CONFORMING STOKES ELEMENTS YIELDING DIVERGENCE–FREE APPROXIMATIONS ON QUADRILATERAL MESHES

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In this dissertation, we propose conforming finite element methods that yield divergence–free velocity approximations for the steady Stokes problem on cubical and quadrilateral meshes. In the first part, we construct the finite element spaces for the two-dimensional problem on rectangular grids. Then in the second part, we extend these spaces to n-dimensional spaces. We use discrete differential forms and smooth de Rham complexes to verify the stability and the conformity of the proposed methods, and the solenoidality of the velocity approximations. In the third part, we shift our focus from a dimensionwise extension to a meshwise improvement by introducing macro elements on general shape–regular quadrilateral meshes. By utilizing a smooth de Rham complex, we prove that the macro finite element method yields divergence–free velocity solutions, and with the construction of a Fortin operator, we validate the stability of the method. To improve the pressure approximation properties, we compute a post–processed pressure solution locally. In addition, we describe the implementation process of the (velocity) macro finite element. We show that the methods developed in this dissertation yield optimal convergence rates and present numerical experiments which are supportive of the theoretical results. Moreover, we provide experimental results of our method for the Navier-Stokes equations and show that the convergence rates are preserved.

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PREFACE

I would like to express my deepest gratitude to my supervisor, Michael Neilan, for his guidance, insightful comments and support throughout my graduate studies at the University of Pittsburgh. I thank him for introducing me to the unconventional and inspiring interplay between different fields of mathematics. I also thank him for always being available, and willing to discuss different viewpoints. I believe working on a classical problem would not be as much fun as it was if I did not work with him. In addition, I appreciate the freedom he has provided me with in my research which helped me become an independent researcher.

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I would like to thank my dear friends for their support and encouragement. My graduate school experience abroad would be really hard without them by my side.

Most of all, I would like to thank my dear parents, Kadriye Sap and Ali Asker Sap, my siblings, Filiz and Hakan, and my uncle, Zeki Sap, for all their support and encouragement. I am truly grateful for having such a loving, understanding and generous family.

I hope that some readers will be inspired by the concepts I shed light on and the methods I introduce, and some readers will enjoy the scenery I paint with symbols and figures as I exhibit the hidden beauty of mathematics in nature.

1.0 INTRODUCTION

Stokes equations describe the motion of the Stokes flow which is a kind of fluid flow where the inertial forces are smaller than the viscous forces. In fluid dynamics, a dimensionless quantity known as the Reynolds number (Re) is used in scaling the relative importance of the inertial (or convective) terms and viscous (or diffusion) terms in a flow equation. By definition, $Re = UL/\nu$ where ν denotes the fluid dynamic viscosity and the constants U and L denote the respective reference length and velocity of the simulated flow. Stokes flow occurs when Re is low, that is, where the fluid velocities are very low, or the viscosities are very large, or the length-scales of the flow are very small [27]. In nature, this kind of flow occurs in the locomotion of microorganisms, and the flow of lava, and in technology, it often occurs in paint, MEMS devices, and in the flow of viscous polymers [17].

The steady Stokes system with no–slip boundary conditions is given by

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, &\quad in \;\; \Omega, \\ \nabla\cdot\mathbf{u} = 0, &\quad in \;\; \Omega, \\ \mathbf{u} = \mathbf{0}, &\quad on \;\; \partial\Omega, \end{aligned} \tag{1.0.1}$$

where $\mathbf{u}$ is the fluid velocity, p is the fluid pressure and $\mathbf{f}$ is the external force applied to the fluid. We assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded, simply–connected domain with a piecewise smooth boundary. For simplicity, we also assume that ν is constant.

In Figure 1.1, an example of the Stokes flow where a solid sphere is moving at a constant speed through a viscous fluid is illustrated [30].

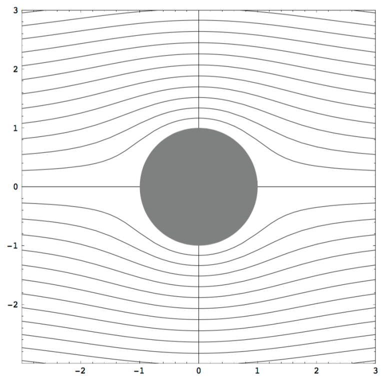


Figure 1.1: Stokes flow around the unit sphere, [30].

In this dissertation, we construct conforming and stable finite element spaces that yield pointwise divergence–free velocity approximations and optimal convergence rates for the discrete version of the Stokes system (1.0.1). However, the methods developed here can be extended for the steady, incompressible Navier-Stokes Equations (NSE) given by

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad in \ \ \Omega, \tag{1.0.2a}$$

$$\nabla\cdot\mathbf{u} = 0, \quad in \ \ \Omega, \tag{1.0.2b}$$

$$\mathbf{u} = \mathbf{0}, \quad on \ \ \partial\Omega, \tag{1.0.2c}$$

where equation (1.0.2a) represents the conservation of momentum, and equation (1.0.2b), known as the continuity equation, represents the conservation of mass.

The NSE play an essential role in modeling fluid flows such as the flows in pipes and channels and the flows around wings of a plane [31]. Solving the NSE for specified boundary conditions provides approximations for the fluid velocity and the fluid pressure in a given geometry. However, due to their complexity, the NSE admit only a limited number of analytical solutions. For example, the NSE for a laminar, steady, two–dimensional flow of an incompressible fluid between two parallel plates or in a circular pipe can be solved analytically [29].

1.1 FINITE ELEMENT METHOD APPLIED TO THE STOKES PROBLEM

Finite element methods are numerical techniques developed for approximating solutions for partial differential equations by piecewise polynomials. A finite element method is based on the division of a mathematical model into disjoint components of simple geometry. These disjoint components are called finite elements. Each element contributes to the model a finite number of degrees of freedom evaluated at a set of points, and then the model is approximated via the discrete model obtained by connecting or assembling these elements.
Consider a partition $\mathcal{T}_h$ of an open, bounded and simply–connected domain Ω for a fixed discretization parameter h.
A finite element space is a triple (T, P_T, Σ_T) that satisfies [10]

- $T \subset \mathbb{R}^n$ is closed and bounded with a piecewise smooth boundary ∂T, (element domain).
- P_T is a finite dimensional space of real–valued functions over T, (basis functions, a.k.a., shape functions).
- Σ_T consists of linear forms over $C^\infty(T)$, (degrees of freedom, a.k.a, nodal points).

In finite element theory, it is assumed that Σ_T is P_T unisolvent, that is, if $\Sigma_T = \{\phi_i\}_{i=1}^N$, then for scalars $\{\alpha_i\}_{i=1}^N$ there exists a unique function $p \in P_T$ that satisfies $\phi_i(p) = \alpha_i$ for $1 \leq i \leq N$. Therefore, $dim(P_T) = N = card(\Sigma_T)$ and there exists a set of functions $\{p_j\}_{j=1}^N$ that forms a basis of P_T and satisfies $\phi_i(p_j) = \delta_{i,j}$ for $1 \leq i, j \leq N$. As a result, any $p \in P_T$ can be written as $p = \sum_{i=1}^{N} \phi_i(p) p_i$.
To approximate $(\mathbf{u}, p)$ that solves the Stokes system given by (1.0.1), we need to construct velocity and pressure finite element spaces.
Let $\mathbf{V}_h \subset \mathbf{H}_0^1(\Omega)$ denote the velocity finite element space and $W_h \subset L_0^2(\Omega)$ denote the pressure finite element space. The finite element discretization of (1.0.1) reads:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ that satisfies

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{u}_h : \nabla \mathbf{v} - \int_\Omega p_h \, div(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in \mathbf{V}_h, \\ \int_\Omega q \, div(\mathbf{u}_h) &= 0, \qquad \forall q \in W_h. \end{aligned} \tag{1.1.1}$$

A necessary criterion for the well–posedness and the stability of the discrete problem (1.1.1) is the discrete inf–sup condition, also known as the Ladyzenskaja-Babuska-Brezzi (LBB) condition, which is given by [9]:

$$\alpha \|q\|_{L^2(\Omega)} \leq \sup_{\mathbf{v} \in \mathbf{V}_h \setminus \{\mathbf{0}\}} \frac{\int_\Omega q \; div(\mathbf{v})}{\|\mathbf{v}\|_{H^1(\Omega)}}, \qquad \forall q \in W_h, \tag{1.1.2}$$

where $\alpha > 0$ is a constant independent of the discretization parameter h.

The discrete inf–sup stability condition is equivalent to the property $W_h \subseteq \mathcal{P}_{W_h} div(\mathbf{V}_h)$ with a bounded right-inverse, where $\mathcal{P}_{W_h} : L^2_0(\Omega) \to W_h$ denotes the L^2–projection onto W_h.

The inf–sup condition is invalid if the space $S_p = \{q \in W_h : \int_\Omega q \; div(\mathbf{v}) = 0, \; \forall \mathbf{v} \in \mathbf{V}_h\}$ is non–trivial. A non–zero element $s \in S_p$ is known as a spurious pressure mode. Every spurious pressure mode s yields a pressure solution $(s + p_h)$ for (1.1.1). Therefore, the existence of spurious pressure modes violates the uniqueness of the pressure solution [9].

Proposition 1 (Proposition 2.1 in [9])**.** *Let $(\mathbf{u}, p)$ be the solution of (1.0.1) and suppose that the inf–sup condition (1.1.2) holds. Then, there exists a unique pair $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ that solves the discretized system given by (1.1.1) and the errors satisfy*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq c \inf_{\{\mathbf{v}_h \in \mathbf{V}_h, \; q_h \in W_h\}} \Big(\|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)} + \|p - q_h\|_{L^2(\Omega)} \Big),$$

where $c > 0$ is a constant independent of the discretization parameter h.

Remark 1. *In the rest of this dissertation, we denote by $c > 0$ a generic constant that may take different values at different instances.*

1.2 PREVIOUS WORKS AND THE ADVANTAGES OF DIVERGENCE–FREE APPROXIMATIONS

Over the past years, several mixed finite element methods for the Stokes problem on triangular meshes have been developed. Although conforming and stable approximations have been derived by most of these methods, the incompressibility condition is usually only weakly satisfied. Taylor-Hood elements, the MINI element [3], the Crouzeix-Raviart elements [15] and the $(\mathbf{P}_2 - P_0^{dc})$ pair in [9] are among the methods that belong to this class.

The discrete velocity solution is divergence–free if the image of the divergence of the discrete velocity finite element space is a subset of the discrete pressure finite element space. On the other hand, the cases where the image of the discrete divergence operator is strictly smaller than the discrete pressure finite element space invalidate both the uniqueness of the discrete pressure solution and the inf–sup stability condition on general meshes. For example, the ($\mathbf{Q}_1 - P_0^{dc}$) element does not satisfy the inf–sup stability condition [9]. The stability of this element depends highly on the mesh and global spurious modes, which can not be eliminated easily, are observed on some regular meshes. Another example is the ($\mathbf{P}_1 - P_0$) element. In this case, the dimension of the spurious modes grows as the mesh size tends to zero [9].

It is known that the finite element spaces that only satisfy the incompressibility condition weakly may lead to instabilities in nonlinear problems [9, 10]. Additionally, since the divergence–free condition models the conservation of mass, the inexact satisfaction of the incompressibility condition also leads to the lack of mass conservation [22]. Additional advantages of divergence–free finite element methods are as follows:

- The invariance $\mathbf{f} \to \mathbf{f} + \nabla\phi \Longrightarrow p \to p + \phi$ is preserved numerically, that is, the velocity approximation is only influenced by the divergence–free part of the source function.
- If the velocity is coupled with transport, then the enforcement of mass conservation is paramount to ensure accurate approximations [8].
- The construction of divergence–free yielding elements results in explicit characterizations of the divergence–free subspaces, possibly leading to efficient iterative solvers [40].
- The velocity error estimates for divergence–free, stable and conforming finite element pairs are decoupled from the pressure error estimates, i.e., [22]:

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)} \leq c \inf_{\mathbf{v} \in \mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}\|_{H^1(\Omega)}$$

  where $\mathbf{Z}_h$ denotes the kernel of the divergence operator acting on $\mathbf{V}_h$.

On simplicial meshes, the first conforming, divergence–free finite elements were introduced by Scott-Vogelius [41]. They showed that the $\mathbf{P}_k - P_{k-1}^{dc}$ pair is stable on simplicial triangulations of two–dimensional polygonal domains if the polynomial degree k satisfies $k \geq 4$ and the triangulation does not contain singular vertices. Their results were extended in [22, 25]. In [6, 38, 45, 47], it was shown that the spaces $\mathbf{P}_k - P_{k-1}^{dc}$ provide mass conservation for smaller values of k if the meshes are Hsieh-Clough-Tocher or Powell-Sabin triangulations. Arnold and Qin [6] showed that $\mathbf{P}_2 - P_1^{dc}$ pair yields an unstable method for some meshes even after the spurious pressure modes are removed. However, for some meshes the method is stable and gives optimal order convergence for velocity once the local spurious pressure modes are removed, and yet for other meshes, the method is stable and yields optimal convergence rates for both the pressure and the velocity approximations. Falk and Neilan [22] developed conforming and stable finite elements that satisfy mass conservation on general triangular meshes by using discrete smooth de Rham complexes. Unlike Scott and Vogelius, they do not require quasiuniform meshes in their analysis and their elements have significantly fewer global degrees of freedom than the Scott-Vogelius elements. The methods developed in the first two chapters of this dissertation are extensions of the Falk-Neilan elements to rectangular grids. Guzmán and Neilan [25] constructed divergence–free, conforming and stable finite element pairs for the three–dimensional Stokes problem on general simplicial triangulations. They used a pressure space consisting of piecewise constants and a velocity space consisting of cubic polynomials augmented with rational functions. Christiansen and Hu [12] presented a general framework for the discretization of de Rham sequences of differential forms with high regularity and some examples of finite element spaces that fit in the framework.

Similar to the simplicial case, the construction of Stokes pairs yielding divergence–free approximations on tensor–product meshes is mostly limited to the two–dimensional case [26, 46, 7]. The first conforming, divergence–free element on a rectangular mesh was proposed by Austin, Manteuffel and McCormick [7]. The finite element space they introduced is a continuous space that is based on a Raviart-Thomas finite element space. The authors constructed a $P_{3,2} \times P_{2,3}$ finite element pair as a direct sum of two L^2–orthogonal spaces and proved that the optimal convergence is obtained in the energy norm for tensor–product grids. Another conforming, divergence–free element on rectangular grids was proposed by Zhang [45]. He showed that the $(P_{k+1,k} \times P_{k,k+1} - P_k^-)$ mixed element, where P_k^- denotes the discontinuous polynomials of separated degree k or less with spurious modes filtered, is stable and yields an optimal order of approximation for the Stokes problem for all $k \geq 2$. Moreover, Buffa, de Falco and Sangalli [11] proposed several choices of spline spaces which can be perceived as extensions of the Taylor-Hood, Nédélec and Raviart-Thomas pairs of finite element spaces for the approximations of the velocity and pressure fields. They studied the stability and convergence of each method and discussed the exact mass conservation of the discrete velocity fields. Evans and Hughes [21, 20] developed B–spline discretizations that produce pointwise divergence–free velocity approximations. Their method is motivated by isogeometric discrete differential forms and can be interpreted as a smooth generalization of Raviart-Thomas elements.

1.3 METHODOLOGY

The construction of the finite element pairs in this dissertation is motivated by smooth de Rham complexes (Stokes complexes) [22]. The Stokes complex we use as a guiding tool in developing our methods is a Hilbert complex used for the Stokes flow model that constitutes Hilbert spaces as scalar and vector potentials, the flow velocity and the flow pressure.
The Stokes complex employs the standard Sobolev spaces:

$$\begin{aligned} L_0^2(\Omega) &= \{v \in L^2(\Omega) : \int_\Omega v \ = 0\}, \\ H^k(\Omega) &= \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega), \quad \forall |\alpha| \le k\}, \\ H_0^k(\Omega) &= \{v \in H^k(\Omega) : D^\alpha v = 0 \text{ on } \partial\Omega,\ \forall |\alpha| \le k-1\}. \end{aligned}$$

The two–dimensional Stokes complex is given by the sequence:

$$\mathbb{R} \xrightarrow{\subset} H^2(\Omega) \xrightarrow{curl} \mathbf{H}^1(\Omega) \xrightarrow{div} L^2(\Omega) \rightarrow 0, \tag{1.3.1}$$

where the *curl* operator acting on a scalar function $z \in H^2(\Omega)$ is defined as $curl(z) = (\frac{\partial z}{\partial y}, -\frac{\partial z}{\partial x})^t$. Provided Ω is simply–connected, the complex (1.3.1) is exact, that is, the range of each map in the complex is the null space of the succeeding map.
Our main goal is to construct finite element spaces $(\Sigma_h, \mathbf{V}_h, W_h)$ which by the sequence (1.3.2) forms an exact subcomplex of the complex (1.3.1).

$$\mathbb{R} \xrightarrow{\subset} \Sigma_h \xrightarrow{curl} \mathbf{V}_h \xrightarrow{div} W_h \rightarrow 0. \tag{1.3.2}$$

Since (1.3.2) is a subcomplex of (1.3.1), there holds $\Sigma_h \subset H^2(\Omega)$, $\mathbf{V}_h \subset \mathbf{H}^1(\Omega)$ and $W_h \subset L^2(\Omega)$, that is, the finite element spaces are conforming. To verify the exactness of (1.3.2), we need to show that for every $q \in W_h$, there exists $\mathbf{v} \in \mathbf{V}_h$ satisfying $div(\mathbf{v}) = q$, and if $\mathbf{v} \in \mathbf{V}_h$ with $div(\mathbf{v}) = 0$, then $\mathbf{v} = curl(z)$ for some $z \in \Sigma_h$. Along with an estimate of the right–inverse, this result also yields inf–sup stability.

In n–dimensions, we view the functions in $\mathbf{H}^1(\Omega)$ and $L^2(\Omega)$ as $(n-1)$-forms and n-forms, respectively. In particular, if $\mathbf{v} \in \mathbf{H}^1(\Omega)$ with $\mathbf{v} = (v^{(1)}, v^{(2)}, \dots, v^{(n)})^t$ and $q \in L^2(\Omega)$, then

$$\mathbf{v} \sim \sum_{j=1}^{n} v^{(j)}\ dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots dx^n, \quad q \sim q\ dx^1 \wedge \dots dx^n,$$

where *hat* indicates a suppressed argument. Let d denote the exterior differentiation operator and set $H\Lambda^l(\Omega)$ as the space of $L^2(\Omega)$ l–forms with exterior derivatives in $L^2(\Omega)$. Then the n–dimensional de Rham complex with minimal $L^2(\Omega)$ smoothness is given by [4, 5]

$$\mathbb{R} \xrightarrow{d} H\Lambda^0(\Omega) \xrightarrow{d} \dots \xrightarrow{d} H\Lambda^{n-2}(\Omega) \xrightarrow{d} H\Lambda^{n-1}(\Omega) \xrightarrow{d} H\Lambda^n(\Omega) \rightarrow 0. \tag{1.3.3}$$

The n–dimensional Stokes complex is obtained by imposing additional regularity in the second–to–last and third–to–last spaces in the sequence as follows:

$$\mathbb{R} \xrightarrow{d} H\Lambda^0(\Omega) \xrightarrow{d} \dots \xrightarrow{d} \hat{H}\Lambda^{n-2}(\Omega) \xrightarrow{d} H^1\Lambda^{n-1}(\Omega) \xrightarrow{d} H\Lambda^n(\Omega) \rightarrow 0, \tag{1.3.4}$$

where $H^1\Lambda^{n-1}(\Omega)$ denotes the space of $(n-1)$-forms with coefficients in $H^1(\Omega)$ and
$\hat{H}\Lambda^{n-2}(\Omega) := \{\omega \in H\Lambda^{n-2}(\Omega) : d\omega \in H^1\Lambda^{n-1}(\Omega)\}$.
Starting with a $H\Lambda^0(\Omega)$–conforming finite element space, we follow the sequence (1.3.4) to derive a finite element pair $\mathbf{V}_h \times W_h$ with the desired properties.
Throughout this dissertation, we refer to the following theorem in obtaining the stability results.

Theorem 1 (Theorem 2.1.2 in [19])**.** *For every $q \in L_0^2(\Omega)$, there exists $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ such that*

$$div(\mathbf{v}) = q,$$

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)},$$

where $c > 0$ is a constant independent of the discretization parameter h.

1.4 OUTLINE

In the next chapter, we describe our method for constructing a pair of conforming, stable and divergence–free finite element spaces for the Stokes problem on rectangular meshes. We discuss the stability and the approximation properties of the element pairs we propose. In the third chapter, we explain the extension of the two–dimensional finite element spaces constructed on rectangular meshes to n–dimensions, and state the stability and the convergence characteristics of the new element pairs. Additionally, we derive reduced elements with less global degrees of freedom and carry out the convergence analysis for the reduced elements. In the fourth chapter, we introduce a macro finite element method on convex quadrilateral meshes. We construct a Fortin operator to prove the inf–sup stability on general shape–regular quadrilateral meshes and derive convergence results. Furthermore, we compute a post–processed pressure solution locally to improve the rate of convergence of our pressure approximations. In addition, we describe the implementation process of the (velocity) macro finite element, and provide numerical experiments that support the theoretical results. Moreover, we provide experimental results of our method for the Navier-Stokes equations.

2.0 FEM FOR THE STOKES PROBLEM ON RECTANGULAR GRIDS

In this chapter, we construct a pair of conforming and stable finite elements that yield divergence–free velocity approximations on rectangular grids. We assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded, simply–connected, polygonal domain with edges parallel to the coordinate axes. We denote by $\mathcal{Q}_h$ the shape–regular rectangular mesh of Ω. To each element $\mathcal{Q} \in \mathcal{Q}_h$, we assign $h_{\mathcal{Q}} = diam(\mathcal{Q})$, and then define the global mesh size $h := \max_{\mathcal{Q} \in \mathcal{Q}_h} h_{\mathcal{Q}}$. Additionally, we let $\mathcal{V}_{\mathcal{Q}} := \{a_i\}_{i=1}^4$ and $\mathcal{E}_{\mathcal{Q}} := \{\mathcal{L}_i\}_{i=1}^4$ denote the vertices and the edges of a rectangular element $\mathcal{Q} \in \mathcal{Q}_h$. Moreover, we assume that the elements of $\mathcal{V}_{\mathcal{Q}}$ are ordered in a counter-clockwise fashion starting at $a_1 = (x_0, y_0)$ and ending at $a_4 = (x_0, y_1)$ (See Figure 2.1).
The space of polynomials of degree less than or equal to m in x and n in y is denoted by $P_{m,n}$. The vector–valued functions and variables are written in bold–face. For example, $\mathbf{H}^1(\Omega) = (H^1(\Omega))^2$.

2.1 THE LOCAL FINITE ELEMENT SPACES

2.1.1 The C^1 Bogner-Fox-Schmidt Finite Element Space

We utilize the Bogner-Fox-Schmidt(BFS) rectangular element in building the finite element space $\Sigma_h(\mathcal{Q}) = P_{3,3}(\mathcal{Q})$ [13]. The degrees of freedom of $\Sigma_h(\mathcal{Q})$ shown in Figure 2.1 are given by

$$S_\Sigma = \{z(a_i), \nabla z(a_i), \frac{\partial^2 z}{\partial x \partial y}(a_i) : \ a_i \in \mathcal{V}_{\mathcal{Q}}\}.$$

Lemma 1. *The degrees of freedom stated in S_Σ are unisolvent on $\Sigma_h(\mathcal{Q})$.*

Proof. Note that the cardinality of the set S_Σ equals the dimension of the polynomial space $P_{3,3}(\mathcal{Q})$. Therefore, it suffices to prove that if z nullifies S_Σ, then $z = 0$ in $\mathcal{Q}$.

If $z \in \Sigma_h(\mathcal{Q})$, then we may write $z(x,y) = s_1(x)s_2(y)$, where s_1 and s_2 are univariate cubic polynomials. Then, if z nullifies S_Σ, we have

$$s_1(x_0)s_2(y_0) = s_1(x_1)s_2(y_0) = s_1(x_0)s_2(y_1) = s_1(x_1)s_2(y_1) = 0,$$
$$s_1'(x_0)s_2(y_0) = s_1'(x_1)s_2(y_0) = s_1'(x_0)s_2(y_1) = s_1'(x_1)s_2(y_1) = 0,$$
$$s_1(x_0)s_2'(y_0) = s_1(x_1)s_2'(y_0) = s_1(x_0)s_2'(y_1) = s_1(x_1)s_2'(y_1) = 0.$$

On $\mathcal{L}_4$, $z(x,y) = s_1(x_0)s_2(y)$, and if $s_1(x_0) \neq 0$, then $s_2'(y_0) = s_2'(y_1) = s_2(y_0) = s_2(y_1) = 0$. This implies that $s_2|_{\mathcal{L}_4} = 0$ since $s_2 \in P_3(\mathcal{Q})$. Therefore, $z|_{\mathcal{L}_4} = 0$.

Similar computations show that $z|_{\partial\mathcal{Q}} = 0$. Thus, $z = \alpha\ b_{\mathcal{Q}}$, where $\alpha \in P_{1,1}(\mathcal{Q})$ and $b_{\mathcal{Q}}$ is a bubble function. More precisely, $b_{\mathcal{Q}}$ is a biquadratic polynomial that vanishes on $\partial\mathcal{Q}$, takes the value one at the center of $\mathcal{Q}$ and is nonnegative in $\mathcal{Q}$. If we denote by b_i a non–trivial linear function that vanishes on $\mathcal{L}_i$ for every $1 \leq i \leq 4$, then we may write $b_{\mathcal{Q}} = b_1b_2b_3b_4$. Moreover, we have $\frac{\partial z}{\partial x \partial y}(a_i) = 0$ for all $1 \leq i \leq 4$, therefore, z has $b_{\mathcal{Q}}^2$ as a factor. Since $b_{\mathcal{Q}}^2 \in P_{4,4}$, this implies that $z = 0$ in $\mathcal{Q}$.

□

2.1.2 The Velocity Space

We define the local velocity space as $\mathbf{V}_h(\mathcal{Q}) = P_{3,2}(\mathcal{Q}) \times P_{2,3}(\mathcal{Q})$.

The degrees of freedom of $\mathbf{V}_h(\mathcal{Q})$ illustrated in Figure 2.1 are given by the set

$$S_v = \{\mathbf{v}(a_i),\ \frac{\partial v_1}{\partial x}(a_i),\ \frac{\partial v_2}{\partial y}(a_i) :\ \mathbf{v} = (v_1, v_2),\ a_i \in \mathcal{V}_{\mathcal{Q}},\ \int_{\mathcal{Q}} \mathbf{v}\cdot\boldsymbol{\kappa} :\ \boldsymbol{\kappa} \in P_{1,0}(\mathcal{Q}) \times P_{0,1}(\mathcal{Q}),$$
$$\int_{\mathcal{L}_i} \mathbf{v}\cdot\mathbf{n} :\ \mathcal{L}_i \in \mathcal{E}_{\mathcal{Q}}\},$$

where $\mathbf{n}$ denotes the (outward) unit normal vector.

Lemma 2. *The degrees of freedom stated in S_v are unisolvent on $\mathbf{V}_h(\mathcal{Q})$.*

Proof. Since $dim(\mathbf{V}_h(Q)) = 24$ equals the cardinality of the set S_v, it suffices to show that if $\mathbf{v}$ nullifies S_v, then $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$.

We may write $v_1(x,y) = s_1(x)s_2(y)$, where s_1 and s_2 are cubic and quadratic polynomials, respectively. Then, if $\mathbf{v}$ nullifies S_v, the following holds

$$v_1(x_0,y_0) = s_1(x_0)s_2(y_0) = v_1(x_0,y_1) = s_1(x_0)s_2(y_1) = 0,$$
$$v_1(x_1,y_0) = s_1(x_1)s_2(y_0) = v_1(x_1,y_1) = s_1(x_1)s_2(y_1) = 0,$$

$$\begin{aligned}
\frac{\partial v_1}{\partial x}(x_0,y_0) = s_1'(x_0)s_2(y_0) &= \frac{\partial v_1}{\partial x}(x_0,y_1) = s_1'(x_0)s_2(y_1) = 0,\\
\frac{\partial v_1}{\partial x}(x_1,y_0) = s_1'(x_1)s_2(y_0) &= \frac{\partial v_1}{\partial x}(x_1,y_1) = s_1'(x_1)s_2(y_1) = 0,\\
\int_{\mathcal{L}_3} v_1 \ &= \int_{\mathcal{L}_4} v_1 \ = 0,\\
\int_{\mathcal{Q}} v_1\ \kappa_1 + v_2\ \kappa_2 = 0, &\quad \text{for all } \boldsymbol{\kappa} = (\kappa_1,\kappa_2) \in P_{1,0}(\mathcal{Q}) \times P_{0,1}(\mathcal{Q}).
\end{aligned}$$

On $\mathcal{L}_1$, $v_1(x,y) = s_1(x)s_2(y_0)$, and if $s_2(y_0) \neq 0$, then $s_1(x_0) = s_1(x_1) = s_1'(x_0) = s_1'(x_1) = 0$. This implies that $s_1|_{\mathcal{L}_1} = 0$ since $s_1 \in P_3(\mathcal{Q})$. Thus, $v_1|_{\mathcal{L}_1} = 0$. Similar arguments show that $v_1|_{\mathcal{L}_2} = 0$.

On $\mathcal{L}_4$, $v_1(x,y) = s_1(x_0)s_2(y)$, and if $s_1(x_0) \neq 0$, then $s_2(y_0) = s_2(y_1) = 0$. Moreover, $\int_{\mathcal{L}_4} v_1 = 0$ implies that $s_2(y^*) = 0$ for some $y^* \in (y_0,y_1)$. As a result, we have $s_2|_{\mathcal{L}_4} = 0$ since $s_2 \in P_2(\mathcal{Q})$. Therefore, $v_1|_{\mathcal{L}_4} = 0$. Similar arguments show that $v_1|_{\mathcal{L}_3} = 0$. Hence, $v_1 \in H_0^1(\mathcal{Q}) \cap P_{3,2}(\mathcal{Q})$. Therefore, we may write $v_1 = q_1 b_{\mathcal{Q}}$, where $b_{\mathcal{Q}} \in P_{2,2}(\mathcal{Q})$ is a bubble function and $q_1 \in P_{1,0}(\mathcal{Q})$.

Similar computations show that $v_2 \in H_0^1(\mathcal{Q}) \cap P_{2,3}(\mathcal{Q})$. Hence, we may write $v_2 = q_2 b_{\mathcal{Q}}$ where $q_2 \in P_{0,1}(\mathcal{Q})$. Thus, we have $\mathbf{v}|_{\partial\mathcal{Q}} = \mathbf{0}$.

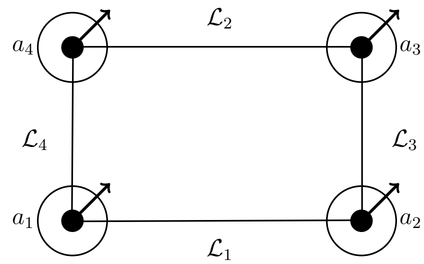
In order to show that $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$, we need to verify that $q_1 = q_2 = 0$ in $\mathcal{Q}$. Note that $\int_{\mathcal{Q}} \mathbf{v}\cdot\boldsymbol{\kappa} = 0$ for all $\boldsymbol{\kappa} = (\kappa_1,\kappa_2) \in P_{1,0}(\mathcal{Q}) \times P_{0,1}(\mathcal{Q})$. Letting $\boldsymbol{\kappa} = (q_1,q_2)$ gives $\int_{\mathcal{Q}} \mathbf{v}\cdot\boldsymbol{\kappa} = \int_{\mathcal{Q}} (q_1^2 + q_2^2)\ b_{\mathcal{Q}} = 0$. Since $b_{\mathcal{Q}} > 0$ in $\mathcal{Q}$, this implies that $(q_1^2 + q_2^2) = 0$ on $\mathcal{Q}$, therefore, $q_1 = q_2 = 0$ in $\mathcal{Q}$. As a result, $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$. □

2.1.3 The Pressure Space

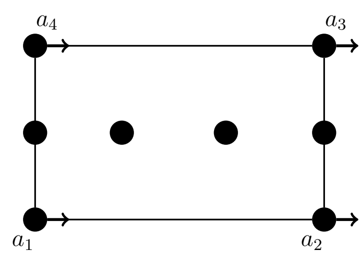
We define the local pressure space as $W_h(\mathcal{Q}) = P_{2,2}(\mathcal{Q})$.

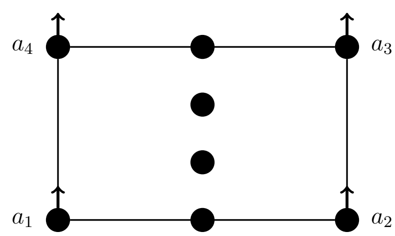
The degrees of freedom of $W_h(\mathcal{Q})$ shown in Figure 2.1 are as follows:

$$S_q = \{q(a_i), \int_{\mathcal{Q}} q\ r\ \ : r \in P_{2,2}(\mathcal{Q}),\ r(a_i) = 0,\ a_i \in \mathcal{V}_{\mathcal{Q}}\}.$$

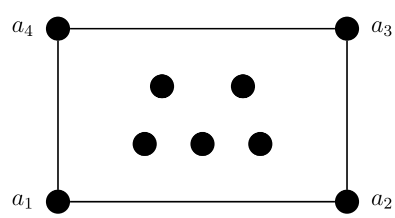


The BFS Element, $\Sigma_h(\mathcal{Q})$.





The Velocity Elements, $V_h(\mathcal{Q})$.



The Pressure Element, $W_h(\mathcal{Q})$.

Figure 2.1: Degrees of freedom on a rectangular element. Solid circles indicate function evaluations, larger circles indicate gradient evaluations and arrows indicate derivative evaluations.

Lemma 3. *The degrees of freedom stated in S_q are unisolvent on $W_h(\mathcal{Q})$.*

Proof. Note that the cardinality of the set S_q equals $dim(W_h(\mathcal{Q})) = 9$. If q nullifies S_q, then $q(a_i) = 0$ and $\int_{\mathcal{Q}} q\, r \; = 0$ for all $r \in P_{2,2}(\mathcal{Q})$ that satisfies $r(a_i) = 0$. Choosing $r = q$ yields $\int_{\mathcal{Q}} q^2 \; = 0$, and this implies $q = 0$ in $\mathcal{Q}$. As a result, the given set of degrees of freedom determines $W_h(\mathcal{Q})$. □

2.1.4 A Local Characterization of the Divergence Operator

In this section, we impose boundary conditions on the local finite element spaces and prove a local inf–sup stability condition. To this end, on each $\mathcal{Q} \in \mathcal{Q}_h$, we define the spaces

- $\Sigma_{h,0}(\mathcal{Q}) = H_0^2(\mathcal{Q}) \cap \Sigma_h(\mathcal{Q})$,
- $\mathbf{V}_{h,0}(\mathcal{Q}) = \mathbf{H}_0^1(\mathcal{Q}) \cap \mathbf{V}_h(\mathcal{Q})$,
- $W_{h,0}(\mathcal{Q}) = \{q \in W_h(\mathcal{Q}) : \int_{\mathcal{Q}} q \; = 0,\ q(a_i) = 0 \text{ at } \forall a_i \in \mathcal{V}_{\mathcal{Q}}\}$.

Lemma 4. *The space $\Sigma_{h,0}(\mathcal{Q})$ is the trivial set, i.e.,*

$$\Sigma_{h,0}(\mathcal{Q}) = \{0\}.$$

Proof. If $z \in \Sigma_{h,0}(\mathcal{Q})$, then $z = b_{\mathcal{Q}}^2 w$ where $b_{\mathcal{Q}} \in H_0^2(\mathcal{Q})$ is a biquadratic bubble function and $w \in P_{1,1}(\mathcal{Q})$. Since $b_{\mathcal{Q}}^2 \in P_{4,4}(\mathcal{Q})$, we conclude that $w = 0$ in $\mathcal{Q}$, therefore, $z = 0$ in $\mathcal{Q}$. □

Lemma 5. *There holds*

$$div(\mathbf{V}_{h,0}(\mathcal{Q})) \subseteq W_{h,0}(\mathcal{Q}).$$

Proof. Let $r \in div(\mathbf{V}_{h,0}(\mathcal{Q}))$. Then, there exists $\mathbf{v} \in \mathbf{V}_{h,0}(\mathcal{Q})$ such that $div(\mathbf{v}) = r$. Since $\mathbf{v} \in \mathbf{H}_0^1(Q)$, the divergence theorem yields

$$\int_{\mathcal{Q}} r \; = \int_{\mathcal{Q}} div(\mathbf{v}) \; = \int_{\partial\mathcal{Q}} \mathbf{v}\cdot\mathbf{n} \; = 0,$$

where $\mathbf{n}$ denotes the unit (outward) normal vector to $\partial\mathcal{Q}$. Moreover, at each vertex $a_i \in \mathcal{V}_{\mathcal{Q}}$,

$$r(a_i) = div(\mathbf{v}(a_i)) = \frac{\partial v_1}{\partial x}(a_i) + \frac{\partial v_2}{\partial y}(a_i) = 0,$$

since $\mathbf{v}|_{\partial\mathcal{Q}} = 0$ implies that $\frac{\partial v_i}{\partial x}(a_i) = \frac{\partial v_i}{\partial y}(a_i) = 0$. Thus, $r \in W_{h,0}(\mathcal{Q})$.

□

Lemma 6. *The kernel of the divergence operator acting on $\mathbf{V}_{h,0}(Q)$ satisfies*

$$Ker(div(\mathbf{V}_{h,0}(\mathcal{Q}))) = curl(\Sigma_{h,0}(\mathcal{Q})) = \{\mathbf{0}\}.$$

Proof. By Lemma 4, we have $curl(\Sigma_{h,0}(\mathcal{Q})) = \{\mathbf{0}\}$. Therefore, it suffices to show that $Ker(div(\mathbf{V}_{h,0}(\mathcal{Q}))) \subseteq curl(\Sigma_{h,0}(\mathcal{Q}))$. Let $\mathbf{v} \in Ker(div(\mathbf{V}_{h,0}(\mathcal{Q})))$, then $\mathbf{v} \in \mathbf{V}_{h,0}(\mathcal{Q})$ and $div(\mathbf{v}) = 0$. This implies that $\mathbf{v} = curl(z)$ for some $z \in H_0^2(\mathcal{Q})$ [24]. In addition, the definitions of the spaces $\mathbf{V}_h(\mathcal{Q})$ and $\Sigma_h(\mathcal{Q})$ imply that $z \in \Sigma_h(\mathcal{Q})$. Thus, $z \in \Sigma_{h,0}(\mathcal{Q})$. This yields the desired inclusion. □

Theorem 2. *The mapping* $div : \mathbf{V}_{h,0}(\mathcal{Q}) \to W_{h,0}(\mathcal{Q})$ *is bijective, i.e.,*

$$div(\mathbf{V}_{h,0}(\mathcal{Q})) = W_{h,0}(\mathcal{Q}) \text{ and } Ker(div(\mathbf{V}_{h,0}(\mathcal{Q}))) = \{\mathbf{0}\}.$$

Proof. Lemmas 5-6 imply that it suffices to show that $dim(div(\mathbf{V}_{h,0}(\mathcal{Q}))) = dim(W_{h,0}(\mathcal{Q}))$ to verify the bijectivity of the divergence map.

Since there are 5 linearly independent constraints imposed on the space $W_h(\mathcal{Q})$ in the definition of $W_{h,0}(\mathcal{Q})$, we have $dim(W_{h,0}) = dim(W_h) - 5 = 4$. Furthermore, by the rank-nullity theorem, Lemma 6 and the definition of $\mathbf{V}_{h,0}$, we have

$$\begin{aligned} dim(div(\mathbf{V}_{h,0}(\mathcal{Q})) &= dim(\mathbf{V}_{h,0}(\mathcal{Q})) - dim(Ker(div(\mathbf{V}_{h,0}(\mathcal{Q}))) \\ &= dim(\mathbf{V}_{h,0}(\mathcal{Q})) - 0 = dim(P_{1,0} \times P_{0,1}) - 0 = 4. \end{aligned} \tag{2.1.1}$$

Hence, $div(\mathbf{V}_{h,0}(\mathcal{Q})) = W_{h,0}(\mathcal{Q})$, therefore, for any $q \in W_{h,0}(\mathcal{Q})$, there exists $\mathbf{v} \in \mathbf{V}_{h,0}(\mathcal{Q})$ such that $div(\mathbf{v}) = q$. Moreover, $Ker(div(V_{h,0}(\mathcal{Q}))) = \{\mathbf{0}\}$ implies that $\mathbf{v}$ is unique. Thus, $div : \mathbf{V}_{h,0}(\mathcal{Q}) \to W_{h,0}$ is bijective. □

We define a reference element $\hat{\mathcal{Q}}$ and an affine transformation that maps $\hat{\mathcal{Q}}$ to a physical element $\mathcal{Q}$. We utilize the unit square as the reference element $\hat{\mathcal{Q}}$ and consider the transformation $\mathcal{F}_\mathcal{Q} : \hat{\mathcal{Q}} \to \mathcal{Q}$ defined as $\mathcal{F}_\mathcal{Q}(\hat{\mathbf{x}}) = \mathbf{B}\hat{\mathbf{x}} + \mathbf{b}$, where $\mathbf{B} \in \mathbb{R}^{2\times 2}$, $\mathbf{b} \in \mathbb{R}^2$ and $\hat{\mathbf{x}} = (\hat{x}, \hat{y}) \in \mathbb{R}^2$. It is easy to see that $D\mathcal{F}_\mathcal{Q} = \mathbf{B}$.

An affine transformation maps points (resp. edges) to points (resp. edges), and preserves parallelism, convexity and colinearity [32]. Moreover, the inverse of an affine transformation is also an affine transformation. Therefore, we may write $\mathcal{F}_\mathcal{Q}$ as follows:

$$\mathcal{F}_\mathcal{Q}(\hat{x}, \hat{y}) = (x_0 + \hat{x}(x_1 - x_0), y_0 + \hat{y}(y_1 - y_0)) = (x, y).$$

Setting $h_x := x_1 - x_0$ and $h_y := y_1 - y_0$ and using these in the definition of $\mathcal{F}_\mathcal{Q}$, we write

$$\mathbf{B} = \begin{pmatrix} h_x & 0 \\ 0 & h_y \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The shape–regularity condition implies that $h_{\mathcal{Q}} \approx h_x \approx h_y$. We note that $\|\mathbf{B}\| \leq ch_{\mathcal{Q}}$, $\|\mathbf{B}^{-1}\| \leq ch_{\mathcal{Q}}^{-1}$, $det(D\mathcal{F}_{\mathcal{Q}}) \leq ch_{\mathcal{Q}}^2$ and $det(D\mathcal{F}_{\mathcal{Q}}^{-1}) \leq ch_{\mathcal{Q}}^{-2}$.
Then, we define the velocity space $\mathbf{V}_{h,0}(\hat{\mathcal{Q}})$ on the reference element. Since $\mathbf{B}$ is diagonal, $\mathbf{v} \in \mathbf{V}_{h,0}(\mathcal{Q})$ implies that $\hat{\mathbf{v}} := \mathbf{v} \circ \mathcal{F}_{\mathcal{Q}} \in \mathbf{V}_{h,0}(\hat{\mathcal{Q}})$.

Lemma 7. $|||\hat{\mathbf{v}}||| = \|\widehat{div}(\hat{\mathbf{v}})\|_{L^2(\hat{\mathcal{Q}})}$ *defines a norm on* $\mathbf{V}_{h,0}(\hat{\mathcal{Q}})$.

Proof. The following arguments show that $|||\cdot|||$ defines a norm on $\mathbf{V}_{h,0}(\hat{\mathcal{Q}})$.

1. *(Positivity)* Theorem 2 implies that $|||\hat{\mathbf{v}}||| = \|\widehat{div}(\hat{\mathbf{v}})\|_{L^2(\hat{\mathcal{Q}})} = 0$ if and only if $\hat{\mathbf{v}} = \mathbf{0}$.
2. *(Scalar multiplication)* $|||\widehat{c\mathbf{v}}||| = \|\widehat{div}(\widehat{c\mathbf{v}})\|_{L^2(\hat{\mathcal{Q}})} = c\|\widehat{div}(\hat{\mathbf{v}})\|_{L^2(\hat{\mathcal{Q}})} = c|||\hat{\mathbf{v}}|||$.
3. *(Triangle inequality)* $|||\hat{\mathbf{v}} + \hat{\boldsymbol{\omega}}||| \leq \|\widehat{div}(\hat{\mathbf{v}})\|_{L^2(\hat{\mathcal{Q}})} + \|\widehat{div}(\hat{\boldsymbol{\omega}})\|_{L^2(\hat{\mathcal{Q}})} = |||\hat{\mathbf{v}}||| + |||\hat{\boldsymbol{\omega}}|||$.

□

Lemma 8. *For* $q \in W_{h,0}(\mathcal{Q})$*, define* $\hat{q}$ *through the relation* $\hat{q} = q \circ \mathcal{F}_{\mathcal{Q}}$*. Then,* $\hat{q} \in W_{h,0}(\hat{\mathcal{Q}})$*.*

Proof. Since $\mathcal{F}_{\mathcal{Q}}$ is affine, $q \in P_{2,2}(\mathcal{Q})$ implies that $\hat{q} \in P_{2,2}(\hat{\mathcal{Q}})$ and $q(a_i) = 0$ yields $\hat{q}(\hat{a}_i) = 0$. Furthermore, by a change of variables, we have

$$0 = \int_{\mathcal{Q}} q(x) = \int_{\hat{\mathcal{Q}}} \hat{q}(\hat{x})|det(D\mathcal{F}_{\mathcal{Q}}(\hat{x}))|.$$

Since $|det(D\mathcal{F}_{\mathcal{Q}}(\hat{x}))|$ is constant, this implies that

$$\int_{\hat{\mathcal{Q}}} \hat{q}(\hat{x}) = 0.$$

Hence, $\hat{q} \in W_{h,0}(\hat{\mathcal{Q}})$. □

Theorem 3. *For every* $q \in W_{h,0}(\mathcal{Q})$*, there exists* $\mathbf{v} \in \mathbf{V}_{h,0}(\mathcal{Q})$ *such that* $div(\mathbf{v}) = q$ *and*

$$\|\mathbf{v}\|_{H^1(\mathcal{Q})} \leq c\|q\|_{L^2(\mathcal{Q})},$$

where $c > 0$ *is a constant independent of the mesh size. Therefore, the (local) inf-sup condition*

$$\inf_{q \in W_{h,0}(\mathcal{Q}) \setminus \{0\}} \sup_{\mathbf{v} \in \mathbf{V}_{h,0}(\mathcal{Q}) \setminus \{\mathbf{0}\}} \frac{\int_{\Omega} q \; div(\mathbf{v})}{\|\mathbf{v}\|_{H^1(\mathcal{Q})}\|q\|_{L^2(\mathcal{Q})}} \geq c > 0$$

holds. This implies that the $\mathbf{V}_{h,0}(\mathcal{Q}) \times W_{h,0}(\mathcal{Q})$ *pair is locally stable.*

Proof. By Theorem 2, for $q \in W_{h,0}(\mathcal{Q})$, there exists $\mathbf{v} \in \mathbf{V}_{h,0}(\mathcal{Q})$ that satisfies $div(\mathbf{v}) = q$. In addition, by scaling, we have

$$\|\widehat{div}(\hat{\mathbf{v}})\|^2_{L^2(\mathcal{Q})} = \int_{\hat{\mathcal{Q}}} |\widehat{div}(\hat{\mathbf{v}})|^2 \; d\hat{x} \leq \int_{\mathcal{Q}} h_{\mathcal{Q}}^2 | \; div(\mathbf{v})|^2 \; |det(D\mathcal{F}_{\mathcal{Q}}^{-1})| \; dx$$

$$\leq c \int_{\mathcal{Q}} |\ div(\mathbf{v})|^2\ dx = c\|div(\mathbf{v})\|^2_{L^2(\mathcal{Q})},$$

where $c > 0$ is an $h_{\mathcal{Q}}$-independent constant. Moreover, by Lemma 7 and the equivalence of norms in finite dimensions, we have

$$\|\hat{\mathbf{v}}\|_{H^1(\hat{Q})} \leq c\|\widehat{div}(\hat{\mathbf{v}})\|_{L^2(\hat{Q})}.$$

Then, again by scaling, we obtain

$$\|\mathbf{v}\|_{H^1(\mathcal{Q})} \leq c\|\hat{\mathbf{v}}\|_{H^1(\hat{Q})} \leq c\|\widehat{div}(\hat{\mathbf{v}})\| \leq c\|div(\mathbf{v})\| = c\|q\|_{L^2(\mathcal{Q})}.$$

□

2.2 THE GLOBAL FINITE ELEMENT SPACES

In this section, we introduce the global finite element spaces. With the help of the Scott-Zhang interpolant [42], trace and inverse inequalities and the Nitsche's method, we verify the global conformity and stability constraints.

We denote by $\mathcal{Q}$, $\mathcal{V}$ and $\mathcal{E}$ the sets of faces, vertices and edges in the mesh. In addition, we let $|\cdot|$ stand for the cardinality measure of a set. For instance, $|\mathcal{V}| = card(\mathcal{V})$.

The local finite element spaces introduced in Section 2.1 induce the global spaces

- $\Sigma_h = \{z \in H^2(\Omega)\colon\ z|_{\mathcal{Q}} \in \Sigma_h(\mathcal{Q}),\ \frac{\partial^2 z}{\partial x \partial y}(a)$ is C^0, $\forall a \in \mathcal{V}$, $\forall \mathcal{Q} \in \mathcal{Q}_h\ \}$,
- $\mathbf{V}_h = \{\mathbf{v} \in \mathbf{H}^1(\Omega)\colon\ \mathbf{v}|_{\mathcal{Q}} \in \mathbf{V}_h(\mathcal{Q}),\ \frac{\partial v_i}{\partial x_i}(a)$ is C^0, $\forall a \in \mathcal{V}$, $\forall \mathcal{Q} \in \mathcal{Q}_h\ \}$,
- $W_h = \{q \in L^2(\Omega) : q|_{\mathcal{Q}} \in W_h(\mathcal{Q}),\ q(a)$ is C^0, $\forall a \in \mathcal{V}$, $\forall \mathcal{Q} \in \mathcal{Q}_h\ \}$.

Lemma 9. *There holds*

$$Ker(div(\mathbf{V}_h)) = curl(\Sigma_h). \tag{2.2.1}$$

Proof. If $\mathbf{v} \in curl(\Sigma_h)$, then there exists $z \in \Sigma_h$ that satisfies $curl(z) = \mathbf{v}$. It is easy to see that $\mathbf{v} \in Ker(div(\mathbf{V}_h))$. On the other hand, if $\mathbf{v} \in Ker(div(\mathbf{V}_h))$, then $\mathbf{v} \in \mathbf{V}_h$ and $div(\mathbf{v}) = 0$. Moreover, since Ω is simply-connected, there exists $z \in H^2(\Omega)$ such that $\mathbf{v} = curl(z)$ [22]. From the definition of the space $\mathbf{V}_h$, it follows that $\mathbf{v}|_{\mathcal{Q}} \in P_{3,2}(\mathcal{Q}) \times P_{2,3}(\mathcal{Q})$ and $\frac{\partial v_i}{\partial x_i}$ is C^0 at the vertices. As a result, $z|_{\mathcal{Q}} \in P_{3,3}(\mathcal{Q})$ and $\frac{\partial^2 z}{\partial x \partial y}$ is C^0 at the vertices, therefore, $z \in \Sigma_h$. Thus, $Ker(div(\mathbf{V}_h)) \subseteq curl(\Sigma_h)$. □

Theorem 4. *The Stokes complex (1.3.2) is exact provided Ω is simply–connected.*

Proof. By Lemma 9, it suffices to show that $div : \mathbf{V}_h \to W_h$ is surjective. Clearly, $div(\mathbf{V}_h) \subseteq W_h$, therefore, we need to show that $\dim(div(\mathbf{V}_h)) = dim(W_h)$ to complete the proof.

By the rank-nullity theorem, and Lemmas 1 and 2, we have

$$\begin{aligned} dim(div(\mathbf{V}_h)) = dim(\mathbf{V}_h) - dim(curl(\Sigma_h)) &= dim(\mathbf{V}_h) - (dim(\Sigma_h) - 1) \\ &= (2(2|\mathcal{Q}| + 2|\mathcal{V}|) + |\mathcal{E}|) - 4|\mathcal{V}| + 1 \\ &= 4|\mathcal{Q}| + 4|\mathcal{V}| + |\mathcal{E}| - 4|\mathcal{V}| + 1 \\ &= 4|\mathcal{Q}| + |\mathcal{E}| + 1. \end{aligned}$$

Thus, by Lemma 3 and the Euler's identity on simply–connected domains (Lemma 4.41 in [28]), we obtain

$$dim(W_h) - dim(div(\mathbf{V}_h)) = (|\mathcal{V}| + 5|\mathcal{Q}|) - 4|\mathcal{Q}| - |\mathcal{E}| - 1 = |\mathcal{V}| + |\mathcal{Q}| - |\mathcal{E}| - 1 = 0.$$

This implies that $div(\mathbf{V}_h) = W_h$, and the result follows. ☐

Lemma 10 (Lemma 2.2 in [22])**.** *For any simply–connected domain S with piecewise smooth boundary ∂S, there exists a constant $c > 0$ such that $\|\mathbf{v}\|_{L^2(\partial S)} \leq c\|\mathbf{v}\|_{L^2(S)}^{1/2}\|\mathbf{v}\|_{H^1(S)}^{1/2}$ for all $\mathbf{v} \in \mathbf{H}^1(S)$.*

By scaling, this implies that for every $\mathbf{v} \in \mathbf{H}^1(\mathcal{Q})$, the estimate

$$\|\mathbf{v}\|_{L^2(\partial\mathcal{Q})}^2 \leq c(h_{\mathcal{Q}}^{-1}\|\mathbf{v}\|_{L^2(\mathcal{Q})}^2 + h_{\mathcal{Q}}\|\mathbf{v}\|_{H^1(\mathcal{Q})}^2),$$

holds on every element $\mathcal{Q} \in \mathcal{Q}_h$.

Lemma 11. *For any $q \in W_h$, there exists $\mathbf{v}^{(1)} \in \mathbf{V}_h$ that satisfies $(q - div(\mathbf{v}^{(1)}))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$ for all $\mathcal{Q} \in \mathcal{Q}_h$. Moreover, $\|\mathbf{v}^{(1)}\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)}$.*

Proof. For $q \in W_h \subset L^2(\Omega)$, there exists $\boldsymbol{\omega} \in \mathbf{H}^1(\Omega)$ such that $div(\boldsymbol{\omega}) = q$ and $\|\boldsymbol{\omega}\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)}$ (Lemma 2.2 in [9]). Let $\mathbf{I}_h\boldsymbol{\omega} : \mathbf{H}^1(\Omega) \to \mathbf{P}_{2,2}(\Omega)$ be the Scott-Zhang interpolant of $\boldsymbol{\omega}$ [42].

Define $\mathbf{v}^{(1)} = (v_1^{(1)}, v_2^{(1)}) \in \mathbf{V}_h$ that satisfies the following conditions:

(i) $\mathbf{v}^{(1)}(a) = \mathbf{I}_h\boldsymbol{\omega}(a), \quad \forall a \in \mathcal{V}.$

(ii) $\frac{\partial v_1^{(1)}}{\partial x}(a) = \frac{\partial v_2^{(1)}}{\partial y}(a) = \frac{q(a)}{2}, \quad \forall a \in \mathcal{V}.$

(iii) $\int_{\mathcal{Q}} \mathbf{v}^{(1)} \cdot \boldsymbol{\kappa} = \int_{\mathcal{Q}} \boldsymbol{\omega} \cdot \boldsymbol{\kappa}, \quad \forall \boldsymbol{\kappa} \in P_{1,0} \times P_{0,1}, \forall \mathcal{Q} \in \mathcal{Q}_h.$

(iv) $\int_e \mathbf{v}^{(1)} \cdot \mathbf{n} = \int_e \boldsymbol{\omega} \cdot \mathbf{n}, \quad \forall e \in \mathcal{E}.$

Note that condition (iv) yields $\int_{\partial\mathcal{Q}} \mathbf{v}^{(1)} \cdot \mathbf{n} \ = \int_{\partial\mathcal{Q}} \boldsymbol{\omega} \cdot \mathbf{n}$ for all $\mathcal{Q} \in \mathcal{Q}_h$ and condition (ii) implies that $div(\mathbf{v}^{(1)}(a)) = q(a)$ at all $a \in \mathcal{V}$.

Then, by the divergence theorem and condition (iv), we deduce that

$$\begin{aligned}\int_{\mathcal{Q}} q - div(\mathbf{v}^{(1)}) \ &= \int_{\mathcal{Q}} q \ - \int_{\mathcal{Q}} div(\mathbf{v}^{(1)}) \ = \int_{\mathcal{Q}} q \ - \int_{\partial\mathcal{Q}} \mathbf{v}^{(1)} \cdot \mathbf{n} \\ &= \int_{\mathcal{Q}} q \ - \int_{\partial\mathcal{Q}} \boldsymbol{\omega} \cdot \mathbf{n} \ = \int_{\mathcal{Q}} q - div(\boldsymbol{\omega}) \ = 0.\end{aligned}$$

Thus, $(q - div(\mathbf{v}^{(1)}))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$. Moreover, by the triangle inequality, we have

$$\|\mathbf{v}^{(1)}\|_{H^1(\mathcal{Q})} \leq \|\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}\|_{H^1(\mathcal{Q})} + \|\mathbf{I}_h\boldsymbol{\omega}\|_{H^1(\mathcal{Q})}. \tag{2.2.2}$$

Since $(\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega})|_{\mathcal{Q}} \in \mathbf{V}_h(\mathcal{Q})$, by Lemma 2 and scaling, we obtain

$$\begin{aligned}\|\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}\|^2_{H^1(\mathcal{Q})} \approx &\sum_{a_j \in \mathcal{V}_{\mathcal{Q}}} \Big(|(\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega})(a_j)|^2 + h^2_{\mathcal{Q}} |\nabla(\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega})(a_j)|^2 \Big) \\ &+ \sum_{\mathcal{L}_i \in \mathcal{E}_{\mathcal{Q}}} h_e^{-1} \Big| \int_{\mathcal{L}_i} (\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}) \cdot \mathbf{n} \ \Big|^2 + h_{\mathcal{Q}}^{-2} \sup_{r \in D} \Big| \int_{Q} (\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}) \cdot \mathbf{r} \ \Big|^2,\end{aligned}$$

where $D = \{\mathbf{r} : \ \mathbf{r}|_{\mathcal{Q}} \in P_{1,0}(\mathcal{Q}) \times P_{0,1}(\mathcal{Q}), \ \|\mathbf{r}\|^2_{L^2(\mathcal{Q})} = 1\}$.

Combining this with the conditions stated in the construction of $\mathbf{v}^{(1)}$, using scaling arguments and the Cauchy Schwarz inequality, we obtain

$$\begin{aligned}\|\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}\|^2_{H^1(\mathcal{Q})} \leq &c\Big(\sum_{a_j \in \mathcal{V}_{\mathcal{Q}}} h^2_{\mathcal{Q}}(|q(a_j)|^2 + |\nabla \mathbf{I}_h\boldsymbol{\omega}(a_j)|^2) + \sum_{L_i \in \mathcal{E}_{\mathcal{Q}}} h_e^{-1} \Big| \int_{L_i} (\boldsymbol{\omega} - \mathbf{I}_h\boldsymbol{\omega}) \cdot \mathbf{n} \ \Big|^2 \\ &+ h_{\mathcal{Q}}^{-2} \sup_{r \in D} \Big| \int_{Q} (\boldsymbol{\omega} - \mathbf{I}_h\boldsymbol{\omega}) \cdot \mathbf{r} \ \Big|^2 \Big) \\ \leq &c\Big(\sum_{a_j \in \mathcal{V}_{\mathcal{Q}}} h^2_{\mathcal{Q}}(|q(a_j)|^2 + |\nabla \mathbf{I}_h\boldsymbol{\omega}|^2) + h_{\mathcal{Q}}^{-1} \|\boldsymbol{\omega} - \mathbf{I}_h\boldsymbol{\omega}\|^2_{L^2(\partial\mathcal{Q})} \\ &+ h_{\mathcal{Q}}^{-2} \|\boldsymbol{\omega} - \mathbf{I}_h\boldsymbol{\omega}\|^2_{L^2(\mathcal{Q})} \Big).\end{aligned} \tag{2.2.3}$$

Again by scaling, we have

$$\sum_{a_j \in \mathcal{V}_{\mathcal{Q}}} h^2_{\mathcal{Q}} |\nabla \mathbf{I}_h\boldsymbol{\omega}(a_j)|^2 \leq c\|\mathbf{I}_h\boldsymbol{\omega}\|^2_{H^1(\mathcal{Q})}, \quad \text{and} \quad \sum_{a_j \in \mathcal{V}_{\mathcal{Q}}} h^2_{\mathcal{Q}} |q(a_j)|^2 \leq c\|q\|^2_{L^2(\mathcal{Q})}. \tag{2.2.4}$$

Moreover, by Lemma 10 and the approximation properties of the Scott–Zhang interpolant [42], we obtain

$$h_{\mathcal{Q}}^{-1} \|\boldsymbol{\omega} - \mathbf{I}_h\boldsymbol{\omega}\|^2_{L^2(\partial\mathcal{Q})} \leq c\Big(h_{\mathcal{Q}}^{-2} \|\boldsymbol{\omega} - \mathbf{I}_h\boldsymbol{\omega}\|^2_{L^2(\mathcal{Q})} + \|\boldsymbol{\omega} - \mathbf{I}_h\boldsymbol{\omega}\|^2_{H^1(\mathcal{Q})} \Big) \leq c\|\boldsymbol{\omega}\|^2_{H^1(\omega(\mathcal{Q}))} \tag{2.2.5}$$

where $\omega(\mathcal{Q})$ denotes the patch of rectangles that touch $\mathcal{Q}$. Using (2.2.4), (2.2.5) and the approximation and the stability properties of the Scott-Zhang interpolant [42] in (2.2.3), we derive the estimate

$$\begin{aligned}\|\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}\|^2_{H^1(\mathcal{Q})} \leq & c\Big(\|\mathbf{I}_h\boldsymbol{\omega}\|^2_{H^1(\mathcal{Q})} + \|q\|^2_{L^2(\mathcal{Q})} + \|\boldsymbol{\omega}\|^2_{H^1(\omega(\mathcal{Q}))}\Big) \\ & \leq c\Big(\|q\|^2_{L^2(\mathcal{Q})} + \|\boldsymbol{\omega}\|^2_{H^1(\omega(\mathcal{Q}))}\Big) \end{aligned} \tag{2.2.6}$$

Combining (2.2.2) and (2.2.6), and summing over $\mathcal{Q} \in \mathcal{Q}_h$ yields

$$\|\mathbf{v}^{(1)}\|_{H^1(\Omega)} \leq \|\boldsymbol{\omega}\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)}.$$

□

Corollary 1. *Let $\mathbf{v}^{(1)}$ and q be defined as in the proof of Lemma 11. Then, there exists $\mathbf{v}^{(2)} \in \mathbf{V}_h$ that satisfies $\mathbf{v}^{(2)}|_{\mathcal{Q}} = (q - div(\mathbf{v}^{(1)}))|_{\mathcal{Q}}$ and $\|\mathbf{v}^{(2)}\|_{H^1(\Omega)} \leq c(\|q\|_{L^2(\Omega)} + \|\mathbf{v}^{(1)}\|_{H^1(\Omega)})$.*

Proof. Since $(q - div(\mathbf{v}^{(1)}))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$, by Theorem 2, there exists $\mathbf{v}^{(2)}_{\mathcal{Q}} \in \mathbf{V}_{h,0}(\mathcal{Q})$ that satisfies $div(\mathbf{v}^{(2)}_{\mathcal{Q}}) = (q - div(\mathbf{v}^{(1)}))|_{\mathcal{Q}}$ and it is easy to see that

$$\|\mathbf{v}^{(2)}_{\mathcal{Q}}\|_{H^1(\mathcal{Q})} \leq c\|q - div(\mathbf{v}^{(1)})\|_{L^2(\mathcal{Q})} \leq c(\|q\|_{L^2(\mathcal{Q})} + \|\mathbf{v}^{(1)}\|_{H^1(\mathcal{Q})}). \tag{2.2.7}$$

Let $\mathbf{v}^{(2)}$ be such that $\mathbf{v}^{(2)}|_{\mathcal{Q}} = \mathbf{v}^{(2)}_{\mathcal{Q}}$. Since $\mathbf{v}^{(2)}_{\mathcal{Q}} \in \mathbf{V}_{h,0}(\mathcal{Q}) \subset \mathbf{H}^1_0(\mathcal{Q})$, $\frac{\partial v^{(2)}_1}{\partial x}$ and $\frac{\partial v^{(2)}_2}{\partial y}$ vanish at $\forall a \in \mathcal{V}_{\mathcal{Q}}$. This implies that the partial derivatives $\frac{\partial v^{(2)}_1}{\partial x}$ and $\frac{\partial v^{(2)}_2}{\partial y}$ are continuous at $\forall a \in \mathcal{V}$. As a result, $\mathbf{v}^{(2)} \in \mathbf{V}_h$. Finally, summing (2.2.7) over $\mathcal{Q} \in \mathcal{Q}_h$ yields $\|\mathbf{v}^{(2)}\|_{H^1(\Omega)} \leq c(\|q\|_{L^2(\Omega)} + \|\mathbf{v}^{(1)}\|_{H^1(\Omega)})$. □

Theorem 5. *For any $q \in W_h$, there exists $\mathbf{v} \in \mathbf{V}_h$ such that $div(\mathbf{v}) = q$ and*

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)}.$$

where c is a constant independent of the discretization parameter h. This implies the global inf–sup stability condition

$$\inf_{q\in W_h\setminus\{0\}} \sup_{\mathbf{v}\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{\int_\Omega q\ div(\mathbf{v})}{\|\mathbf{v}\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}} \geq c > 0.$$

Proof. For $q \in W_h$, let $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ be defined as in Lemma 11 and Corollary 1, respectively, and set $\mathbf{v} := \mathbf{v}^{(1)} + \mathbf{v}^{(2)}$. Then, we note that $\mathbf{v} \in \mathbf{V}_h$ and $div(\mathbf{v}) = q$. Combining the results of Lemma 11 and Corollary 1, we derive the desired estimate

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq c(\|\mathbf{v}^{(1)}\|_{H^1(\Omega)} + \|\mathbf{v}^{(2)}\|_{H^1(\Omega)}) \leq c(\|\mathbf{v}^{(1)}\|_{H^1(\Omega)} + \|q\|_{L^2(\Omega)}) \leq \|q\|_{L^2(\Omega)}.$$

□

2.3 THE GLOBAL FINITE ELEMENT SPACES WITH IMPOSED BOUNDARY CONDITIONS

In this section, we impose homogeneous boundary conditions on the finite element spaces defined in Section 2.2. We let $\mathcal{V}_b \subset \mathcal{V}$ denote the boundary vertices and $\mathcal{V}_c$ denote the corner vertices of the mesh. In addition, we denote by $\mathcal{E}^b$ the boundary edges, and we let $\mathcal{E}^b_x \subset \mathcal{E}^b$ and $\mathcal{E}^b_y \subset \mathcal{E}^b$ stand for the vertical and horizontal boundary edges of the mesh, respectively. Moreover, we denote by $\mathcal{V}_{b,x}$ (resp. $\mathcal{V}_{b,y}$) the non–corner boundary vertices where x (resp. y) is constant. Hence, we may decompose the boundary vertices and the boundary edges as follows: $\mathcal{V}_b = \mathcal{V}_{b,x} \cup \mathcal{V}_{b,y} \cup \mathcal{V}_c$ and $\mathcal{E}^b = \mathcal{E}^b_x \cup \mathcal{E}^b_y$.

A candidate list of global finite element spaces with boundary conditions is:

- $\Sigma^0_h = \Sigma_h \cap H^2_0(\Omega)$,
- $\mathbf{V}^0_h = \mathbf{V}_h \cap \mathbf{H}^1_0(\Omega)$,
- $W^0_h = W_h \cap L^2_0(\Omega)$.

We note that $z \in \Sigma_h$ is in Σ^0_h if and only if:

(i) $z(a) = 0, \quad \forall a \in \mathcal{V}_b$,

(ii) $\nabla z(a) = 0, \quad \forall a \in \mathcal{V}_b$,

(iii) $\frac{\partial^2 z}{\partial x \partial y}(a) = 0, \quad \forall a \in \mathcal{V}_b$.

The number of constraints imposed on Σ_h in (i)-(iii) is $4|\mathcal{V}_b|$. Thus, we have

$$dim(\Sigma^0_h) = dim(\Sigma_h) - 4|\mathcal{V}_b| = 4|\mathcal{V}| - 4|\mathcal{V}_b|.$$

Simiarly, we note that $\mathbf{v} \in \mathbf{V}_h$ is in $\mathbf{V}^0_h$ if and only if:

(i) $\mathbf{v}(a) = 0, \quad \forall a \in \mathcal{V}_b$,

(ii) $\frac{\partial v_1}{\partial x}(a) = 0$, $\forall e \in \mathcal{V}_{b,y} \cup \mathcal{V}_c$,

(iii) $\frac{\partial v_2}{\partial y}(a) = 0$, $\forall e \in \mathcal{V}_{b,x} \cup \mathcal{V}_c$,

(iv) $\int_e v_1 = 0$, $\forall e \in \mathcal{E}^b_x$,

(v) $\int_e v_2 = 0$, $\forall e \in \mathcal{E}^b_y$.

The number of constraints imposed on $\mathbf{V}_h$ in (i)-(v) is $3|\mathcal{V}_b| + |\mathcal{V}_c| + |\mathcal{E}^b|$. We then have

$$dim(\mathbf{V}^0_h) = dim(\mathbf{V}_h) - 3|\mathcal{V}_b| - |\mathcal{V}_c| - |\mathcal{E}^b| = (4|\mathcal{Q}| + 4|\mathcal{V}| + |\mathcal{E})|) - 3|\mathcal{V}_b| - |\mathcal{V}_c| - |\mathcal{E}^b|.$$

On the other hand, there is only one constraint imposed on W_h in the definition of the space W^0_h. Therefore, $dim(W^0_h) = dim(W_h) - 1 = (4|\mathcal{Q}| + |\mathcal{E}| + 1) - 1 = 4|\mathcal{Q}| + |\mathcal{E}|$.

Lemma 12. *There holds* $Ker(div(\mathbf{V}_h^0)) = curl(\Sigma_h^0)$.

Proof. The inclusion $curl(\Sigma_h^0) \subseteq Ker(div(\mathbf{V}_h^0))$ is trivial, thus, it suffices to prove the inclusion $Ker(div(\mathbf{V}_h^0)) \subseteq curl(\Sigma_h^0)$. If $\mathbf{v} \in Ker(div(\mathbf{V}_h^0))$, then $\mathbf{v} \in \mathbf{V}_h^0$ and $div(\mathbf{v}) = 0$. Moreover, there exists $z \in H^2(\Omega)$ such that $\mathbf{v} = curl(z)$. Since $\mathbf{v}|_{\mathcal{Q}} \in \mathbf{V}_h(\mathcal{Q})$, we have $z|_{\mathcal{Q}} \in \Sigma_h(\mathcal{Q})$. In addition, the conditions (i)-(v) imposed on $\mathbf{V}_h$ in the definition of the space $\mathbf{V}_h^0$ imply that z satisfies the conditions (i)-(iii) imposed on Σ_h in the definition of the space Σ_h^0. Hence, $z \in \Sigma_h^0$, and this implies that $Ker(div(\mathbf{V}_h^0)) \subseteq curl(\Sigma_h^0)$. □

By Lemma 12 and the rank-nullity theorem, we have

$$\begin{aligned} dim(W_h^0) - dim(div(\mathbf{V}_h^0)) &= dim(W_h^0) - (dim(\mathbf{V}_h^0) - dim(Ker(div(\mathbf{V}_h^0)))) \\ &= dim(W_h^0) - (dim(\mathbf{V}_h^0) - dim(curl(\Sigma_h^0))) \\ &= dim(W_h^0) - (dim(\mathbf{V}_h^0) - dim(\Sigma_h^0)). \end{aligned} \tag{2.3.1}$$

From the dimension analysis, it is easy to see that (2.3.1) implies

$$\begin{aligned} dim(\Sigma_h^0) + dim(W_h^0) - dim(\mathbf{V}_h^0) =& (4|\mathcal{V}| - 4|\mathcal{V}_b|) + (4|\mathcal{Q}| + |\mathcal{E}|) - 4|\mathcal{Q}| - 4|\mathcal{V}| - |\mathcal{E}| \\ &+ 3|\mathcal{V}_b| + |\mathcal{V}_c| + |\mathcal{E}^b| = -|\mathcal{V}_b| + |\mathcal{V}_c| + |\mathcal{E}^b| \\ =& |\mathcal{V}_c| > 0, \end{aligned}$$

since $|\mathcal{E}^b| = |\mathcal{V}_b|$. As a result, $(\Sigma_h^0, \mathbf{V}_h^0, W_h^0)$ does not form an exact sequence. Moreover, this result implies that the pressure space is larger than desired.

Therefore, we define the global finite element spaces as follows:

(i) $\Sigma_{h,0} = \Sigma_h \cap H_0^1(\Omega)$,

(ii) $\mathbf{V}_{h,0} = \{\mathbf{v}_h \in \mathbf{V}_h : (\mathbf{v}_h \cdot \mathbf{n})|_{\partial\Omega} = 0\}$,

(iii) $W_{h,0} = W_h \cap L_0^2(\Omega)$.

We note that $z \in \Sigma_h$ is in $\Sigma_{h,0}$ if and only if:

(i) $z(a) = 0, \quad \forall a \in \mathcal{V}_b$,

(ii) $\frac{\partial z}{\partial x}(a) = 0, \quad \forall a \in \mathcal{V}_{b,y} \cup \mathcal{V}_c$,

(iii) $\frac{\partial z}{\partial y}(a) = 0, \quad \forall a \in \mathcal{V}_{b,x} \cup \mathcal{V}_c$.

Thus, the number of constraints imposed on Σ_h is: $2|\mathcal{V}_b| + |\mathcal{V}_c|$.

As a result, $dim(\Sigma_{h,0}) = dim(\Sigma_h) - 2|\mathcal{V}_b| - |\mathcal{V}_c| = 4|\mathcal{V}| - 2|\mathcal{V}_b| - |\mathcal{V}_c|$.

Similarly, we note that $\mathbf{v}_h \in \mathbf{V}_h$ is in $\mathbf{V}_{h,0}$ if and only if:

(i) $v_1(a) = 0$, $\forall a \in \mathcal{V}_{b,x} \cup \mathcal{V}_c$,

(ii) $\int_e v_1 \ = 0$, $\forall e \in \mathcal{E}^b_x$,

(iii) $v_2(a) = 0$, $\forall a \in \mathcal{V}_{b,y} \cup \mathcal{V}_c$,

(iv) $\int_e v_2 \ = 0$, $\forall e \in \mathcal{E}^b_y$.

The number of constraints imposed on $\mathbf{V}_{h,0}$ is: $|\mathcal{V}_b| + |\mathcal{V}_c| + |\mathcal{E}^b|$.

Thus, $dim(\mathbf{V}_{h,0}) = dim(\mathbf{V}_h) - |\mathcal{V}_b| - |\mathcal{V}_c| - |\mathcal{E}^b| = 4|\mathcal{Q}| + 4|\mathcal{V}| + |\mathcal{E}| - |\mathcal{V}_b| - |\mathcal{V}_c| - |\mathcal{E}^b|$.

Lemma 13. *There holds $Ker(div(\mathbf{V}_{h,0})) = curl(\Sigma_{h,0})$.*

Proof. As in the proof of Lemma 12, the inclusion $curl(\Sigma_{h,0}) \subseteq Ker(div(\mathbf{V}_{h,0}))$ is trivial. Thus, it suffices to show that $Ker(div(\mathbf{V}_{h,0})) \subseteq curl(\Sigma_{h,0})$. Suppose $\mathbf{v} \in Ker(div(\mathbf{V}_{h,0}))$, then $\mathbf{v} \in \mathbf{V}_{h,0}$ such that $div(\mathbf{v}) = 0$. Moreover, there exists $z \in H^1(\Omega)$ such that $\mathbf{v} = curl(z)$. Since $\mathbf{v}|_{\mathcal{Q}} \in \mathbf{V}_h(\mathcal{Q})$, we have $z|_{\mathcal{Q}} \in \Sigma_h(\mathcal{Q})$. Furthermore, the continuity of the partial derivatives of $\mathbf{v} \in \mathbf{V}_h$ at the vertices implies that $\frac{\partial^2 z}{\partial x \partial y}$ is continuous at the vertices. The conditions (i) and (iii) imposed on $\mathbf{V}_h$ in the definition of $\mathbf{V}_{h,0}$ yield the conditions (ii) and (iii) imposed on Σ_h in the definition of $\Sigma_{h,0}$. In addition, the conditions (ii)–(iv) imposed on $\mathbf{V}_h$ imply that the condition (i) imposed on Σ_h holds. Therefore, $z \in \Sigma_{h,0}$, and this implies that $Ker(div(\mathbf{V}_{h,0})) \subseteq curl(\Sigma_{h,0})$. □

Remark 2. *By Lemma 3 and the Euler's formula [28],*

$$\begin{aligned} dim(W_{h,0}) = dim(W_h) - 1 &= |\mathcal{V}| + 5|\mathcal{Q}| - 1 \\ &= |\mathcal{V}| + 5|\mathcal{Q}| - (|\mathcal{V}| + |\mathcal{Q}| - |\mathcal{E}|) \\ &= 4|\mathcal{Q}| + |\mathcal{E}|. \end{aligned} \tag{2.3.2}$$

Theorem 6. *$div : \mathbf{V}_{h,0} \to W_{h,0}$ is a surjective map.*

Proof. Note that $div(\mathbf{V}_{h,0}) = \{r : r|_{\mathcal{Q}} \in P_{2,2}(\mathcal{Q}),\ div(\mathbf{v}) = r \text{ for } \mathbf{v} \in \mathbf{V}_{h,0}\}$. Let $r \in div(\mathbf{V}_{h,0})$. Then, there exists $\mathbf{v} \in \mathbf{V}_{h,0}$ such that $div(\mathbf{v}) = r$. Then, by the divergence theorem,

$$\int_\Omega r = \int_\Omega div(\mathbf{v}) = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \ = 0.$$

Moreover, r is continuous at the vertices since $\mathbf{v} \in \mathbf{V}_h$, therefore, $r \in W_{h,0}$. As a result, $div(\mathbf{V}_{h,0}) \subseteq W_{h,0}$. Furthermore, by the rank–nullity theorem and Lemma 13, we have

$$dim(div(\mathbf{V}_{h,0})) = dim(\mathbf{V}_{h,0}) - dim(Ker(div(\mathbf{V}_{h,0})) = dim(\mathbf{V}_{h,0}) - dim(\Sigma_{h,0})$$

$$= (4|\mathcal{Q}| + 4|\mathcal{V}| + |\mathcal{E}| - |\mathcal{V}_b| - |\mathcal{V}_c| - |\mathcal{E}^b|) - (4|\mathcal{V}| - 2|\mathcal{V}_b| - |\mathcal{V}_c|)$$
$$= 4|\mathcal{Q}| + |\mathcal{V}_b| - |\mathcal{E}^b| + |\mathcal{E}|.$$

Thus, $dim(div(\mathbf{V}_{h,0})) = 4|\mathcal{Q}| + |\mathcal{E}|$. From (2.3.2), it follows that $div(\mathbf{V}_{h,0}) = W_{h,0}$. □

Now, we define a discrete H^1–type norm on $\mathbf{V}_{h,0}$ by

$$\|\mathbf{v}\|_h^2 := \|\nabla \mathbf{v}\|_{L^2(\Omega)}^2 + \sum_{e\in\mathcal{E}^b} h_e \|\frac{\partial \mathbf{v}}{\partial n_e}\|_{L^2(e)}^2 + \sum_{e\in\mathcal{E}^b} \frac{1}{h_e}\|\mathbf{v}\|_{L^2(e)}^2. \tag{2.3.3}$$

Lemma 14. *The finite element pair $\mathbf{V}_{h,0} \times W_{h,0}$ satisfies the inf-sup stability condition*

$$\inf_{q\in W_{h,0}\setminus\{0\}} \sup_{\mathbf{v}\in\mathbf{V}_{h,0}\setminus\{\mathbf{0}\}} \frac{\int_\Omega q\ div(\mathbf{v})}{\|\mathbf{v}\|_h \|q\|_{L^2(\Omega)}} \geq c > 0,$$

where $c > 0$ is an h-independent constant.

Proof. Let $\mathbf{v}$, $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\boldsymbol{\omega}$ and $\mathbf{I}_h\boldsymbol{\omega}$ be defined as in the proof of Lemma 11, Corollary 1 and Theorem 5. Then, the proof is similar to that of Lemma (11). By the divergence theorem, we note that

$$\int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n} = \int_{\partial\Omega} \mathbf{v}^{(1)}\cdot\mathbf{n} + \int_{\partial\Omega} \mathbf{v}^{(2)}\cdot\mathbf{n} = \int_\Omega div(\mathbf{v}^{(1)}) + \int_\Omega div(\mathbf{v}^{(2)})$$
$$= \int_\Omega div(\mathbf{v}^{(1)}) + \int_\Omega (q - div(\mathbf{v}^{(1)})) = \int_\Omega q = 0,$$

since $q \in W_{h,0}$. Therefore, $\mathbf{v} \in \mathbf{V}_{h,0}$. By the triangle inequality and scaling, we have

$$\|\mathbf{v}\|_h^2 \leq c(\|\mathbf{v}^{(1)}\|_h^2 + \|\mathbf{v}^{(2)}\|_h^2). \tag{2.3.4}$$

Moreover, Corollary 1 and the equivalence of norms imply

$$\|\mathbf{v}^{(2)}\|_h \leq c\|\mathbf{v}^{(2)}\|_{H^1(\Omega)} \leq c(\|q\|_{L^2(\Omega)} + \|\mathbf{v}^{(1)}\|_{H^1(\Omega)}) \tag{2.3.5}$$

By using the following result of the trace inequality:

$$\sum_{e\in\mathcal{E}^b} h_e \|\frac{\partial \mathbf{v}}{\partial \mathbf{n}_e}\|_{L^2(e)}^2 \leq c\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2 \tag{2.3.6}$$

in the definition of the $\|\cdot\|_h$ norm given by (2.3.3), we derive a bound on the $\|\cdot\|_h$ norm

$$\|\mathbf{v}\|_h^2 \leq c(\|\mathbf{v}\|_{H^1(\Omega)}^2 + \sum_{e\in\mathcal{E}^b} \frac{1}{h_e}\|\mathbf{v}\|_{L^2(e)}^2).$$

This implies that $\|\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}\|_h^2 \leq c(\|\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}\|_{H^1(\Omega)}^2 + \sum_{e\in\mathcal{E}^b} \frac{1}{h_e}\|\mathbf{v}^{(1)} - \mathbf{I}_h\boldsymbol{\omega}\|_{L^2(e)}^2)$.
Then, by the same arguments used in the proof of Lemma 11, we deduce that

$$\|\mathbf{v}^{(1)}\|_h^2 \leq c\|\boldsymbol{\omega}\|_{H^1(\Omega)}^2 \leq c\|q\|_{L^2(\Omega)}^2. \tag{2.3.7}$$

Using (2.3.7) and (2.3.5) in (2.3.4) yields $\|\mathbf{v}\|_h \leq c\|q\|_{L^2(\Omega)}$. As a result, the inf–sup condition

$$\inf_{q\in W_{h,0}\setminus\{0\}} \sup_{\mathbf{v}\in \mathbf{V}_{h,0}\setminus\{\mathbf{0}\}} \frac{\int_\Omega q\ div(\mathbf{v})}{\|\mathbf{v}\|_h \|q\|_{L^2(\Omega)}} \geq c > 0$$

holds for an h-independent constant $c > 0$. □

2.4 NITSCHE'S METHOD AND THE CONVERGENCE ANALYSIS

In this section, we apply Nitsche's method [37] to prove the existence and the uniqueness of the numerical solution $(\mathbf{u}_h, p_h)$ for the discrete problem and analyze the approximation properties.

Recall the Stokes system

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad in\ \Omega, \tag{2.4.1a}$$

$$div(\mathbf{u}) = 0, \quad in\ \Omega, \tag{2.4.1b}$$

$$\mathbf{u} = \mathbf{0}, \quad on\ \partial\Omega. \tag{2.4.1c}$$

Multiplying (2.4.1a) by $\mathbf{v}_h \in \mathbf{V}_{h,0}$ and integrating over Ω, we obtain

$$\nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}_h \ - \nu\int_{\partial\Omega}\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\mathbf{v}_h \ - \int_\Omega p\ div(\mathbf{v}_h) \ + \int_{\partial\Omega} p\ (\mathbf{v}_h\cdot\mathbf{n}) \ = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h, \tag{2.4.2}$$

where $\mathbf{n}$ denotes the (outward) unit normal vector of $\partial\Omega$ and : denotes the Frobenius inner product which provides $\nabla\mathbf{u} : \nabla\mathbf{v}_h = \sum\limits_{i,j}(\nabla\mathbf{u})_{ij}(\nabla\mathbf{v}_h)_{ij}$.

Since $\mathbf{v}_h \in \mathbf{V}_{h,0}$, we have $(\mathbf{v}_h\cdot\mathbf{n})|_{\partial\Omega} = 0$, therefore, $\int_{\partial\Omega} p\ (\mathbf{v}_h\cdot\mathbf{n}) \ = 0$.

Let $\mathbf{t}$ denote the unit tangent vector to $\partial\Omega$. Then, we may write $\mathbf{v}_h = (\mathbf{v}_h\cdot\mathbf{n})\cdot\mathbf{n} + (\mathbf{v}_h\cdot\mathbf{t})\cdot\mathbf{t} = (\mathbf{v}_h\cdot\mathbf{t})\cdot\mathbf{t}$. Moreover, we may write

$$(\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\mathbf{v}_h)|_{\partial\Omega} = \frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot(\mathbf{v}_h\cdot\mathbf{t})\mathbf{t} = ((\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\mathbf{n})\mathbf{n} + (\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\mathbf{t})\mathbf{t})(\mathbf{v}_h\cdot\mathbf{t})\mathbf{t} = (\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\mathbf{t})(\mathbf{v}_h\cdot\mathbf{t}).$$

Thus, (2.4.2) becomes:

$$\nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}_h - \nu\int_{\partial\Omega}(\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\mathbf{t})(\mathbf{v}_h\cdot\mathbf{t}) \ - \int_\Omega p\ div(\mathbf{v}_h) \ = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h\,. \tag{2.4.3}$$

Let $b : \mathbf{H}^1(\Omega)\times L^2(\Omega) \to \mathbb{R}$ denote the bilinear form and $F : \mathbf{H}^1(\Omega) \to \mathbb{R}$ denote the bounded linear functional defined by the following expressions:

$$b(\mathbf{v}_h, p) = -\int_\Omega p\, div(\mathbf{v}_h),$$

$$F(\mathbf{v}_h) = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h.$$

Then (2.4.3) can be written as:

$$\nu\int_{\Omega}\nabla\mathbf{u}:\nabla\mathbf{v}_h\ -\nu\int_{\partial\Omega}(\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\mathbf{t})(\mathbf{v}_h\cdot\mathbf{t})\ +b(\mathbf{v}_h,p)=F(\mathbf{v}_h). \tag{2.4.4}$$

Since $\mathbf{u}|_{\partial\Omega=0}$, we may symmetrize and stabilize (2.4.4) as follows:

$$\nu\int_{\Omega}\nabla\mathbf{u}:\nabla\mathbf{v}_h\ -\nu\sum_{e\in\mathcal{E}^b}\int_e\big((\frac{\partial\mathbf{u}}{\partial\mathbf{n}_e}\cdot\mathbf{t})(\mathbf{v}_h\cdot\mathbf{t})+(\frac{\partial\mathbf{v}_h}{\partial\mathbf{n}_e}\cdot\mathbf{t})(\mathbf{u}\cdot\mathbf{t})-\frac{\sigma}{h_e}\mathbf{u}\cdot\mathbf{v}_h\big)+b(\mathbf{v}_h,q)=F(\mathbf{v}_h), \tag{2.4.5}$$

where σ is an h-independent penalization parameter.

Let $a_h:(\mathbf{H}^2(\Omega)+\mathbf{V}_{h,0})\times(\mathbf{H}^2(\Omega)+\mathbf{V}_{h,0})\to\mathbb{R}$ denote the bilinear form defined by

$$a_h(\mathbf{v},\boldsymbol{\omega})=\nu\Big(\int_{\Omega}\nabla\mathbf{v}:\nabla\boldsymbol{\omega}\ -\sum_{e\in\mathcal{E}^b}\int_e((\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\cdot\mathbf{t})(\boldsymbol{\omega}\cdot\mathbf{t})+(\frac{\partial\boldsymbol{\omega}}{\partial\mathbf{n}_e}\cdot\mathbf{t})(\mathbf{v}\cdot\mathbf{t})-\frac{\sigma}{h_e}\mathbf{v}\cdot\boldsymbol{\omega})\Big).$$

The weak formulation of (2.4.1) is given by

$$a_h(\mathbf{u},\mathbf{v}_h)+b(\mathbf{v}_h,p)=F(\mathbf{v}_h),\quad\forall\mathbf{v}_h\in\mathbf{H}_0^1(\Omega), \tag{2.4.6a}$$

$$b(\mathbf{u},q_h)=0,\qquad\qquad\forall q_h\in L_0^2(\Omega). \tag{2.4.6b}$$

The finite element method for (2.4.6) reads:

Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_{h,0}\times W_{h,0}$ that satisfies

$$a_h(\mathbf{u}_h,\mathbf{v}_h)+b(\mathbf{v}_h,p_h)=F(\mathbf{v}_h),\quad\forall\mathbf{v}_h\in\mathbf{V}_{h,0}, \tag{2.4.7a}$$

$$b(\mathbf{u}_h,q_h)=0,\qquad\qquad\forall q_h\in W_{h,0}. \tag{2.4.7b}$$

To prove the existence and uniqueness of the solution to (2.4.7), we need to show that $a_h(\cdot,\cdot)$ is continuous and coercive on $\mathbf{V}_{h,0}$.

Lemma 15. *With respect to the H^1–type norm given by (2.3.3), $a_h(\cdot,\cdot)$ is a continuous bilinear form on $(\mathbf{H}^2(\Omega)+\mathbf{V}_{h,0})$.*

Moreover, $a_h(\cdot,\cdot)$ is coercive on $\mathbf{V}_{h,0}$ provided σ is sufficiently large.

Proof. To prove the coercivity of $a_h(\cdot,\cdot)$ on $\mathbf{V}_{h,0}$, we need to show that $a_h(\mathbf{v},\mathbf{v})\geq c\nu\|\mathbf{v}\|_h^2$ holds for every $\mathbf{v}\in\mathbf{V}_{h,0}$. By the definition of $a_h(\cdot,\cdot)$, we have

$$a_h(\mathbf{v},\mathbf{v})=\nu\Big(\int_{\Omega}|\nabla\mathbf{v}|^2-2\sum_{e\in\mathcal{E}^b}\int_e\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\mathbf{v}+\sum_{e\in\mathcal{E}^b}\int_e\frac{\sigma}{h_e}|\mathbf{v}|^2\Big).$$

Using (2.3.6), for any $\epsilon>0$, we have

$$\begin{aligned}2|\sum_{e\in\mathcal{E}^b}\int_e\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\mathbf{v}|&\leq2\sum_{e\in\mathcal{E}^b}|\int_e(h_e^{-1/2}\mathbf{v})(h_e^{1/2}\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e})|\\&\leq2(\sum_{e\in\mathcal{E}^b}h_e^{-1}\|\mathbf{v}\|^2_{L^2(e)})^{1/2}(\sum_{e\in\mathcal{E}^b}h_e\|\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\|^2_{L^2(e)})^{1/2}\end{aligned}$$

$$\leq 2c(\sum_{e\in\mathcal{E}^b} h_e^{-1}\|\mathbf{v}\|^2_{L^2(e)})^{1/2}\|\nabla\mathbf{v}\|_{L^2(\Omega)}$$
$$\leq 2c(\frac{1}{2\epsilon}\sum_{e\in\mathcal{E}^b} h_e^{-1}\|\mathbf{v}\|^2_{L^2(e)} + \frac{\epsilon}{2}\|\nabla\mathbf{v}\|^2_{L^2(\Omega)})$$
$$= \frac{c}{\epsilon}\sum_{e\in\mathcal{E}^b} h_e^{-1}\|\mathbf{v}\|^2_{L^2(e)} + c\epsilon\|\nabla\mathbf{v}\|^2_{L^2(\Omega)}.$$

Applying this estimate to the definition of $a_h(\cdot,\cdot)$ yields

$$a_h(\mathbf{v},\mathbf{v}) \geq \nu\Big(\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + \sigma\sum_{e\in\mathcal{E}^b}\frac{1}{h_e}\|\mathbf{v}\|^2_{L^2(e)} - 2|\sum_{e\in\mathcal{E}^b}\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\mathbf{v}ds|\Big)$$
$$\geq \nu\Big((1-c\epsilon)\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + (\sigma - \frac{c}{\epsilon})\sum_{e\in\mathcal{E}^b}\frac{1}{h_e}\|\mathbf{v}\|^2_{L^2(e)}\Big).$$

If we choose $\epsilon = \frac{1}{2c}$, then we have $\sigma - \frac{c}{\epsilon} = \sigma - 2c^2$. Thus, for $\sigma \geq 4c^2$,

$$a_h(\mathbf{v},\mathbf{v}) \geq \nu\Big(\frac{1}{2}\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + 2c^2\sum_{e\in\mathcal{E}^b}\frac{1}{h_e}\|\mathbf{v}\|^2_{L^2(e)}\Big)$$
$$\geq \nu\min\{\frac{1}{2}, 2c^2\}(\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + \sum_{e\in\mathcal{E}^b}\frac{1}{h_e}\|\mathbf{v}\|^2_{L^2(e)})\Big) \geq c\nu\|\mathbf{v}\|^2_h.$$

As a result, $a_h(\cdot,\cdot)$ is coercive on $\mathbf{V}_{h,0}$ provided σ is sufficiently large.

To prove the continuity of $a_h(\cdot,\cdot)$ on $\mathbf{V}_{h,0} + \mathbf{H}^2(\Omega)$, we need to show that $|a_h(\mathbf{v},\boldsymbol{\omega})| \leq c\nu\|\mathbf{v}\|_h\|\boldsymbol{\omega}\|_h$ for all $\mathbf{v}$, $\boldsymbol{\omega} \in \mathbf{V}_{h,0} + \mathbf{H}^2(\Omega)$.

By the Cauchy-Schwarz inequality, for $\mathbf{v}$, $\boldsymbol{\omega} \in \mathbf{V}_{h,0} + \mathbf{H}^2(\Omega)$, we have

$$|a_h(\mathbf{v},\boldsymbol{\omega})|^2 \leq c\nu^2\Big(|\int_\Omega \nabla\boldsymbol{\omega} : \nabla\mathbf{v}|^2 + \sum_{e\in\mathcal{E}^b}|\int_e(\frac{\partial\boldsymbol{\omega}}{\partial\mathbf{n}_e}\mathbf{v} + \frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\boldsymbol{\omega} + \frac{\sigma}{h_e}\boldsymbol{\omega}\cdot\mathbf{v})|^2\Big)$$
$$\leq c\nu^2\Big(\|\nabla\mathbf{v}\|^2_{L^2(\Omega)}\|\nabla\boldsymbol{\omega}\|^2_{L^2(\Omega)} + \sum_{e\in\mathcal{E}^b}|\int_e\frac{\partial\boldsymbol{\omega}}{\partial\mathbf{n}_e}\mathbf{v}|^2 + \sum_{e\in\mathcal{E}^b}|\int_e\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\boldsymbol{\omega}|^2$$
$$+ \sum_{e\in\mathcal{E}^b}|\int_e\frac{\sigma}{h_e}\boldsymbol{\omega}\cdot\mathbf{v})|^2\Big)$$
$$\leq c\nu^2\Big((\|\nabla\mathbf{v}\|^2_{L^2(\Omega)}\|\nabla\boldsymbol{\omega}\|^2_{L^2(\Omega)} + \sum_{e\in\mathcal{E}^b}(h_e\|\frac{\partial\boldsymbol{\omega}}{\partial\mathbf{n}_e}\|^2_{L^2(e)})(\frac{1}{h_e}\|\mathbf{v}\|^2_{L^2(e)})$$
$$+ \sum_{e\in\mathcal{E}^b}(h_e\|\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\|^2_{L^2(e)})(\frac{1}{h_e}\|\boldsymbol{\omega}\|^2_{L^2(e)}) + \sum_{e\in\mathcal{E}^b}|\frac{\sigma}{h_e}|^2\|\mathbf{w}\|^2_{L^2(e)}\|\mathbf{v}\|^2_{L^2(e)})\Big)$$
$$\leq c\nu^2(\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + \sum_{e\in\mathcal{E}^b}h_e\|\frac{\partial\mathbf{v}}{\partial\mathbf{n}_e}\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}^b}\frac{1}{h_e}\|\mathbf{v}\|^2_{L^2(e)})(\|\nabla\boldsymbol{\omega}\|^2_{L^2(\Omega)}$$
$$+ \sum_{e\in\mathcal{E}^b}h_e\|\frac{\partial\boldsymbol{\omega}}{\partial\mathbf{n}_e}\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}^b}\frac{1}{h_e}\|\boldsymbol{\omega}\|^2_{L^2(e)})$$
$$= c\nu^2\|\mathbf{v}\|^2_h\|\boldsymbol{\omega}\|^2_h.$$

Thus, $a_h(\cdot,\cdot)$ is continuous on $\mathbf{V}_{h,0} + \mathbf{H}^2(\Omega)$. □

The equations (2.4.6b) and (2.4.7b) suggest that the systems of equations given by (2.4.6) and (2.4.7) can be reduced by utilizing the continuous and discrete kernels of the bilinear form $b(\cdot,\cdot)$. The continuous and discrete kernels of $b(\cdot,\cdot)$ are given by the following sets $\mathbf{Z}$ and $\mathbf{Z}_h$:

$$\mathbf{Z} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : b(\mathbf{v}, q) = 0,\ (\mathbf{v}\cdot\mathbf{n})|_{\partial\Omega} = 0,\quad \forall q \in L^2_0(\Omega)\},$$

$$\mathbf{Z}_h = \{\mathbf{v}_h \in \mathbf{V}_{h,0} : b(\mathbf{v}_h, q_h) = 0,\quad \forall q_h \in W_{h,0}\}.$$

Therefore, the system given by (2.4.7) reduces to the system:

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = F(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{Z}_h. \tag{2.4.8}$$

Since $a_h(\cdot,\cdot)$ is a symmetric, coercive and continuous bilinear form on $\mathbf{Z}_h$, and F is a continuous linear form, by the Lax-Milgram theorem [10], there exists a unique solution $\mathbf{u}_h \in \mathbf{Z}_h$ satisfying (2.4.8).

Suppose $\mathbf{u}$ is the velocity solution of the system given by (2.4.1). Then, since $\mathbf{Z}_h \subset \mathbf{Z}$, for all $\mathbf{v}_h \in \mathbf{Z}_h$, we have $a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0$. Furthermore, by the coercivity and the continuity of $a_h(\cdot,\cdot)$ on $\mathbf{Z}_h$, for arbitrary $\mathbf{v}_h \in \mathbf{Z}_h$, we have

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_h^2 &\le a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) = a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h)\\ &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) \le c\|\mathbf{u} - \mathbf{u}_h\|_h \|\mathbf{u} - \mathbf{v}_h\|_h.\end{aligned}$$

This implies that $\|\mathbf{u} - \mathbf{u}_h\|_h \le c\|\mathbf{u} - \mathbf{v}_h\|_h$ for all $\mathbf{v}_h \in \mathbf{Z}_h$. Therefore,

$$\|\mathbf{u} - \mathbf{u}_h\|_h \le c \inf_{\mathbf{v}_h \in \mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}_h\|_h. \tag{2.4.9}$$

From Lemma 14, it follows that [13],

$$\|\mathbf{u} - \mathbf{u}_h\|_h \le c \inf_{\mathbf{v}_h \in \mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}_h\|_h \le c \inf_{\mathbf{v}_h \in \mathbf{V}_{h,0}} \|\mathbf{u} - \mathbf{v}_h\|_h. \tag{2.4.10}$$

Lemma 16. *Suppose that the pair $(\mathbf{u}, p)$ that solves (2.4.6) satisfies the regularity condition $\mathbf{u} \in \mathbf{H}^3(\Omega)$ and $p \in H^2(\Omega)$, and let $(\mathbf{u}_h, p_h)$ be the solution of the discrete problem (2.4.7). Then, there holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \le ch^2\|\mathbf{u}\|_{H^3(\Omega)}, \quad \|p - p_h\|_{L^2(\Omega)} \le ch^2(\ \|p\|_{H^2(\Omega)} + \nu\|\mathbf{u}\|_{H^3(\Omega)}),$$

where $\|\cdot\|_h$ is the H^1–type norm given by (2.3.3).

Proof. Let $\mathbf{\Pi}_h : \mathbf{H}^3(\Omega) \to \mathbf{V}_{h,0}(\Omega)$ be the nodal interpolant and $P_h : L^2(\Omega) \to W_{h,0}$ be the

L^2–projection operator. Then, the following holds [10].

$$\|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\|_{H^\ell(\Omega)} \leq ch^{3-\ell} |\mathbf{u}|_{H^3(\Omega)}, \qquad 0 \leq \ell \leq 3, \tag{2.4.11}$$

$$\int_\Omega P_h p \; q \; = \int_\Omega p \; q \; , \qquad \forall q \in W_{h,0}, \tag{2.4.12}$$

$$\|p - P_h p\|_{L^2(\Omega)} \leq ch^s \|p\|_{H^s(\Omega)}, \qquad 1 \leq s \leq 2. \tag{2.4.13}$$

By the definition of the H^1–type norm given by (2.3.3), (2.3.6), the trace inequalities and (2.4.11), we obtain

$$\begin{aligned}
\|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\|_h^2 =& \|\nabla(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u})\|_{L^2(\Omega)}^2 + \sum_{e \in \mathcal{E}^b} h_e \|\frac{\partial(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u})}{\partial \mathbf{n}_e}\|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}^b} h_e^{-1} \|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\|_{L^2(e)}^2 \\
\leq& \|\nabla(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u})\|_{L^2(\Omega)}^2 + c\|\nabla(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u})\|_{L^2(\Omega)} \\
&+ \sum_{e \in \mathcal{E}^b} h_e^{-1} (h^{-1} \|(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u})\|_{L^2(\Omega)}^2 + h|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}|_{H^1(\Omega)}^2) \\
\leq& ch^4 \|\mathbf{u}\|_{H^3(\Omega)}^2,
\end{aligned}$$

Thus, we have

$$\|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\|_h \leq ch^2 \|\mathbf{u}\|_{H^3(\Omega)}.$$

Since $\mathbf{\Pi}_h \mathbf{u} \in \mathbf{V}_{h,0}$, (2.4.10) implies that the error in the velocity approximation of this scheme satisfies

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq ch^2 \|\mathbf{u}\|_{H^3(\Omega)}. \tag{2.4.14}$$

On the other hand, by the triangle inequality and (2.4.13), we have

$$\begin{aligned}
\|p - p_h\|_{L^2(\Omega)} &\leq \|p - P_h p\|_{L^2(\Omega)} + \|P_h p - p_h\|_{L^2(\Omega)} \\
&\leq ch^s \|p\|_{H^s(\Omega)} + \|P_h p - p_h\|_{L^2(\Omega)}, \quad 1 \leq s \leq 2.
\end{aligned} \tag{2.4.15}$$

Then, Lemma 14 yields

$$c\|p_h - P_h p\|_{L^2(\Omega)} \leq \sup_{\mathbf{v}_h \in \mathbf{V}_{h,0} \backslash \{\mathbf{0}\}} \frac{b(\mathbf{v}_h, p_h - P_h p)}{\|\mathbf{v}_h\|_h}. \tag{2.4.16}$$

Since $div(\mathbf{v}_h) \in W_{h,0}$, (2.4.12) and (2.4.6a) imply

$$a_h(\mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, P_h p) = F(\mathbf{v}_h). \tag{2.4.17}$$

Subtracting (2.4.7a) from (2.4.17) yields $a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, P_h p - p_h) = 0$.

Thus, $b(\mathbf{v}_h, P_h p - p_h) = -a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)$.

By the continuity of $a_h(\cdot, \cdot)$ in the H^1–type norm, we obtain

$$|b(\mathbf{v}_h, P_h p - p_h)| = |a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)| \leq \nu \|\mathbf{u} - \mathbf{u}_h\|_h \|\mathbf{v}_h\|_h. \tag{2.4.18}$$

Using (2.4.18) in (2.4.16), by (2.4.14), we derive the following result:

$$\|P_h p - p_h\|_{L^2(\Omega)} \leq \frac{c\nu\|\mathbf{u} - \mathbf{u}_h\|_h \|\mathbf{v}_h\|_h}{\|\mathbf{v}_h\|_h} \leq c\nu\|\mathbf{u} - \mathbf{u}_h\|_h \leq c\nu h^2\|\mathbf{u}\|_{H^3(\Omega)}$$

Consequently, (2.4.15) with $s = 2$ yields $\|p - p_h\|_{L^2(\Omega)} \leq ch^2(\,\|p\|_{H^2(\Omega)} + \nu\|\mathbf{u}\|_{H^3(\Omega)})$. □

2.5 NUMERICAL EXPERIMENTS

In this section, we perform some numerical experiments on the domain $\Omega = (0,1)^2$. We assume that the viscosity constant $\nu = 1$.

2.5.1 Experiment 1

In this experiment, the exact solutions are given by

$$\mathbf{u}(x,y) = \begin{pmatrix} 2x^2y(x-1)^2(y-1)^2 + x^2y^2(2y-2)(x-1)^2 \\ -2xy^2(x-1)^2(y-1)^2 - x^2y^2(2x-2)(y-1)^2 \end{pmatrix},$$

$$p = 0.$$

2.5.2 Experiment 2

This experiment is a modified version of Experiment 1. The exact solutions are given by

$$\mathbf{u}(x,y) = \begin{pmatrix} 2x^2y(x-1)^2(y-1)^2 + x^2y^2(2y-2)(x-1)^2 \\ -2xy^2(x-1)^2(y-1)^2 - x^2y^2(2x-2)(y-1)^2 \end{pmatrix},$$

$$p = x - y.$$

Table 2.1: Experiment 1, Convergence results on rectangular meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$3.44e-05$		$1.13e-03$		$1.77e-05$	2.7491
1/16	$4.48e-06$	2.9408	$2.79e-04$	2.0180	$2.50e-06$	2.8237
1/32	$5.70e-07$	2.9745	$6.97e-05$	2.0010	$3.35e-07$	2.8997
1/64	$7.19e-08$	2.9867	$1.74e-05$	2.0017	$4.34e-08$	2.9489

Table 2.2: Experiment 2, Convergence results on rectangular meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/4	$1.5166e-04$		$4.5308e-03$		$1.3212e-03$	
1/8	$1.8236e-05$	3.0560	$1.1174e-03$	2.0196	$3.7532e-04$	1.8157
1/16	$2.2548e-06$	3.0157	$2.7862e-04$	2.0038	$7.2013e-05$	2.3818
1/32	$2.8105e-07$	3.0041	$6.9613e-05$	2.0009	$1.3184e-05$	2.4495

2.5.3 Discussion

Numerical experiments show that the velocity solution has second-order convergence in H^1–norm as the theory suggests (See Lemma 16). However, the pressure solutions exhibit superconvergence (See Table 2.1, Table 2.2).

For the case with mesh size $h = \frac{1}{32}$, the discrete solutions for the Experiment 2 are illustrated in Figures 2.2–2.4.

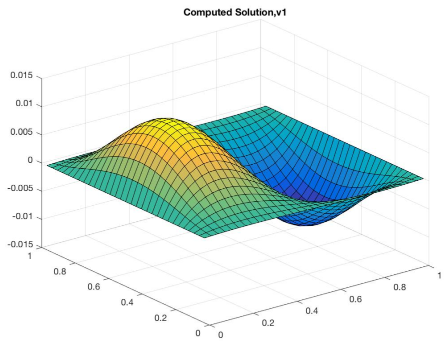


Figure 2.2: Velocity solution, u_1.

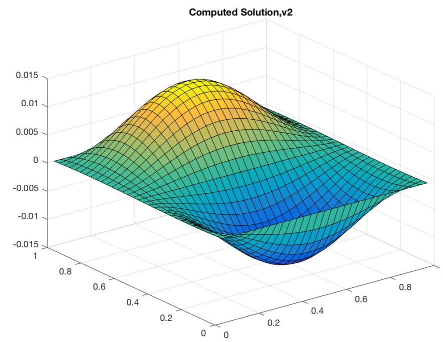


Figure 2.3: Velocity solution, u_2.

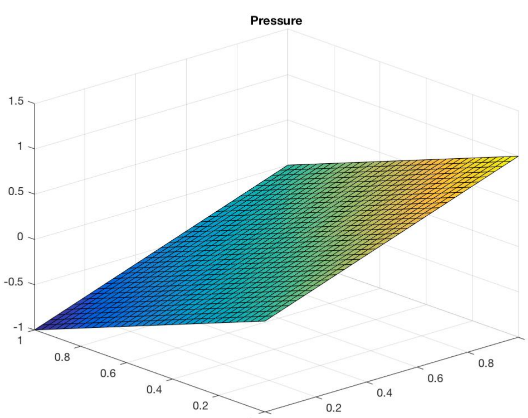


Figure 2.4: Pressure solution.

3.0 FEM FOR THE STOKES PROBLEM ON CUBICAL GRIDS

In this chapter, we extend the method we introduced in Chapter 2 to n–dimensions. We assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded, simply–connected domain with boundary parallel to the coordinate axes, and define a conforming cubical partition $\mathcal{Q}_h$ of Ω. We further assume that every element $\mathcal{Q} \in \mathcal{Q}_h$ has boundaries parallel to the coordinate axes. For $\mathcal{Q} \in \mathcal{Q}_h$, $h_{\mathcal{Q}}$ denotes its diameter and $\square_s(\mathcal{Q})$ stands for the set of its s-dimensional faces with $(n-s)$ coordinates equal to one of two constant values. We denote the subset of $\square_s(\mathcal{Q})$ with x_i constant by $\square_s^i(\mathcal{Q})$ and set $\hat{\square}_s^i(\mathcal{Q}) := \square_s(\mathcal{Q}) \setminus \square_s^i(\mathcal{Q})$.

We write the space of polynomials on a domain $D \subset \mathbb{R}^n$ of degree less than or equal to k_i in x_i as $P_{\vec{k}}(D) := P_{k_1,k_2,\ldots,k_n}(D)$ and set $\Lambda_k(D) := P_{\vec{k}}(D)$ with $k_i = k$ for all i. Then, we define the vector–valued space

$$\mathbf{\Lambda}_k^-(\mathcal{Q}) := \{\mathbf{v} \in (\Lambda_k(\mathcal{Q}))^n : v^{(i)} \in P_{\vec{k}}(\mathcal{Q}),\ k_i = k,\ k_j = k-1 \text{ for } i \neq j\}. \tag{3.0.1}$$

For instance, for $n = 3$, $\mathbf{\Lambda}_k^-(\mathcal{Q}) = P_{k,k-1,k-1}(\mathcal{Q}) \times P_{k-1,k,k-1}(\mathcal{Q}) \times P_{k-1,k-1,k}(\mathcal{Q})$. Note that for $n = 2$ and $k = 3$, $\mathbf{\Lambda}_3^-(\mathcal{Q}) = P_{3,2}(\mathcal{Q}) \times P_{2,3}(\mathcal{Q})$ which is the polynomial space forming the local velocity space defined in Section 2.1.

For $\mathcal{Q} \in \mathcal{Q}_h$ with face $S \in \square_s(\mathcal{Q})$ for $1 \leq s \leq n$, b_S denotes the bubble function with respect to S. We note that $\nabla b_S \neq \mathbf{0}$ on ∂S, however, ∇b_S vanishes on $(s-2)$–dimensional subfaces of S.

Lemma 17 ([2])**.** *A function $q \in \Lambda_k(\mathcal{Q})$ is uniquely determined by*

$$\{\int_S q\ \lambda :\ \lambda \in \Lambda_{k-2}(S), \quad S \in \square_s(\mathcal{Q}), \quad s = 0, 1, \ldots, n\},$$

where $\int_S q$ with $S \in \square_0(\mathcal{Q})$ stands for the value of q at the vertex S.

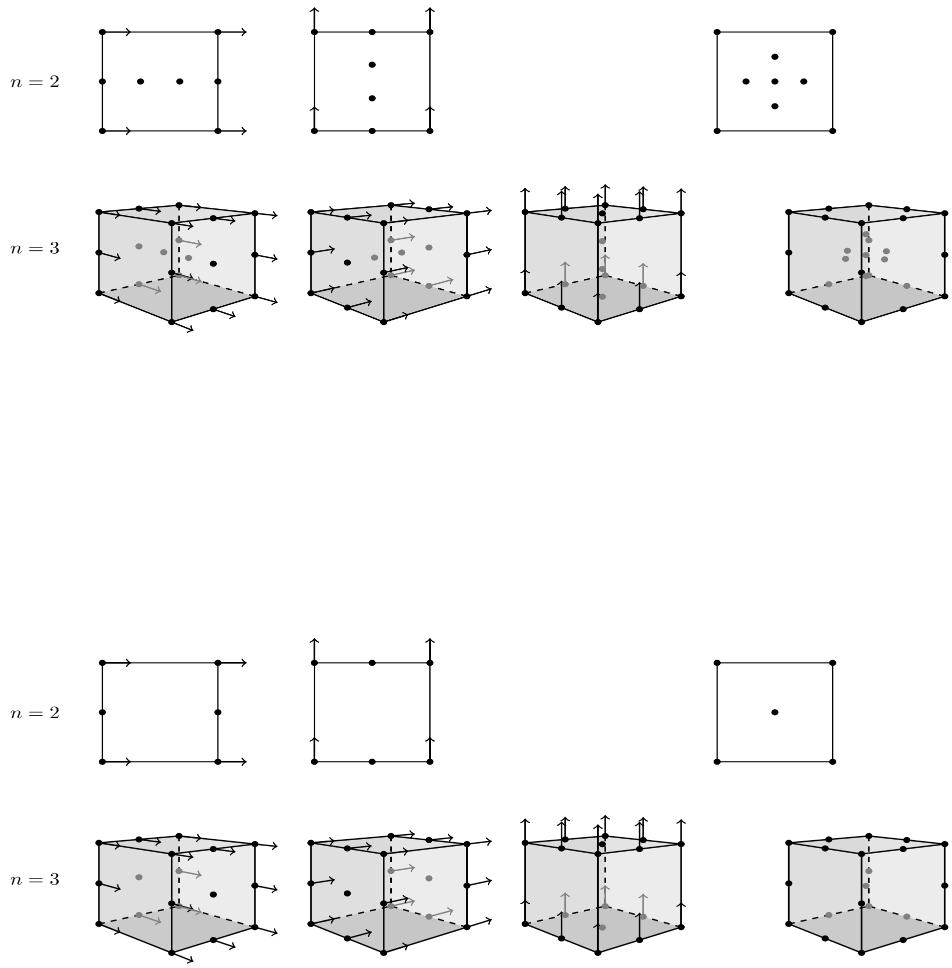


Figure 3.1: Degrees of freedom on the cubical mesh.
Degrees of freedom of the velocity (left) and pressure (right) elements in two and three dimensions (Top half). Degrees of freedom of the reduced velocity (left) and reduced pressure (right) elements in two and three dimensions (Bottom half).
Solid circles indicate function evaluations and the lines indicate directional derivatives.

3.1 THE LOCAL FINITE ELEMENT SPACES

In this section, we define the local velocity and pressure finite elements. Additionally, we derive a characterization of the divergence operator acting on the local velocity space which is an essential tool in the stability analysis of the global spaces.

Lemma 18. *Suppose* $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)}) \in \mathbf{\Lambda}_3^-(\mathcal{Q})$ *satisfies*

$$\int_S v^{(i)} = 0, \quad \int_S \frac{\partial v^{(i)}}{\partial x_i} = 0, \qquad S \in \square_s^{(i)}(\mathcal{Q}), \quad s = 0, 1, \ldots, m, \tag{3.1.1}$$

for all $1 \leq i \leq n$ *and for some* $0 \leq m \leq n-2$. *Then,* $\mathbf{v} = \mathbf{0}$, $\frac{\partial v^{(i)}}{\partial x_i} = 0$ *on* $\square_s(\mathcal{Q})$ *for all* $1 \leq i \leq n$ *and* $0 \leq s \leq m$.

Also, if

$$\int_S v^{(i)} = 0, \qquad S \in \square_{m+1}^{(i)}(\mathcal{Q}),$$

then $\mathbf{v} = \mathbf{0}$ *on* $\square_{m+1}(\mathcal{Q})$.

Proof. The proof is by mathematical induction on m. Note that $\square_0^{(i)}(\mathcal{Q}) = \square_0(\mathcal{Q})$ for all $1 \leq i \leq n$, therefore, the case $m = 0$ is trivial.

Assume that $\mathbf{v} = \mathbf{0}$ and $\frac{\partial v^{(i)}}{\partial x_i} = 0$ on all $S \in \square_m(\mathcal{Q})$ for some $m \in \{0, 1, \ldots, n-3\}$ and all $1 \leq i \leq n$. Let $S \in \square_{m+1}(\mathcal{Q})$. Then, by the induction hypothesis, we have $\mathbf{v} = \mathbf{0}$ and $\frac{\partial v^{(i)}}{\partial x_i} = 0$ on ∂S for all $1 \leq i \leq n$. If x_j is constant on S, then by definition, $S \in \square_{m+1}^{(j)}(\mathcal{Q})$ and $v^{(j)}|_S$, $\frac{\partial v^{(j)}}{\partial x_j}|_S \in \Lambda_2(S)$. Therefore, we may write $v^{(j)}|_S = b_S q^{(j)}$ and $\frac{\partial v^{(j)}}{\partial x_j}|_S = b_S p^{(j)}$ for some $q^{(j)}, p^{(j)} \in \mathbb{R}$, and (3.1.1) implies that $v^{(j)} = \frac{\partial v^{(j)}}{\partial x_j} = 0$ on S.

If $S \in \widehat{\square}_{m+1}^{(j)}(\mathcal{Q})$, then there exist exactly two m-dimensional faces $S^{(1)}, S^{(2)} \subset \partial S$ on which x_j is constant. We denote by ∇_S and $\nabla_{S^{(i)}}$ the surface gradient of S and $S^{(i)}$, respectively. Then, we have $\nabla_S(\cdot) = (\nabla_{S^{(i)}}(\cdot), \frac{\partial(\cdot)}{\partial x_j})$. Therefore, since $v^{(j)}$ and $\frac{\partial v^{(j)}}{\partial x_j}$ vanish on $S^{(1)}$ and $S^{(2)}$, we have $\nabla_S v^{(j)} = 0$ on $S^{(1)}$ and $S^{(2)}$. In addition, since $v^{(j)}|_S \in \Lambda_3(S)$, and $v^{(j)}|_{\partial S} = 0$, we may write $v^{(j)}|_S = b_S q$ for some $q \in \Lambda_1(S)$. As a result,

$$0 = \nabla_S v^{(j)}|_{S^{(i)}} = q \nabla_S b_S|_{S^{(i)}}. \tag{3.1.2}$$

Therefore, $q = 0$ on $S^{(i)}$, and since $q \in \Lambda_1(S)$, $q = 0$ in S. Thus, $v^{(j)} = 0$ and $\frac{\partial v^{(j)}}{\partial x_j} = 0$ on S. Thus, the proof of the first assertion follows. The same arguments can be used to prove the second assertion. □

3.1.1 The Velocity Space

Lemma 19. *A unisolvent set of degrees of freedom of the local velocity space* $\mathbf{\Lambda}_3^-(\mathcal{Q})$ *are given by*

$$S_v = \{\int_S v^{(i)}, \int_S \frac{\partial v^{(i)}}{\partial x_i} : \ S \in \square_s^{(i)}(\mathcal{Q}), \quad \int_S v^{(i)} : \ S \in \square_{n-1}^{(i)}(\mathcal{Q}), \quad \int_{\mathcal{Q}} \mathbf{v} \cdot \boldsymbol{\kappa} : \ \boldsymbol{\kappa} \in \mathbf{\Lambda}_1^-(\mathcal{Q})\},$$

where $0 \le s \le n-2$ *and* $1 \le i \le n$ *(See Figure 3.1).*

Proof. The cardinality of S_v is

$$\begin{aligned} 2n \sum_{s=0}^{n-2} |\square_s^{(i)}(\mathcal{Q})| + n|\square_{n-1}^{(i)}(\mathcal{Q})| + dim(\mathbf{\Lambda}_1^-(\mathcal{Q})) &= 2n \sum_{s=0}^{n-2} 2^{n-s} \binom{n-1}{s} + 2n + 2n \\ &= -4n + 4n \sum_{s=0}^{n-1} 2^{n-s-1} \binom{n-1}{s} + 4n \\ &= 4n3^{n-1} = \mathbf{\Lambda}_3^-(\mathcal{Q}), \end{aligned}$$

by the binomial formula. Thus, it suffices to show that $\mathbf{v}$ nullifies S_v if and only if $\mathbf{v} = \mathbf{0}$. Suppose that $\mathbf{v}$ vanishes on S_v. Then, $\frac{\partial v^{(i)}}{\partial x_i} = 0$ on $\square_{n-2}(\mathcal{Q})$ and $\mathbf{v} = \mathbf{0}$ on $\square_{n-1}(\mathcal{Q})$ by Lemma 18. If we denote by $b_{\mathcal{Q}}$ the bubble function of $\square_n(\mathcal{Q}) = \{\mathcal{Q}\}$, then we may write $\mathbf{v} = b_{\mathcal{Q}}\mathbf{q}$ for some $\mathbf{q} \in \mathbf{\Lambda}_1^-(\mathcal{Q})$. After we set $\boldsymbol{\kappa} = \mathbf{q}$ in $\int_{\mathcal{Q}} \mathbf{v} \cdot \boldsymbol{\kappa} = 0$, we deduce that $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$. □

Remark 3. *For* $S \in \square_s(\mathcal{Q})$, *we define an orthonormal set of vectors* $\{\mathbf{n}_S^{(j)}\}_{j=1}^{n-s}$ *orthogonal to the tangent space of* S. *Then, we may redefine the set of degrees of freedom as follows:*

$$S_v^\star = \{\int_S \mathbf{v} \cdot \mathbf{n}_S^{(j)}, \int_S \frac{\partial \mathbf{v}}{\partial \mathbf{n}_S^{(j)}} \cdot \mathbf{n}_S^{(j)} : \ S \in \square_s(\mathcal{Q}), \tag{3.1.3a}$$

$$\int_S \mathbf{v} \cdot \mathbf{n}_S : \quad S \in \square_{n-1}(\mathcal{Q}), \tag{3.1.3b}$$

$$\int_{\mathcal{Q}} \mathbf{v} \cdot \boldsymbol{\kappa} : \ \boldsymbol{\kappa} \in \mathbf{\Lambda}_1^-(\mathcal{Q})\}, \tag{3.1.3c}$$

where $0 \le s \le n-2$ *and* $1 \le j \le n-s$.

3.1.2 The Pressure Space

We form the local pressure space by the tensor–product quadratic polynomials $\Lambda_2(\mathcal{Q})$. By Lemma 17, any function $q \in \Lambda_2(\mathcal{Q})$ is uniquely determined by the set

$$\{\int_S q : \ S \in \square_s(\mathcal{Q}), \quad s = 0, 1, \ldots, n\}.$$

We define a subspace of $\Lambda_2(\mathcal{Q})$ by

$$\breve{\Lambda}_2(\mathcal{Q}) := \{q \in \Lambda_2(\mathcal{Q}) : q = 0 \text{ on all } (n-2) \text{ dimensional faces of } \mathcal{Q}\}. \tag{3.1.4}$$

Remark 4. (3.1.4) *implies that a function $q \in \breve{\Lambda}_2(\mathcal{Q})$ is uniquely determined by its average over each $(n-1)$ dimensional face and its average over $\mathcal{Q}$, therefore, dim$(\breve{\Lambda}_2(\mathcal{Q})) = (2n+1)$.*

Lemma 20 (Lemma 3.4 in [35])**.** *Any function $q \in \Lambda_2(\mathcal{Q})$ is uniquely determined by the set of values*

$$\{ \int_S q : \ S \in \square_s(\mathcal{Q}), \ s = 0, 1, \dots, (n-2), \quad \int_{\mathcal{Q}} q \ \lambda : \ \lambda \in \breve{\Lambda}_2(\mathcal{Q})\}.$$

3.1.3 A Local Characterization of the Divergence Operator

Lemma 21. *Let $\mathbf{v} \in \mathbf{\Lambda}_3^-(\mathcal{Q})$. If $div(\mathbf{v}) = 0$ and $\mathbf{v}|_{\partial\mathcal{Q}} = 0$, then $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$.*

Proof. Let $\bar{v}$ denote the $(n-1)$-form with the vector proxy $\mathbf{v} = (v^{(1)}, v^{(2)}, \dots, v^{(n)})$, i.e.,

$$\bar{v} = \sum_{i=1}^{n} v^{(i)} dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots dx^n,$$

where the *hat* indicates a suppressed argument. Moreover, for a vector-valued function $\boldsymbol{\kappa}$, denote by $\underline{\kappa}$ the one-form given by

$$\underline{\kappa} = \sum_{i=1}^{n} \kappa^{(i)} dx^i.$$

We may state the divergence–free condition on $\mathbf{v}$ as $d\bar{v} = 0$, where d denotes the exterior differentiation operator. Additionally, the boundary condition $\mathbf{v}|_{\partial\mathcal{Q}} = 0$ implies that $\bar{v}|_{\partial\mathcal{Q}} = 0$. Moreover, by the exactness of the complex (1.3.3), we deduce that there exists $\phi \in \mathring{H}\Lambda^{n-2}(\mathcal{Q})$ such that $\bar{v} = d\phi$ [4, 5], where $\mathring{H}\Lambda^{n-2}(\mathcal{Q})$ denotes the space of $L^2(\mathcal{Q})$ $(n-2)$-forms with exterior derivative in $L^2(\mathcal{Q})$ and vanishing trace. By the Stokes Theorem, for any $\boldsymbol{\kappa} \in \mathbf{\Lambda}_1^-(\mathcal{Q})$, we have

$$\int_{\mathcal{Q}} \mathbf{v} \cdot \boldsymbol{\kappa} = \int_{\mathcal{Q}} \bar{v} \wedge \underline{\kappa} = \int_{\mathcal{Q}} d\phi \wedge \underline{\kappa} = (-1)^{n-1} \int_{\mathcal{Q}} \phi \wedge d\underline{\kappa} = 0. \tag{3.1.5}$$

since $d\underline{\kappa} = 0$ for $\boldsymbol{\kappa} \in \mathbf{\Lambda}_1^-(\mathcal{Q})$ as shown by the following:

$$d\underline{\kappa} = \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial \kappa^{(j)}}{\partial x_k} dx^k \wedge dx^j = \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\partial \kappa^{(j)}}{\partial x_k} dx^k \wedge dx^j = 0.$$

Thus, (3.1.5) and Lemma 19 imply that $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$. □

Theorem 7. *Define*

$$\mathbf{V}_{h,0}(\mathcal{Q}) \ = \mathbf{\Lambda}_3^-(\mathcal{Q}) \cap \mathbf{H}_0^1(\mathcal{Q}), \qquad W_{h,0}(\mathcal{Q}) \ = \breve{\Lambda}_2(\mathcal{Q}) \cap L_0^2(\mathcal{Q})$$

Then, $div : \mathbf{V}_{h,0}(\mathcal{Q}) \to W_{h,0}(\mathcal{Q})$ is bijective. Moreover,

$$\|\mathbf{v}\|_{H^1(\mathcal{Q})} \leq c\|q\|_{L^2(\mathcal{Q})},$$

where $c > 0$ is a constant independent of $h_{\mathcal{Q}}$.

Therefore, $\mathbf{V}_{h,0}(\mathcal{Q}) \times W_{h,0}(\mathcal{Q})$ forms a locally inf–sup stable pair.

Proof. The definitions of $\mathbf{V}_{h,0}(\mathcal{Q})$ and $W_{h,0}(\mathcal{Q})$ imply that $div(\mathbf{V}_{h,0}(\mathcal{Q})) \subseteq W_{h,0}(\mathcal{Q})$. Therefore, it suffices to show that $dim(div(\mathbf{V}_{h,0}(\mathcal{Q}))) = dim(W_{h,0}(\mathcal{Q}))$ to prove the bijectivity of the divergence operator. By Lemma 21, $dim(Ker(div(\mathbf{V}_{h,0}(\mathcal{Q})))) = \mathbf{0}$.

Then, by the rank nullity theorem, we have

$$dim(div(\mathbf{V}_{h,0}(\mathcal{Q}))) = dim(\mathbf{V}_{h,0}(\mathcal{Q})) = dim(\mathbf{\Lambda}_1^-(\mathcal{Q})) = 2n$$

On the other hand, by Remark 4 we have $dim(W_{h,0}(\mathcal{Q})) = dim(\breve{\Lambda}_2(\mathcal{Q})) - 1 = 2n$. Thus, $dim(div(\mathbf{V}_{h,0}(\mathcal{Q}))) = dim(W_{h,0}(\mathcal{Q}))$. Therefore, for every $\mathbf{v} \in \mathbf{V}_{h,0}$, there exists a unique $q \in W_{h,0}$ that satisfies $div(\mathbf{v}) = q$.

Now, we define an affine transformation $\mathcal{F} : \hat{\mathcal{Q}} \to \mathcal{Q}$ by $\mathcal{F}(\hat{x}) = \mathbf{B}\hat{x} + \mathbf{b}$, where $\hat{\mathcal{Q}} = (0,1)^n$ is the reference element, $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ is a diagonal matrix with entries proportional to $h_{\mathcal{Q}}$. We define the velocity functions $\hat{\mathbf{v}} : \hat{\mathcal{Q}} \to \mathbb{R}^n$ on the reference element by the Piola transform $\hat{\mathbf{v}} := \mathbf{B}^{-1}(\mathbf{v} \circ \mathcal{F})det(\mathbf{B})$ [33]. This definition yields $\widehat{div}(\hat{\mathbf{v}}) = div(\mathbf{v} \circ \mathcal{F})det(\mathbf{B})$. Then, by Lemma 21, scaling and the equivalence of norms in finite dimensions, we deduce that

$$\|\mathbf{v}\|_{H^1(\mathcal{Q})} \leq ch_{\mathcal{Q}}^{-\frac{n}{2}}\|\hat{\mathbf{v}}\|_{H^1(\hat{\mathcal{Q}})} \leq ch_{\mathcal{Q}}^{-\frac{n}{2}}\|\widehat{div}(\hat{\mathbf{v}})\|_{L^2(\hat{\mathcal{Q}})} \leq c\|div(\mathbf{v})\|_{L^2(\mathcal{Q})} = c\|q\|_{L^2(\mathcal{Q})}.$$

□

3.2 THE GLOBAL FINITE ELEMENT SPACES

In this section, we define the global finite element spaces and derive the inf–sup stability condition needed for the well–posedness of the discrete problem.

By Lemma 19 and Lemma 20, the induced global finite element spaces are given by

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\mathcal{Q}} \in \mathbf{\Lambda}_3^-(\mathcal{Q}) : \mathcal{Q} \in \mathcal{Q}_h,\ \frac{\partial v_i}{\partial x_i} \in C^0 \text{ across } (n-2) \text{ dimensional faces}\},$$

$$W_h = \{q \in L^2(\Omega) : q|_{\mathcal{Q}} \in \Lambda_2(\mathcal{Q}) : \mathcal{Q} \in \mathcal{Q}_h,\ q \in C^0 \text{ across } (n-2) \text{ dimensional faces}\}.$$

Lemma 22. *For any $q \in W_h$, there exists $\mathbf{v}_1 \in \mathbf{V}_h$ such that $(q - div(\mathbf{v}_1))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$ for all $\mathcal{Q} \in \mathcal{Q}_h$. Moreover, $\|\mathbf{v}_1\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)}$.*

Proof. Let $\boldsymbol{\omega} \in \mathbf{H}^1(\Omega)$ satisfy $div(\boldsymbol{\omega}) = q$ and $\|\boldsymbol{\omega}\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)}$[24]. We denote by $\mathbf{I}_h\boldsymbol{\omega}$ the Scott-Zhang interpolant [42] of $\boldsymbol{\omega}$ such that $\mathbf{I}_h\boldsymbol{\omega}|_{\mathcal{Q}} \in \boldsymbol{\Lambda}_1(\mathcal{Q})$. Then, the error of the interpolant in H^m-norm satisfies

$$\|\boldsymbol{\omega} - \mathbf{I}_h\boldsymbol{\omega}\|_{H^m(\mathcal{Q})} \leq c h_{\mathcal{Q}}^{2-m}\|\boldsymbol{\omega}\|_{H^1(w_{\mathcal{Q}})}, \qquad m = 0, 1,$$

where $w_{\mathcal{Q}}$ denotes the patch of elements that touch $\mathcal{Q}$. Then, we define $\mathbf{v}_1 \in \mathbf{V}_h$ uniquely by the following iterative process. At the vertices, we set

$$\mathbf{v}_1(S) = \mathbf{I}_h\boldsymbol{\omega}(S), \quad \frac{\partial v_1^{(i)}}{\partial x_i}(S) = \frac{q(S)}{n}, \qquad \text{for } S \in \square_0, \quad 1 \leq i \leq n. \tag{3.2.1}$$

Then, we build $\mathbf{v}_1$ that satisfies

$$\int_S \mathbf{v}_1 \cdot \mathbf{n}_S^{(j)} = \int_S \mathbf{I}_h\boldsymbol{\omega} \cdot \mathbf{n}_S^{(j)}, \quad \int_S \frac{\partial \mathbf{v}_1}{\partial \mathbf{n}_S^{(j)}} \cdot \mathbf{n}_S^{(j)} = \frac{1}{n-m-1}\int_S (q - \sum_{i=1}^{m+1} \frac{\partial \mathbf{v}_1}{\partial \mathbf{t}_S^{(i)}} \cdot \mathbf{t}_S^{(i)}), \ S \in \square_{m+1},$$

where $1 \leq j \leq n - m - 1$ and $\mathbf{t}_S^{(i)}$ denotes the unit tangent vector to the surface S with $1 \leq i \leq m + 1$. Repeating this process up to $m = n - 3$, we deduce that

$$\int_S \mathbf{v}_1 \cdot \mathbf{n}_S^{(j)} = \int_S \mathbf{I}_h\boldsymbol{\omega} \cdot \mathbf{n}_S^{(j)}, \qquad \int_S div(\mathbf{v}_1) = \int_S q, \quad S \in \square_S, \quad 0 \leq s \leq n-2. \tag{3.2.2}$$

After that, we impose the last set of conditions on $\mathbf{v}_1$:

$$\int_S \mathbf{v}_1 \cdot \mathbf{n}_S = \int_S \boldsymbol{\omega} \cdot \mathbf{n}_S, \qquad S \in \square_{n-1}, \tag{3.2.3}$$

$$\int_{\mathcal{Q}} \mathbf{v}_1 \cdot \boldsymbol{\kappa} = \int_{\mathcal{Q}} \mathbf{I}_h\boldsymbol{\omega} \cdot \boldsymbol{\kappa}, \qquad \boldsymbol{\kappa} \in \Lambda_1^-(\mathcal{Q}), \ \mathcal{Q} \in \mathcal{Q}_h. \tag{3.2.4}$$

By Lemma 19, (3.2.1)–(3.2.4) uniquely define $\mathbf{v}_1 \in \mathbf{V}_h$. Moreover, by the second identity in (3.2.2), we have $div(\mathbf{v}_1) = q$ on $(n-2)$-dimensional faces. Also, by the Stokes theorem and (3.2.3), we have

$$\int_{\mathcal{Q}} div(\mathbf{v}_1) = \int_{\mathcal{Q}} div(\boldsymbol{\omega}) = \int_{\mathcal{Q}} q, \qquad \forall \mathcal{Q} \in \mathcal{Q}_h. \tag{3.2.5}$$

As a result, $(q - div(\mathbf{v}_1))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$ for all $\mathcal{Q} \in \mathcal{Q}_h$. It remains to derive the stability estimate $\|\mathbf{v}_1\|_{H^1(\Omega)} \leq \|q\|_{L^2(\Omega)}$.

Since $S_v^\star$ unisolves $\boldsymbol{\Lambda}_3^-(\mathcal{Q})$ and $(\mathbf{v}_1 - \mathbf{I}_h\boldsymbol{\omega})|_{\mathcal{Q}} \in \boldsymbol{\Lambda}_3^-(\mathcal{Q})$, we may write

$$\begin{aligned}\|\mathbf{v}_1 - \mathbf{I}_h\boldsymbol{\omega}\|_{H^1(\mathcal{Q})}^2 \approx &\sum_{s=0}^{n-2} \sum_{S \in \square_s(\mathcal{Q})} \sum_{j=1}^{n-s} h_{\mathcal{Q}}^{n-2s} \Big| \int_S (\frac{\partial \mathbf{v}_1}{\partial \mathbf{n}_S^{(j)}} \cdot \mathbf{n}_S^{(j)} - \frac{\partial \mathbf{I}_h\boldsymbol{\omega}}{\partial \mathbf{n}_S^{(j)}} \cdot \mathbf{n}_S^{(j)}) \Big|^2 \\ &+ \sum_{S \in \square_{n-1}(\mathcal{Q})} h_{\mathcal{Q}}^{-n} \Big| \int_S (\mathbf{v}_1 - \mathbf{I}_h\boldsymbol{\omega}) \cdot \mathbf{n}_S \Big|^2\end{aligned}$$

Then, by scaling arguments, we have

$$\begin{aligned}\|\mathbf{v}_1-\mathbf{I}_h\boldsymbol{\omega}\|^2_{H^1(\mathcal{Q})}\leq&\sum_{s=0}^{n}\sum_{S\in\square_s(\mathcal{Q})}h_{\mathcal{Q}}^{n-2s}|\int_S q|^2+\sum_{s=0}^{n-2}\sum_{S\in\square_s(\mathcal{Q})}\sum_{j=1}^{n-s}h_{\mathcal{Q}}^{n-2s}|\int_S\frac{\partial\mathbf{I}_h\boldsymbol{\omega}}{\partial\mathbf{n}_S^{(j)}}\cdot\mathbf{n}_S^{(j)}|^2\\&+\sum_{s=0}^{n-2}\sum_{i=1}^{s}h_{\mathcal{Q}}^{n-2s}|\int_S\frac{\partial\mathbf{v}_1}{\partial\mathbf{t}_S^{(i)}}\cdot\mathbf{t}_S^{(i)}|^2+\sum_{S\in\square_{n-1}}h_{\mathcal{Q}}^{-n}|\int_S(\boldsymbol{\omega}-\mathbf{I}_h\boldsymbol{\omega})\cdot\mathbf{n}_S|^2.\end{aligned}\tag{3.2.6}$$

For a face $S\in\square_s(\mathcal{Q})$ with $1\leq s\leq n-2$ and unit tangent vector $\mathbf{t}_S^{(i)}$ with $1\leq i\leq s$, let S_1, $S_2\in\square_{s-1}(\mathcal{Q})$ be the unique $(s-1)$ dimensional faces such that $\mathbf{t}_S^{(i)}=\pm\mathbf{n}_{S_1}^{(j_1)}$ and $\mathbf{t}_S^{(i)}=\pm\mathbf{n}_{S_2}^{(j_2)}$ for some $1\leq j_1,j_2\leq n-s+1$. Then, by the Fundamental Theorem of Calculus and (3.2.2), we have

$$\begin{aligned}\int_S\frac{\partial\mathbf{v}_1}{\partial\mathbf{t}_S^{(i)}}\cdot\mathbf{t}_S^{(i)}&=\pm\int_{S_1}\mathbf{v}_1\cdot\mathbf{n}_{S_1}^{(j_1)}\pm\int_{S_2}\mathbf{v}_1\cdot\mathbf{n}_{S_2}^{(j_2)}\\&=\pm\int_{S_1}\mathbf{I}_h\boldsymbol{\omega}\cdot\mathbf{n}_{S_1}^{(j_1)}\pm\int_{S_2}\mathbf{I}_h\boldsymbol{\omega}\cdot\mathbf{n}_{S_2}^{(j_2)}=\int_S\frac{\partial\mathbf{I}_h\boldsymbol{\omega}}{\partial\mathbf{t}_S^{(i)}}\cdot\mathbf{t}_S^{(i)}.\end{aligned}\tag{3.2.7}$$

By applying (3.2.7) to (3.2.6) and scaling, we obtain

$$\begin{aligned}\|\mathbf{v}_1-\mathbf{I}_h\mathbf{w}\|^2_{H^1(\mathcal{Q})}\leq&\sum_{s=0}^{n}\sum_{S\in\square_s(\mathcal{Q})}h_{\mathcal{Q}}^{n-2s}|\int_S q|^2+\sum_{s=0}^{n-2}\sum_{S\in\square_s(\mathcal{Q})}\sum_{j=1}^{n-s}h_{\mathcal{Q}}^{n-2s}|\int_S\frac{\partial\mathbf{I}_h\boldsymbol{\omega}}{\partial\mathbf{n}_S^{(j)}}\cdot\mathbf{n}_S^{(j)}|^2\\&+\sum_{s=0}^{n-2}\sum_{i=1}^{s}h_{\mathcal{Q}}^{n-2s}|\int_S\frac{\partial\mathbf{I}_h\boldsymbol{\omega}}{\partial\mathbf{t}_S^{(i)}}\cdot\mathbf{t}_S^{(i)}|^2+\sum_{S\in\square_{n-1}(\mathcal{Q})}h_{\mathcal{Q}}^{-n}|\int_S(\boldsymbol{\omega}-\mathbf{I}_h\boldsymbol{\omega})\cdot\mathbf{n}_S|^2\\&\leq c(\|q\|^2_{L^2(\mathcal{Q})}+\|\mathbf{I}_h\boldsymbol{\omega}\|^2_{H^1(\mathcal{Q})}+h_{\mathcal{Q}}^{-1}\|\boldsymbol{\omega}-\mathbf{I}_h\boldsymbol{\omega}\|^2_{L^2(\partial\mathcal{Q})}\leq c\|\boldsymbol{\omega}\|^2_{H^1(\omega_{\mathcal{Q}})}.\end{aligned}\tag{3.2.8}$$

Finally, by the triangle inequality, the stability of the interpolant and summing over $\mathcal{Q}\in\mathcal{Q}_h$, we deduce that

$$\|\mathbf{v}_1\|_{H^1(\Omega)}\leq c(\|\mathbf{I}_h\boldsymbol{\omega}\|_{H^1(\Omega)}+\|\boldsymbol{\omega}\|_{H^1(\Omega)})\leq c\|\boldsymbol{\omega}\|_{H^1(\Omega)}\leq c\|q\|_{L^2(\Omega)}.$$

□

Theorem 8. *For any $q\in W_h$, there exists $\mathbf{v}\in\mathbf{V}_h$ such that $div(\mathbf{v})=q$ and $\|\mathbf{v}\|_{H^1(\Omega)}\leq c\|q\|_{L^2(\Omega)}$. As a result, we derive the inf–sup stability condition*

$$\sup_{\mathbf{v}\in\mathbf{V}_h\setminus\{\mathbf{0}\}}\frac{\int_\Omega q\ div(\mathbf{v})}{\|\mathbf{v}\|_{H^1(\Omega)}}\geq c\|q\|_{L^2(\Omega)},\qquad\forall q\in W_h.$$

Proof. By Lemma 22, for any $q\in W_h$, there exists $\mathbf{v}_1\in\mathbf{V}_h$ that satisfies $(q-div(\mathbf{v}_1))|_{\mathcal{Q}}\in W_{h,0}(\mathcal{Q})$ for all $\mathcal{Q}\in\mathcal{Q}_h$ and $\|\mathbf{v}_1\|_{H^1(\mathcal{Q})}\leq c\|q\|_{L^2(\mathcal{Q})}$. Therefore, by Theorem 7, for every $\mathcal{Q}\in\mathcal{Q}_h$, there exists $\mathbf{v}_{2,\mathcal{Q}}\in\mathbf{V}_{h,0}(\mathcal{Q})$ that satisfies $div(\mathbf{v}_{2,\mathcal{Q}})=(q-div(\mathbf{v}_1))|_{\mathcal{Q}}$ and $\|\mathbf{v}_{2,\mathcal{Q}}\|_{H^1(\mathcal{Q})}\leq$

$c\|q-div(\mathbf{v}_1)\|_{L^2(\mathcal{Q})}$. We define $\mathbf{v}_2$ such that $\mathbf{v}_2|_{\mathcal{Q}} = \mathbf{v}_{2,\mathcal{Q}}$, and set $\mathbf{v} := \mathbf{v}_1+\mathbf{v}_2$. Note that since $\mathbf{v}_2|_{\mathcal{Q}} \in \mathbf{V}_{h,0}(\mathcal{Q})$ for all $\mathcal{Q} \in \mathcal{Q}_h$, $\mathbf{v}_2|_{\mathcal{Q}} \in \mathbf{H}_0^1(\mathcal{Q})$ and this implies that $\nabla \mathbf{v}_2|_S = 0$ on all $S \in \square_s$ with $0 \leq s \leq n-2$. Thus, $\mathbf{v} \in \mathbf{V}_h$ with $div(\mathbf{v}) = div(\mathbf{v}_1)+div(\mathbf{v}_2) = q$. As a result, we obtain the stability estimate $\|\mathbf{v}\|_{H^1(\Omega)} \leq \|\mathbf{v}_1\|_{H^1(\Omega)}+\|\mathbf{v}_2\|_{H^1(\Omega)} \leq c(\|\mathbf{v}_1\|_{H^1(\Omega)}+\|q-div(\mathbf{v}_1)\|_{H^1(\Omega)}) \leq c\|q\|_{L^2(\Omega)}$. □

3.3 THE GLOBAL FINITE ELEMENT SPACES WITH IMPOSED BOUNDARY CONDITIONS

Imposing boundary conditions on finite element spaces while preserving the surjectivity of the divergence is a significant matter [22]. For instance, if $\mathbf{v}$ is a globally continuous function on Ω and vanishes on $\partial\Omega$, then the derivatives of $\mathbf{v}$ vanish at the corners (if $n = 2$) and the edges (if $n = 3$) of $\partial\Omega$. Thus, the divergence operator is not surjective from $\mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$ to $W_h \cap L_0^2(\Omega)$, and this violates the inf–sup condition.

On simplicial meshes, by imposing mesh and regularity conditions locally on the boundary, this issue may be eased [22]. However, on cubical meshes, such procedures are not applicable. For instance, in two–dimensions, there will always be elements in $\mathcal{Q}_h$ that have at least two boundary edges. Therefore, we impose the weaker boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$ on the velocity finite element space and impose the tangential boundary condition weakly in the finite element method via the Nitsche's method [37] as we did in Chapter 2. As a result, we specify the boundary conditions on the global finite element spaces as follows:

$$\mathbf{V}_{h,0} = \{\mathbf{v} \in \mathbf{V}_h : (\mathbf{v} \cdot \mathbf{n})|_{\partial\Omega} = 0\},$$
$$W_{h,0} = W_h \cap L_0^2(\Omega).$$

3.4 INF–SUP STABILITY

In this section, we define a discrete norm on $\mathbf{V}_{h,0}$ and prove the inf-sup stability condition. We denote by $\square_s^l$ the set of open s-dimensional faces that do not intersect with $\partial\Omega$ and set

$\square_s^b := \square_s \setminus \square_s^l$. Then, we define the discrete H^1–type norm as follows:

$$\|\mathbf{v}\|_h^2 := \|\nabla \mathbf{v}\|_{L^2(\Omega)}^2 + \sum_{S\in\square_{n-1}^b} (h_S\|\frac{\partial \mathbf{v}}{\partial \mathbf{n}_S}\|_{L^2(S)}^2 + \frac{1}{h_S}\|\mathbf{v}\|_{L^2(S)}^2). \tag{3.4.1}$$

Theorem 9. *The inf–sup condition*

$$c\|q\|_{L^2(\Omega)} \le \sup_{\mathbf{v}\in\mathbf{V}_{h,0}\setminus\{\mathbf{0}\}} \frac{\int_\Omega div(\mathbf{v})q}{\|\mathbf{v}\|_h}, \qquad \forall q \in W_{h,0}, \tag{3.4.2}$$

holds for a constant $c > 0$ independent of h.

Proof. As in the proof of Theorem 8, for any $q \in W_{h,0}$ given, we use the degrees of freedom of $\mathbf{V}_h$ to construct $\mathbf{v} \in \mathbf{V}_{h,0}$ that satisfies $div(\mathbf{v}) = q$ and $\|\mathbf{v}\|_{H^1(\Omega)} \le c\|q\|_{L^2(\Omega)}$. By Theorem 1, for any $q \in \mathbf{W}_{h,0} \subseteq L_0^2(\Omega)$, there exists $\boldsymbol{\omega} \in \mathbf{H}_0^1(\Omega)$ that satisfies $div(\boldsymbol{\omega}) = q$ and $\|\boldsymbol{\omega}\|_{H^1(\Omega)} \le c\|q\|_{L^2(\Omega)}$. We set $\mathbf{I}_h\boldsymbol{\omega}$ as the Scott-Zhang interpolant [41] of $\boldsymbol{\omega}$ that satisfies $\mathbf{I}_h\boldsymbol{\omega}|_{\mathcal{Q}} \in \boldsymbol{\Lambda}_1(\mathcal{Q})$ on each $\mathcal{Q} \in \mathcal{Q}_h$. Note that $\mathbf{I}_h\boldsymbol{\omega} \in \mathbf{H}_0^1(\Omega)$, in particular, $\mathbf{I}_h\boldsymbol{\omega}|_S = 0$ for all boundary faces $S \in \square_s^b$ where $0 \le s \le n-1$. Then, by repeating the procedure followed in the proof of Lemma 22, we determine $\mathbf{v}_1 \in \mathbf{V}_h$ uniquely via the conditions (3.2.1)-(3.2.4). This construction yields $(q - div(\mathbf{v}_1))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$ for all $\mathcal{Q} \in \mathcal{Q}_h$ and $\|\mathbf{v}_1\|_{H^1(\Omega)} \le c\|q\|_{L^2(\Omega)}$.

Furthermore, we let $S \in \square_{n-1}^b$, and denote by $\square_s(S)$ with $0 \le s \le n-1$ the set of s–dimensional subfaces of S. Then, for a subface $S' \in \square_s(S)$, we have $S' \in \square_s^b$. Moreover, the (outward) unit normal of S is an (outward) unit normal of S' (up-to-sign), i.e., $\mathbf{n}_S = \pm\mathbf{n}_{S'}^{(j)}$, for some $j \in \{1, 2, \dots, n-s\}$. Therefore, by (3.2.1)-(3.2.4), we have

$$\int_{S'} \mathbf{v}_1 \cdot \mathbf{n}_S = \int_{S'} \mathbf{I}_h\boldsymbol{\omega} \cdot \mathbf{n}_S = 0, \qquad \forall S' \in \square_s(S), \quad 0 \le s \le n-1.$$

Since $(\mathbf{v}_1 \cdot \mathbf{n}_S) \in \Lambda_2(S)$ on S, Lemma 17 implies that $\mathbf{v}_1 \cdot \mathbf{n}_S = 0$ on $S \in \square_{n-1}^b$. Thus, $\mathbf{v}_1 \in \mathbf{V}_{h,0}$.

To validate the stability estimate $\|\mathbf{v}_1\|_h \le c\|q\|_{L^2(\Omega)}$, we set $\hat{\mathbf{v}}(\hat{x}) = \mathbf{v}(x)$, where $x = \mathcal{F}(\hat{x})$ and $\mathcal{F} : \hat{\mathcal{Q}} \to \mathcal{Q}$ denotes an affine transformation for $\mathcal{Q} \in \mathcal{Q}_h$.

Then, by scaling, we have

$$\begin{aligned}\|\nabla(\mathbf{v}_1 - \mathbf{I}_h\boldsymbol{\omega})\|_{L^2(\mathcal{Q})}^2 + \sum_{S\in\square_{n-1}^b\cap\square_{n-1}(\mathcal{Q})} \frac{1}{h_S}\|\mathbf{v}_1 - \mathbf{I}_h\boldsymbol{\omega}\|_{L^2(S)} \le Ch_{\mathcal{Q}}^{n-2}\Big(\|\hat{\nabla}(\hat{\mathbf{v}}_1 - \widehat{\mathbf{I}_h\boldsymbol{\omega}})\|_{L^2(\hat{\mathcal{Q}})}^2 \\ + \sum_{\hat{S}\in\square_{n-1}(\hat{\mathcal{Q}})} \|\hat{\mathbf{v}}_1 - \widehat{\mathbf{I}_h\boldsymbol{\omega}}\|_{L^2(\hat{S})}^2\Big) \le Ch_{\mathcal{Q}}^{n-2}\|\hat{\mathbf{v}}_1 - \widehat{\mathbf{I}_h\boldsymbol{\omega}}\|_{H^1(\hat{\mathcal{Q}})}^2 \le C\|\mathbf{v}_1 - \mathbf{I}_h\boldsymbol{\omega}\|_{H^1(\mathcal{Q})}^2\end{aligned} \tag{3.4.3}$$

Therefore, since $\mathbf{I}_h\boldsymbol{\omega}|_{\partial\Omega} = \mathbf{0}$, (3.2.8) yields

$$\|\nabla \mathbf{v}_1\|^2_{L^2(\mathcal{Q})} + \sum_{S\in\square_{n-1}(\mathcal{Q})\cap\square^b_{n-1}} \frac{1}{h_S}\|\mathbf{v}_1\|^2_{L^2(S)} \le c\|\boldsymbol{\omega}\|^2_{H^1(\omega_{\mathcal{Q}})}.$$

Finally, summing over $\mathcal{Q} \in \mathcal{Q}_h$, we obtain

$$\sum_{\mathcal{Q}\in\mathcal{Q}_h} \|\nabla \mathbf{v}_1\|^2_{L^2(\mathcal{Q})} + \sum_{S\in\square^b_{n-1}} \frac{1}{h_S}\|\mathbf{v}_1\|^2_{L^2(S)} \le c\|\boldsymbol{\omega}\|^2_{H^1(\Omega)} \le c\|q\|_{L^2(\Omega)}.$$

By scaling and the equivalence of norms in finite–dimensions, this implies that $\|\mathbf{v}_1\|_h \le c\|q\|_{L^2(\Omega)}$. Then, we consider $\mathbf{v}_{2,\mathcal{Q}} \in \mathbf{V}_{h,0}(\mathcal{Q})$ that satisfies $div(\mathbf{v}_{2,\mathcal{Q}}) = (q - div(\mathbf{v}_1))|_{\mathcal{Q}}$ and define $\mathbf{v}_2 \in \mathbf{V}_h$ such that $\mathbf{v}_2|_{\mathcal{Q}} = \mathbf{v}_{2,\mathcal{Q}}$ on each $\mathcal{Q} \in \mathcal{Q}_h$. Note that $\mathbf{v}_2 \in \mathbf{V}_h \cap \mathbf{H}^1_0(\Omega)$, therefore, $\mathbf{v}_2 \in \mathbf{V}_{h,0}$. Then, setting $\mathbf{v} := \mathbf{v}_1 + \mathbf{v}_2 \in \mathbf{V}_{h,0}$ and using the properties of $\mathbf{v}_1$ and $\mathbf{v}_2$, we deduce that $div(\mathbf{v}) = q$ and $\|\mathbf{v}\|_{H^1(\Omega)} \le c\|q\|_{L^2(\Omega)}$. □

3.5 THE REDUCED ELEMENTS WITH CONTINUOUS PRESSURE APPROXIMATIONS

In this section, we reduce the global degrees of freedom of our finite element pair by restricting the range of the divergence operator. In forming the reduced finite element spaces, we denote by $B_s(\mathcal{Q}) \subset \Lambda_2(\mathcal{Q})$ the space spanned by the bubble functions associated with $\square_s(\mathcal{Q})$, i.e.,

$$B_s(\mathcal{Q}) = \bigoplus_{S\in\square_s(\mathcal{Q})} < b_S >,$$

where $< b_S >$ denotes the span of b_S. By Lemma 17, we have

$$\Lambda_2(\mathcal{Q}) = \Lambda_1(\mathcal{Q}) \oplus \bigoplus_{s=1}^{n} B_s(\mathcal{Q}).$$

Then, we define the reduced local pressure space $W_{r,h}(\mathcal{Q})$ by removing the $(n-1)$ dimensional face bubbles of $\Lambda_2(\mathcal{Q})$ as follows:

$$W_{r,h}(\mathcal{Q}) := \Lambda_1(\mathcal{Q}) \oplus \Big(\bigoplus_{s=1}^{n-2} B_s(\mathcal{Q})\Big) \oplus B_n(\mathcal{Q}).$$

Note that in two–dimensions, $W_{r.h}(\mathcal{Q})$ is the space of bilinear polynomials enriched with face bubbles, whereas in three dimensions, $W_{r,h}(\mathcal{Q})$ is the space of trilinear polynomials enriched with edge and volume bubbles.

By Lemma 17, the dimension of $W_{r,h}(\mathcal{Q})$ is $(3^n - 2n)$, and a function $q \in W_{r,h}(\mathcal{Q})$ is uniquely determined by the set of values

$$\{\int_S q: \quad S \in \square_s(\mathcal{Q}), \quad s = 0, 1, \dots, n-2, n\}.$$

We define the reduced local velocity element as

$$\mathbf{V}_{r,h}(\mathcal{Q}) := \{\mathbf{v} \in \mathbf{\Lambda}_3^-(\mathcal{Q}) : div(\mathbf{v}) \in W_{r,h}(\mathcal{Q})\}. \tag{3.5.1}$$

Lemma 23. *A function $\mathbf{v} \in \mathbf{V}_{r,h}(\mathcal{Q})$ is uniquely determined by the set of values (See Figure 3.1).*

$$S_v^r = \{\int_S v^{(i)}, \int_S \frac{\partial v^{(i)}}{\partial x_i} : \ S \in \square_s^{(i)}(\mathcal{Q}), \ 0 \le s \le n-2, \quad \int_S v^{(i)} : \ S \in \square_{n-1}^{(i)}(\mathcal{Q})\},$$

where $1 \le i \le n$.

Proof. Note that the number of constraints imposed on $\mathbf{\Lambda}_3^-(\mathcal{Q})$ in the definition of the space $\mathbf{V}_{r,h}(\mathcal{Q})$ is $2n$, therefore, $dim(\mathbf{V}_{r,h}(\mathcal{Q})) = dim(\mathbf{V}_{r,h}(\mathcal{Q})) - 2n = 4n3^{n-1} - 2n = |S_v^r|$. Therefore, it suffices to show that if $\mathbf{v} \in \mathbf{V}_{r,h}(\mathcal{Q})$ nullifies S_v^r, then $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$.
If $\mathbf{v} \in \mathbf{V}_{r,h}(\mathcal{Q}) \subset \mathbf{\Lambda}_3^-(\mathcal{Q})$ nullifies the functionals in S_v^r, then $\mathbf{v} \in \mathbf{H}_0^1(\mathcal{Q})$ by Lemma 19. Moreover, Lemma 19 also implies that $div(\mathbf{v})|_S = 0$ for all $S \in \square_s(\mathcal{Q})$ with $0 \le s \le n-2$. Since $W_{r,h}(\mathcal{Q}) \subset \Lambda_2(\mathcal{Q})$, this implies that $div(\mathbf{v}) = c\, b_{\mathcal{Q}}$ for some $c \in \mathbb{R}^n$. Then, integration by parts yields

$$c \int_{\mathcal{Q}} b_{\mathcal{Q}} = \int_{\mathcal{Q}} div(\mathbf{v}) = \int_{\partial\mathcal{Q}} \mathbf{v} \cdot \mathbf{n} = 0$$

Thus, $div(\mathbf{v}) = 0$ and therefore by Lemma 21, we have $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$. □

Let $\mathbf{H}^1(div;\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) : div(\mathbf{v}) \in H^1(\Omega)\}$. Then, we define the reduced global spaces as follows:

$$\mathbf{V}_{r,h} := \{\mathbf{v} \in \mathbf{H}^1(div;\Omega) \cap \mathbf{V}_h : \mathbf{v}|_{\mathcal{Q}} \in \mathbf{V}_{r,h}(\mathcal{Q}), \ (\mathbf{v} \cdot \mathbf{n})|_{\partial\Omega} = 0\},$$
$$W_{r,h} := \{q \in H^1(\Omega) \cap L_0^2(\Omega) : q|_{\mathcal{Q}} \in W_{r,h}(\mathcal{Q})\}.$$

Theorem 10. *For every $q \in W_{r,h}$ there exists $\mathbf{v} \in \mathbf{V}_{r,h}$ that satisfies $div(\mathbf{v}) = q$ and*

$$\|\mathbf{v}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}.$$

Moreover, $div(\mathbf{V}_{r,h}) = W_{r,h}$. Therefore, the reduced pair $\mathbf{V}_{r,h} \times W_{r,h}$ is inf–sup stable.

Proof. By Theorem 9, for $q \in W_{r,h} \subset W_{h,0}$ given, there exists $\mathbf{v} \in \mathbf{V}_{h,0}$ such that $div(\mathbf{v}) = q$ and $\|\mathbf{v}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$. Since $q \in W_{r,h}$, $\mathbf{v}|_{\mathcal{Q}} \in \mathbf{V}_{r,h}(\mathcal{Q})$ on all $\mathcal{Q} \in \mathcal{Q}_h$, and $div(\mathbf{v}) \in H^1(\Omega)$. Therefore, $\mathbf{v} \in \mathbf{V}_{r,h}$. This implies that $W_{r,h} \subseteq div(\mathbf{V}_{r,h})$. Thus, the second assertion follows from the inclusion $div(\mathbf{V}_{r,h}) \subseteq W_{r,h}$. □

3.6 CONVERGENCE ANALYSIS

Let $\mathbf{X}_{h,0} \times Y_{h,0}$ denote either the pair $\mathbf{V}_{h,0} \times W_{h,0}$ or the reduced pair $\mathbf{V}_{r,h} \times W_{r,h}$. Then, the finite element method reads: Find $(\mathbf{u}_h, p_h) \in \mathbf{X}_{h,0} \times Y_{h,0}$ that satisfies

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}) - \int_\Omega div(\mathbf{v})\, p_h &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\,, \qquad &\forall \mathbf{v} \in \mathbf{X}_{h,0} \\ \int_\Omega div(\mathbf{u}_h)\, q &= 0, &\forall q \in Y_{h,0}, \end{aligned} \tag{3.6.1}$$

where the bilinear form $a_h(\cdot,\cdot) : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ is defined as

$$a_h(\mathbf{v}, \boldsymbol{\omega}) = \nu \int_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\omega} - \nu \sum_{S \in \square_{n-1}^b} \int_S \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}_S} \boldsymbol{\omega} + \frac{\partial \boldsymbol{\omega}}{\partial \mathbf{n}_S} \mathbf{v} - \frac{\sigma}{h_S} \mathbf{v} \cdot \boldsymbol{\omega}\right),$$

where $\sigma > 0$ is an h–independent penalty parameter.

Theorem 11. *There exists a unique pair $(\mathbf{u}_h, p_h) \in \mathbf{X}_{h,0} \times Y_{h,0}$ satisfying (3.6.1). Moreover, for $1 \leq s \leq 2$, there holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq ch^s \|\mathbf{u}\|_{H^{s+1}(\Omega)}, \qquad \|p - p_h\|_{L^2(\Omega)} \leq ch^s(\, \|p\|_{H^{s+1}(\Omega)} + \nu \|\mathbf{u}\|_{H^{s+1}(\Omega)}).$$

where the constant $C > 0$ is independent of h or the viscosity ν.

Proof. Following the same procedure described in Section 2.4, it is easy to show that if σ is sufficiently large, then $a_h(\cdot,\cdot)$ is coercive on $\mathbf{X}_{h,0}$ with respect to the discrete H^1–type norm given by (3.4.1). In addition, $a_h(\cdot,\cdot)$ is continuous on $(\mathbf{H}^2(\Omega) + \mathbf{V}_{h,0})$. Thus, there exists a unique $(\mathbf{u}_h, p_h)$ pair that satisfies (3.6.1) by Theorem 9, Theorem 10 and standard theory [9, 24]. Moreover, the velocity approximation $\mathbf{u}_h$ is independent of the choice of the finite element space $\mathbf{X}_{h,0} = \mathbf{V}_{h,0}$ or $\mathbf{X}_{h,0} = \mathbf{V}_{r,h}$ since the kernel of the divergence operator acting on each of these spaces is $\mathbf{Z}_h = \{\mathbf{v} \in \mathbf{X}_{h,0} : div(\mathbf{v}) = 0\}$. Restricting (3.6.1) to the divergence–free space $\mathbf{Z}_h$, and using the consistency of the bilinear form $a_h(\cdot,\cdot)$, we deduce by Cea's Lemma, $\|\mathbf{u} - \mathbf{u}_h\|_h \leq c \inf_{\mathbf{v} \in \mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}\|_h$ provided $\mathbf{u} \in \mathbf{H}^s(\Omega)$ for $s > 3/2$. As in Chapter 2, we estimate the approximation properties of $\mathbf{Z}_h$ by following the arguments given in Theorem 12.5.17 in [10]. To this end, for an arbitrary function $\mathbf{v} \in \mathbf{X}_{h,0}$, we let $\boldsymbol{\omega} \in \mathbf{X}_{h,0}$ satisfy $div(\boldsymbol{\omega}) = -div(\mathbf{v}) \in Y_{h,0}$, therefore, $\boldsymbol{\omega} + \mathbf{v} \in \mathbf{Z}_h$. Moreover, $\|\boldsymbol{\omega}\|_h \leq c\|div(\mathbf{v})\|_{L^2(\Omega)} = \|div(\mathbf{u} - \mathbf{v})\|_{L^2(\Omega)} \leq c\|\mathbf{u} - \mathbf{v}\|_h$. This yields $\|\mathbf{u} - (\boldsymbol{\omega} + \mathbf{v})\|_h \leq \|\mathbf{u} - \mathbf{v}\|_h + \|\boldsymbol{\omega}\|_h \leq c\|\mathbf{u} - \mathbf{v}\|_h$. As a result,

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq c \inf_{\mathbf{v} \in \mathbf{Z}_h} \|\mathbf{u} - \mathbf{v}\|_h \leq c \inf_{\mathbf{v} \in \mathbf{X}_{h,0}} \|\mathbf{u} - \mathbf{v}\|_h.$$

Thus, the velocity error estimate is decoupled from the pressure error estimate and independent of the viscosity. By standard approximation theory and scaling arguments, we derive the estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_h \le c \inf_{\mathbf{v} \in \mathbf{X}_{h,0}} \|\mathbf{u} - \mathbf{v}\|_h \le ch^s \|\mathbf{u}\|_{H^{s+1}(\Omega)}$$

for $1 \le s \le 2$. Note that since $\mathbf{u}_h$ is independent of the choice of the global velocity finite element space, $\|\mathbf{u} - \mathbf{u}_h\|_h$ satisfies the same estimate in both of the cases $\mathbf{X}_{h,0} = \mathbf{V}_{r,h}$ and $\mathbf{X}_{h,0} = \mathbf{V}_{h,0}$. To derive an error estimate on the pressure solution, we define the L^2–projection $\mathcal{P}_h : L^2(\Omega) \to Y_{h,0}$. Since $div(\mathbf{X}_{h,0}) = Y_{h,0}$, it follows from Theorem 9, Theorem 10 and the properties of $\mathcal{P}_h$ that

$$\begin{aligned} c\|p_h - \mathcal{P}_h p\|_{L^2(\Omega)} &\le \sup_{\mathbf{v} \in \mathbf{V}_{h,0} \setminus \{\mathbf{0}\}} \frac{\int_\Omega div(\mathbf{v})(p_h - \mathcal{P}_h p)}{\|\mathbf{v}\|_h} = \sup_{\mathbf{v} \in \mathbf{V}_{h,0} \setminus \{\mathbf{0}\}} \frac{\int_\Omega div(\mathbf{v})(p_h - p)}{\|\mathbf{v}\|_h} \\ &= \sup_{\mathbf{v} \in \mathbf{V}_{h,0} \setminus \{\mathbf{0}\}} \frac{a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v})}{\|\mathbf{v}\|_h} \le c\nu \|\mathbf{u} - \mathbf{u}_h\|_h. \end{aligned}$$

Therefore, we have

$$\|p - p_h\|_{L^2(\Omega)} \le \|p - \mathcal{P}_h p\|_{L^2(\Omega)} + c\nu\|\mathbf{u} - \mathbf{u}_h\|_h \le ch^s(\|p\|_{H^s(\Omega)} + \nu\|\mathbf{u}\|_{H^{s+1}(\Omega)})$$

for $1 \le s \le 2$. □

Remark 5. *The velocity error estimate has optimal order of convergence whereas the pressure error estimate is of optimal order provided* $Y_{h,0} = W_{r,h}$.

4.0 FEM FOR THE STOKES PROBLEM ON QUADRILATERAL MESHES

In Chapter 2, we propose a finite element method that is based on building solution spaces on rectangular meshes. However, the domains that could be considered in that framework are limited since we assume that the boundaries of both the domain and the mesh elements are parallel to the coordinate axes. In this chapter, we propose a method that gives a pair of stable and conforming finite element spaces yielding pointwise divergence–free velocity approximations on general shape–regular quadrilateral meshes [36].

Conforming finite element pairs that yield divergence–free approximations tend to be high-order or conforming and stable only on certain meshes. Moreover, the construction of conforming finite elements that yield divergence–free approximations are not extended to general convex quadrilaterals defined by bilinear mappings. In this chapter, we address some of these issues by introducing a low-order, conforming and stable finite element pair yielding divergence–free approximations on general shape–regular quadrilateral meshes. First, we define the H^2 finite element space Σ_h as the de Veubeke–Sanders macro element, which is a globally C^1 piecewise–cubic spline [16, 14, 28, 39]. Then, via the subcomplex given by (1.3.2), we induce a piecewise–quadratic macro velocity space and a piecewise–constant pressure space. The dimension of our global spaces is comparable to the lowest–order serendipity Taylor–Hood spaces [43]. Our analysis shows that the velocity error is decoupled from the pressure error and a locally–computed, post–processed pressure solution has the same rate of convergence as the velocity solution.

A nonconforming finite element method that enforce the divergence–free constraint pointwise on each quadrilateral element has recently been done in [48]. The method proposed there is low-order and applicable to convex quadrilaterals. However, the error estimates are still coupled with a negative scaling of the viscosity as a consequence of its nonconformity. Macro elements have recently been used on simplicial partitions in [1, 12]. Here, we present a study that complements and extends these results to quadrilateral meshes.

4.1 THE QUADRILATERAL MESH

In this chapter, $\mathcal{Q}_h$ denotes a shape-regular quadrilateral partition of $\Omega \subset \mathbb{R}^2$ consisting of convex quadrilateral elements. For each quadrilateral element $\mathcal{Q} \in \mathcal{Q}_h$, $K_r^{\mathcal{Q}} = \{K_i^{\mathcal{Q}}\}_{i=1}^4$ denotes the triangular partition of $\mathcal{Q}$ obtained by drawing in the two diagonals of $\mathcal{Q}$. In addition, $c_{\mathcal{Q}}$ denotes the point of intersection of these two diagonals. We assume that the triangular elements in the partition $K_r^{\mathcal{Q}}$ are ordered in a counterclockwise fashion (See Figure 4.1). As in the previous chapters, the sets of vertices and boundary vertices are given by $\mathcal{V}$ and $\mathcal{V}_b$, respectively, and the set of vertices (resp. edges) of an element $\mathcal{Q}$ is given by $\mathcal{V}_{\mathcal{Q}}$ (resp. $\mathcal{E}_{\mathcal{Q}}$). Moreover, for a vertex $a \in \mathcal{V}$, $\mathcal{Q}_a$ (resp. $\mathcal{E}_a$) stands for the set of quadrilaterals (resp. edges) that share a. Furthermore, we let $\mathcal{E}_a^b$ denote the set of boundary edges in $\mathcal{E}_a$. The space of piecewise polynomials with respect to the triangular partition of $\mathcal{Q}$ is given by $P_k(K_r^{\mathcal{Q}})$. For example, $p \in P_k(K_r^{\mathcal{Q}})$ if and only if $p|_{K_i^{\mathcal{Q}}} \in P_k(K_i^{\mathcal{Q}})$ for every $K_i^{\mathcal{Q}} \in K_r^{\mathcal{Q}}$. We set $P_k(K_r^{\mathcal{Q}}) := \prod_{i=1}^{4} P_k(K_i^{\mathcal{Q}})$ and $P_k(\mathcal{Q}_h) := \prod_{\mathcal{Q}\in\mathcal{Q}_h} P_k(\mathcal{Q})$.

Let $\mathcal{Q}_\pm \in \mathcal{Q}_h$ be two elements in the mesh sharing an edge $e = \partial\mathcal{Q}_+ \cap \partial\mathcal{Q}_-$. For a function v, we set $v_\pm = v|_{\mathcal{Q}_\pm}$. Assuming that the global labeling number of $\mathcal{Q}_+$ is smaller than that of $\mathcal{Q}_-$, we define the jump of a scalar or a vector-valued function v as $[v]|_e := v_+ - v_-$. For a boundary edge $e = \partial\mathcal{Q}_+ \cap \partial\Omega$, we set $[v]|_e = v_+$.

Additionally, we define the local L^2–projection $\mathcal{P}_{h,\mathcal{Q}} : \mathbf{L}^2(\mathcal{Q}) \to \mathbf{P}_1(\mathcal{Q})$ and introduce $\mathcal{P}_h : \mathbf{L}^2(\Omega) \to \mathbf{P}_1(\mathcal{Q}_h)$ via the relation $\mathcal{P}_h|_{\mathcal{Q}} = \mathcal{P}_{h,\mathcal{Q}}$ for every $\mathcal{Q} \in \mathcal{Q}_h$.

By definition, $\mathcal{P}_{h,\mathcal{Q}}$ satisfies

$$\int_{\mathcal{Q}} \mathcal{P}_{h,\mathcal{Q}}\mathbf{v} \cdot \boldsymbol{\omega} = \int_{\mathcal{Q}} \mathbf{v} \cdot \boldsymbol{\omega} \ , \quad \text{for all} \quad \boldsymbol{\omega} \in \mathbf{P}_1(\mathcal{Q}).$$

For $\mathbf{v} \in \mathbf{H}^s(\mathcal{Q})$ and $\ell = \min\{2, s\}$, the L^2–projection $\mathcal{P}_{h,\mathcal{Q}}$ satisfies [18]

$$\|\mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|_{L^2(\mathcal{Q})} \le c h_{\mathcal{Q}}^{\ell} |\mathbf{v}|_{H^\ell(\mathcal{Q})}. \tag{4.1.1}$$

Lemma 24. *The local L^2–projection $\mathcal{P}_{h,\mathcal{Q}} : \mathbf{L}^2(\mathcal{Q}) \to \mathbf{P}_1(\mathcal{Q})$ satisfies*

$$\|\nabla(\mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v})\|_{L^2(\mathcal{Q})} \le Ch_{\mathcal{Q}}^{s-1}|\mathbf{v}|_{H^s(\mathcal{Q})}, \qquad \forall \mathbf{v} \in \mathbf{H}^s(\mathcal{Q}),\ s = 1, 2. \tag{4.1.2}$$

Proof. By the triangle inequality and inverse estimates, for an arbitrary $\boldsymbol{\omega} \in \mathbf{P}_1(\mathcal{Q})$, there holds

$$\begin{aligned}\|\nabla(\mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v})\|_{L^2(\mathcal{Q})} &\le \|\nabla(\mathbf{v} - \boldsymbol{\omega})\|_{L^2(\mathcal{Q})} + Ch_{\mathcal{Q}}^{-1}\|\boldsymbol{\omega} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|_{L^2(\mathcal{Q})} \\ &\le \|\nabla(\mathbf{v} - \boldsymbol{\omega})\|_{L^2(\mathcal{Q})} + Ch_{\mathcal{Q}}^{-1}\big(\|\mathbf{v} - \boldsymbol{\omega}\|_{L^2(\mathcal{Q})} + \|\mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|_{L^2(\mathcal{Q})}\big).\end{aligned}$$

Let $\mathbf{w}$ be an interpolant of $\mathbf{v}$. Then, standard approximation theory [18, 13] and (4.1.1) yield (4.1.2). □

Lemma 25. *$\mathcal{P}_h$ is piecewise H^1–stable on $\mathbf{H}^1(\Omega)$.*

Proof. By the definition of the H^1–norm and the L^2–projection, we have

$$\|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|^2_{H^1(\mathcal{Q})} = \|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|^2_{L^2(\mathcal{Q})} + |\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}|^2_{H^1(\mathcal{Q})}, \tag{4.1.3}$$

$$\|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|^2_{L^2(\mathcal{Q})} \le \|\mathbf{v}\|^2_{L^2(\mathcal{Q})}. \tag{4.1.4}$$

In addition, using the triangle inequality and Lemma 24 with $s = 1$, we obtain

$$\begin{aligned}|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}|^2_{H^1(\mathcal{Q})} &\le (|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v} - \mathbf{v}|_{H^1(\mathcal{Q})} + |\mathbf{v}|_{H^1(\mathcal{Q})})^2 \le c(|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v} - \mathbf{v}|^2_{H^1(\mathcal{Q})} + |\mathbf{v}|^2_{H^1(\mathcal{Q})}), \\ &\le c|\mathbf{v}|^2_{H^1(\mathcal{Q})}.\end{aligned} \tag{4.1.5}$$

Combining (4.1.3), (4.1.4) with (4.1.5) yields

$$\|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|_{H^1(\mathcal{Q})} \le c\|\mathbf{v}\|_{H^1(\mathcal{Q})}. \tag{4.1.6}$$

Summing over $\mathcal{Q} \in \mathcal{Q}_h$, we deduce the H^1–stability condition $\|\mathcal{P}_h\mathbf{v}\|_{H^1(\Omega)} \le c\|\mathbf{v}\|_{H^1(\Omega)}$. □

4.2 THE LOCAL FINITE ELEMENT SPACES

In building a divergence–free conforming finite element pair, we utilize the local C^1 macro element constructed by de Veubeke and Sander [16, 39]

$$\Sigma_h(\mathcal{Q}) = \{\psi \in P_3(K_r^{\mathcal{Q}}) \cap H^2(\mathcal{Q})\} = \{\psi \in P_3(K_r^{\mathcal{Q}}) \cap C^1(\mathcal{Q})\}.$$

Lemma 26. *The dimension of $\Sigma_h(\mathcal{Q})$ is 16, and its degrees of freedom are given by*

$$S_\Sigma = \{D^\alpha\psi(a) : \ |\alpha| \le 1,\ a \in \mathcal{V}_{\mathcal{Q}},\ \int_e \frac{\partial\psi}{\partial\mathbf{n}_e} : \quad e \in \mathcal{E}_{\mathcal{Q}}\}, \tag{4.2.1}$$

where $\mathbf{n}_e$ denotes the (outward) unit normal of e (See Figure 4.1).

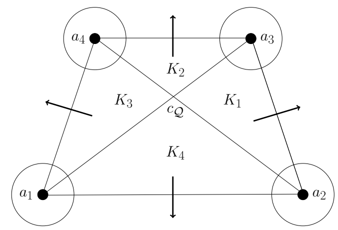


Figure 4.1: Degrees of freedom of the macro element $\Sigma_h(\mathcal{Q})$. Solid circles indicate function evaluations, larger circles indicate gradient evaluations and the lines indicate the means of the normal derivative.

Proof. First, we note that the cardinality of S_Σ is 28 and the dimension of $P_3(K_i^{\mathcal{Q}})$ is 10 for every $1 \leq i \leq 4$, therefore, the dimension of $P_3(K_r^{\mathcal{Q}})$ is 40. Since the point $c_{\mathcal{Q}}$ is a singular vertex with respect to the partition $K_r^{\mathcal{Q}}$, C^1–continuity at this point imposes 8 constraints on $P_3(K_r^{\mathcal{Q}})$ [34]. In addition, C^1–continuity imposed at the interior edge midpoints yield 4 more constraints on $P_3(K_r^{\mathcal{Q}})$. As a result, S_Σ and the C^1–continuity constraints provide 40 equations in total. Hence, it suffices to show that $\phi \in \Sigma_h(\mathcal{Q})$ nullifies S_Σ if and only if $\phi = 0$. Let $\mu \in P_1(K_r^{\mathcal{Q}}) \cap H_0^1(\mathcal{Q})$ be the unique continuous, piecewise–linear polynomial that takes the value one at $c_{\mathcal{Q}}$. If $\phi \in \Sigma_h(\mathcal{Q})$ vanishes on S_Σ, then $\phi = \mu^2 p$ for some $p \in P_1(K_r^{\mathcal{Q}}) \cap H^1(\mathcal{Q})$. Let $\mu_i, p_i \in P_1(K_i^{\mathcal{Q}})$ denote the restrictions of μ and p to $K_i^{\mathcal{Q}}$. Denote by $\ell_i = \partial K_i \cap \partial K_{i+1}$ the interior edge shared by the triangles K_i and K_{i+1}. Then, the C^1–continuity of ϕ yields

$$\nabla\phi|_{\ell_i} = 2\mu p \nabla\mu_i + \mu^2 \nabla p_i = 2\mu p \nabla \mu_{i+1} + \mu^2 \nabla p_{i+1}.$$

Note that $\mu = 0$ at the vertices of $\mathcal{Q}$ and $\nabla\mu_i$ is parallel to the normal direction of the edge $\partial\mathcal{Q} \cap \partial K_i$, especially, $\nabla\mu_i \neq \nabla\mu_{i+1}$. Then, $\big(2p\nabla(\mu_i - \mu_{i+1}) + \mu\nabla(p_i - p_{i+1})\big)|_{\ell_i} = 0$ implies that p vanishes at the vertices of $\mathcal{Q}$, therefore, $p = c\mu$ for some $c \in \mathbb{R}$. Thus, $\phi = c\mu^3$.

However, since $\mu^3 \notin C^1(\mathcal{Q})$, we have $c = 0$. As a result, $\phi = 0$.

Alternative proofs of Lemma 26 are provided in [16, 39]. $\square$

Corollary 2. *There holds* $\Sigma_h(\mathcal{Q}) \cap H_0^2(\mathcal{Q}) = \{0\}$.

Proof. If $\phi \in \Sigma_h(\mathcal{Q}) \cap H_0^2(\mathcal{Q})$, then ϕ vanishes on the degrees of freedom. By Lemma 26, we deduce that $\phi = 0$. $\square$

The smooth de Rham complex and the local space $\Sigma_h(\mathcal{Q})$ suggests that a natural candidate for the local Stokes pair is $(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})) \times P_1(K_r^{\mathcal{Q}})$. Indeed, it is clear that $div(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})) \subseteq P_1(K_r^{\mathcal{Q}})$, and $curl(\Sigma_h(\mathcal{Q})) \subseteq \mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})$. However, the dimension arguments we state in Lemma 27 below show that $(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})) \times P_1(K_r^{\mathcal{Q}})$ is not inf-sup stable, therefore, the global pair induced by this local pair is also not inf-sup stable.

Lemma 27. *The space* $div(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q}))$ *has dimension* 11*. Therefore,* $div : \mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q}) \to P_1(K_r^{\mathcal{Q}})$ *is not surjective.*

Proof. Let $\mathbf{Z}_\star$ denote the kernel of the divergence operator acting on $\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})$, i.e.,

$$\mathbf{Z}_\star = \{\mathbf{v} \in \mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q}) : \; div(\mathbf{v}) = 0\}.$$

If $\mathbf{v} \in \mathbf{Z}_\star$, then $\mathbf{v} = curl\, \phi$ for some $\phi \in H^2(\mathcal{Q})$. Since $\mathbf{v}$ is a piecewise–quadratic polynomial, ϕ is a piecewise–cubic polynomial. As a result, $\phi \in \Sigma_h(\mathcal{Q})$. Therefore, $\mathbf{Z}_\star = curl(\Sigma_h(\mathcal{Q}))$. Then, the rank–nullity theorem and Lemma 26 yield

$$\begin{aligned} dim(div(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})) &= dim(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})) - dim(\mathbf{Z}_\star) \\ &= dim(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})) - dim(curl(\Sigma_h(\mathcal{Q}))) \\ &= dim(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})) - dim(\Sigma_h(\mathcal{Q})) + 1 \\ &= 2\; dim(P_2(K_r^{\mathcal{Q}}) \cap H^1(\mathcal{Q})) - 16 + 1 = 11. \end{aligned}$$

However, $dim(P_1(K_r^{\mathcal{Q}})) = 12 > dim(div(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q}))$. This implies that there exists $q \in P_1(K_r^{\mathcal{Q}}) \setminus \{0\}$ such that $\int_K div(\mathbf{v})\; q = 0$ for all $\mathbf{v} \in \mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})$. Therefore, the divergence operator mapping $\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})$ to $P_1(K_r^{\mathcal{Q}})$ is not surjective and the pressure solution is not unique. $\square$

Remark 6. (Proposition 2.1 in [44]) *The range of the divergence operator is characterized as*

$$div(\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})) = \{q \in P_1(K_r^{\mathcal{Q}}) : \sum_{i=1}^{4} (-1)^i q|_{K_i}(c_K) = 0\}.$$

Lemma 27 implies that the dimension of the candidate pressure space $P_1(K_r^{\mathcal{Q}})$ is larger than the dimension of the range of the divergence operator acting on the candidate velocity space $\mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})$. Therefore, we restrict the range of the divergence operator by defining the velocity space as

$$\mathbf{V}_h(\mathcal{Q}) := \{\mathbf{v} \in \mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q}) : div(\mathbf{v}) \in P_0(\mathcal{Q})\}.$$

Lemma 28. *The dimension of $\mathbf{V}_h(\mathcal{Q})$ is 16, and a function $\mathbf{v} \in \mathbf{V}_h(\mathcal{Q})$ is uniquely determined by the values (See Figure 4.2)*

$$S_v = \{\mathbf{v}(a) : \ a \in \mathcal{V}_{\mathcal{Q}}, \quad \int_e \mathbf{v} \ : \ e \in \mathcal{E}_{\mathcal{Q}}\}. \tag{4.2.2}$$

Proof. It is easy to see that the kernel of the divergence operator acting on $\mathbf{V}_h(\mathcal{Q})$ satisfies $Ker(div(\mathbf{V}_h(\mathcal{Q})) = curl(\Sigma_h(\mathcal{Q}))$. Therefore, by the rank nullity theorem, we have

$$\dim \mathbf{V}_h(\mathcal{Q}) = \dim(curl(\Sigma_h(\mathcal{Q}))) + \dim P_0(\mathcal{Q}) = \dim \Sigma_h(\mathcal{Q}) - 1 + \dim P_0(\mathcal{Q}) = 16.$$

Since the cardinality of S_v is also 16, it suffices to show that if $\mathbf{v} \in \mathbf{V}_h(\mathcal{Q})$ vanishes on S_v, then $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$. If $\mathbf{v} \in \mathbf{V}_h(\mathcal{Q})$ vanishes on S_v, then $\mathbf{v} \in \mathbf{V}_h(\mathcal{Q}) \cap \mathbf{H}_0^1(\mathcal{Q})$ since $\mathbf{v}$ is a piecewise–quadratic polynomial. Therefore, by the divergence theorem, we have

$$\int_{\mathcal{Q}} div(\mathbf{v}) \ = \int_{\partial\mathcal{Q}} \mathbf{v} \cdot \mathbf{n} \ = 0.$$

Since $div(\mathbf{v}) \in P_0(\mathcal{Q})$, this result implies that $\mathbf{v}$ is divergence–free. Thus, $\mathbf{v} = curl\, \phi$ for some $\phi \in \Sigma_h(\mathcal{Q})$, and since $\mathbf{v} \in \mathbf{H}_0^1(\mathcal{Q})$, we may assume that $\phi \in \Sigma_h(\mathcal{Q}) \cap H_0^2(\mathcal{Q})$. However, Corollary 2 implies that $\phi = 0$ in $\mathcal{Q}$, therefore, $\mathbf{v} = \mathbf{0}$ in $\mathcal{Q}$. $\square$

We define the local pressure space as $W_h(\mathcal{Q}) := P_0(\mathcal{Q})$ to ensure that the velocity approximations are pointwise divergence–free.

4.3 THE GLOBAL FINITE ELEMENTS WITH IMPOSED BOUNDARY CONDITIONS

The local spaces defined in Section 4.2 induce the following global finite element spaces:

$$\Sigma_h := \{z \in H_0^2(\Omega) : \ z|_{\mathcal{Q}} \in \Sigma_h(\mathcal{Q}), \ \forall \mathcal{Q} \in \mathcal{Q}_h\},$$

$$\mathbf{V}_h := \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \ \mathbf{v}|_{\mathcal{Q}} \in \mathbf{V}_h(\mathcal{Q}), \ \forall \mathcal{Q} \in \mathcal{Q}_h\},$$

$$W_h := \{q \in L_0^2(\Omega) \ : \ q|_K \in W_h(\mathcal{Q}), \ \forall \mathcal{Q} \in \mathcal{Q}_h\}.$$

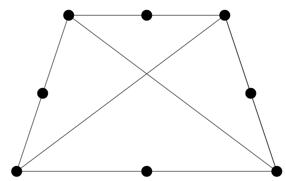


Figure 4.2: Degrees of freedom of the macro velocity element $\mathbf{V}_h(\mathcal{Q})$. Solid circles indicate function evaluations.

Lemma 29. *The sequence stated in* (1.3.2) *is exact provided* Ω *is a simply–connected domain.*

Proof. Clearly, the definitions of the finite element spaces imply that $Ker(div(\mathbf{V}_h)) = curl(\Sigma_h)$. We denote by $|\mathring{\mathcal{V}}|$, $|\mathring{\mathcal{E}}|$ and $|\mathcal{Q}|$ the number of interior vertices, interior edges and quadrilaterals in the mesh. Then, by Lemma 26, $dim(\Sigma_h) = 3|\mathring{\mathcal{V}}| + |\mathring{\mathcal{E}}|$, and by Lemma 28, $dim(\mathbf{V}_h) = 2|\mathring{\mathcal{V}}| + 2|\mathring{\mathcal{E}}|$. Therefore, the rank–nullity theorem yields

$$dim(div(\mathbf{V}_h)) = dim(\mathbf{V}_h) - dim(curl(\Sigma_h)) = dim(\mathbf{V}_h) - dim(\Sigma_h) = |\mathring{\mathcal{E}}| - |\mathring{\mathcal{V}}|.$$

Moreover, by the Euler identity [28], we have $|\mathring{\mathcal{E}}| = |\mathcal{Q}| + |\mathring{\mathcal{V}}| - 1$, and the definition of W_h implies that $dim(W_h) = |\mathcal{Q}| - 1$. Thus, we obtain $dim(div(\mathbf{V}_h)) = dim(W_h)$. Since $div(\mathbf{V}_h) \subseteq W_h$, this implies that $div(\mathbf{V}_h) = W_h$. As a result, the complex (1.3.2) is exact. □

4.4 INF-SUP STABILITY

In this section, we carry out the stability analysis and verify the well–posedness of the discrete problem and the uniqueness of the numerical solution. We show that the finite element pair $\mathbf{V}_h \times W_h$ forms an inf–sup stable finite element pair, therefore, the proposed method yields a unique solution.

For $\mathcal{Q} \in \mathcal{Q}_h$, we define

$$\|\mathbf{v}\|_{h,\mathcal{Q}}^2 := \sum_{a\in\mathcal{V}_\mathcal{Q}} |\mathbf{v}(a)|^2 + \sum_{e\in\mathcal{E}_\mathcal{Q}} h_e^{-2} |\int_e \mathbf{v}|^2. \tag{4.4.1}$$

Lemma 30. *$\|\cdot\|_{h,\mathcal{Q}}$ is a H^1–type norm on $\mathbf{V}_h(\mathcal{Q})$.*

Proof. The result follows by the unisolvency of S_v over $\mathbf{V}_h(\mathcal{Q})$ (See Lemma 28). □

Lemma 31. *Let $\mathcal{P}_W : L_0^2(\Omega) \to W_h$ denote the L^2–projection onto W_h. Then, there exists a Fortin operator $\Pi_h : \mathbf{H}_0^1(\Omega) \to \mathbf{V}_h$ such that $div(\Pi_h \mathbf{v}) = \mathcal{P}_W div(\mathbf{v})$ and $\|\Pi_h \mathbf{v}\|_{H^1(\Omega)} \leq C\|\mathbf{v}\|_{H^1(\Omega)}$ for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$. Consequently, $\mathbf{V}_h \times W_h$ forms an inf–sup stable finite element pair. Moreover, the error of the Fortin interpolant satisfies*

$$|\Pi_h \mathbf{v} - \mathbf{v}|_{H^1(\Omega)} \leq Ch|\mathbf{v}|_{H^2(\Omega)}. \tag{4.4.2}$$

Proof. Define $\Pi_h : \mathbf{H}_0^1(\Omega) \to \mathbf{V}_h$ such that it satisfies the following conditions:

$$\int_e \Pi_h \mathbf{v} = \int_e \mathbf{v}, \qquad \forall e \in \mathcal{E}_\mathcal{Q}, \tag{4.4.3a}$$

$$\Pi_h(\mathbf{v})(a) = \frac{1}{|\mathcal{Q}_a|} \sum_{\mathcal{Q}'\in\mathcal{Q}_a} \mathcal{P}_{h,\mathcal{Q}'}\mathbf{v}(a), \qquad \forall a \in \mathcal{V} \setminus \mathcal{V}_b, \tag{4.4.3b}$$

$$\Pi_h(\mathbf{v})(a) = 0, \qquad \forall a \in \mathcal{V}_b, \tag{4.4.3c}$$

where $|\mathcal{Q}_a|$ denotes the cardinality of the set $\mathcal{Q}_a$.

To prove the inf–sup stability, it suffices to show that Π_h is a Fortin operator, that is, it suffices to verify that Π_h satisfies [9]:

$$\int_\Omega div(\mathbf{v}) = \int_\Omega div(\Pi_h \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{4.4.4a}$$

$$\|\Pi_h \mathbf{v}\|_{H^1(\Omega)} \leq c\|\mathbf{v}\|_{H^1(\Omega)}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega). \tag{4.4.4b}$$

By the divergence theorem and (4.4.3a),

$$\int_\mathcal{Q} div(\Pi_h \mathbf{v}) = \int_{\partial\mathcal{Q}} (\Pi_h \mathbf{v} \cdot \mathbf{n}) = \int_{\partial\mathcal{Q}} (\mathbf{v} \cdot \mathbf{n}) = \int_\mathcal{Q} div(\mathbf{v}), \quad \forall \mathcal{Q} \in \mathcal{Q}_h.$$

Thus, (4.4.4a) holds. This implies that $div(\Pi_h \mathbf{v}) = \mathcal{P}_W div(\mathbf{v})$.

By the triangle inequality and (4.1.6), we have

$$\begin{aligned}\|\Pi_h \mathbf{v}\|_{H^1(\Omega)}^2 &\leq c \sum_{\mathcal{Q}\in\mathcal{Q}_h} (\|\Pi_h \mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|_{H^1(\mathcal{Q})}^2 + \|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|_{H^1(\mathcal{Q})}^2) \\ &\leq c \sum_{\mathcal{Q}\in\mathcal{Q}_h} (\|\Pi_h \mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|_{H^1(\mathcal{Q})}^2 + \|\mathbf{v}\|_{H^1(\mathcal{Q})}^2).\end{aligned} \tag{4.4.5}$$

Since $\mathbf{P}_1(\mathcal{Q}) \subset \mathbf{V}_h(\mathcal{Q})$, Lemma 30 implies

$$\|\Pi_h \mathbf{v} - \mathcal{P}_h \mathbf{v}\|^2_{H^1(\mathcal{Q})} \approx \|\Pi_h \mathbf{v} - \mathcal{P}_h \mathbf{v}\|^2_{h,\mathcal{Q}}$$
$$= \sum_{a\in\mathcal{V}_\mathcal{Q}} |\Pi_h \mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2 + \sum_{e\in\mathcal{E}_\mathcal{Q}} h_e^{-2} |\int_e (\Pi_h \mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v})|^2. \tag{4.4.6}$$

For $a \in \mathcal{V}_\mathcal{Q} \setminus \mathcal{V}_b$, applying the triangle inequality to the first expression in (4.4.6) and using (4.4.3b), we obtain

$$|\Pi_h \mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2 = |\frac{1}{|\mathcal{Q}_a|} \sum_{\mathcal{Q}'\in\mathcal{Q}_a} (\mathcal{P}_{h,\mathcal{Q}'}\mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a))|^2$$
$$\leq \frac{C}{|\mathcal{Q}_a|^2} \sum_{\mathcal{Q}'\in\mathcal{Q}_a} |\mathcal{P}_{h,\mathcal{Q}'}\mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2, \tag{4.4.7}$$

where $C > 0$ depends only on the shape–regularity of the mesh.

Then, for $\mathcal{Q}, \mathcal{Q}' \in \mathcal{Q}_a$, there exists $\{\mathcal{Q}_i\}_{i=0}^m \subset \mathcal{Q}_a$ such that $\mathcal{Q}_i$ and $\mathcal{Q}_{i+1}$ share a common edge. Setting $\mathcal{Q}_0 = \mathcal{Q}$, $\mathcal{Q}_m = \mathcal{Q}'$, and using the inverse inequality, we obtain

$$|\mathcal{P}_{h,\mathcal{Q}'}\mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2 \leq \sum_{i=0}^m |\mathcal{P}_{h,\mathcal{Q}_{i+1}}\mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}_i}\mathbf{v}(a)|^2$$
$$\leq \sum_{e\in\mathcal{E}_a} \|[\mathcal{P}_h \mathbf{v}]\|^2_{L^\infty(e)}$$
$$\leq \sum_{e\in\mathcal{E}_a} h_e^{-1} \|[\mathcal{P}_h \mathbf{v}]\|^2_{L^2(e)}, \tag{4.4.8}$$

where $[\mathcal{P}_h \mathbf{v}]|_e = |(\mathcal{P}_{h,\mathcal{Q}_{i+1}}\mathbf{v})|_e - (\mathcal{P}_{h,\mathcal{Q}_i}\mathbf{v})|_e|$ for $e \in \mathcal{E}_a$ with $e = \partial\mathcal{Q}_{i+1} \cap \partial\mathcal{Q}_i$. We note that

$$\sum_{e\in\mathcal{E}_a} h_e^{-1} \|[\mathcal{P}_h \mathbf{v}]\|^2_{L^2(e)} \leq \sum_{e\in\mathcal{E}_a} h_e^{-1} \|[\mathcal{P}_h \mathbf{v} - \mathbf{v}]\|^2_{L^2(e)}. \tag{4.4.9}$$

Finally, combining (4.4.7)-(4.4.9), we obtain

$$|\Pi_h \mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2 \leq C \sum_{e\in\mathcal{E}_a} h_e^{-1} \|\mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|^2_{L^2(e)}. \tag{4.4.10}$$

For $a \in \mathcal{V}_b$, since $\Pi_h \mathbf{v}(a) = 0$ at all $a \in \mathcal{V}_b$, (4.4.8) and (4.4.9) yield

$$|\Pi_h \mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2 = |\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2 \leq \sum_{e\in\mathcal{E}_a^b} h_e^{-1} \|\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|^2_{L^2(e)},$$
$$\leq \sum_{e\in\mathcal{E}_a^b} h_e^{-1} \|\mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|^2_{L^2(e)}. \tag{4.4.11}$$

Combining (4.4.10)-(4.4.11) and using the trace inequality, for $a \in \mathcal{V}_\mathcal{Q}$, we obtain

$$|\Pi_h \mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2 \leq C \sum_{\mathcal{Q}\in\mathcal{Q}_a} (|\mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}|^2_{H^1(\mathcal{Q})} + h^{-2} \|\mathbf{v} - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|^2_{L^2(\mathcal{Q})}). \tag{4.4.12}$$

Then, (4.1.1) and (4.1.2) imply

$$\sum_{a\in\mathcal{V}_{\mathcal{Q}}} |\Pi_h \mathbf{v}(a) - \mathcal{P}_{h,\mathcal{Q}}\mathbf{v}(a)|^2 \le C|\mathbf{v}|^2_{H^1(\omega_{\mathcal{Q}})}, \tag{4.4.13}$$

where $\omega_{\mathcal{Q}} = \bigcup\limits_{a\in\mathcal{V}_{\mathcal{Q}}} \bigcup\limits_{\mathcal{Q}'\in\mathcal{Q}_a} \mathcal{Q}'$. Now consider the second expression in (4.4.6).

By (4.4.3a), the Cauchy-Schwarz, the trace inequality, (4.1.1) and (4.1.2), respectively, we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_a} h_e^{-2}|\int_e(\Pi_h\mathbf{v}-\mathcal{P}_h\mathbf{v})|^2 &= \sum_{e\in\mathcal{E}_a} h_e^{-2}|\int_e(\mathbf{v}-\mathcal{P}_h\mathbf{v})|^2 \le \sum_{e\in\mathcal{E}_a} h_e^{-1}\|\mathbf{v}-\mathcal{P}_h\mathbf{v}\|^2_{L^2(e)} \\ &\le c\sum_{\mathcal{Q}\in\mathcal{Q}_a}(\|\nabla(\mathbf{v}-\mathcal{P}_{h,\mathcal{Q}}\mathbf{v})\|^2_{L^2(\mathcal{Q})} + h^{-2}\|\mathbf{v}-\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|^2_{L^2(\mathcal{Q})}) \\ &\le c|\mathbf{v}|^2_{H^1(\omega_{\mathcal{Q}})}. \end{aligned} \tag{4.4.14}$$

Using (4.4.13) and (4.4.14) in (4.4.6) yields

$$\|\Pi_h\mathbf{v}-\mathcal{P}_h\mathbf{v}\|^2_{H^1(\mathcal{Q})} \le C|\mathbf{v}|^2_{H^1(\omega_{\mathcal{Q}})}. \tag{4.4.15}$$

Consequently, summing over $\mathcal{Q}\in\mathcal{Q}_h$, then using (4.4.5), we deduce that

$$\|\Pi_h\mathbf{v}\|_{H^1(\Omega)} \le c\|\mathbf{v}\|_{H^1(\Omega)}.$$

Thus, (4.4.4b) holds. As a result, Π_h is a Fortin operator. It suffices to prove the estimate (4.4.2) to complete the proof.

For $\mathbf{v}\in\mathbf{H}^2(\mathcal{Q})$, (4.1.2) and (4.1.1) yield

$$|\mathbf{v}-\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}|_{H^1(\mathcal{Q})} \le ch|\mathbf{v}|_{H^2(\mathcal{Q})} \tag{4.4.16a}$$

$$\|\mathbf{v}-\mathcal{P}_{h,\mathcal{Q}}\mathbf{v}\|_{L^2(\mathcal{Q})} \le ch^2|\mathbf{v}|_{H^2(\mathcal{Q})} \tag{4.4.16b}$$

Using (4.4.16) in (4.4.12) and (4.4.14), then summing (4.4.6) over $\mathcal{Q}\in\mathcal{Q}_h$, we obtain

$$\|\Pi_h\mathbf{v}-\mathcal{P}_h\mathbf{v}\|_{H^1(\Omega)} \le ch|\mathbf{v}|_{H^2(\Omega)}. \tag{4.4.17}$$

Then, by the triangle inequality and scaling, we have

$$\begin{aligned}|\Pi_h\mathbf{v}-\mathbf{v}|^2_{H^1(\Omega)} &\le c(|\Pi_h\mathbf{v}-\mathcal{P}_h\mathbf{v}|^2_{H^1(\Omega)} + |\mathcal{P}_h\mathbf{v}-\mathbf{v}|^2_{H^1(\Omega)}) \\ &\le c(\|\Pi_h\mathbf{v}-\mathcal{P}_h\mathbf{v}\|^2_{H^1(\Omega)} + |\mathcal{P}_h\mathbf{v}-\mathbf{v}|^2_{H^1(\Omega)})\end{aligned} \tag{4.4.18}$$

Therefore, using (4.1.2) and (4.4.17) in (4.4.18) yield

$$|\Pi_h\mathbf{v}-\mathbf{v}|_{H^1(\Omega)} \le ch|\mathbf{v}|_{H^2(\Omega)}.$$

□

4.5 CONVERGENCE ANALYSIS

Recall the weak formulation of the Stokes system (1.0.1)

$$\begin{aligned} \nu \int \nabla \mathbf{u} : \nabla \mathbf{v} - \int div(\mathbf{v})\ p &= \int \mathbf{f} \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ \int div(\mathbf{u})\ q &= 0, \qquad\qquad \forall q \in L_0^2(\Omega), \end{aligned} \tag{4.5.1}$$

and the discretized version of (4.5.1) given by

$$\begin{aligned} \nu \int \nabla \mathbf{u}_h : \nabla \mathbf{v}_h - \int div(\mathbf{v}_h)\ p_h &= \int \mathbf{f} \cdot \mathbf{v}_h, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \int div(\mathbf{u}_h)\ q_h &= 0, \qquad\qquad \forall q_h \in W_h, \end{aligned} \tag{4.5.2}$$

As a consequence of Lemma 31, the discrete problem (4.5.2) is well-posed.

We define the continuous and the discrete divergence–free spaces

$$\mathbf{Z} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : div(\mathbf{v}) = 0\}. \tag{4.5.3a}$$

$$\mathbf{Z}_h = \{\mathbf{v} \in \mathbf{V}_h : div(\mathbf{v}) = 0\}. \tag{4.5.3b}$$

Restricting the domains of the velocity spaces in (4.5.1) and (4.5.2) to $\mathbf{Z}$ and $\mathbf{Z}_h$ reduces the systems of equations given by (4.5.1) and (4.5.2) to the equations (4.5.4a) and (4.5.4b) as follows:

$$\nu \int \nabla \mathbf{u} : \nabla \mathbf{v} = \int \mathbf{f} \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in \mathbf{Z}, \tag{4.5.4a}$$

$$\nu \int \nabla \mathbf{u}_h : \nabla \mathbf{v}_h = \int \mathbf{f} \cdot \mathbf{v}_h, \qquad \forall \mathbf{v}_h \in \mathbf{Z}_h. \tag{4.5.4b}$$

Theorem 12. *Suppose that $(\mathbf{u}, p)$ solves (4.5.4a) and $(\mathbf{u}_h, p_h)$ solves (4.5.4b), then for a convex polygonal domain Ω, the error estimates satisfy the following:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)} \le ch^s |\mathbf{u}|_{H^{s+1}(\Omega)}, \qquad \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \le ch^{s+1} \|\mathbf{u}\|_{H^{s+1}(\Omega)},$$

$$\|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)} \le ch(|p|_{H^1(\Omega)} + \nu h^{s-1} |\mathbf{u}|_{H^{s+1}}), \qquad for \quad s = 1, 2,$$

where the constant $c > 0$ is independent of the discretization parameter and the viscosity.

Proof. Since $\mathbf{Z}_h$ is divergence–free conforming, by Cea's Lemma, we have

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)} \le \inf_{\mathbf{v}_h \in \mathbf{Z}_h} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L^2(\Omega)}. \tag{4.5.5}$$

Additionally, since $\mathbf{Z}_h = curl(\Sigma_h)$, there exists $\sigma_h \in \Sigma_h$ for every $\mathbf{v}_h \in \mathbf{Z}_h$ such that $curl(\sigma_h) = \mathbf{v}_h$. Thus, (4.5.5) is equivalent to

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)} \le \inf_{\sigma_h \in \Sigma_h} \|\nabla(\mathbf{u} - curl(\sigma_h))\|_{L^2(\Omega)}. \tag{4.5.6}$$

Writing $\mathbf{u}$ in terms of its stream function $\psi_h \in H_0^2(\Omega)$, we have $\mathbf{u} = curl(\psi_h)$. Then, (4.5.6) becomes

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)} \leq \inf_{\sigma_h\in\Sigma_h} \|\nabla(curl(\psi_h) - curl(\sigma_h))\|_{L^2(\Omega)} = \inf_{\sigma_h\in\Sigma_h} |\psi_h - \sigma_h|_{H^2(\Omega)}. \quad (4.5.7)$$

By Theorem 6.18 in [28], for every $\psi_h \in H^{m+1}(\Omega)$, there exists $\sigma_h \in \Sigma_h$ that satisfies

$$\|D^\alpha(\sigma_h - \psi_h)\|_{L^2(\Omega)} \leq ch^{m+1-\alpha}|\psi_h|_{H^{m+1}(\Omega)}, \qquad 0 \leq m \leq 3,\ 0 \leq \alpha \leq m, \quad (4.5.8)$$

where c is an h–independent constant. Since (4.5.7) implies that $\alpha = 2$, $m \geq 2$ in (4.5.8), and we may rewrite (4.5.7) as

$$\begin{aligned}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)} &\leq \inf_{\sigma_h\in\Sigma_h} |\psi_h - \sigma_h|_{H^2(\Omega)} \leq ch^s|\psi_h|_{H^{s+2}(\Omega)} \\ &= ch^s|\mathbf{u}|_{H^{s+1}(\Omega)}, \quad s = 1, 2.\end{aligned} \quad (4.5.9)$$

To derive an error estimate for the velocity solution in the L^2-norm, we use the Aubin–Nitsche duality technique [10]. Setting $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and letting $\boldsymbol{\gamma} \in \mathbf{H}^2(\Omega)$ be the solution of the dual problem on the convex polygonal domain Ω corresponding to (4.5.4b), we obtain

$$\nu(\nabla\mathbf{e}, \nabla\boldsymbol{\gamma}) = -\nu(\mathbf{e}, \Delta\boldsymbol{\gamma}) = (\mathbf{e}, \mathbf{e}). \quad (4.5.10)$$

Since $\Pi_h\boldsymbol{\gamma} \in \mathbf{Z}_h \subseteq \mathbf{Z}$, (4.5.4) implies

$$\nu(\nabla\mathbf{e}, \nabla\Pi_h\boldsymbol{\gamma}) = 0. \quad (4.5.11)$$

Therefore, we have

$$\nu(\nabla\mathbf{e}, \nabla\boldsymbol{\gamma}) = \nu(\nabla\mathbf{e}, \nabla(\boldsymbol{\gamma} - \Pi_h\boldsymbol{\gamma})). \quad (4.5.12)$$

By (4.4.2) and (4.5.9)–(4.5.12), we have

$$\|\mathbf{e}\|^2_{L^2(\Omega)} \leq |\mathbf{e}|_{H^1(\Omega)}|\boldsymbol{\gamma} - \Pi_h\boldsymbol{\gamma}|_{H^1(\Omega)} \leq c|\mathbf{e}|_{H^1(\Omega)}h|\boldsymbol{\gamma}|_{H^2(\Omega)}.$$

Since Ω is convex [9], the regularity estimate $\|\boldsymbol{\gamma}\|_{H^2(\Omega)} \leq c\|\mathbf{e}\|_{L^2(\Omega)}$ holds, and this yields $\|\mathbf{e}\|_{L^2(\Omega)} \leq ch|\mathbf{e}|_{H^1(\Omega)}$. Combining this result with (4.5.9), we obtain the L^2 error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \leq ch^{s+1}\|\mathbf{u}\|_{H^{s+1}(\Omega)} \quad (4.5.13)$$

In addition, by the inf–sup condition and standard arguments [9], we have

$$\|p_h - \mathcal{P}_W p\|_{L^2(\Omega)} \leq \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)} \leq C\nu h^s|\mathbf{u}|_{H^{s+1}(\Omega)}, \quad s = 1, 2. \quad (4.5.14)$$

Then, the triangle inequality and the properties of the L^2–projection imply that the error in the pressure approximation satisfies

$$\begin{aligned}\|p - p_h\|_{L^2(\Omega)} &\leq \|p - \mathcal{P}_W p\|_{L^2(\Omega)} + \|p_h - \mathcal{P}_W p\|_{L^2(\Omega)} \\ &\leq ch(|p|_{H^1(\Omega)} + \nu h^{s-1}|\mathbf{u}|_{H^{s+1}(\Omega)}), \qquad s = 1, 2.\end{aligned}$$

□

4.5.1 Convergence Analysis for the Post-Processed Pressure Solution

Theorem 12 shows that the velocity and pressure approximations have different orders of convergence. In this section, we develop a locally–computed, post–processed pressure solution that has the same quadratic convergence as the velocity error by making use of the super convergence property (4.5.14).

For $\mathcal{Q} \in \mathcal{Q}_h$, we define $p^*_{\mathcal{Q}} \in P_1(\mathcal{Q})$ that satisfies

$$\int_{\mathcal{Q}} \nabla p^*_{\mathcal{Q}} \cdot \nabla q = \nu \int_{\mathcal{Q}} \Delta_h \mathbf{u}_h \cdot \nabla q + \int_{\mathcal{Q}} \mathbf{f} \cdot \nabla q, \qquad \forall q \in P_1(\mathcal{Q}), \tag{4.5.15a}$$

$$\int_{\mathcal{Q}} p^*_{\mathcal{Q}} = \int_{\mathcal{Q}} p_h, \tag{4.5.15b}$$

where the discrete laplacian operator Δ_h is defined piecewise with respect to $\mathcal{Q}$, that is, $\Delta_h \mathbf{u}_h|_{K_i} = \Delta \mathbf{u}_h|_{K_i}$ for $1 \le i \le 4$ and $K_i \in K_r^{\mathcal{Q}}$. Since $\nabla p^*_{\mathcal{Q}}$ and ∇q are both constant, (4.5.15a) yields

$$\nabla p^*_{\mathcal{Q}} \cdot \nabla q = \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} (\nu \Delta_h \mathbf{u}_h + \mathbf{f}) \cdot \nabla q \ , \qquad \forall q \in P_1(\mathcal{Q}). \tag{4.5.16}$$

By letting $q = x_i$, we derive a formula for the post-processed pressure solution

$$p^*_{\mathcal{Q}} = c^*_{\mathcal{Q}} + \frac{\mathbf{x}}{|\mathcal{Q}|} \cdot \int_{\mathcal{Q}} (\nu \Delta_h \mathbf{u}_h + \mathbf{f}) \ , \tag{4.5.17}$$

with the constant $c^*_{\mathcal{Q}}$ chosen such that (4.5.15b) is satisfied.

Theorem 13. *Suppose that the solution of the weak problem* $(\mathbf{u}, p) \in \mathbf{H}^3(\Omega) \times \mathbf{H}^2(\Omega)$*. Let* $p^* \in P_1(\mathcal{Q}_h)$ *satisfy* (4.5.15) *on each* $\mathcal{Q} \in \mathcal{Q}_h$*, i.e.,* $p^*|_{\mathcal{Q}} = p^*_{\mathcal{Q}}$*, where* $p^*_{\mathcal{Q}}$ *is defined by* (4.5.17)*. Then, there holds*

$$\|p - p^*\|_{L^2(\Omega)} \le ch^2(\nu \|\mathbf{u}\|_{H^3(\Omega)} + \|p\|_{H^2(\Omega)}).$$

Proof. Since p, $\mathcal{P}_W p \in L^2_0(\Omega)$, we have

$$\int_{\Omega} (p - \mathcal{P}_W p) \ = 0.$$

Fix $\mathcal{Q}_1 \in \mathcal{Q}_h$ and choose $\mathcal{Q} \in \mathcal{Q}_h$ with $\mathcal{Q} \neq \mathcal{Q}_1$. Let $q|_{\mathcal{Q}_1} = 1$, $q|_{\mathcal{Q}} = -\frac{|\mathcal{Q}_1|}{|\mathcal{Q}|}$, and $q = 0$ otherwise. Thus, $q \in W_h \subset L^2_0(\Omega)$, and by the definition of the L^2–projection,

$$\int_{\mathcal{Q}} (p - \mathcal{P}_W p) \ = \frac{|\mathcal{Q}|}{|\mathcal{Q}_1|} \int_{\mathcal{Q}_1} (p - \mathcal{P}_W p).$$

This identity holds for all $\mathcal{Q} \in \mathcal{Q}_h$, and it follows that

$$0 = \int_{\Omega} (p - \mathcal{P}_W p) = \sum_{\mathcal{Q} \in \mathcal{Q}_h} \int_{\mathcal{Q}} (p - \mathcal{P}_W p)$$

$$= \sum_{\mathcal{Q}\in\mathcal{Q}_h} \frac{|\mathcal{Q}|}{|\mathcal{Q}_1|} \int_{\mathcal{Q}_1} (p - \mathcal{P}_W p) \ = \frac{|\Omega|}{|\mathcal{Q}_1|} \int_{\mathcal{Q}_1} (p - \mathcal{P}_W p) \ .$$

Thus, we have $\int_{\mathcal{Q}}(p - \mathcal{P}_W p) \ = 0$ for all $\mathcal{Q} \in \mathcal{Q}_h$. Then, applying Poincaré–Friedrich's inequality on $\mathcal{Q} \in \mathcal{Q}_h$, we obtain

$$\|p - p^*_{\mathcal{Q}}\|_{L^2(\mathcal{Q})} \leq c(\|\overline{p - p^*_{\mathcal{Q}}}\|_{L^2(\mathcal{Q})} + h_{\mathcal{Q}}\|\nabla(p - p^*_{\mathcal{Q}})\|_{L^2(\mathcal{Q})}), \tag{4.5.18}$$

where $(\overline{p - p^*_{\mathcal{Q}}})$ denotes the mean of $(p - p^*_{\mathcal{Q}})$ over $\mathcal{Q}$. The definition of $\mathcal{P}_W p$ and (4.5.15b) imply

$$\overline{p - p^*_{\mathcal{Q}}} = \overline{p} - \overline{p^*_{\mathcal{Q}}} = \overline{\mathcal{P}_W p} - \overline{p_h} = \overline{\mathcal{P}_W p - p_h}.$$

Therefore,

$$\|\overline{p - p^*_{\mathcal{Q}}}\|_{L^2(\mathcal{Q})} = \|\overline{\mathcal{P}_W p - p_h}\|_{L^2(\mathcal{Q})} \leq \|\mathcal{P}_W p - p_h\|_{L^2(\mathcal{Q})}. \tag{4.5.19}$$

By the triangle inequality, (4.5.15a) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\nabla(p - p^*_{\mathcal{Q}})\|^2_{L^2(\mathcal{Q})} &= \int_{\mathcal{Q}} \nabla(p - p^*_{\mathcal{Q}}) \cdot \nabla(p - q) + \int_{\mathcal{Q}} \nabla(p - p^*_{\mathcal{Q}}) \cdot \nabla(q - p^*_{\mathcal{Q}}) \\ &= \int_{\mathcal{Q}} \nabla(p - p^*_{\mathcal{Q}}) \cdot \nabla(p - q) + \nu \int_{\mathcal{Q}} \Delta_h(\mathbf{u} - \mathbf{u}_h) \ \nabla(q - p^*_{\mathcal{Q}}) \\ &\leq \|\nabla(p - p^*_{\mathcal{Q}})\|_{L^2(\mathcal{Q})}\|\nabla(p - q)\|_{L^2(\mathcal{Q})} \\ &\quad + \nu\|\Delta_h(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\mathcal{Q})}\|\nabla(q - p^*_{\mathcal{Q}})\|_{L^2(\mathcal{Q})}, \end{aligned} \tag{4.5.20}$$

for all $q \in P_1(\mathcal{Q})$. Again, by the triangle inequality, we have

$$\|\nabla(q - p^*_{\mathcal{Q}})\|_{L^2(\mathcal{Q})} \leq \|\nabla(q - p)\|_{L^2(\mathcal{Q})} + \|\nabla(p - p^*_{\mathcal{Q}})\|_{L^2(\mathcal{Q})} \tag{4.5.21}$$

Note that $\mathcal{P}_W p|_{\mathcal{Q}} \in P_0(\mathcal{Q}) \subset P_1(\mathcal{Q})$ and letting $q = \mathcal{P}_W p$ in (4.5.21) yields:

$$\|\nabla(q - p^*_{\mathcal{Q}})\|_{L^2(\mathcal{Q})} \leq \|\nabla(p - p^*_{\mathcal{Q}})\|_{L^2(\mathcal{Q})}. \tag{4.5.22}$$

Using (4.5.22) in (4.5.20), we obtain

$$\|\nabla(p - p^*_{\mathcal{Q}})\|_{L^2(\mathcal{Q})} \leq c(\|\nabla(p - q)\|_{L^2(\mathcal{Q})} + \nu\|\Delta_h(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\mathcal{Q})}). \tag{4.5.23}$$

Combining (4.5.18), (4.5.19) and (4.5.23) results in

$$\begin{aligned} \|p - p^*_{\mathcal{Q}}\|_{L^2(\mathcal{Q})} \leq & c(\|p_h - \mathcal{P}_W p\|_{L^2(\mathcal{Q})} + \nu h_{\mathcal{Q}}\|\Delta_h(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\mathcal{Q})} \\ & + \inf_{q\in P_1(\mathcal{Q})} h_{\mathcal{Q}}\|\nabla(p - q)\|_{L^2(\mathcal{Q})}). \end{aligned} \tag{4.5.24}$$

In addition, by the Bramble-Hilbert lemma and the regularity of p, we have

$$\inf_{q\in P_1(\mathcal{Q})} h_{\mathcal{Q}}\|\nabla(p - q)\|_{L^2(\mathcal{Q})} \leq ch^2_{\mathcal{Q}}\|p\|_{H^2(\mathcal{Q})}. \tag{4.5.25}$$

Furthermore, by the inverse inequality and the triangle inequality, for any $\mathbf{v} \in \mathbf{P}_2(K^{\mathcal{Q}}_r) \cap$

$\mathbf{H}^1(\mathcal{Q})$, there holds

$$\begin{aligned} h_{\mathcal{Q}}\|\Delta_h(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\mathcal{Q})} &\le h_{\mathcal{Q}}\|\Delta_h(\mathbf{u}-\mathbf{v})\|_{L^2(\mathcal{Q})} + c\|\mathbf{u}_h-\mathbf{v}\|_{H^1(\mathcal{Q})} \\ &\le c(h_{\mathcal{Q}}\|\Delta_h(\mathbf{u}-\mathbf{v})\|_{L^2(\mathcal{Q})} + \|\mathbf{u}-\mathbf{v}\|_{H^1(\mathcal{Q})} + \|\mathbf{u}-\mathbf{u}_h\|_{H^1(\mathcal{Q})}). \end{aligned} \tag{4.5.26}$$

Let $\mathbf{v}$ be the nodal interpolant of $\mathbf{u}$. Then, (4.5.26) yields

$$h_{\mathcal{Q}}\|\Delta(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\mathcal{Q})} \le c(h_{\mathcal{Q}}^2\|\mathbf{u}\|_{H^3(w(\mathcal{Q}))} + \|\mathbf{u}-\mathbf{u}_h\|_{H^1(\mathcal{Q})}), \tag{4.5.27}$$

where $w(\mathcal{Q})$ denotes the set of quadrilaterals that intersect with $\mathcal{Q}$.

Applying (4.5.27) and (4.5.25) to (4.5.24), we obtain

$$\|p-p_{\mathcal{Q}}^*\|_{L^2(\mathcal{Q})} \le c(\|p_h-\mathcal{P}_W p\|_{L^2(\mathcal{Q})} + \nu\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\mathcal{Q})} + h_{\mathcal{Q}}^2(\nu\|\mathbf{u}\|_{H^3(\mathcal{Q})} + \|p\|_{H^2(\mathcal{Q})})).$$

Summing over $\mathcal{Q}\in\mathcal{Q}_h$ and using (4.5.9),

$$\begin{aligned} \|p-p^*\|_{L^2(\Omega)} &\le c(\nu\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)} + h_{\mathcal{Q}}^2(\nu\|\mathbf{u}\|_{H^3(\Omega)} + \|p\|_{H^2(\Omega)})) \\ &\le ch_{\mathcal{Q}}^2(\nu\|\mathbf{u}\|_{H^3(\Omega)} + \|p\|_{H^2(\Omega)}). \end{aligned}$$

□

4.6 IMPLEMENTATION

In this section, we describe how we build the velocity space $\mathbf{V}_h(\mathcal{Q})$ in detail. More precisely, we present the computation and the implementation of a basis of $\mathbf{V}_h(\mathcal{Q})$. We note that, unlike the serendipity Taylor-Hood pair, our local velocity space is defined on a physical element of the mesh, and it is not invariant under bilinear maps. These restrictions may suggest that the basis must be solved locally on each quadrilateral $\mathcal{Q}\in\mathcal{Q}_h$, and this leads to solving a (16×16) linear system for each element. Here, we discuss an alternative construction which is more efficient and may possibly be extended to isoparametric elements.

4.6.1 Construction of a Canonical Basis

Our goal is to construct an affine, bijective map between a physical element $\mathcal{Q}$ and a reference element. In Chapter 2, we have used the unit square as the reference element. However, since there does not exist an affine map between a quadrilateral and the unit square, here we can not use the unit square as the reference element, therefore, we define a skewed version of the unit square as the reference element and denote it by $\hat{\mathcal{Q}}_{\hat{A}}$ [23]. In the construction of $\hat{\mathcal{Q}}_{\hat{A}}$, we employ three vertices of $\mathcal{Q}$ to build an affine map and then we use this affine map to identify $\hat{A}$ that characterizes $\hat{\mathcal{Q}}_{\hat{A}}$.

For a quadrilateral $\mathcal{Q} \in \mathcal{Q}_h$, denote by T_1, T_2 the two triangles obtained by splitting $\mathcal{Q}$ from opposite vertices. Let $A = (A_1, A_2)$ be the unique vertex of T_2 that is not a vertex of T_1 and denote by $\hat{T}$ the reference triangle with the vertices $(0,0)$, $(1,0)$, $(0,1)$. Then, define an affine bijection $\mathcal{F} : \hat{T} \to T_1$, and set $\hat{T}_2 = \mathcal{F}^{-1}(T_2)$ and $\hat{A} = (\hat{A}_1, \hat{A}_2) := \mathcal{F}^{-1}(A)$. Moreover, set $\hat{T}_1 := \hat{T}$. Then, the union $\hat{T}_1 \cup \hat{T}_2$ gives a convex quadrilateral with the vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(\hat{A}_1, \hat{A}_2)$ (See Figure 4.3). We define the skewed reference element as $\hat{\mathcal{Q}}_{\hat{A}} := \hat{T}_1 \cup \hat{T}_2$. We further note that $\mathcal{F} : \hat{\mathcal{Q}}_{\hat{A}} \to \mathcal{Q}$ is an affine, bijective transformation.
Let $\mathbf{V}(\hat{\mathcal{Q}}_{\hat{A}})$ denote the velocity space defined on the reference quadrilateral $\hat{\mathcal{Q}}_{\hat{A}}$. Denote the set of vertices and edge midpoints of $\hat{\mathcal{Q}}_{\hat{A}}$ by $\{\hat{a}_j\}_{j=1}^{8}$ as shown in Figure 4.4. Then, define $\{\hat{\mathbf{v}}_i^{(k)}\} \subset \mathbf{V}(\hat{\mathcal{Q}}_{\hat{A}})$ that satisfy $\hat{\mathbf{v}}_i^{(k)}(\hat{a}_j) = \delta_{i,j}\mathbf{e}_k$, where $\mathbf{e}_1 = (1,0)^t$ and $\mathbf{e}_2 = (0,1)^t$, for $1 \le i,\ j \le 8$ and $1 \le k \le 2$. Thus, $\{\hat{\mathbf{v}}_i^{(k)}\}$ forms a canonical basis of $\mathbf{V}(\hat{\mathcal{Q}}_{\hat{A}})$.
By using a modified Piola transform, we define the following velocity functions on $\mathcal{Q}$:

$$\mathbf{v}_i^{(k)} = \mathbf{B}(\beta_1^{(k)}\hat{\mathbf{v}}_i^{(1)} + \beta_2^{(k)}\hat{\mathbf{v}}_i^{(2)}),$$

where

$$\begin{pmatrix} \beta_1^{(k)} \\ \beta_2^{(k)} \end{pmatrix} := \mathbf{B}^{-1}\mathbf{e}_k, \quad x = \mathcal{F}(\hat{x}), \quad \hat{x} \in \hat{\mathcal{Q}}.$$

Clearly, $\mathbf{v}_i^{(k)} \in \mathbf{P}_2(K_r^{\mathcal{Q}}) \cap \mathbf{H}^1(\mathcal{Q})$. Moreover, the divergence of $\mathbf{v}_i^{(k)}$ satisfies

$$div(\mathbf{v}_i^{(k)}) = \beta_1^{(k)}\widehat{div}\hat{\mathbf{v}}_i^{(1)} + \beta_2^{(k)}\widehat{div}\hat{\mathbf{v}}_i^{(2)}.$$

Furthermore, at $a_j = \mathcal{F}(\hat{a}_j)$ with $\hat{a}_j \in \hat{\mathcal{Q}}$, we have

$$\begin{aligned} \mathbf{v}_i^{(k)}(a_j) &= \mathbf{B}(\beta_1^{(k)}\hat{\mathbf{v}}_i^{(1)}(\hat{a}_j) + \beta_2^{(k)}\hat{\mathbf{v}}_i^{(2)}(\hat{a}_j)) \\ &= \mathbf{B}(\beta_1^{(k)}\delta_{i,j}\mathbf{e}_1 + \beta_2^{(k)}\delta_{i,j}\mathbf{e}_2) \\ &= \delta_{i,j}\mathbf{B}\begin{pmatrix} \beta_1^{(k)} \\ \beta_2^{(k)} \end{pmatrix} \\ &= \delta_{i,j}\mathbf{e}_k. \end{aligned}$$

Thus, $\{\mathbf{v}_i^{(k)}\}$ is a canonical basis for $\mathbf{V}_h(\mathcal{Q})$.

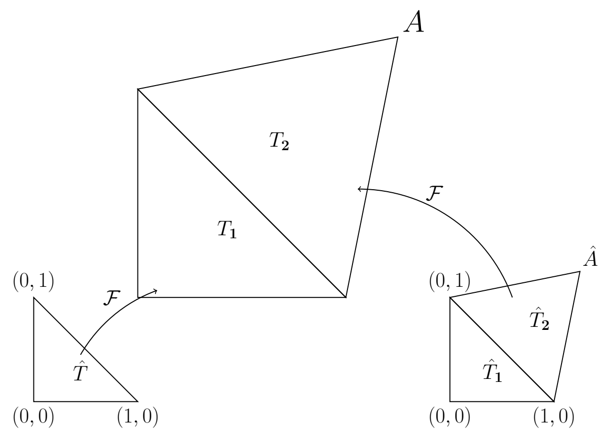


Figure 4.3: Construction of the skewed reference element. Affine transformation $\mathcal{F}$ mapping the reference element $\hat{\mathcal{Q}}_{\hat{A}}$ (right) to the physical element $\mathcal{Q}$ (middle).

4.6.2 Derivation of the Reference Basis Functions

In this section, we describe an efficient way of computing the basis on the skewed reference element $\hat{\mathcal{Q}}_{\hat{A}}$. In the two–diagonal split of $\hat{\mathcal{Q}}_{\hat{A}}$, we let $\{\hat{c}_j\}_{j=1}^4$ denote the midpoints of the interior edges and $\hat{c}_5$ be the point of intersection of the two diagonals (See Figure 4.4).

Moreover, we let $\{\hat{w}_i\}_{i=1}^5 \subset P_2(\hat{K}_r^{\hat{\mathcal{Q}}}) \cap H^1(\hat{\mathcal{Q}}_{\hat{A}})$ satisfy $\hat{w}_i(\hat{c}_j) = \delta_{i,j}$ and $\hat{w}_i(\hat{a}_j) = 0$, and $\{\hat{v}_i\}_{i=1}^8 \subset P_2(\hat{K}_r^{\hat{\mathcal{Q}}}) \cap H^1(\hat{\mathcal{Q}}_{\hat{A}})$ satisfy $\hat{v}_i(\hat{c}_j) = 0$ and $\hat{v}_i(\hat{a}_j) = \delta_{i,j}$. Thus, $\{\hat{w}_i, \hat{v}_i\}$ form the canonical basis of $P_2(\hat{K}_r^{\hat{\mathcal{Q}}}) \cap H^1(\hat{\mathcal{Q}})$. Note that $\hat{w}_i|_{\partial\hat{\mathcal{Q}}} = 0$, therefore, $\hat{w}_i$ are quadratic bubble functions. Furthermore, $\{\hat{w}_i \mathbf{e}_k, \hat{v}_i \mathbf{e}_k\}$ forms the canonical basis for $\mathbf{P}_2(\hat{K}_r^{\hat{\mathcal{Q}}}) \cap \mathbf{H}^1(\hat{\mathcal{Q}})$.

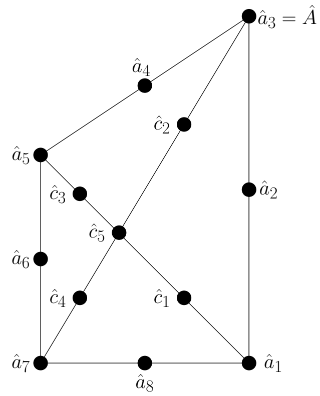


Figure 4.4: Labeling of the skewed reference element.

The basis of $\mathbf{V}(\hat{\mathcal{Q}}_{\hat{A}})$ is regarded as the Lagrange subbasis $\{\hat{v}_i\mathbf{e}_k\}$ corrected by the functions $\{\hat{w}_i\}$ to enforce the constant divergence constraint. In particular, the degrees of freedom and the definition of $\mathbf{V}(\hat{\mathcal{Q}}_{\hat{A}})$ yield the following result.

Lemma 32. *Let $\hat{\mathbf{v}}_i^{(k)} \subset \mathbf{V}(\hat{\mathcal{Q}}_{\hat{A}})$ denote the canonical basis of $\mathbf{V}(\hat{\mathcal{Q}}_{\hat{A}})$. Then, for each i and k, there exist unique vectors $\{\mathbf{b}_{i,j}^{(k)}\}_{j=1}^{5} \subset \mathbb{R}^2$ such that $\hat{\mathbf{v}}_i^{(k)} = \mathbf{e}_k\hat{v}_i + \sum_{j=1}^{5} \mathbf{b}_{i,j}^{(k)}\hat{w}_j$. In particular, the vectors are uniquely determined by the following constraint, which represents a (10×10) system.*

$$\widehat{div}(\hat{\mathbf{v}}_i^{(k)}) = \mathbf{e}_k \cdot \hat{\nabla}\hat{v}_i + \sum_{j=1}^{5} \mathbf{b}_{i,j}^{(k)} \cdot \hat{\nabla}\hat{w}_j \in P_0(\hat{\mathcal{Q}}_{\hat{A}}). \tag{4.6.1}$$

Remark 7. *The Lagrange basis $\{\hat{v}_i, \hat{w}_i\}$ are easily computable, even on the skewed quadrilateral $\hat{\mathcal{Q}}_{\hat{A}}$. The (10×10) system given in (4.6.1) can be solved symbolically in terms of the point $\hat{A}$. We assume that $\hat{A}$ is opposite to the origin as shown in Figure 4.3.*

Remark 8. *The condition $\widehat{div}(\hat{\mathbf{v}}_i^{(k)})|_{\hat{T}_i} \in P_0(\hat{T}_i)$ imposes two linearly–independent constraints for each $\hat{T}_i$, therefore, yields 8 constraints in total. The additional continuity condition $\widehat{div}(\hat{\mathbf{v}}_i^{(k)}) \in P_0(\hat{\mathcal{Q}}_{\hat{A}})$ only adds 2 constraints due to the singular vertex.*

4.7 NUMERICAL EXPERIMENTS

In this section, we present some numerical results supporting the theoretical results and compare the convergence rates of the divergence–free macro elements with the convergence rates of the serendipity elements for some test cases on both tensor–product and general convex quadrilateral meshes (See Figure 4.5). We note that the degrees of freedom on quadrilateral meshes are less than the degrees of freedom on tensor–product meshes (See Tables 4.2, 4.3).

4.7.1 Experiment 1: Stokes Problem

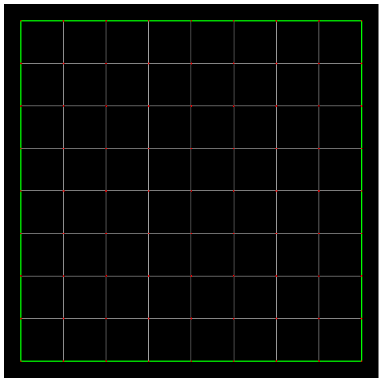
We compute the finite element method (4.5.2) on a sequence of refined meshes obtained from tensor–product and quadrilateral grids. In this experiment, we consider the domain $\Omega = (0,1)^2$ and take the viscosity constant as $\nu = 1$. We choose the data such that the velocity solution $\mathbf{u}$ and the pressure solution p are as follows:

$$\mathbf{u}(x,y) = \begin{pmatrix} 2x^2y(2y-1)(y-1)(x-1)^2 \\ -2xy^2(y-1)^2(2x-1)(x-1) \end{pmatrix},$$
$$p(x,y) = x^2 - y^2.$$

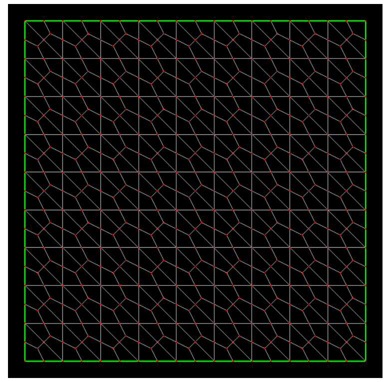
The error estimates with the rates of convergence on the tensor–product and quadrilateral meshes are listed in Tables 4.4, 4.5 and Tables 4.6, 4.7, respectively. Moreover, for comparison, we list the errors obtained by using the serendipity elements on the same sequence of meshes in Tables 4.8, 4.9.

4.7.2 Discussion on the Experimental Results for the Stokes Problem

Experiment 1 shows that the velocity solution obtained from the divergence–free macro element method have optimal order convergence on both the tensor-product and quadrilateral meshes (See Tables 4.4, 4.6). In addition, second–order convergent pressure solutions are derived via post-processing (See Tables 4.5, 4.7). These results support the theoretical results stated in Section 4.5. On the other hand, we observe that the velocity solutions obtained by using serendipity elements on the quadrilateral meshes are suboptimal and the pressure solutions are only first–order convergent (See Table 4.9).



(a) Tensor-product mesh.



(b) Quadrilateral mesh.

Figure 4.5: Mesh with $h = 0.125$.

Table 4.1: Mesh statistics for $h = 0.125$.

Mesh Type	Tensor-Product	Quadrilateral
Elements	64	486
Vertices	81	523
Boundary Vertices	32	72
Edges	144	1008
Boundary Edges	32	72

Table 4.2: Node statistics for the tensor–product mesh with $h = 0.125$.

FEM	Div–Free Macro	Serendipity
Nodes	514	531
Interior nodes (dofs)	385	402
Velocity nodes	450	450
Pressure nodes	64	81
Interior velocity nodes (dofs)	322	322
Interior pressure nodes (dofs)	63	80

Table 4.3: Node statistics for the quadrilateral mesh with $h = 0.125$.

FEM	Div–Free Macro	Serendipity
Nodes	3548	3585
Interior nodes (dofs)	3259	3296
Velocity nodes	3062	3062
Pressure nodes	486	523
Interior velocity nodes (dofs)	2774	2774
Interior pressure nodes (dofs)	485	522

Table 4.4: Experiment 1, Convergence results for the Div–Free Macro Stokes elements on tensor–product meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|div(\mathbf{u}_h)\|_{L^\infty(\Omega)}$
1/8	$2.4205E-05$		$1.4663E-03$		$3.0531E-16$
1/16	$3.0107E-06$	3.0071	$3.6793E-04$	1.9947	$6.9562E-16$
1/32	$3.7554E-07$	3.0031	$9.2048E-05$	1.9990	$1.5838E-15$
1/64	$4.6913E-08$	3.0009	$2.3016E-05$	1.9997	$3.0271E-15$

Table 4.5: Experiment 1, Convergence results for the Div–Free Macro Stokes elements on tensor–product meshes.

h	$\|p-p_h\|_{L^2(\Omega)}$	Rate	$\|p-p^*\|_{L^2(\Omega)}$	Rate
1/8	$5.8833E-02$		$1.7754E-03$	
1/16	$2.9451E-02$	0.9983	$4.2075E-04$	2.0771
1/32	$1.4730E-02$	0.9996	$1.0351E-04$	2.0232
1/64	$7.3655E-03$	0.9999	$2.5770E-05$	2.0060

Table 4.6: Experiment 1, Convergence results for the Div–Free Macro Stokes elements on quadrilateral meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|div(\mathbf{u}_h)\|_{L^\infty(\Omega)}$
1/8	$5.5600E-06$		$5.3695E-04$		$5.6483E-15$
1/16	$8.2511E-07$	2.7524	$1.4937E-04$	1.8459	$7.2720E-15$
1/32	$1.1228E-07$	2.8775	$3.9240E-05$	1.9285	$2.1866E-14$
1/64	$1.4632E-08$	2.9399	$1.0043E-05$	1.9661	$8.8635E-14$

Table 4.7: Experiment 1, Convergence results for the Div–Free Macro Stokes elements on quadrilateral meshes.

h	$\|p-p_h\|_{L^2(\Omega)}$	Rate	$\|p-p^*\|_{L^2(\Omega)}$	Rate
1/8	$2.7710E-02$		$4.7204E-04$	
1/16	$1.4674E-02$	0.9172	$1.2760E-04$	1.8873
1/32	$7.5599E-03$	0.9568	$3.2827E-05$	1.9587
1/64	$3.8382E-03$	0.9779	$8.2920E-06$	1.9851

Table 4.8: Experiment 1, Convergence results for the Serendipity elements on tensor–product meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$2.25254e-05$		$1.18497e-03$		$1.65257e-03$	
1/16	$2.70148e-06$	3.0597	$2.81811e-04$	2.0721	$4.11852e-04$	2.0045
1/32	$3.35915e-07$	3.0076	$6.97705e-05$	2.0140	$1.02940e-04$	2.0003
1/64	$4.19576e-08$	3.0011	$1.74091e-05$	2.0028	$2.57347e-05$	2.0000

Table 4.9: Experiment 1, Convergence results for the Serendipity elements on quadrilateral meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$5.20096e-06$		$6.47006e-04$		$2.63414e-04$	
1/16	$1.06298e-06$	2.2907	$2.74659e-04$	1.2361	$8.40444e-05$	1.6481
1/32	$2.42658e-07$	2.1311	$1.29238e-04$	1.0876	$2.84605e-05$	1.5622
1/64	$5.94236e-08$	2.0298	$6.37548e-05$	1.0194	$1.09533e-05$	1.3776

4.7.3 Experiment 2: NSE with Homogeneous Boundary Conditions

In this experiment, we approximate a finite element solution for the NSE given by (4.7.1) defined on $\Omega = (0,1)^2$ with the viscosity constant $\nu = 1$.

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, &\quad \text{in} \quad \Omega, \\ div(\mathbf{u}) = 0, &\quad \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{0}, &\quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.7.1}$$

We choose the data such that the velocity solution $\mathbf{u}$ and the pressure solution p are as follows:

$$\mathbf{u}(x,y) = \begin{pmatrix} 2a\pi\ sin^2(a\pi x)\ sin(a\pi y)\ cos(a\pi y) \\ -2a\pi\ sin(a\pi x) sin^2(a\pi y)\ cos(a\pi x) \end{pmatrix},$$

$$p(x,y) = 5000\ (x^2 - y^2).$$

The error estimates with the rates of convergence on the tensor–product and quadrilateral meshes are listed in Table 4.10 and Table 4.11, respectively. Moreover, for comparison, we list the errors obtained by using the serendipity elements on the same sequence of meshes in Tables 4.12, 4.13.

4.7.4 Experiment 3: NSE with Nonhomogeneous Boundary Conditions

In this experiment, homogeneous boundary conditions are not imposed on the velocity of the fluid. We provide a finite element solution for the NSE defined on $\Omega = (0,1)^2$ with the viscosity constant $\nu = 1$. We choose the data such that the velocity solution $\mathbf{u}$ and the pressure solution p are as follows:

$$\mathbf{u}(x,y) = \begin{pmatrix} 4x\Big(\pi(y+0.5)cos(2\pi y) + 0.5sin(2\pi y)\Big) \\ (-2y-1)sin(2\pi y) \end{pmatrix},$$

$$p(x,y) = 100\ (x^4 - \frac{3y^2}{5}).$$

The error estimates with the rates of convergence on the tensor–product and quadrilateral meshes are listed in Table 4.14 and Table 4.15, respectively. Moreover, for comparison, we list the errors obtained by using the serendipity elements on the same sequence of meshes in Tables 4.16, 4.17.

Table 4.10: Experiment 2, Convergence results for the Div–Free Macro elements on tensor–product meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$3.8053E-01$		$2.3205E+01$		$2.9417E+02$	
1/16	$5.2011E-02$	2.8711	$6.2894E+00$	1.8834	$1.4726E+02$	0.9983
1/32	$6.7061E-03$	2.9553	$1.6108E+00$	1.9651	$7.3650E+01$	0.9996
1/64	$8.4546E-04$	2.9877	$4.0524E-01$	1.9910	$3.6828E+01$	0.9999

Table 4.11: Experiment 2, Convergence results for the Div–Free Macro elements on quadrilateral meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$8.7070E-02$		$8.7116E+00$		$1.3856E+02$	
1/16	$1.4875E-02$	2.5493	$2.6659E+00$	1.7083	$7.3372E+01$	0.9172
1/32	$2.1467E-03$	2.7927	$7.2930E-01$	1.8701	$3.7800E+01$	0.9568
1/64	$2.8518E-04$	2.9122	$1.8935E-01$	1.9455	$1.9191E+01$	0.9780

Table 4.12: Experiment 2, Convergence results for the Serendipity elements on tensor–product meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$6.8818E-01$		$3.3271E+01$		$1.5469E+01$	
1/16	$6.2157E-02$	3.4688	$6.7530E+00$	2.3007	$2.0737E+00$	2.8990
1/32	$7.8048E-03$	2.9935	$1.6346E+00$	2.0466	$5.1515E-01$	2.0092
1/64	$9.7815E-04$	2.9962	$4.0662E-01$	2.0071	$1.2868E-01$	2.0012

Table 4.13: Experiment 2, Convergence results for the Serendipity elements on quadrilateral meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$7.1858E-02$		$8.2505E+00$		$1.4904E+00$	
1/16	$1.2636E-02$	2.5077	$2.7464E+00$	1.5869	$4.4017E-01$	1.7595
1/32	$2.1439E-03$	2.5592	$9.8155E-01$	1.4844	$1.5426E-01$	1.5127
1/64	$4.1461E-04$	2.3704	$4.1411E-01$	1.2450	$6.4232E-02$	1.2640

Table 4.14: Experiment 3, Convergence results for the Div–Free Macro elements on tensor–product meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$1.7628E-02$		$9.5541E-01$		$5.9564E+00$	
1/16	$2.2195E-03$	2.9896	$2.4098E-01$	1.9872	$2.9949E+00$	0.9919
1/32	$2.7824E-04$	2.9958	$6.0397E-02$	1.9964	$1.4995E+00$	0.9980
1/64	$3.4809E-05$	2.9988	$1.5109E-02$	1.9990	$7.5004E-01$	0.9995

Table 4.15: Experiment 3, Convergence results for the Div–Free Macro elements on quadrilateral meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$1.4531E-03$		$1.8033E-01$		$2.6512E+00$	
1/16	$2.1610E-04$	2.7494	$5.0425E-02$	1.8384	$1.4054E+00$	0.9157
1/32	$2.9512E-05$	2.8723	$1.3351E-02$	1.9172	$7.2423E-01$	0.9564
1/64	$3.8567E-06$	2.9359	$3.4360E-03$	1.9582	$3.6772E-01$	0.9778

Table 4.16: Experiment 3, Convergence results for the Serendipity elements on tensor–product meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$1.8497E-02$		$9.5349E-01$		$3.2482E-01$	
1/16	$2.3105E-03$	3.0010	$2.3925E-01$	1.9947	$8.0246E-02$	2.0171
1/32	$2.8879E-04$	3.0001	$5.9869E-02$	1.9986	$2.0024E-02$	2.0027
1/64	$3.6098E-05$	3.0000	$1.4971E-02$	1.9997	$5.0041E-03$	2.0006

Table 4.17: Experiment 3, Convergence results for the Serendipity elements on quadrilateral meshes.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)}$	Rate	$\|p-p_h\|_{L^2(\Omega)}$	Rate
1/8	$2.0689E-03$		$2.8715E-01$		$6.8657E-02$	
1/16	$4.6722E-04$	2.1467	$1.2884E-01$	1.1562	$2.4849E-02$	1.4662
1/32	$1.1389E-04$	2.0365	$6.2162E-02$	1.0515	$9.5961E-03$	1.3727
1/64	$2.8575E-05$	1.9948	$3.0851E-02$	1.0107	$3.9694E-03$	1.2735

4.7.5 Discussion on the Experimental Results for the NSE

Experiment 2 shows that the divergence–free macro element method provides optimal rates of convergence for the velocity solution of the NSE with homogeneous boundary conditions on both the tensor–product and quadrilateral meshes (See Table 4.10, 4.11). We note that these rates coincide with the rates we obtained for the Stokes problem (See Table 4.4, 4.5, 4.6, 4.7). Although the serendipity element method exhibit optimal convergence on tensor–product meshes (See Table 4.12), the convergence of this method is suboptimal on quadrilateral meshes (See Table 4.13).

In Experiment 3, we see that for the NSE with nonhomogeneous boundary conditions, divergence–free macro element method yields optimally–convergent velocity solutions on both kinds of meshes as in the case of non–slip boundary conditions (See Table 4.14, 4.15). Moreover, the serendipity element method yields velocity solutions with suboptimal convergence rates as in Experiment 3 (See Table 4.17).

5.0 CONCLUSION

In this dissertation, we built conforming finite elements that yield divergence–free velocity solutions for the steady Stokes problem on cubical and quadrilateral meshes of open, bounded, simply–connected polygonal domains. First, we constructed the finite element spaces with the desired properties for the two-dimensional problem on rectangular meshes. Then, we extended these spaces to spaces on n-dimensional cubical meshes. We proved the stability, the conformity of the methods we propose and the solenoidality of the velocity solutions through the use of discrete differential forms and smooth de Rham complexes. We also developed macro elements for the two–dimensional problem on quadrilateral meshes. By utilizing the tools of differential calculus, we showed that our method yields divergence–free velocity solutions, and with the construction of a Fortin operator, we validated the stability of our method. We verified that the methods we develop here yield optimal convergence rates and present some numerical experiments which are supportive of these theoretical results. Furthermore, we applied our divergence–free macro element method to the Navier-Stokes equations and provided numerical experiments which show that the convergence rates are preserved.

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