effect on learners' challenges with mathematical symbols:

H_0: There are no gender differences in terms of learners' experiences/ difficulties with mathematical symbolisation.

H_1: There are gender differences in terms of learners' experiences/ difficulties with mathematical symbolisation.

The results for the test are shown in Table 3-6 ($df = 109, t = 36.762, p = 0.00$). The null hypothesis was rejected since the p-value is less than 0.05. Hence we conclude that there is a significant difference with regard to the challenges experienced by males and females due to mathematics symbols.

Table 3-7: Analysis of Variance (ANOVA)

Demographic Variables		Sum of Squares	df	Mean Square	F	Sig.
Household Size	Between groups	.077	2	.038	.044	.957
	Within groups	93.278	107	.872		
	Total	93.355	109			
Gender	Between groups	.471	2	.235	1.046	.355
	Within groups	24.084	107	.225		
	Total	24.555	109			
Age	Between groups	.057	2	.028	.205	.815
	Within groups	14.898	107	.139		
	Total	14.955	109			
Home Language	Between groups	.068	2	.034	.218	.804
	Within groups	16.705	107	.156		
	Total	16.773	109			
Grade	Between groups	.164	2	.082	.268	.765
	Within groups	32.600	107	.305		
	Total	32.764	109			
Residential Area	Between groups	1.235	2	.618	1.549	.217
	Within groups	42.665	107	.399		
	Total	43.900	109			

Analysis of Variance (ANOVA) tests were conducted to provide statistical evidence of whether or not the means of several extraneous variables are equal. The following hypotheses were envisaged:

H_0: The various extraneous variables have no effect on learners' difficulties with mathematical symbolisation.

H_1: The various extraneous variables have an effect on learners' difficulties with mathematical symbolisation.

The results of the tests are shown in table 3.7.The p-values for household size, gender, age, home language, ethnicity, grade and residential area were all greater than 0.05. Therefore, the null hypothesis is retained and we conclude that these demographic differences have no effect on the challenges learners experience during engagement with mathematical symbols. However, this is not consistent with Pope and Sharma (2001) who observed that the ability to use symbols with understanding, appears to increase with maturity and experience. Their findings echo the findings of English and Warren (1998) and suggest that greater experience with symbols and algebra in particular, assists in developing confidence. The ability to move flexibly between graphical and symbolic relationships improved with age; that was not found in the pilot study. The direction and strength of relationship between social economic statuses and learners' understanding of mathematical symbols can be established using correlation.

Table 3-8: Correlations

Demographic Variables		Age	Home Language	Gender	Ethnicity	SES	Grade	Res/Area	H/Size
Symbols Difficulty	Pearson Correlation	−.039	.057	−.068	−.101	−.246	−.068	−.028	−.016
	Sig. (2tailed)	.685	.551	.481	.292	.009	.483	.772	.871
	N	110	110	110	110	110	110	110	110

The correlations between symbols difficulty and all the variables except home language are negative and weak. However, the p-value for home-language is greater than 0.50 implying that the two variables have no significant relationship. The p-value for social economic status (0.009) is less than 0.05 suggesting that there is a relationship between social economic status and the challenges experienced by learners due to mathematics symbols. Thus, further research is needed to establish how the two variables are related.

3.9.1 Open-ended questions

Participants were asked to respond to open-ended questions in order to answer in their own words in short phrases or paragraphs. The aim of including these questions was to find out the respondents' views and opinions apart from those suggested by the researcher. Open questions usually provide qualitative data, where the respondent answers the question in as much detail as they want. Open questions add a richness to survey results that are difficult, if not impossible, to achieve with closed questions, so including some as follow-ups to closed items can yield significant benefits (Krosnick &

Presser, 2010). In order to analyse the answers to open questions, the researcher classified learners' responses into four categories: textbooks and problem solving, informal versus formal mathematics and instructional strategies.

Textbooks and problem-solving

The main theme that emerged from this category of questions was that textbooks do not fully provide thorough explanations pertaining to how the symbols are used to develop mathematical concepts and problem solving procedures. Learners indicated that they were not capable of using conventional mathematics symbols that they have learned in class to represent problem solving situations, procedures and concepts. Learners also indicated that there are many symbols to learn in a single topic and sometimes they forget others. Learners expressed limited ability to initiate a mathematical expression or symbol or sign as demanded by a given mathematical problem. Learners confessed lack of symbol sense.

Informal versus formal mathematics

Participants indicated that they make attempts to foster connections between their informal ways of thinking and the actual mathematical symbols. One of the core concepts in all dynamic views on mathematics is the concept of a symbol. Symbols function as means for regulation of the thinking process. However, participants indicated that they do not think in connection with mathematical symbols and pay little attention to their meanings during mathematics lessons. Learners' informal ways of thinking about mathematical symbols were also evident from their responses on the role of mathematical symbols in the learning of mathematical concepts. Common responses to this question were that symbols make learning easier and shorten the amount of writing. None of the responses was formal.

3.10 Instructional strategies

One of the questions requires learners to suggest instructional strategies that teachers should employ to eliminate the negative influences of the challenges posed by mathematical symbols. Participants suggested that teachers should teach mathematics concepts in ways that promote retention and even trying to link symbol with their references. Another interesting response from the participants was that they do not see the relevance of some symbols to what they are learning and it makes difficult for them to think in terms of such symbols during problem solving. Learners also blamed teachers for quoting and substituting into formula without explaining the meanings of the symbols they used in the formula.

3.11 Teaching and Learning Approaches, Methods and Tools

Participants confirmed that the current textbooks use familiar symbols and notation though the teacher is needed to offer clarification on some of the unfamiliar symbols. Learners indicated that textbooks symbols and notations are relevant after the teacher explanations. It is crucial that mathematics teachers should emphasise and develop learners' abilities to understand and connect meanings to mathematical symbols. Teachers should avoid concentrating on teaching learners what to do (procedure) when they see certain symbols or situations. Meaningful mathematics teaching requires teachers to help learners to construct concepts for spoken mathematical words and written symbols.

It is common practice amongst teachers to ask learners to use symbols very early while they are still trying to understand a topic. Mathematical symbols are an abstract communication. The symbols are associated with many mathematical words, so teachers need to guide the learners to become familiar with the mathematics vocabulary and references associated with them. Most mathematical concepts in these grade levels are modelled at the abstract level using only numbers and mathematical symbols. Learners should be provided with a variety of opportunities to practice and demonstrate mastery before moving to a new math skill. As suggested by Clement (2004) teachers should introduce symbols after learners have made connections among the other representations, so that they have multiple ways to connect the symbols to mathematical ideas.

Teachers' responses

Table 3-9: Demographic variables

Variable	Categories	Frequency(f)	Percentage (%)
Gender	Female	6	60
	Male	4	40
Age	26 − 35 years	2	20
	35 − 50 years	6	60
	51 years and above	2	20
Home Language	Sepedi	7	70
	Tshivenda	2	20
	Other languages	1	10
Residential Area	Urban	1	10
	Semi-Urban	2	20
	Rural	7	70
Highest Academic Level (mathematics)	Post-matric diploma/certificate	4	40
	Undergraduate Degree	6	60
	Post –graduate degree	0	0
Teaching Experience	5 years and below	2	20
	6 − 10 years	1	10
	11 − 15 years	4	40
	16 − 20 years	3	30

The data in Table 3-9 shows the demographic composition of teachers who participated in the study. The sample had 10 mathematics teachers who were purposively selected to provide information pertaining to the challenges they experience when teaching mathematical concepts using symbols and signs. A purposive sample is one that is selected based on characteristics of a population and the purpose of the study. Six (6) female and 4 male teachers participated in the pilot survey. Most of the participants (6(60%)) were in the 35-50 years age category and 90 (90%) black. The dominant language was Sepedi (70%) and most participants (70%) were drawn from rural schools. Participants' highest academic qualifications were post-matric diplomas and undergraduate degrees majoring in Mathematics. Most of the participants (70%) had more than 10 years' experience of teaching mathematics in the targeted grades.

3.12 Findings and Discussions

C1: Challenges of teaching mathematical symbolisation

Teachers indicated that learners take time to familiarise themselves with mathematical symbols. They also indicated that there is a big gap between the senior, Further Education, and Training (FET) phase in terms of content, level of abstract concepts and symbol-rich mathematical concepts. One participant cited geometry and trigonometry as topics that have challenging symbols that confuse learners. Teachers revealed that the learners have many misconceptions about the use of symbols in the topics and this has a bearing on their learning of concepts. Teachers claimed that problems encountered by the learners with mathematical symbols have a connection with their lack of conceptual knowledge. Other mathematics teachers blame textbooks for not presenting content in an elaborate way that provides sufficient information for learners to develop their relational knowledge and conceptual understanding.

C2: Mathematical symbolisation affects classroom teaching

Teachers revealed that learners have difficulty in translating word problems into algebraic symbolic forms. This is further compounded by learners' lack of mathematical language. The symbolic language of math is a distinct special-purpose language. It has its own rules of grammar that are quite different from those of English. The symbolic language consists of symbolic expressions written in a way in which symbols are arranged according to specific rules. Another participant reported that learners' failure to understand mathematical symbols forces them to memorise the symbols and the procedures identified with the symbols instead of the actual mathematics concept.

Teachers also pointed out the danger of prematurely focusing on symbols before learners grasp the concept. The same sentiments were shared by Sloutsky, Kaminski and Heckler (2005) who recommended that concepts must be understood first before symbols, otherwise symbols themselves have no meaning. Learners should grasp concepts before they can read and write in symbolic forms. Starting with the abstract nature of symbols assuredly leads to low retention and consequently failure. Teachers also suggested that successful teaching in mathematics depends on learners' symbol processing ability, that is, the ability to derive meaning from symbols whether they could be numbers, figures, words, formulas or any string of symbols. These findings were consistent with Cragg and Gilmore (2014).

C3: Instruction strategies to enforce verbalising, writing and reading symbols

Teacher participants indicated that they cater for learners who struggle to verbalise, write or read symbols in the various mathematical activities. Teachers pointed out that some of the problems have their roots in the learners' prior experiences that cannot be corrected or learned at FET phase. Some teachers argued that these skills should be taught at primary school level. Teachers also suggested that learners experiencing challenges with mathematical symbols and concepts should be identified and encouraged to engage in discourse during classroom deliberations. Mathematics is construed through the use of mathematical symbolism, graphs, diagrams, and language. In both written mathematical texts and classroom discourse, these codes alternate as the primary resource for meaning, and also interact with each other to construct meaning.

C4: Linking mathematical symbols and their meanings

Teachers expressed the opinion that using language and symbols enables exchange of information, experiences, and ideas through modes such as written and spoken language, symbols, gestures and body language. These modes make meaning, create and maintain relationships with the goal of building a common understanding. Teachers pointed out that learners should be guided to become competent language and symbol users through activities that intimately link language and communication. Teachers can foster the effective use of language and symbols by opening opportunities for learners to create, analyse, interpret, and reflect on ideas. This can be done through written, oral, visual, and digital forms for informative and imaginative purposes; and both formally and informally within literacy, mathematical, scientific, social and artistic contexts. Another issue that emerged from teachers' observations was that many learners regard mathematics as symbol manipulation rules, and methods for solving problems. They do not adequately link symbolic rules to mathematical concepts.

C5: Mathematics Symbols and problem solving

Participants confirmed that the symbolic structure of some problems directs and influences their goals. Learners' goals in problem solving are determined by their prior experiences with the symbols in the problem. Learners' problem solving abilities are determined by what they are "seeing" in the symbolic structure of a problem. This observation coincides with the findings of Kenney (2009) who observed that learners have preconceived ideas about what math symbols are supposed to represent, and often base their interpretations on these experiences. Teachers argued that this difficulty has its origins in learners who apply personal and informal meaning to symbols. Thus, good teaching entails the ability to foster connections between the learner's informal symbols and the formal abstract and arbitrary system of symbolism.

C6: Symbol precedence and conceptual development in textbooks

One of the questions requires teachers to analyse textbooks, focusing on the sequence of problem-solving activities. Participants confirmed that textbooks present symbolic problems prior to verbal problems. Thus, teachers confirm that the textbooks used by learners are of such a standard that they can gain mathematical knowledge from reading.

SUMMARYOF THEMES

Table 3-10: Pilot Survey Themes

i	Theme(T_i)	Summary of Attributes
1	Teaching Methods and learners' conceptual understanding	Symbols, concepts and meanings are not properly linked during instruction. Instruction emphasises procedural and surface learning instead of conceptual understanding.
2	Symbol and sense and problem-solving.	Learners lack symbol sense. Learners lack algebraic insight required for problem-solving.
3	Timing and syllabus coverage	Learning mathematical symbolisation requires time Time allocated for teaching and learning mathematics is inadequate. Learners are accelerated to complete the syllabus with little conceptual understanding.
4	Contexts in which symbols are used	Mathematical symbols assume different meanings in different contexts. Dual roles of symbols as concepts or processes (procept).
5	Multiple meanings of mathematical Symbols	Mathematical symbols have multiple meanings Multiple meanings confuse learners.

		Symbols represent both processes and objects
6	Learners' conceptions of mathematical symbols	There many symbols to learn in a mathematics topic Unfamiliar notation jeopardises conceptual understanding Symbols make mathematics more complicated
7	Mathematical concepts, symbols and Representation	Math concepts and symbols have multiple representations Multiple representations of the same concept confuse learners
8	Formal and informal mathematical knowledge	Learners have informal mathematical conceptions about symbols. There is a gap between learners' informal strategies and formal mathematical symbolism
9	Mathematical language	Mathematical language is unique and technical. Mathematical language is different from English
10	Reading mathematics text	Reading mathematics text is difficult due to unfamiliar symbols. Mathematics instruction should provide opportunities for classroom reading, writing and discourse.

3.13 SUMMARY

This chapter described the research methodology; sampling, data collection, instruments as well as data analysis. Strategies used to ensure the ethical standards; reliability and validity of the study are observed. The reasons for using both qualitative and quantitative approaches to conduct the research were discussed in this chapter. The chapter also captures the pilot survey that was conducted to set the stage for data collection for the main study.

CHAPTER 4: DATA ANALYSIS

The discussion in the previous chapter focuses on research methods and techniques used to collect data for the study. This discussion entailed a mixed method research, hence data collection strategies, site selection, sampling, credibility, a pilot study and ethical issues were based on both qualitative and quantitative paradigms. This chapter discusses how data were analysed and presented. Data were collected, processed and analysed in response to the problems posed in chapter one. Two fundamental research questions guided data collection goals and subsequently data analysis. These goals were to explore the difficulties learners and teachers experience with mathematics symbols during teaching and learning and the possible instructional strategies to mitigate the effects of symbolic obstacles.

4.1 Data Analysis

The researcher presents the findings resulting from an exploration of difficulties learners and teachers experience with mathematical symbols during teaching and learning. Data were gathered from two main of sources: questionnaires and interviews. Additional data were collected by compiling field notes during the observation with comments written after the field. In addition, discussions in the interviews were recorded with a digital voice recorder. The formal conversational focus group interviews were mainly conducted between the researcher and learners with interview scripts. Due to time constraints, learners at each grade level were engaged in focus group interviews. Focus group interviews are used when it is better to obtain information from a group rather than individuals (Gill et al, 2008). Focus group interviews were chosen as they can reveal a lot of detailed information and deep insight since several perspectives about the same topic can be drawn from the group participants simultaneously. The researcher created a conducive discussion environment where participants were ease to discuss their views, allowing then to respond to questions in their own words and add meaning to their answers. The benefits of focus group interviews research include gaining insights into participants' shared understandings of the phenomena (Anderson, 2010). The group sizes were restricted to a maximum of 12 learners, which was deemed large enough to generate rich discussions. The responses of learners were audio recorded and transcribed.

4.2 Analysing Qualitative Data

The analysis of data in this study follows mainly a qualitative approach. The aim of using a qualitative approach is to uncover hidden details of a phenomenon and understand the big picture by using the data to describe the phenomenon and what it means (Cassidy et al, 2011). Qualitative data analysis strategies for fall into three main groups: Categorising strategies such as coding and thematic analysis; connecting strategies (such as narrative analysis and individual case studies); and memoranda and displays (Maxwell, 2005). These methods can be combined. The strategies used to analyse the data in this study were inductive analysis “bottom up” (Braun & Clarke, 2006), and typological “top down” (deductive) analysis (Buckley, Halbesleben & Wheele, 2015). The use of inductive analysis is to code the data without fitting it into a pre-existing coding frame, or the researcher’s analytic pre-conceptions, themes identified are strongly linked to the data themselves. This type of analysis is derived from the collected data.

Typological (deductive) analysis on the other hand involves splitting the data set into several groups or categories based on pre-determined categories which are generated from theory, common sense, and research objectives (Hsieh and Shannon, 2005). A topological analysis is normally driven by the researcher’s theoretical or analytic interest in the area. This was the case after obtaining preliminary results from the closed- ended questions. The research utilised interviews that were analysed using typological analysis. The researcher was careful in order to avoid bias in the whole analysis process as the coding framework has been decided in advance, thus severely limiting theme and theory development. Furthermore, the use of typological analysis usually blinds the researcher from looking into other important dimensions in the data. This weakness was counter-balanced by the use of inductive analysis that analyses actual data without taking a predetermined theory into consideration. This approach is deemed comprehensive and most suitable when there is little prior knowledge about the phenomenon of interest (Ghauri & Grønhaug, 2005).

4.3 Inductive Analysis

Inductive analysis is a data processing approach that uses raw data to derive concepts, categories and themes (Bernauer, Lichtman, Jacobs & Robinson, 2013). This inductive procedure for analysing qualitative data is guided by specific objectives that are determined by the researcher in advance (Thomas, 2003). The main reason for selecting inductive approach is that it allows research findings to emerge from frequently occurring responses inherent in raw data. Furthermore, the approach is not affected by the restraints as in structured methodologies. Structured methodologies use a formal methodical approach to the analysis and design of information systems. To carry out inductive analysis data were scanned for categories and relationships among those categories were further grouped into typologies, allowing themes to emerge from the data (Scruggs, Mastropieri & McDuffie, 2007). The main idea was to allow research findings to emerge from the frequent, dominant or significant themes inherent in raw data, without the restraints imposed by structured methodologies. The benefit of utilising induction analysis is that key themes, which are often obscured, reframed or left invisible because of the preconceptions in the data collection and data analysis procedures, can emerge. An inductive approach helps to understand meanings of complex data by developing a summary of themes from the raw data (data reduction). Inductive analysis was used to derive nodes from closed questions that were later envisaged using focus group interviews.

4.4 Transcription of verbal data

Verbal data collected from group interviews with learners and individual teacher interviews, was transcribed from the voice recorder into written form in order to conduct a thematic analysis. Transcription is the first step towards familiarisation with the data (Hart, Brannan & De Chesnay, 2014). Bird (2009) also argues that the process of transcription should be taken as:

"…a key phase of data analysis within interpretative qualitative methodology and recognised as an interpretative act, where meanings are created, rather than simply translating spoken words into written statements" (p.227).

Green, Franquiz, and Dixon (1997) also interpreted interview transcripts as a form of data. They focused mainly on their constructed quality and echoed the following sentiments:

"*…. a transcript is a text that "re"-presents an event; it is not the event itself. Following this logic, what is re-presented is data constructed by a researcher for a particular purpose, not just talk written down". (p. 172*)

This study utilised thematic analysis to analyse transcribed data. The researcher read the transcripts several times to locate categories and later uses these categories to extract broad themes. The researcher developed a coding frame that was used to code the transcripts. If new codes emerged, the coding frame was changed and the transcripts were reread according to the new structure. This process was used to develop categories, which were then conceptualized into broad themes afterwards. The themes were categorized. The researcher checks the transcripts against the original audio recordings for accuracy in order to ensure the validity of the data.

4.5 Coding

Codes were used to organise and sort data. Coding involves combining the data for themes, ideas and categories. This is done by marking similar passages of text with a code label or code so that they can easily be retrieved at a later stage for further comparison and analysis. Coding allows the researcher to mark the data, in such a way that it becomes easier to search the data, to make comparisons and to identify any patterns that require further investigation (Taylor & Gibbs, 2010). Codes were used to develop to label or identify issues raised by learners about their encounters with mathematical symbols. Codes also assisted in compiling and organising data. The coding becomes the basis for developing the analysis. It is generally understood, then, that "coding is analysis". The codes for this study were derived from keywords, ideas and concepts raised by participants as recommended by Ryan and Bernard (2003b). The researchers read learners' texts and identify passages, phrases and keywords that were judged to represent the same, theme or concept and assigned to a code.

The identified codes were assigned names that give an indication of the idea or concept that underpins the theme or category. Any part of the data that relates to a code topic was appropriately labelled. This process of coding involves close reading of the text. If a theme is identified from, the data that does not quite fit the codes already existing then a new code is created. As the researcher read participants' responses, new codes evolved and grew as more topics or themes become apparent.

The Coding Process

The coding process was derived from two basic sources: *a-priori* ideas from literature review, pre-existing theories and those that emerge from the data set during analysis (grounded theory). Research questions that were addressed by the study and issues from the interview schedule also informed the coding process. The researcher used his knowledge of mathematics, classroom experience and subject expertise in creating the codes.

Phase 1: Open coding

Open coding was the first step in the coding process. The researcher looked for unique and distinct concepts and categories emerging from data. These concepts form the basic units or first-level categories. To do this, the researcher was guided by the following questions: What conditions caused or influenced concepts and categories?

Phase 2: Axial coding

In the axial coding phase, the researcher used categories from open coding while re-reading the text in order to: confirm that the concepts and categories accurately represent interview responses and to explore how these concepts and categories were related. To do this, the researcher was guided by the following questions: What was the context in which the participant responded to the question? What are the associated effects or consequences of participants' responses? What is the meaning of participants' response?

Development of Codes

Table 4-1 below shows how the codes were developed. Coding becomes the basis for developing the data analysis process as well as linking data collection and interpretation.

Table 4-1: Development of Codes

Number	What can be coded (C_i)	Examples
1	Reading mathematical symbols	Learners struggle to read mathematical symbols Teaching instructions exclude reading. Reading text is not part of mathematics teaching.
2	Teaching Methods and conceptual understanding	Symbols, concepts and meanings are not properly linked during instruction Instruction emphasises procedural and surface learning
3	Symbol and sense and symbol manipulation	Learners lack symbol sense Learners lack algebraic insight required for problem-solving
4	Timing and syllabus coverage	Learning mathematical symbolisation requires time Time allocated for teaching and learning mathematics is inadequate Learners are accelerated to complete the syllabus with little conceptual understanding
5	Contexts in which symbols are used	Math symbols assume different meanings in different contexts Dual roles of symbols as concepts or processes
6	Multiple meanings of mathematical Symbols	Math symbols have multiple meanings Multiple meanings confuse learners Symbols represent both processes and objects
7	Learners' conceptions of mathematical symbols	There many symbols to learn in mathematics symbols Unfamiliar notation jeopardise conceptual understanding Symbols make mathematics even more complicated
8	Mathematical concepts, symbols and Representation	Math concepts and symbols have multiple representations Multiple representations confuse learners
9	Formal and informal mathematical knowledge	Learners bring informal mathematics understanding to learning There is a gap between learners' informal strategies and formal mathematical symbolism
10	Mathematical language	Math language is unique. Math language is different from other languages.

4.6 Thematic Analysis

Thematic analysis was utilised in this study. Thematic analysis, derived from grounded theory approach (Heath & Cowley, 2004), is a qualitative analytic method for identifying, analysing and reporting patterns (themes) within data. The process involves analysing transcripts and identifying themes within the data. It minimally organises and describes data set in (rich) detail. However, it goes further and interprets various aspects of the research topic' (Braun & Clarke, 2006:79). A theme captures something important about the data in relation to the research question and represents some level of patterned response or meaning within the data set. Thematic analysis is essentially independent of theory and epistemology and is firmly rooted in the essentialist and constructionist paradigms (Braun & Clarke, 2006). It can be applied across a range of theoretical and epistemological approaches. Thus, this theoretical freedom makes thematic analysis a flexible and useful research tool, which can provide a rich and detailed, yet complex account of data. Constructionists identify common themes and cluster them together, while essentialists consider meanings across the whole data set, semantic themes and cluster data according to semantic themes. In summary, thematic analysis involves searching the data set for repeated patterns of meaning.

Steps in Inductive Analysis

1. Read the data and identify frames of analysis
2. Create domains based on semantic relationships discovered within frames of analysis
3. Identify salient domains, assign them codes and put others aside.
4. Re-read the data, refining salient domains and keeping a record of where relationships are found in the data.
5. Decide if domains are supported by the data and search data for examples that do not fit with or run counter to the relationships in your domains.
6. Complete an analysis within domains
7. Search for themes across domains
8. Create a master outline expressing relationships within and among domains
9. Select data excepts to support the elements of your outline

Textbox 4-1: Inductive analysis (*Adapted from Hatch, 2002*)

4.7 Problems encountered in data collection

The researcher originally proposed to interview all mathematics teachers teaching in the FET phase at each of the selected schools, but this was not possible, as some of them could not avail themselves for the interview due to work commitments. The researcher also resorted to focus group interviews with the learners since the time allocated for research activities was not adequate for individual interviews. Language problems were also encountered despite thorough revisions of the instrument after the pilot survey. Learners at times failed to interpret the questions or struggled to express themselves clearly in their responses. Nevertheless, participants managed to provide adequate data for purposes of the study. Thus, the results were not affected.

4.8 Response rate

The response rate for learners' questionnaire was 70.63% since participants completed the questionnaire in the presence of the researcher. Fifteen, 15(83%) out of a possible 18 teachers completed questionnaires which is greater than the threshold recommended by Fincham and Draugalis (2013).

Demographic data for learner participants

Table 4-2: Demographic Data

	Variable	Frequency(N)	Percentage (%)
Gender	Male	245	43.4
	Female	320	56.6
	11-15	137	24.3
Age (Years)	16-20	428	75.7
	Sepedi	499	88.3
	Sesotho	10	1.8
Home Language	Tshivenda	11	1.9
	Xitsonga	20	3.5
	Other Languages	25	4.4
	10	200	35.4
Grade	11	215	38.1
	12	150	28.5
	Urban	180	31.0
	Semi-urban	199	35.2
Residential Area	Rural	161	33.8
	Deep Rural	14	2.5

Participants

Demographic data about learner respondents shows that 245 (43, 4%) were males and 320(56.6%) were females. Only two age groups were observed. The majority 428 (75%) of the participants were in the 16-20 years category while the 11-15 years age category had 137(24, 3%). Sepedi was the dominant home language with 499(88.3%) while 66(11.7%) speak other local languages. The other languages had an insignificant combined representation. The initial plan was to draw an equal number (200) of learners from each grade level; however, it was not possible to obtain 200 learners from the sampled schools due to the dwindling number of learners taking mathematics at Grade 12. Thus 200(35.4%) learners were drawn from Grade 10, 215(38.1%) were drawn from

Grade 11 while 150(28.5%) were drawn from Grade 12. The participants were drawn from rural schools 175(31%, semi urban 199(35.2%) and urban 195(33.8%). Most participants (435(76.9%) were drawn from families with household sizes ranging from 4 to 5 people.

4.9 Cluster Analysis

Cluster analysis is a segmentation or taxonomy technique that is used to identify homogenous groups of participants based on their responses or experiences of a given phenomenon (Gopichandran & Chetlapalli, 2013). Cluster analysis methods provide means for classifying a given population into groups (clusters), based on similarity or closeness measures (Ragno, De Luca & Loele, 2007). A cluster analysis identifies what homogeneous groups exist among learners (for example learners can be classified according to their challenges with mathematical symbols as mild, moderate and severe difficulties). Cluster analysis is a grouping technique that identifies cases when the groups cannot be determined in advance. Cluster analysis is also interpreted as a multivariate analysis that divides data into groups or "clusters" of objects (sample plots) that are "similar" to each other (Lance & Williams, 1966). Two-step cluster analysis was preferred in this study since it is quick and automatically selects the number of clusters and groups or clusters based on their experience of the phenomenon.

4.9.1 Demographic variables

To check if demographic variables can be used as predictors of learners' competency with mathematical symbols, an SPSS two-step cluster analysis procedure was used to analyse the importance of the each of the variables. The results of the analysis are show in figure 4.1 below.

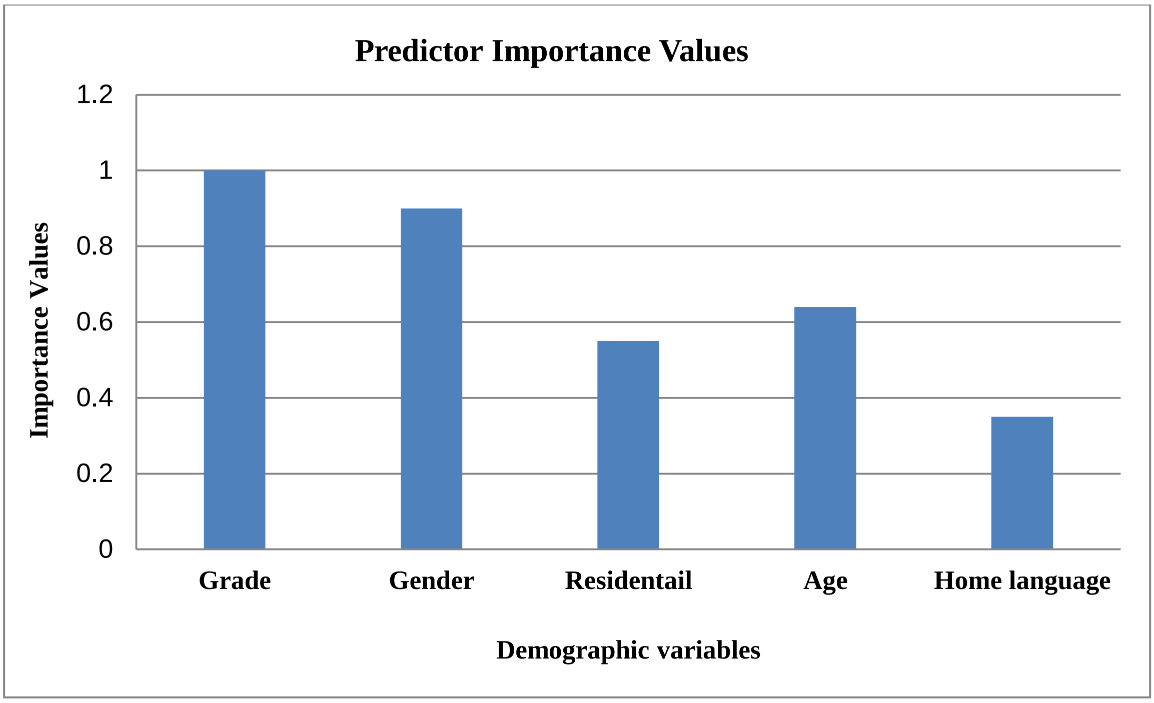


Figure 4-1: **Predictor importance indicators**

SPSS Predictor Importance view shows the relative importance of each demographic variable in explaining how these variables affect learners' level of competence with mathematical symbolism. This indicates how well the variable can differentiate different clusters. The view shows that variables such as grade, gender, residential area and age have a significant effect on the learners' understanding of mathematical concepts together with their symbols.

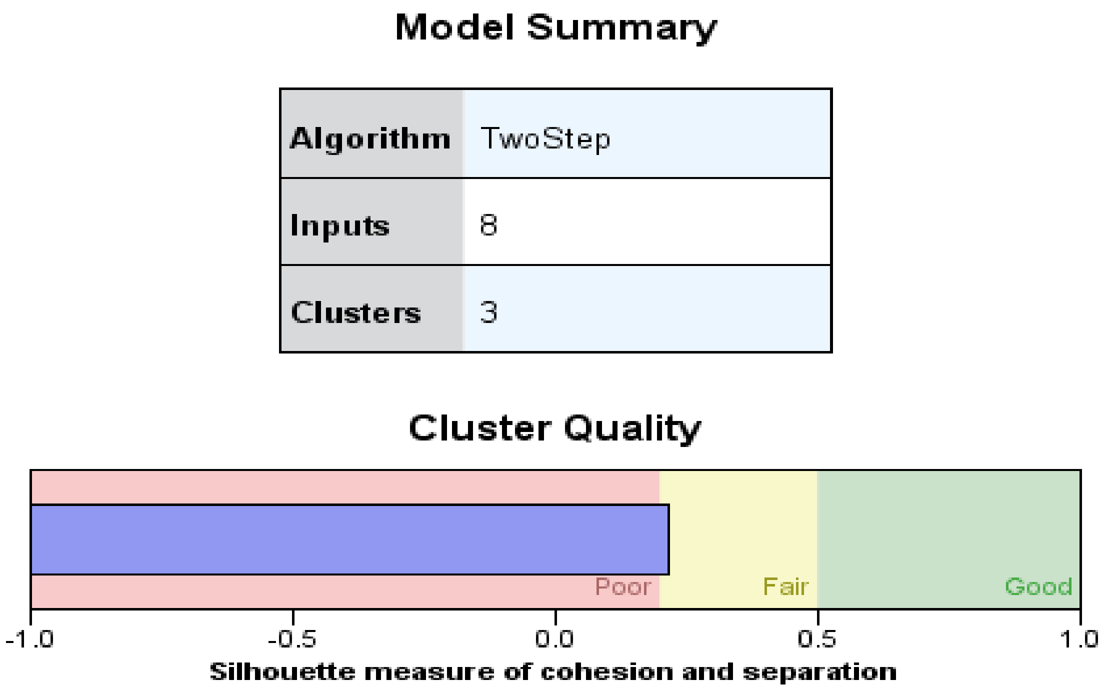


Figure 4-2: Model summary

Figure 4.2 shows a summary of the cluster model and a positive Silhouette measure of cluster cohesion and separation. This measure lies in the 'Fair' category, which implies that the model is unbiased cluster. Eight demographic variables were clustered into three clusters. This summary also shows that the cluster quality was fair. The assessment of the quality of clusters was based on the criteria suggested by Kaufman and Rousseeuw (1990). In the model summary view, a good result equates to data that reflects Kaufman and Rousseeuw's (1990) rating as either reasonable or strong evidence of cluster structure, fair reflects their rating of weak evidence, and poor reflects their rating of no significant evidence.

A silhouette coefficient of 1 means that all cases are located directly on their cluster centres. A silhouette coefficient of −1 means all cases are located on the cluster centres of some other cluster. A silhouette coefficient of 0 means, on average, cases are equidistant between their own cluster centres and the nearest other cluster. In this data set, the Silhouette coefficient is approximately 0.35 suggesting that the structure is weak and the researcher's prior classification can be allowed in clustering the demographic variables.

Demographic Clusters

Table 4-3: Demographic clusters

Clusters

Input (Predictor) Importance

1.0 0.8 0.6 0.4 0.2 0.0

Cluster	1	2	3
Label			
Description			
Size	37.9% (214)	32.6% (184)	29.6% (167)
Inputs	GRADE Grade 10 (83.6%)	GRADE Grade 11 (51.1%)	GRADE Grade 11 (61.7%)
	Gender Female (51.9%)	Gender Female (100.0%)	Gender Male (85.0%)
	SES Middle SES (80.8%)	SES Middle SES (100.0%)	SES Middle SES (62.9%)
	R/ AREA Urban (39.3%)	R/ AREA Rural (46.7%)	R/ AREA Rural (70.1%)
	Age 11-15 years (50.0%)	Age	Age 16-20 years (82.0%)
	H/L Sepedi (75.2%)	H/L Sepedi (94.6%)	H/L Sepedi (98.2%)
	Ethnicity Black (87.4%)	Ethnicity Black (98.9%)	Ethnicity Black (99.4%)
	H/Size	H/Size	H/Size

Table 4-3 above shows three clusters of demographic variables:

- Grade level and gender were the main inputs (predictor) importance variables while ethnicity and home size were the least inputs (predictor) importance variables. This means that ethnicity and home size can be dropped from the list of demographic variables.
- Cluster 1 consists of 214(37.9%) learners, cluster 2 consists of 184(32.6%) learners and cluster 3 was made up of 167(29.6%) learners.

- Learners in cluster 1 were mostly Grade 10(83.6%) while 16.4% were drawn from Grades 11 and 12, with 51.9% females and 48.1% males, from middle social economic status families (80.8%), 39.3% are from urban areas while 60.7% are from rural and urban backgrounds, 107(50%) learners in this cluster are in the 11-15 years age category. One hundred and sixty–one (161) speak Sepedi while 35(24.8%) speak other languages. The dominant ethnicity group in this cluster was black (87.4%) while the other ethnic groups constitute 12.6% of the cluster.

- Learners in cluster 2 were mainly Grade 11(51.1%), 167(100% females, all from middle social economic status backgrounds, 46.7% drawn from rural backgrounds while 53.3% were from semi- urban and urban backgrounds, 94.6% of the learners speak Sepedi while other languages constitute 5.4% of the group. The dominant ethnicity group in cluster 2 was black (99.4%) (172) while the other ethnic groups constitute 0.6% (12) of the cluster.

- Learners in cluster 3 consisted of 104 (61.7%) Grade11 learners and 63 Grade 10 and 12 learners. The dominant gender in this cluster was male (142), which accounts for 85% of the cluster size. One hundred and six 106 (62.9%) learners in this cluster were drawn from middle social economic status families and 70% of the learners were from rural areas. The dominant (81%) age group was 16-20 years. Sepedi (98.2%) and black Africans (99.4%) dominated home language and ethnicity.

These cluster compositions can be used to understand the relative contribution of each demographic variable to learners' the overall experiences with mathematical symbols as well as their understanding of mathematical concepts. Strong contributors (importance predictors) for these clusters were Grade level (1.0), gender (0.8) and the social economic status (0.6) of the learners. Weak contributors such as home language, ethnicity (0.4) and household size (0.2) may be deemed to have less effect on learners understanding of mathematical symbolisation. Two observations predicted by clustering worth mentioning: the effects of grade level and gender on learners' experiences with mathematical symbolisation. These were envisaged further in the proceeding sections.

T- Tests (Gender differences)

Hypothesis

H_0: There are no gender differences in terms of learners' experiences/ difficulties with mathematical symbolisation.

H_1: There are gender differences terms of learners' experiences/ difficulties with mathematical symbolisation.

Table 4-4: T-test

					Levene's Test for Equality of Variances		t-test for Equality of Means			95% Confidence Interval of the Difference	
	Gender	N	Mean	SD	F	Sig.	t	df	Sig. (2-tailed)	Lower	Upper
Mean	Male	245	2.998	.4579	3.802	.052	-.010	563	0.992	-.0802	.0794
	Female	320	2.999	.4942							

The results in table 4.4 indicate that the Levene's value for testing if k samples have equal variances shows that there is no significant difference in learners' in terms of variation between males and females. Equal variances across samples are called homogeneity of variance. The p-value (0.052) of Levene's test is more than the significance level $(p = 0.05)$, the obtained differences in sample variances are due to random sampling from a population with equal variances. Thus, the null hypothesis of equal variances is not rejected and we concluded that there is no difference between the variances in the population.

T-test results indicate, $t\ (263) = -0.10$; $p = 0.992 > 0.05$. We do not reject the null hypothesis and conclude that there are no gender differences in terms of learners' experiences/difficulties with mathematical symbolisation. Though gender emerged as one of (predictor) importance variables, further tests indicate that it has no effect on learners' perceptions about mathematical symbols.

4.9.2 Analysis of Variance (ANOVA)

A one-way analysis of variance (ANOVA) was used to determine whether there were any significant differences between the means of demographic variables. The results are shown in the ANOVA table 4-5 below:

Table 4-5: ANOVA

		Sum of Squares	DF	Mean Square	F	Sig.
Grade	Between Groups	36.958	62	.596	1.143	.002
	Within Groups	261.846	502	.522		
	Total	298.804	564			
Age	Between groups	16.408	62	.265	1.478	.014
	Within groups	89.885	502	.179		
	Total	106.294	564			
H/L	Between groups	63.152	62	1.019	1.011	.457
	Within groups	505.602	502	1.007		
	Total	568.754	564			
Ethnicity	Between groups	27.076	62	.437	.907	.677
	Within groups	241.834	502	.482		
	Total	268.910	564			
SES	Between groups	14.052	62	.227	1.283	.081
	Within groups	88.649	502	.177		
	Total	102.701	564			
R/ AREA	Between groups	31.514	62	.508	.800	.001
	Within groups	319.140	502	.636		
	Total	350.655	564			
H/Size	Between groups	80.074	62	1.292	1.319	.060
	Within groups	491.614	502	.979		
	Total	571.688	564			

The p–values for grade, age and residential area were less than $p = 0.05$, implying that these variables were statistically significant in explaining learners' different experiences in dealing with mathematical symbols. The same variables emerged as the main predictors of importance on the previous section. Demographic variables such family size, ethnicity, home language and social economic status were statistically insignificant since their $p -$values were all greater than 0.05. The direction and strength of relationship between grades, gender, age and residential area and learners' understanding of mathematical symbols can be established using correlation.

4.9.3 Correlations

Table 4-6: Correlations

Demographic Variable		**Age**	**H/Lang**	**Gender**	**Ethnicity**	**SES**	**Grade**	**R. Area**	**H/size**
Symbols Difficulty	Pearson Correlation	-0.129	-0.029	0.000	-0.056	-0.030	0.019	0.130	-0.007
	Sig. (2tailed)	0.005	0.489	0.992	0.186	0.473	0.654	.002	.865
	N	565	565	565	565	565	565	565	565

**. Correlation is significant at the 0.01 level (2-tailed)

The correlations between symbols difficulty and all the variables except residential area and age are negative and weak. However, the p −values for age and residential area are less than 0.05 suggesting that age and residential area have significant relationships with leaners' difficulties with symbolisation. The other variables have $p-$ values greater than 0.05 implying that that they have no significant relationship with learners' difficulties with mathematical symbols. A correlation coefficient of zero was recorded between gender and learners' difficulties with mathematics symbols suggesting that the two variables do not have a correlational association of any kind. Home size has a Pearson correlation coefficient (r) of 0.007 and a p −value of 0.865 which is greater than 0.05 suggesting that there is almost no relationship between home size and the challenges experienced by learners due to mathematics symbols.

The impact of environmental location of the school, that is, the urban/rural location of the community was also reported by Zanolla (2014) as having a significant impact on learners' mathematical competency. The same observation was made by Owoeye and Yara (2011) who noted that learners in urban areas had better academic achievement than their rural counterparts. Unlike in the pilot survey, social economic status (SES) was insignificant. This is not consistent with Spaull (2013) who noted more than 30 percent of the variation in mathematics achievement in South Africa is explained by socio-economic status alone. Studies conducted in poor countries such as Tanzania, Kenya and

Swaziland show that poor performance in high schools is not predicted by a disadvantaged background as is the case in South Africa (Schleicher, 2009).

4.9.4 Descriptive Statistics

Table 4-7: Frequencies of responses

Key:
1 Strongly Disagree
2 Disagree
3 Neutral
4 Agree
5 Strongly Agree

	Questionnaire Item	Strongly Disagree 1	Disagree 2	Agree 4	Strongly Agree 5	Mean μ	Skewness S	Kurtosis k
		Disagree		**Agree**				
C1	Mathematical symbols affect conceptual understanding	192		331		3.26	-.437	-1.21
C2	Symbols and formulae in the current textbooks are confusing	187		355		3.24	-.576	-1.11
C3	I am able to express word problems compactly using appropriate symbols.	308		248		2.63	.147	-1.61
C4	When I fail to cope with some symbol, I seek help instead of taking them as they are.	323		236		2.55	.333	-1.62
C5	I am able to handle expressions and equations using appropriate symbols.	292		260		2.70	.096	-1.63
C6	I struggle to assign meanings to the symbols and this negatively affects my conceptualisation.	226		323		3.32	-.291	-1.47
C7	Unfamiliar mathematical symbols in a concept/topic negatively affect my understanding of the topic.	147		402		3.81	-.780	.103
C8	I am able to learn how to use all symbols and language that is used in the textbooks.	312		248		2.65	.231	-1.64
C9	Navigating through the symbols and their meanings is easy to do.	370		186		2.38	.491	-1.39
C10	Mathematical symbols strongly affect my understanding of Algebra and related topics.	156		398		2.30	.622	-1.25
C11	My own meanings of mathematical symbols contradict with the actual meanings.	158		390		3.72	-.738	-.916
C12	My interpretation and use of mathematical symbols affect my competence in mathematics.	261		284		3.01	-.075	-1.50
C13	The symbols in a formula sometimes contradict with my thinking.	251		289		2.98	-.147	-1.46
C14	Linking concepts and appropriate symbols is easy.	227		213		3.20	-.291	-1.37
C15	I am flexible to move from one formula to another in relation to the demands of task using appropriate symbols.	280		323		2.78	.098	-1.55
C16	The teaching and learning methods used by my current teacher enhance my understanding of mathematical concepts and symbols.	224		309		3.13	-.262	-1.44
C17	Mathematics teachers who taught me in lower grades attempted to foster the connection between symbols and their meanings.	227		316		3.16	-.272	-1.44
C18	I get my mathematics tasks done quickly with clear understanding of the symbols.	269		240		2.83	.048	-1.41
C19	Discovering new symbols and features with their meanings is easy.	333		243		2.44	.433	-1.31
C20	Mathematical symbols and formula strings are satisfying to use	212		411		3.16	-.329	-1.32
C21	The symbols in a mathematical problem have a	188		446		3.28	-.484	-1.16

	significant influence on my attempt to solve a problem					
C22	The symbols in a mathematical problem influence my goals, activities and organisation of results during problem solving.	207	418	3.16	-.379	-1.32
C23	I am able to switch representations from geometric to algebraic and vice-versa.	270	309	2.84	.048	-1.43
C24	I am able to define the meaning of symbols introduced to solve problems, including specifying units and distinguishing among the three main uses of variables(unknowns, placeholders, parameters)	248	275	2.95	-.119	-1.47
C25	I am able to read expressions, formulae in different ways.	310	240	2.60	.280	-1.66
C26	I read the question several times to gain the meaning of the problem together with the symbols before solving it.	140	415	3.75	-.935	-.670

After analysing the frequencies for each aspect of mathematical symbolisation, the following items (C1, C2, C3, C6, C9, C11, C21, C22 and C24) were observed to have high frequencies of learners disagreeing or agreeing with the statements. Table 4-8 below shows the skewness and kurtosis values for each of the items. Items C1, C2, C3, C6, C9, C11, C21, C22 and C24 are negatively skewed meaning that the distributions were concentrated on the right (agree and strongly agree). Learners were in agreement with most of the statements that they encounter challenges with symbols in those items. Items C3 and C9 are positively skewed meaning that the distributions were concentrated on the left (disagree and strongly disagree). Table 4-8 below shows the further details of their distributions in terms of skewness, standard error of skewness, kurtosis and standard error of kurtosis.

Table 4-8 : Skewness and Kurtosis

Statistics

	C1	C2	C3	C6	C9	C11	C21	C22	C24
N	565	565	565	565	565	565	565	565	565
Skewness	-.437	-.576	.147	-.291	.491	-.738	-.484	-.379	-.119
Std. Error of Skewness	.103	.103	.103	.103	.103	.103	.103	.103	.103
Kurtosis	-1.214	-1.11	-1.610	-1.474	-1.391	-.916	-1.169	-1.326	-1.439
Std. Error of Kurtosis	.205	.205	.205	.205	.205	.205	.205	.205	.205

Skewness measures the level of symmetry or non-symmetry. If the distribution of the data is symmetric then skewness will be close to 0 (zero). The further from 0, the more skewed the data. A negative skewness value indicates a skew to the left. In order to tell if the skewness is large enough to cause concern, a measure of the standard error of skewness can be calculated as $=\sqrt{\frac{6}{565}}$ = 0.103 and compare with the standard value $=\sqrt{\frac{6}{50}}$ = 0.346. If the skewness is more than twice this amount, then it indicates that the distribution of the data is non-symmetric. In this case the standard error of skewness is less than 0.346 the data can be assumed to be fairly symmetric although somewhat marginally so. However, this does not indicate that the data are normally distributed. The distributions of C1, C2, C6, C11, C21, C22 and C24 are all negatively skewed while C3 and C9 are positively skewed.

Kurtosis is a measure of the peakedness of the data. Again, for normally distributed data the kurtosis is 0 (zero). As with skewness, if the value of kurtosis is too big or too small, there is concern about the normality of the distribution. In this case the estimate of standard error for kurtosis is $(\boldsymbol{k} = \sqrt{\frac{\mathbf{24}}{\mathbf{565}}} = \mathbf{0.205})$. This value is less than the standard value $(\boldsymbol{k}=\sqrt{\frac{\mathbf{24}}{\mathbf{50}}} \quad = \mathbf{0\ 692})$, hence the value of kurtosis falls within two standard error, the data may be considered to meet the criteria for normality by this measure. The distributions of C1, C2, C3, C6, C9, C11, C21, C22 and C24 have a constant standard error for kurtosis, which is less than 0.692, suggesting that they all satisfy the normality criteria.

C1: Mathematical symbols affect understanding of mathematical concepts

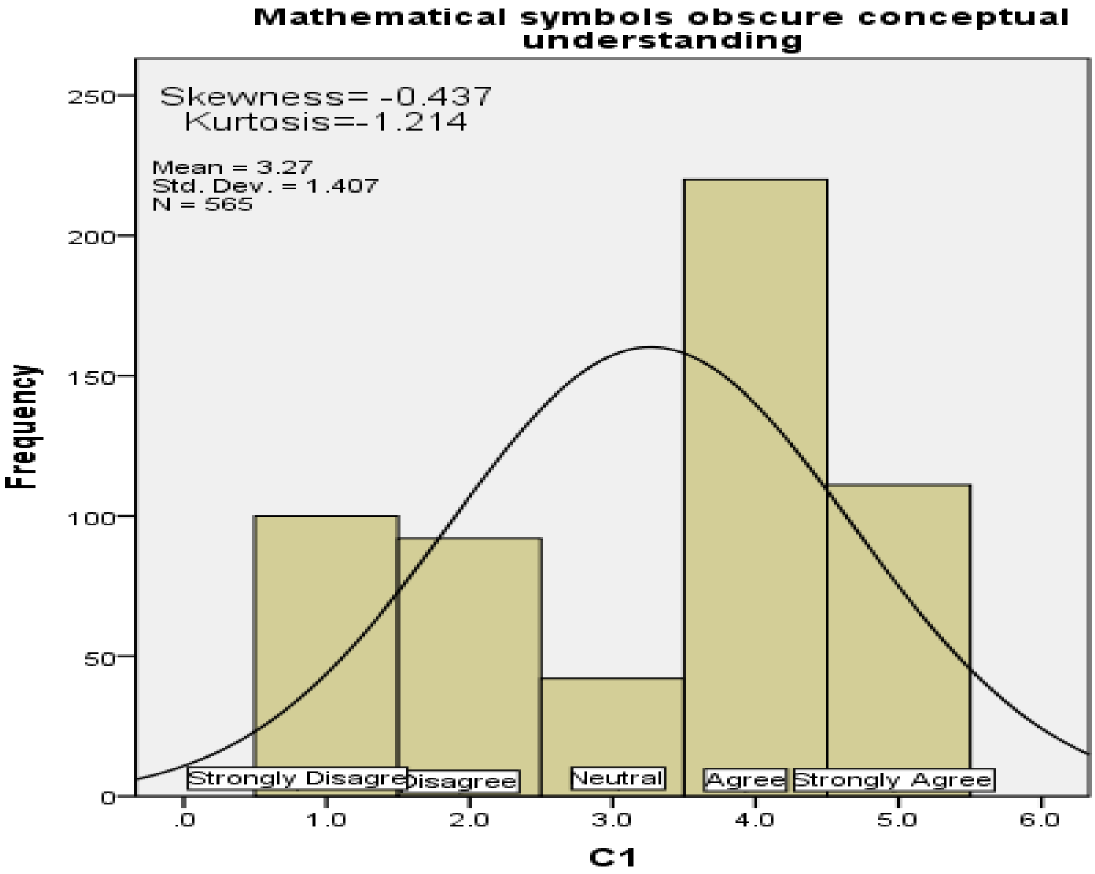


Figure 4-3: Symbols affect understanding of mathematical concepts

Three hundred and thirty-one (58.5%) participants indicated that they do not understand mathematical symbols that are used during engagement with various topics and activities in the current mathematics curriculum. One hundred and ninety- two (34.1%) learners indicated that they understand the symbols used in mathematics textbooks. Five (7.4%) learners indicated that they could cope with mathematics symbols depending on the topic under discussion. Further probing into the issue indicated that most learners familiarise themselves with symbols used in a particular topic and associate the symbols with the concept. These findings are consistent with Worthington and Carruthers (2003) who observed that learners find it challenging to traverse the symbol system of a given topic. This observation was also found by Yetkin (2003) who noted that learners experience difficulties in constructing mathematical meanings of standard written symbols. Learners indicate that they are not able to build understanding for written symbols by making connections within the symbol system.

C2: Symbols and formulae in the current textbooks are confusion

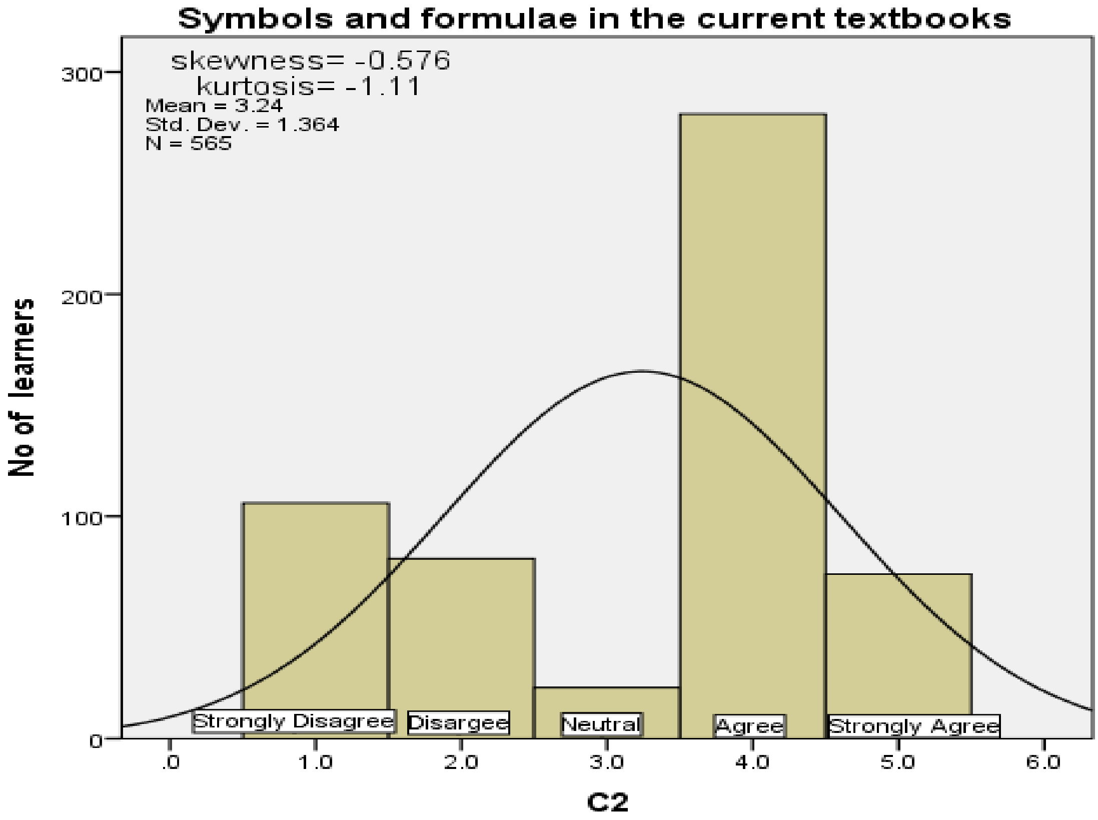


Figure 4-4: Symbols and formulae in the current textbooks

Participants indicted that they do not understand the symbols and formulae in their current mathematics textbooks. Three hundred and twenty-eight (328 (58.1%)) of the participants confirmed that they experience difficulties in understanding the symbols and formulae when reading mathematics textbooks. These findings are consistent with Murray (2009) who revealed that many learners find mathematics difficult because they have trouble learning and understanding symbols in mathematics formulas. So before learners can understand new mathematics topics and their formulae they need to learn what each of the symbols are and what they mean. Only 187 (37.8%) indicated that they understand the symbols and formulae in the current textbooks and can use the textbook as a learning resource. Learners reiterated that textbooks do not presenting content in a way that provides them with the opportunity to develop relational knowledge and conceptual understanding of Algebra.

C3: Expressing word problems compactly using appropriate symbols

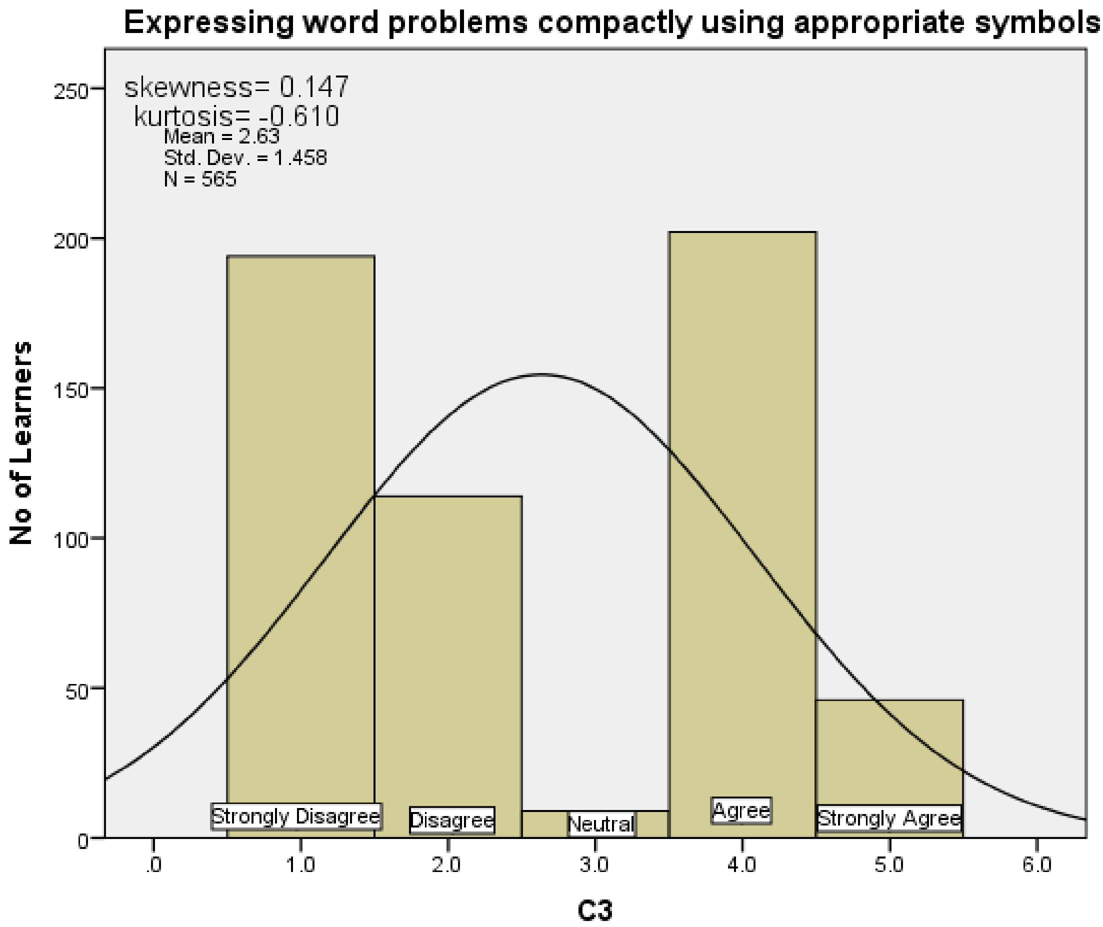


Figure 4-5: Expressing word problems compactly using appropriate symbols

Participants indicted that they struggle to express word problems compactly using appropriate symbols. Three hundred and eight (308 (54.5%)) participants confirmed that they experience difficulties in converting word problems into algebraic statements using appropriate symbols while 43.9% indicate that they can successfully handle the transition from word problems to algebraic statements with little difficulty. The challenge of translating the word problems into symbolic expressions was also documented by Isik and Kar (2012). They observed that the transition from verbal to the algebraic expressions is a challenge for many learners.

Molina, Rodríguez-Domingo, Cañadas and Castro (2016) also conducted a study with primary school learners that indicated that learners had difficulties in forming the algebraic statements with appropriate meanings for letters; they believe that algebraic expressions involving operation signs simplify to give a “single answer" without an operation sign. Another common misconception was to think of algebraic symbols as abbreviations or labels of objects, for example the letter h represents height. Letters represent different meanings in different contexts. For an example, in arithmetic, 5cm means five centimetres, that is, 5 times a centimetre. However, in algebra, 3m mean three times an unknown number m. Therefore, the letters carry two different meanings depending on the context.

C6: Mathematical symbols negatively affect conceptualisation

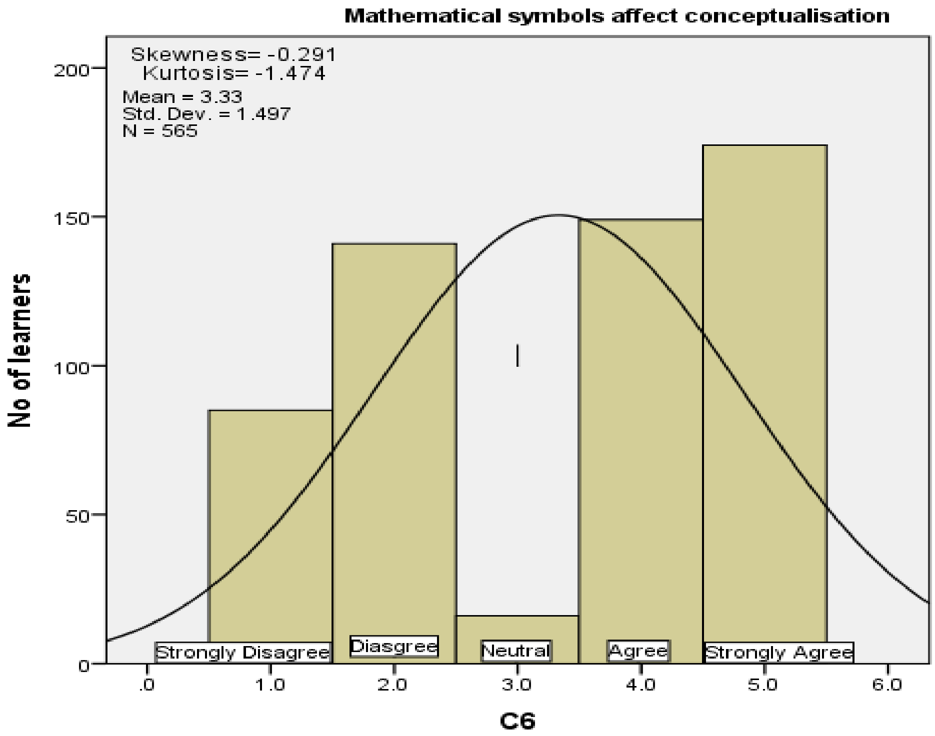


Figure 4-6: Mathematical symbols and this negatively affect conceptualisation

Almost three quarters of the participants (323(75.2%) indicated their major challenge during classroom engagement is to assign meanings to mathematical symbols and this has negatively affected their conceptualisation of mathematics concepts. Learners indicated they just use symbols in the same way they use numbers without understanding their meanings. Sepeng and Madzorera (2014) highlighted the same problem in which learners fail to embed a mathematical message in the context of a three-way relationship involving mathematical words, symbols and numerals. According to Boulet (2007) these three components define the mathematical language. Thus learners should familiarise themselves with words, symbols and numerals used in a given mathematics topic.

C9: Linking mathematical symbols and their meanings

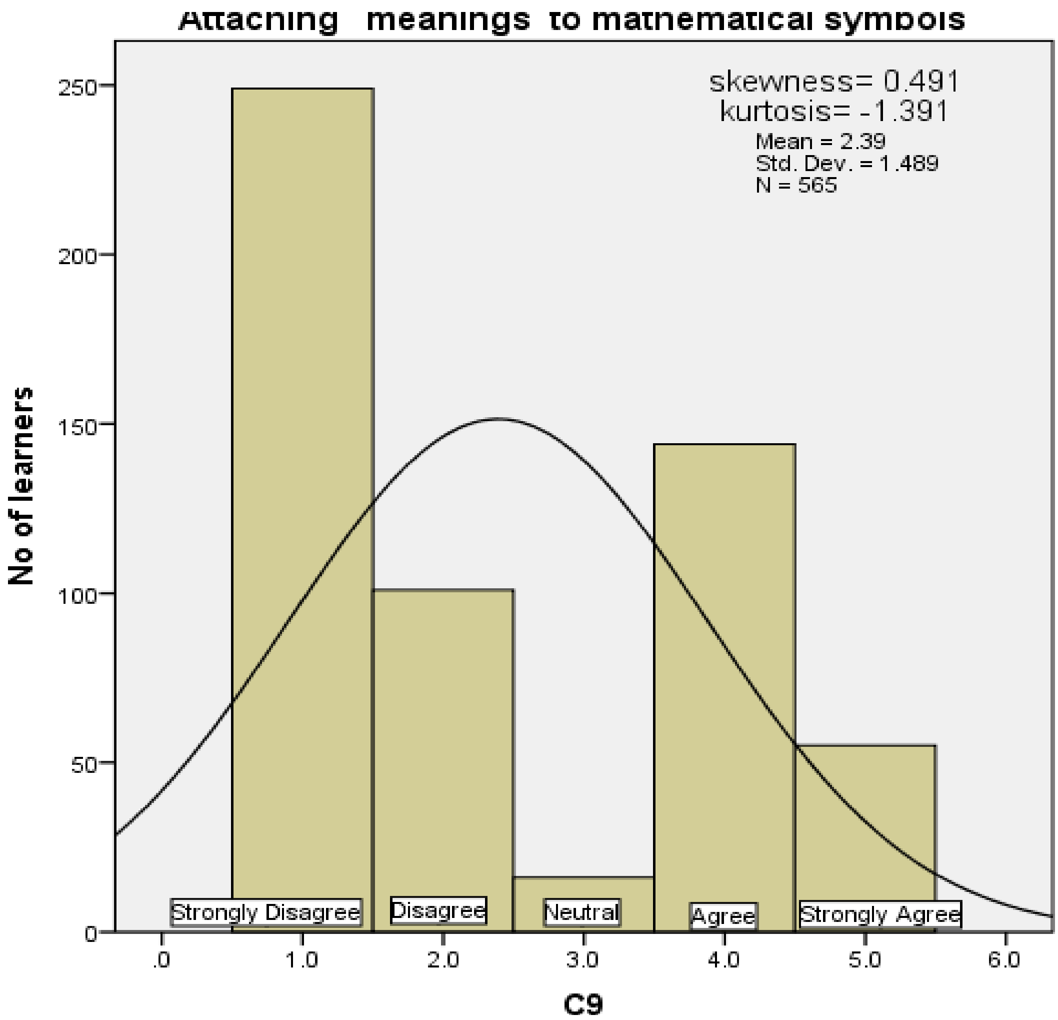


Figure 4-7: Linking mathematical symbols and their meanings

Majority 350(61.9%) indicted that they experience difficulties in navigating through mathematical symbols and their meanings. Learners need to understand how important it is to be precise about the symbols they use. Thus a well-developed symbol sense is a necessary and sufficient requirement for a learner to be able to operate across different representations. Thus, for a learner to understand mathematical symbols and their meaning there are two things that can be considered: the context of the problem, or the particular topics being studied and the conventions that have been decided about particular symbols.

C11: Informal mathematical conceptions contradict actual meanings

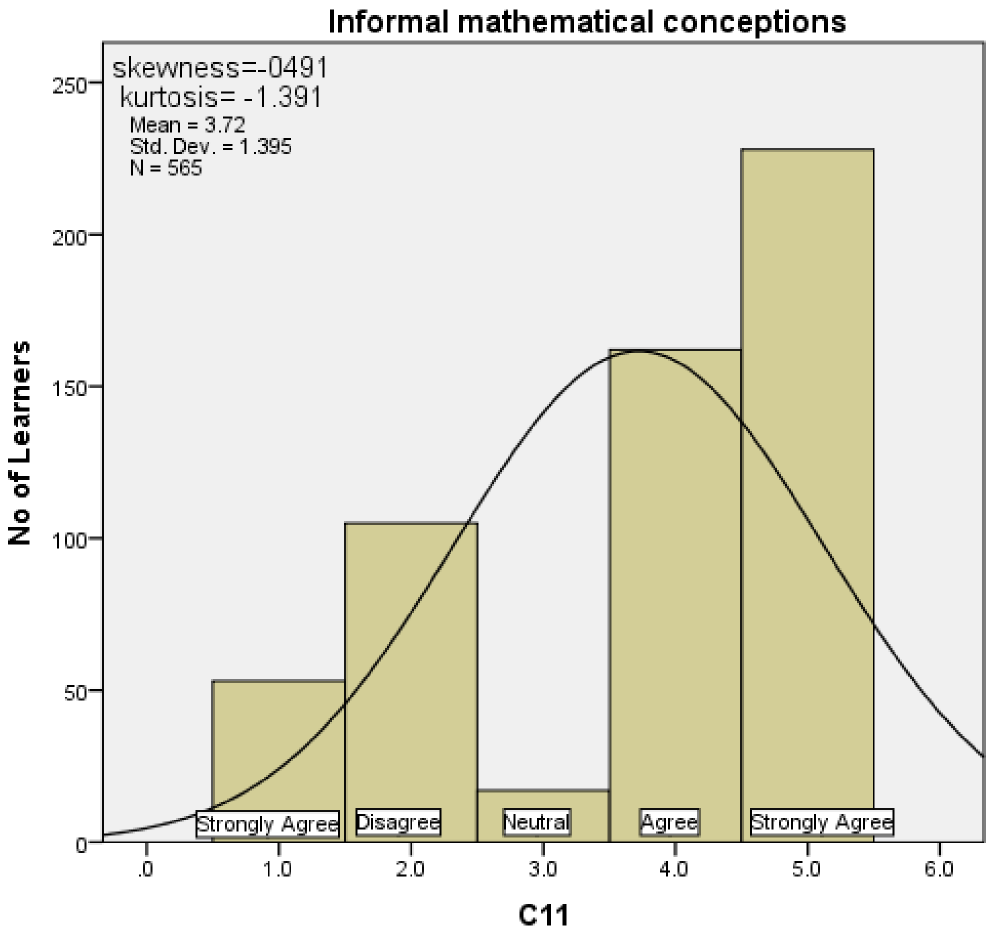


Figure 4-8: Informal mathematical conceptions contradict actual meanings

Three hundred and ninety (390(69.1%)) participants confirmed that their own meanings of mathematical symbols contradict with formal meanings and this has derailed their progress in problem solving**.** However, 158(28%) indicated that they are able to operate using formal symbols. This observation is consistent with the findings by Howard (2008) who revealed that one reason for learners' difficulty with symbols and symbolic structures comes from the way in which individuals apply personal meanings to symbols.

Similarly, Tambychik and Meerah (2010) also noted that there is overwhelming evidence confirming the notion that many learners who experience difficulties with mathematics actually bring to school a strong foundation of informal mathematical understanding. Consequently, they encounter difficulties in connecting this knowledge to the more formal procedures, language, and symbolic notation system of school mathematics. Teachers should desist from the habit of rushing learners into symbols as this practice impoverish the background experience that is needed in further mathematics learning and leads to trouble later. In addition to the provision of manipulatives, it is essential to avail time for classroom discourse where learners can talk about their activities and developing their own informal records before meeting the formal symbols of adult mathematicians. One explanation for this for this difficulty is that individuals differ in the way they apply personal meaning to symbols. Learners' interpretations of mathematical symbols and concepts are based on the prior experiences that they bring to the classroom. Schleppegrell (2007) points out that learners have their own ideas about the uses of letters and symbols and their prior experiences often hinder understanding of mathematical language and notation.

C21: Mathematical symbols problem affect problem-solving goals

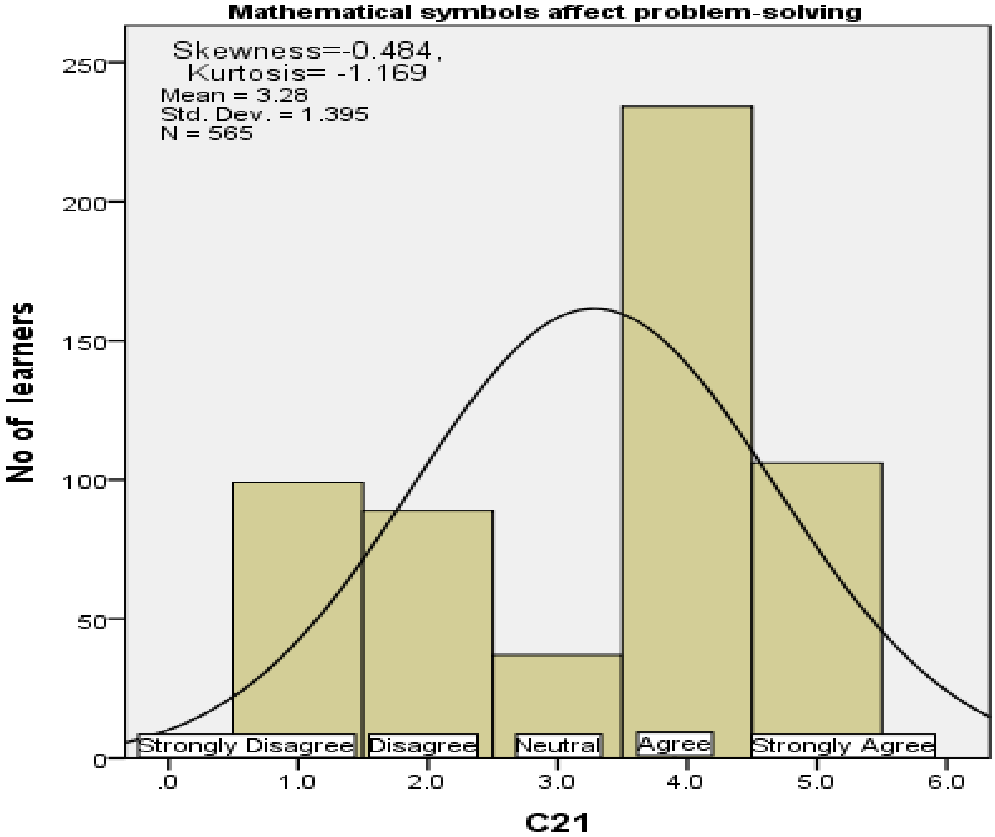


Figure 4-9: Mathematical symbols problem affect problem-solving goals

Majority (340(60.2%)) of the participants indicated that the symbols in a mathematical problem have a significant influence on their attempt to solve a mathematical problem while 188(33%) had a contrary view, arguing that symbols in a mathematical problem can help as cues that influence their thinking. The same observations were made by Shepherd, Selden and Selden (2012) who reported reading difficulties arise because of unfamiliar symbols, syntax and notation that are different from natural language and English. This is caused by symbol load and density of meaning. Some mathematical symbols raise learners' emotions. For example, learners who failed to further with mathematics reported that the sight of x and y or angles θ and β brings unpleasant memories and feelings about mathematics. These feelings may trigger unpleasant memories of when the symbols were first encountered. Learners also struggle to understand some symbols due to their shapes. The use of visual images such as graphical models may help to develop thought processes and concepts about symbols. However, learners see graphs as another form representation different from the algebraic format.

C22: Mathematical symbols affect problem-solving processes

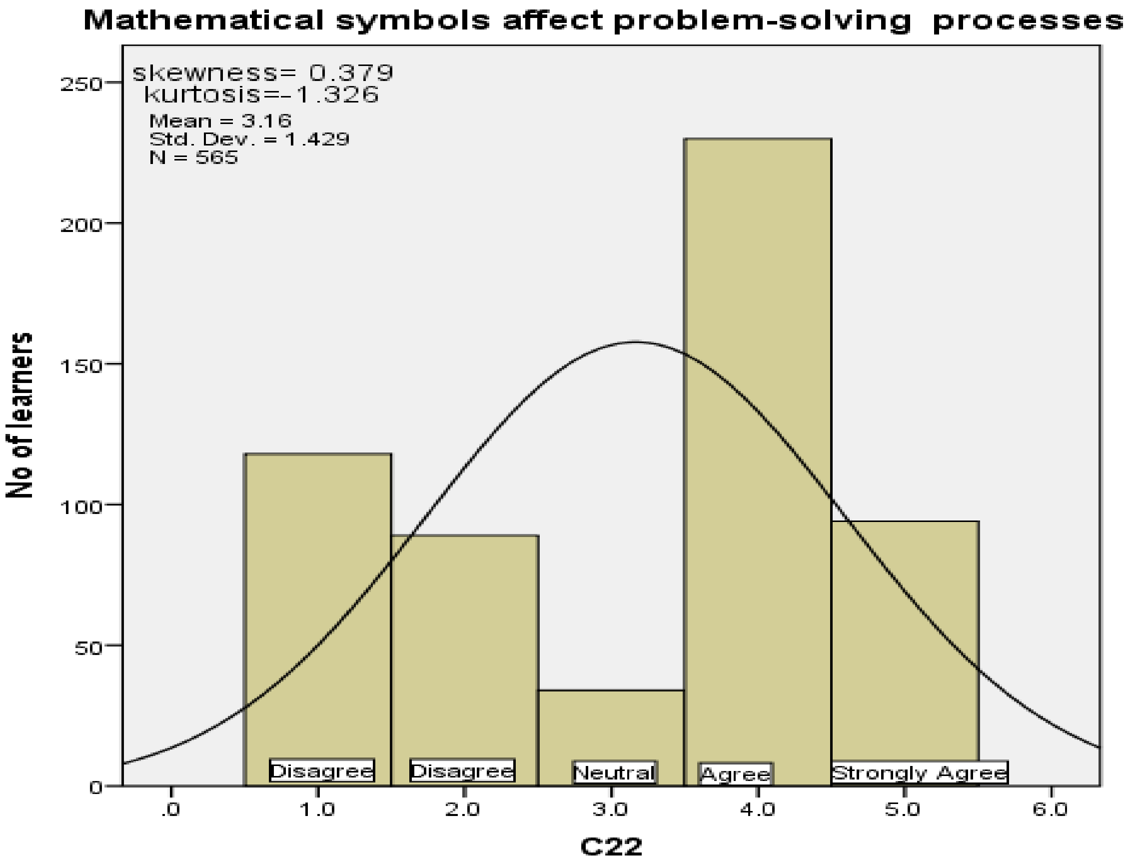


Figure 4-10: Mathematical symbols affect problem-solving processes

Three hundred and twenty-four participants (324(57.3%)) indicated that the symbols in a mathematical problem influence their goals, activities and organisation of results when solving a mathematical problem while 207(48, 7%) provided evidence that their actions and goals are influenced by other variables other than mathematical symbols. These finding are complementary to Kenney (2008) who conducted a study which focused on how learners' mathematical thinking about symbols in order to find what they see in the symbolic structure of a problem and how it influences their goals, activities, and organisation of the solution. The findings also concur well with Arcavi (2005) who noted that many learners fail to see algebra and its symbols as a tool for understanding, communicating, and making connections. Thus learners in this study lack symbol sense that is a necessary component of sense making in mathematics. Learners with a developed symbol sense are able to read into the meaning of a problem and to check the reasonableness of choices of symbolic expressions.

C24: Initiating and contextualising symbols during problem solving

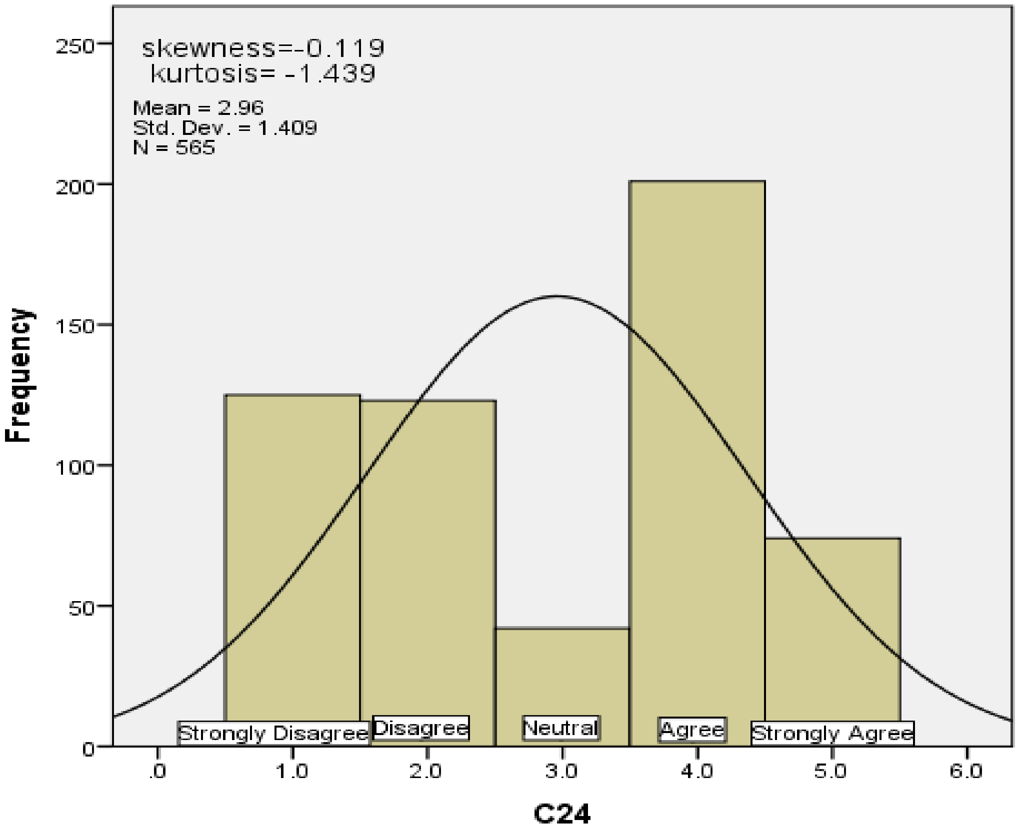


Figure 4-11: Initiating and contextualising symbols during problem solving

Participants were evenly divided in terms of their ability to initiate appropriate mathematical symbols when confronted with word problems. Two hundred and seventy-five (48.7%) indicted that they struggle to initiate symbols in order to solve problems while 248(43, 9%) confirmed that they are able to select and distinguish among unknowns, placeholders, parameters and use them in symbolic sentences. The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation, if learners are not able to use these tools, think abstractly, move across symbolic representations and manipulate symbols in problem solution processes then they are deemed to lack symbol sense. The findings are consistent with Martinez, Brizuela and Superfine (2011) who observed that learners have challenges with working with parameters. They suggested that parameters should be conceived as general numbers that require a clear algebraic meaning.

4.9.5 Medoid Cluster Analysis

Following data entry and cleaning, the first step in this analysis involved medoid clustering of the data in Section B of the questionnaire This section (see Appendix A) requires the respondents to indicate how they agree or disagree with the given statements in relation to their experiences with learning mathematical concepts through symbolisation. Participants were asked to rate each of these statements using a 5-point Likert scale, wherein, $1 =$ strongly disagree, $2 =$ disagree, $3 =$ neutral, and $4 =$ agree and $5 =$ strongly agree. Cluster analysis is an efficient method of classifying large data sets, has the ability to create groups using categorical and continuous variables and it provides an automatic selection of number of clusters. Cluster analysis produce partitions that reflect the internal structure of the data and identify natural groups (Lee, 2009). Two-step cluster analysis is a method of the statistical software package SPSS used for large databases (Garson, 2009). Cluster analysis is based on the assumption that the sample is large $(n > 200)$. This criteria was met since the sample size for this study was 565.

Two-Step Cluster Analysis Results

Table 4-9: Cluster analysis results

Algorithm	Inputs	clusters	Cluster Quality	Silhouette value
Two-step	26	3	Good	0.55

The summary in table 4-9 shows decomposition of 26 inputs into 3 clusters. A two-step cluster analysis algorithm was used to obtain a Silhouette measure of cluster cohesion and separation of 0.55, which indicates a good cluster quality. The silhouette value is in the interval $[-1, +1]$; where values close to -1 indicate that the point is very likely in the wrong cluster. Points whose silhouette value is close to $+1$ are likely to have been correctly clustered (Salo, Salmi, Czink & Vainikainen, 2005). Twenty-six (26) variables were clustered into three clusters. The Silhouette value for the model was above zero indicating that cluster assignment was satisfactory (Larose & Larose, 2015).

To check if variables can be used as predictors of learners' competency with mathematical symbols, SPSS two-step cluster analysis procedure was used to analyse the importance of the each of the variables. The results of the analysis are show in the figure below.

Predictor Importance Indicators

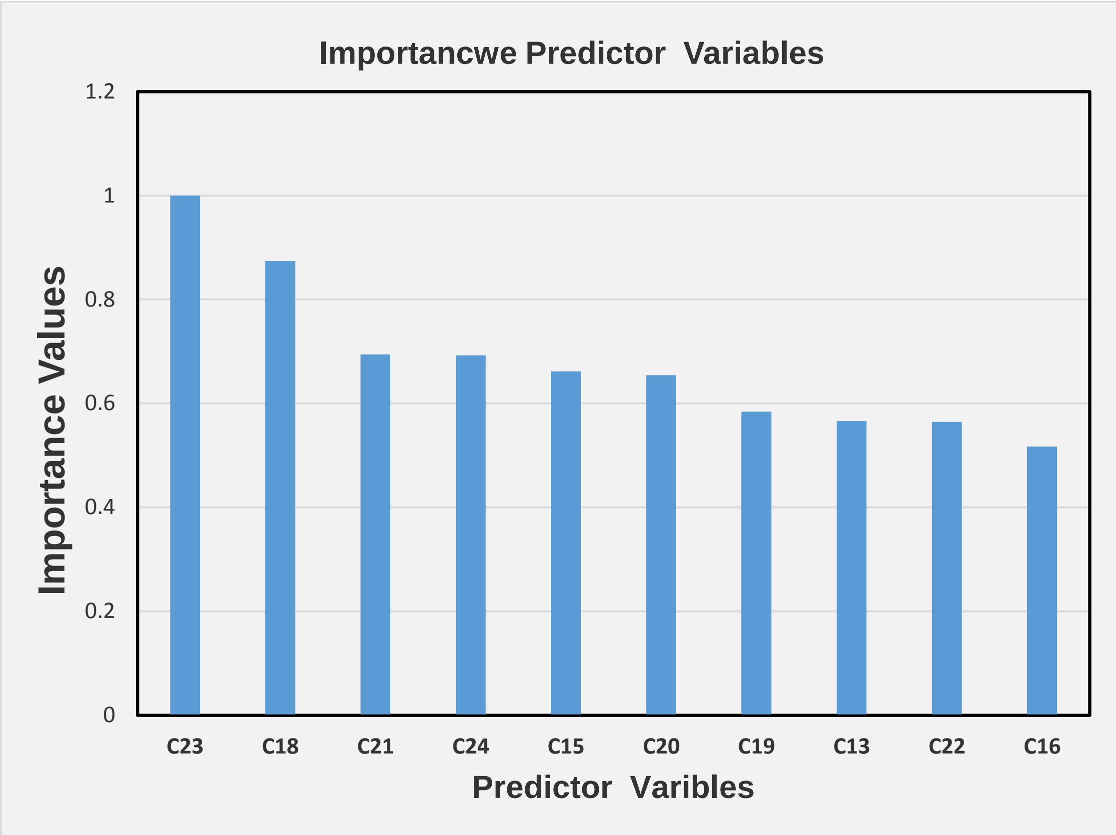


Figure 4-12: Importance predictor variables

SPSS Predictor Importance view in figure 4-12 shows the importance of each variable (C_i) in describing attributes that make mathematical symbolisation difficult to learners. This indicates how well the variable can contribute or can be used as a predictor for explaining why learners' experience challenges with symbolisation. The predictor importance view shows that variables such as C23, C18, C21, C24 and C15 have a significant effect on the learners' understanding of mathematical concepts while the least important variables were C4, C10, C9 and C8 though they are not indicated on the graph. The importance values of the 10 most predictor variables are shown in the table 4-10 below.

Table 4-10: Importance predictor nodes

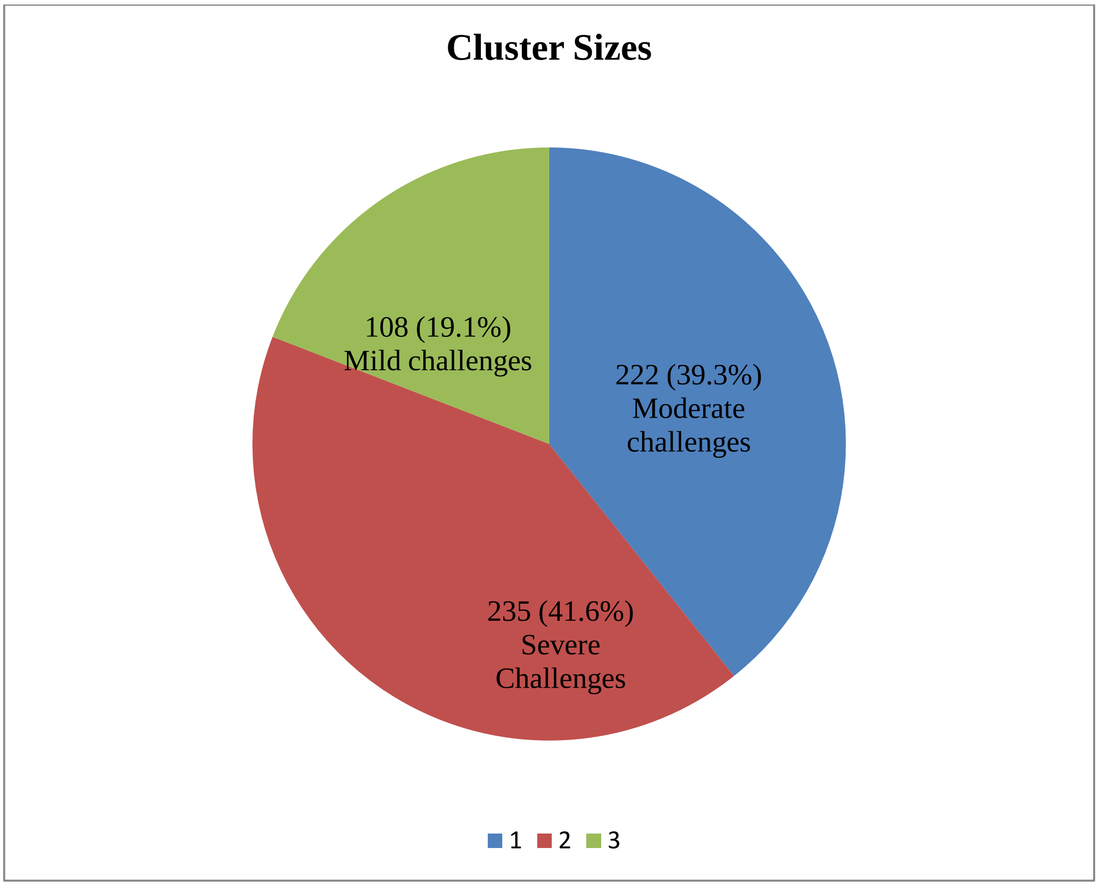
Nodes	Description	Importance Value
C23	Switching representations using appropriate symbols	1
C18	Symbols obscure conceptual understanding	0.8744
C21	Symbols affect problem solving	0.6946
C24	Decoding symbol meanings is important for problem solving	0.6927
C15	Learners lack flexibility with mathematical symbols	0.6618
C20	Mathematical symbols are not satisfying to use	0.6546
C19	Unfamiliar symbols are an obstacle to understanding	0.5841
C13	Informal symbols and formal symbols are contradictory	0.5661
C22	Symbols influence problem solving goals and activities	0.5643
C16	Symbols affect my problem solving goals	0.5173

Table 4-10 above shows the predictor importance values of the most dominant variables in the data set. These nodes were used as leads to thematic analysis during categorising of qualitative data. Focus group interview questions were also built around these nodes. Thus the three clusters suggested by the model were based on C23, C18 and C21.Learners indicated that the most challenging aspect of mathematical symbolisation was switching representations using appropriate symbols, moving from geometric representations to geometric or vice versa. Learners indicated that they struggle to link the abstract mathematics with real world representations to which they relate. This challenge was also revealed by Eva (2006) who noted that failure to provide a representation for mathematical symbolism means that one does not truly understand that which the symbolism portrays. Many learners consistently look for meaning based upon the symbolic representation of concepts. Teachers should therefore provide a representation from the real world to illuminate mathematical abstractions, and it is their responsibility to identify such representations and use them to assist conceptual understanding.

Another highly ranked challenge raised by learners was that mathematical symbols obscure conceptual understanding (C18). In particular, learners indicated that symbols obscure them from understanding the concepts represented by the symbols. This observation confirms that learners find it difficult to associate symbols with related concepts. Thus learners need particularly strong conceptual and symbolic understandings in order to make conceptual sense of the mathematical concepts.

Learners also indicated that symbols affect problem-solving competences (C21). Learners reported that they experience challenges in using symbols productively in problem-solving. Learners indicated that they are not capable of making sense of quantities and their relationships in problem situations. Learners lack the skills of abstracting mathematical situations, represent them symbolically, and manipulate the representing symbols without necessarily attending to their referents.

Cluster Distribution



Size of Smallest Cluster	108 (19.1%)
Size of Largest Cluster	235 (41.6%)
Ratio of Sizes: Largest Cluster to Smallest Cluster	2.18

Figure 4-13: Cluster distribution

The cluster sizes are shown in a pie chart in figure 4-13 above. The number of learners and percentage of each cluster is shown on each slice. Table 4-11 below summarises the information in the SPSS output pie chart and table.

Table 4-11: Cluster distribution frequencies

	Cluster Size (N)	% of Combined	% of Total
Cluster 1	222	39.3%	39.3%
Cluster 2	235	41.6%	41.6%
Cluster 3	108	19.1%	19.1%
Total	565	100%	100%

Table 4-11 above shows cluster sizes suggested by the model. The smallest cluster (cluster 3) has a size of 108(19.1%) learners. Cluster 2 has a size of 235 (41, 6%) while cluster 1 has a size of 222(39.3%). The ratio of the largest cluster to the smallest cluster was 2.18. This ratio indicates that there are at least twice the number of learners who experience challenges with mathematical symbolism than those who are competent and comfortable with the use symbols.

Cluster Composition

Items in the questionnaire were grouped into clusters. The Cronbach's alpha coefficient was calculated for each cluster to ensure all the items in a cluster measure the same attribute (Masitsa, 2011). Tavakol and Dennick (2011) pointed that researchers should estimate this quantity to add validity and accuracy to the interpretation of their data. This is to ensure that each item measures the same latent trait on the same scale. The Cronbach alpha values were calculated each cluster and results are shown in table 4-12 below.

Table 4-12: Cluster composition

Cluster	2	1	3
Size(n)	**235(41.6%)**	**222(39.3%)**	**108(19.1%)**
	23SD	23A	23N
	18SD	18SA	18N
	21SD	21SA	21A
	24SD	24SA	24A
	15SD	15SA	15A
	20SD	20SA	20A
	19SD	19D	19N
	13SD	13SA	13A
	22SD	22A	22A
	16SD	16A	16A
Inputs	17SD	17A	17A
	14SD	14A	14A
	12SD	12A	12SA
	5SD	5A	5A
	11SD	11A	11SA
	26SD	26SA	26SA
	3SD	3A	3SD
	7SA	7SA	7SA
	6SA	6A	6SA
	25SD	25SD	25SD
	8SD	8A	8SD
	9 SD	9 A	9 SD
	10 SD	10 SD	10 SD
	4 SD	4 SD	4 SD
Cronbach's alpha coefficient	**0.761**	**0.654**	**0.781**
Key	SD: Strongly Disagree, D:Disagree	A: Agree, N: Neutral,	SA: Strongly Agree

The following conclusions were arrived at after analysing the results provided by two-step cluster analysis.

Cluster 1

Cluster 1 fills 222(39.3%) of the sample, consisted mainly of learners who indicated limited instances of symbol sense. It consists of learners who are able to switch representations from one form to another (45%). It also consists of learners who have little difficulty in doing mathematical tasks despite lack of proficiency in symbol use (45.5%).

Cluster 2

This is the largest cluster 235(41.6%) containing mostly learners who indicated that they strongly agree that they experience challenges in handling, manipulating and using mathematical symbols to understand mathematical concepts. The cluster consists of learners who struggle to switch representations from one form to another (46%). It also consists of learners who struggle to do mathematical tasks due to lack of proficiency in symbol use (47.7%). Learners (34.5%) in this cluster also indicated that symbols in a mathematics problem have a strong influence on their attempt to solve the problem. This is strongly linked to another concern in which learners (47.6%) indicated that they struggle to initiate symbols in order to solve problems. Another difficulty associated with learners in this cluster is the lack of flexibility to switch from one formula/ structure to another in relation to the demands of task and the symbols used in a mathematics problem. Learners in this cluster could not link symbolic and algebraic representations to graphical forms. Thus, the learners (235(41.9%)) in cluster 2 lack symbol sense as most of the aspects in this cluster indicate instances of symbol sense.

Cluster 3

The third cluster, which fills 108 (19.1%), contains mostly a mixture of learners whose understanding of mathematical concepts and symbols ranges from agree to strongly agree. Learners in this cluster indicated that they could confidently manipulate mathematical symbols with understanding. About 39.8% of the participants indicated that the symbols in a mathematical problem have a significant influence on their attempt to solve a mathematics problem. The cluster also contains learners who understand mathematical concepts and are able to initiate symbols to solve problems, including specifying units and distinguishing among the three main uses of variables (unknowns, placeholders, parameters). Learners in this cluster are also flexible to move from one formula to another in relation to the demands of task and the symbols used in the question and formulae do not affect their understanding of concepts. Thus out of the 565 learners surveyed only 108 do not have severe difficulties with mathematical symbols, instead symbols to them are tools to aid understanding.

4.10 Typological Analysis of Learners' responses

Section C of the questionnaire consists of open–ended questions. Open-ended questions on questionnaires require participants to further elaborate responses to closed questions and offer insights or issues not captured in the closed questions (Harris & Brown, 2010). The questions were formulated in such a way as to encourage the explanation of the answers and reactions to the question with a sentence, phrase or a paragraph. This allows the researcher to better access the respondents' true feelings, opinions and perceptions on an issue (Popping, 2008).

Typological analysis was used to analyse learners' responses in section C and the responses from interview discussions. Typological analysis is a qualitative or quantitative strategy for describing a set of related but distinct categories within a phenomenon (Given, 2008). Typology analysis requires the researcher to carefully analyse raw data before deciding the category in which it can be classified. The categories used in a typology should be mutually exclusive to reduce ambiguity in classifying data (Babbie, 1998). Typological analysis technique serves three functions: descriptive, classificatory and explanatory (Bennett & Elman, 2006). The descriptive function defines and describes the various types, distinguishing, for example, semiotic and instructional challenges of mathematical symbolisation.

Hammersley and Atkinson (1995) confirm that typologies are a useful way of displaying associations in qualitative data by displaying how particular views or experiences may attach to particular groups or sections of the population. Several typologies were identified after collecting, coding and analysing learners' and teachers' interviews and questionnaire data. After scrutinising data and typologies selected, certain patterns and relationships begin to emerge. These patterns were subsequently clustered into themes. Babchuk (2009) defines themes as integrating concepts that can be defined as statements of meaning that run through all or most of the pertinent issues in the data.

Participants were required to indicate their experiences in learning mathematical concepts through symbolisation. Through the analysis, the researcher examines patterns and trends in the responses to reach certain conclusions. The researcher prepared a coding scheme using word frequencies. The coding scheme was also informed by literature and theoretical frameworks. A word frequency was utilised to identify the most frequently used words that provide indications of the most frequently expressed difficulties. The researcher read each response carefully at least twice with the aim of identifying common themes emerging from the responses. Themes were built around these commonly used words. The researcher took cognisant of contexts in which some of the most frequently used words appear in order to assess their suitability for inclusion in a given category.

4.10.1 Category 1: Reading mathematical symbols

Participants in this study indicated that they struggle to read words and mathematical symbols appropriately. Reading mathematics and science requires special reading skills, skills that learners have not acquired. Mathematics textbooks are written in a concise manner using symbols, diagrams and graphs. The conceptual density of mathematics text is one of the major reasons for learners' difficulties (Barton, Heidema & Jordan, 2002). In addition to comprehending text passages, learners need to decode and comprehend scores of scientific and mathematical signs, symbols, and graphics. The theme emerging from this typology is that learners have difficulties in interpreting or understanding meanings of certain mathematical symbols due to the way in which they are taught to read those symbols. Furthermore, there is ample evidence that learners struggle to study mathematics alone at home since they do not know how to read mathematical symbols because they seldom hear them being spoken. During focus group interviews one of the learners said:

> *"……. textbooks use complicated language and symbols that are unfamiliar and this puts me off when I am reading on my own at home, I have to wait and seek help from my teacher"*

Thus, learners confirmed that current textbooks use unfamiliar symbols and learning takes place when the teacher explains the meanings of these symbols to the learners. Without a teacher, reading a mathematics textbook is a difficult task for many learners. Another learner had this to say:

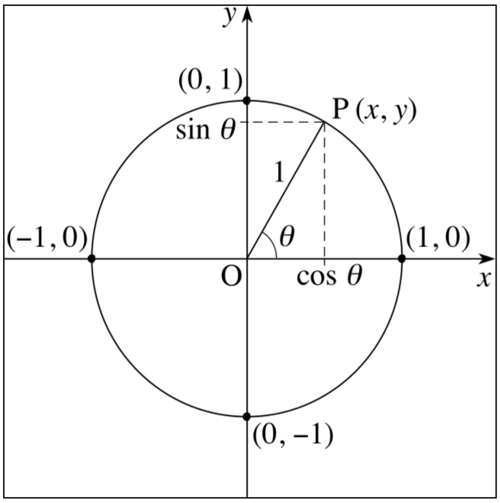
> *"...some symbols often look alike or one symbol represents many things and this confuses me".*

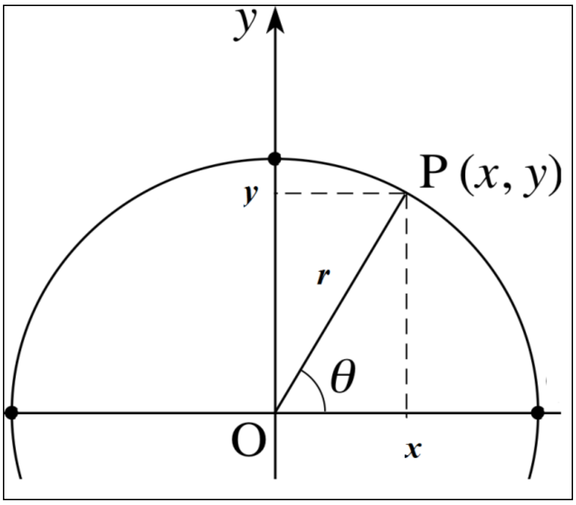
When asked to elaborate further the learner mentioned trigonometry as one of the topics which is difficult for a novice learner to read and understand concepts due to many confusing symbols. Learners indicated that there are many symbols that are used to represent an angle$(\alpha, \beta, A, B, P, \theta\ , X)$. The angles are labelled either using vertices or Greek letters. The letter ‘X’ used in the introduction to introduce trigonometric ratios $(cos\,\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$ and $sin\theta\ = \frac{y}{r})$ but latter in the chapter learners are asked to solve for x if $cos\,x = 0.5$. Learners indicated that the introduction to trigonometry brings confusion as teachers try to relate angles and distances. The following vignette captures the conversation between the researcher and Grade 11 learners in one of the focus group interviews:

Learner A: We have just stated a new topic "Trigonometry" and the following diagram used in the textbook to introduce the topic confused me (the learner draws the attention of the class to a diagram in a textbook which looks like diagram B in figure 4.1 below).

Researcher: (Looking at the diagram) … ok what is your problem with that illustration? Where and what is the problem in this diagram?

A B





Learner A: The coordinates of point P, are (x, y), yet the x- value in the x-axis is $\cos\ \theta$ and the y-value is $\sin\theta$, so does it mean that $x = \cos\theta$ and $y = \sin\theta$?

Learner B: … the same diagram confused me since it is a bit different from the one we were introduced to in Grade 10.

Textbox 4-2 : Learners' Conceptions of trigonometric ratios

θ θ

Theme 1: Learners have difficulties in understanding mathematical concepts due to pedagogical strategies that exclude reading symbols.

Theme 2: Mathematical symbols assume different meanings in different contexts and therefore confuse learners.

These findings lend support to what Brown (2006) found in her study of learners' understanding of sine and cosine functions. The study revealed that learners have partial understanding of the ways to view sine and cosine: as coordinates of a point on the unit circle, as a horizontal and vertical distances that are graphical entailments of those coordinates, and as ratios of sides of a reference triangle.

4.10.2 Category 2: Mathematical concepts and symbols linkages

Another observation that emerged from data analysis is that learners experience difficulties with the dual role of mathematics symbols. Symbols are regarded either as processes themselves or as products of the process, depending on the context. Thus, a compact mathematical symbolic structure can represent a complex concept that may also be mentally manipulated as a single entity. Learners lack the ability to select appropriate representation for a given stage of a mathematical problem, cannot move flexibly between representations. It is this versatility to move between representations and choose the most appropriate that gives them strong symbol sense and algebraic insight.

Theme 1: Learners tend to cling to procedurally driven symbolic approaches, which are less flexible and impose greater cognitive strain on them.

4.10.3 Category 3: Time allocated for individual mathematics practice

Majority of the learner respondents (81%) confessed that they do not allocate time to study mathematics like other curriculum subjects. Learners indicated that they do not consider mathematics as a reading subject. Reading mathematics requires skills different from from curriculum subjects. Reading mathematics is objective, uses unique procedures, involves symbols and has a vocabulary system of its own. Consequently, learners need to develop study skills peculiar to mathematics. Doing mathematics is an active process that requires routine studying and frequent learning. One of the learners interviewed had this to say,

"….. I just don't have time for mathematics, its frustrates and takes a lot of time without making progress while trying to figure out something that I don't know".

Participants indicated that they learn mathematics during class time and are not patient to read and understand mathematical concepts on their own. Respondents who spare time to study mathematics indicated that unfamiliar symbols coupled with language barriers derail their progress. The theme emerging from typology is that learners do not devote adequate time to understand mathematical concepts but they stick to procedures that lead to correct answers without conceptual understanding.

Theme 1: Learning mathematical symbolisation requires time and is frustrating if the time invested yields no understanding of mathematical concepts.

Theme 2: The 4.5 hours allocated per week (CAPS) for teaching and learning mathematics is inadequate.

4.10.4 Category 4: Symbols and Problem solving

In response to the question, **"**What challenges do you encounter in your attempt to understand mathematical concepts symbols and problem solving procedures through interpreting mathematical symbols?" one learner has this to say:

> *"……. I often encounter contradictions between the explanations and examples given"*

Another learner indicted that,

> *"…… I find symbols very difficult, because whenever I find an unfamiliar symbol, I lose confidence in doing my work and switch to other subjects and reserve the questions for the teacher"*

In response to the question, "Are you able to initiate a mathematical expression or symbol or sign as demanded by a given mathematical problem?" Learners indicated that they need an example similar to the one asked so that they can imitate the steps when solving the new problem. Thus, learners prefer to solve the problem by referring to a similar worked problem using symbols without actually understanding their actual meaning. Another group of learners indicate that translating word problems into algebraic is not an obvious thing to get at the first attempt, one has to go through a series of attempts in order to come up with the right concept and appropriate symbols. Participants indicated that transforming word problems into symbolic statements is not an easy task; one has to be familiar with many symbols and topics make connections between topics, a well-developed symbol sense as well as having enough practice of the topics. One learner recommends that teachers should help learners to interpret mathematical symbols at the beginning of the topic, explain their meanings, and refer to them during the course of teaching so that learners can retain them. Conceptual density should also be kept to a minimal as one learner has this to say:

> *"…my main challenge with using mathematical symbols is that I cannot retain many symbols in my mind and I quickly get confused when I am asked to recall symbols and concepts from other topics, especially Trigonometry and Euclidean geometry"*

The researcher made a follow up to these statements made by learners from a Grade 10 class and probes them until they presented the following questions:

1. Solve for x if $x \in [0^0 \ , \ 90^0]$ correct to one decimal place

a) $2\cos x = 0.817$

b) $\cos(2x - 20) = 0.917$

2. If $15\tan\theta = 8$ and $0^0 \le \theta \le 180^0$, use a sketch (not a calculator) to calculate the value of:

a) $2\tan^2\theta - 2$

b) $\sin\theta - \cos\theta$

Researcher: (Looking at the two problems) … ok what are your problems with the two questions?

Learner 1: Sir, I do not understand the two questions fully except that I should 'solve for x", and what is the meaning of $x \in [0^0 \ , \ 90^0]$ and $0^0 \le \theta \le 180^0$?

Researcher: The notations $x \in [0^0 \ , \ 90^0]$ and $0^0 \le \theta \le 180^0$ are the same, it's only that they are expressed differently but they all refer to the domain.

Learner 2: May you clarify what you mean give us a clear picture of how we use it when solving the problems. Why do you say they are the same yet they do not look the same?

Researcher: The notation $x \in [0^0 \ , \ 90^0]$ means the interval of values of angle x from 0^0 to 90^0 including the end points, while $0^0 \le \theta \le 180^0$ means the interval of values of angle θ from 0^0 to 180^0 including the end points.

Learner 3: Sir, so are you saying the brackets and the inequalities means the same?

Researcher: Yes, $x \in [0^0, \ 90^0]$ is the same as $0^0 \le \theta \le 90^0$ while $0^0 \le \theta \le 180^0$ is the same as $\theta \in [0^0, \ 180^0]$ (writing on the chalkboard). Can you proceed and solve the two equations?

Textbox 4-3: Learners' Conceptions of Domains

(a) (b)

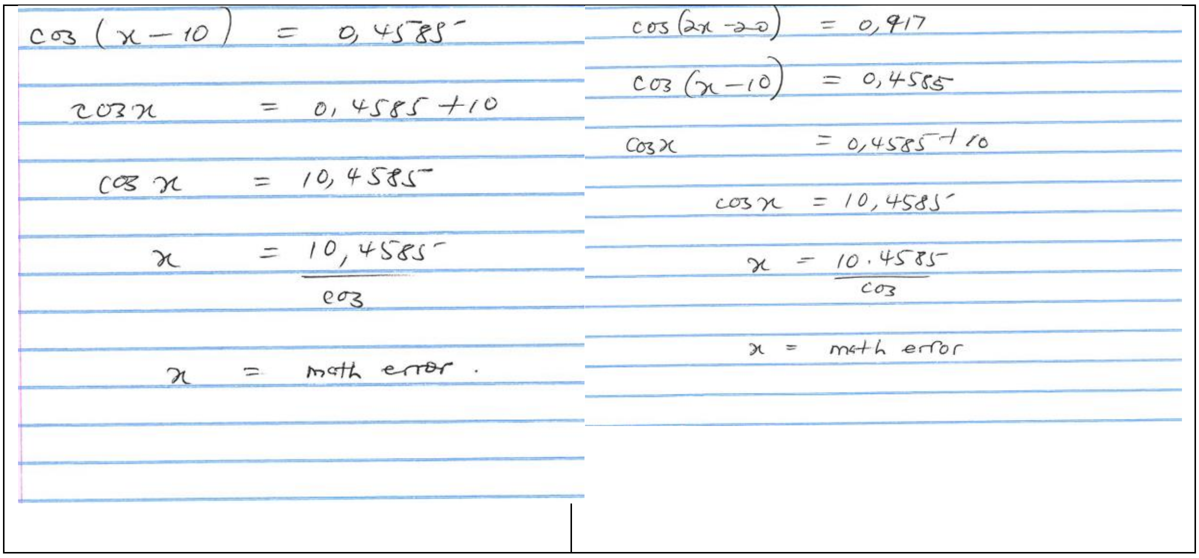


Figure 4-14: Learners' sample solutions

The two solution samples indicate that learners did interpret x as angle despite the fact that in the discussion the intention was to find the value of x as an angle rather than as a variable. The confusion was that learners treated cos as a number that can be used in calculations. Learners used algebraic applications that are not suitable for the problem context, like dividing by 2 throughout.

One of the groups could not distinguish between trigonometric equations and expressions. They failed to interpret the question and equate the expressions to zero to make them equations. The procedure for solving the created equations was correct though it was applied wrongly. The meaning of symbols, especially the angle was well understood. Learners ignored the graphical approach suggested by the question and proceed with the algebraic approach, which failed them at end. Sample solutions for this item are shown in figure 4.15 below:

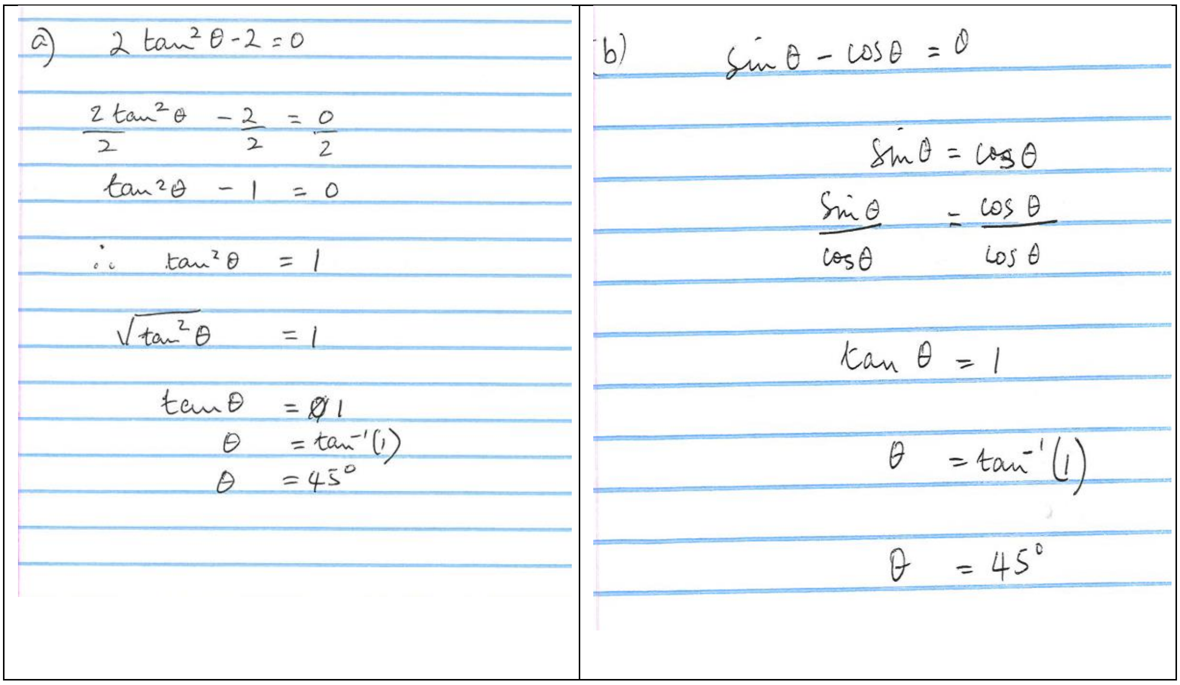


Figure 4-15: Learners' Sample solutions

Thus as a recommendation, teachers should try to keep the number of mathematical symbols to a minimal as many symbols tend to confuse learners. However, this is not practical as the number of symbols used in a mathematics lesson cannot be determined in advance. Cowan (2006) suggested that a thoughtful approach to lesson-

planning can reduce confusion. The same sentiments were echoed by Totten, Eaton and Dirst (2009) who concurred that if many symbols are introduced at the same time, learners find them difficult to understand or relate to the mathematical concept they represent. Teachers need to help learners to recognise and interpret symbols so that they become part of their mathematics language. Teachers should draw a grade-appropriate list of symbols together with their corresponding meanings. Learners must be drawn into discussions involving the history of some the symbols as a way of enhancing their retention.

Theme 1: Learners lack symbol sense and algebraic insight required for problem-solving.

Theme 2: Unfamiliar notation jeopardise conceptual understanding and learners seek solace in other subjects.

Theme 3: There are too many symbols to learn in mathematics and recalling them is difficult, hence proper planning is needed to reduce symbol load.

4.10.5 Category 5: Are symbols important in mathematics?

Questions 10 and 12 require learners to indicate whether mathematical symbols are useful as well as their role in conceptual cognition. Some of the responses provided by the learners were:

Learner A:

> *"... symbols are used to shorten problems and to enable the solution process but unfortunately sometimes they give me challenges".*

Learner B:

> *"……. I do not appreciate the role of mathematics symbols in enhancing my understanding of mathematical concepts; instead they make mathematics even more complicated because I have to understand two things: understand the concept and its symbols together with their meanings".*

From these learners' sentiments, one gets the notion that symbols are not merely a short-hand of expressing mathematical ideas and concepts, but they constitute the concepts themselves. Mathematical symbols make possible to express ideas physically on the paper and perform mathematical operations that are otherwise impossible to do mentally. In this way, learners were referring to mathematical symbols as epistemic tools, which can be manipulated to produce actions (processes). Thus, learners also share the view that symbols are external way of representing abstract mathematical ideas (De Cruz & De Smedt, 2013). Participants' concerns were also echoed by Askew (2014) who pointed out that mathematical symbols do not carry meaning and their role cannot be established until they are mediated through suitable reference contexts. Thus, it is important for a mathematics teacher to select contexts in which mathematical concepts and symbols can be easily understood. Confusion and misconceptions emanating from improper use of mathematical symbols are detrimental to learner's understanding of content presented in a given learning environment. Rubenstein and Thompson (2001) raised the same issue and warned that learners who cannot appreciate the role and use of standard mathematical symbols will at some stage be hindered in their mathematical development.

Theme 1: Mathematical symbols allow concepts to be expressed compactly and are used to present the solution processes.

Theme 2: Mathematical symbols make mathematics even more complicated since the concept and its symbols need to be learned and understood as if they are separate entities.

4.10.6 Typology 6: Intervention Strategies

One of the items in the questionnaire requires learners to suggest the kind of support and a strategic intervention that they can be used to improve the use and interpretation of mathematical symbols. One of the respondents has this to say:

> *"... all new symbols in a topic should be written on the board and their meanings should be explained. The teacher should demonstrate how these symbols are used in conceptual development as well as in problem solving."*

The general consensus among the participants was that they clearly understand the mathematical concept using other forms of representations before symbolic representation. Symbols should be introduced after the learners have conceptually grasped the content. A great deal of time should be devoted to teaching the concept using manipulatives and other concrete objects and later introduce symbolism as a means of representing the idea in a compact manner. Another interesting response from participants was:

> *"... if we were introduced to these symbols at an early stage it would be easier for us to use them without any difficulty".*

This suggestion is consistent with Barcelos and Silveira (2012) who observed that learners have the potential to understand symbols but need guidance to understand their role within a system. Fricke et al. (2008**)** also emphasised that good teaching should attempt to foster connections between the learner's informal knowledge and formal system of symbolism. A study conducted by Worthington and Carruthers (2003) also revealed that learners possess informal knowledge of mathematics by the time they start schooling school. Teachers need to support learners' informal knowledge and use it as a foundation to construct knowledge algebraic of symbols.

Theme 1: Teachers' should demonstrate knowledge of pedagogy by linking symbols, concepts and meanings during lesson delivery as well as in problem- solving.

Theme 2: A timeous introduction of mathematical symbolisation gives all learners more opportunities to operate with abstract concepts in later mathematics.

4.11 Pedagogy and Instructional Materials

Section D of the questionnaire consists of open–ended questions. The items in this section requires learners to provide information about how current teaching and learning methods and resources incorporate and address challenges emanating from the use of mathematical symbols. The aim of these open-ended questions was to give learners an opportunity to suggest instructional strategies that can help them to understand mathematical concepts together with their symbolic representations. The items in this section fall into four main categories, which are: (1) reading materials, (2) instructional strategies, (3) context and multiple representations.

4.11.1 Category 1: Reading materials

Items C1, C6 and C9 address issues related to learning materials. Participants indicated that current reading materials are relevant and use familiar symbols; however, some of the symbols do not carry the same meanings from previous grades. Some of the learners indicated that they consult the glossary section of the textbook to get the meanings of some of the symbols. One learner had this to say:

"…the current textbooks we use especially (name of the textbook mentioned) actually explain everything to you and their worked examples are well explained".

However, this was not the case with the learner who said:

> *"…. current textbooks use unfamiliar symbols, but can be learned from once you understand the concept; this is where I need my teacher the most".*

Learners from urban schools indicate that textbooks explain symbols better and are very friendly to use, however this is possible with the help from the teacher and other experts in the subject. The same could not be said by learners from rural settings who indicate that the symbols at their current grade levels are too many, complex and confusing. This is consistent with Navsaria et al (2011) who found that learners from impoverished backgrounds have reading challenges. Another interesting issue raised by participants is that some of them revert to old textbooks of previous syllabi because they explain concept clearly, use simple mathematical language and have fewer mistakes compared to current textbooks. Learners also indicated the importance of prior knowledge of some of the concepts together with their symbolic representations taught in the previous grades. One learner had this to say:

> *"... yes, because some topics make sense when I link them with what I learnt in the previous grades"*

This statement is consistent with Tall (2008) who elaborated that the learners' understanding of symbols is determined by the prior mathematical knowledge they bring to a new learning situation classroom. Thus, understanding is a cognitive matter, which depends on what the learner brings to the learning situation. These ideas lend support from Bruner (1990) who believes that learners construct new knowledge by building on prior knowledge. Teaching involves processing information, deriving meaning from experience, forming hypothesis and making decisions. To do this, teachers need to scaffold learners by providing guidance, support, connections and assistance when first introducing a new mathematical concept or skill to learners.

Theme 1: Teachers need to scaffold learners by providing guidance, support, connections and assistance to read new mathematical symbols and concepts based on their current knowledge.

4.11.2 Category 2: Instructional Strategies

Items C2, C4 and C5 address issues related to teachers' instructional practices. Learners were asked whether current instructional practices help them to understand mathematical concepts through the process of symbolisation. Item C2 requires learners to evaluate whether their teacher helps them to understand new symbols in a new topic. One of the learners had this to say:

> *"…very seldom. Our class is a bit fast–paced because we have to complete the syllabus. Sometimes when we ask why, he says, "That's how it is" and from that as a learner I start to lose hope in maths and that's the reason why I say mathematics is hard".*

From the quotation above one gets the impression that teachers' explanations sometimes do not satisfy learners and learners' concerns are not given the impetus they deserve. Learners also alleged that teachers teach to complete the syllabus instead of conceptual understanding. Another observation made by learners is that teachers at times select a method which may not be the ideal one for a given classroom context and insist that learners should understand novel concepts. One of the participants had this to say:

> *"...sometimes the teacher wants us to understand even though we cannot understand some of his untraceable methods".*

Learners also pointed out that some of the topics are treated as revision, yet learners are experiencing the concepts for the first time. Another issue raised by learners is that teachers sometimes make wrong assumptions of thinking that learners know a certain concept while in actual fact they do not. All learners in a fast class are assumed to have the same competency, which is not always the case. Thus, at times teachers' strategies of approaching certain mathematical concepts lack detailed explanations that convince a learner who is learning the concept for the first time. Teachers need make their choices of teaching methods as well as selecting the method based on learners' needs. Teachers need to think systemically about instruction. Instruction should be planned around establishing relationships among concepts and processes (Petrina, 2009). Classroom instructional systems involve decisions related to what (content) learners will learn and how (methodology) they will be taught and how learning will be assessed. Learners suggest that the teacher should assess them thoroughly to check if they grasp the concepts before going deeper into the topic.

Grevholm (2008) revealed that every learner carries his or her own personal concept image of a mathematical concept. Learners' concept images often align poorly with the concept's standard mathematical definition, and many difficulties arise when this is the case. Mathematics teachers should encourage developing meaning for a mathematical concept before it is symbolised. Later, when the symbol is introduced, learners will view it simply as an abbreviation of the concept rather than as an indicator of an algorithm or object.

Christiansen, Howson and Otte (2012) recommended that teaching activities solely pitched at the iconic and symbolic levels need to be restricted considerably, and concrete modes of instruction should be explored first. English and Halford (2012) recommended that teachers should move from concrete manipulatives to semi-abstract representations such as pictures, diagrams and finally to abstract mathematical symbols. Abstract mathematical symbols should be introduced informally and learners should be allowed to use their own informal symbols. The transition to formal symbols should allow learners to map the new symbols onto their existing understanding of the concept through a series of experiences during problem solving.

Theme 1: Teachers do not adequately address learners' problems but instead accelerate them to complete the syllabus without conceptual understanding.

Theme 2: Teachers should take cognisant of the fact that learners do not assimilate mathematical knowledge at the same pace.

4.11.3 Category 3: Problem Context and Multiple Meanings

Item C10 addresses issues related to learners' competency in distinguishing the use of given symbols in different contexts. Learners were asked if they are able to realise that a mathematical symbol can assume different meanings in a variety of contexts. One learner confessed that this is only possible if it can be done under the supervision of the teacher. The prevalence of multiple meaning symbols carries the potential to impart confusion and disorientation to the learners who attempt to comprehend concepts in mathematics (Rouhani & Kowsary, 2014). The use of the same symbol to represent a variable and a mathematical operation has the potential to cause learner confusion and misunderstanding. For example, if we see the + symbol written in the sum $3 + 5$, we understand that the context is one of adding the two numbers, 3 and 5, to give a sum of 8. When studying directed numbers the symbol $(+5)$ shows the position of 5 on a number line. Thus, the $(+)$ sign can be regarded as an addition sign in the first context and as a position sign in when studying directed numbers.

Theme 1: Symbols have different meanings in different context, this causes cognitive discomfort for learners, and teachers need to be precise about the symbols they use, emphasise the importance of context for some mathematical symbols.

Teachers' responses regarding the use of mathematics symbols

Table 4-13: Demographic variables

Variable	Categories	Frequency(f)	Percentage (%)
Gender	Female	6	64
	Male	9	60
Age	26-35 years	7	46.6
	35-50 years	6	40
	51 years and above	2	13.4
Home Language	Sepedi	11	73.4
	Tshivenda	2	13.3
	Other languages	2	13.3
Residential Area	Urban	5	33.3
	Semi-Urban	4	26.7
	Rural	6	40
Highest Academic Level	Post-diploma/certificate	9	60
	Undergraduate Degree	4	26.7
	Post –graduate degree	2	13.3
Teaching Experience	5 years and below	2	13.3
	6-10 years	3	20
	11-15 years	4	26.6
	16-20 years	6	40

Table 4-13 shows demographic data for teachers who participated in the study. The sample consisted of 15 mathematics teachers who were purposively selected to provide information pertaining to the challenges they experience when teaching mathematical concepts using symbols and signs. Six (40%) female and 9(60%) male teachers participated in the pilot survey. The majority of the participants (13 (86.7%)) were in the

26-50 years age category and mostly (86.7%) black. The dominant language was Sepedi (73.4%) and most participants (40%) were drawn from rural schools. Participants' highest academic qualifications were post-matric diplomas and undergraduate degrees majoring in mathematics. Participants had vast teaching experience as most (66.7%) of them had more than 10 years of teaching mathematics in the targeted grades.

4.11.4 Categories and themes emerging from teachers' responses

The questionnaire for teachers consists of open–ended questions. The questions were framed in such a way as to encourage the explanation of the responses and reactions to the questions with a sentence, a paragraph. This allows researcher to better access the respondents' views on an issue (Popping, 2008). Typological analysis was used to analyse teachers' responses. This section of the survey instrument (see Appendix) requires respondents to indicate their experiences when teaching mathematical concepts through symbolisation.

Through the analysis, the researcher examines patterns and trends in the responses so as to reach certain conclusions. The researcher developed a coding scheme to classify teachers' responses by recording words or phrase that were frequent. The researcher read each response carefully at least twice with the aim of identifying common themes emerging from the responses. Themes were built around these commonly used words.

4.11.5 Category 1: Challenges of mathematical symbolisation

Teachers indicated that learners take time to familiarise themselves with mathematical symbols. One of the teachers had this to say:

> *"…there is a big gap between the Senior and Further education and training (FET) phase in terms of content, level of abstract concepts and symbol-rich mathematical concepts".*

Teachers pointed out that in some cases one concept has to be repeated several times for learners to understand. However, there is limited time as the syllabus is long and scheduled. The time allocated for some concepts is so short considering the rate at which learners understand the concepts. One participant cited Geometry and Trigonometry as topics that have challenging abstract symbols. The participant had this to say:

“…. There are too many challenging symbols and representations that learners have to learn in Geometry and Trigonometry”.

This is consistent with Yusha'u (2013) who reported that the teaching and learning of Trigonometry presents learners with challenges emanating from the mixture of specialised words and technical language coupled with a host of unfamiliar symbols especially at Grade 11 level. Participants indicate that there is a growing tendency amongst teachers to shy away from difficult topics such as trigonometry and geometry with learners running away from Mathematics classes on the traditional belief that it is abstract and difficult.

Teachers revealed that learners have many misconceptions about the use of symbols in these topics. Symbols are used to build mathematical concepts; if a learner does not know symbols it is difficult to understand concepts. Teachers submit that learners’ difficulties with mathematical symbolisation have connection with their roots in lack of conceptual knowledge. The teaching they received in previous grades created these cognitive gaps. The use of other forms of representation such as graphs, pictures and making connections with other symbols further compound the problems.

Teachers indicate that the selection of referents to some mathematical concepts must be done with thorough thought since some of them may not well represent the mathematical concept. Another issue raised by teachers was that mathematical symbolisation is problematic if the context of the problem is not well understood. Understanding the problem situation helps learners to select appropriate written mathematical symbols that represent the ideas. The multiple meanings nature of mathematical symbols also presents cognitive difficulties for learners; good teaching entails equipping earners with the skills of analysing the problem contexts and select relevant symbols. Teachers indicate that the problems encountered by the learners with mathematical symbols stems from lack of conceptual knowledge that can be possibly explained by the teaching they experience in lower grades. Teachers’ limited content knowledge that influences heavily on their choice of symbols in conceptual development.

Theme1: Learners have a low retention rate hence take time to grasp new mathematical concepts, schemes and work schedules, pace setters are too congested and fast-paced.

Theme 2: Learners have many misconceptions about the use of symbols from past learnings and this hinders conceptual understanding.

Theme 3: Inadequate correspondence between the represented mathematical concept and the representing world is an obstacle to conceptual understanding.

4.11.6 Category 2: Pedagogy and Symbolisation

Most teachers reiterated that they make efforts to align their teaching with real-life situations; however, they face the problem of linking real–life experiences to mathematical forms expressed in symbolic forms. Learners encounter the problem of treating the same mathematical concept expressed in different forms as two separate entities, thereby failing to make conceptual links. Mathematics teaching at lower secondary school levels and primary school should not resonate too much on the manipulation of concrete objects, but the foundation of abstract mathematical thinking should be laid. One of the teachers indicated that the basic meanings of abstract symbols such as x^2 should be taught using simple English. For example, x^2 should be interpreted as "a certain number multiplied by itself twice", x^3 as "a certain number multiplied by itself three times', x^n "a certain number multiplied by itself n times". Thus from this learners will be able to generalise, treat x as variable that can assume different variables, exponent tells us to multiply the base by itself n times. Learners will begin to observe the emerging pattern and symbol burden will be outweighed by conceptual understanding. Crooks and Labial (2013) submit that it is unethical and psychologically unhealthy for teachers to encourage learners to manipulate mathematical symbols without or with little or no associated meaning.

Good teaching entails providing relevant, illuminating and understandable examples of abstraction that encourages the integration of symbolism and experience in a learner's mind. A sensitive mathematics teacher should seek to close the gap between symbols and their meanings. Such gaps when allowed to occur obstruct learning and reduce learners' retention. Thus teachers should timeously introduce symbolism. Learners should be assisted to develop symbol systems that support their mathematical activities and tasks. Teachers should be alert to symbolisation problems, place them in a proper perspective for their learners, and help to head off symbolisation difficulties.

Another interesting issue raised by teachers was use of a variety of representations to enhance learners' understanding. To be effective in imparting mathematical knowledge teachers should use a variety of representational tools to support learners' mathematical development. The spectrum of these representations includes algebraic symbolism, graphs, diagrams, models, equations, notations, images, analogies and technology. These tools provide means for communicating, reflecting, and as basis for mathematical argumentation. They are most useful when they become integral parts of learners' mathematical reasoning.

Theme 1: Linking real –life experiences to mathematical forms expressed in symbolic forms is a challenge since some mathematical forms are purely abstract.

Theme 2: Sophisticated mathematical symbolism should be timeously introduced after conceptual understanding.

4.11.7 Category 3: Instructional Strategies for Symbolisation

The majority of participants agreed that teachers should be aware of the difficulties that symbolism creates for learners. They stressed that symbolism should be treated as a form of mathematical language and learners must be encouraged to acquire it in order to be competent in the subject. There is limited use of mathematical language, symbols and concepts outside the mathematics classroom, therefore mathematics classrooms should be dominated by these three aspects. Teachers strongly recommended that learning should proceed from concrete instances to abstract. Teaching mathematical symbolism and mathematical concepts should be done simultaneously as these two are interconnected. Conceptual understanding should precede symbolisation, that learners should grasp the mathematical concept before it can be symbolised. Mathematics teachers need to engage learners in contexts, problems, and activities that move them from known to unknown mathematical ideas. Learners should be guided and motivated so that they can see the value of being fluent with mathematical symbols. 'Symbol pushing', that is the use of symbols without understanding their meanings should be avoided, learners should be discouraged from reproducing symbols without attaching meaning to them. One of the teacher participants had this to say:

> *".... when introducing a new concept and its symbols we need to be careful to emphasise the new symbol, demonstrate how it is written and used, and give learners a chance to verbalise it, read it, write it, and practice its use".*

Teachers also revealed that learners' prior knowledge should be established before introducing a new concept and new symbols. This helps the teacher to attach new symbols to referents that are familiar and meaningful to learners. A variety of meaning-making activities should be included such as encouraging learners to think aloud so that teachers may understand how learners interpret symbols; translating symbols into words, diagrams, word problems into symbols. Most mathematics classroom are characterised by learners spending a lot of class time listening and watching the teacher demonstrating some mathematical procedures, they are not given opportunities to practice reading, writing, and speaking mathematics. Mathematical ideas presented in oral and written communication about should be recognised as an important part of mathematics learning. The dual role of mathematical symbols as instruments of communication and thought should be emphasised in the structural features of mathematical objects such as expressions, equations and functions.

Teachers also emphasised the creation of links between ordinary English and the symbolic language that dominates mathematical notation. Word problems should be used to introduce Algebraic sentences. Learners should possess skills of transforming word problems into symbolic expressions. Learners should be encouraged to seek meaning when dealing with algebraic expressions, equations, and functions. Learners need exposure to different contexts and a variety of representational systems that represent the same construct. Learners should be able to switch across the translational shifts in representation, that is, from words to symbols, from symbols to words, from symbols to graphs, and within the system of symbols.

Theme 1: Learners should grasp the mathematical concept before it can be symbolised, thus, conceptual understanding should precede symbolisation. Alternative representations such graphical should be used for teaching the concept before it can be symbolised.

Theme 2: Learners' prior knowledge of mathematical symbols and concepts must be established and used as a basis for new knowledge.

4.11.8 Category 4: Textbook content development and layout

Textbooks serve as major learning resources, but at times their composition and organisation can be a source of difficulty. Most mathematics textbooks follow the symbol precedence where symbolic problems are presented prior to verbal problems. New concepts are introduced in a symbolic format and word problems and story problems are presented as challenging activities. Learning from a textbook is complex and depends on a number of interacting factors such as the problem context, the prior knowledge and familiarity with the notation. Teachers confirmed that the structural aspects of current textbooks such coherence both at the microstructural and macro-structural levels are up to standard though there is need to include side notes to cater for learners who are reading the content for the first time. The same recommendation was made by learners. Nathan, Long and Alibali (2002) emphasised that text coherence also helps to activate the readers' prior knowledge to improve comprehension and inference making.

Theme 1: Authors of mathematics textbooks should include side notes that explain concepts, meanings of symbols and cater for learners who are reading the content for the first time or reading at home.

4.11.9 Category 5: Learners' conceptions of mathematical symbols

Teachers indicate that a learner's prior knowledge often confounds the teacher's efforts to deliver concepts accurately. Generally learning proceeds primarily from prior knowledge, and only secondarily from the presented materials. If prior knowledge is at odds with the presented material learners will distort presented material. Teachers should not neglect learners' prior knowledge otherwise teaching would be fruitless no matter how well the lessons are executed. Prior knowledge can be viewed as the bane of transmission-absorption models of learning. All the 15 teacher participants subscribe to the notion that prior knowledge influences learning, and that learners construct concepts from what they already know. Thus teachers need to change their view of prior knowledge from the view that learning is absorption of transmitted knowledge, to the view that learning is conceptual change. The learner's prior experience is re-enacted on a daily base. Teachers confirmed that the majority of learners lack adequate prior knowledge to extract meaning from instruction. Teachers confessed that they often make erroneous assumptions that learners come to class possessing the skills and information to learn what they teach. Attention, then, needs to be paid to what the learner brings to the learning process.

The teacher can engage learners in a teaching and learning approach called scaffolding. The guidance that the teacher extends to the learners is termed scaffolding. It is assumed that through scaffolding, learners can become independent learners. Scaffolding techniques such as clarifying doubts, inviting responses, focusing on task, reinforcing important facts and evaluating learners' works can be used by teachers to enhance understanding. The teacher initially provides extensive instructional support, or scaffolding, necessary to help learners build their own understanding of new concepts or skills. Scaffolding is a term in the world of education that exists in modern constructivist theory of learning. In learning, scaffolding takes a very important role in the development of learner learning. Each time the learners reach a certain developmental stage in learning which is characterized by the fulfilment of indicators in certain aspects, the learners will require scaffolding.

Vygotsky in (Nur, 2004) suggests that the scaffolding is the concept of learning with assistance (assisted learning). According to Vygotsky (1986), the functions of higher mental, including memory and the ability to direct attention to specific goals and the ability to think in symbols, is a behaviour that requires assistance, especially in the form of media. Scaffolding is derived from the view that learning mathematics needs a multiway interaction, teacher-learner, learner-learner, learner-teaching materials so that learners-based on experience- can develop mathematical knowledge and strategies to respond to mathematical problem given. Allowing learners more time to work out mathematical problems using symbols initially and then discussing the reasoning may also be an effective way to scaffold mathematical understanding. Gibbons (2009) views scaffolding as a form of support in which learners to take increased responsibility for their learning. Vygotsky (1986) coined "the zone of proximal development" which is the gap between what a learner can do independently and what they can do with help.

Teachers need to consider how they can provide high levels of support when necessary while ensuring that learners are challenged enough to make progress.

Theme 1: Scaffolding is strongly recommended as a teaching strategy in order to give learners guidance and support until they can work independently.

4.11.10 Category 6: Teacher pedagogical content knowledge

Another important aspect emerging from teachers' responses was teachers' pedagogical content knowledge. Teacher's knowledge of how to present academic content is a crucial factor for learners' understanding. A mathematics teacher should possess in-depth knowledge of concepts and procedures that build proficiency with mathematical symbolism. This knowledge is essential for making decisions concerning classroom tasks, resources and activities that feed into learning process. Competent teachers make and create wider connections between facts, concepts, structures, and practices. To teach mathematics content for effective understanding requires teachers to be grounded with the knowledge of how learners learn. Such understanding can help them to correctly anticipate learners' conceptions and misconceptions of some mathematical concepts. This awareness helps to make informed choice of instructional decisions that support learners' conceptual understanding. Teachers should play a role in helping learners to use symbols effectively, and should teach mathematical language which is dominated by symbols. One of the participants had this to say:

> *"….as teachers we need to use simple English to unpack and explain the meanings of complex mathematics symbols to help learners draw on the different meaning making modes for understanding".*

Another strategy suggested by participants was to support learners to move from the informal symbols into formal mathematics register by having learners talk about mathematics using technical language and symbols as they solve problems, encouraging them to articulate patterns and make generalisations. Teachers should engage learners in mathematical discussions and conversations in classrooms. The discussions will allow teachers to understand better whether learners are making appropriate conceptual connections between symbols and their mathematical meanings. Teachers should also lead learners to understand the dual role of symbols standing for both process and concept (applying the notion of procept).

Theme1: Mathematics teachers should possess knowledge of explanations and representations, learners' thinking, and multiple solutions to mathematical tasks.

4.12 Thematic Analysis

Thematic analysis was the last step in the analysis of data. It involves searching for themes and patterns of meaning across a dataset in relation to research questions (Braun & Clarke, 2013). Thematic analysis involves a search for relationships among domains, as well as a search for how these relationships are linked to the overall phenomena under study (Onwuegbuzie, Leech & Collins, 2012). Thematic analysis starts with specific data that is then transformed into categories and themes. Repetition of terms and typologies may assist in generating analytic patterns or themes (Ryan & Bernard, 2003). Thematic analysis identifies patterns or themes in dataset. Themes are important for the description of a phenomenon and are associated to a specific research question. This study is guided by two main research questions: semiotic challenges and instructional strategies. The researcher used his own judgement to determine the themes.

Thematic analysis was utilised as a categorising strategy for qualitative data. All the emerging themes from the learners and teachers' typologies were further analysed to discover patterns and developing new themes. The researcher uses words and phrases that serve as labels for the groups of themes that can be grouped together to form a common theme.

INITIAL THEMES

Table 4-14: Initial themes

Theme	Description	Category
T1	Pedagogical strategies for mathematics teaching and learning exclude reading	Reading Mathematics Symbols
T2	Mathematical symbols assume different meanings in different contexts and therefore confuse learners	Problem Context and multiple meanings
T3	Learners tend to cling to procedurally driven symbolic approaches, which are less flexible and imposes greater cognitive strain on them.	Problem solving
T4	Learning mathematical symbolisation requires time and is frustrating if the time invested yields no understanding of mathematical concepts.	Time allocated for mathematics practice
T5	Time (4.5hrs) allocated per week (CAPS) for teaching and learning mathematics is inadequate.	
T6	Learners lack symbol sense and algebraic insight required for problem-solving.	Symbols sense and Problem solving
T7	Unfamiliar notation jeopardise conceptual understanding and problem solving	
T8	There are too many symbols to learn in mathematics and recalling them is difficult.	
T9	Mathematical symbols allow concepts to be expressed compactly and are used to present the solution process	The importance of symbols enhance problem solving
T10	Mathematical symbols are mistakenly understood as concepts	
T11	Teachers' need to improve their PCK for linking symbols, concepts and meanings during problem solving.	Instructional Strategies
T12	Timeous introduction of mathematical symbols enable learners to operate with abstract concepts in future mathematics.	Timeous introduction of symbols
T13	Learners need guidance and support to develop fluency in reading mathematical symbols and concepts based on their current grade level.	Reading materials and Pedagogy
T14	Teachers do not adequately address learners' problems but instead accelerate them to complete the syllabus without conceptual understanding.	Instructional Strategies and Timing
T15	Symbols have different meanings in different context, hence the need for precision.	Problem Context and Multiple meanings
T16	Learners take time to grasp new concepts but work schedules are too congested and fast-paced.	Timing
T17	Some mathematical symbols are purely abstract and cannot be linked to real–life experiences.	Pedagogy and Symbolisation
T18	Mathematical symbolism should be timeously introduced after conceptual understanding.	Instruction and Prior knowledge
T19	Representational tools should be carefully selected to support learners' conceptual understanding and problem-solving.	
T20	Learners should grasp the mathematical concept before it can be symbolised, thus, conceptual understanding should precede symbolisation.	
T21	Learners' prior knowledge of mathematical symbols and concepts must be established and used as a basis for new knowledge.	
T22	Learners do not assimilate mathematical knowledge at the same pace.	Teaching and learning resources
T23	Textbooks should explain unfamiliar concepts and symbols to cater for individual reading and learning.	

Reviewing Themes: Second Phase Themes

This phase involves the refining and reviewing of themes. In this phase, the researcher searches for data that supports or refutes the proposed themes. The researcher further expands and revises the themes as they emerge. The phase also involves reworking of the initial themes to make sure that they suit the categories in which they are classified. Some of the themes were collapsed into each other; while other themes were condensed into smaller units or regrouped to form a new theme common to the previous themes. Overlapping themes were identified and connected or lead the researcher to examine the possibility of creating new patterns and dimensions on the data.

4.12.1 Themes Emerging

Two broad sets of themes seem to emerge from further analysis of final themes: semiotic challenges and instructional challenges. Semiotics challenges are concerned with meaning; representation, sense (language, images, and objects) or the processes by which mathematical meaning is attributed. Instructional challenges are difficulties or obstacles that hinder learners and teachers as they attempt to achieve specific learning outcomes. The main themes emerging from semiotic challenges are reading mathematical text, mathematics pedagogy, under-developed symbol sense, limited time for mathematics teaching and learning and the context in which symbols are used during teaching and problem solving. Timing and pedagogical limitations constitute instructional challenges. Lack of reading proficiency, under-developed symbol sense and inability to contextualise mathematical symbols and contexts were the major challenges experienced by learners while teaching and learning situations characterised by insufficient time and poor pedagogical strategies compounds learners' understanding of mathematical concepts together with their symbols. Teaching strategies for dealing with the above-mentioned difficulties can be derived from challenges identified.

Theme 1: Reading mathematical text

Table 4-15: Reading mathematical text

T_i	Item	Categories	Theme
T1	Pedagogical strategies for mathematics teaching and learning exclude reading	Reading mathematics Symbols	Reading math text is challenging
T7	Unfamiliar notation jeopardise conceptual understanding and problem solving	Instructional Strategies for Symbolisation	
T15	Symbols have different meanings in different context, hence the need for precision.		
T23	Textbooks should explain unfamiliar concepts and symbols to cater for individual reading and learning.	Reading materials	
T12	Textbooks use symbols without giving thorough explanations and clear examples.		

Being able to think mathematically is reflected by the ability to read and comprehend mathematical symbolism. Proficiency in reading mathematical text entails the ability to pronounce, verbalise and having base knowledge to derive meaning from what is being read. Reading helps learners to verbalise, and assign meaning to abstract mathematical symbols. Barton, Heidema and Jordan (2002) state that learning to read mathematics is essential in understanding the meaning of the problem and being able to implement a solution effectively. Reading shares some common elements with symbol sense and mathematical context since both require a working knowledge of the interaction of numeric discrete skills. From this theme, teachers should try to help learners to read and interpret mathematics text. Learners need to be taught how to read for understanding, that is, interact with text, and interpret text and to reason. Teachers should be trained in reading instruction and should recognise literacy as part of their skills set.

Mathematics teachers should take recognisant of the need to train learners to read mathematical sentences: equations and inequalities. Reading, writing and mathematics should be inseparable. Reading should be incorporated in mathematical instruction. Incorporating reading in the mathematics class enhances learners to use the symbolic language to focus and work through problems and communicate ideas coherently. In order to succeed in mathematics should be able to read and understand the language of mathematics. The teacher should possess the skills to help the learners to understand ways of interpreting mathematics text (formula, textbook and symbol), and understand it. Teachers need to encourage learners to "read, create, use, and comprehend numerous mathematical representations as a way of demonstrating mathematical literacy. For a learner who is struggling with both mathematics and reading, it is advisable to focus on reading skills first which ultimately may fix both problems.

Theme 2: Context of mathematical symbols and words

Table 4-16: Context of mathematical symbols and words

T_i	Item	Categories	Theme
T2	Mathematical symbols assume different meanings in different contexts and therefore confuse learners	Reading Mathematics Symbols	The context in which some mathematical symbols are used confuses learners
T3	Learners tend to cling to procedurally driven symbolic approaches, which are less flexible and imposes greater cognitive strain on them.	Problem solving	
T15	Symbols have different meanings in different context, hence the need for precision.	Problem Context and Multiple meanings	
T17	Some mathematical symbols are purely abstract and cannot be linked to real–life experiences.	Pedagogy and Symbolisation	

The context in which mathematical symbols are used emerged as one of the difficulties that learners experience with mathematical symbols. Though there are on-going debates about whether a context makes a problem easier or harder for learners, it has emerged from this study that the contexts in which mathematical symbols are used have a strong bearing on learners' conceptual understanding. Educators should take note of the context in which some mathematical symbols and words are used by learners and teach them in their correct mathematical situations. Symbols and words with multiple meanings in mathematical texts were cited as hindrances to learners' conceptual understanding and achievement levels in solving word problems. Teachers suggested that they should create learning environments where learners can maximally exploit authentic experiences to investigate and understand formal representations. They also recommended that symbolisations should go hand in hand with the development of the mathematical conceptualisation of the problem situation. On one hand, the symbolisation derives its meaning from the situation that it describes. The form or situation in which a mathematical concept is perceived is highly influenced by the symbols through which the situation is viewed. This scenario requires teachers to take a dynamic activity-oriented view of learning, which encourages symbolisations and meaning- making to co-evolve in a dialectic process.

Theme 3: Symbol Sense

Table 4-17: Symbol Sense

T_i	Item	Categories	Theme
T6	Learners lack symbol sense and algebraic insight required for problem-solving.	Symbols sense & Problem solving	Learners lack Symbol sense
T7	Unfamiliar notation jeopardise conceptual understanding and problem solving		
78	There are too many symbols to learn in mathematics and recalling them is difficult.		
T9	Mathematical symbols allow concepts to be expressed compactly and are used to present the solution process	The importance of symbols enhance problem solving	
T10	Mathematical symbols make mathematics even more complicated since the concept and its symbols need to be learned and understood as if they are separate entities.		
T15	Symbols have different meanings in different context, hence the need for precision.	Problem Context and Multiple meanings	
T19	Representational tools should be carefully selected to support learners' conceptual understanding and problem-solving.	Pedagogy, problem – solving & symbol sense.	

Symbol sense emerged as one of the difficulties that learners experience with understanding mathematical concepts. It refers to the ability to select the correct notation and symbols to represent a mathematical situation and to judge when and where the use of certain symbols or representation is appropriate. Learners indicated they struggle to use and make sense of symbols in different mathematical contexts. The development of reading skills for symbols and acquiring their meanings require careful thought and attention during teaching and learning. Teachers should encourage learners to desist from jumping to symbols before understanding the problem. Learners should be encouraged to make sense of the problem, draw a graph or a picture, and describe what they see and to reason about it. Teachers should select learning materials and classroom practices that nurture the search for symbols and their meanings. The study indicates that learners have misconceptions about the use of symbols which have an effect on their understanding of mathematics concepts. Learners lack strong symbolic and conceptual understandings in order to make conceptual sense of the mathematical ideas.

Theme 4: Timing

Table 4-18: Timing

T_i	Item	Categories	Themes
T4	Learning mathematical symbolisation requires time and is frustrating if the time invested yields no understanding of mathematical concepts.	Time allocated for mathematics practice	Time allocated for learning is inadequate.
T5	Time (4.5hrs) allocated per week (CAPS) for teaching and learning mathematics is inadequate.		
T12	Timeous introduction of mathematical symbols enable learners to operate with abstract concepts in future mathematics.	Timeous introduction of symbols	Some mathematical symbols and concepts are not timeously introduced.
T14	Teachers do not adequately address learners' problems but instead accelerate them to complete the syllabus without conceptual understanding.	Instructional Strategies & Timing	
T16	Learners take time to grasp new concepts but work schedules are too congested and fast-paced.	Timing	
T18	Mathematical symbolism should be timeously introduced after conceptual understanding.	Time allocated for individual mathematics practice	

There are two aspects pertaining to timing as observed from the findings of this study. Firstly, timing refers to the stage at which some mathematical concepts and symbols are introduced during the teaching and learning phase. Mathematical symbols should be introduced after learners have understood the mathematical concepts. Symbols should be introduced to enhance problem solving and not as means for conceptual understanding.

Sullivan (2012) also made a claim that symbols develops over time. A significant amount of time needs to be availed for learners to become familiar and comfortable enough with mathematical symbols to extract meaningful from them (Arcavi, 1994). The same sentiments were echoed by Gray and Tall (1994) who argue that it takes time working with new content for learners to step back and reason about the symbolic representation in a conceptual manner. Webster et al (2012) argued that the ability to recognise and process symbols develops over time, and becomes a fundamental tool in mathematics education.

Research in cognitive psychology has shown that the mind processes letters and symbols differently. Tydgat and Grainger (2009) demonstrate that reaction times vary when learners attempt to recognise various images in an array of letters, symbols or digits. They showed that learners manipulate symbols differently from letters and digits, and the processing of symbols take longer than that of letters and digits. Understanding these differences may help explain why learners find symbols intimidating in mathematics classrooms (Cobb et al., 2000). An increased cognitive load caused by processing the symbols could contribute to the challenges learners face working with mathematical symbols. Berends and van Lieshout (2009) showed that the additional cognitive load caused by accompanying illustrations had a detrimental effect on both the speed and accuracy of learners' performance in solving arithmetic word problems. It is possible that the additional load required to process mathematical symbols, could also distract learners from focusing on the underlying mathematical concepts.

Secondly, timing refers to the time allocated for teaching and learning specific mathematical concepts. The study found that the time allocated for mathematics teaching and learning is not adequate. An examination driven curriculum tends to promote procedural learning at the expense of conceptual understanding. Teachers rush the syllabus and teach superficial content to make learners ready for examinations. Teachers are frustrated by the pressure to “teach to the test,” due to fear of non-proficient scores, complaints from parents and school administrators when learners fail. Capraro and Joffrion (2006) support the claim that more time should be spent on the meaning of a symbol as opposed to developing procedural familiarity with the symbol itself. If the individual symbols have meaning, then the learners will better understand how to perform the process for which the symbol represents.

Theme 5: Pedagogical strategies for teaching mathematical symbols

Table 4-19: Pedagogical strategies for teaching mathematical symbols

T_i	Item	Categories	Theme
T11	Teachers' have PCK for linking symbols, concepts and meanings during problem solving.	Instructional Strategies	
T13	Learners need guidance and support to develop fluency in reading mathematical symbols and concepts based on their current grade level.	Instructional Strategies	Pedagogical strategies for teaching mathematical symbols
T14	Teachers do not adequately address learners' problems but instead accelerate them to complete the syllabus without conceptual understanding.	Instructional Strategies	
T17	Some mathematical symbols are purely abstract and cannot be linked to real–life experiences.	Pedagogy and Symbolisation	
T18	Mathematical symbolism should be timeously introduced after conceptual understanding.		
T19	Representational tools should be carefully selected to support learners' conceptual understanding and problem-solving.		
T20	Learners should grasp the mathematical concept before it can be symbolised, thus, conceptual understanding should precede symbolisation.	Instruction and Prior Knowledge	
T21	Learners' prior knowledge of mathematical symbols and concepts must be established and used as a basis for new knowledge.		

Classroom teaching approaches should be modelled around a teaching framework that allows learners to create meaningful connections between concrete, representational and abstract levels of thinking and understanding. Hands-on experiences allow learners to understand how numerical symbols and abstract equations operate at a concrete level. Learning should start with visual, tangible, and kinaesthetic experiences to establish basic understanding, and progress to pictorial representations (drawings, diagrams, or sketches) and finally the abstract level of thinking, where mathematical symbols are exclusively used to represent and model problems. Mathematical symbolism and mathematical understanding are intertwined, but meaning must generally precede symbolisation. Teachers should highlight the need for learners' need to understand the value of being fluent with mathematical symbols. Learners should be fluent in reading, writing, verbalising symbols and attaching correct meanings to them.

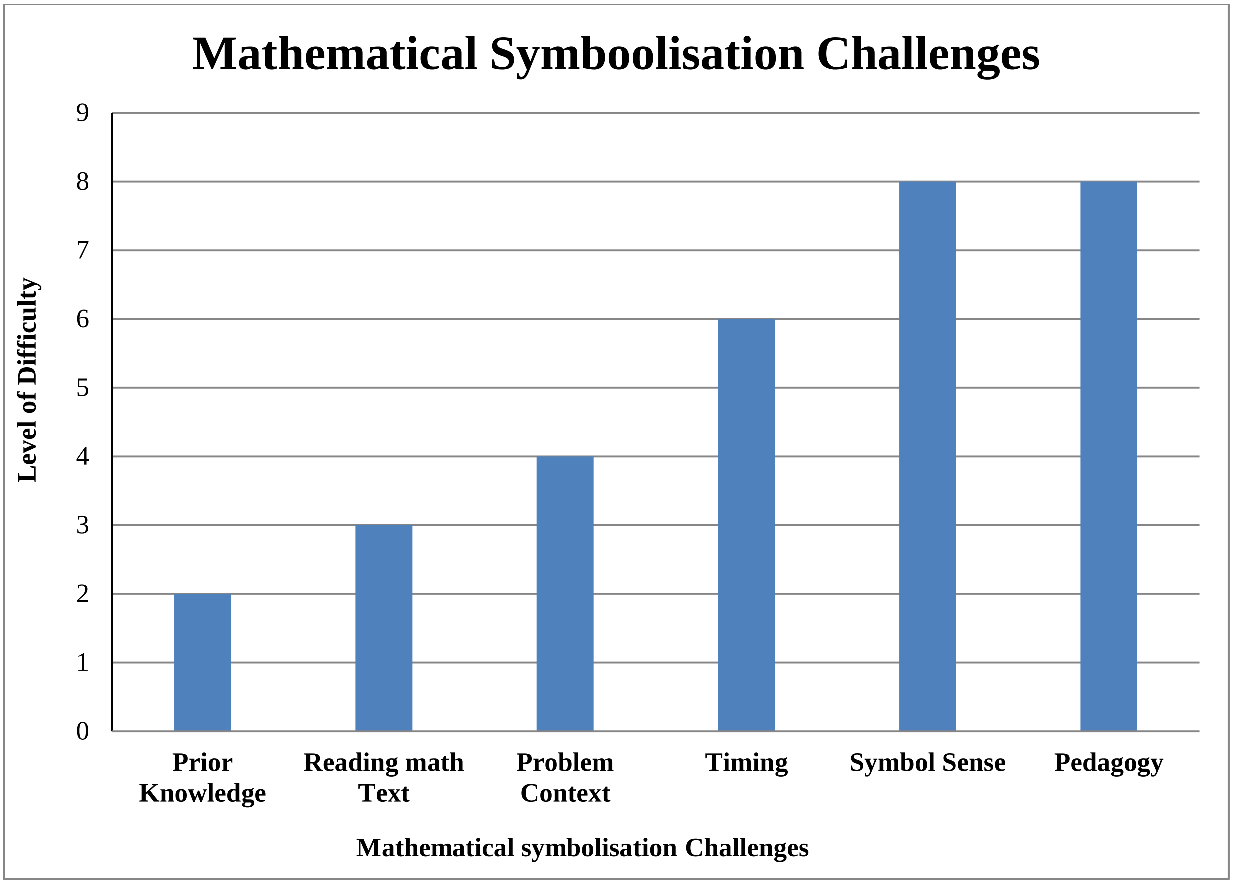


Figure 4-16: Symbolisation Challenges

Figure 4-16 summarises the mathematical symbolisation challenges experienced by learners. Six (6) main challenges emerged from the analysis. Pedagogy and symbol sense emerged as the dominant challenges. Current pedagogical or instructional approaches used by teachers do not adequately support development of learners' mathematical symbolisation. Mathematics teachers should use symbols precisely and consciously attend to symbols during classroom instruction. Learners fail to grasp mathematical concepts because teachers do not provide explanations for the meanings and proper uses of the symbols. Teachers should avoid emphasising symbolism without understanding the relations it represents.

Learners' lack symbol sense obscures them from understanding and decoding mathematical concepts during teaching and learning situations. Learners struggle with interpreting and using mathematical symbols in describing mathematical situations and problem solving procedures. Learners also have limited choice of symbols, limited skills of manipulating symbols, and limited choice of symbols based on the context. Thus, teaching or learning mathematics with understanding requires learners to be fluent and flexible in their use of mathematical symbols. At times learners fail to distinguish symbols as objects or ideas and as processes.

Another challenge of mathematical symbolisation emerging from the analysis of data is timing. Mathematical symbols should be timeously introduced after learners have acquired the mathematical concepts. As recommended by Chirume (2012) teachers should introduce symbols only after a satisfactory explanation of the concept has been given. Learning mathematics by symbolisation requires time, but the time set aside for mathematics in the South African mathematics curricula is inadequate for learners to become fluent at using the notation. The process of linking symbols to meanings is complex and cannot be learnt in a once-off experience. The transition from arithmetic to algebra takes times and requires a wise teacher to transform and change learners' orientation from concrete to abstract algebraic reasoning.

The context in which symbols are used emerged as one of the challenges of mathematical symbolisation. The contexts in which some symbols are used confuse learners. Teachers should emphasise situations and mathematical contexts in some of the symbols are used.

The reading of mathematical text also emerges as one of the challenges of mathematical symbolisation. Learners lack skills of verbalising the symbols in a text. Mathematics teachers do not consider reading as part of their classroom instruction. Reading a mathematics text requires specific strategies unique to mathematics. Teachers compensate for this gap by rewording or interpreting mathematics problems for their learners. They do not devote time to work specifically on reading and interpreting mathematics text. Mathematics text is compact, dense contains and has little redundancy. The text is a mixture of numeric and non-numeric symbols that reader needs to decode. The readers also need to interpret tables and graphics to make sense of text. Reading mathematics is also a challenge since many mathematics textbooks are written above the grade level of learners. Prior knowledge is also an essential requirement for reading text. Learners indicated that they have insufficient prior knowledge to make inferences required to construct meaning from text. Thus, before introducing new concepts, teachers need to establish level of prior knowledge and use it as a baseline for what needs to be taught or read.

Table 4-20: Final phase themes

Sub-Theme	Description	New Theme
2	Context of mathematical symbols	The design of teaching and learning instruction should consider the context of mathematical symbols and timing
4	Timing	
5	Teaching and learning styles	
1	Reading mathematical text	Instruction should include reading of mathematical text to improve symbol sense.
3	Symbol Sense	
6	Prior knowledge	

Table 4-20 shows the final two broad sets of themes emerging from the final analysis of themes: semiotic challenges and instructional challenges. Semiotics challenges identified were reading mathematical text and symbol sense. These challenges involve meanings; representations, sense (language, images, and objects) or the processes by which mathematical meaning is attributed. Instructional challenges are difficulties or obstacles that hinder learners and teachers as they attempt to achieve specific learning outcomes. The main instructional challenges emerging from the analysis are limited time for mathematics teaching and learning and the context in which symbols are used during teaching and learning as well as teaching and learning styles. Lack of reading proficiency, under-developed symbol sense and inability to contextualise mathematical situations and contexts were the major challenges experienced by learners while teaching and learning situations characterised by insufficient time and poor pedagogical strategies compounds learners' understanding of mathematical concepts together with their symbols. Teaching strategies for dealing with the above-mentioned difficulties can be derived from challenges identified.

4.12.2 Validating the themes

The validation of themes was informed by Patton's (1990) dual criteria for judging categories: internal homogeneity and external heterogeneity. In order to comply with this dual criterion, the researcher screened within and across themes and ensures that they cohere together meaningfully, while there should be clear and identifiable distinctions between themes. The researcher assessed how accurately the emerging themes reflect the experiences of the participants. To do that the researcher read the data several times to determine if the themes capture all the concerns from participants.

4.13 Summary

Data were collected in order to explore the difficulties learners and teachers experience with mathematical symbols during teaching and learning and the possible instructional strategies to remove or reduce the effects of symbolic obstacles. Data were collected using questionnaires and semi-structured interviews. SPSS version 23 was used to analyse close- ended questions by utilising cluster analysis while typological and thematic analyses were used to generate themes from open-ended questions. A thematic analysis of participants' responses highlighted factors that had a positive or negative impact on learners' understanding of mathematical concepts through symbolisation. Data have shown that demographic variables such as grade, gender and residential area have significant effects on learners' understanding of mathematical symbols. Learners indicated that they experience the following symbolisation challenges: switching representations from one form to another, they struggle to do mathematics tasks due to unfamiliar symbols and features used in the task, symbols in a mathematical problem affects the solution strategy, lack of symbol sense and the task of learning concepts and symbols at the same time. However, participants indicated that they do not experience challenges with the following aspects of mathematical symbolisation: symbol pushing (learners ask their teachers for clarification), textbook language and symbols at each grade level are appropriate and use of mathematical symbols to understand Algebra. Learners prefer an instructional approach in which teachers effectively link new concepts and symbols to what they already know, taught them at a reasonably slow pace, prefer to be given tasks where they demonstrate understanding and understanding concepts before symbolising them.

Teachers demonstrate knowledge of mathematical content and pedagogical techniques at the grade level they are teaching. However, most teachers do not integrate these two domains of knowledge effectively due to limited time allocated for the topics on work schedules. More specifically, most teachers do not present the learners with a well-sequenced series of activities that help learners acquire the underlying mathematical concept. Teachers revealed that there many cognitive gaps between primary and secondary school mathematics. They indicated that there is too much reliance on concrete teaching at the expense of algebraic teaching where symbols can be utilised. Learners are restricted to the concrete level of mathematical understanding. Teachers recommended that learners should be exposed to symbolic language at an early age and a variety of representations ranging from tables to graphs should be part of a learner's mathematical development tools.

Teachers recommended a teaching approach in which concepts are conceptualised before being symbolised. Learners' prior knowledge should be invoked and teaching should proceed from known to unknown. Teachers also indicated that learners do not read textbooks to understand mathematical concepts; instead, they see that as the responsibility of the teacher to read and interpret text for them. To this effect, teachers need to equip learners with the skills to use a mathematical text as a learning resource. The next chapter discusses of the research finding and implications.

CHAPTER 5: DISCUSSION AND IMPLICATIONS

This chapter discusses the findings emerging from the analysis of data. The chapter also discusses the implications of these findings on classroom practice, mathematics education and research. The study sought to gain insights into the ways learners perceive mathematical symbols with a view to identify ways of strengthening their understanding of symbols and ability to solve mathematical problems. The study also sought to raise teachers' awareness of how learners develop understanding of mathematical concepts through symbolisation. It also investigated the ways in learners use symbols to understand mathematical concepts and processes. The chapter also presents the limitations of the study.

5.1 Discussion

Cluster, typological and thematic analysis were the main strategies used to analyse the data for this research. Themes related to learners' challenges with mathematical symbols were derived from the researcher' interpretation of learners' and teachers' experiences. Clusters analysis was used to group the emerging views from participants. The themes, which emerged from the participants' responses, are discussed and contrasted by findings from other researchers. The research was guided by the following research questions:

a) What challenges do secondary school learners encounter with symbols when interpreting and solving mathematical problems?
b) What instructional strategies do mathematics teachers use to remove or reduce the effects of mathematical symbolisation obstacles?

5.1.1 Challenges of mathematical symbolisation

The first research question focused on the challenges experienced by educators and learners when teaching and learning mathematical concepts through symbolisation. The research question sought to enquire about the challenges of mathematical symbolisation from the perspectives of teachers and learners. The participants expressed the following views:

- Learners do not understand mathematical concepts due to symbols that are used during teaching and learning.
- Symbols and formulae in textbooks are confusing.
- Learners cannot utilise textbooks as learning resources due to reading challenges that originate from unfamiliar symbols.
- Learners face the challenge of translating word-problems into symbolic sentences.
- Mathematical symbols negatively affect learners' understanding of mathematical concepts.
- Learners face the challenge of establishing signifier and signified relationships
- Learners' informal conceptions contradict with formal conceptions of mathematical processes and conceptions
- Symbols in a mathematical problem affect learners' goals, methods, and at times, they struggle to initiate the solution process.
- Learners face the challenge of switching representations (symbols) during problem solving.

Learners indicated that mathematical symbols present them with barriers to conceptual understanding. Mathematical symbols can be confusing and can act as real barriers to learning and understanding basic mathematical concepts. This finding is consistent with Chirume (2012) who argued that most learners fail to understand mathematical concepts due to inability to interpret or decode the meaning of math symbols. Bardinia and Pierce (2015) made the same conjecture, adding that the increase in symbol load due to unfamiliarity and increased density may cause learners to lose confidence and subsequently choose a study path that minimises their need for mathematics. Mathematics derives much of its power from the use of symbols (Arcavi, 2005), but research at secondary school level has shown that their conciseness and abstraction can be a barrier to learning (Kilhamn, 2011; Bardini & Pierce, 2015).

The findings are also similar to those of Torigoe and Gladding (2007) who found that learners' performance in mathematics is highly correlated to their understanding of symbols. The issues of reading, recognising and understanding symbols underpin all mathematics topics. Serfati (2005) recommended that learners should familiarise themselves with the following attributes of mathematical symbols: materiality (what it looks like), syntax (how it is combined with other symbols), and meaning. The materiality of a symbol focuses on its physical attributes (what it looks like), including the category to which the symbol belongs (a letter, a numeral and a specific shape). The syntax of a symbol relates to the rules it must obey in the symbolic writing. The meaning of the symbol is the concept being conveyed (the representation of an unknown, of a given operation). In order to understand mathematical notations, learners have to take into account both the syntactical aspect of a symbol and the underpinning mathematical concept(s) conveyed.

Translating word problems into symbolic forms is one of the difficulties highlighted by learners in this study. Learners indicated that they lack proficiency and have limited understanding of letters, variables and objects. As noted by Verzosa and Mulligan (2013), the process of translating words to symbols is the first step towards solution processes. This is the the critical stage of problem solving. Word problems require that learners read and comprehend the text, identify the problem that needs to be solved and select appropriate mathematical symbols to solve the problem. Learners in this study indicated that they have difficulties in reading and comprehending the written content expressed as word problems. Word problems require learners to apply the knowledge of concepts that they have learned in the topic. Real life contexts dominate word problems and are usually discussed after mathematical concepts have been understood. On the other hand, word problems may be taught in context, that is, they may be used to teach a mathematical idea or process in a context familiar to learners' background.

Learners also indicated that they have difficulties in reading symbols and formulae in textbooks. They indicated that textbooks are confusing and use unfamiliar notation. Before studying a mathematics topic, it is mandatory to familiarise learners with a variety of symbols in that topic (Sepeng & Madzorera, 2014). In mathematics, concepts are generally learned in a sequence. Reading a mathematics textbook is like reading a foreign language since mathematical language is not used in everyday communication. Mathematics text is overloaded with symbols that are complex, have multiple meanings and learners need to decode the context in which these symbols are used. Mathematics is a language that has its own language structure, symbols, definitions, and theorems. Learners and teachers should strive to be as precise as possible. Lack of precision often leads to confusion and frustration if one is not familiar with what the different symbols mean. One way of reading mathematics text suggested by Fan and Kaeley (2000) is to go through the worked examples. This could be a great way to solidify one's understanding of a concept. However, this is possible if the learner has adequate prior knowledge of the text.

Another challenge expressed by learners is that mathematical symbols negatively affect their conceptualisation of mathematics concepts. The conceptualisation of mathematical concepts begins by making personal meaning of a defined mathematical object. This is derived from the study of advanced mathematical thinking (Tall, 2006). Mathematical objects are understood through the various symbols of the definition. Berger (2006) recommended that learning occurs by manipulating and using knowledge of previously learned objects to form new actions. These actions become incorporated into processes hence, objects. Processes and objects are then converted into schemas. Communicating with peers and with teachers using new mathematical object (symbols) gives initial access to the new objects (Anthony & Walshaw, 2009). According to Berger (2005), symbols focus–one's attention on selecting distinctive features and analysing and synthesising a new concept.

The same observation was made by Rojano (2002) who revealed that differences in meaning of the same symbols and symbol chains present difficulties for secondary school learners when learning algebra. Schleppegrell (2007) also made a similar claim that learners' difficulties with symbols also depend on the contexts in which the symbols are taught, manipulated in mathematics classes, and on the teachers' choices of mathematical tasks.

Another semiotic difficulty revealed by teachers during teaching was the informal conceptions that learners have about some mathematical symbols. These informal conceptions contradict with formal conceptions of some mathematical symbols. Learners indicated that connecting informal and formal reasoning is difficult. Learners tend to take time to change from their informal to formal ways of thinking. Most learners indicated that their approaches to a number of mathematical concepts and use of certain symbols do not agree with the formal conceptions. Learners indicated that it takes time to adapt to new symbols especially in a new topic. New knowledge, concepts, symbols and ideas always contradict their informal ways of thinking. The same observation was made by Viholainen (2008) who examined informal and formal understanding of the concepts of a derivative and differentiability. He reported that connecting informal and formal reasoning is difficult for the learners. This study, like other several studies revealed that crossing the cognitive divide between informal and formal representation systems is difficult for many learners (Koedinger, Alibali & Nathan, 2008). This inability restricts learners' reasoning. In order to improve this, the teaching of mathematics should support the development of the coherence of learners' knowledge structure. Among other things, it should strengthen the understanding of connections between informal and formal representations. This recommendation is also consistent with Lesh and Sriraman (2005) who argues that mathematics should be considered firstly as a formal, deductive rigorous body of knowledge and secondly as a human activity in which informal actions leads to some mathematical processes.

Despite the formal nature of mathematical knowledge, Viholainen (2007) argued that learning mathematics is informal. In the process of inventing new mathematical ideas and applying mathematical processes, a learner needs to reason intuitively, making associations and building mental images of the process. This description is the informal part of mathematics. This is consistent with the recommendation made by Weinberg, Wiesner and Pfaff (2010). They suggested that learners should develop an informal understanding of the ideas that underlie inference before learning the concepts formally. Thus, teachers can tap into this informal understanding to build formal conceptions.

Another challenge experienced by learners in this study is that symbols in a mathematical problem affect their goals, method, and their initiative to solve mathematical problems. Learners indicated that they struggle to initiate solution processes to problems due to lack of symbol sense. Although this is a challenge for learners, it is one of the aims of teaching mathematics. The South African Department of Basic Education (DBE, 2011) recognises mathematics as a language that uses symbols and notations for describing numerical, geometric and graphical relationships.

Solving mathematical problems requires learners to have a strong symbol sense and conceptual understanding (Duval, 2006). However, there are challenges with regard to problem solving in South African mathematics classrooms. Teachers generally complain that the South African mathematics curriculum is overloaded and there is very little or no time to pursue real problem solving during mathematics lessons (Govender, 2015).

Ganal and Guiab (2014) suggested that certain pedagogical misconceptions on critical thinking and problem solving need to be clarified before learners are engaged in problem solving. This includes prescribing rules such as finding key words, symbols and concepts in a problem which assist in determining the appropriate operations or algorithm for a given problem. Teachers should bear in mind that developing critical and analytical thinking through problem solving takes time and requires commitment, passion and dedication from the teacher (Limjap, 2001). Rai, Khan and Chauhan (2014) suggested the idea of situated cognition in which the learner has to actively participate in the formation of mathematical concepts. Learners should not passively receive knowledge from the teacher but should be actively involved in the construction of knowledge. Rai, Khan, Chauhan and Chauhan (2014) suggest that teachers should apply the constructivist theory of learning to enhance learners' thinking skills that are best developed within a constructivist framework.

When solving mathematical problems learners need to develop a cognitive schema called problem-type schemata. According to Zodik and Zaslavsky (2008), learners need to gather information and knowledge about the problem such as the underlying principles, symbols, concepts, relations, procedures, rules, operations and so on. Problem-type schemata are acquired through some inductive or generalization process involving comparisons among similar or analogous problems of one type. Learners should be able to represent, categorize and associate problems to be able to determine the appropriate solution. The teacher's schematic processing leads to an accurate analysis of the problem that the novice hardly achieves. Limjap (2001) claims that learners' schemata include mainly typical surface-level information associated with a problem type such as form or symbols, whereas expert's schemata include mainly statements of abstract principles that are relevant to the problem type. Learners often struggle to recognise the mathematical information in the material being read, thus hindering the processing of information that leads to the correct solution. This means they are not able to make a schema or a visual form of the concept.

Learners and teachers in this study identified switching representations and symbols during problem solving as one the challenges. Switching representations from a graphical to algebraic and vice-versa was identified as a major challenge. Participants reported that the use of multiple representations to denote the same mathematical concept is confusing. Representation is a term used to denote the form with which we present mathematical objects and processes, and which we find essential for defining, explaining, visualising, recording and communicating mathematics knowledge. The learner's ability to represent mathematical concepts in different forms is a measure of power of symbolisation and abstraction. In this study such abilities are linked to the main conceptual framework of this study; symbol sense. Rachman and Levesque (2004) acknowledged the importance of representation by pointing out that it is impossible to study knowledge related phenomena without recourse to the notation of representation. He emphasised that no knowledge can be mobilised by a learner without activating representation. Learners revealed that they lack the abilities to switch from one representation to another whenever the other is more efficient for the next step one wants to take (another instance of symbol sense). Translation between representations and transformation within the representations are other important skills that learners in this study lack.

Mathematics problems employ three different symbol systems namely: symbolic, graphic and table. It is important for mathematics problems to be presented in all three ways in order to understand the problem structure. Symbolic representation is a quantitative representation of a problem that solving a problem is a procedure that has to be memorised, practiced and habituated, and it emphasises answer getting not meaning making. Understanding the deep structure of a problem relies on qualitative (semantic) representation of the problem. Anthony and Walshaw (2010) warn teachers that if learners are taught mostly using symbols, which is a quantitative problem representation approach; they will lack the deep-level structural characteristics of mathematics.

5.1.2 Teachers' role in the reading of mathematical text

Teachers do not appreciate that reading a mathematics text is essential. Reading a mathematics text is different from other types of reading. It requires special reading skills and strategies unique to mathematics. Learners indicated that they rely on their teachers for interpretations of problems. Teachers play the role of an interpreter and reader in a mathematics class. They read and help learners to interpret mathematics text as well as leading discussions on problem-solving strategies. Many mathematics textbooks are written above the grade level for which they are intended (Carter & Dean, 2006). One strategy of solving this problem is to encourage learners to model their thinking aloud as they read and figure out what a problem is asking them to do (Funke, 2010). Another strategy suggested by Taplin (2006) involves dialoguing with learners about any difficulties they may have in understanding a problem. The National Council of Teachers of Mathematics (1996) states that,

"... because mathematics is so often conveyed in symbols, oral and written communication about mathematical ideas is not always recognised as an important part of mathematics education. Learners do not necessarily talk about mathematics naturally; teachers need to help them to do so" (p. 60).

If learners are to understand mathematical concepts rather than produce specific performances, they must engage meaningfully with mathematics texts. Reading a read mathematics text is a transaction in which the reader is required to unpack the ideas that the text presents (Adams, Pegg & Case, 2015). The interpretation of the reader depends largely on meaning that reader' draws from the text based on his /her prior knowledge of the information (Draper, 2002). Learners need explicit scaffolding experiences to help learners connect the text and new symbols to their prior knowledge and to build new knowledge.

5.1.3 Fostering mathematical symbolism

It is important to understand the crucial role that mathematics teachers play in defining, approaching and conveying the meanings of various mathematical symbols to learners. According to Boulet (2007), teachers should strive to foster the symbolic feature of mathematics; though they should be very cautious about how they approach the symbols of mathematics. Symbols themselves bear no meaning nor signify any purpose until someone endows such meaning or purpose through relational conveyance. In mathematics, teachers are the agents of endowment. Teachers often depend on their own expertise, background experience and textbook recommendations to assign meanings to symbols. However, researchers noted that the assignment of such meanings requires deeper thought and analysis (Monroe & Orme, 2002; Phillips, 2008). Radford (2000) asserts that mathematical symbols do not possess meaning until they are mediated into suitable reference contexts. Askew (2015) observes that learners tend to construe mathematical symbols as mathematics if the link between a symbol and a mathematical concept is not well established. For example, the number "6" is arbitrary unless it is associated with six physical objects such as 6 counters.

5.1.4 Instructional Strategies for Mathematical Symbolisation

Instructional strategies for teaching mathematical symbolisation can be classified into two main categories: syntactic and semantic. If the aim is to teach the learner the syntactic or convention, then teaching should aim at showing the new symbol, demonstrate how it is written and used, and give learners a chance to say it, read it, write it, and practice its use. In most cases teachers concentrate on the semantics. In this approach symbols need to be attached to referents that are already meaningful to learners (Drews, 2007). Learners need to be engaged in a variety of meaning-making activities: appreciating the purpose of the symbolisation, thinking aloud so teachers may understand how learners interpret symbols; translating symbols into words, diagrams, word problems (Goldin, 2003); confronting theirs and other learners' errors and "debugging" those productions; contrasting similar but distinct expressions; and others.

Learners indicated they are rarely taught how to read, verbalise and write mathematical text. This is often overlooked during instruction. K'Odhiambo and Gunga (2010) recommend that teachers need to encourage learners to read, write, and verbalise mathematical terms and symbols. These skills are important for learners to understand and communicate mathematical ideas. Learners must appreciate the value of being fluent in the language of mathematics. Learners should be comfortable in the use of mathematical symbols in order to gain the most from their mathematical education experience. Diagramming as a teaching strategy is helpful in assisting learners to speak, read, and write the language of mathematics is. Liu, Chen and Chan (2010) suggest that the use of diagrams as symbols can be utilised to make connections between different mathematical vocabularies and representations.

Teachers indicated that learning environments are not conducive for learning. Most teachers complained of overcrowded classes. Teachers reported that they find it difficult to evaluate learners' learning, assessing weaknesses and strengths, and ideally prescribe the proper strategies to achieve optimal learning. The National Council of Teachers of Mathematics (NCTM, 2000) recommends teachers to create a classroom environment where learners learn mathematics by being engaged in discussions that enhances their articulation of mathematical concepts. Driscoll (2013) argue that real understanding occurs when learners are able to use their prior knowledge and to understand new situations. Learners who understand concepts well, are able to grasp subsequent concepts more efficiently (Alagic, 2003) while learners with learning gaps struggle in this endeavour. According to Alagic (2003) learners who have mastered concepts demonstrate understanding by:

> *"…being able to carry out a variety of actions or performances with the topic by the ways of critical thinking: explaining, applying, generalizing, representing in new ways, making analogies and metaphors" (p. 384).*

For this reason, it is important for teachers to create classroom environments that provide learners the opportunity to demonstrate their understanding in a wide range of contexts.
Teachers should support learning of abstract mathematical concepts. Although most learners are able to acquire mathematical knowledge using concrete objects, learners should demonstrate correct use of symbols and other mathematical notation. Symbols present cognitive difficulties to learners, especially to learners who have not yet fully understood the concepts they represent (Fraser, Murray, Hayward & Erwin, 2004). Most high school learners in the FET phases cannot operate in the abstract phase of mathematical thinking. Linking abstractly involves moving from a concrete mode of mathematical thinking of a concept to a more abstract understanding of the concept (Sfard, 2000). Ross and Willson (2012) proposed the use of multiple representations to support abstraction. Teachers should select representations that scaffold the learners' understanding, moving them from concrete to abstract representational forms.

To promote learners' understanding of mathematical concepts, as opposed to the blind memorisation of rules without reason, it is crucial that teachers link abstract mathematical symbolism with representations from everyday world whenever this is possible (Arnawa, Kartasasmita & Baskoro, 2012). Concrete materials should be included in classroom instruction to facilitate discussion and assist the understanding of mathematical symbolism. Manipulatives should be used to complement the real world phenomenon that makes sense to the learner thereby enriching the learner's schema for representing abstract mathematical symbolism.

As recommended by the Evans, Leija and Falkner (2001), teachers should search for strategies and contexts that enhance learners' understanding of mathematical symbolism and appreciate the significance of linking it with reality. To change the way mathematics is taught in schools, teachers should be given the opportunity to construct new mathematical frameworks for themselves so that they will be able to link the abstract mathematics that they have to teach with real world representations to which they and their learners can relate. Learners' social and cultural context should be considered when planning and organising mathematical instruction (Godino, Batanero & Font, 2007). Learners have the potential to understand mathematics if given opportunities to develop connections between symbolic mathematics and appropriate real world representations. If learners cannot provide a real world context for an abstract mathematical concept, teachers should be wary of the learners' true understanding and explore the concept further. It is the responsibility of teachers to provide representations from the real world to illuminate mathematical abstractions, to search out such representations whenever they exist, and use them to assist understanding.

Another strategy suggested by teachers is the use of a variety of representations during teaching and learning of mathematics. The use of different representational modes may ease communication of mathematical ideas. For instance the words "one-half" symbolised by the notation "$1/2$" or picture of half of an object. The use of different representational forms may provide a range of options that learners can to communicate their mathematical thinking. Learners prefer concrete modes in which mathematical concepts are linked to reality (Cox & Sagor, 2013). Concrete modes of mathematical representation are more suitable when introducing learners to new mathematical ideas. The use of concrete tools has a potential to ease the learners' passage into the concept and the basic connections needed for them to progress to the abstract phase of a mathematical idea (Ross & Willson, 2012). Using concrete materials make mathematics more accessible while the use of abstract mathematical ideas demonstrates deeper knowledge. It is therefore important to corporate both approaches in a mathematics lesson to allow learners to understand a mathematical concept deeply.

Another attribute of learning that teachers should bear in mind is that learning is gradual and incrementally connective. It is a step-by step progression. Learners must be provided with adequate time to grasp mathematical concepts. Instruction can be modelled by scaffolding and guiding learners to move gradually and progressively to abstract forms (Alagic & Palenz, 2006).

Teachers can also incorporate Dienes (1960)'s five levels of mathematical understanding: free play, generalization, representation, symbolisation and formalization. In free play, learners work with manipulatives to discover basics about the concept. In generalization, learners notice patterns and form mental images in the representation level. They then describe their representations using mathematical language and symbols. This process culminates in a set of rules and algorithms to match their understanding of the concept. The sequence of these levels describes a form of scaffolding. A learner who has mastered earlier levels of mathematical understanding has the potential to progress to higher levels of understanding.

5.1.5 Themes emerging from data analysis

Thematic analysis was used to identify, analyse, and report themes that emerged from the data. Thematic synthesis guided the researcher to combine learners' and teachers' views about mathematical symbolisation and identified key themes related to challenges and intervention strategies. Three phases of thematic analysis were conducted with initial themes drawn directly from participants during focus group interviews, responses to open- ended questions and face-to- face interviews with teachers. Initial themes were drawn from the 15 categories in which responses were classified. The second phase of the analysis produced five (5) themes, namely: challenges with reading mathematical text, context of mathematical symbols and language, learners' level of symbol sense and reasoning, timing and pedagogical strategies for teaching mathematical symbols. The final phase of thematic analysis produced two broad sets of themes: semiotic challenges and instructional challenges. From this analysis, the researcher arrived at the following conclusions:

a) The main challenges experienced by secondary school mathematics learners in understanding mathematical concepts and problem solving procedures are reading mathematics text and lack symbol sense to decode mathematical situations.

b) Learners have weak symbol sense levels that prevent them from decoding meanings of mathematical concepts and processes.

c) Teachers' choice of instructional strategies should be informed by the context in which mathematical symbols are used, timing the stage at which symbols are introduced, allow more time for learners to synthesise and conceptualise mathematical concepts together with relevant symbols.

d) Instructional approaches should be modelled around a teaching framework that allows learners to make meaningful connections between concrete, representational, and abstract levels of thinking and understanding.

5.2 Implications

Teachers should be aware of the challenges experienced by learners when learning mathematics through symbolisation. This study found that many learners lack the desired attributes of symbol sense suggested by Arcavi (2005). These attributes contribute to the broader theme of sense making in mathematics. Teachers must engage learners in classroom conversations that invoke the meanings of mathematical concepts together with their symbols and further link meanings between learners' prior experiences and specific problems. A strong symbol sense helps learners to build fluency with the complicated language of mathematics. Including more symbols, talking about symbols in classroom discourse are some of the ways of building symbol sense in learners (Arcavi, 2005).

Learners in this study indicated that the use of more than one symbol to represent the same concept was confusing. For example, the following three questions use different symbols but they represent the same concept (derivative) and the same process (differentiation), determine $f^{\prime}(x)$ if $f(x) = 1 - 3x^2$, determine $D_x\left[4 - \frac{4}{x^3} - \frac{1}{x^4}\right]$ and determine $\frac{dy}{dx}$ if $y = \left(1 + \sqrt{x}\right)^2$. A critical view into the ways learners interpret mathematical symbols could be a useful strategy of identifying ways to strengthen their understanding of symbols. It provides information into learners' networks of understandings.

Teachers should provide a variety of options for mathematical language and symbols. Learners with a well–developed symbol sense are able to clarify concepts and provide clear explanations of mathematical processes. A graphical illustration of an algebraic relationship between two variables may be informative to one learner and be inaccessible to another. It is therefore important for learners to be able to switch from a graphical representation to an algebraic representation and vice versa. Equally important is the ability to form an algebraic equation that represents a simple graph (such as a linear graph). A picture or image that carries meaning for some learners may carry very different meanings for learners with different mathematical backgrounds. As a result, differential understanding arises when information is presented to all learners through a single form of representation. It is therefore important for teachers to vary instructional strategies to ensure that alternative representations are used to improve accessibility, clarity and comprehensibility for all learners.

Teaching must define vocabulary and symbols in all their optional forms**.** Learners understand words, symbols, and icons differently, depending on their background lexical knowledge, and disabilities. To ensure that all learners understand mathematical concepts the following aspects need attention:

- Teachers should try to link mathematical concepts, processes and symbols, to connect to the learners' social and cultural experiences and prior knowledge.
- Clarify new symbols within the text.

The teacher must provide opportunities for decoding mathematical notation. The ability to fluently understand words, numbers and symbols that are in an encoded format (for example symbols for text or algebraic numbers in place of a quantity) requires practice for any learner, and some learners never full master it. Failure to achieve fluency or automaticity makes it difficult to understand encoded text. To ensure that all learners understand mathematical language, it is important for teachers to provide instruction that reduces decoding barriers for learners who are dysfluent with mathematical symbols.

Text often dominates classroom materials. However, it is difficult to present many concepts for learners who have text or language-related problems. Learners who struggle to use the textbooks can be provided with alternative materials such as:

- expository text or a math equation can be presented as an illustration, diagram, or animation.
- Textual information should be linked with illustrations, graphs or diagrams.

These suggestions were deduced from responses of mathematics teachers and learners.

5.2.1 Work Schedules and assessment

Time allocated for mathematics teaching and learning in work schedules and assessments is not adequate for learners to adapt to new symbols and their meanings. The sequencing and timing of concepts in the syllabus should be revised to provide more time for new experiences to be assimilated and practised. Learners should be exposed to a variety of representations and contexts in a given topic. Work schedules and school accountability systems are organised and centred on examinations. This encourages teachers to teach to the test and focus on procedural approach to mathematics (ACME, 2009). The scheduling of teaching units should take into account learners' natural ways of dealing with new perceptual and verbal information including those ways that are helpful for new mathematical ideas and those that obstruct their learning.

Learners should be guided to know when to apply formal, informal or situated methods and symbols. Teachers should possess adequate content and pedagogic knowledge. Subject content should be presented in a coherent, cognitive and progressive in order to enable learners to develop all aspects of mathematical proficiency. To this effect, Higgins

(2011) suggested a curriculum review cycle that is long enough to develop a coherent, informed, package of assessment, textbooks and teacher knowledge.

A better conceptual understanding of symbols by teachers will prepare them for possible difficulties that learners may experience in the classrooms. An important question for mathematics education is: "when and how to introduce symbolism within the curriculum?" If it is introduced too early, learners may lack the maturity to understand and reason symbolically. If it is introduced too late, some mathematical concepts, methods and processes cannot be taught as they rely heavily on symbolism (Heeffer, 2013). Teachers should be aware of the fact that symbolism does not act in a completely neutral and abstract way. An insight into how perceptual processes direct learners' understanding of symbolism prepares teachers for possible mistakes and difficulties in classroom practice. There is a need to highlight the importance of paying attention to potential barriers to learning because of discontinuity, unchartered extension and heightened complexity in the use of symbols when learners progress in mathematics from one grade level to the next.

5.2.2 Learning materials

The use of textbooks dominates most classroom learning materials. However, Spook (2009) argued that text is a weak format for presenting many concepts and for explicating most processes. Text is a weak form of representation for learners who have text or language-related disabilities (Kamhi, 2009). Providing alternatives especially illustrations, simulations, images or interactive graphics can make the information in text more comprehensible for learners. Teaching through concrete-to-representational-to-abstract sequence of instruction can help learners to understand abstract concepts (Lane, 2010). Teaching through a concrete-to-representational-to-abstract sequence of instruction ensures that learners have a thorough understanding of the mathematics concepts or skills they are learning before they are symbolised. If learners with mathematics learning problems are allowed to first develop a concrete conception of the math concept, they are more likely to perform that math skill and understand math concepts at the abstract level.

5.2.3 Language of Instruction

Hugo and Nieman (2010) pointed out that many rural South African secondary school learners lack knowledge of academic English yet it is the language used in schools. Learners find it difficult to learn using content-specific vocabulary. Terms such as equation, algebraic and other everyday terms may have different meanings when used in mathematical contexts. Some mathematical terms also do not translate well into everyday language thus leaving the learner unable to understanding the content taught. The effective mathematics teacher needs to be aware of these barriers and address them during instruction. Mathematics teachers need to assist learners to acquire mathematical language by using appropriate terms, symbols, concepts, and communicating their meaning in ways that learners understand (Adams, 2003).

5.2.4 Making connections

Effective teaching should support learners in making connections between mathematical concepts and their representations, so learners appreciate mathematical symbols as part of their own lives. This requires a deep understanding of mathematics on the part of teachers in order to facilitate responsive and productive mathematical discussions (Stein, Engle, Smith & Hughes, 2008). Big mathematical ideas need to be expertly developed and explained. Connections need to be made within and across concepts and similar mathematical ideas. In this way, learners can develop connections and deep mathematical understanding.

5.2.5 Classroom dialogue

Teachers should facilitate argumentation as part of classroom dialogue. In such an environment learners, learn how to communicate mathematical thought, justify their standpoints and thinking. This focuses the learners less on the answers and more on the processes. Learners also learn to critically consider other intellectual views and debate. Learners should be actively engaged in collaborative conversations (Parsons & Taylor, 2011).

5.2.6 Teaching and learning resources

In the classroom, charts with the symbol, word equivalents, worked examples and illustration examples can help learners to reinforce a correct interpretation of symbols. Research at secondary level has shown that learners can struggle immensely when it comes to shifting their perception from an unknown to a variable (Bardini, Pierce & Stacey, 2004). Learning and transfer of learning occur when multiple representations are used, because they allow learners to make connections within and across concepts (Laurillard, 2013). In short, there is not one means of representation that is optimal for all learners; providing options for representation is essential.

Information should be presented through a variety of representational forms. Teachers should ensure that alternative representations are provided not only for accessibility, but also for clarity and comprehensibility across all learners. A variety of tools should be available to teachers, including the number system, symbols, diagrams, models, notation, stories, technologies and a range of 'concrete' materials. Young and Loveridge (2010) highlighted the importance of instructional tools in assisting learners to make connections between operations, concepts and their symbolic representations. They recommended that instructional resources must be carefully selected and that teachers should ensure that connections between concepts and representations are explicit.

5.2.7 Teacher knowledge

Effective mathematics teachers develop and use knowledge as a basis for responding to mathematical needs of all their learners (Anthony & Walshaw, 2010). König and Pflanzl (2016) reported that teacher's pedagogic knowledge impacts directly on the way they plan and present their mathematics instruction in the classroom and upon the learning experiences for their learners. Teachers at all levels need to know their learners too, they need to be able to anticipate the difficulties that their learners may encounter in their mathematics learning, to challenge and extend their learners' knowledge, and they should be able to describe learning trajectories and next learning steps. This demands a skilful response to teaching situations rather than simply an adherence to scripts or texts.

5.2.8 Attending to Precision

One essential feature of mathematics is precision (Wu, 2009). Precision means being specific in the ways teachers and learners present and communicate mathematical concepts, processes, symbols and representations. Communication in mathematics includes oral and written communication, vocabulary, notation and symbols. Terms, words and phrases used in mathematics communication have intentional and rich definitions; notation and symbols have specific meanings and uses. When incorrect vocabulary is used in mathematics, notation and symbols, learning is lost (Schleppegrell, 2007). Precise use of mathematical symbols, notation and vocabulary supports learners' development of the critical nuances of mathematical ideas with content specific terms. Learners often emulate their teachers when it comes to precision in terms of definitions, general language, terminology, symbols and ideas related to mathematics (Meiers, 2010). Thus, teachers need to ensure that they are modelling precision in their classroom instruction. Teachers should be precise in terms of notation, symbols and models they use during teaching and learning. Precision also arises in representing situations with equations, graphs, variables, labelling of axes, scale and representing discrete and continuous data using correct symbols.

5.2.9 Learners' Prior Knowledge

Symbols are designed to make mathematics easier as long as the meaning of the symbol is established prior to using it (Heath, 2010). Learners' experiences are valuable in the interpretation of mathematical symbols and concepts. Prior knowledge influences learning and learners construct concepts from prior experiences (Rouhani & Kowsary, 2014). Learners who have a great deal of background knowledge in a given subject area are likely to learn new information readily and quite well. The converse is also true. Having prior knowledge is important factor influencing learning and learner achievement. Hailikari et al (2008) posted that an essential consideration in developing an integrated knowledge framework is to create a learning environment in which learners actively construct knowledge and skills based on prior knowledge. Inadequate or fragmented prior knowledge is an important issue to consider because if there is a mismatch between the teachers' expectations of learner knowledge and the learners' actual knowledge base, learning may be hindered. Trying to learn something without having adequate prior knowledge or, worse, having misconceptions, may result in rote memorisation.

Teachers can assess learners' prior-knowledge as a tool for evaluating the level of support learners needs prior to a new topic. Prior knowledge is the knowledge base learners bring to a lesson. Do they know enough to move forward? It is a critical requirement for teachers to establish learners' prior knowledge in order to prescribe support building on what learners already know for new experience. It is therefore important for teachers to determine what a learner actually understands about a concept as they prepare for new instruction (Ball, 2000). Understanding new concepts involves juxtaposing new information with prior knowledge held by the learner. The process of building on *a priori* knowledge involves accommodation into an already existing schema that is inadequate to assimilate new phenomena (Nashon, Anderson & Nielsen, 2009). This results in cognitive imbalance and the learner is forced to restructure the existing knowledge or schema.

5.2.10 Timeous introduction of mathematical symbols

Leaners should be accorded ample time to familiarise with symbols and comfortable in using them with the extensive mathematical vocabulary that is associated with them. For effective teaching to take place, these considerations must be planned for. For many learners the ability to make sense of mathematical symbols is crucial to their development of a particular mathematical concept (Nunes, Bryant & Watson, 2007). The teacher must provide adequate reference systems to make sure that learners are able to make correct connections between the symbols and the mathematical concept or process.

5.3 Summary

In this chapter, the major challenges of mathematical symbolisation during teaching and learning were discussed. The chapter also discusses the implications of these findings on classroom practice, mathematics education and research. Classroom implications for dealing with challenges related to the teaching and learning of mathematical concepts through symbolisation were also discussed. The chapter also identified and discusses several implications for teaching and learning mathematical concepts by paying attention to symbols and their meanings.

CHAPTER 6: SUMMARY OF THE STUDY

The previous chapter discussed the research findings and their implications. This chapter provides an overview of the study, together with the conclusions drawn and the resulting recommendations. The limitations encountered in conducting the study are outlined and possible avenues for future research are discussed. The chapter concludes with a summary of the value of this research study.

6.1 Summary of the study

In South Africa, the history of mathematics education in secondary schools spans from early apartheid years to the newly improved National Curriculum Statement (NCS) which was recently replaced by the Curriculum Assessment Policy Statements (CAPS) curriculum (Department of Education, 2011). The CAPS curriculum is the fourth wave of curriculum reforms in the post-apartheid South Africa. In reviewing research on curriculum reforms, one observes that while the South African mathematics curriculum reforms have been shaped and changed by a number of factors including international and national trends and developments in mathematics education, theory and practice. However, very little evidence exists that research played any significant role in the direction or form taken by the curriculum over time (Vithal & Volmink, 2005).

The pass rate in mathematics in rural secondary schools remains unacceptably high (Steyn, 2006; Tachie & Chireshe, 2013; Spaull, 2013). In 2015 national matric examinations, KwaZulu-Natal, Eastern Cape and Limpopo were the worst performers in mathematics with just 20%, 21.8% and 32.4% of learners in each province respectively achieving a mark of 40% (Gavin, 2016). These statistics represents a downward trend from previous results. Currently the pass rate in mathematics is 46.3%, which is still far below the expected national standard (Department of Basic Education, 2016). Another feature of the South African educational system is that the national average mathematics achievement score for different grade levels is similar and stable; around 30% to 40% (HSRC Review, 2011).

Spaull (2013) reported that many learners in rural and township secondary schools do not sufficiently master the content and thinking skills needed in learning and problem solving. Feza-Piyose (2012: 62) asserts that, "one of the contributing factors to the poor performance of learners in mathematics is quality of instruction received by the majority of South African learners". Thus, learners often acquire deficient, superficial (Maree & De Boer, 2003) and rote knowledge of basic concepts (Maree & Steyn, 2001). A number of researchers (Howie, 2001; Mosibudi, 2012) have investigated the problem of mathematics underachievement in South African rural secondary schools. The causative factors found range from poor social and economic backgrounds of the learners, lack of appropriate learner support materials, and generally impoverished school environments, poor quality of teaching and language of instruction.

Despite efforts by researchers to get to the root causes of poor performance in mathematics at secondary school level, no attempts have been made to assess learners' challenges in the different mathematics syllabi for the various curricula. Very little evidence in terms of how research has been used to look into the specific challenges that teachers and learners face when implementing the curriculum. The high mathematics failure in rural secondary school could be attributed to learners' failure to acquire the language system of mathematics using the various symbols and notations.

Mathematics presents many challenges and barriers during teaching and learning. The most noticeable barrier to communication is that mathematics is a subject heavily laden with symbolism. Mathematics is one of the most unpopular subjects in South African secondary schools. Learners do not achieve well in the subject. There are several variables to explain why learners lag behind in the subject. The range of causes of poor performance stretches from deficits in learning behaviour to poor instruction (Chisholm, 2008). The researcher observed that most learners have trouble in grasping mathematical skills and concepts. The researcher speculates that the reason for this failure could be the symbols which are unfamiliar, confusing and sometimes contradictory. Learners struggle with understanding mathematical concepts and problem solving due to the way in which symbolism is used to develop the concepts. The study was guided by the following research questions:

a) What challenges do secondary school learners encounter with symbols when interpreting and solving mathematical problems?
b) What instructional strategies do mathematics teachers use to mitigate the effects of mathematical symbolisation obstacles?

The rationale of the study is to provide insights into learners' challenges, experiences with mathematical symbolism and recommend instructional strategies and practices to address learners' shortcomings. In particular, the educational purpose of this study is to inform mathematics teachers on how learners construct meanings for mathematical concepts and learn mathematical concepts with symbols. The study also sought to inform mathematics teachers, district curriculum advisors and managers, and the Department of Education (DoE) on how symbols and conceptual understanding should be emphasised during teaching and learning.

A number of conceptual frameworks provided a lens for looking at both learners' challenges in understanding mathematical concepts and symbols and teachers' instructional strategies to enhance competence with mathematical symbols. This study adopted and was guided by a combination of Arcavi's (1994) symbol sense; Pierce and Stacey's (2001) Algebraic insight framework; Dubinsky and McDonald's (2002) APOS theory; and Tall's (2004) Procept Theory. Working fluently within the multiple semiotic systems of the language of mathematics requires developing strong symbol sense and connecting meaning of symbols to meanings in natural language.

Symbol sense involves having an awareness that one can successfully create symbolic relationships that represent written information; experiencing different roles played by symbols; and appreciating the power of symbols to display and explain relationships expressed in natural language (Arcavi, 2005). In problem solving, Arcavi (1994) describes symbol sense as:

"…a quick or accurate appreciation, understanding, or instinct regarding symbols" that is involved at all stages of mathematical problem solving" (p.31).

Algebraic insight framework is a sub-framework of symbol sense that is involved in problem solving. Algebraic insight framework forms the basis from which to plan, assess or reflect on learners' algebraic insight. Algebraic insight may be divided into two: the insight required in symbolic representation, and the insight required for and gained from linking symbolic to graphic and numeric representations.

This study is also guided by the APOS theory (Dubinsky & McDonald, 2001). APOS theory postulates that a mathematical concept develops as one tries to transform existing physical or mental objects. The mental structures refer to actions, processes, and schema required for learning the concept. According to Parameswaran (2010), concepts can be abstract, considered in two fundamentally different ways: as processes (operationally) or objects (structurally). In APOS theory, action and process are operational conceptions, while objects and schema are structural. The theory requires that for a given concept, the appropriate mental structures need to be detected, and then suitable learning activities should be designed to support the construction of these mental structures. According to this theory, the goal for teaching should consist of strategies for helping learners build appropriate mental structures, and guiding them to apply these structures to construct their own understanding of mathematical concepts.

The Procept Theory (Gray & Tall, 1994), was also adopted as a theoretical framework for this study to explain the relationship between a mathematical object or process and meaning in constructing and communicating a mathematical object. Gray and Tall (1994) adopt the term *procept* to describe a combination of three components: a mathematical concept; a process, and a symbol. If a symbol is used as a signifier, a learner should be able to observe the process acting on an input to produce an output as a concept. The APOS theory explains how learners understand mathematical concepts and informs the choice of pedagogic interventions.

A mixed approach was used to address the research problem. The participants for the study consist of 565 learners (Grade 10 -12) and 15 teachers drawn from rural, semi-urban and urban schools in Limpopo Province, South Africa. Multistage random sampling (learners) and purposive sampling (for teachers) were used to select the sample of participants for the study. The research instruments for the study consist of questionnaires and interviews. Questionnaires were administered to learners and teachers. Learners were group interviewed while teachers were individually interviewed.

A pilot study was conducted to check the feasibility of the study. Three schools from three geographical cohorts participated in the pilot study and were not considered in the main study to prevent data contamination. One hundred and ten (110) learners drawn from the FET band 10 mathematics teachers participated in the study. The outcomes of the pilot study addressed two things: practical considerations and assessment of instruments. Practical considerations were things like length of questionnaires, the time limit per group interview session; and keeping the interview session active. The time limit per group interview session was set at 15-20 minutes initially. It emerged that this time limit was inadequate and was changed to 20-30 minutes; thereafter the interview process was adjusted in courtesy, clarity, pace and relevance of the content.

In order to assess the reliability and relevance of the instruments, a pilot study identified mathematical symbolisation as a challenge that hinders successful understanding of mathematical concepts. Four categories of themes related to learners' difficulties with mathematical symbols were observed: textbooks and problem-solving, problems with transition from informal to formal mathematical symbols, context and multiple meanings, and instructional strategies. Teachers indicated that they face the following difficulties when teaching mathematics: the challenge of introducing unfamiliar notation in a new topic; teaching reading, writing and verbalising symbols; signifier and signified connections; and teaching both symbolisation and conceptual understanding simultaneously. Further consultations with experts in mathematics education were also conducted. Feedback from these experts on the operational feasibility, clarity, length, content and relevance of the main instruments was positive.

Data were collected using two research instruments, namely, questionnaire and interviews. Five hundred and sixty-five (565) learners and 15 teachers completed the questionnaires, 15 teachers were all interviewed, and 12 group interviews were held with learners. All the interviews were digitally recorded and transcribed. The audio records were replayed to participants for respondent validation and comment on the interviews.

SPSS version 23 was used to analyse closed questions that had a five point Likert scale. A two – stage cluster analysis was utilised to cluster learners according to the level of difficulties with mathematical symbolisation. The data was classified into three clusters. Cluster 1 was made up of 222(39.3%) of the participants. Learners in this cluster expressed mild difficulties with mathematical symbolisation. They indicated that mathematical symbols were neither difficulty nor easy for them. It consists of learners who are able to switch representations from one form to another. It also consists of learners who have little difficulty in doing mathematical tasks despite lack of proficiency in symbol use. The largest cluster (2) accounted for 235(41.6%) of the participants. The learners in this cluster can be described as having a weak symbol sense, cannot manipulate mathematical symbols with understanding, cannot switch representations, consists struggle to do mathematical tasks due to lack of symbol manipulation proficiency, symbols affect their problem solving abilities and they struggle to initiate symbols in problem solving. Cluster 3, which fills 108(19.1%) consists of learners who can confidently manipulate mathematical symbols with understanding, mathematical concepts and initiate symbols to solve problems, move within and across representation and conceive symbols as tools for understanding concepts.

Two demographic variables: gender and grade level emerged as the main predictor importance values in distinguishing and separating learners' challenges with symbolisation. However, further enquiry using inferential tests (T-tests and ANOVA) indicate that there were no gender differences in terms of learners' experiences and difficulties with mathematical symbolisation. Analysis of variance (ANOVA) indicates that grade, age and residential area were statistically significant in explaining learners' different experiences in dealing with mathematical symbols. Cluster analysis also ranked learners' challenges with mathematical symbols as: (1) switching representations from geometric to algebraic and vice-versa, (2) symbols create barriers to conceptual understanding and (3) symbols negatively affect problem solving. The least ranked items were: (24) symbols affect learners' understanding of algebra related topics, (25) navigating through the symbols and their meanings without a mathematical concept is easy, (26) mathematics textbooks use unfamiliar symbols and language.

Typological and thematic analysis were used to cluster teachers and learners' challenges and instructional into categories and themes. The following learning themes emerge from learners' responses: reading mathematical symbols is a challenge, linking mathematical concepts, symbols and their meanings is complex, time allocated for effective learning and teaching is not adequate, symbols and problem solving present double problems, the role of symbols in maths is not well understood, and instructional strategies used by teachers do not adequately address learners' needs. The following themes emerge from teachers' responses: learners' prior knowledge is shallow, textbooks lack explicit explanations, symbols present cognitive load in learners' minds, symbols are too abstract and above learners' cognitive level and teacher pedagogical content knowledge. Further, one more step of thematic analysis to produce two sets of themes: semiotic and instructional challenges. The two semiotic challenges were lack of symbol sense and difficulties in reading mathematical text. Themes related to instructional challenges include contexts in which mathematical symbols are used, the timing or stages at which some mathematical symbols and concepts are introduced and the teaching and learning styles that do address learners' cognitive needs.

6.2 Conclusions

In conclusion, the data for this study presents two major findings. Firstly, there are challenges connected to the use of mathematical symbols in the teaching and learning of mathematical concepts. Secondly, instructional strategies to curb mathematical symbolisation challenges are not yet available and teachers are still in the trying phase. Learners indicated that mathematical symbols obscure them from understanding mathematical concepts and to present solutions to problems. Learning of mathematics is hindered by the use of unfamiliar notation that textbooks and at times teachers cannot explain to learners' satisfaction. Most learners confirmed that they face the challenge of familiarising themselves with symbols used in some topics and struggle to associate the symbols with the concepts.

Mathematics classes are still characterised by meaningless symbol manipulation; learners use symbolic expressions without understanding their meanings. Learners revealed that navigating through the symbols and their meanings is a complex process due to multiple meanings of some symbols. Consequently, Algebraic topics are unpopular because of symbols and the rules for manipulating and combining them.

A large proportion of the learners complained of lack of link between their informal meanings of mathematical symbols with the actual mathematical meanings. Learners indicated that their meanings of mathematical symbols are contradictory to formal meanings. An important issue emerging from the study is that the majority of the learners lack the ability to generate symbols and use them in problem solving. Thus, most of the learners in this study lacked symbol sense, that is, they cannot use symbols correctly. The study showed that learners' poor understanding of mathematical concepts is based on poor conceptualisation. Misconceptions and poor conceptions in the interpretation of mathematical symbol result in learners failing to link mathematical symbols and formulae with appropriate concepts. Thus, classroom interactions should focus on making sense of mathematical symbols; rules and formulae to assist learners to develop meaningful understanding of mathematical concepts.

Another challenge raised by learners was that the symbolic representation of mathematics concepts is abstract and therefore more difficult to learn than concrete representations. It requires learners to operate at the abstract level of thinking and use higher order skills of mathematical thinking. However, most of the learners in the FET band have not yet acquired this level. It is the responsibility of teachers to ascertain the actual developmental level of the learners they teach, and work to build the learner's insight to accommodate new horizons. Teachers tend to encourage learners to manipulate symbols without the proper conceptual foundation that limits their progress into higher mathematics.

Learners indicated that textbooks use unfamiliar symbols and notations that are difficult to understand. They are relevant and make sense after the teacher explanations. The flow of mathematical concepts is not linear. Reading a mathematics textbook requires careful understanding of each word as suggested by Simonson (2011). Textbooks do not take learners' background knowledge into account. Reading mathematics text requires learners to be active and competent users of mathematics textbooks, including all parts of textbooks. There is need for teachers to make reading an integral part of mathematics instruction.

6.3 Recommendations

In view of these conclusions, the following recommendations regarding the teaching of mathematical concepts by paying attention to symbolisation were made:

6.3.1 Teaching for understanding

Borrowing from the cognitive revolution in education, mathematics teachers should teach for understanding. It is inadequate for learners to be competent in manipulating mathematical symbols, solving problems or answering certain questions; instead learners should have conceptual understanding that under guides such abilities. Understanding entails being able to think and act flexibly within a topic or concept. It goes beyond knowing; it is more than a collection of information, facts, or data. It is more than being able to reproduce steps in a solution procedure. Meaningless manipulation of symbols is detrimental for further mathematics learning. Thompson, Cheepurupalli, Hardin, Lienert and Selden (2010) discouraged the use of use symbols without understanding. A learner-centred approach can be used to promote understanding by planning classroom instruction based on where "*learners are"*, in terms of mathematical ideas. Learners should approach tasks in ways that make sense to them. Learners should be able to explain, provide evidence, create examples, generalize, analyse, predict, apply concepts, represent ideas in diverse ways, and articulating relationships between the different ideas.

6.3.2 Conducive learning environments

According to the symbolic interactionist perspective, meaning is constructed as a result of interaction with one another, it is therefore essential to consider the nature of the interactions that occur in a mathematics classroom (Tamene, 2015). Giving attention to classroom socio-mathematical standards and to classroom discourse can result in supporting learners' development of mathematical argumentation. Engaging learners in active classroom discourse helps them to explain and justify their thinking to others. In the process, they develop intellectual autonomy, mathematical power. Learners also share experiences in constructing connections between symbols and their references. Challenges experienced with written symbols, concepts and procedures can be reduced by creating interactive learning environments that help learners build connections between their formal and informal mathematical knowledge. Such an environment uses the problem context to make appropriate representations to connect procedural and conceptual knowledge (Yetkin, 2003). Therefore, teachers need to create collaborative and learner-centred environments, where learners have opportunities to reason and construct their understanding as part of a community of learners

Teachers should create an emotionally safe learning environment that helps learners feel secure and willing to take risks and consider mistakes as positive learning steps. The teacher should help learners to set realistic and manageable goals based on the learners' ability. Learners should be actively engaged in the teaching and learning process. Eison (2010) observed that effective learning results then learners participate in the activity; discussion, practice, review, or application. This contrasts with traditional styles of teaching, where learners passively receive information from the teacher.

6.3.3 Meaningful representations

Teachers should choose meaningful representations in which the objects and actions available to the learner directly link to the mathematical objects (ideas, symbols) and actions (processes or algorithms) they wish learners to understand. There is need for teachers to guide learners to make connections between representations and ideas. By using multiple representations, learners can deepen conceptual understanding and skills by switching form from one representational form to another (Bal, 2014). It also helps learners to relate them to the real world, justify their thought processes and clarify their thinking. The representation form should be used to strengthen the connections made in knowledge construction.

6.3.4 Linking symbols and meanings

It is crucial that mathematics teachers should emphasise and develop learners' abilities to understand and connect meanings to mathematical symbols. Teachers should desist from concentrating on teaching learners what to do (procedure) when they see certain symbols or situations. Meaningful mathematics teaching requires teachers to help learners to construct concepts for mathematical words and written symbols (Adams, 2003; Schleppegrell, 2007). Teachers should desist from asking learners to use symbols very early while they are still trying to understand a topic. Symbols should be should be appropriate to the learner's grade level.

6.3.5 Mathematical language and vocabulary

Mathematical symbols are associated with numerous mathematical words, so teachers need to guide learners to become familiar with the mathematics vocabulary and references associated with them. Most mathematical concepts in these grade levels are presented at abstract level using only numbers and mathematical symbols. In order to reinforce learning, learners should practice with a variety of opportunities and demonstrate mastery at the abstract level before moving to a new mathematical concept. Teachers should introduce symbols after learners have had opportunities to make connections among the other representations, so that learners have multiple ways to connect the symbols to mathematical ideas, thus increasing the likelihood that the symbols will be comprehensible to learners.

Learners have challenges in the learning new mathematical concepts due to symbolic notations. Teachers need to use symbolic notations in flexible and applicable ways in order to approach algebraic structures, so learners can recognise the different meanings of symbols in different algebraic situations. If learners explore real life problems systematically build the concept of notation, it takes learners away from focusing on processes while disregarding the general idea hidden behind each representation. Researcher (Egodawatte, 2011; Verzosa & Mulligan, 2013; Madzorera, 2014) suggests that, if learners actively articulate the meaning of symbols in writing, they develop a better sense of symbolic notations, are able to correct some of their own misconceptions and use symbols appropriately in many different algebraic situations.

6.3.6 Closing the gap between primary and secondary mathematics

Mathematics teaching at primary school level does not thoroughly prepare learners for secondary school mathematics (Mapolelo & Mojeed, 2015). Primary school teachers do not lay a proper foundation for mathematics, learners have gaps in knowledge. The net result is that learners do not attain the basic foundation of learning mathematics at primary school. This becomes a problem at the secondary school level. Mathematics is a more logical subject that requires continuity in learning. Once there is a gap in learning, one cannot learn further. To overcome this, the research recommends the re-organisation of the syllabus and more abstract concepts should be introduced at higher levels.

Specialised and expert mathematics teachers should be deployed at primary school level to teach mathematics, this will narrow the gap.

6.3.7 Instructional approaches to symbolisation

Mathematics teachers should be cognisant of their instructional approaches to symbolisation for it is the learner's interpretation of such instruction that conveys mathematical meaning to a symbol and not simply the presentation of the symbol itself. As recommended by Phillips (2008), mathematical symbols can only be effectively used and understood as mathematics communication tools if learners grasp their meanings. This recommendation is consistent with the concerns of this study in observing that the confusion generated from faulty basic understanding of the meaning of a symbol presents major obstacles to learning further mathematical concepts. As indicated earlier on, many difficulties that learners experience are rooted in the multiple meanings or roles that the same symbol carries in different contexts.

In order to help learners to overcome these difficulties, the algebraic rules, formulae, and definitions should be explored in various contexts so that learners can understand symbols in different situations. A new concept should be incrementally built on existing knowledge focusing on the structure rather than on pure calculation process. Learners should be given opportunities to develop and make sense of symbolic expressions, notations, and representations by participating in exploratory processes instead of teacher presentations. Teachers need to inculcate the skills of reading symbols for meaning and careful attention should be paid when teaching. As recommended by Ball (2003), teachers should anticipate the difficulties learners can possibly encounter with some symbols, how to study with them and choosing an appropriate definition to study.

6.3.8 Support decoding mathematical text, notation and symbols

Learners need consistent and meaningful exposure to symbols so that they can comprehend, decode and use them effectively. The lack of fluency for decoding symbols greatly reduces the capacity for information processing and comprehension. To ensure that all learners have equal access to knowledge, it is important to provide options that reduce the barriers that decoding raises for learners who are unfamiliar or dysfluent with mathematical symbols. Mathematics teachers should develop individual strategies to assist learners in the reading phase of mathematics instruction.

6.4 Limitations of the Study

The generalizability of the research findings of this study is limited because they were generated mainly in an exploratory qualitative inquiry. The research design was a mixed method design. One of the drawbacks of this design is that quantifying qualitative data causes it to lose its flexibility and depth. Qualitative codes are multidimensional while quantitative codes are one-dimensional and fixed so basically changing rich qualitative data to dichotomous variables produces one dimensional immutable data. Quantifying qualitative data was done to save time and avoid complex process as it requires analysing, coding and integrating data from unstructured to structured data.

The intention of the study was not to produce results that classify learners' experiences with symbolism as most experimental, hypothesis-testing studies do. However, the inquiry generated a relatively clear and specific grounded theory that can be applied to practical experiences. It should be relatively easy to design a series of focused hypothesis-testing studies to experimentally verify and expand the theory generated here. These studies are more likely to produce findings generalisable to larger populations.

Furthermore, time and budget constraints made it impractical to assess learners on given algebraic topics to explore further specific challenges they experience with mathematical symbols instead of relying on narratives of their experiences. A longitudinal study spread over multiple topics and months could have been more viable. Relevant literature suggests that many of the insights that come from participants may not show up until long after learners realise complex situations where symbols obscure them from understanding certain mathematical concepts. Thus, it is possible that participants may be experiencing more symbolisation problems than what was actually collected. Collecting such data was beyond the scope of the current study. Future studies might consider topics such as Trigonometry, Financial mathematics and Euclidean Geometry in which learners indicated that they experience learning difficulties. Instructional interventions that were suggested could be pursued with a longitudinal study to explore further symbolisation challenges and instructional interventions.

6.5 Suggestions for further research

Researchers unanimously considered symbols as driving forces of algebraic thinking (Zazkis & Liljedahk, 2002; Arcavi, 2005). This study is consistent with several research results that reveal that learners' difficulties in algebra have their roots in misinterpretation of symbolic notations. However, there are different views and ongoing debates on the best strategy to overcome the symbol difficulties through appropriate teaching. It can be argued that not all difficulties in representing or interpreting symbolic notations are due to teaching approaches. There are varied explanations about sources of symbolic misconceptions ranging from the multiple meanings to roles that the same symbol plays in algebraic contexts. Therefore, to improve learners' algebraic learning ability there is still need for further research into dialectical relationships between symbols and algebraic thinking. This study creates opportunities for further research in the South African context.

The experience gained from this study raises the following questions:

a. What are teachers' perceptions on incorporating reading in mathematics instruction?
b. What strategies can teachers use to help learners overcome difficulties experienced during the transition from concrete to abstract representations of mathematical concepts?
c. In what ways can mathematics syllabi be re-organised in order to close learning gaps between primary and secondary mathematics?

In addition, researchers who may wish to extend knowledge on mathematical symbolisation should take into account the fact that the data in this study were collected from six secondary schools in two districts. Engaging learners and teachers from a number of districts over a long period may allow researchers to gain more insights into the envisaged phenomena.

6.6 Closing Remarks

If mathematics teachers want learners to understand mathematical concepts, as opposed to the rote memorisation and manipulation of rules without reason, it is crucial that they link abstract mathematical symbolism with representations from learners' everyday world. The focus of mathematics teaching should aim to enable learners to construct cognitive links between abstract mathematical concepts and their symbolic representations. Furthermore, the context or the topic in which symbols are used should be understood in order to get appropriate meanings. Without such links, mathematical abstractions remain mysterious, unattainable and learners will continue to fail to grasp the importance and power of mathematical symbolisation in solving mathematics problems.

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APPENDICES

Participant Number

8.1 Appendix A: QUESTIONNAIRE FOR LEARNERS

P O Box 199, Sovenga 0727
Tel: + 27 71 757 9859 / +27152683619

July 2015

Dear sir/madam

I, Mutodi Paul, wish to undertake a research project to explore and examine the nature of challenges and obstacles faced by secondary school learners as they try to use and verbalise various mathematical symbols written in grade 10-12 South African secondary school textbooks. To this end I kindly request that you complete the following short questionnaire regarding your views about the challenges and obstacles experienced by secondary school learners in their attempt to use mathematical symbols and the various instructional strategies that can be employed to solve the identified difficulties. It should take no longer than 30 minutes of your time. Although your response is of the utmost importance to me, your participation in this survey is entirely voluntary.

Please do not enter your name or contact details on the questionnaire. It remains anonymous. Information provided by you remains confidential and will be reported in summary format only.

Kindly return the completed questionnaire to me in the postage paid return envelope on or before 30 August 2015.Should you have any queries or comments regarding this survey, you are welcome to contact me telephonically at 071 757 9859 or e-mail me at paurosmutodi@yahoo.com

Yours sincerely

Paul Mutodi.

Please answer the following questions by marking (√) the relevant block or writing down your answer responses in the spaces provided.

Your opinion is extremely important in this survey, as it seeks to better understand the experiences of all learners in relation to mathematical symbolisation. To ensure that the system of education of South Africa is continually improving, particularly in the teaching and learning of mathematics, I would like to invite you to share your thoughts and experiences about the challenges faced by secondary school learners in their attempt use various mathematics symbols. All results will be aggregated and kept anonymously. Please be assured that your individual responses will be used specifically for the purpose of this study and for no other purpose and will be treated with the strictest confidence it deserves. Thanks in advance.

SECTION A

BACKGROUND INFORMATION

For each of the items below, please indicate the option that applies to you with a circle or supply the required detail.

EXAMPLE of how to complete this questionnaire:

Respond by circling the response that applies to you as illustrated below:

Your gender?

If you are female:

Male	1
Female	(2)

1. Gender

Male	1	Female	2

2. Age in years

10 years and below	1	11-15 years	2	16-20 years	3	21 years and above	4

3. Home Language

Sepedi	1	Sesotho	2	Tshivenda	3	Xitsonga	4	Other (Specify)	5

4. Ethnicity

Black	1	White	2	Coloured	3	Indian or Asian	4	Other (Specify)	5

5. How would you describe your social economic status (SES)? Please note that SES refers to family's economic and social position in relation to others, based on income, education, and occupation.

High SES	1	Middle SES	2	Low SES	3

6. Your current grade level?

Grade 10	1	Grade 11	2	Grade 12	3

7. How would you describe the area in which you are residing?

Urban	1	Semi-Urban	2	Rural	3	Deep Rural	4

8. Size of your household, i.e. the number of people, including yourself, who live in your house/dwelling for at least three months of the year.

Live Alone	1	Two	2	Three	3	Four	4	Above 5	5

SECTION B

This section of the questionnaire explores your experiences, challenges and obstacles, encounters if any, with regard to the use of mathematical symbols.

With respect to your understanding, experiences and difficulties in dealing with the various mathematical symbols and formulae please indicate by means of an X directly under the number the extent to which you agree or disagree with the following statements:

	Key: **1 = Strongly Disagree 2 = Disagree** **3 = Neutral 4 = Agree** **5 = Strongly Agree**	**Strongly Disagree**	**Disagree**	**Neutral**	**Agree**	**Strongly Agree**
C1	Mathematical symbols affect my understanding of mathematics concepts.	1	2	3	4	5
C2	I understand the symbols and formulae in the current textbooks	1	2	3	4	5
C3	I am able to express word problems compactly using appropriate symbols.	1	2	3	4	5
C4	When I fail to cope with some symbol I seek help instead of taking them as they are.	1	2	3	4	5
C5	I am able to handle expressions and equations using appropriate symbols.	1	2	3	4	5
C6	I struggle to assign meanings to the symbols and this negatively affects my conceptualisation.	1	2	3	4	5
C7	Unfamiliar mathematical symbols in a concept/topic often mark the point where I fail to understand the topic.	1	2	3	4	5
C8	I am able to learn how to use all symbols and language that is used in the textbooks.	1	2	3	4	5
C9	Navigating through the symbols and their meanings is easy to do.	1	2	3	4	5
C10	Mathematical symbols strongly affect my understanding of Algebra and related topics.	1	2	3	4	5
C11	Sometimes my own meanings of mathematical symbols often contradicts with the actual meaning and this often hampers my progress in problem solving	1	2	3	4	5
C12	My interpretation and use of mathematical symbols affect my competence in mathematics.	1	2	3	4	5
C13	The symbols in a formula sometimes contradict with my thinking.	1	2	3	4	5
C14	Linking concepts and appropriate symbols is easy.	1	2	3	4	5
C15	I am flexible to move from one formula to another in relation to the demands of task using appropriate symbols.	1	2	3	4	5
C16	The teaching and learning methods used by my current teacher enhance my understanding of the use of the various mathematical symbols	1	2	3	4	5
C17	Mathematics teachers who taught me in lower grades made attempts to foster the connection between symbols and their meanings.	1	2	3	4	5
C18	I get my mathematics tasks done quickly with clear understanding of the symbols and features used in the task.	1	2	3	4	5
C19	Discovering new symbols and features with their meanings is easy.	1	2	3	4	5
C20	Mathematical symbols and formula strings are satisfying to use	1	2	3	4	5

C21	The symbols in a mathematical problem have a significant influence on my attempt to solve a problem	1	2	3	4	5
C22	The symbols in a mathematical problem influence my goals, activities and organisation of results when solving a mathematical problem.	1	2	3	4	5
C23	I am able to switch representations from geometric situations to algebraic and algebraic situations to geometric.	1	2	3	4	5
C24	I am able to define the meaning of symbols introduced to solve problems, including specifying units and distinguishing among the three main uses of variables(unknowns, placeholders, parameters)	1	2	3	4	5
C25	I am able to read expressions, formulae in different ways.	1	2	3	4	5
C26	I read the question several times to gain the meaning of the problem together with the symbols before solving it.	1	2	3	4	5

SECTION C

1. What challenges do you encounter in your attempt to understand mathematical concepts and problem solving procedures through interpreting mathematical symbols?

----

2. Do you attempt to make connections between what you think and the actual mathematical concept in its system of symbolism?

----

3. Are you capable of making adequate use of the conventional mathematics symbols you have learned in class to represent problem solving situations, procedures and concepts?

----

4. What challenges do you normally experience when using or reading mathematical symbols?

----

5. What do you think is the major cause of your inability to effectively use mathematical symbols?

----

6. What kinds of support do think you need in order to improve your use of and interpretation of mathematical symbols?

----

7. In your own opinion, what instructional strategies can the teachers employ to eliminate the negative influences of the challenges posed by mathematical symbols?

----

8. Do you use mathematical symbols and endow them with meaning during mathematics lessons and activities?

----

9. Are you able to engineer a mathematical expression or symbol or sign as demanded by a given mathematical problem?

----

10. What do you think is the role of mathematical symbols in the learning of Mathematical concepts?

----

11. Do you think your grade 8 and 9 teachers and current teachers were or are making effort to make sure that you grasp the symbols and their meanings?

----

12. Do you appreciate the role of mathematical symbols in enhancing your understanding of mathematical concepts?

SECTION D

CURRENT TEACHING AND LEARNING APPROACHES, METHODS AND TOOLS

1. Do you think the current textbooks use familiar symbols? Explain your response fully.

----

2. Do you think your teacher helps you to understand new symbols in a new topic? Explain your response fully.

----

3. Do you make an attempt to understand new symbols when attempting a new topic? Explain your response fully.

----

4. Do the teaching and learning methods used by the teacher(s) help you to understand and grasp the concepts together with symbolic strings?

----

5. How do you expect the teachers to engage you so that you can link concepts and their symbol as well as their meanings? Explain your response fully.

What are your alternative sources of understanding mathematical symbols and concepts? Explain your response fully.

----

6. Are you able to link symbols from other topics and apply them appropriately to the new topic? Explain your response fully.

----

7. Are you able to apply a strategy or reference system that draws on previous learning in another context? Explain your response fully.

----

8. Are you able to make connections between new and prior knowledge to make sense of what you are learning? Explain your response fully.

Are you able to distinguish the use a given symbol in different contexts? Explain your response fully.

----

9. Are you able to make connections between different representations, e.g., numeric, graphical, and/or algebraic? Explain your response fully.

Thank you for taking the time to complete this questionnaire

8.2 Appendix B: Questionnaire for Teachers

Participant Number

INTERVIEW SCHEDULE FOR SECONDARY SCHOOL MATHEMATICS TEACHERS

P O Box 199, Sovenga 0727,South Africa
Tel: + 27 71 757 9859 / +27152683619

July 2015

In this interview I would like to hear your views about the aspects of mathematical symbolisation you find easy or difficult to apply in your classes and the reasons for this. This discussion will be confidential and the reporting will be anonymous, there is no physical or emotional harm resulting from participating in the interview although some psychological discomfort may result from the nature of some questions. Should you wish to discuss any aspect related to the study, you are free to use my contact details below. It is in your best interest to decide independently, without any coercion, whether or not to participate in this study. You also have the right not to respond to questions that can cause any form of discomfort, to disclose or not to disclose personal information and to ask for clarification about any aspect that caused some uncertainty. So I hope you can be as frank as possible.

SECTION A

BACKGROUND INFORMATION

This section of the questionnaire refers to background or biographical information. Although I am aware of the sensitivity of the questions in this section, the information will allow me to compare groups of respondents. Once again, I assure you that your response will remain anonymous. Your co-operation is appreciated.

1. Gender

Male	1	Female	2

2. Age in years

25 years and below	1	26-35 years	2	35-50years	3	51 years and above	4

3. Home Language

Sepedi	1	Sesotho	2	Tshivenda	3	Xitsonga	4	Other (Specify)	5

4. Ethnicity

Black	1	White	2	Coloured	3	Indian or Asian	4	Other (Specify)	5

4. How would you describe your social economic status?

High SES	1	Middle SES	2	Low SES	3

5. Your highest academic level?

	Mathematics		Any other Subject	
Post-Matric Diploma or certificate	1		2	
Undergraduate Degree(s)	1		2	
Highest level of mathematics i.e. have you done Maths 1, Maths 2, Maths 3, etc.	Maths 1	Maths2	Maths 3	Maths 4
Post- Graduate Degree(s)	1		2	
Other(s)(Specify): ----------------------------	1		2	

6. How would you describe the area in which your school is located?

Urban	1	Semi-Urban	2	Rural	3	Deep rural	4

7. Mathematics Teaching Experience at FET Phase

5years and below	1
6-10 years	2
11-15years	3
16-20years	4
21 years & above	5

SECTION B

This section of explores your experiences, challenges and obstacles, encounters if any, with regard to the use of mathematical symbols. Try to provide as much information as possible.

1. Based on your experience as a mathematics teacher, what do you think are the critical challenges of mathematical symbolisation?

----

2. In what ways do mathematical symbolisation affects your classroom teaching? Do you sometimes design strategies to help learners who struggle to verbalise, write or read symbols in the various mathematical activities? What strategies do you use to enforce these activities?

3. Do you think an instruction in reading; writing and verbalising mathematical symbols improve learners' understanding of mathematical concepts?

4. What challenges do you encounter when dealing with mathematical symbols to develop concepts in the teaching and learning process?

5. How do you foster the connection between symbols and their meanings in the teaching and learning process?

6. How could it be made easier for you to integrate this in your mathematics teaching?

7. What do you think are the instructional strategies (practices) that can be implemented in order to remove or reduce the effects symbolic obstacles?

8. Do the current teaching and learning methods used by the teacher's address the

9. difficulties learners experience as a result of mathematical symbolisation? Which methods do you use to address these challenges?

10. What alternatives did you consider and why did you reject the current teaching and learning practices?

11. In what ways do the symbols in a mathematics problem influence the learner's attempt to solve the problem?

12. In what ways do the symbols in a mathematical problem influence the learners' goals, activities and organization of results when solving a mathematical problem?

13. Do you sometimes make an attempt to foster connections between learners' informal knowledge and the abstract and arbitrary system of symbolism?

14. Explain how you do it.

15. How could it be made easier for you to integrate this in your teaching?

16. What can you do as mathematics teacher to ensure that learners are capable of using conventional mathematics symbols they have learned in class to represent problem - solving situations, procedures and concepts?

17. Do your learners use mathematical symbols and endow them with meaning during mathematics lessons and activities?

18. Do mathematics textbooks used by learners exhibit a symbol precedence of mathematical development, that is, do they present symbol problems prior to verbal problems?

19. What additional support do you think will strengthen the implementation of the method that you suggested will solve the problem of mathematical symbolisation in classroom teaching?

20. Is there anything else you want to add about the ways mathematical symbolisation has impacted on your mathematics teaching?

SECTION C

Instructional Practices

i. When monitoring learners' progress toward mathematical symbolisation goals, do you check how each learner is progressing in relation to learning goals?

----

ii. Which assessment tool do you use to check learners' progress in the attainment of mathematical symbolisation goals?

----

iii. How do you handle learners' misconceptions related to the use of specific symbols for a particular concept?

----

2. When making instructional decision, do you sometimes use evidence about learners' progress with mathematical symbolisation to make instructional decisions?

----

3. When evaluating learners' achievement— do you keep records of how each learner's understanding of mathematical symbols at any stage compares with the goals that learner is expected to achieve?

----

4. When evaluating the curriculum—does the CAPS curriculum meet the goals and expectations of learners' mathematical symbolisation?

---

5. When writing math problems, formulas, and other information on a chalk board or flip chart, do you think it is important to write large, neat, and specific symbols for a particular mathematical concept?

---

6. i. Do you ensure that the use of "real-world" contexts for teaching mathematics maintains a focus on mathematical ideas and emphasise the selection of appropriate symbols?

----

ii. Do you make attempts to link mathematical symbols and their referrals in real –life situation? In which ways do you do that?

7.Mathematics should be taught using multiple strategies; however, the teacher is responsible for selecting appropriate an strategy and symbols for a specific concept .Do you select a strategy together with the appropriate symbols in order to teach a specific concept?

8.Do you think it is important for learners to understand the underlying meaning and justifications for ideas and their respective symbols and be able to make connections among topics?

----

9. Do you think or believe that competence with mathematical symbols precedes verbal reasoning?

----

10. Do you use a variety of continuous assessment programmes designed to evaluate both learner progress and teacher effectiveness in acquiring proficiency with mathematical symbolisation?

----

11. Do you create learning environments where learners are active participants as well as members of collaborative groups?

----

12. Do you create a safe environment in which high, clear expectations and positive relationships are fostered; active learning is promoted in order to enhance understanding of mathematical concepts with their appropriate symbols? How?

----

13. Do you think the provision of access to the common core curriculum by utilizing differentiated teaching strategies, interventions, manipulatives, calculators and information technology can help learners to understand abstract mathematics?

----

14. Do you as mathematics teacher effectively allocate time for learners to engage in hands-on experiences, discuss and process content and make meaningful connections?

----

15. Do you design lessons that allow learners to participate in empowering activities in which they understand that learning is a process and mistakes are a natural part of learning? Illustrate

----

----

16. Do you create an environment where learner work is valued, appreciated and used as a learning tool?

ADDITIONAL QUESTIONS

17. Do you emphasise the importance of using and manipulating symbols when doing operations ranging from simple basic addition to algebraic equations?

18. Do you integrate hands-on activities and verbal explanations into the learning of spatially based concepts which are in symbolic forms such as graphs and pictures?

19. Do you select and use examples of familiar situations, or analogies, to talk and think about mathematics concepts and link them to abstract ideas together with their appropriate symbols?

20. Do you encourage learners to communicate mathematics both orally and in writing in order to deepen their understanding of the mathematics?

21. How do you harmonise learners' use of informal symbols and formal symbols when solving mathematical problems?

Thank you for participating in the interview

8.3 Appendix C: Informed Consent Form for Conducting Research

Research Topic: Mathematical Symbolisation: Challenges and Instructional Strategies for Limpopo Province Secondary School learners

Parents' Informed Consent

1. I hereby confirm that I have been informed by the researcher, Mr. P. Mutodi about the nature of the study.
2. I have also received, read and understood the Information and Consent sheets regarding the educational study.
3. I am aware that the information my child gives will be processed without mentioning his/her real name.
4. In view of the requirements of the research, I agree that the data collected during this study can be processed in a computerized system by the researcher.
5. My child can at any stage, without prejudice, withdraw his/her participation in the study.
6. I have had sufficient time to ask questions and (of my free will) allow my child to join the study.

Name: ____

Signature: ____

Date: ____

Consent Form Learners' Interview

Please fill in the reply slip below if you agree to be interviewed. I will use your answers to my questions for my study called:

Mathematical Symbolisation: Challenges and Instructional Strategies for Limpopo Province Secondary School learners

Permission for interview

My name is: ____________________

I would like to be interviewed for this study.	YES/NO
I know that Mr Mutodi will keep my information confidential.	YES/NO
I know that I can stop the interview at any time and don't have to answer all the questions asked.	YES/NO

Sign: ______________________________ Date: __________________

Contact Details:

NAME: Mr P Mutodi

ADDRESS: University of Limpopo
Turf loop Campus
P.O. Box X1106, Sovenga

TEL No: 015 265 3619/ 0717579859