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Optimality Conditions for Convex Semi-Infinite Programming Problems with Finitely Representable Compact Index Sets

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Optimality conditions for convex SIP

Olga Kostyukova · Tatiana Tchemisova

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Abstract In the present paper, we analyze a class of convex Semi-Infinite Programming problems with arbitrary index sets defined by a finite number of nonlinear inequalities. The analysis is carried out by employing the constructive approach, which, in turn, relies on the notions of immobile indices and their immobility orders. Our previous work showcasing this approach includes a number of papers dealing with simpler cases of semi-infinite problems than the ones under consideration here. Key findings of the paper include the formulation and the proof of implicit and explicit optimality conditions under assumptions, which are less restrictive than the constraint qualifications traditionally used. In this perspective, the optimality conditions in question are also compared to those provided in the relevant literature. Finally, the way to formulate the obtained optimality conditions is demonstrated by applying the results of the paper to some special cases of the convex semi-infinite problems.

Olga Kostyukova
Institute of Mathematics, National Academy of Sciences of Belarus,
Surganov str. 11, 220072, Minsk, Belarus
kostyukova@im.bas-net.by

Tatiana Tchemisova, Corresponding author
Center for Research and Development in Mathematics and Applications,
Department of Mathematics, University of Aveiro,
Campus Universitário Santiago, 3810-193, Aveiro, Portugal
tatiana@ua.pt

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1 Introduction

In Semi-Infinite Programming (SIP), one has to minimize functions of finite-dimensional variables, which are subject to infinitely many constraints. SIP problems often arise in mathematics as well as in diverse engineering and economical applications of the latter (see [1–5], and the references therein). A large class of distributionally robust optimization problems can be described and solved with the help of convex SIP [6]. A number of SIP control-related challenges, to be met with in practical applications, can be found in [7–9], among others. In recent years, machine learning methods are gaining popularity because of their reliability and efficiency in dealing with "real-life" problems. In [10], an innovative method is proposed for generating infinitely many kernel combinations with the help of infinite and semi-infinite optimization.

In the study of optimization problems, in general, and the SIP ones, in particular, many important issues are associated to an eventual valid choice of efficient optimality conditions. The relevant literature on SIP and generalized SIP features a number of approaches to optimality conditions (cf. [1,2, 11–17]). Very often optimality conditions are based on the topological study of inequality systems (e.g. [18–20] et al.) and use different Constraint Qualifications (CQ) [12–14,18]. Various CQs and assorted questions on regularity and stability of the feasible sets in semi-infinite optimization are studied in [21–24] and the references therein.

The methodology, which will be described below, is often followed in order to verify the optimality of a given feasible solution. Using the information about a given problem and its feasible solution x^0, we formulate an auxiliary Nonlinear Programming (NLP) problem with a finite number of constraints. This problem is constructed in such a way that, under special additional conditions, the optimality property of x^0 in the original SIP problem should be connected with the optimality of x^0 in the auxiliary NLP problem. This allows for the use of a rich arsenal of tools, provided by the theory of NLP, and permits to derive explicit necessary and sufficient optimality conditions for SIP. This methodology affords two main approaches to optimality. The first one, the *discretization approach*, as its name suggests, uses a simple idea of approximation of the infinite index set by a finite grid to formulate a rather simple auxiliary NLP problem, a discretized one (NLP_D). The main drawback of this approach is that, for the optimality conditions of the original SIP problem to be formulated in terms of the optimality conditions for the auxiliary problem (NLP_D), rather strong additional conditions (CQs) should prevail, which is not often the case. The second approach, under the term *reduction*, takes into account the specific properties and the structure of the original SIP problem more accurately. This is made possible by the use of more sophisticated auxiliary (finite) problems, which are denoted here as reduced problems ($\mathrm{NLP_R}$). The reduction approach has the advantage of permitting the formulation of the optimality conditions for SIP in terms of optimality conditions for the reduced problems under weaker CQs. For the discretization and reduction approaches, as well as another less frequently used ones, see [1,2], and others.

It occurs that even more efficient auxiliary problems can be formulated for some classes of SIP problems. Thus, in the authors' papers [17,25,26], among others, the notion of *immobile* (or *carrier* as in [19]) indices is employed to construct auxiliary NLP problems of a new type for certain number of classes of convex SIP problems with polyhedral index sets. These auxiliary problems represent more accurate approximations of the original SIP problems, thus allowing for the proof of new (explicit and implicit) optimality conditions under weaker additional conditions. This certainly, will expand the scope of applications of the theory and methods of convex SIP.

The paper can be seen as a significant step forward in the study launched in our previous works, its natural, but not trivial expansion to one of the most general classes of convex SIP problems, the class of problems with compact index sets defined by finite numbers of functional inequalities. Our studies has been started in [27], where we introduced an auxiliary finite problem (let us qualify it here by (NLP_*)), performed an in-depth study of its properties, and validated a number of technical statements, which are necessary for further development of the new approach. The main aim of this paper is to apply the results from [27] to the study of the optimality in the convex SIP problems with finitely representable index sets. We will formulate and prove new optimality conditions in the form of implicit optimality criteria, explicit necessary and sufficient optimality conditions. These conditions do not necessitate any CQ and can be met under rather weak assumptions. We will compare the optimality conditions, thus obtained, with those known from the literature and prove the accrued efficiency of the former over the latter from the following point of view: *a*) the new optimality conditions do not use any constraint qualification (are *CQ-free*); *b*) a more restrained subset of the feasible solutions satisfies the necessary optimality conditions, obtained in the paper; and *c*) the new sufficient conditions describe a wider subset of optimal solutions.

It should be emphasized here that a simple translation of the optimality results from [17] and [25] to the more general class of convex SIP problems, considered in this paper, is impossible since the more complex geometry of index sets requires a non- trivial review of concepts and methods lying in the basis of our approach. It is worth mentioning that the class of compact sets, which are finitely representable in the form of systems of functional inequalities, is much wider than that of the convex polyhedra. Therefore, it is very important from both, the theoretical and practical points of view, to develop new tools, which allow to obtain efficient optimality conditions for the convex SIP problems considered in the paper.

The paper is organized as follows. Section 1 hosts the Introduction. In Section 2, we state the convex SIP problem with finitely representable index set, formulate the auxiliary problem (NLP$_*$), and recall some of the results obtained in [27]. In Section 3, we introduce a parametric problem (P(ε)) and study its properties, which are used in Section 4 to prove implicit optimality criteria and explicit optimality conditions for the original SIP problem. Several special cases are considered in Section 5: a case of SIP problems satisfying the Slater CQ; another one, where the lower level problem satisfies certain additional conditions; yet another case, where the index set is a polyhedron and, finally, the case of linear constraints. For each of these cases, we explicitly formulate optimality conditions. An example in Section 6 illustrates the applicability of the theoretical results obtained in the paper, the efficiency of the theorems proved here, and the usefulness of information about the immobile indices for numerical implementations. We use this example also to compare the optimality conditions obtained in the paper with other results known from the literature. In section 7, we discuss perspectives for future research and some open problems. The final Section 8 contains the conclusions and final remarks.

2 Convex SIP Problem with Finitely Representable Index Set

In this section, we formulate the problem, give the basic notations, and present some results from [27], which will be used in what follows.

Consider the following SIP problem:

$$\text{(SIP)}: \qquad \min_{x\in\mathbb{R}^n} c(x), \ \text{ s.t. } \ f(x,t) \le 0 \ \ \forall t \in T, \tag{1}$$

where $T \subset \mathbb{R}^p$ is a compact index set defined by a finite system of inequalities:

$$T := \{t \in \mathbb{R}^p : \ g_s(t) \le 0, \ s \in S\}, \ |S| < \infty. \tag{2}$$

Suppose that the cost function $c(x)$ and the constraint function $f(x,t)$, for all $t \in T$, are convex w.r.t. $x \in \mathbb{R}^n$. Hence, the problem (SIP) is convex. Suppose also that functions $c(x)$, $f(x,t)$ and $g_s(t)$ are sufficiently smooth w.r.t. $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^p$, which means here that the (partial) derivatives of these functions of all orders, that will be needed in sequel, exist and are continuous for all respective variables. The main aim of this study is to apply our approach developed in the previous papers, to the convex SIP problem (1) with the index set in the form (2), and obtain new optimality conditions for this problem.

Let us, first, reformulate some definitions introduced in [27].

Denote by X the set of feasible solutions (the feasible set) in the problem (SIP), $X := \{x \in \mathbb{R}^n : f(x,t) \le 0 \ \forall t \in T\}$. Suppose that the problem is consistent, i.e. $X \neq \emptyset$.

Definition 2.1 Problem (SIP) is said to satisfy the Slater condition (the Slater CQ) iff the interior of its feasible set is not empty:

$$\text{(SCQ)}: \qquad \exists\, \bar{x} \in \mathbb{R}^n : \ f(\bar{x},t) < 0 \ \ \forall t \in T.$$

Definition 2.2 An index $t \in T$ is said to be immobile in the problem (SIP) iff $f(x,t) = 0$ for all $x \in X$.

From Definition 2.2, it follows that any immobile index is an optimal solution of the *lower level problem*

$$(\text{LLP}(x)): \qquad \max_{t \in \mathbb{R}^p} f(x,t), \text{ s.t. } t \in T := \{t \in \mathbb{R}^p,\ g_s(t) \leq 0,\ s \in S\},$$

for all $x \in X$.

Consider an index $t \in T$. Denote by $S_a(t)$ the set of active indices in problem (LLP(x)): $S_a(t) := \{s \in S :\ g_s(t) = 0\}$, and by $L(t)$ the linearized tangent cone to the set T at t: $L(t) := \{l \in \mathbb{R}^p :\ \frac{\partial g_s^T(t)}{\partial t} l \leq 0,\ s \in S_a(t)\}$.

In [27], the necessary optimality conditions for the lower level problem were formulated under the *Mangasarian-Fromovitz CQ*, which is the most well known and widely used regularity condition.

Definition 2.3 Given the lower level problem (LLP(x)), the Mangasarian-Fromovitz CQ is said to hold at $\bar{t} \in T$ iff

$$(\text{MFCQ}): \qquad \exists\, l \in \mathbb{R}^p :\ \frac{\partial g_s^T(\bar{t})}{\partial t} l < 0,\ s \in S_a(\bar{t}).$$

Note that (MFCQ) is supposed to fulfill at $\bar{t} \in T$ if $S_a(\bar{t}) = \emptyset$.

Denote by $T^* \subset T$ the set of all immobile indices in (SIP). For $\bar{t} \in T^*$, $x \in \mathbb{R}^n$, and $l \in L(\bar{t})$, consider a parametric Linear Programming (LP) problem

$$(\text{LP}(x,\bar{t},l)): \quad \max_{w \in \mathbb{R}^p} \frac{\partial f^T(x,\bar{t})}{\partial t} w, \quad \text{s.t.} \quad \frac{\partial g_s^T(\bar{t})}{\partial t} w \leq -l^T \frac{\partial^2 g_s(\bar{t})}{\partial t^2} l, \quad s \in S_a(\bar{t}).$$

Suppose that $x \in X$ and (MFCQ) holds at $\bar{t} \in T^*$. Then problem (LP($x,\bar{t},l$)) has an optimal solution for all $l \in L(\bar{t})$.

Denote by $val(P)$ the optimal value of the cost function of an optimization problem (P) and consider the functions defined for $x \in \mathbb{R}^n$, $t \in T^*$, and $l \in L(t)$,

$$F_1(x,t,l) := \frac{\partial f^T(x,t)}{\partial t} l, \ \ F_2(x,t,l) := l^T \frac{\partial^2 f(x,t)}{\partial t^2} l + val(LP(x,t,l)). \quad (3)$$

Then, given $x \in X$, the first and the second order necessary optimality conditions for $\bar{t} \in T^*$ in the problem (LLP(x)) can be formulated in terms of functions (3), as follows (see [27]), respectively:

$$F_1(x,\bar{t},l) \leq 0 \ \ \forall l \in L(\bar{t}), \quad F_2(x,\bar{t},l) \leq 0 \ \ \forall l \in C(x,\bar{t}), \qquad (4)$$

where $C(x,\bar{t}) := \{l \in L(\bar{t}) :\ \frac{\partial f^T(x,\bar{t})}{\partial t} l = 0\}$ is the *cone of critical directions* at the point $\bar{t}$ in the lower level problem (LLP(x)).

Given immobile index $\bar{t} \in T^*$, taking into account conditions (4), which should be fulfilled by all $x \in X$, let us give the next definition.

Definition 2.4 Let $\bar{t} \in T^*$ satisfy (MFCQ) and $\bar{l} \in L(\bar{t}),\ \bar{l} \neq 0$. Define the immobility order $q(\bar{t}, \bar{l})$ of the immobile index $\bar{t}$ along the direction $\bar{l}$ as follows:

- $q(\bar{t}, \bar{l}) = 0$, if $\exists\ \bar{x} = x(\bar{t}, \bar{l}) \in X$ such that $F_1(\bar{x}, \bar{t}, \bar{l}) < 0$;
- $q(\bar{t}, \bar{l}) = 1$, if $F_1(x, \bar{t}, \bar{l}) = 0, \forall x \in X$, and $\exists\ \bar{x} = x(\bar{t}, \bar{l}) \in X$ such that $F_2(\bar{x}, \bar{t}, \bar{l}) < 0$;
- $q(\bar{t}, \bar{l}) > 1$, if $F_1(x, \bar{t}, \bar{l}) = 0$ and $F_2(x, \bar{t}, \bar{l}) = 0,\ \forall x \in X$.

It is seen from the definition, that in the case $F_1(x, \bar{t}, \bar{l}) = 0, F_2(x, \bar{t}, \bar{l}) = 0$, for all $x \in X$, the immobility order of the index $\bar{t}$ is greater than one. It is easy to specify the value of $q(\bar{t}, \bar{l})$ for this case, but we will not do it here, since our study is based on the following assumptions.

Assumption 1. *Given a feasible solution $x \in X$ of the convex* SIP *problem* (1), *the lower level problem* (LLP(x)) *meets the regularity condition* (MFCQ) *at any immobile index $\bar{t} \in T^* \subset T$.*

Assumption 2. *Given problem* (SIP), *for all $\bar{t} \in T^*$, it holds $q(\bar{t}, l) \leq 1$ for all $l \in L(\bar{t}),\ l \neq 0$.*

Assumptions 1 and 2 are supposed to be trivially satisfied if $T^* = \emptyset$.

In [27], it was proved that, under Assumptions 1 and 2, the set T^* of immobile indices in the convex problem (SIP) is finite and, therefore, admits a presentation $T^* := \{t^*_j, j \in J_*\}$, where $0 \leq |J_*| < \infty$.

Consider $j \in J_*$ and the corresponding immobile index $t^*_j \in T^*$. The set $L(j) := L(t^*_j)$ (the linearized tangent cone to the index set T in the point t^*_j) admits an alternative representation in terms of *extremal rays* (see [25]):

$$\begin{aligned} &L(j) := \{l \in \mathbb{R}^p : \exists\ \beta_i, i \in P(j),\ \alpha_i \geq 0, i \in I(j) \\ &\text{such that } l = \sum_{i \in P(j)} \beta_i b_i(j) + \sum_{i \in I(j)} \alpha_i a_i(j)\}, \end{aligned} \tag{5}$$

where $b_i(j), i \in P(j)$, are bidirectional extremal rays, and $a_i(j), i \in I(j)$, are unidirectional extremal rays of the cone $L(j)$. The extremal rays can be constructed using the procedure described in [28]. Note here that the extremal rays satisfy the following properties:

$$\sum_{i \in P(j)} |\beta_i| + \sum_{i \in I(j)} \alpha_i > 0 \Rightarrow \quad l = \sum_{i \in P(j)} \beta_i b_i(j) + \sum_{i \in I(j)} \alpha_i a_i(j) \neq 0, \tag{6}$$

$$b_i^T(j) a_m(j) = 0,\ i \in P(j),\ m \in I(j). \tag{7}$$

Given $j \in J_*$ and the corresponding cone $L(j)$, denote by $I_0(j)$ and $I_*(j)$ the indices of the unidirectional extremal rays $a_i(j), i \in I(j)$, such that $q(t^*_j, a_i(j)) \geq 1$ and $q(t^*_j, a_i(j)) = 0$, respectively:

$$I_0(j) := \{i \in I(j) :\ \tfrac{\partial f^T(x, t^*_j)}{\partial t} a_i(j) = 0,\ \forall x \in X\},\ I_*(j) := I(j) \setminus I_0(j). \tag{8}$$

Let $C_0(j) := \{l \in \mathbb{R}^p : \ l = \sum\limits_{i \in P(j)} \beta_i b_i(j) + \sum\limits_{i \in I_0(j)} \alpha_i a_i(j), \ \alpha_i \geq 0, \ i \in I_0(j)\}$.

It is shown in [27] that, given $t^*_j \in T^*$, the set $C_0(j) \setminus \{0\}$ consists of all directions $l \in L(j)$, whose immobility orders are greater that one. Therefore,

$$F_1(x, t^*_j, l) := \frac{\partial f^T(x, t^*_j)}{\partial t} l = 0, \ \forall l \in C_0(j), \ \forall x \in X. \tag{9}$$

By construction, $C_0(j) \subset C(x, t^*_j) \subset L(j), \ \ \forall x \in X$. In what follows, for simplicity, we will use notation $F_{1j}(x, l) := F_1(x, t^*_j, l)$, $F_{2j}(x, l) := F_2(x, t^*_j, l), j \in J_*$ for the functions defined in (3). In [27], the following result was proved.

Theorem 2.1 [Theorem 2 in [27], under additional Assumption 2] *Given problem* (SIP)*, let Assumptions* 1, 2 *be fulfilled. If for $x^0 \in X$, there exist subsets of indices and vectors*

$$\begin{gathered} \{t_j, j \in J_a\} \subset T_a(x^0) \setminus T^*, \\ \{l_k(j), k = 1, ..., m(j)\} \subset \{l \in C_0(j) : F_{2j}(x^0, l) = 0\}, j \in J_*, \\ \textit{with } |J_a| + \sum_{j \in J_*} m(j) < \infty, \end{gathered} \tag{10}$$

such that the point x^0 is optimal in the following NLP *problem:*

$$\begin{aligned} & \min \ c(x), \\ (\text{NLP}_*): \quad & s.t. \quad f(x, t^*_j) = 0, \ F_{1j}(x, b_i(j)) = 0, i \in P(j), \\ & F_{1j}(x, a_i(j)) = 0, i \in I_0(j), \ \ F_{1j}(x, a_i(j)) \leq 0, i \in I_*(j), \\ & F_{2j}(x, l_k(j)) \leq 0, k = 1, ..., m(j), j \in J_*; \ f(x, t_j) \leq 0, j \in J_a, \end{aligned} \tag{11}$$

then x^0 is an optimal solution in the problem (SIP)*.*

Here and in what follows, $T_a(x)$ is the active index set at a feasible solution $x \in X$: $T_a(x) := \{t \in T : f(x, t) = 0\}$.

Denote by $Q = Q(T^*) \subset \mathbb{R}^n$, the set defined by the equality constraints of the problem (NLP$_*$): $Q = \{x \in \mathbb{R}^n : \ f(x, t^*_j) = 0, F_{1j}(x, b_i(j)) = 0, i \in P(j), F_{1j}(x, a_i(j)) = 0, i \in I_0(j), j \in J_*\}$.

Then problem (NLP$_*$) can be written in the form

$$\begin{aligned} & \min \ c(x), \\ (\text{NLP}_*): \quad & \text{s.t.} \ \ x \in \bar{Q} := \{x \in Q : F_{1j}(x, a_i(j)) \leq 0, i \in I_*(j), j \in J_*\}, \\ & F_{2j}(x, l_k(j)) \leq 0, k = 1, ..., m(j), j \in J_*; \ \ f(x, t_j) \leq 0, j \in J_a. \end{aligned} \tag{12}$$

It follows from Lemmas 3 and 5, and Corollary 4 in [27], that, under Assumptions 1 and 2, the problem (NLP$_*$) possesses the following properties.

Property 2.1 The set $Q = Q(T^*)$ is convex.

Property 2.2 There exists a point $\tilde{x} \in X$ such that

$$F_{1j}(\tilde{x}, l) < 0, \ \forall l \in L(j) \setminus C_0(j), \ \|l\| = 1; \tag{13}$$

$$F_{2j}(\tilde{x}, l) < 0, \ \forall l \in C_0(j), \ \|l\| = 1, \ j \in J_*; \tag{14}$$

$$f(\tilde{x}, t) < 0, \ t \in T \setminus T^*. \tag{15}$$

Property 2.3 For all $j \in J_*$, the auxiliary functions $F_{1j}(x, l)$ with $l \in L(j)$ are convex w.r.t. x in Q and the functions $F_{2j}(x, l)$ with $l \in C_0(j)$, are convex w.r.t. x in the convex set $\bar{Q}$ defined in (12).

Here and in what follows, we use the Euclidean norm $|| \cdot ||$.

Basing on Theorem 2.1 and the properties above, we can conclude that, under Assumptions 1 and 2, the sufficient optimality conditions for a feasible solution x^0 in the problem (SIP) can be substituted by the optimality conditions for x^0 in the auxiliary problem (NLP$_*$), which is convex and satisfies the Slater type CQ (Property 2.2).

3 Parametric Problem (P(ε)) and its Properties

In this section, using the constraints of the problem (NLP$_*$), we introduce a special parametric problem and study its properties, which are crucial for the proof of the necessary optimality conditions for the problem (SIP).

Suppose that the problem (SIP) has an optimal solution x^0. Given $\varepsilon > 0$, define the set $T(\varepsilon) := T \setminus \bigcup_{j \in J_*} \text{int } T_\varepsilon(j)$, where $T_\varepsilon(j) := \{t \in T : \|t - t_j^*\| \le \varepsilon\}$, $j \in J_*$, and consider a problem

$$(\text{P}(\varepsilon)) : \qquad \min c(x), \qquad \text{s.t.} \quad x \in Y \cap B, \ f(x, t) \le 0, \ \forall t \in T(\varepsilon),$$

where $Y = Y(T^*) := \{x \in \bar{Q} : F_{2j}(x, l) \le 0, \forall l \in C_0(j), j \in J_*\}$,
$B = B(\varepsilon_0, x^0) := \{x \in \mathbb{R}^n : \ \|x - x^0\| \le \varepsilon_0\}$, ε_0 being an arbitrary fixed number satisfying the inequality $||\tilde{x} - x^0|| > \varepsilon_0$, and $\tilde{x} \in X$ a point satisfying relations (13)-(15).

It is easy to see that the set Y is defined by the constraints of the problem (NLP$_*$) corresponding to the immobile indices of the original problem (SIP). In the case $T^* = \emptyset$, we have $Y = \mathbb{R}^n$ and $T(\varepsilon) := T$. It follows from Properties 2.1 and 2.3, that the set Y is convex, hence the set $Y \cap B$ is convex as well. Since the feasible set of the problem (P(ε)) is bounded, closed and not empty (vector x^0 is feasible), this problem has an optimal solution.

The main properties of the parametric problem (P(ε)) can be derived from the following proposition.

Proposition 3.1 *Suppose that Assumptions* 1 *and* 2 *are satisfied. Consider a vector* $\tilde{x} \in X$ *satisfying inequalities* (13)-(15)*, and let* $z \in Y$*. Then, for any sufficiently small* $\varepsilon > 0$*, there exists a number* $\lambda(\varepsilon) \in [0, 1]$ *such that*

$$f(x(\lambda(\varepsilon), z), t) \le 0, t \in T_\varepsilon(j), j \in J_*, \ \text{and } \lambda(\varepsilon) \to 0 \ \text{as } \varepsilon \downarrow 0, \tag{16}$$

where $x(\lambda, z) := (1 - \lambda)z + \lambda\tilde{x} = z + \lambda(\tilde{x} - z), \ \lambda \in [0, 1]$.

Proof. Given $z \in Y$, set $x(\lambda) := x(\lambda, z)$. Let $\varepsilon > 0$ be a sufficiently small positive number. Denote

$$\lambda_j(t) := \begin{cases} \dfrac{f(z,t)}{f(z,t) - f(\tilde{x},t)}, & \text{if } f(z,t) > 0, \\ 0, & \text{if } f(z,t) \leq 0, \end{cases} \qquad t \in T_\varepsilon(j), j \in J_*. \tag{17}$$

Since the function $f(x,t)$ is convex w.r.t. x, then $f(x(\lambda),t) \leq 0$ for $\lambda \in [\lambda_j(t), 1], t \in T_\varepsilon(j), j \in J_*$. Note that if $f(z,t) \leq 0$ for all $t \in T_{\bar{\varepsilon}}(j),\ j \in J_*$, and some $\bar{\varepsilon} > 0$, then $\lambda_j(t) = 0$ for all $t \in T_\varepsilon(j),\ j \in J_*$, and all $0 < \varepsilon \leq \bar{\varepsilon}$. Consequently, relations (16) take place with $\lambda(\varepsilon) \equiv 0,\ 0 < \varepsilon \leq \bar{\varepsilon}$, and the proposition is proved for this case.

Let $j \in J_*$ be an arbitrary index such that $T_\varepsilon(j) \cap T^+(z) \neq \emptyset$, where $T^+(z) := \{t \in T :\ f(z,t) > 0\}$. By construction, $f(z,t) > 0, t \in T_\varepsilon(j) \cap T^+(z)$. Hence

$$0 \leq \lambda_j(t) \leq -\frac{f(z,t)}{f(\tilde{x},t)} =: \mu_j(t),\ t \in T_\varepsilon(j) \cap T^+(z). \tag{18}$$

To show that

$$\mu_j(t) \leq O_j(\varepsilon) \ \text{ for } t \in T_\varepsilon(j) \cap T^+(z), \tag{19}$$

where $O_j(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, suppose that (19) is not satisfied. Hence, there exist sequences $\varepsilon_i > 0,\ t_i \in T_{\varepsilon_i}(j) \cap T^+(z),\ i = 1, 2, \ldots$, such that

$$||t_i - t_j^*|| = \varepsilon_i,\ \lim_{i \to \infty} \varepsilon_i = 0,\ \lim_{i \to \infty} \mu_j(t_i) = \bar{\mu}_j > 0. \tag{20}$$

For any $i = 1, 2, ...$, the index $t_i \in T_{\varepsilon_i}(j) \cap T^+(z) \subset T$ can be represented in the form $t_i = t_j^* + \Delta t_i$, where $\|\Delta t_i\| = \varepsilon_i$. Therefore,

$$g_s(t_i) = \frac{\partial g_s^T(t_j^*)}{\partial t} \Delta t_i + o(\|\varepsilon_i\|) \leq 0,\ s \in S_a(t_j^*). \tag{21}$$

Evidently, the sequence $\frac{\Delta t_i}{||\Delta t_i||},\ i = 1, 2, ...$, possesses a convergent subsequence $\frac{\Delta t_{k_i}}{||\Delta t_{k_i}||},\quad i = 1, 2, ...$. Denote $\bar{l} := \lim_{i \to \infty} \frac{\Delta t_{k_i}}{||\Delta t_{k_i}||}$. From (21), it follows that $\bar{l} \in L(j),\ \|\bar{l}\| = 1$. To simplify the exposition, without loss of generality, we assume here that $k_i = i$ for $i = 1, 2, ...$ From the considerations above, it follows that Δt_i admits representation:

$$\Delta t_i = \varepsilon_i \cdot (\bar{l} + w_i(\Delta t_i)), \tag{22}$$

where $w_i(\Delta t)$ is a function satisfying the property $w_i(\Delta t) \to 0$ as $\|\Delta t\| \to 0$. Recall that any $\bar{l} \in L(j)$, can be presented in the form

$$\bar{l} = \gamma_* l^{(*)} + \gamma_0 l^{(0)},\ \gamma_* \geq 0,\ \gamma_0 \geq 0, \tag{23}$$

where

$$l^{(*)} := A_*\bar{\alpha}_*,\ \ l^{(0)} := (A_0, B)\begin{pmatrix}\bar{\alpha}_0\\ \bar{\beta}\end{pmatrix}; \bar{\alpha}_0 \geq 0,\ \bar{\alpha}_* \geq 0; \|l^{(*)}\| = |l^{(0)}\| = 1,$$
$$A_0 = A_0(j) := (a_i(j), i \in I_0(j));\ \ A_* = A_*(j) := (a_i(j), i \in I_*(j));$$
$$B = B(j) := (b_i(j), i \in P(j));\ \bar{\alpha}_0 = (\bar{\alpha}_i(j), i \in I_0(j)); \tag{24}$$
$$\bar{\alpha}_* = (\bar{\alpha}_i(j), i \in I_*(j));\ \bar{\beta} = (\bar{\beta}_i(j), i \in P(j));$$

and the coefficients $\bar{\alpha}_i$ and $\bar{\beta}_i$ are associated with the representation of $\bar{l} \in L(j)$ in terms of the extremal rays (see (5)). Here we took into account (7).

From (6), it follows that the sets $\{\alpha_* : \alpha_* \geq 0, \alpha_*^T A_*^T A_* \alpha_* = 1\}$ and $\{(\beta, \alpha_0) : \alpha_0 \geq 0,\ \beta^T B^T B\beta + \alpha_0^T A_0^T A_0\alpha_0 = 1\}$ are closed and bounded. Here $\alpha_0 \in \mathbb{R}^{|I_0(j)|}$, $\alpha_* \in \mathbb{R}^{|I_*(j)|}$, $\beta \in \mathbb{R}^{|P(j)|}$.

One of two following cases can occur in (23): A. $\gamma_* > 0$, B. $\gamma_* = 0$.

Let us, first, assume that the case A holds. Since $t_i = t_j^* + \bar{l}\varepsilon_i + o(\varepsilon_i)$ and $\frac{\partial f^T(\tilde{x}, t_j^*)}{\partial t} l^{(0)} = 0$, then $f(\tilde{x}, t_i) = f(\tilde{x}, t_j^*) + \varepsilon_i \frac{\partial f^T(\tilde{x}, t_j^*)}{\partial t}\bar{l} + o(\varepsilon_i) =$
$\varepsilon_i \frac{\partial f^T(\tilde{x}, t_j^*)}{\partial t}(\gamma_* l^{(*)} + \gamma_0 l^{(0)}) + o(\varepsilon_i) = \varepsilon_i \frac{\partial f^T(\tilde{x}, t_j^*)}{\partial t} l^{(*)}\gamma_* + o(\varepsilon_i) \leq \varepsilon_i\gamma_* C_1 + o(\varepsilon_i)$, where $l^{(*)} = A_*\bar{\alpha}_*$ (see (24)) and C_1 is the optimal value of the cost function in the problem $\max\limits_{\alpha_*} \dfrac{\partial f^T(\tilde{x}, t_j^*)}{\partial t} A_*\alpha_*$, s.t. $\alpha_*^T A_*^T A_*\alpha_* = 1,\ \alpha_* \geq 0$.

As $A_*\alpha_* \notin C_0(j)$ for any $\alpha_* \geq 0,\ \alpha_* \neq 0$, inequalities (13) hold. Hence, $C_1 < 0$ and for a sufficiently small $\varepsilon_i > 0$ we have

$$-f(\tilde{x}, t_i) \geq -C_1\varepsilon_i\gamma_* + o(\varepsilon_i) > 0. \tag{25}$$

Taking into account that, by construction, $f(z, t_j^*) = 0$, $\frac{\partial f^T(z, t_j^*)}{\partial t} l^{(0)} = 0$, $\frac{\partial f^T(z, t_j^*)}{\partial t} l^{(*)} \leq 0$, it holds $f(z, t_i) = \varepsilon_i \frac{\partial f^T(z, t_j^*)}{\partial t} l^{(*)}\gamma_* + o_1(\varepsilon_i) > 0$, wherefrom, with respect to the inequality $\varepsilon_i \frac{\partial f^T(z, t_j^*)}{\partial t} l^{(*)}\gamma_* \leq 0$, we get

$$0 < f(z, t_i) \leq o_1(\varepsilon_i). \tag{26}$$

From (25) and (26), it follows $\mu_j(t_i) = -\frac{f(z,t_i)}{f(\tilde{x},t_i)} \leq -\frac{o_1(\varepsilon_i)}{C_1\varepsilon_i\gamma_* + o(\varepsilon_i)} = O_j(\varepsilon_i)$, that contradicts assumption (20).

Now, let us consider case B: $\gamma_* = 0$ in (23). By assumption, $z \in Y$, and taking into account $l^{(0)} \in C_0(j)$, one gets

$$F_{2j}(z, l^{(0)}) = (l^{(0)})^T \frac{\partial^2 f(z, t_j^*)}{\partial t^2} l^{(0)} + val(LP(z, t_j^*, l^{(0)})) \leq 0. \tag{27}$$

The constraints of the problem (LP$(z, t_j^*, l^{(0)})$) are consistent and it follows from (27) that $val(LP(z, t_j^*, l^{(0)})) < +\infty$. Hence, this problem has a dual solution, i.e. there exist numbers $y_s = y_s(z),\ s \in S_a(t_j^*)$, such that

$$\sum_{s \in S_a(t_j^*)} y_s \frac{\partial g_s(t_j^*)}{\partial t} = \frac{\partial f(z, t_j^*)}{\partial t};\ y_s \geq 0, s \in S_a(t_j^*), \tag{28}$$

$$val(LP(z,t_j^*,l^{(0)})) = -\sum_{s\in S_a(t_j^*)} y_s(l^{(0)})^T\frac{\partial^2 g_s(t_j^*)}{\partial t^2}l^{(0)}. \tag{29}$$

Since $t_i = t_j^* + \Delta t_i$ with Δt_i defined in (22), then the inequalities

$$g_s(t_i) = g_s(t_j^*) + \frac{\partial g_s^T(t_j^*)}{\partial t}\Delta t_i + \frac{1}{2}\Delta t_i^T\frac{\partial^2 g_s(t_j^*)}{\partial t^2}\Delta t_i + o(\varepsilon_i^2) \le 0,\ s\in S_a(t_j^*),$$

can be rewritten in the form

$$\varepsilon_i\frac{\partial g_s^T(t_j^*)}{\partial t}(\bar l + w_i(\Delta t_i)) + \frac{1}{2}\varepsilon_i^2\bar l^T\frac{\partial^2 g_s(t_j^*)}{\partial t^2}\bar l + o(\varepsilon_i^2) \le 0,\ s\in S_a(t_j^*). \tag{30}$$

Similarly, we have

$$f(z,t_i) = \varepsilon_i\frac{\partial f^T(z,t_j^*)}{\partial t}(\bar l + w_i(\Delta t_i)) + \frac{1}{2}\varepsilon_i^2\bar l^T\frac{\partial^2 f(z,t_j^*)}{\partial t^2}\bar l + o(\varepsilon_i^2). \tag{31}$$

From (MFCQ), one can conclude that the set of vectors $y_s, s\in S_a(t_j^*)$ satisfying (28), is bounded. Multiply each inequality in (30) by the corresponding value $y_s \ge 0, s\in S_a(t_j^*)$, and sum the resulting inequalities:

$$\varepsilon_i\sum_{s\in S_a(t_j^*)} y_s\frac{\partial g_s^T(t_j^*)}{\partial t}(\bar l + w_i(\Delta t_i)) \le -\frac{1}{2}\varepsilon_i^2\sum_{s\in S_a(t_j^*)}\bar l^T\frac{\partial^2 g_s(t_j^*)}{\partial t^2}\bar l + o(\varepsilon_i^2). \tag{32}$$

From (28) and (31), it follows

$$f(z,t_i) = \varepsilon_i\sum_{s\in S_a(t_j^*)} y_s\frac{\partial g_s^T(t_j^*)}{\partial t}(\bar l + w_i(\Delta t_i)) + \frac{1}{2}\varepsilon_i^2\bar l^T\frac{\partial^2 f(z,t_j^*)}{\partial t^2}\bar l + o(\varepsilon_i^2).$$

This relation, together with (32), implies

$$f(z,t_i) \le \frac{1}{2}\varepsilon_i^2\bar l^T\left(-\sum_{s\in S_a(t_j^*)} y_s\frac{\partial^2 g_s(t_j^*)}{\partial t^2} + \frac{\partial^2 f(z,t_j^*)}{\partial t^2}\right)\bar l + o(\varepsilon_i^2),$$

wherefrom, w.r.t. the equality (29), the inequality $0 < f(z,t_i)$, and the fact that $\bar l = l^{(0)}$ in the case B, we get $0 < f(z,t_i) \le \frac{1}{2}\varepsilon_i^2 F_{2j}(z,l^{(0)}) + o(\varepsilon_i^2)$. Taking into account (27), from the the last inequalities we get

$$0 < f(z,t_i) \le o(\varepsilon_i^2). \tag{33}$$

Similarly, we have $f(\tilde x,t_i) \le \frac{1}{2}\varepsilon_i^2 F_{2j}(\tilde x,l^{(0)}) + \tilde o(\varepsilon_i^2) \le \frac{1}{2}\varepsilon_i^2 C_2 + \tilde o(\varepsilon_i^2)$, where $f(\tilde x,t_i) < 0$ and C_2 denotes the optimal value of the cost function in the problem

$$\max_{\beta,\alpha_0} F_{2j}(\tilde x, B\beta + A_0\alpha_0),\quad \text{s.t. } \beta^T B^T B\beta + \alpha_0^T A_0^T A_0\alpha_0 = 1,\ \alpha_0 \ge 0.$$

Since $B\beta + A_0\alpha_0 \in C_0(j)$ for any $(\beta,\alpha_0) \ne 0,\ \alpha_0 \ge 0$, the inequalities (14) take place. Hence, for C_2, defined above, and $\varepsilon_i > 0$ sufficiently small, it holds

$C_2 < 0$ and $-f(\tilde{x}, t_i) \geq -\frac{1}{2}\varepsilon_i^2 C_2 + \tilde{o}(\varepsilon_i^2) > 0$. From the last inequality together with (25) and (33), we get $\mu_j(t_i) = \frac{f(z,t_i)}{-f(\tilde{x},t_i)} \leq \frac{o(\varepsilon_i^2)}{\frac{1}{2}\varepsilon_i^2 C_2 + \tilde{o}(\varepsilon_i^2)} = \tilde{O}_j(\varepsilon_i)$. But this again contradicts our assumption (20). The contradictions obtained in the cases A and B, prove that relations (19) take place.

Set $\lambda(\varepsilon) := \max\{\lambda_j(\varepsilon), j \in J_*\}$, where

$$\lambda_j(\varepsilon) := \begin{cases} 0, & \text{if} \quad T_\varepsilon(j) \cap T^+(z) = \emptyset, \\ \max\limits_{t \in T_\varepsilon(j) \cap T^+(z)} \mu_j(t), & \text{if} \quad T_\varepsilon(j) \cap T^+(z) \neq \emptyset. \end{cases}$$

It follows from (18) and (19) that $\lambda(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and, by construction, $\lambda(\varepsilon) \geq \lambda_j(t)$ for $t \in T_\varepsilon(j) \cap T^+(z)$. Hence, relations (16) are fulfilled. □

It is worth mentioning that the proof of Proposition 3.1 (for SIP problems with finitely representable index sets) at the root is different from that of Proposition 5 in [17] (for SIP problems with the box constrained index sets). This is due to the fact that, in spite of the external similarity, the parametric problem (P(ε)) fundamentally differs from the parametric problem, which was introduced in [17]. This difference is explained by the more complex geometry of the finitely representable index set, and makes it impossible to simply transfer the evidence of [17] on the more complex case.

The following two corollaries, that can be proved in a similar way as Corollary 3 and Proposition 6 in [17], are obtained on the basis of Proposition 3.1.

Corollary 3.1 *Suppose that Assumptions* 1 *and* 2 *are satisfied for the convex problem* (SIP). *Then* $\lim\limits_{\varepsilon \downarrow 0} c(z^0(\varepsilon)) = c(x^0)$, *where* x^0 *is an optimal solution of problem* (SIP) *and* $z^0(\varepsilon)$ *is an optimal solution of the problem* (P(ε)).

Corollary 3.2 *Suppose that Assumptions* 1 *and* 2 *are satisfied for the convex problem* (SIP). *Consider a vector function* $x(\lambda, z) = (1 - \lambda)z + \lambda\tilde{x}$, $\lambda \in [0, 1]$, *where vector* $\tilde{x}$ *satisfies* (13)-(15), *and* $z \in Y$. *Then for all* $\lambda \in]0, 1]$, *there exists* $\Delta = \Delta(\lambda, z) > 0$ *such that* $f(x(\lambda, z), t) \leq 0$, $t \in T_\Delta(j)$, $j \in J_*$.

4 Optimality Conditions for the Convex SIP Problems with Finitely Representable Index Sets

In this section, we will use the properties of the parametric problem (P(ε)) proved in the previous section, to obtain new optimality conditions for the problem (SIP), that is the main goal of the paper.

4.1 Implicit Optimality Criteria

Using Corollaries 3.1 and 3.2, and following the main steps the proof of Theorem 1 from [17], we can prove the following theorem.

Theorem 4.1 *Suppose that Assumptions* 1 *and* 2 *are satisfied for the convex problem* (SIP). *Then a feasible solution* $x^0 \in X$ *is optimal in this problem if and only if there exists a set* $\{t_j,\ j \in J_a\} \subset T_a(x^0)\backslash T^*,\quad |J_a| \leq n$, *such that* x^0 *is an optimal solution of the auxiliary problem*

$$\text{(AP)}: \qquad \min c(x), \quad \text{s.t. } x \in Y,\ f(x,t_j) \leq 0, j \in J_a.$$

Now, let us rewrite the problem (AP) in the form

$$\min_{x \in \bar{Q} \subset \mathbb{R}^n} c(x),$$
$$\text{s.t. } F_{2j}(x,l) \leq 0,\ \forall l \in C_0(j),\ \|l\| = 1,\ j \in J_*; \quad f(x,t_j) \leq 0,\ j \in J_a,$$

where the set $\bar{Q} \subset \mathbb{R}^n$ is defined in (12). Under Assumptions 1 and 2, problem (AP) possesses the Properties 2.1 - 2.3 and, therefore, satisfies the conditions of Theorem 1 from [29]. Applying this result together with Theorem 4.1, one can prove the following theorem.

Theorem 4.2 [Implicit Optimality Criterion] *Suppose that Assumptions* 1 *and* 2 *are satisfied for the convex problem* (SIP). *Then the feasible solution* $x^0 \in X$ *is optimal iff there exist a set of indices* $\{t_j, j \in J_a\} \subset T_a(x^0) \setminus T^*$ *and a set of vectors* $l_k(j),\ k = 1, \ldots, m(j), j \in J_*$, *defined in* (10), *such that*

$$|J_a| + \sum_{j \in J_*} m(j) \leq n, \tag{34}$$

and the vector x^0 *is an optimal solution of the convex* NLP *problem* (11).

Note that the optimality conditions given by this theorem, are both necessary and sufficient, and the Assumptions 1 and 2 are not too restrictive.

According to Theorem 4.2, given feasible x^0, instead of testing its optimality in the **infinite dimension SIP problem** (SIP), one can test the optimality of x^0 in a **finite dimension** NLP problem (NLP_*). The transition to a simpler and more studied problem allows us to obtain new explicit optimality conditions for convex SIP. In fact, having applied Theorem 4.2 and any optimality conditions for the convex problem (NLP_*) (either some conditions already known from the theory of NLP, or new ones, that are specially formulated for the case), one gets new optimality conditions for SIP. Some of such conditions are presented in the next section.

4.2 Explicit Optimality Conditions

In the previous section, we have proven the implicit optimality criteria for the problem (SIP). Now we will formulate and prove new *explicit* sufficient and necessary optimality conditions for this problem. These conditions differ from the known ones and are formulated under assumptions that are less restrictive than the usually used CQs.

Denote $S_a^0(t_j^*) := \{s \in S_a(t_j^*) : \exists i_0 \in I_0(j)$ such that $\frac{\partial g_s^T(t_j^*)}{\partial t} a_{i_0}(j) \neq 0\}$, $S_a^*(t_j^*) := S_a(t_j^*) \setminus S_a^0(t_j^*)$. For $j \in J_*$, consider LP problem

$$(\mathrm{LP_j}(x)) : \qquad \max_w \frac{\partial f^T(x,t_j^*)}{\partial t} w, \quad \text{s.t.} \quad \frac{\partial g_s^T(t_j^*)}{\partial t} w \leq 0 \;\; s \in S_a^*(t_j^*).$$

The following lemma states some important properties of this problem.

Lemma 4.1 *Given $x \in \bar{Q}$ and $j \in J_*$, any feasible solution of the problem* $(\mathrm{LP_j}(x))$ *admits a representation*

$$\mu = \sum_{i \in P(j)} b_i(j)\beta_i + \sum_{i \in I_0(j) \cup I_*(j)} a_i(j)\alpha_i, \; \alpha_i \geq 0, \; i \in I_*(j). \tag{35}$$

Moreover, this problem has an optimal solution and $val(LP_j(x)) = 0$.

Proof. For $x \in \bar{Q}$ and $j \in J_*$, consider the problem $(\mathrm{LP_j}(x))$. Let μ be its feasible solution. It follows from the definition of the sets $S_a^*(t_j^*)$ and $S_a^0(t_j^*)$, that there exist numbers $\tilde{\alpha}_i \geq 0, i \in I_0(j)$, such that the vector

$$\bar{\mu} := \mu + \sum_{i \in I_0(j)} a_i(j)\tilde{\alpha}_i \tag{36}$$

satisfies the relations $\frac{\partial g_s^T(t_j^*)}{\partial t}\bar{\mu} \leq 0, \; s \in S_a(t_j^*), \;\; \frac{\partial f^T(x,t_j^*)}{\partial t}\bar{\mu} = \frac{\partial f^T(x,t_j^*)}{\partial t}\mu.$

Then $\bar{\mu} \in L(j)$ and the following representation is possible:

$$\bar{\mu} = \sum_{i \in P(j)} b_i(j)\bar{\beta}_i + \sum_{i \in I_0(j) \cup I_*(j)} a_i(j)\bar{\alpha}_i, \; \bar{\alpha}_i \geq 0, \; i \in I_0(j) \cup I_*(j). \tag{37}$$

From the inclusion $x \in \bar{Q}$, it follows $\frac{\partial f^T(x,t_j^*)}{\partial t}\bar{\mu} \leq 0$ and, hence, $\frac{\partial f^T(x,t_j^*)}{\partial t}\mu \leq 0$ for each feasible solution μ of the problem $(\mathrm{LP_j}(x))$. Then, evidently, vector $\mu = 0$ is an optimal solution and $val(LP_j(x)) = 0$.

Moreover, from equalities (36) and (37), it follows: $\mu = \bar{\mu} - \sum\limits_{i \in I_0(j)} a_i(j)\tilde{\alpha}_i$ $= \sum\limits_{i \in P(j)} b_i(j)\bar{\beta}_i + \sum\limits_{i \in I_0(j)} a_i(j)(\bar{\alpha}_i - \tilde{\alpha}_i) + \sum\limits_{i \in I_*(j)} a_i(j)\bar{\alpha}_i, \;\; \bar{\alpha}_i \geq 0, \;\; i \in I_*(j).$ This proves that every feasible solution μ of the problem $(\mathrm{LP_j}(x))$ admits representation (35). □

Let $sol(P)$ denote the set of the optimal solutions of a given optimization problem (P), and suppose that $\sum_{k=1}^m \cdots = 0$ if $m = 0$.

Theorem 4.3 [Explicit sufficient optimality conditions] *Let Assumptions* 1, 2 *hold true and $x^0 \in X$ be a feasible solution of the convex problem* (SIP). *Suppose that there exist active indices $t_j, j \in J_a$, vectors $l_k(j), \; k = 1, \dots, m(j)$, $j \in J_*$, defined in* (10), (34) *as well as vectors*

$$\mu_k(j) \in sol(LP(x^0, t_j^*, l_k(j))), k = 1, \dots, m(j); \;\; \mu_j \in sol(LP_j(x^0)), j \in J_*, \tag{38}$$

and numbers $\lambda_j, j \in J_*,\ \nu_j \geq 0, j \in J_a$, *such that*

$$\frac{\partial c(x^0)}{\partial x} + \sum_{j \in J_a} \nu_j \frac{\partial f(x^0, t_j)}{\partial x} + \sum_{j \in J_*} \Big[\lambda_j \frac{\partial f(x^0, t_j^*)}{\partial x} + \frac{\partial^2 f(x^0, t_j^*)}{\partial x \partial t} \mu_j \tag{39}$$

$$+ \sum_{k=1}^{m(j)} \Big(\frac{\partial^2 f(x^0, t_j^*)}{\partial x \partial t} \mu_k(j) + \frac{\partial}{\partial x}[(l_k(j))^T \frac{\partial^2 f(x^0, t_j^*)}{\partial t^2} l_k(j)] \Big) \Big] = 0.$$

Then x^0 *is an optimal solution of the problem* (SIP).

Note that here and in what follows, it may happen that $m(j_0) = 0$ for some $j_0 \in J_*$. This means that the set $\{l_k(j_0), k = 1, ..., m(j_0)\}$ is empty.

Proof. For $x^0 \in \bar{Q}$ and $j \in J_*$, let us consider the problem $(\mathrm{LP_j}(x^0))$. It follows from Lemma 4.1, that $0 = \frac{\partial f^T(x^0, t_j^*)}{\partial t} \mu = \frac{\partial f^T(x^0, t_j^*)}{\partial t} \sum_{i \in I_*(j)} a_i(j) \alpha_i$ for $\mu = \mu_j \in sol(LP_j(x^0))$. Taking into account the last equality and the inequalities $\frac{\partial f^T(x^0, t_j^*)}{\partial t} a_i(j) \leq 0,\ \alpha_i \geq 0,\ i \in I_*(j)$, we obtain

$$\alpha_i \geq 0, \text{ if } \frac{\partial f^T(x^0, t_j^*)}{\partial t} a_i(j) = 0; \quad \alpha_i = 0, \text{ if } \frac{\partial f^T(x^0, t_j^*)}{\partial t} a_i(j) < 0,\ i \in I_*(j). \tag{40}$$

Let $\bar{x}$ be a feasible solution in (11). Since problem (11) is convex, then for all $\lambda \in [0, 1]$, the vector $x(\lambda) := x^0(1 - \lambda) + \bar{x}\lambda = x^0 + \lambda \Delta x$ with $\Delta x := \bar{x} - x^0$ is its feasible solution as well. Hence, for Δx it holds:

$$\Delta x^T \frac{\partial f(x^0, t_j^*)}{\partial x} = 0,\ j \in J_*;\ \Delta x^T \frac{\partial f(x^0, t_j)}{\partial x} \leq 0,\ j \in J_a; \tag{41}$$

$$\Delta x^T \frac{\partial^2 f(x^0, t_j^*)}{\partial x \partial t} b_i(j) = 0, i \in P(j),\ j \in J_*; \tag{42}$$

$$\Delta x^T \frac{\partial^2 f(x^0, t_j^*)}{\partial x \partial t} a_i(j) \begin{cases} = 0, \text{ if } i \in I_0(j), \\ \leq 0, \text{ if } i \in I_*(j) \text{ and } \frac{\partial f^T(x^0, t_j^*)}{\partial t} a_i(j) = 0, \end{cases} j \in J_*, \tag{43}$$

$$\Delta x^T \frac{\partial}{\partial x} \Big(l_k^T(j) \frac{\partial^2 f(x^0, t_j^*)}{\partial t^2} l_k(j) \Big) + \max_{\mu \in sol(LP(x^0, t_j^*, l_k(j)))} \Delta x^T \frac{\partial^2 f(x^0, t_j^*)}{\partial x \partial t} \mu \leq 0, \qquad k = 1, \ldots, m(j),\ j \in J_*. \tag{44}$$

Since $\mu_k(j) \in sol(LP(x^0, t_j^*, l_k(j)))$, the inequalities (44) imply

$$\Delta x^T \frac{\partial}{\partial x} \Big(l_k^T(j) \frac{\partial^2 f(x^0, t_j^*)}{\partial t^2} l_k(j) \Big) + \Delta x^T \frac{\partial^2 f(x^0, t_j^*)}{\partial x \partial t} \mu_k(j) \leq 0, \quad k = 1, \ldots, m(j),\ j \in J_*. \tag{45}$$

Let Δx be a vector satisfying conditions (42), (43) and $\mu_j \in sol(LP_j(x^0))$. Taking into account (35), (40), and (43), we obtain

$$\Delta x^T \frac{\partial^2 f(x^0, t^*_j)}{\partial x \partial t}\mu_j = \sum_{i \in I_*(j)} \Delta x^T \frac{\partial^2 f(x^0, t^*_j)}{\partial x \partial t} a_i(j)\alpha_i \leq 0. \tag{46}$$

By assumption, equality (39) holds true. Let us multiply both sides of this equality by Δx^T and take into account (41)-(46). As a result, we get

$$\Delta x^T \frac{\partial c(x^0)}{\partial x} = -\sum_{j \in J_a} \nu_j \Delta x^T \frac{\partial f(x^0, t_j)}{\partial x} - \sum_{j \in J_*} \lambda_j \Delta x^T \frac{\partial f(x^0, t^*_j)}{\partial x} - \sum_{j \in J_*} \Big[\Delta x^T \frac{\partial^2 f(x^0, t^*_j)}{\partial x \partial t}\mu_j$$
$$+ \sum_{k=1}^{m(j)} \Delta x^T \Big(\frac{\partial^2 f(x^0, t^*_j)}{\partial x \partial t}\mu_k(j) + \frac{\partial}{\partial x}[l^T_k(j) \frac{\partial^2 f(x^0, t^*_j)}{\partial t^2} l_k(j)] \Big) \Big] \geq 0.$$

Thus, we have shown that for every feasible solution $\bar{x}$ of problem (11) the inequality $\frac{\partial c^T(x^0)}{\partial x}(\bar{x} - x^0) \geq 0$ holds true. Note that since the function $c(x)$ is convex, then $\frac{\partial c^T(x^0)}{\partial x}(\bar{x} - x^0) \leq c(\bar{x}) - c(x^0)$. The last two inequalities imply the inequality $c(x^0) \leq c(\bar{x})$, which has to be satisfied by all feasible solutions $\bar{x}$ of problem (11). This means that the vector $x^0 \in X$ is an optimal solution of this problem. Taking into account that the set of feasible solutions X of the original SIP problem is a subset of the set of feasible solutions of problem (11), we conclude that the vector x^0 solves the original SIP problem as well. The theorem is proved. □

Following [30], let us introduce the following definition.

Definition 4.1 The Constant Rank Constraint Qualification (CRCQ) is said to be held at $\bar{x} \in X$ in the NLP problem (11) iff there exists a neighborhood $\Omega(\bar{x}) \subset \mathbb{R}^n$ of $\bar{x}$ such that the system of vectors

$$\Big\{ \frac{\partial f(x, t^*_j)}{\partial x}, \frac{\partial^2 f(x, t^*_j)}{\partial x \partial t} b_i(j), i \in P(j), \frac{\partial^2 f(x, t^*_j)}{\partial x \partial t} a_i(j), i \in I_0(j), j \in J_* \Big\}, \tag{47}$$

has a constant rank for every $x \in \Omega(\bar{x})$.

Theorem 4.4 [Explicit optimality criterion] *Let Assumptions* 1 *and* 2 *hold true for the convex problem* (SIP). *Suppose that* (CRCQ) *is satisfied at* $x^0 \in X$. *Then the vector* x^0 *is an optimal solution of problem* (SIP) *iff there exist indices* $t_j, j \in J_a$, *and vectors* $l_k(j), k = 1, \dots, m(j), j \in J_*$, *defined in* (10) *and* (34), *as well as vectors* $\mu_k(j)$, $k = 1, \dots, m(j)$, *and* $\mu_j, j \in J_*$, *defined in* (38), *and numbers* $\lambda_j, j \in J_*$, $\nu_j \geq 0, j \in J_a$, *such that equality* (39) *takes place.*

Proof. $\Rightarrow$ It follows from Theorem 4.2, that there exist indices $t_j, j \in J_a$, and vectors $\bar{l}_k(j), k = 1, \dots, m(j), j \in J_*$, defined in (10) such that the vector x^0

is optimal in problem (11). Rewrite the last problem in the form

$$
\begin{aligned}
&\min\ c(x),\\
\text{s.t.}\quad & f(x,t_j^*)=0,\ \frac{\partial f^T(x,t_j^*)}{\partial t}b_i(j)=0, i\in P(j),\\
&\frac{\partial f^T(x,t_j^*)}{\partial t}a_i(j)=0, i\in I_0(j), \frac{\partial f^T(x,t_j^*)}{\partial t}a_i(j)\le 0, i\in I_*(j), \qquad (48)\\
&(\bar{l}_k(j))^T\frac{\partial^2 f(x,t_j^*)}{\partial t^2}\bar{l}_k(j)+val(LP(x,t_j^*,\bar{l}_k(j))\le 0, k=1,...,m(j),\ j\in J_*,\\
&f(x,t_j)\le 0, j\in J_a.
\end{aligned}
$$

It follows from the assumptions of the theorem, that problem (48) possesses the Properties 2.1 - 2.3 (see section 2) and the following one: there exists a neighborhood $\Omega(x^0)\subset\mathbb{R}^n$ of x^0 such that the system of vectors (47) has a constant rank for every $x\in\Omega(x^0)$. According to [31], under fulfillment of these properties, a feasible vector x^0 is an optimal solution in problem (48) if and only if there exist numbers and vectors

$$
\begin{aligned}
&\nu_j\ge 0, j\in J_a;\ \lambda_j,\ \omega_i(j), i\in P(j),\ \gamma_i(j), i\in I_0(j),\ \gamma_i(j)\ge 0, i\in I_*(j);\\
&\lambda_{kj}\ge 0,\ \bar{\mu}_{kj}\in sol(LP(x^0,t_j^*,\bar{l}_k(j))), k=1,...,m(j),\ j\in J_*,
\end{aligned}
$$

such that $\gamma_i(j)\frac{\partial f^T(x^0,t_j^*)}{\partial t}a_i(j)=0, i\in I_*(j), j\in J_*$, and the equality

$$
\begin{aligned}
&\frac{\partial c(x^0)}{\partial x}+\sum_{j\in J_a}\nu_j\frac{\partial f(x^0,t_j^*)}{\partial x}+\sum_{j\in J_*}\Big[\lambda_j\frac{\partial f(x^0,t_j^*)}{\partial x}+\frac{\partial^2 f(x^0,t_j^*)}{\partial x\partial t}\mu_j \qquad (49)\\
&+\sum_{k=1}^{m(j)}\lambda_{kj}\Big(\frac{\partial^2 f(x^0,t_j^*)}{\partial x\partial t}\bar{\mu}_{kj}+\frac{\partial}{\partial x}[(\bar{l}_k(j))^T\frac{\partial^2 f(x^0,t_j^*)}{\partial t^2}\bar{l}_k(j)]\Big)\Big]=0
\end{aligned}
$$

holds true with

$$
\mu_j:=\Big(\sum_{i\in P(j)}b_i(j)\omega_i(j)+\sum_{i\in I_0(j)\cup I_*(j)}a_i(j)\gamma_i(j)\Big), j\in J_*. \qquad (50)
$$

It follows from Lemma 4.1, that for $j\in J_*$, the vector μ_j is feasible in problem $(LP_j(x^0))$ and $\frac{\partial f^T(x^0,t_j^*)}{\partial t}\mu_j=0$. Hence, $\mu_j\in sol(LP_j(x^0))$.

Basing on Theorems 4.2 and equality (49), without loss of generality, we can suppose that $\lambda_{kj}>0, k=1,...,m(j),\ j\in J_*$, since in the case when $\lambda_{kj}=0$, we may exclude from consideration the vector $\bar{l}_k(j)$ and the corresponding constraint of problem (48). Denote

$$
l_k(j):=\sqrt{\lambda_{kj}}\ \bar{l}_k(j),\ \ \mu_k(j):=\lambda_{kj}\bar{\mu}_k(j),\ k=1,...,m(j), j\in J_*. \qquad (51)
$$

Evidently, $l_k(j)\in\{l\in C_0(t):\ F_{2j}(x^0,l)=0\}$, $\mu_k(j)\in sol(LP(x^0,t_j^*,l_k(j)))$, $k=1,...,m(j), j\in J_*$. Hence, equality (49) implies equality (39) with vectors $\mu_j\in sol(LP_j(x^0))$ defined in (50) and $l_k(j)$, $\mu_k(j), k=1,...,m(j)$, defined in (51) for $j\in J_*$. The necessary part of the theorem is proved.

$\Leftarrow$ The sufficient part of the proof follows from Theorem 4.3. □

It is worth mentioning that the optimality conditions proved above are of the first order w.r.t. x. For the convex SIP problems, Theorems 4.3 and 4.4, provide more efficient optimality conditions when compared with the ones, which can be found in the literature. Indeed, the necessary optimality conditions from [11] (Theorems 5.113, 5.118) and [1] (Theorem 5.1) are trivially fulfilled for any $x \in X$, if the constraints of the problem (SIP) do not satisfy the Slater CQ. Hence, these conditions are useless in such situation. But this does not happen under the conditions of Theorem 4.4. In fact, suppose that for the problem (SIP), the Slater CQ fails. Then the set of the immobile indices $T^* = \{t^*_j, j \in J_*\}$ is nonempty. Let x^* be any feasible solution of the problem (SIP) and $T_a(x^*)$ be the corresponding active index set. By construction, $T^* \subset T_a(x^*)$. Note that, since the indices $t^*_j, j \in J_*$ are immobile, it is easy to show that for any $x^* \in X$, there exist numbers $\lambda^*_j = \lambda^*_j(x^*) \geq 0$, $j \in J_*$, such that $\sum\limits_{j \in J_*} \lambda^*_j \frac{\partial f(x^*, t^*_j)}{\partial x} = 0$, $\sum\limits_{j \in J_*} \lambda^*_j > 0$. Consider the multiplies $\lambda_0 = 0,\ \lambda_j = \lambda^*_j,\ j \in J_*;\ \ \lambda(t) = 0$ for $t \in T_a(x^*) \setminus T^*$. For definiteness, let us consider the necessary optimality conditions from Theorems 5.113 and 5.118 in [11]. It is easy to verify that the chosen above multiplies satisfy condition (5.284) in [11]. Note that for the convex SIP problems, condition (5.316) in [11] is always satisfied since $h^T \frac{\partial^2 f(x^*, t)}{\partial x^2} h \geq 0$ and $\vartheta(t, h) \geq 0$ for $h \in C(x^*)$, $t \in T_a(x^*)$ (see (5.302) in [11]). Hence, we have shown that the necessary optimality conditions from Theorems 5.113 and 5.118 in [11] are fulfilled for any feasible $x^* \in X$ of the problem (SIP).

If consider Theorem 4.4 proven above, it should be noted that it provides the optimality criterion under the assumption that (CRCQ) is satisfied. Therefore, in this case **only the optimal** solutions of the problem (SIP) satisfy the conditions of the theorem. In Sections 5 and 6, we will present some situations, where the necessary conditions of Theorem 4.4 are not trivially satisfied even when the Slater CQ fails. The example from [27] along with one another, which will be discussed in Section 6, shows that, given a convex SIP problem, the set of the feasible solutions satisfying the sufficient conditions proved in Theorem 4.3 can be wider, when compared to the set of the feasible solutions satisfying the sufficient conditions from [11,12]. Therefore, we can conclude that the first order optimality conditions presented in this paper, are stronger than the known first order optimality conditions.

5 Special Cases

We will consider here some special cases of SIP problems, for which the optimality conditions from the previous sections can be reformulated in a simpler form.

Case 1. Problem (SIP) satisfies the Slater condition

Suppose that the problem (SIP) satisfies the Slater condition. Then $T^* = \emptyset$ and, hence, the (CRCQ) is trivially fulfilled. Then Theorem 4.4 takes the form of one well known result from [1].

Theorem 5.1 *Let the convex problem* (SIP) *satisfy* (SCQ). *A feasible point* $x^0 \in X$ *is an optimal solution of* (SIP) *iff there exist a set of indices* $\{t_j,\ j \in J_a\} \subset T_a(x^0),\ |J_a| \leq n$, *and numbers* $\nu_0 = 1, \nu_j \geq 0, j \in J_a$, *such that*

$$\nu_0 \frac{\partial c(x^0)}{\partial x} + \sum_{j \in J_a} \nu_j \frac{\partial f(x^0, t_j)}{\partial x} = 0. \tag{52}$$

The following observations should be made here:

- The statement of Theorem 5.1 continues to be true in its sufficient part even when the problem (SIP) does not satisfy (SCQ). But such a sufficient optimality condition for convex SIP is too restrictive.
- Without (SCQ), the first order necessary optimality conditions from [1, 11] are as follows: *Let* $x^0 \in X$ *be an optimal solution of* (SIP). *Then there exist active indices* $\{t_j,\ j \in J_a\} \subset T_a(x^0),\quad |J_a| \leq n$, *and numbers* $\nu_0 \geq 0, \nu_j \geq 0, j \in J_a$, *such that equality* (52) *takes place.* It is easy to show that if $T^* \neq \emptyset$, then these conditions are fulfilled for all $x \in X$.

From the observations above, we can conclude that the optimality conditions formulated in the Theorems 4.3 and 4.4 coincide with the classical first order optimality conditions for the problems (SIP) satisfying (SCQ), and they are more efficient than the classical conditions, if (SCQ) is not satisfied.

Case 2. The lower level problem satisfies some additional conditions

It was shown above that, given an optimal solution x^0 of the convex problem (SIP), its immobile indices solve the corresponding lower level problem (LLP(x^0)), i.e. $t^*_j \in sol(LLP(x^0)),\ j \in J_*$. Consider the following condition:

$$F_{2j}(x^0, l) < 0\ \forall l \in C_0(j) \setminus \{0\},\ j \in J_*. \tag{53}$$

Note that condition (53) is weaker then the classical second order sufficient optimality conditions (SOSOC) for $t^*_j,\ j \in J_*$, in the problem (LLP(x^0)):

$$\text{(SOSOC)}: \qquad F_{2j}(x^0, l) < 0\ \forall l \in C(x^0, t^*_j) \setminus \{0\},\ j \in J_*.$$

Condition (53) implies that $m(j) = 0\ \forall j \in J_*$ in (10), and Theorem 4.3 takes the form

Theorem 5.2 *Let Assumptions* 1, 2 *hold true,* $x^0 \in X$, *and condition* (53) *be satisfied. Suppose that there exist a subset of the set of active indices* $\{t_j,\ j \in J_a\} \subset T_a(x^0) \backslash T^*,\ |J_a| \leq n$, *vectors* $\mu_j \in sol(LP_j(x^0, 0))$ *and numbers* $\lambda_j, j \in J_*,\ \nu_j \geq 0, j \in J_a$, *such that*

$$\frac{\partial c(x^0)}{\partial x} + \sum_{j \in J_a} \nu_j \frac{\partial f(x^0, t_j)}{\partial x} + \sum_{j \in J_*} \Big[\lambda_j \frac{\partial f(x^0, t^*_j)}{\partial x} + \frac{\partial^2 f(x^0, t^*_j)}{\partial x \partial t} \mu_j \Big] = 0.$$

Then x^0 *is an optimal solution of problem* (SIP).

Case 3. The index set $T \subset R^p$ is a polyhedron

Suppose that the functions $g_s(t), s \in S$, in (2) are linear: $g_s(t) = h_s^T t + \Delta h_s$, $s \in S$. In this case, the inclusions $sol(LP(x^0, t_j^*, l_j^{(k)})) \subset sol(LP_j(x^0))$, $k = 1, \dots, m(j), j \in J_*$, take place and Assumption 1 is not mandatory. Hence, Theorem 4.3 takes the form

Theorem 5.3 *Let Assumption* 2 *hold true for the convex problem* (SIP) *with polyhedral index set* T*, and* $x^0 \in X$*. Suppose that there exist a subset of the set of active indices* $\{t_j,\ j \in J_a\} \subset T_a(x^0)\backslash T^*$*, a set of vectors* $l_k(j),\ k = 1, \dots, m(j), j \in J_*$*, defined in* (10) *and* (34)*, vectors* $\mu_j \in sol(LP_j(x^0)), j \in J_*$*, and numbers* $\lambda_j, j \in J_*,\ \nu_j \geq 0, j \in J_a$*, such that*

$$\frac{\partial c(x^0)}{\partial x} + \sum_{j \in J_a} \nu_j \frac{\partial f(x^0, t_j)}{\partial x} +$$
$$\sum_{j \in J_*} \Big[\lambda_j \frac{\partial f(x^0, t_j^*)}{\partial x} + \frac{\partial^2 f(x^0, t_j^*)}{\partial x \partial t}\mu_j + \sum_{k=1}^{m(j)} \frac{\partial}{\partial x}[l_k^T(j) \frac{\partial^2 f(x^0, t_j^*)}{\partial t^2} l_k(j)]\Big] = 0.$$

Then x^0 *is an optimal solution of problem* (SIP).

More detailed considerations can be found in [25], where SIP problems with polyhedral index set T and linear w.r.t. x constraint function $f(x,t)$ are considered.

Case 4. The constraint function $f(x,t)$ is linear w.r.t. $x \in \mathbb{R}^n$

Suppose that in the problem (SIP), the constraint function $f(x,t)$ is linear w.r.t. $x \in \mathbb{R}^n$. Then (CRCQ) is fulfilled and Theorem 4.4 gives us a new optimality **criterion** for a feasible $x^0 \in X$ in problem (SIP).

6 Example

In [27], the efficiency of the implicit optimality conditions formulated in Theorem 2.1, was illustrated with the help of an example in which the lower level problem satisfies the additional conditions (SOSOC) (see subsection 5.2). Now we will slightly modify this example to illustrate the efficiency of the explicit optimality conditions proposed in Theorem 4.3 also in the case, when the conditions (SOSOC) are not satisfied.

Let $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4,\ t = (\tau_1, \tau_2)^T \in \mathbb{R}^2$, and

$$f_1(x,t) = -\tau_1^2 x_1 + \tau_1\tau_2 x_1 + \tau_1 x_2 + \sin(\tau_1)x_3 + \tau_1 x_4 - \tau_2^2,$$
$$f_2(x,t) = \tau_2 x_1 + (\tau_2+1)^2 x_2 + (1-\tau_2)x_3 + x_4 - (\tau_1 - 3)^2 + (\tau_1 - 3)\tau_2;$$
$$T_1 = \{t \in \mathbb{R}^2 : -(\tau_1+1)^2 - (\tau_2 - 1)^2 \leq -2, -0.5 \leq \tau_1 \leq 1, -0.5 \leq \tau_2 \leq 0.5\},$$
$$T_2 = \{t \in \mathbb{R}^2 : (\tau_1 - 2.5)^2 + (\tau_2 - 0.5)^2 \leq 0.5\}.$$

Note here that the set T_2 is convex but not polyhedral, and the set T_1 is not convex. Consider the following convex SIP problem:

$$\min\ x_1^2,\ \text{ s.t. } f_1(x,t) \leq 0\ \forall t \in T_1,\ \ f_2(x,t) \leq 0\ \forall t \in T_2. \qquad (54)$$

Problem (54) admits a feasible solution $x^0 = (x^0_1, x^0_2, x^0_3, x^0_4)^T$ such that $x^0_1 = -2a - 2\sqrt{a^2 - b} \approx 0.0695$, $x^0_2 = -0.25$, $x^0_3 = x^0_1 + 2x^0_2 \approx -0.4305$, $x^0_4 = -x^0_1 - 3x^0_2 \approx 0.6805$, where $a = -2 + \sin(1)$, $b = -0.5(\sin(1) - 1)$.

Let us, first, test the optimality of x^0 in problem (54) using the approach suggested in the paper. Denote $t_1 := (0, 0)^T \in T_1$, $t_2 := (3, 0)^T \in T_2$, and $t_3 := (1, x^0_1/2)^T \in T_1$. It can be checked that the indices t_1, t_2, and t_3 form the active index set in x^0: $f_1(x^0, t_1) = f_2(x^0, t_2) = f_1(x^0, t_3) = 0$, and two of these indices, $t^*_1 = t_1$ and $t^*_2 = t_2$, are immobile (hence $J_* = \{1, 2\}$). By construction, the immobile index t^*_1 is situated in the locally non-convex part of the index set T_1. Note here that (MFCQ) is fulfilled at both immobile indices, t^*_1 and t^*_2, and there exists a feasible $\tilde{x} \in \mathbb{R}^4$:

$$\tilde{x}_1 = 2\sin(1), \tilde{x}_3 = \frac{-(\tilde{x}_1)^2/4 + \tilde{x}_1}{\sin(1) - 1}, \tilde{x}_2 = 0.5(\tilde{x}_3 - \tilde{x}_1), \tilde{x}_4 = -\tilde{x}_2 - \tilde{x}_3, \quad (55)$$

such that the following inequalities hold: $l^T \frac{\partial^2 f_1(\tilde{x}, t^*_1)}{\partial t^2} l < 0$, $l^T \frac{\partial^2 f_2(\tilde{x}, t^*_2)}{\partial t^2} l < 0$, $\forall l \in \mathbb{R}^2 \setminus \{0\}$. Hence Assumptions 1 and 2 are satisfied for problem (54).

For the index t^*_1, we have $S_a(t^*_1) = \{1\}$. The cone $L(t^*_1) = \{l \in \mathbb{R}^2 : -l_1 + l_2 \leq 0\}$ can be represented by one bidirectional ray $b_1(1) = (1, 1)^T$ and one unidirectional ray $a_1(1) = (1, -1)^T$ with $q(t^*_1, b_1(1)) = 1$ and $q(t^*_1, a_1(1)) = 1$. Then the sets in (8) are as follows: $I_*(1) = \emptyset$, $I_0(1) = \{1\}$.

For the index $t^*_2 = (3, 0)^T \in T^*$, we have $S_a(t^*_2) = \{1\}$, and the cone $L(t^*_2) = \{l \in \mathbb{R}^2 : l_1 - l_2 \leq 0\}$ is represented by $b_1(2) = (1, 1)^T$ and $a_1(2) = (-1, 1)^T$ with $q(t^*_2, b_1(2)) = 1$; $q(t^*_2, a_1(2)) = 1$. Hence, the sets in (8) are given by $I_*(2) = \emptyset$, $I_0(2) = \{1\}$.

One can show that

$$l^T \frac{\partial^2 f_1(x^0, t^*_1)}{\partial t^2} l < 0, \quad l^T \frac{\partial^2 f_2(x^0, t^*_2)}{\partial t^2} l \leq 0 \ \forall l \in \mathbb{R}^2 \setminus \{0\}, \quad (56)$$

and there exists unique (up to a positive multiplier) vector $\bar{l} = (0.5, 1)^T, \bar{l} \in L_1(t^*_2)$ such that $\|\bar{l}\| \neq 0$ and $\bar{l}^T \frac{\partial^2 f_2(x^0, t^*_2)}{\partial t^2} \bar{l} = 0$. Hence, according to the optimality criterion formulated in Theorem 4.2, vector x^0 is optimal in problem (54) iff it is optimal in the following Quadratic Programming (QP) problem:

$$\min \ x_1^2,$$

$$\text{s.t.} \qquad f_i(x, t^*_i) = 0, \frac{\partial f_i^T(x, t^*_i)}{\partial t} b_1(i) = 0, \frac{\partial f_i^T(x, t^*_i)}{\partial t} a_1(i) = 0, \quad i = 1, 2;$$

$$\bar{l}^T \frac{\partial^2 f_2(x, t^*_2)}{\partial t^2} \bar{l} + val(LP(x, t^*_2, \bar{l})) \leq 0, \quad f_1(x, t_3) \leq 0,$$

where

$$(\text{LP}(x, t, l)) : \max_{(\omega_1, \omega_2)} (x_1 + 2x_2 - x_3)\omega_2, \ \text{s.t.} \ \frac{\partial g_2(t)}{\partial \tau_1}\omega_1 + \frac{\partial g_2(t)}{\partial \tau_2}\omega_2 \leq -l^T \frac{\partial^2 g_2(t)}{\partial \tau^2} l.$$

Taking into account that $\frac{\partial f_2^T(x,t_2^*)}{\partial t}b_1(2) = \frac{\partial f_2^T(x,t_2^*)}{\partial t}a_1(2) = x_1 + 2x_2 - x_3$, the QP problem can be rewritten in the form

$$\begin{gathered} \min\ x_1^2, \\ \text{s.t. } x_2 + x_3 + x_4 = 0,\ x_1 + 2x_2 - x_3 = 0,\ 2x_2 + 0.5 \le 0, \\ x_1(0.5x_1^0 - 1) + x_2 + \sin(1)x_3 + x_4 - 0.25(x_1^0)^2 \le 0. \end{gathered} \tag{57}$$

Applying the known optimality criterion for convex QP, it is easy to check that vector x^0 is optimal in problem (57) and, therefore, (see Theorem 4.2) it is optimal in the SIP problem (54). One can show that the statements of Theorems 4.3 and 4.4n are fulfilled as well.

It was shown above that the necessary optimality conditions from [11] (Theorems 5.113, 5.118) and [1] (Theorem 5.1) are trivially fulfilled for any $x \in X$ if the constraints of the convex SIP problem (SIP) do not satisfy the Slater CQ. In our example, the constraints of the SIP problem (54) do not satisfy the Slater condition. Therefore, the necessary conditions from [1,11] are not informative for problem (54). A similar situation can not happen in the case of the necessary optimality conditions formulated in Theorem 4.4, since these conditions are satisfied not for all feasible, but **only for the optimal solutions**. For example, one can check that the vector $\tilde{x}$ (defined in (55) and feasible in problem (54)) does not satisfy the necessary optimality conditions from Theorem 4.4.

Now, let us show that the second order sufficient optimality conditions from [11,12] are not fulfilled for x^0. For definiteness, we will consider the conditions from [12]. For our example, relations (5.6) from [12] are as follows:

$$\bar{\lambda}_0\frac{\partial c(x^0)}{\partial x}+\bar{\lambda}_1\frac{\partial f_1(x^0,t_1)}{\partial x}+\bar{\lambda}_2\frac{\partial f_2(x^0,t_2)}{\partial x}+\bar{\lambda}_3\frac{\partial f_1(x^0,t_3)}{\partial x} = 0, \bar{\lambda}_i \ge 0, i = 0, ..., 3.$$

Since the system above admits a solution $\bar{\lambda}_0 = 0,\ \bar{\lambda}_1 \ge 0,\ \bar{\lambda}_2 = 0,\ \bar{\lambda}_3 = 0$, relations (5.7) from [12] take the form

$$-\bar{\lambda}_1\Big[(\eta(\xi))^T\frac{\partial^2 f_1(x^0,t_1)}{\partial t^2}\eta(\xi) + 2\xi^T\frac{\partial^2 f_1(x^0,t_1)}{\partial x \partial t}\eta(\xi)\Big] < 0 \quad \forall \xi \in \mathcal{K}, \xi \ne 0, \tag{58}$$

where (see [12]) $\eta(\xi)$ is a solution to the following auxiliary problem:

$$(\mathrm{Q}_{\mathrm{t}_1}(\xi)): \quad \max \frac{1}{2}\eta^T\frac{\partial^2 f_1(x^0,t_1)}{\partial t^2}\eta + \xi^T\frac{\partial^2 f_1(x^0,t_1)}{\partial x \partial t}\eta,\ \text{s.t. } (-1,1)\eta \le 0,$$

and $\mathcal{K} = \{\xi \in \mathbb{R}^4 : \xi^T\frac{\partial c(x^0)}{\partial x} \le 0, \xi^T\frac{\partial f_1(x^0,t_1)}{\partial x} \le 0, \xi^T\frac{\partial f_2(x^0,t_2)}{\partial x} \le 0,$ $\xi^T\frac{\partial f_1(x^0,\ t_3)}{\partial x} \le 0\}$. It is easy to check that $\bar{\xi} = (\frac{1-\sin(1)}{0.5x_1^0-1}, -\frac{1}{2}, 1, -\frac{1}{2})^T \in \mathcal{K}$, and $\bar{\xi}^T\frac{\partial^2 f_1(x^0,t_1)}{\partial x \partial t} = (0,0)$. Then, taking into account relations (56), we conclude that the problem $(\mathrm{Q}_{\mathrm{t}_1}(\bar{\xi}))$ admits an optimal solution $\eta(\bar{\xi}) = 0$. Consequently, conditions (58) (as well as conditions (5.7) from [12]) are not fulfilled for the feasible x^0 in problem (54). In other words, the optimality conditions from [12] are not able to recognize the optimality of x^0 in the convex problem (54).

Remind once again that the given vector x^0 satisfies the explicit sufficient optimality conditions formulated in Theorem 4.3.

It was shown above, how the additional information about the properties of the immobile indices permits to obtain the optimality conditions, which are more efficient than the known ones. This additional information can be useful for numerical methods as well. Let us illustrate this with an example.

One of the methods for solving SIP problems (*discretization approach*) consists in overlaying a rather dense grid on the index set and constructing a corresponding discretized problem ($\mathrm{NLP_D}$). A solution of the discretised problem is considered as an approximate solution of the original SIP problem.
We will apply this method to problem (54). Let $\mu_s > 0, \nu_s > 0$, be the discretezation steps in the corresponding directions for the index sets T_s, $s = 1, 2$. Denote: $a_s = \min\limits_{t\in T_s} \tau_1$, $\bar{a}_s = \max\limits_{t\in T_s} \tau_1$, $b_s = \min\limits_{t\in T_s} \tau_2$, $\bar{b}_s = \max\limits_{t\in T_s} \tau_2$,

$$N_s = \left[\frac{\bar{a}_s - a_s}{\mu_s}\right] + 2,\ M_s = \left[\frac{\bar{b}_s - b_s}{\nu_s}\right] + 2,\ \ \alpha_s(1) = a_s,\ \ \beta_s(1) = b_s,$$

$$\alpha_s(i+1) = \alpha_s(i)+\mu_s, i = 1, ..., N_s-1;\ \ \beta_s(j+1) = \beta_s(j)+\nu_s, j = 1, ..., M_s-1;$$

$$U_s = \{(i, j) : (\alpha_s(i), \beta_s(j)) \in T_s,\ i = 1, ..., N_s,\ j = 1, ..., M_s\},\ \ s = 1, 2.$$

Choose the following grids in the index sets T_1 and T_2:

$$\tau_s(i, j) = (\alpha_s(i), \beta_s(j)),\ (i, j) \in U_s;\ s = 1, 2,$$

and solve the following *discretized problem* (NLP problem):

$$\min\ x_1^2, \quad \text{s.t. } f_s(x, \tau_s(i, j)) \le 0,\ (i, j) \in U_s,\ s = 1, 2. \tag{59}$$

For the step values

$$\mu_1 = 0.0367,\ \nu_1 = 0.0069,\ \mu_2 = 0.0067,\ \nu_2 = 0.0069, \tag{60}$$

problem (59) admits a solution $x_D^1 = (0, -0.0061, -0.0210, 0.0237)$. For another step values, $\mu_1 = 0.0061$, $\nu_1 = 0.0013$, $\mu_2 = 0.0011$, $\nu_2 = 0.0013$, a solution of the discretized problem (59) is $x_D^2 = (0, -0.0109, -0.0036, 0.0139)$. Both vectors x_D^1 and x_D^2 considerably differ from the optimal solution x^0. This example shows that even for a very dense greed, the optimal solution of the discretized problem can be very far from that of the original SIP problem.

Now, let us add to the discretized problem (59) the additional constraints $\frac{\partial f_1^T(x,t_1^*)}{\partial t} b_1(1) = 0$, $\frac{\partial f_2^T(x,t_2^*)}{\partial t} b_1(2) = 0$, obtained as the result of the analysis of the immobile indices. These constraints, as it was shown above, should be satisfied for any solution of (54). Having solved the obtained problem on the grid with step values (60), we get $x_{newD}^1 =(0.0694, -0.2500, -0.4305, 0.6805)$. It is easy to see that this solution is almost identical to the optimal solution of the original SIP problem (54). Therefore, we can conclude that the discretization methods may be improved by introducing the new additional constraints, which are obtain on the base of the notion of the immobile indices.

7 Perspectives

We would like to complete the article by a short discussion about the prospects open to researchers of SIP and connected problems, when using a new approach to optimality conditions, described here.

As a rule, a non-compliance of the KKT type necessary optimality conditions in SIP is related with the fact that SIP problems may possess hidden additional constraints. Those are the consequence of the *full continuum* system of the constraints, but are not a consequence of any of its *finite* subsystems. The analysis of the properties of the immobile indices of constraints has allowed us to formulate these additional constraints in an explicit form. This made it possible to derive new optimality conditions.

The obtained results permit to conclude that the further research, which is aimed at identification and accounting the *immobile indices* and the corresponding additional constraints, is promising and may lead to new findings. It inspires us to continue investigation in this area, and below we discuss some possible topics of new studies. At the outset, recall that in the present paper

a) the *convex* SIP problems were considered;
b) it was assumed that the (infinite) index set T is *compact*;
c) the set of immobile indices T^* was assumed to consist of a *finite* number of elements;
d) for all $t \in T^*$, the immobility orders were supposed to be *less or equal to one.*

Now, let us outline a few directions for the future research.

Our efforts will be aimed at weakening the assumptions $b) - d)$. Namely, it is planned to investigate the problems, in which:

b^*) the index set T is *not compact*,
c^*) an *infinite* number of immobile indices is possible;
d^*) the immobility orders may be *greater than one.*

When the convex SIP problems are being studied, it is usually assumed in the literature, that the mentioned above situations do not take place. However, in many important applications of SIP the situations $b^*) - d^*)$ are typical. Let us list some of them.

Firstly, there are important for different applications problems of Copositive Programming (CP) (see, e.g., [32]). For these problems, situations c*) and d*) may occur. In [33], we have already successfully applied our approach to the Semidefinite Programming (SDP) problems, which can be considered as a particular case of CP problems. It should be emphasized that, in a general, CP problems are much more complex than those of SDP.

Secondly, there are problems of Semi-Infinite Polynomial Programming (SIPP) and, in particular, the Linear SIPP problems, which have recently emerged in the spotlight in literature (see, e.g., [34]). For these problems, the situations $b^*) - d^*)$ are typical. In study of SIPP problems with noncompact index set T, a special technique called *homogenization*, is used [34]. This technique allows, under some generic assumptions, to reduce the original SIPP problem (with noncompact set T) to the equivalent SIPP problem with a compact one. However, the use of homogenization technique does not guarantee that the Slater condition is fulfilled for the new equivalent SIPP problem, even when the original problem satisfies this condition. In fact, consider the following simple Linear SIPP problem:

$$\begin{aligned} &\min_{x\in\mathbb{R}^n,\rho\in\mathbb{R}} \; c^T x - \rho, \\ \text{s.t.} \quad & t^T Dt + (d^T + x^T A)t \geq \rho \;\; \forall t \in K = \{t \in \mathbb{R}^p : Bt \geq 0\}, \end{aligned} \tag{61}$$

where $c \in \mathbb{R}^n$, $d \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times p}$, $B \in \mathbb{R}^{m\times p}$ are given. Having applied the homogenization technique, one gets the equivalent problem

$$\begin{aligned} &\min_{x\in\mathbb{R}^n,\rho\in\mathbb{R}} \; c^T x - \rho, \\ \text{s.t.} \quad & t^T Dt + t_0(d^T + x^T A)t \geq \rho t_0^2, \\ & \forall \bar{t} \in \{\bar{t} = (t, t_0) \in \mathbb{R}^{p+1} : Bt \geq 0, t_0 \geq 0, ||\bar{t}|| = 1\}. \end{aligned} \tag{62}$$

Evidently, if the feasible set of problem (61) is nonempty, then the constraints of this problem satisfy the Slater condition. At the same time, this condition is violated for problem (62), when the set $\Delta K = \{\tau \in K \setminus \{0\} : \tau^T D\tau = 0\}$ is nonempty. Note that all indices $\bar{t} = (\tau, 0)$, with $\tau \in \Delta K$, are immobile and for them, as a rule, the situations $c^*)$ and $d^*)$ occur. In [35,36], for SIP problems with noncompact index set, the KKT necessary optimality conditions are formulated under the Farkas-Minkowski CQ. In problem (61), this CQ does not hold true and, therefore, the KKT conditions may be not fulfilled. Thus, further study of SIPP problems, on the basis of the proposed in the paper approach, is relevant and promising.

It may also be interesting and auspicious to use our approach to reveal the "hidden" constraints both in general and specific non-convex SIP problems. For example, we can apply it to SIP problems with disjunctive index sets in the form

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} c(x), \\ \text{s.t. } & f_1(t^{(1)}, x) \leq 0 \;\; \forall t^{(1)} \in T_1 \;\; \vee \;\; f_2(t^{(2)}, x) \leq 0 \;\; \forall t^{(2)} \in T_2 \;\; \vee \\ & \ldots \vee \; f_m(t^{(m)}, x) \leq 0 \;\; \forall t^{(m)} \in T_m, \end{aligned}$$

to fractional SIP problems in the form

$$\min_{x\in\mathbb{R}^n} \inf_{\tau\in\mathcal{T}} \frac{g_1(\tau,x)}{g_2(\tau,x)} \quad \text{s.t. } f(t,x) \leq 0 \;\; \forall t \in T; \;\; g_2(\tau, x) \geq 0 \;\; \forall \tau \in \mathcal{T},$$

and to various types of min max and multi-objective SIP problems [37].

The identification and accounting of the "hidden" constraints in the generalized SIP problems are also of interest.

The information about the "hidden" constraints can be used for development of the duality theory in SIP.

The illustrative example, described in the paper, shows that the use of the "hidden" constraints has a positive impact on the effectiveness of the numerical methods. Therefore, it is relevant to

— create and justify efficient algorithms, which constructively describe the set of immobile indices, and formulate, with the help of these indices, new additional constraints satisfied by all feasible solutions of the original SIP problem;

— develop the numerical methods for solving the arising auxiliary problems which contain these additional constraints.

The results of this paper can serve as a good theoretical and constructive basis for work in the above-mentioned directions.

8 Conclusions

In the present paper, we have considered the convex SIP problems with finitely representable compact index sets under Assumptions 1 and 2, which are less restrictive than the known CQs. Using the notion of immobile indices, we obtained new efficient optimality conditions in implicit and explicit forms and showed that these conditions are more efficient than the known ones, when applied to the considered class of problems. We discussed perspectives in the study of various classes of optimization problems, which a new approach opens, and indicated some directions for future research.

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References

1. Hettich R., Kortanek K.O.: Semi-Infinite Programming: Theory, Methods and Applications. SIAM Rev. 35, 380-429 (1993)
2. López M., Still G.: Semi-Infinite Programming. EJOR 180 (2), 491–518 (2007)
3. Kirst P., Stein O.: Solving Disjunctive Optimization Problems by Generalized Semi-Infinite Optimization Techniques. J. Optim. Theory Appl. 169, 1079-1109 (2016)
4. Ahmed F., Dür M., Still G.: Copositive Programming via Semi-Infinite Optimization. J. Optim. Theory Appl. 159, 322-349 (2013)

5. Weber G.-W., Kropat E., Alparslan Gök S.Z.: Semi-Infinite and Conic Optimization in Modern Human Life and Financial Sciences under Uncertainty. In: ISI Proceedings of 20th Mini-EURO conference, 180-185 (2008)
6. Mehrotra S., Papp D.: A Cutting Surface Algorithm for Semi-Infinite Convex Programming with an Application to Moment Robust Optimization. SIAM Journal on Optimization. 24(4), 1670-1697 (2014)
7. Sachs E.W.: Semi-Infinite Programming in Control. In Reemtsen R. and Rückmann J.-J. (eds.): Semi-Infinite Programming, Springer US, Boston, MA, pp. 389–411 (1998)
8. Weber G.-W.: Generalized Semi-Infinite Optimization: Theory and Applications in Optimal Control and Discrete Optimization, Journal of Statistics and Management Systems 5, 359-388 (2002)
9. Kropat E., Pickl S., Rössler A., and Weber G.-W.: A New Algorithm from Semi-Infinite Optimization for a Problem of Time-Minimal Control. Vycislitel'nye technologii (Computational Technologies) 5(4), 67-81 (2000)
10. Özöğür-Akyüz S., Weber G.-W.: Learning with Infinitely Many Kernels via Semi-Infinite Programming, Optim. Methods Softw. 25 (6), 937-970 (2010)
11. Bonnans J.F., Shapiro A.: Perturbation Analysis of Optimization Problems. Springer-Verlag, New-York (2000)
12. Hettich R., Still G.: Second Order Optimality Conditions for Generalized Semi-Infinite Programming Problems. Optimization 34, 195–211 (1995)
13. Rückmann J.-J., Shapiro A.: First-Order Optimality Conditions in Generalized Semi-Infinite Programming, J. Optim. Theory Appl. 101, 677-691 (1999)
14. Klatte D.: Stable Local Minimizers in Semi-Infinite Optimization: Regularity and Second-Order Conditions. J. Comput. Appl. Math. 56 (1–2), 137–157 (1994)
15. Stein O, Still G.: On Optimality Conditions for Generalized Semi-Infinite Programming Problems. J. Optim. Theory Appl. 104 (2), 443-458 (2000)
16. Yang X., Chen Z., Zhou J.: Optimality Conditions for Semi-Infinite and Generalized Semi-Infinite Programs via Lower Order Exact Penalty Functions. J. Optim. Theory Appl. 169(3), 984-1012 (2016)
17. Kostyukova O.I., Tchemisova T.V.: Implicit Optimality Criterion for Convex SIP problem with Box Constrained Index Set. TOP 20 (2), 475–502 (2012)
18. Li C., Ng K. F., Pong T. K.: Constraint Qualifications for Convex Inequality Systems with Applications in Constrained Optimization. SIAM J. Optim. 19 (1), 163–187 (2008)
19. Fajardo M.D., López M.A.: Some Results about the Facial Geometry of Convex Semi-Infinite Systems. Optimization 55(5-6), 661–684 (2006)
20. Cánovas M.J., López M.A, Mordukhovich B.S., Parra J.: Variational Analysis in Semi-Infinite and Infinite Programming, II: Necessary Optimality Conditions. Research Report MI 48202, Wayne State University (2009)
21. Stein, O.: Bi-level Strategies in Semi-infinite Programming, Kluwer, Boston (2003)
22. Jongen H.Th., Twilt F., and Weber G.-W.: Semi-Infinite Optimization: Structure and Stability of the Feasible Set, J. Optim. Theory Appl. 72, 529–552 (1992)
23. Rückmann, J.-J.: On Existence and Uniqueness of Stationary Points in Semi-Infinite Optimization, Math. Programming 86, 387–415 (1999)
24. Klatte, D.: On Regularity and Stability in Semi-Infinite Optimization, Set-Valued Analysis, 3, 101–111 (1995)
25. Kostyukova O.I., Tchemisova T.V.: On a Constructive Approach to Optimality Conditions for Convex SIP Problems with Polyhedral Index Sets. Optimization 63(1), 67–91 (2014)
26. Kostyukova O.I., Tchemisova T.V., and Yermalinskaya S.A. Convex Semi-Infinite Programming: Implicit Optimality Criterion Based on the Concept of Immobile Indices. J. Optim. Theory Appl. 145 (2), 325-342 (2010)
27. Kostyukova O.I., Tchemisova T.V.: Convex SIP Problems with Finitely Representable Compact Index Sets: Immobile Indices and the Properties of an Auxiliary NLP Problem. Set-Valued and Variational Analysis. 23(3), 519–546 (2015)
28. Kostyukova O.I., Tchemisova T.V.: A Constructive Algorithm of Determination of the Sets of Immobile Indices in Convex SIP Problems with Polyhedral Index Sets. Working paper of the University of Aveiro, (2012)

29. Levin V.L.: Application of E. Helly's Theorem to Convex Programming, Problems of the Best Approximation and Related Questions. Math. USSR Sbornik. 8 (2), 235–247 (1969)
30. Kruger A., Minchenko L., Outrat J.: On Relaxing the Mangasarian-Fromovitz Constraint Qualification. Positivity. 18, 171 – 189 (2014)
31. Kostyukova O.I.: Optimality Conditions for Nonsmooth Convex Programming Problems. Dokladi Nationalnoi Akademii Nauk Belarusi 57 (5), 22–27 (2013)
32. Bomze I.M., Dür M., de Klerk E., Roos C., Quist A.J., Terlaky T.: On copositive programming and standard quadratic optimization problems. J. Global Optim. 18(4) 301-320 (2000)
33. Kostyukova O. I., Tchemisova T. V.: Optimality criteria without constraint qualifications for linear SDP problems. J. Math. Sci. 182(2) 126-143 (2012)
34. Wang L., Guo F.: Semidefinite relaxations for semi-infinite polynomial programming. Computational Optimization and Applications 58(1) 133-159 (2013)
35. Goberna M.A., and López M. A.: Linear Semi-Infinite Optimization. Wiley, Chichester, (1998)
36. Mordukhovich B., Nghia T. T. A.: Constraint qualifications and optimality conditions for nonconvex semi-infinite and infinite programs. Math.Program. 139(1-2) 271-300 (2013)
37. Parpas P., Rustem B.: An Algorithm for the Global Optimization of a Class of Continuous Minimax Problems. J. Optim. Theory Appl. 141(2) 461-473 (2009)