Realizable lists via the spectra of structured matrices

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Abstract

A square matrix of order n with $n \geq 2$ is called a *permutative matrix* or permutative when all its rows (up to the first one) are permutations of precisely its first row. In this paper, the spectra of a class of permutative matrices are studied. In particular, spectral results for matrices partitioned into 2-by-2 symmetric blocks are presented and, using these results sufficient conditions on a given list to be the list of eigenvalues of a nonnegative permutative matrix are obtained and the corresponding permutative matrices are constructed. Guo perturbations on given lists are exhibited.

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1. Introduction

We present here a short overview related with the nonnegative inverse eigenvalue problem (NIEP) that is the problem of determining necessary and sufficient conditions for a list of complex numbers

$$\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{1}$$

to be the spectrum of a n-by-n entrywise nonnegative matrix A. If a list σ is the spectrum of a nonnegative matrix A, then σ is *realizable* and the matrix A *realizes* σ, (or, that is a realizing matrix for the list). This problem attracted the attention of many authors over 50+ years and it was firstly considered by Suleĭmanova [25] in 1949. Although some partial results were obtained the NIEP is an open problem for $n \geq 5$. In [12] this problem was solved for $n = 3$ and for matrices of order $n = 4$ the problem was solved in [14] and [15]. It has been studied in its general form in e.g. [2, 6, 8, 9, 12, 22, 23, 26]. When the realizing nonnegative matrix is required to be symmetric (with, of course, real eigenvalues) the problem is designated by symmetric nonnegative inverse eigenvalue problem (SNIEP) and it is also an open problem. It has also been the subject of considerable attention e.g [3, 7, 11, 24]. The problem of which lists of n real numbers can occur as eigenvalues of an n-by-n nonnegative matrix is called real nonnegative inverse eigenvalue problem (RNIEP), and some results can be seen in e.g. [1, 4, 17, 20, 21]. In what follows $\sigma(A)$ denotes the set of eigenvalues of a square matrix A. Below are listed some necessary conditions on a list of complex numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be the spectrum of a nonnegative matrix.

1. The Perron eigenvalue $\max\{|\lambda| : \lambda \in \sigma(A)\}$ belongs to σ.
2. The list σ is closed under complex conjugation.
3. $s_k(\sigma) = \sum_{i=1}^{n} \lambda_i^k \geq 0$.
4. $s_k^m(\sigma) \leq n^{m-1} s_{km}(\sigma)$ for $k, m = 1, 2, \ldots$.

The first condition listed above follows from the Perron-Frobenius theorem, which is an important theorem in the theory of nonnegative matrices. The last condition was proved by Johnson [6] and independently by Loewy and London [12]. The necessary conditions that were presented for the NIEP are sufficient only when the list σ has at most three elements. The solution for NIEP was also found for lists with four elements, while the problem for lists with five or more elements is still open.

Definition 1. The list σ in (1) is a Suleĭmanova spectrum if the $\lambda's$ are real numbers, $\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $s_1(\sigma) \geq 0$.

Suleĭmanova, [25] stated (and loosely proved) that every such spectrum is realizable. Fiedler [3] proved that every Suleĭmanova spectrum is symmetrically realizable (i.e. realizable by a symmetric nonnegative matrix).

One of the most promising attempts to solve the NIEP is to identify the spectra of certain structured matrices with known characteristic polynomials. Friedland in [4] and Perfect in [18] proved Suleĭmanova's result via companion matrices of certain polynomials. However, constructing the companion matrix of a Suleĭmanova's spectrum is computationally difficult. Recently, Paparella [16] gave a constructive proof of Suleĭmanova's result. The author defined *permutative* matrix as follows.

Definition 2. [16] Let $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$. Let $P_2, \ldots, P_n$ be permutation matrices. A *permutative matrix* P is a matrix which takes the form

$$P = \begin{pmatrix} \mathbf{x}^T \\ (P_2\mathbf{x})^T \\ \vdots \\ (P_{n-1}\mathbf{x})^T \\ (P_n\mathbf{x})^T \end{pmatrix}.$$

In [16], explicit permutative matrices which realize Suleĭmanova spectra were found. A few remarks concerning the brief history of permutative matrices are in order.

1. Ranks of permutative matrices were studied by Hu et al. [5]
2. Moreover, the author [16] proposed the interesting problem which asks if all realizable spectra can be realizable by a permutative matrix or by a direct sum of permutative matrices. An equivalent problem communicated to the author by R. Loewy is to find an extreme nonnegative matrix [8] with real spectrum that can not be realized by a permutative matrix or a direct sum of permutative matrices. Loewy [13] resolved this problem in the negative by showing that the list $\sigma = \left(1, \frac{8}{25} + \frac{\sqrt{51}}{50}, \frac{8}{25} + \frac{\sqrt{51}}{50}, -\frac{4}{5}, -\frac{21}{25}\right)$ is realizable but cannot be realized by a permutative matrix or by a direct sum of permutative matrices.

In this paper we call the problem as PNIEP when the NIEP involves permutative matrices. Note that the lists considered along the paper are equivalent (up to a permutation of its elements). Therefore, unless we say the contrary, we call a given n-tuple σ or any permutation resulting from it, as "the list". In consequence, any of these lists can be used.

In this work we will find spectral results for partitioned into 2-by-2 blocks matrices and using these results sufficient conditions on given lists to be the list of eigenvalues of a nonnegative permutative matrix are obtained. The paper is organized as follows: At Section 2 some definitions and facts related to permutative matrices are given. At Section 3 spectral results for matrices partitioned into 2-by-2 blocks are presented and the results are applied to NIEP, SNIEP and PNIEP. Some illustrative examples are provided. At Section 4 results for matrices with odd order are presented. Finally, at Section 5 Guo perturbations on lists of eigenvalues of this class of permutative matrices (in order to obtain a new permutative matrix) are studied.

2. Permutatively equivalent matrices

In this section some auxiliary results from [16] and some new definitions are introduced. In [16] the following results were proven.

Lemma 3. *[16, Lemma 3.1] For* $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$, *let*

$$X = \begin{pmatrix} x_1 & x_2 & \ldots & x_i & \ldots & x_{n-1} & x_n \\ x_2 & x_1 & \ldots & x_i & \ldots & x_{n-1} & x_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x_i & x_2 & \ddots & x_1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & x_2 & \ldots & \vdots & \vdots & x_1 & x_n \\ x_n & x_2 & \ldots & x_i & \ldots & x_{n-1} & x_1 \end{pmatrix}. \tag{2}$$

Then, the set of eigenvalues of X *is given by*

$$\sigma(X) = \left\{ \sum_{i=1}^{n} x_i, x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n \right\}. \tag{3}$$

Theorem 4. *[16] Let $\sigma = (\lambda_1, \ldots, \lambda_n)$ be a Suleĭmanova spectrum and consider the n-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where*

$$x_1 = \tfrac{\lambda_1 + \cdots + \lambda_n}{n} \quad \text{and} \quad x_i = x_1 - \lambda_i,\ 2 \leq i \leq n,$$

then the matrix in (2) realizes σ. In particular, if $\lambda_1 + \cdots + \lambda_n = 0$ the solution matrix, X_0 becomes

$$X_0 = \begin{pmatrix} 0 & |\lambda_2| & \ldots & |\lambda_i| & \ldots & |\lambda_{n-1}| & |\lambda_n| \\ |\lambda_2| & 0 & \ldots & |\lambda_i| & \ldots & |\lambda_{n-1}| & |\lambda_n| \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ |\lambda_i| & |\lambda_2| & \ddots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ |\lambda_{n-1}| & |\lambda_2| & \vdots & \vdots & \vdots & 0 & |\lambda_n| \\ |\lambda_n| & |\lambda_2| & \ldots & |\lambda_i| & \ldots & |\lambda_{n-1}| & 0 \end{pmatrix}.$$

Remark 5. By previous results and the proof of above Theorem 4 in [16], it is clear that for any set $\sigma = \{\alpha_1, \ldots, \alpha_n\}$ there exists a permutative matrix with the shape of X in (2) whose set of eigenvalues is σ.

The following notions will be used in the sequel.

Definition 6. Let $\tau = (\tau_1, \ldots, \tau_n)$ be an n-tuple whose elements are permutations in the symmetric group S_n, with $\tau_1 = id$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Define the row-vector,

$$\tau_j(\mathbf{a}) = \left(a_{\tau_j(1)}, \ldots, a_{\tau_j(n)}\right)$$

and consider the matrix

$$\tau(\mathbf{a}) = \begin{pmatrix} \tau_1(\mathbf{a}) \\ \tau_2(\mathbf{a}) \\ \vdots \\ \tau_{n-1}(\mathbf{a}) \\ \tau_n(\mathbf{a}) \end{pmatrix}. \tag{4}$$

An n-by-n matrix A, is called *τ-permutative* if $A = \tau(\mathbf{a})$ for some n-tuple $\mathbf{a}$.

Remark 7. Although the statement in Definition 2 is precisely the statement found in [16, Definiton 2.1], it is clear that Definition 6 of this work is the proper definition of a permutative matrix (indeed, since every permutation matrix is a permutative matrix, it is not ideal to define the latter with the former). Thus, Definition 6 is a better definition of a permutative matrix than the one given at Definition 2.

Definition 8. If A and B are τ-permutative by a common vector $\tau = (\tau_1, \ldots, \tau_n)$ then they are called *permutatively equivalent.*

Definition 9. Let $\varphi \in S_n$ and the n-tuple $\tau = (id, \varphi, \varphi^2, \ldots, \varphi^{n-1}) \in (S_n)^n$. Then a τ-permutative matrix is called *φ-permutative.*

It is clear from the definitions that two φ-permutative matrices are permutatively equivalent matrices.

Remark 10. If permutations are regarded as bijective maps from the set $\{0, 1, \cdots, n-1\}$ to itself, then a circulant (respectively, left circulant) matrix is a φ-permutative matrix where $\varphi(i) \equiv i - 1 (\text{mod } n)$ (resp. $\varphi(i) \equiv i + 1 (\text{mod } n)$). Indeed, notice that the 3-by-3 circulant matrix

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

is φ- permutative with

$$\varphi = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

and the 3-by-3 left circulant matrix

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

is φ- permutative with

$$\varphi = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Remark 11. A permutative matrix A defines the class of permutatively equivalent matrices. Let σ_1, σ_2 be two Suleĭmanova spectra, then the corresponding realizing matrices X_{σ_1} and X_{σ_2} given by Theorem 4 are permutatively equivalent matrices. Furthermore, by Lemma 3 and Remark 5 it is easy to check that given two arbitrary inverse eigenvalue problems (not necessarily NIEP) there exist a solution which is permutatively equivalent to the matrix X in (2).

For τ-permutative matrices an analogous property related with circulant matrices is given below.

Proposition 12. *Let $\{A_i\}_{i=1}^k$ be a family of permutatively equivalent matrices in $\mathbb{C}^{n\times n}$. Let $\{\gamma_i\}_{i=1}^k$ be a set of complex numbers. Consider*

$$A = \sum_{i=1}^{k} \gamma_i A_i. \tag{5}$$

Then A_1 and A are permutatively equivalent matrices.

Proof. Let $\tau = (\tau_1, \ldots, \tau_n)$ be an n-tuple whose elements are permutations in the symmetric group S_n and suppose that the family $\{A_i\}$ are permutatively equivalent by τ. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the canonical row vectors in $\mathbb{C}^n$. The result is an immediate consequence of the fact that for any $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$ the matrix $\tau(\mathbf{a})$ in (4) can be decomposed as

$$\tau(\mathbf{a}) = a_1\tau(\mathbf{e}_1) + a_2\tau(\mathbf{e}_2) + \cdots + a_n\tau(\mathbf{e}_n),$$

where

$$\tau(\mathbf{e}_j) = \begin{pmatrix} \mathbf{e}_j \\ \tau_2(\mathbf{e}_j) \\ \vdots \\ \tau_{n-1}(\mathbf{e}_j) \\ \tau_n(\mathbf{e}_j) \end{pmatrix}.$$

□

3. Eigenpairs for some into block matrices

In this section we exhibit spectral results for matrices that are partitioned into 2-by-2 symmetric blocks and we apply the results to NIEP, SNIEP and PNIEP. The next theorem is valid in an algebraic closed field K of characteristic 0. For instance, $K = \mathbb{C}$.

Theorem 13. *Let K be an algebraically closed field of characteristic 0 and suppose that $A = (A_{ij})$ is a block matrix of order $2n$, where*

$$A_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix},\ a_{ij},\, b_{ij} \in K. \tag{6}$$

If

$$s_{ij} = a_{ij} + b_{ij},\ 1 \leq i, j \leq n$$

and

$$c_{ij} = a_{ij} - b_{ij},\ 1 \leq i, j \leq n$$

Then

$$\sigma(A) = \sigma(S) \cup \sigma(C)$$

where

$$S = (s_{ij}) \text{ and } C = (c_{ij}).$$

Proof. Let (λ, v) be an eigenpair of S, with $v = (v_1, \ldots, v_n)^T$, and consider the $2n$-by-1 block vector $w = (w_j)$, where $w_j := v_j\mathbf{e}$ and $\mathbf{e} = (1,1)^T$. Since

$$A_{ij}w_j = \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix} v_j\mathbf{e} = \begin{pmatrix} a_{ij} + b_{ij} \\ b_{ij} + a_{ij} \end{pmatrix} v_j = s_{ij}v_j\mathbf{e},$$

notice that, for every $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} A_{ij}w_j = \left(\sum_{j=1}^{n} s_{ij}v_j\right)\mathbf{e} = (\lambda v_i)\,\mathbf{e} = \lambda\,(v_i\mathbf{e}) = \lambda w_i$$

i.e (λ, w) be an eigenpair of A. Thus $\sigma(S) \subseteq \sigma(A)$.

Similarly, let (μ, x) be an eigenpair of C, with $x = (x_1, \ldots, x_n)^T$ and consider the $2n$-by-1 block vector $y = (y_j)$, where $y_j := x_j\mathbf{f}$ and $\mathbf{f} = (1, -1)^T$. Since

$$A_{ij}y_j = \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix} x_j\mathbf{f} = \begin{pmatrix} a_{ij} - b_{ij} \\ b_{ij} - a_{ij} \end{pmatrix} x_j = c_{ij}x_j\mathbf{f},$$

notice that, for every $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} A_{ij} y_j = \left(\sum_{j=1}^{n} c_{ij} x_j \right) \mathbf{f} = (\lambda x_i)\, \mathbf{f} = \lambda\, (x_i \mathbf{f}) = \lambda y_i$$

i.e (μ, y) is also an eigenpair of A. Thus $\sigma(C) \subseteq \sigma(A)$. Suppose that

$$\Theta_s = \left\{ (x_{1i}, x_{2i}, \ldots, x_{ni})^T : i = 1, \ldots, n \right\}$$

and

$$\Theta_c = \left\{ (y_{1i}, y_{2i}, \ldots, y_{ni})^T : i = 1, \ldots, n \right\}$$

are bases formed with eigenvectors of S and C, respectively. The result will follow after proving the linear independence of the set $\Upsilon = \Upsilon_1 \cup \Upsilon_2$, where:

$$\Upsilon_1 = \left\{ \left(x_1 \mathbf{e}_2^T, x_2 \mathbf{e}_2^T, \ldots, x_n \mathbf{e}_2^T \right)^T : (x_1, x_2, \ldots, x_n)^T \in \Theta_s \right\}$$

and

$$\Upsilon_2 = \left\{ \left(y_1 \mathbf{f}_2^T, y_2 \mathbf{f}_2^T, \ldots, y_n \mathbf{f}_2^T \right)^T : (y_1, y_2, \ldots, y_n)^T \in \Theta_c \right\}.$$

Therefore, we consider the following determinant,

$$d = \begin{vmatrix} y_{11} & \ldots & y_{1n} & x_{11} & \ldots & x_{1n} \\ -y_{11} & \ldots & -y_{1n} & x_{11} & \ldots & x_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{n1} & \ldots & y_{nn} & x_{n1} & \ldots & x_{nn} \\ -y_{n1} & \ldots & -y_{nn} & x_{n1} & \ldots & x_{nn} \end{vmatrix}.$$

Note that d stands for the determinant of a $2n$-by-$2n$ matrix obtained from the coordinates of the vectors in Υ. By adding rows and after making suitable row permutations we conclude that the absolute value of d coincides with the absolute value of the following determinant

$$\begin{vmatrix} y_{11} & \ldots & y_{1n} & x_{11} & \ldots & x_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{n1} & \ldots & y_{nn} & x_{n1} & \ldots & x_{nn} \\ 0 & \ldots & 0 & 2x_{11} & \ldots & 2x_{1n} \\ 0 & \ldots & 0 & \vdots & \ldots & \vdots \\ 0 & \ldots & 0 & 2x_{n1} & \ldots & 2x_{nn} \end{vmatrix}$$

which is nonzero by the linear independence of the sets Θ_s and Θ_c respectively. ∎

Theorem 14. *Let $S = (s_{ij})$ and $C = (c_{ij})$ be matrices of order n whose spectra (counted with their multiplicities) are $\sigma(S) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\sigma(C) = (\mu_1, \mu_2, \ldots, \mu_n)$, respectively. Let $0 \leq \gamma \leq 1$. If*

$$|c_{ij}| \leq s_{ij},\ 1 \leq i, j \leq n, \tag{7}$$

(or equivalently if S, $S + C$ and $S - C$ are nonnegative matrices), then the matrices $\frac{1}{2}(S + \gamma C)$ and $\frac{1}{2}(S - \gamma C)$ are nonnegative and the nonnegative matrices

$$M_{\pm\gamma} = \left(M_{ij_{\pm\gamma}}\right), \text{with } M_{ij_{\pm\gamma}} = \begin{pmatrix} \frac{s_{ij}\pm\gamma c_{ij}}{2} & \frac{s_{ij}\mp\gamma c_{ij}}{2} \\ \frac{s_{ij}\mp\gamma c_{ij}}{2} & \frac{s_{ij}\pm\gamma c_{ij}}{2} \end{pmatrix}, 1 \leq i, j \leq n \tag{8}$$

realize, respectively, the following lists

$$\sigma(S) \cup \gamma\sigma(C) := (\lambda_1, \lambda_2, \ldots, \lambda_n, \gamma\mu_1, \gamma\mu_2, \ldots, \gamma\mu_n)$$

and

$$\sigma(S) \cup (-\gamma\sigma(C)) := (\lambda_1, \lambda_2, \ldots, \lambda_n, -\gamma\mu_1, -\gamma\mu_2, \ldots, -\gamma\mu_n).$$

Proof. By the definitions of $\frac{1}{2}(S + \gamma C)$ and $\frac{1}{2}(S - \gamma C)$ and the condition in (7) it is clear that $M_{\pm\gamma}$ in (8) are nonnegative matrices. By conditions of Theorem 13 one can see that each (i, j)-block of the matrix takes the form $\begin{pmatrix} x_{ij} & y_{ij} \\ y_{ij} & x_{ij} \end{pmatrix}$ and its spectrum is partitioned into the union of the spectra of the n-by-n matrices $(x_{ij} + y_{ij})_{i,j=1}^n$ and $(x_{ij} - y_{ij})_{i,j=1}^n$. If we impose that $S = (x_{ij} + y_{ij})_{i,j=1}^n$ and $\pm\gamma C = (x_{ij} - y_{ij})_{i,j=1}^n$ we obtain $s_{ij} = x_{ij} + y_{ij}$ and $\pm\gamma c_{ij} = x_{ij} - y_{ij}$. Thus, for both cases $x_{ij} = \frac{s_{ij}\pm\gamma c_{ij}}{2}$ and $y_{ij} = \frac{s_{ij}\mp\gamma c_{ij}}{2}$, as it is required for the respective realization of the spectra $\sigma(S) \cup \pm\gamma\sigma(C)$. ∎

Remark 15. Note that in the previous result if $S = (s_{ij})$ and $C = (c_{ij})$, then

$$M_{\pm\gamma} = \begin{pmatrix} \frac{s_{11}\pm\gamma c_{11}}{2} & \frac{s_{11}\mp\gamma c_{11}}{2} & \cdots & \cdots & \frac{s_{1n}\pm\gamma c_{1n}}{2} & \frac{s_{1n}\mp\gamma c_{1n}}{2} \\ \frac{s_{11}\mp\gamma c_{11}}{2} & \frac{s_{11}\pm\gamma c_{11}}{2} & \cdots & \cdots & \frac{s_{1n}\mp\gamma c_{1n}}{2} & \frac{s_{1n}\pm\gamma c_{1n}}{2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{s_{n1}\pm\gamma c_{n1}}{2} & \frac{s_{n1}\mp\gamma c_{n1}}{2} & \cdots & \cdots & \frac{s_{nn}\pm\gamma c_{nn}}{2} & \frac{s_{nn}\mp\gamma c_{nn}}{2} \\ \frac{s_{n1}\mp\gamma c_{n1}}{2} & \frac{s_{n1}\pm\gamma c_{n1}}{2} & \cdots & \cdots & \frac{s_{nn}\mp\gamma c_{nn}}{2} & \frac{s_{nn}\pm\gamma c_{nn}}{2} \end{pmatrix}. \tag{9}$$

The next corollary establishes the result when the matrices S and C are symmetric, both with prescribed list of eigenvalues.

Corollary 16. *Let $S = (s_{ij})$ and $C = (c_{ij})$ be symmetric matrices of orders n whose spectra (counted with their multiplicities) are $\sigma(S) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\sigma(C) = (\mu_1, \mu_2, \ldots, \mu_n)$, respectively. Let $0 \leq \gamma \leq 1$. Moreover, suppose that $|c_{ij}| \leq s_{ij}$ for all $1 \leq i, j \leq n$. Then $\frac{1}{2}(S + \gamma C)$ and $\frac{1}{2}(S - \gamma C)$ are symmetric nonnegative matrices and*

$$M_{\pm\gamma} = \left(M_{ij_{\pm\gamma}}\right) \text{ with } M_{ij_{\pm\gamma}} = \begin{pmatrix} \frac{s_{ij} \pm \gamma c_{ij}}{2} & \frac{s_{ij} \mp \gamma c_{ij}}{2} \\ \frac{s_{ij} \mp \gamma c_{ij}}{2} & \frac{s_{ij} \pm \gamma c_{ij}}{2} \end{pmatrix}, \text{ for } 1 \leq i, j \leq n$$

are symmetric nonnegative matrices such that, respectively, realize the following lists

$$\sigma(S) \cup (\pm\gamma\sigma(C)) = (\lambda_1, \lambda_2, \ldots, \lambda_n, \pm\gamma\mu_1, \pm\gamma\mu_2, \ldots, \pm\gamma\mu_n).$$

Proof. It is an immediate consequence of Theorem 14 that if the matrices S and C are symmetric, then the matrices $M_{\pm\gamma}$ obtained in (9) are also symmetric. □

Remark 17. We remark that for two permutatively equivalent n-by-n matrices $S = (s_{ij})$ and $C = (c_{ij})$ whose first row, are the an n-tuple $(s_1, \ldots, s_n)$, and $(c_1, \ldots, c_n)$, respectively, the inequalities $|c_{ij}| \leq s_{ij}$ hold if and only if $|c_i| \leq s_i, 1 \leq i \leq n$.

Theorem 18. *Let $S = (s_{ij})$ and $C = (c_{ij})$ be permutatively equivalent matrices whose first row are the n-tuples $(s_1, \ldots, s_n)$ and $(c_1, \ldots, c_n)$, respectively, such that $|c_i| \leq s_i, 1 \leq i \leq n$. Moreover, their spectra (counted with their multiplicities) are the lists $\sigma(S) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\sigma(C) = (\mu_1, \mu_2, \ldots, \mu_n)$, respectively. Let $0 \leq \gamma \leq 1$. Then, $\frac{1}{2}(S + \gamma C)$ and $\frac{1}{2}(S - \gamma C)$ are nonnegative matrices, permutatively equivalent matrices and the following matrices:*

$$M_{\pm\gamma} = \left(M_{ij_{\pm\gamma}}\right) \text{ with } M_{ij_{\pm\gamma}} = \begin{pmatrix} \frac{s_{ij} \pm \gamma c_{ij}}{2} & \frac{s_{ij} \mp \gamma c_{ij}}{2} \\ \frac{s_{ij} \mp \gamma c_{ij}}{2} & \frac{s_{ij} \pm \gamma c_{ij}}{2} \end{pmatrix}, \ 1 \leq i, j \leq n \tag{10}$$

are permutative and realize, respectively, the following lists

$$\sigma(S) \cup (\pm\gamma\sigma(C)) = (\lambda_1, \lambda_2, \ldots, \lambda_n, \pm\gamma\mu_1, \pm\gamma\mu_2, \ldots, \pm\gamma\mu_n).$$

In particular, if the list $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *is a Suleĭmanova's type list and* $(\mu_1, \mu_2, \ldots, \mu_n)$ *satisfies the condition*

$$\mu_1 + \mu_2 + \cdots + \mu_n \leq \lambda_1 + \lambda_2 + \cdots + \lambda_n, \tag{11}$$

and

$$\left|\frac{\mu_1 + \mu_2 + \cdots + \mu_n}{n} - \mu_i\right| \leq \left(\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{n} - \lambda_i\right) \text{ for } 2 \leq i \leq n, \tag{12}$$

then, the lists $\sigma(S) \cup (\pm\gamma\sigma(C))$ *are respectively, realizable by the matrices* $M_{\pm\gamma}$ *in (10), where* $S = (s_{ij})$ *and* $C = (c_{ij})$ *are the corresponding permutative matrices obtained from Theorem 4 and Lemma 3 by replacing with the lists of eigenvalues.*

Proof. Is an immediate consequence of the fact that if the matrices S and C in Theorem 14 are considered to be permutatively equivalent matrices then, by Proposition 12, both $\frac{1}{2}(S + \gamma C)$ and $\frac{1}{2}(S - \gamma C)$ are permutatively to S. Therefore, by the shape of the matrices in (9) the matrices $M_{\pm\gamma}$ in (10) become permutative matrices. In particular, if the matrices S and C and its spectra $\sigma(S)$ and $\sigma(C)$, respectively, are as in the statement, by last statement of Remark 11 the matrices S and $\pm\gamma C$ that realize the spectra $\sigma(S)$ and $\pm\gamma\sigma(C)$ are permutatively equivalent matrices. Then $\frac{1}{2}(S + \gamma C)$ and $\frac{1}{2}(S - \gamma C)$ are nonnegative permutative matrices, implying, by the above reasoning that the matrices $M_{\pm\gamma}$ in (9), with the given description by (10), will be also nonnegative permutative matrices. The conditions in (11) and (12) are derived from the condition $|c_i| \leq s_i$, for all $1 \leq i \leq n$ when the n-tuples $(s_1, \ldots, s_n)$ and $(c_1, \ldots, c_n)$ are the first row of S and C, respectively, where the corresponding descriptions of $(s_1, \ldots, s_n)$ and $(c_1, \ldots, c_n)$ are obtained from Lemma 3 and Theorem 4 . □

Note that it is important that the matrices S and C are permutatively equivalent otherwise we can not guarantee that the matrices $M_{\pm\gamma}$ are permutative. In fact consider the following example:

Example 19. Both matrices S and C are not permutatively equivalent and

M constructed as in the previous theorem is not permutative.

$$S = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \tag{13}$$

$$C = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \tag{14}$$

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 2 & 0 & \frac{3}{2} & \frac{3}{2} & 1 & 0 \\ 0 & 2 & \frac{3}{2} & \frac{3}{2} & 0 & 1 \end{pmatrix} \tag{15}$$

In the next examples S and C are permutatively equivalent.

Example 20. Let S and C be the following circulant matrices, in consequence, they are permutatively equivalent matrices

$$S = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

whose spectra, respectively, are the following lists

$$\left(5, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right) \text{ and } \left(1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right).$$

It is easy to see that the conditions of Theorem 18 are verified. In consequence, the 6-by-6 matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

is a permutative matrix and realizes the list

$$\left(5, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}, 1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right).$$

Example 21. Let S and C be the following circulant matrices, in consequence, they are permutatively equivalent matrices

$$S = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

whose spectra, respectively, are

$$\left(5, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right) \text{ and } \left(-1, \frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}\right).$$

It is easy to see that the conditions of Theorem 18 are verified. In consequence, the 6-by-6 matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

is nonnegative permutative and realizes the spectrum

$$\left(5, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}, -1, \frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}\right).$$

Example 22. Let $\sigma = (10, 7, -3, -3, -2, -2, -2 - 1)$. The following Suleĭmanova sub-lists $(7, -3, -2, -2)$ and $(10, -3, -2, -1)$ can be obtained from σ. Thus, the conditions of Theorem 18 hold and by Theorem 4 the matrix that realizes $(10, -3, -2, -1)$ is

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 3 & 2 & 1 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

and the matrix that realizes $(7, -3, -2, -2)$ is

$$C = \begin{pmatrix} 0 & 2 & 2 & 3 \\ 2 & 0 & 2 & 3 \\ 2 & 2 & 0 & 3 \\ 3 & 2 & 2 & 0 \end{pmatrix}.$$

Therefore, the matrix M in (9) becomes

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 2 & 0 & \frac{5}{2} & \frac{1}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 2 & \frac{1}{2} & \frac{5}{2} & \frac{1}{2} & \frac{7}{2} \\ 2 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{1}{2} & \frac{7}{2} & \frac{1}{2} \\ 0 & 2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{1}{2} & \frac{7}{2} \\ \frac{5}{2} & \frac{1}{2} & 2 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} & 0 & 2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{7}{2} \\ \frac{7}{2} & \frac{1}{2} & 2 & 0 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{7}{2} & 0 & 2 & \frac{1}{2} & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

which is a permutative matrix and realizes the initial list.

4. Real odd spectra

We now present spectral results for matrices partitioned into blocks and with odd order. We start with the following spectral result that is presented in an algebraic closed field, K, for instance $K = \mathbb{C}$.

Theorem 23. *Let K be an algebraically closed field of characteristic 0 and suppose that $A = (A_{ij})$ is an into block square matrix of order $2n+1$, where*

$$A_{ij} = \begin{cases} \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix} & 1 \le i, j \le n \\ \begin{pmatrix} a_{ij} \\ a_{ij} \end{pmatrix} & 1 \le i \le n, \ j = n+1 \\ \begin{pmatrix} a_{ij} & b_{ij} \end{pmatrix} & i = n+1, \ 1 \le j \le n \\ a_{ij} & i = n+1, \ \ j = n+1. \end{cases}$$

If

$$s_{ij} = \begin{cases} a_{ij} + b_{ij} & 1 \le i, j \le n \\ a_{ij} & 1 \le i \le n, \ j = n+1 \\ a_{ij} + b_{ij} & i = n+1, \ 1 \le j \le n \\ a_{ij} & i = n+1, \ \ j = n+1 \end{cases}$$

and

$$c_{ij} = a_{ij} - b_{ij},\ 1 \le i, j \le n.$$

Then

$$\sigma(A) = \sigma(S) \cup \sigma(C),$$

where

$$S = (s_{ij}) \text{ and } C = (c_{ij}).$$

Proof. Let (λ, v) be an eigenpair of S, with $v = (v_1, \ldots, v_n, v_{n+1})^T$, and consider the $(2n+1)$-by-1 block vector $w = \begin{pmatrix} (w_j) \\ w_{n+1} \end{pmatrix}$, where by an abuse of notation, we have

$$w_j = \begin{cases} v_j \mathbf{e} & 1 \le j \le n \\ v_{n+1} & j = n+1. \end{cases}$$

Since

$$\begin{aligned} A_{ij} w_j &= \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix} v_j \mathbf{e} = \begin{pmatrix} a_{ij} + b_{ij} \\ b_{ij} + a_{ij} \end{pmatrix} v_j = s_{ij} v_j \mathbf{e}, \quad 1 \le i, j \le n \\ A_{i,n+1} w_{n+1} &= \begin{pmatrix} a_{ij} \\ a_{ij} \end{pmatrix} w_{n+1} = s_{i,n+1} v_{n+1} \mathbf{e}, \quad 1 \le i \le n \\ A_{n+1,j} w_j &= \begin{pmatrix} a_{n+1,j} & b_{n+1,j} \end{pmatrix} w_j = \begin{pmatrix} a_{n+1,j} & b_{n+1,j} \end{pmatrix} (v_j \mathbf{e}) \\ &= (a_{n+1,j} + b_{n+1,j}) v_j = s_{n+1,j} v_j, \quad 1 \le j \le n \end{aligned}$$

Finally,

$$A_{n+1,n+1}\, w_{n+1} = s_{n+1,n+1} v_{n+1}.$$

Notice that, for every $i \in \{1, \ldots, n\}$,

$$\begin{aligned} \sum_{j=1}^{n+1} A_{ij} w_j &= \sum_{j=1}^{n} A_{ij} w_j + A_{i,n+1} w_{n+1} \\ &= \left(\sum_{j=1}^{n} s_{ij} v_j + s_{i,n+1} v_{n+1} \right) \mathbf{e} = (\lambda v_i)\, \mathbf{e} = \lambda (v_i \mathbf{e}) = \lambda w_i \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{n+1} A_{n+1,j} w_j &= \sum_{j=1}^{n} s_{n+1,j} v_j + A_{n+1,n+1} w_{n+1} \\ &= \left(\sum_{j=1}^{n} s_{n+1,j} v_j + s_{n+1,n+1} v_{n+1} \right) = \lambda v_{n+1} = \lambda w_{n+1} \end{aligned}$$

i.e (λ, w) is an eigenpair of A. Thus $\sigma(S) \subseteq \sigma(A)$.

Similarly, let (μ, x) be an eigenpair of C, with $x = (x_1, \ldots, x_n)^T$ and consider the $(2n+1)$-by-1 block vector $y = \begin{pmatrix} (y_j) \\ 0 \end{pmatrix}$, where $y_j := x_j \mathbf{f}$ and $\mathbf{f} = (1, -1)^T$. Since

$$A_{ij} y_j = \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix} x_j \mathbf{f} = \begin{pmatrix} a_{ij} - b_{ij} \\ b_{ij} - a_{ij} \end{pmatrix} x_j = c_{ij} x_j \mathbf{f},$$

notice that, for every $i = 1, \ldots, n$,

$$\sum_{j=1}^{n+1} A_{ij} y_j = \left(\sum_{j=1}^{n} c_{ij} x_j \mathbf{f} + A_{i,n+1} y_{n+1} \right) = (\lambda x_i) \mathbf{f} = \lambda (x_i \mathbf{f}) = \lambda y_i$$

i.e (μ, y) be an eigenpair of A. Thus $\sigma(C) \subseteq \sigma(A)$. Suppose that

$$\Theta_s = \left\{ (x_{1i}, x_{2i}, \ldots, x_{ni}, x_{n+1,i})^T : i = 1, \ldots, n+1 \right\}$$

and

$$\Theta_c = \left\{ (y_{1i}, y_{2i}, \ldots, y_{ni})^T : i = 1, \ldots, n \right\}$$

are bases of eigenvectors of S and C, respectively. The result will follow after proving the linear independence of the following set $\Upsilon = \Upsilon_1 \cup \Upsilon_2$ where

$$\Upsilon_1 = \left\{ \left(x_1 \mathbf{e}_2^T, x_2 \mathbf{e}_2^T, \ldots, x_n \mathbf{e}_2^T, x_{n+1} \right)^T : (x_1, x_2 \ldots, x_n, x_{n+1})^T \in \Theta_s \right\}$$

and

$$\Upsilon_2 = \left\{ \left(y_1 \mathbf{f}_2^T, y_2 \mathbf{f}_2^T, \ldots, y_n \mathbf{f}_2^T, 0 \right)^T : \left(\eta, (y_1, y_2, \ldots, y_n)^T \right) \in \Theta_c \right\}.$$

To this aim, we study the next determinant:

$$d = \begin{vmatrix} y_{11} & \ldots & y_{1n} & x_{11} & \ldots & x_{1n} & x_{1,n+1} \\ -y_{11} & \ldots & -y_{1n} & x_{11} & \ldots & x_{1n} & x_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{n1} & \ldots & y_{nn} & x_{n1} & \ldots & x_{nn} & x_{n,n+1} \\ -y_{n1} & \ldots & -y_{nn} & x_{n1} & \ldots & x_{nn} & x_{n,n+1} \\ 0 & \ldots & 0 & x_{n+1,1} & \ldots & x_{n+1,n} & x_{n+1,n+1} \end{vmatrix}.$$

Note that d stands for the determinant of a $(2n+1)$-by-$(2n+1)$ matrix obtained from the coordinates of the vectors in Υ. As before, adding rows and making suitable row permutations we conclude that the absolute value of d coincides with the absolute value of the following determinant

$$\begin{vmatrix} y_{11} & \dots & y_{1n} & x_{11} & \dots & x_{1n} & x_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{n1} & \dots & y_{nn} & x_{n1} & \dots & x_{nn} & x_{n,n+1} \\ 0 & \dots & 0 & 2x_{11} & \dots & 2x_{1n} & 2x_{1,n+1} \\ 0 & \dots & 0 & \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 2x_{n,1} & \dots & 2x_{n,n} & 2x_{n,n+1} \\ 0 & \dots & 0 & x_{n+1,1} & \dots & x_{n+1,n} & x_{n+1,n+1} \end{vmatrix}$$

which is nonzero by the linear independence of the set Θ_s and Θ_c. Thus the statement follows.

∎

Theorem 24. *Let $S = (s_{ij})$ be a matrix of order $n+1$ and $C = (c_{ij})$ a matrix of order n whose spectra (counted with their multiplicities) are $\sigma(S) = (\lambda_1, \lambda_2, \dots, \lambda_{n+1})$ and $\sigma(C) = (\mu_1, \mu_2, \dots, \mu_n)$, respectively. Moreover, suppose that $s_{ij} \geq |c_{ij}|$ for all $1 \leq i, j \leq n$, $s_{i,n+1} \geq 0$, for $i = 1, \dots, n+1$ and $\varphi^{(j)}_{n+1,i} \geq 0$, for $j = 1, 2$ and for $i = 1, \dots, n$. Then, for all $0 \leq \gamma \leq 1$, the nonnegative matrices*

$$M_{\pm\gamma} = \begin{pmatrix} \frac{s_{11}\pm\gamma c_{11}}{2} & \frac{s_{11}\mp\gamma c_{11}}{2} & \dots & \dots & \frac{s_{1n}\pm\gamma c_{1n}}{2} & \frac{s_{1n}\mp\gamma c_{1n}}{2} & s_{1,n+1} \\ \frac{s_{11}\mp\gamma c_{11}}{2} & \frac{s_{11}\pm\gamma c_{11}}{2} & \dots & \dots & \frac{s_{1n}\mp\gamma c_{1n}}{2} & \frac{s_{1n}\pm\gamma c_{1n}}{2} & s_{1,n+1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \frac{s_{n1}\pm\gamma c_{n1}}{2} & \frac{s_{n1}\mp\gamma c_{n1}}{2} & \dots & \dots & \frac{s_{nn}\pm\gamma c_{nn}}{2} & \frac{s_{nn}\mp\gamma c_{nn}}{2} & s_{n,n+1} \\ \frac{s_{n1}\mp\gamma c_{n1}}{2} & \frac{s_{n1}\pm\gamma c_{n1}}{2} & \dots & \dots & \frac{s_{nn}\mp\gamma c_{nn}}{2} & \frac{s_{nn}\pm\gamma c_{nn}}{2} & s_{n,n+1} \\ \varphi^{(1)}_{n+1,1} & \varphi^{(2)}_{n+1,1} & \dots & \dots & \varphi^{(1)}_{n+1,n} & \varphi^{(2)}_{n+1,n} & s_{n+1,n+1} \end{pmatrix}, \quad (16)$$

$$\varphi^{(1)}_{n+1,i} + \varphi^{(2)}_{n+1,i} = s_{n+1,i}, \;\; i = 1, 2, \dots, n$$

have spectra

$$\sigma(S) \cup \pm\gamma\sigma(C) = (\lambda_1, \lambda_2, \dots, \lambda_{n+1}, \pm\gamma\mu_1, \pm\gamma\mu_2, \dots, \pm\gamma\mu_n)\,.$$

Proof. The result follows from a direct application of Theorem 23 to the matrix M in (16). ∎

Example 25. Let $S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be matrices which spectrum are $(1,0,3)$ and $(1,1)$, respectively. Let

$$M = \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

Then, M has eigenvalues

$$\sigma(M) = (3,0,1,1,1)\,.$$

Theorem 26. *Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be matrices of order n. Moreover, consider the n-tuples*

$$\mathbf{x} = (x_1, \ldots, x_n)^T$$

and

$$\mathbf{y}^T = (y_1, \ldots, y_n)^T\,.$$

Let

$$S = \left(\begin{array}{c|c} A+B & \mathbf{x} \\ \hline 2\mathbf{y} & u \end{array}\right) \text{ and } C = A - B \tag{17}$$

with $A+B$ and $A-B$ nonnegative matrices and with $\mathbf{x}$, $\mathbf{y}$ and u also nonnegative and, consider the matrix partitioned into blocks

$$M = \begin{pmatrix} M_{11} & M_{12} & \ldots & \ldots & M_{1n} & \mathbf{x}_1 \\ M_{21} & M_{22} & \ldots & \ldots & M_{2n} & \mathbf{x}_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ M_{n1} & M_{n2} & \ldots & \ldots & M_{nn} & \mathbf{x}_n \\ \mathbf{y}_1 & \mathbf{y}_2 & \ldots & \ldots & \mathbf{y}_n & u \end{pmatrix}, \tag{18}$$

where, for $1 \leq i, j \leq n$

$$M_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{pmatrix}, \; \mathbf{x}_i = (x_i, x_i)^T \;\; \textit{and}\; \mathbf{y}_i = (y_i, y_i)\,.$$

Then

$$\sigma(M) = \sigma(S) \cup \sigma(C).$$

Moreover, the matrix M *is nonnegative symmetric when* A, B *are symmetric matrices and* $\mathbf{x}^T = \mathbf{y}$.

Proof. This result is a clear consequence of Theorem 23. □

Example 27. Let us consider the list $\sigma = \left(2, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$.

If we want to apply the known sufficient conditions of Laffey and Smigoc, [10], it is not possible to obtain a partition of σ where each of its subset has cardinality three. Nevertheless, the matrices

$$S = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

have respectively, the following list of eigenvalues

$$\left(2, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right) \text{ and } \left(\frac{-1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right).$$

In consequence, we consider the matrix M in (18)

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and by Theorem 26 this matrix M realizes the list

$$\left(2, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right).$$

In [22] it was proven that if $\sigma = (\lambda_1, \ldots, \lambda_n)$ is a list of complex numbers whose Perron root is λ_1 and with $\lambda_i \in \Upsilon = \left\{ z \in \mathbb{C} \colon \operatorname{Re} z < 0,\ \left|\sqrt{3}\operatorname{Re} z\right| \geq |\operatorname{Im} z| \right\}$, for $i = 2, \ldots, n$, then there exists a nonnegative matrix realizing the list σ if and only if $\sum_{i=1}^{n} \lambda_i \geq 0$. The example below shows that the set Υ can widen out.

Example 28. Let

$$S = \begin{pmatrix} 4 & 3 & 5 \\ 5 & 4 & 3 \\ 3 & 5 & 4 \end{pmatrix} \text{ and } C = \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}$$

whose spectra are

$$\sigma(S) = \left(12, i\sqrt{3}, -i\sqrt{3}\right) \text{ and } \sigma(C) = (4 + 3i, 4 - 3i)\,. \tag{19}$$

Both matrices satisfy the conditions of Theorem 24 and, the matrix M obtained from S and C with the techniques above

$$M = \begin{pmatrix} 4 & 0 & 3 & 0 & 5 \\ 0 & 4 & 0 & 3 & 5 \\ 1 & 4 & 4 & 0 & 3 \\ 4 & 1 & 0 & 4 & 3 \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{5}{2} & 4 \end{pmatrix}$$

realizes the complex list

$$\sigma(M) = \left(12, i\sqrt{3}, -i\sqrt{3}, 4 + 3i, 4 - 3i\right) \text{ where, } i\sqrt{3}, -i\sqrt{3}, 4+3i, 4-3i \notin \Upsilon.$$

With this example we illustrate the fact that it is possible to find a nonnegative matrix that realizes a certain list of complex numbers that are not only in Υ. Moreover, note that the list at the example also verifies the condition that the sum of its elements is greater or equal than zero.
The next example shows that accordingly to Theorem 24 the next matrix M also realizes the complex list and, therefore it is worth to notice that there is more than one matrix that realizes it.

Example 29. Let

$$S = \begin{pmatrix} 4 & 3 & 5 \\ 5 & 4 & 3 \\ 3 & 5 & 4 \end{pmatrix} \text{ and } C = \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}$$

be the matrices as previous example, whose spectra are as in (19). Both matrices satisfy the conditions of Theorem 24 and, the matrix M obtained from S and C with the techniques above (recall that the construction of M in (16).

$$M = \begin{pmatrix} 4 & 0 & 3 & 0 & 5 \\ 0 & 4 & 0 & 3 & 5 \\ 1 & 4 & 4 & 0 & 3 \\ 4 & 1 & 0 & 4 & 3 \\ 3 & 0 & 5 & 0 & 4 \end{pmatrix}$$

realizes the same complex list.

5. Guo Perturbations

In what follows the lists are considered as ordered an n-tuples. Guo [26], in a partial continuation of a work by Fiedler extended some spectral properties of symmetric nonnegative matrices to general nonnegative matrices. Moreover, he introduced the following interesting question:

If the list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is symmetrically realizable (that is, σ is the spectrum of a symmetric nonnegative matrix), and $t > 0$, whether (or not) the list $\sigma_t = (\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n)$ is also symmetrically realizable?

In [19] the authors gave an affirmative answer to this question in the case that the realizing matrix is circulant or left circulant.

They also presented a necessary and sufficient condition for σ to be the spectrum of a nonnegative circulant matrix. The following result was presented.

Theorem 30. *[19] Let $\sigma = \left(\lambda_1, \lambda_2, \lambda_3, \ldots, \overline{\lambda}_3, \overline{\lambda}_2\right)$ be the spectrum of an n-by-n nonnegative circulant matrix. Let $t \geq 0$ and $\theta \in \mathbb{R}$. Then*

$$\sigma_t = \left(\lambda_1 + 2t, \lambda_2 \pm t \exp\left(i\theta\right), \lambda_3, \ldots, \overline{\lambda}_3, \overline{\lambda}_2 \pm t \exp\left(-i\theta\right)\right)$$

is also the spectrum of an n-by-n nonnegative circulant matrix. Moreover, if $n = 2m + 2$, then

$$\sigma_t = \left(\lambda_1 + t, \lambda_2, \lambda_3, \ldots, \lambda_{m+1}, \lambda_{m+2} \pm t, \overline{\lambda}_{m+1}, \ldots, \overline{\lambda}_3, \overline{\lambda}_2\right)$$

is also the the spectrum of an n-by-n nonnegative circulant matrix.

Theorem 31. *Let $n = 2m + 2$ and consider the n-tuples $\sigma_1 = \sigma(S) = \left(\lambda_1, \lambda_2, \lambda_3, \ldots, \overline{\lambda}_3, \overline{\lambda}_2\right)$ and $\sigma_2 = \sigma(C) = \left(\beta_1, \beta_2, \beta_3, \ldots, \overline{\beta}_3, \overline{\beta}_2\right)$ with, respectively, realizing matrices S and C being ciculant matrices and such that the matrices S, $S + C$ and $S - C$ are nonnegative matrices (see necessary and sufficient conditions to this fact, for instance, in [19]). Let t_1 and t_2 such that*

$$t_1 \geq |t_2|, \tag{20}$$

then, there exists a nonnegative permutative matrix M realizing the list $\sigma_{s,t_1} \cup \sigma_{c,t_2}$, where

$$\sigma_{s,t_1} = \left(\lambda_1 + t_1, \lambda_2, \lambda_3, \ldots, \lambda_{m+1}, \lambda_{m+2} \pm t_1, \overline{\lambda}_{m+1}, \ldots, \overline{\lambda}_3, \overline{\lambda}_2\right)$$

and

$$\sigma_{c,t_2} = \left(\beta_1 + t_2, \beta_2, \beta_3, \ldots, \beta_{m+1}, \beta_{m+2} \pm t_2, \overline{\beta}_{m+1}, \ldots, \overline{\beta}_3, \overline{\beta}_2\right).$$

Proof. Let $r_s = (s_1, \ldots, s_n)^T$ and $r_c = (c_1, \ldots, c_n)^T$ be the first row of matrices S and C, respectively. In [19], it is shown that these rows satisfy

$$r_s = \frac{1}{n}\overline{F}\sigma_1^T \text{ and } r_c = \frac{1}{n}\overline{F}\sigma_2^T$$

where F is the n by n matrix,

$$F = \left(\omega^{(k-1)(j-1)}\right)_{1\leq k,j\leq n} \text{ and } \omega = \exp\left(\tfrac{2\pi i}{n}\right)$$

and $\overline{F}$ is the matrix conjugate of F. Then, if $\widetilde{r}_s$ and $\widetilde{r}_c$ are the first row of the realizing matrices of the spectra σ_{s,t_1} and σ_{c,t_2} those rows satisfy

$$\widetilde{r}_s = \frac{1}{n}\overline{F}\sigma_{s,t_1}^T \text{ and } \widetilde{r}_c = \frac{1}{n}\overline{F}\sigma_{c,t_2}^T. \tag{21}$$

Let $\mathbf{e}_1$ and $\mathbf{e}_{m+2}$ be the first and the $(m+2)$-nd canonical vectors of $\mathbb{C}^n$. Adding, at first, and after taking difference on the expressions in (21) we obtain

$$\begin{aligned}\widetilde{r}_s + \widetilde{r}_c &= \frac{1}{n}\overline{F}\left(\sigma_{s,t_1}^T + \sigma_{c,t_2}^T\right)\\ &= r_s + r_c + (t_1 + t_2)\,\overline{F}\mathbf{e}_1 \pm (t_1 + t_2)\,\overline{F}\mathbf{e}_{m+2}\end{aligned}$$

and

$$\widetilde{r}_s - \widetilde{r}_c = r_s - r_c + (t_1 - t_2)\,\overline{F}\mathbf{e}_1 \pm (t_1 - t_2)\,\overline{F}\mathbf{e}_{m+2}.$$

Since $S + C$ and $S - C$ are nonnegative then $r_s + r_c$ and $r_s - r_c$ are nonnegative. Moreover both $t_1+t_2 \geq 0, t_1-t_2 \geq 0$ (due to (20)) by Theorem 30, therefore both $\widetilde{r}_s+\widetilde{r}_c$ and $\widetilde{r}_s-\widetilde{r}_c$ are nonnegative columns. In consequence the circulant matrices $\widetilde{S}$, $\widetilde{S+C}$ and $\widetilde{S-C}$, whose first rows, respectively, are $\widetilde{r}_s$, $\widetilde{r}_s + \widetilde{r}_c$ and $\widetilde{r}_s - \widetilde{r}_c$, are nonnegative matrices and by Theorem 30 they are still circulant matrices and nonnegative. In consequence, using the techniques from the above section the matrix $\widetilde{M}$ obtained from the circulant matrices $\widetilde{S+C}$ and $\widetilde{S-C}$ is a permutative circulant by blocks matrix whose spectrum is $\sigma_{s,t_1} \cup \sigma_{c,t_2}$ as required. □

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