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Riemann–Hilbert Problems for Monogenic Functions on Upper Half Ball of $\mathbb{R}^4$

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Abstract. In this paper we are interested in finding solutions to Riemann–Hilbert boundary value problems, for short Riemann–Hilbert problems, with variable coefficients in the case of axially monogenic functions defined over the upper half unit ball centred at the origin in four-dimensional Euclidean space. Our main idea is to transfer Riemann–Hilbert problems for axially monogenic functions defined over the upper half unit ball centred at the origin of four-dimensional Euclidean spaces into Riemann–Hilbert problems for analytic functions defined over the upper half unit disk of the complex plane. Furthermore, we extend our results to axially symmetric null-solutions of perturbed generalized Cauchy–Riemann equations.

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1. Introduction

Bernhard Riemann was the first to consider a version of the problem which is nowadays generally called a Riemann–Hilbert problem. But his statement of the problem and its solution were very geometrically. David Hilbert rewrote the problem using a singular integral operator, i.e. the singular Cauchy integral operator or Hilbert transform. Later on, the theory of Riemann–Hilbert problems has been well developed by many authors, see, e.g. [4,17,22,30,34]. Recently, this theory has also been applied to the investigation of other boundary value problems, including Dirichlet, Neumann, Schwarz, Robin, and Riemann–Hilbert–Poincaré BVPs for analytic functions, which are either special cases of Riemann–Hilbert boundary value problems or closely linked to them. Apart from the theoretical significance of Riemann–Hilbert

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boundary value problems, their study is closely connected with the theory of singular integral equations [17,30] and has a wide range of applications in other fields, such as in the theory of cracks and elasticity [22,30], in quantum mechanics and of statistical physics [9] as well as in the theory of linear and nonlinear partial differential equations [14], in the theory of orthogonal polynomials and asymptotic analysis [12], and in the theory of time-frequency analysis [1]. In addition, their methods and related problems have been extended to null-solutions to complex partial differential equations (PDEs) on the complex plane like poly-analytic functions, meta-analytic functions, and poly-harmonic functions, see, e.g. [5]. Moreover, also recently, the boundary value theory of null-solutions to complex PDEs has been explored on special domains of the complex plane, like the unit disk, upper half plane, half disks, circular rings, triangles, and sectors, and so on, see, e.g. [5,6,36].

In parallel, there have been many attempts of generalizing the classical boundary value theory of Riemann–Hilbert problems into higher dimensions, mainly by considering two principal ways, i.e., the theory of several complex variables and Clifford analysis, in particular quaternionic analysis. The latter is an elegant generalization of the classical theory of complex analysis. It concentrates on the study of the theory of so-called monogenic functions, and refines real harmonic analysis in the sense that its principal operator, the Dirac or generalized Cauchy–Riemann operator, factorizes the higher-dimensional Laplace operator, see, e.g. [7,13,18]. In the setting of quaternionic and Clifford analysis, Riemann–Hilbert problems were discussed by many authors, see, e.g. [2,8,16,18–21,23–29,32,33,35]. In Refs. [2,8,16,19,26,27,29,32,33,35], the authors studied Riemann–Hilbert problems with constant coefficients. Their solutions were given explicitly in terms of Cauchy-type integral operators together with power series expansions. Moreover, these problems are closely connected with applications, like the theory of fluid mechanics [18] and signal processing in higher dimensions [16,32]. To the authors' knowledge in Refs. [20,21] the authors made the first attempt in the direction of solving Riemann–Hilbert problems with variable coefficients for axially monogenic functions (see Sect. 2) defined on four dimensional domains, whose projection onto the corresponding complex plane is contained in the upper half plane. The basic method is to apply Fueter's theorem, see e.g. [10,11,15,31] for the four dimensional case, and to transfer the Riemann–Hilbert problem to an equivalent Riemann–Hilbert problem for analytic functions over the complex plane. However, so far, there are no results about Riemann–Hilbert problems with variable coefficients for axially monogenic functions defined over special domains in four dimensions, like the upper half unit ball, upper half space, and the quadrant. Thus, it is natural to look into these cases. This is not just a theoretical question for Riemann–Hilbert problems because such problems are intimately related to problems in mechanics and mathematical physics, in particular problems like fluid and hydrodynamic mechanics [18]. Moreover, solving such problems will provide a possible tool to study linear and of nonlinear partial differential equations in higher dimensions [14] as well as orthogonal polynomials and their asymptotic analysis [12,13]. Motivated by these considerations, the aim of this paper is to study Riemann–Hilbert problems on the upper half ball centred at the origin of $\mathbb{R}^4$. Our idea is to transfer them to Riemann–Hilbert problems for holomorphic functions in complex plane by applying Fueter's theorem in four dimensions, and to derive explicit integral expressions for the solution to the Riemann–Hilbert problem under investigation. Besides the focus on different domains the difference from [20,21] lies in the fact that in this paper the Riemann–Hilbert problems are studied for continuous boundary data instead of Hölder continuous boundary data like in [20,21], which leads us to the construction of different kernel functions in the involved integral representation formulae. Afterwards, these integral formulae will be used to prove our results. We are confident that our results will help to further develop the theory of Riemann–Hilbert problems for monogenic functions with variable coefficients in four dimensions.

The paper is organized as follows. In Sect. 2, we will recall the necessary facts about quaternionic analysis. In Sect. 3, we will provide a detailed exposition of Riemann–Hilbert problems for axially monogenic functions with variable coefficients defined the upper half unit ball centred at the origin of $\mathbb{R}^4$, and link them to Riemann–Hilbert problems for analytic functions over the upper half disk in the complex plane. Then, we will derive solutions to them with continuous boundary data in terms of a integral formulae by solving the equivalent Riemann–Hilbert problems for analytic functions over the upper half disk in the complex plane. In final section, we extend the results obtained in Sect. 3 to null-solutions to the Riemann–Hilbert problems for $(\mathcal{D}-\alpha)\phi = 0, \alpha \in \mathbb{R}$ with axial symmetry, where $\mathbb{R}$ denotes the field of real numbers.

2. Preliminaries

For the convenience of the reader we review some of the standard facts about Quaternions and quaternionic analysis [7,13,18,28].

Let $\{e_0, e_1, e_2, e_3\}$ be the standard basis of $\mathbb{H}$. They satisfy the relationships as follows:

$$e_ie_j + e_je_i = -2\delta_{i,j}, \quad i,j = 1,2,3, \qquad e_1e_2 = e_3, \quad e_2e_3 = e_1, \quad e_3e_1 = e_2,$$

where δ denotes the Kronecker delta, and $e_0 = 1$ denotes the identity element of the algebra of quaternions $\mathbb{H}$. Thus, $\mathbb{H}$ is a real linear, associative, but non-commutative algebra.

An arbitrary quaternion $x \in \mathbb{H}$ can be written as $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \triangleq x_0 + \underline{x}, x_j \in \mathbb{R}$, where $\mathrm{Sc}(x) \triangleq x_0$ and $\mathrm{Vec}(x) \triangleq \underline{x}$ are the scalar and vector part of $x \in \mathbb{H}$, respectively. Elements $x \in \mathbb{R}^4$ can be identified with quaternions $x \in \mathbb{H}$. The conjugation is defined by $\overline{x} = \sum_{j=0}^{3} \overline{x}_j\overline{e}_j,$ with $\overline{e}_0 = e_0$ and $\overline{e}_j = -e_j, j = 1,2,3$, and hence, $\overline{xy} = \bar{y}\bar{x}$. The algebra of $\mathbb{H}$ possesses an inner product $(x,y) = \mathrm{Sc}(x\overline{y}) = (x\overline{y})_0$ for all $x, y \in \mathbb{H}$.

The corresponding norm is $|x| = (\sum_{j=0}^{3} |x_j|^2)^{\frac{1}{2}} = \sqrt{(x,x)}$. Any element $x \in \mathbb{R}^4\backslash\{0\}$ is invertible with inverse element $x^{-1} \triangleq \overline{x}|x|^{-2}$, i.e., $xx^{-1} = x^{-1}x = 1$. Furthermore, we can introduce the set

$$[x] = \{y : y = \mathrm{Sc}(x) + \mathcal{I}|\underline{x}|, \mathcal{I} \in S^2\},$$

where $S^2 = \{\underline{x} \subset \mathbb{R}^3 : |\underline{x}| = 1\}$.

In this paper we will consider the generalised Cauchy–Riemann operator $\mathcal{D} = \sum_{j=0}^{3} e_j \partial_{x_j}$ in $\mathbb{R}^4$. The generalised Cauchy–Riemann operator factorizes the Laplacian in the sense $\overline{\mathcal{D}}\mathcal{D} = \sum_{j=0}^{3} \partial_{x_j}^2 = \Delta$, where Δ denotes the Laplacian in $\mathbb{R}^4$.

Without loss of generality, in the remainder of this paper we will restrict our attention to the upper unit ball $\mathbf{B}_+ = \{x \in \mathbb{R}^4 | (|x| < 1) \wedge (x_3 > 0)\}$ with its boundary $\partial\mathbf{B}_+ = \{x \in \mathbb{R}^4 | [(|x| = 1) \wedge (x_3 > 0)] \vee [(|x| < 1) \wedge (x_3 = 0)]\}$. The study of the case of other balls centred at the origin of $\mathbb{R}^4$ is similar. Following Definition 2.1 in [20,21,31], we say that a nonempty open set $\Omega \subset \mathbb{R}^4$ is axially symmetric if for arbitrary $x \in \Omega$, the subset of $[x]$ is contained in $\Omega \subset \mathbb{R}^4$. It is obvious that $\mathbf{B}_+$ is an axially symmetric domain. An $\mathbb{H}$-valued function $\phi = \sum_{j=0}^{3} \phi_j e_j$ is continuous, Hölder continuous, p-integrable, continuously differentiable and so on if all components $\phi_j : \mathbf{B}_+ \cup \partial\mathbf{B}_+ \to \mathbb{R}, j = 0, 1, 2, 3$, have that property. The corresponding function spaces, considered as either right-Banach or right-Hilbert modules, are denoted by $C(\Omega, \mathbb{H})$, $H^\mu(\Omega, \mathbb{H})(0 < \mu \leq 1)$, $L_p(\Omega, \mathbb{H})(1 < p < +\infty)$, $C^1(\Omega, \mathbb{H})$, respectively.

Referring to Fueter's theorem (see, [10,11,15,20,21,31]) a function of axial type is given by

$$\phi(x) = A(x_0, r) + \underline{\omega} B(x_0, r), \tag{1}$$

where $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in \mathbb{R}^4, r = |\underline{x}|, \underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$, $A(x_0, r)$ and $B(x_0, r)$ are real-valued functions. It should be mentioned that in [15,25,31] a function of axial type is also called a function with axial symmetry.

In what follows, any functions defined on $\mathbf{B}_+ \cup \partial\mathbf{B}_+$ taking values in $\mathbb{H}$ are supposed to be of axial type unless otherwise stated.

Definition 2.1. A function $\phi \in C^1(\mathbf{B}_+, \mathbb{H})$ is called (left-) monogenic if and only if $\mathcal{D}\phi = 0$. A monogenic function of axial type is called axially monogenic. It is obvious that the set of all axially monogenic functions defined in $\mathbf{B}_+$ forms a right-module, denoted by $\mathcal{M}(\mathbf{B}_+, \mathbb{H})$.

Definition 2.2. For a function of axial type $\phi : \mathbf{B}_+ \to \mathbb{H}$, we define the real part as $\mathrm{Re}\, \phi = A$.

Remark 2.3. From [7,13,20,21], a special type of Vekua system on the unit ball $\mathbf{B}_+$ is derived from the equation $\mathcal{D}\phi = 0$ for functions of axial type belonging to $C^1(\mathbf{B}_+, \mathbb{H})$. That is, associated with Term (1), $\mathcal{D}\phi = 0$ leads to

$$(*) \begin{cases} \partial_{x_0} A - \partial_r B = \frac{2}{r} B, \\ \partial_{x_0} B + \partial_r A = 0, \end{cases}$$

where $\partial_{x_0}, \partial_r$ denote $\frac{\partial}{\partial_{x_0}}, \frac{\partial}{\partial_r}$ respectively, and Term $(*)$ is a special type of Vekua system with respect to A, B.

In what follows we denote by $D_+ = \{z = x_0 + ir \big| |z| < 1, r > 0\} \subset \mathbb{C}_+$ the projection of $\mathbf{B}_+$ into the (x_0, r)-plane, where $\mathbb{C}_+$ is the upper half of the (x_0, r)-plane. This projection corresponds to consider ω fixed and replacing it with the imaginary unit since $\omega^2 = 1$. Moreover, since $\mathbf{B}_+$ is of axial type the projection does not depend on the actual choice of ω.

3. Riemann–Hilbert Problems for Axially Monogenic Functions

In this section we will proceed with the study of the Riemann–Hilbert problems with variable coefficients for axially monogenic functions on upper half unit ball of $\mathbb{R}^4$. Our Riemann–Hilbert problem is solved by taking account into an equivalent Riemann–Hilbert problem on upper half disc for complex analytic functions. When the boundary value belongs to a $C(\mathbf{B}_+, \mathbb{H})$ space, we will get its solution in terms of explicit integral representation formulae. As a corollary we get the solution to the corresponding Schwarz problem.

Problem I. Find a function $\phi \in C^1(\mathbf{B}_+, \mathbb{H})$ of axial type, which satisfies the condition

$$\begin{cases} \mathcal{D}\phi(x) = 0, & x \in \mathbf{B}_+, \\ \mathrm{Re}\{\lambda(t)\phi(t)\} = g(t), & t \in \partial\mathbf{B}_+, \end{cases} \tag{2}$$

where $g : \partial\mathbf{B}_+ \to \mathbb{R}$ and λ is a $\mathbb{R}$-valued function defined on $\partial\mathbf{B}_+$.

Let us start with the following lemma, which links Riemann–Hilbert problems for axially monogenic functions on upper half unit ball of $\mathbb{R}^4$ with Riemann–Hilbert problems for analytic functions over upper half disc of the complex plane. More details about the construction principle can be found in [10, 11, 15, 20, 21, 31].

Lemma 3.1. *The Riemann–Hilbert problem* (2) *is equivalent to the following problem: Find a complex-valued analytic function* h, *satisfying one of the following conditions*:

(i) *When* $\lambda \in C(\partial\mathbf{B}_+, \mathbb{R})$, *and* $\lambda \neq 0$ *for arbitrary* $x \in \partial\mathbf{B}_+$,

$$\begin{cases} \partial_{\bar{z}} h(z) = 0, & z \in D_+, \\ Re\{h(z)\} = \frac{r}{2}\frac{g(z)}{\lambda(z)}, & z \in \partial D_+, \end{cases} \tag{3}$$

where $z = x_0 + ir, \partial_{\bar{z}} = \frac{1}{2}(\partial_{x_0} + i\partial_r), h = \partial_r f$, f *and* r *given as in Lemma* 3.1, $\lambda, g : \partial D_+ \to \mathbb{R}$ *are both scalar-valued functions, and* $D_+ = \{z = x_0 + ir \big| |z| < 1, r > 0\}$ *with boundary* ∂D_+.

(ii) *When* $\lambda = \Pi\widehat{\lambda}$ *with* $\Pi(x) = \Pi_{i=1}^m (x - \widehat{\alpha}_i)^{\nu_i}$, $\widehat{\alpha}_i \in \partial\mathbf{B}_+, \nu_i \in \mathbb{N}$, *and* $\widehat{\lambda} \neq 0$ *for arbitrary* $x \in \partial\mathbf{B}_+$,

$$\begin{cases} \partial_{\bar{z}} \hat{h}(z) = 0, & z \in D_+, \\ Re\{\hat{h}(z)\} = \frac{r}{2}\frac{g(z)}{\hat{\lambda}(z)}, & z \in \partial D_+, \end{cases} \tag{4}$$

where $z = x_0 + ir, \partial_{\bar{z}} = \frac{1}{2}(\partial_{x_0} + i\partial_r), \hat{h} = \Pi\partial_r f$, f *and* r *given as in Lemma* 3.1, $\lambda, g : \partial D_+ \to \mathbb{R}$ *are both scalar-valued functions, and* $D_+ = \{z = x_0 + ir \big| |z| < 1, r > 0\}$ *with boundary* ∂D_+.

Proof. By either Theorem 3.1 in [20] or directly applying Fueter's theorem we know that the Riemann–Hilbert problem (2) is equivalent to the problem

$$\begin{cases} \partial_{\bar{z}} h(z) = 0, & z \in D_+, \\ \mathrm{Re}\{\lambda(z)h(z)\} = \frac{r}{2} g(z), & z \in \partial D_+. \end{cases} \tag{5}$$

Hereby, the uniqueness is imposed by the boundary condition and the maximum principle for holomorphic functions.

Since $\lambda \in C(\partial \mathbf{B}_+, \mathbb{R})$, we get that $\lambda \in C(\partial D_+, \mathbb{R})$. Hence, it is necessary and sufficient for us to consider all zero points of λ on ∂D_+.

(i) When $\lambda \in C(\partial \mathbf{B}_+, \mathbb{R})$, and $\lambda \neq 0$ for arbitrary $x \in \partial \mathbf{B}_+$, then its projection onto (x_0, r)-plane belongs to $C(\partial D_+, \mathbb{R})$, and $\lambda \neq 0$ for arbitrary $x \in \partial D_+$. So, one obtains the Riemann–Hilbert problem (3).

(ii) When $\lambda = \Pi \widehat{\lambda}$ with $\Pi(x) = \Pi_{i=1}^{m} (x - \widehat{\alpha}_i)^{\nu_i}$, $\widehat{\alpha}_i \in \partial \mathbf{B}_+, \nu_i \in \mathbb{N}$, and $\widehat{\lambda} \neq 0$ for arbitrary $x \in \partial \mathbf{B}_+$, it is obvious that $\widehat{\lambda} \in C(\partial \mathbf{B}_+, \mathbb{R})$. Therefore, on the (x_0, r)-plane, $\lambda = \Pi \widehat{\lambda}$ with $\Pi(z) = \Pi_{i=1}^{m} (z - \widehat{\alpha}_i)^{\nu_i}$, $\widehat{\alpha}_i \in \partial D_+, \nu_i \in \mathbb{N}$, and $\widehat{\lambda} \neq 0$ for arbitrary $x \in \partial D_+$, $\widehat{\lambda} \in C(\partial D_+, \mathbb{R})$.

Since $\Pi(z) = \Pi_{i=1}^{m} (z - \widehat{\alpha}_i)^{\nu_i}$ is entire function with respect to z, where $\widehat{\alpha}_i \in \partial D_+, \nu_i \in \mathbb{N}$.

Let $\hat{h} = h\Pi$. The Riemann–Hilbert problem (5) is changed into the case

$$\begin{cases} \partial_{\bar{z}} \hat{h}(z) = 0, & z \in D_+, \\ \mathrm{Re}\{\widehat{\lambda}(z)\hat{h}(z)\} = \frac{r}{2} g(z), & z \in \partial D_+. \end{cases} \tag{6}$$

Applying the case (i), we derive the Riemann–Hilbert problem (4). Thus, the proof is complete. □

It is worth pointing out that Lemma 3.1 allows us to study the solvability of our original Riemann–Hilbert problem on upper half unit ball $\mathbf{B}_+$ of $\mathbb{R}^4$ by discussing the equivalent Riemann–Hilbert problem over the upper half disc of complex plane.

Now, we give the following theorem.

Theorem 3.2. *Let $g \in C(\partial \mathbf{B}_+, \mathbb{R})$, and D_+ with boundary ∂D_+.*

(i) *If $\lambda \in C\big(\partial \mathbf{B}_+, \mathbb{R}\big)$, and $\lambda \neq 0$ for arbitrary $x \in \partial \mathbf{B}_+$, then the Riemann–Hilbert problem (2) is solvable and its solution is given by*

$$\phi(x) = \Delta(Re(f)(x_0, |\underline{x}|) + \underline{\omega} Im(f)(x_0, |\underline{x}|)), \quad x \in \mathbf{B}_+,$$

where $Re(f)$ and $Im(f)$ denote the real and imaginary part of the complex-valued function f, respectively. f itself is given by

$$f(z) = \frac{1}{2\pi} \int_0^z \int_\Gamma \left(\frac{\zeta + \xi}{\zeta - \xi} - \frac{\bar{\zeta} + \xi}{\bar{\zeta} - \xi} \right) \frac{\widetilde{g}_1(\zeta)}{\zeta} d\zeta d\xi,$$
$$+ \frac{1}{\pi} \int_0^z \int_{-1}^1 \left(\frac{1}{t - \xi} - \frac{\xi}{1 - \xi t} \right) \widetilde{g}_1(t) dt d\xi, \quad z \in D_+,$$

with $\widetilde{g}_1 = \frac{r}{2} \lambda^{-1} g$, where $\Gamma = \{\zeta | \ |\zeta| = 1, \mathrm{Re}\ \zeta > 0\}$.

(ii) *If* $\lambda = \Pi\widehat{\lambda}$ *with* $\Pi(x) = \Pi_{i=1}^{m}(x-\widehat{\alpha}_i)^{\nu_i}$, $\widehat{\alpha}_i \in \partial\mathbf{B}_+$ *and* $\nu_i \in \mathbb{N}$, *furthermore, if* $\widehat{\lambda} \in C(\partial\mathbf{B}_+, \mathbb{R})$ *and* $\widehat{\lambda} \neq 0$ *for arbitrary* $x \in \partial\mathbf{B}_+$, *then the Riemann–Hilbert problem* (2) *is solvable, and its solution is given again by*

$$\phi(x) = \Delta(Re(f)(x_0, |\underline{x}|) + \underline{\omega} Im(f)(x_0, |\underline{x}|)), \quad x \in \mathbf{B}_+,$$

where f *is given by*

$$f(z) = \frac{1}{2\pi}\int_0^z \frac{1}{\Pi(\xi)}\int_\Gamma \left(\frac{\zeta+\xi}{\zeta-\xi} - \frac{\bar{\zeta}+\xi}{\bar{\zeta}-\xi}\right)\frac{\widetilde{g}_2(\zeta)}{\zeta}d\zeta d\xi,$$
$$+\frac{1}{\pi}\int_0^z \frac{1}{\Pi(\xi)}\int_{-1}^{1}\left(\frac{1}{t-\xi} - \frac{\xi}{1-\xi t}\right)\widetilde{g}_2(t)dtd\xi, \quad z \in D_+,$$

with $\widetilde{g}_2 = \frac{r}{2}\widehat{\lambda}^{-1}g$, *where* $\Gamma = \{\zeta | \; |\zeta| = 1, \mathrm{Re}\,\zeta > 0\}$.

Proof. Since $D_+ = \{z : |z| < 1, \mathrm{Im} z > 0\}$ is the projection of $\mathbf{B}_+$ into (x_0, r)-plane, in virtue of Lemma 3.1, we get that the problem (2) is equivalent to the following Riemann–Hilbert problem.

(i) When $\lambda \in C\big(\partial\mathbf{B}_+, \mathbb{R}\big)$, and $\lambda \neq 0$ for arbitrary $x \in \partial\mathbf{B}_+$,

$$\begin{cases} \partial_{\bar{z}} h(z) = 0, & z \in D_+, \\ \mathrm{Re}\{h(z)\} = \widetilde{g}_1, & z \in \partial D_+, \end{cases} \tag{7}$$

where $z = x_0 + ir, \partial_{\bar{z}} = \frac{1}{2}(\partial_{x_0} + i\partial_r), h = \partial_r f$, f and r given as in Lemma 3.1, $\lambda, \widetilde{g}_1 : \partial D_+ \to \mathbb{R}$ with $\widetilde{g}_1 = \frac{r}{2}\frac{g}{\lambda}$ are both scalar-valued functions, and $D_+ = \{z = x_0 + ir \big| |z| < 1, r > 0\}$ with boundary ∂D_+.

(ii) When $\lambda = \Pi\widehat{\lambda}$ with $\Pi(x) = \Pi_{i=1}^{m}\big(x - \widehat{\alpha}_i\big)^{\nu_i}$, $\widehat{\alpha}_i \in \partial\mathbf{B}_+, \nu_i \in \mathbb{N}$, and $\widehat{\lambda} \neq 0$ for arbitrary $x \in \partial\mathbf{B}_+$,

$$\begin{cases} \partial_{\bar{z}} \hat{h}(z) = 0, & z \in D_+, \\ \mathrm{Re}\{\hat{h}(z)\} = \widetilde{g}_2, & z \in \partial D_+, \end{cases} \tag{8}$$

where $z = x_0 + ir, \partial_{\bar{z}} = \frac{1}{2}\big(\partial_{x_0} + i\partial_r\big), \hat{h} = \Pi\partial_r f, f$ and r given as in Lemma 3.1, $\lambda, g : \partial D_+ \to \mathbb{R}$ with $\widetilde{g}_2 = \frac{r}{2}\frac{g}{\lambda}$ are both scalar-valued functions, and $D_+ = \big\{z = x_0 + ir \big| |z| < 1, r > 0\big\}$ with boundary ∂D_+.

This shows that solving the Riemann–Hilbert problem (2) is equivalent to finding solutions to the Schwarz problems (3) and (4) over the upper half disc of complex plane.

For the first case (i), since $\lambda, g \in C(\partial D_+, \mathbb{R})$, and $\lambda \neq 0$ for arbitrary $x \in \partial D_+$, we get

$$\widetilde{g}_1 = \frac{r}{2}\frac{g}{\lambda} = \frac{1}{4i}(z - \bar{z})\frac{g}{\lambda} \in C(\partial D_+, \mathbb{R}). \tag{9}$$

Then, the solution to the Riemann–Hilbert problem (7) is expressed by

$$h(z) = \frac{1}{2\pi i}\int\limits_{\Gamma}\left(\frac{\zeta+z}{\zeta-z}-\frac{\bar{\zeta}+z}{\bar{\zeta}-z}\right)\widetilde{g}_1(\zeta)\frac{d\zeta}{\zeta} + \frac{1}{\pi i}\int\limits_{-1}^{1}\left(\frac{1}{t-z}-\frac{z}{1-zt}\right)\widetilde{g}_1(t)dt, \quad z\in D_+, \tag{10}$$

where $\widetilde{g}_1$ is given by (9).

Noticing that $h=\partial_r f$ and h is analytic in D_+ with respect to z, one derives $h=i\partial_z f$ in D_+. Hence, associating with (10), we get

$$f(z) = \frac{1}{2\pi}\int\limits_{0}^{z}\int\limits_{\Gamma}\left(\frac{\zeta+\xi}{\zeta-\xi}-\frac{\bar{\zeta}+\xi}{\bar{\zeta}-\xi}\right)\widetilde{g}_1(\zeta)\frac{d\zeta}{\zeta}d\xi, + \frac{1}{\pi}\int\limits_{0}^{z}\int\limits_{-1}^{1}\left(\frac{1}{t-\xi}-\frac{\xi}{1-\xi t}\right)\widetilde{g}_1(t)dtd\xi, \quad z\in D_+, \tag{11}$$

where $\widetilde{g}_1$ is given by (9).

Therefore, we end up with the solution to the Riemann–Hilbert problem (2) by

$$\phi(x) = \Delta(\mathrm{Re}(f)(x_0,|\underline{x}|)+\underline{\omega}\mathrm{Im}(f)(x_0,|\underline{x}|)), \quad x\in\mathbf{B}_+.$$

For the second case (ii), from the Riemann–Hilbert problem (8), we know that $\widehat{\lambda}, g\in C(\partial D_+,\mathbb{R})$, and $\lambda\neq 0$ for arbitrary $x\in\partial D_+$. So,

$$\widetilde{g}_2 = \frac{r}{2}\frac{g}{\widehat{\lambda}} = \frac{1}{4i}(z-\bar{z})\frac{g}{\widehat{\lambda}} \in C(\partial D_+,\mathbb{R}). \tag{12}$$

Thus, the solution to the Riemann–Hilbert problem (8) is written as

$$h(z) = \frac{1}{2\pi i\Pi(z)}\int\limits_{\Gamma}\left(\frac{\zeta+z}{\zeta-z}-\frac{\bar{\zeta}+z}{\bar{\zeta}-z}\right)\widetilde{g}_2(\zeta)\frac{d\zeta}{\zeta} + \frac{1}{\pi i\Pi(z)}\int\limits_{-1}^{1}\left(\frac{1}{t-z}-\frac{z}{1-zt}\right)\widetilde{g}_2(t)dt, \quad z\in D_+, \tag{13}$$

where $\widetilde{g}_2$ is given by (12), and Π is given by the Riemann–Hilbert problem (8).

Therefore, similar to (11), one has

$$f(z) = \frac{1}{2\pi}\int\limits_{0}^{z}\frac{1}{\Pi(\xi)}\int\limits_{\Gamma}\left(\frac{\zeta+\xi}{\zeta-\xi}-\frac{\bar{\zeta}+\xi}{\bar{\zeta}-\xi}\right)\widetilde{g}_2(\zeta)\frac{d\zeta}{\zeta}d\xi, + \frac{1}{\pi}\int\limits_{0}^{z}\frac{1}{\Pi(\xi)}\int\limits_{-1}^{1}\left(\frac{1}{t-\xi}-\frac{\xi}{1-\xi t}\right)\widetilde{g}_2(t)dtd\xi, \quad z\in D_+, \tag{14}$$

where $\widetilde{g}_2$ is given by (12), and Π is given by the Riemann–Hilbert problem (8).

This allows us to write the solution to the Riemann–Hilbert problem (2) as

$$\phi(x) = \Delta(\mathrm{Re}(f)(x_0, |\underline{x}|) + \underline{\omega}\mathrm{Im}(f)(x_0, |\underline{x}|)), \quad x \in \mathbf{B}_+,$$

This finishes the proof. □

As a special case of Problem I, we can consider the following Schwarz problem. **Schwarz problem** Find a function $\phi \in C^1(\mathbf{B}_+, \mathbb{H})$ which satisfies the system

$$\begin{cases} \mathcal{D}\phi(x) = 0, & x \in \mathbf{B}_+, \\ Re\{\phi(t)\} = g(t), & t \in \partial\mathbf{B}_+, \end{cases} \tag{15}$$

where $g : \partial\mathbf{B}_+ \to \mathbb{R}$ is a $\mathbb{R}$-valued function defined on $\partial\mathbf{B}_+$.

Thanks to Theorem 3.2 we can deduce the following theorem.

Theorem 3.3. *If $g \in C(\partial\mathbf{B}_+, \mathbb{R})$ then the Schwarz problem (15) is solvable, and its solution is given by*

$$\phi(x) = \Delta(Re(f)(x_0, |\underline{x}|) + \underline{\omega}Im(f)(x_0, |\underline{x}|)), \quad x \in \mathbf{B}_+,$$

where f is given by

$$f(z) = \frac{1}{2\pi}\int_0^z \int_\Gamma \left(\frac{\zeta+\xi}{\zeta-\xi} - \frac{\bar{\zeta}+\xi}{\bar{\zeta}-\xi}\right)\frac{\widehat{g}_1(\zeta)}{\zeta}d\zeta d\xi,$$
$$+\frac{1}{\pi}\int_0^z \int_{-1}^1 \left(\frac{1}{t-\xi} - \frac{\xi}{1-\xi t}\right)\widehat{g}_1(t)dtd\xi, \quad z \in D_+,$$

with $\widehat{g}_1 = \frac{r}{2}g$, where $\Gamma = \{\zeta |\ |\zeta| = 1, \mathrm{Re}\ \zeta > 0\}$.

Remark 3.4. What is the key point of Theorem 3.2: we solve the Riemann–Hilbert problems with variable coefficients for axially monogenic functions over upper half unit ball centred at the origin of $\mathbb{R}^4$ by considering the equivalent Riemann–Hilbert problems on upper half disc for analytic functions over the complex plane. This provides a way of overcoming the obstacle that the multiplication of two axially monogenic functions is not axially monogenic.

The Riemann–Hilbert problem on upper half disc for analytic functions, see, e.g. [5,6,36] is a trivial case of the Riemann–Hilbert problem (2) when the space dimension is equivalent to 2. Moreover, the Schwarz problem on upper half disc for analytic functions [5,6,36] is the case of our Riemann–Hilbert problem (2) when the coefficient λ is a constant equal to 1.

Remark 3.5. It is known that in general it is difficult to give explicit analytic solutions to Riemann–Hilbert problems of the Vekua system in terms of integral representation formulas. However, from Theorem 3.3, the explicit solutions to the Schwarz problem for a special type of Vekua system defined over the upper half disc of complex plane [13,18] can be derived.

4. RHBVPs for Perturbed Generalised Cauchy–Riemann Operator

In this section, the approach used in Sect. 3 will be adapted to the case of null-solutions to the equation $(\mathcal{D}-\alpha)\phi = 0, \alpha \in \mathbb{R}$. We will consider RHBVPs with variable coefficients for functions of axial type, i.e., null-solutions to the equation $(\mathcal{D}-\alpha)\phi = 0, \alpha \in \mathbb{R}$, defined over upper half unit ball centred at the origin of $\mathbb{R}^4$. We start with the following extension of Problem I.

Problem II. Find a function $\phi \in \mathcal{C}^1(\mathbf{B}_+, \mathbb{H})$ of axial type which satisfies the condition

$$\begin{cases} (\mathcal{D}-\alpha)\phi = 0, \alpha \in \mathbb{R}, & x \in \mathbf{B}_+, \\ \mathrm{Re}\{\lambda(t)\phi(t)\} = g(t), & t \in \partial\mathbf{B}_+, \end{cases} \tag{16}$$

where α is understood as αI, with I being the identity operator, $g : \partial\mathbf{B}_+ \to \mathbb{R}$ and λ is a $\mathbb{R}$-valued function defined on $\partial\mathbf{B}_+$.

Theorem 4.1. *Let $g \in C(\partial\mathbf{B}_+, \mathbb{R})$, and $D_+ = \{z : |z| < 1, Im z > 0\}$ with boundary ∂D_+.*

(i) *If $\lambda \in C(\partial\mathbf{B}_+, \mathbb{R})$, and $\lambda \neq 0$ for arbitrary $z \in \partial\mathbf{B}_+$, then the Riemann–Hilbert problem (16) is solvable, and its solution is given by*

$$\phi(x) = e^{\alpha x_0}\Delta(Re(f)(x_0, |\underline{x}|) + \underline{\omega} Im(f)(x_0, |\underline{x}|)), \quad x \in \mathbf{B}_+,$$

where $Re(f)$ and $Im(f)$ denote the real and imaginary part of the complex-valued function f, respectively. f itself is given by

$$f(z) = \frac{1}{2\pi}\int_0^z \int_\Gamma \left(\frac{\zeta+\xi}{\zeta-\xi} - \frac{\bar{\zeta}+\xi}{\bar{\zeta}-\xi}\right)\frac{e^{-\alpha x_0}\widetilde{g}_1(\zeta)}{\zeta} d\zeta d\xi,$$

$$+\frac{1}{\pi}\int_0^z \int_{-1}^1 \left(\frac{1}{t-\xi} - \frac{\xi}{1-\xi t}\right) e^{-\alpha x_0}\widetilde{g}_1(t) dt d\xi, \quad z \in D_+,$$

with $\widetilde{g}_1 = \frac{r}{2}\lambda^{-1}g$, where $\Gamma = \{\zeta | \ |\zeta| = 1, \mathrm{Re}\ \zeta > 0\}$.

(ii) *Let $\lambda = \Pi\widehat{\lambda}$ with $\Pi(x) = \Pi_{i=1}^m (x-\widehat{\alpha}_i)^{\nu_i}$, $\widehat{\alpha}_i \in \partial\mathbf{B}_+$, $\nu_i \in \mathbb{N}$, and $\widehat{\lambda} \in C(\partial\mathbf{B}_+, \mathbb{R})$, $\widehat{\lambda} \neq 0$ for arbitrary $x \in \partial\mathbf{B}_+$, then the Riemann–Hilbert problem (16) is solvable, and its solution is given again by*

$$\phi(x) = e^{\alpha x_0}\Delta(Re(f)(x_0, |\underline{x}|) + \underline{\omega} Im(f)(x_0, |\underline{x}|)), \quad x \in \mathbf{B}_+,$$

where f is given by

$$f(z) = \frac{1}{2\pi}\int_0^z \frac{1}{\Pi(\xi)}\int_\Gamma \left(\frac{\zeta+\xi}{\zeta-\xi} - \frac{\bar{\zeta}+\xi}{\bar{\zeta}-\xi}\right)\frac{e^{-\alpha x_0}\widetilde{g}_2(\zeta)}{\zeta} d\zeta d\xi,$$

$$+\frac{1}{\pi}\int_0^z \frac{1}{\Pi(\xi)}\int_{-1}^1 \left(\frac{1}{t-\xi} - \frac{\xi}{1-\xi t}\right) e^{-\alpha x_0}\widetilde{g}_2(t) dt d\xi, \quad z \in D_+,$$

with $\widetilde{g}_2 = \frac{r}{2}\widehat{\lambda}^{-1}g$, where $\Gamma = \{\zeta | \ |\zeta| = 1, \mathrm{Re}\ \zeta > 0\}$.

Proof. Noticing the fact $(\mathcal{D}-\alpha)\phi=\mathcal{D}(e^{-\alpha x_0}\phi), \alpha\in\mathbb{R}$, then

$$(\mathcal{D}-\alpha)\phi=0 \text{ is equivalent to } \mathcal{D}(e^{-\alpha x_0}\phi)=0, \quad \alpha\in\mathbb{R}. \tag{17}$$

Therefore, Problem (16) is equivalent to the case

$$\begin{cases} \mathcal{D}(e^{-\alpha x_0}\phi)=0, & \alpha\in\mathbb{R},\ x\in\mathbf{B}_+,\\ \mathrm{Re}\{\lambda(t)e^{-\alpha x_0}\phi(t)\}=e^{-\alpha x_0}g(t), & t\in\partial\mathbf{B}_+,\end{cases}$$

where $g:\partial\mathbf{B}_+\to\mathbb{R}$ and λ is a $\mathbb{R}$-valued function defined on $\partial\mathbf{B}_+$.

Noting that $g\in C(\partial\mathbf{B}_+,\mathbb{R})$, we have $e^{-\alpha x_0}g\in C(\partial\mathbf{B}_+,\mathbb{R})$.

Applying Theorem 3.2, (i) If $\lambda\in C\big(\partial\mathbf{B}_+,\mathbb{R}\big)$, and $\lambda\neq 0$ for arbitrary $z\in\partial D_+$, then the Riemann–Hilbert problem (16) is solvable, and its solution is written as

$$\phi(x)=e^{\alpha x_0}\Delta(\mathrm{Re}(f)(x_0,|\underline{x}|)+\underline{\omega}\mathrm{Im}(f)(x_0,|\underline{x}|)), \quad x\in\mathbf{B}_+,$$

where $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ denote the real and imaginary part of the complex-valued function f, respectively. f itself is given by

$$\begin{aligned} f(z)=&\frac{1}{2\pi}\int_0^z\int_\Gamma\left(\frac{\zeta+\xi}{\zeta-\xi}-\frac{\bar\zeta+\xi}{\bar\zeta-\xi}\right)\frac{e^{-\alpha x_0}\widetilde{g}_1(\zeta)}{\zeta}d\zeta d\xi,\\ &+\frac{1}{\pi}\int_0^z\int_{-1}^1\left(\frac{1}{t-\xi}-\frac{\xi}{1-\xi t}\right)e^{-\alpha x_0}\widetilde{g}_1(t)dtd\xi, \quad z\in D_+,\end{aligned}$$

with $\widetilde{g}_1=\frac{r}{2}\lambda^{-1}g$, where $\Gamma=\{\zeta|\ |\zeta|=1,\mathrm{Re}\ \zeta>0\}$.

(ii) If $\lambda=\Pi\widehat{\lambda}$ with $\Pi(x)=\Pi_{i=1}^m(x-\widehat{\alpha}_i)^{\nu_i}$, $\widehat{\alpha}_i\in\partial\mathbf{B}_+$ and $\nu_i\in\mathbb{N}$, and if $\widehat{\lambda}\in C(\partial\mathbf{B}_+,\mathbb{R})$ and $\widehat{\lambda}\neq 0$ for arbitrary $x\in\partial\mathbf{B}_+$, then the Riemann–Hilbert problem (16) is solvable, and its solution is given again by

$$\phi(x)=e^{\alpha x_0}\Delta(\mathrm{Re}(f)\big(x_0,|\underline{x}|\big)+\underline{\omega}\mathrm{Im}(f)\big(x_0,|\underline{x}|\big)\big), \quad x\in\mathbf{B}_+,$$

where f is given by

$$\begin{aligned} f(z)=&\frac{1}{2\pi}\int_0^z\frac{1}{\Pi(\xi)}\int_\Gamma\left(\frac{\zeta+\xi}{\zeta-\xi}-\frac{\bar\zeta+\xi}{\bar\zeta-\xi}\right)\frac{e^{-\alpha x_0}\widetilde{g}_2(\zeta)}{\zeta}d\zeta d\xi,\\ &+\frac{1}{\pi}\int_0^z\frac{1}{\Pi(\xi)}\int_{-1}^1\left(\frac{1}{t-\xi}-\frac{\xi}{1-\xi t}\right)e^{-\alpha x_0}\widetilde{g}_2(t)dtd\xi, \quad z\in D_+,\end{aligned}$$

with $\widetilde{g}_2=\frac{r}{2}\widehat{\lambda}^{-1}g$, where $\Gamma=\{\zeta|\ |\zeta|=1,\mathrm{Re}\ \zeta>0\}$. It follows the result. □

Remark 4.2. The Riemann–Hilbert problem for the first order meta-analytic functions defined over upper half disc of complex plane [5,6] is the special case of Problem (16) when the space dimension is 2.

Similarly to the Schwarz Problem (15), take $\lambda=1$ in Problem (16), we can take account into the following problem.

Problem III. Find a function $\phi\in C^1\big(\mathbf{B}_+,\mathbb{H}\big)$ which satisfies the system

$$\begin{cases} (\mathcal{D}-\alpha)\phi=0, & \alpha\in\mathbb{R},\ x\in\mathbf{B}_+,\\ Re\{\phi(t)\}=g(t), & t\in\partial\mathbf{B}_+,\end{cases} \tag{18}$$

where α is understood as αI, with I being the identity operator, and $g : \partial\mathbf{B}_+ \to \mathbb{R}$ is a $\mathbb{R}$-valued function defined on $\partial\mathbf{B}_+$.

According to Theorem 4.1 we can derive the following theorem.

Theorem 4.3. *If $g \in C\big(\partial\mathbf{B}_+, \mathbb{R}\big)$ then Problem (18) is solvable, and its solution is given by*

$$\phi(x) = e^{\alpha x_0}\Delta(Re(f)(x_0, |\underline{x}|) + \underline{\omega} Im(f)(x_0, |\underline{x}|)), \quad x \in \mathbf{B}_+,$$

where f is given by

$$f(z) = \frac{1}{2\pi}\int_0^z \int_\Gamma \left(\frac{\zeta+\xi}{\zeta-\xi} - \frac{\bar{\zeta}+\xi}{\bar{\zeta}-\xi}\right)\frac{e^{-\alpha x_0}\widehat{g}_1(\zeta)}{\zeta} d\zeta d\xi,$$

$$+\frac{1}{\pi}\int_0^z\int_{-1}^1 \left(\frac{1}{t-\xi} - \frac{\xi}{1-\xi t}\right)e^{-\alpha x_0}\widehat{g}_1(t) dt d\xi, \quad z \in D_+,$$

with $\widehat{g}_1 = \frac{r}{2}g$, where $\Gamma = \{\zeta|\ |\zeta| = 1, \mathrm{Re}\,\zeta > 0\}$.

Remark 4.4. Thanks to [31], the following term appeared in Theorems 3.2, 3.3, 4.1 and 4.3

$$\Delta(\mathrm{Re}(f)(x_0, |\underline{x}|) + \underline{\omega}\mathrm{Im}(f)(x_0, |\underline{x}|)) \tag{19}$$

could be replaced by the explicit integral formula. Since it will be a direct replacement, here, we will not develop the details.

Remark 4.5. Given the continuous boundary data, we have a study of the Riemann–Hilbert problems (2), (15), (16) and (18) with variable coefficients for monogenic functions of axial type defined upper half unit ball centred at the origin of $\mathbb{R}^4$ and for null-solutions to $(\mathcal{D} - \alpha)\phi = 0, \alpha \in \mathbb{R}$. Hereby, we single out that Problem (2) is the special case of Problem (16) when α equals 0, and that Problem (15) is the special case of Problem (18).

Remark 4.6. In this context we are concerned about the Riemann–Hilbert problems on upper half ball of $\mathbb{R}^4$. Compared to the results contained in Refs. e.g. [5,6,36], we can consider the Riemann–Hilbert Problems I and II on other sectors, like the upper half space of $\mathbb{R}^4$. For example, let the case be $\mathbb{R}^4_+ = \{x|x \in \mathbb{R}^4, x_0 > 0\}, \mathbb{R}^4_0 = \{x|x \in \mathbb{R}^4, x_0 = 0\}$, and then analogous to Problem I allow us to consider the following Riemann–Hilbert Problem IV.

Problem IV. To find a function $\phi \in C^1(\mathbb{R}^4_+, \mathbb{H})$ of axial type, which satisfies the condition

$$\begin{cases} \mathcal{D}\phi(x) = 0, & x \in \mathbb{R}^4_+, \\ \mathrm{Re}\{\lambda(t)\phi(t)\} = g(t), & t \in \mathbb{R}^4_0, \end{cases} \tag{20}$$

where $g : \mathbb{R}^4_0 \to \mathbb{R}$ and λ is a $\mathbb{R}$-valued function defined on $\mathbb{R}^4_0$.

Similar to Problem I, Problem (21) is equivalent to the problem

$$\begin{cases} \partial_{\bar{z}}h(z) = 0, & z \in Q_I, \\ \mathrm{Re}\{\lambda(z)h(z)\} = \frac{r}{2}g(z), & z \in \partial Q_I, \end{cases} \tag{21}$$

where $Q_I = \{z = x_0 + ir|(r > 0) \wedge (x_0 > 0)\}$ and $\partial Q_I = \{z = x_0 + ir\big|[(r = 0) \wedge (x_0 > 0)] \vee [(r > 0) \wedge (x_0 = 0)]\}$. Hereby, the uniqueness is also imposed by the boundary condition and the maximum principle for holomorphic functions. This leads us to consider a Riemann–Hilbert Problem (21) on the first quadrant of (x_0, r)-plane.

In fact, since in the Problem (20) $\lambda \neq 0, t \in \mathbb{R}^4_0$, then $\lambda \neq 0, z \in \partial Q_I$. Therefore, the Problem (21) is equivalent to the problem

$$\begin{cases} \partial_{\bar{z}} h(z) = 0, & z \in Q_I, \\ \mathrm{Re}\{h(z)\} = \frac{r}{2}\frac{g(z)}{\lambda(z)}, & z \in \partial Q_I. \end{cases} \tag{22}$$

If $g, \lambda^{-1} \in L^2(\mathbb{R}^4_0, \mathbb{R}) \cap C(\mathbb{R}^4_0, \mathbb{R})$, then the Riemann–Hilbert Problem (21) is solvable and its solution is given by

$$\phi(x) = \Delta(\mathrm{Re}(f)(x_0, |\underline{x}|) + \underline{\omega}\mathrm{Im}(f)(x_0, |\underline{x}|)), \quad x \in Q_I, \tag{23}$$

where f is given by

$$f(z) = \frac{1}{\pi} \int\limits_{\partial Q_I} \left(\frac{1}{\zeta - z} + \frac{1}{\zeta + z} \right) g_3(\zeta) d\zeta, \quad z \in Q_I, \tag{24}$$

where $g_3 = \frac{r}{2}\frac{g}{\lambda}$.

Moreover, similar to Problem II, associated with Problem IV, we could derive the solution to a Riemann–Hilbert problem on the upper space $\mathbb{R}^4_+ = \{x | x \in \mathbb{R}^4, x_0 > 0\}$ of $\mathbb{R}^4$. But here we omit the details.

Furthermore, we point out that from (23), (24), in order to solve Problem IV a kernel function should be constructed for the Riemann–Hilbert Problem (21) and the different boundary data are considered, which is different from that of Problem I.

Remark 4.7. The Schwarz problem for analytic functions defined over upper half plane of complex plane [3] is the special case of Problem (20) when the space dimension is 2 and $\lambda = 1$.

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