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Selection of alternatives using fuzzy networks with rule base aggregation

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Abstract

This paper introduces a novel extension of the Technique for Ordering of Preference by Similarity to Ideal Solution (TOPSIS) method. The method is based on aggregation of rules with different linguistic of the output of fuzzy networks to solve multi-criteria decision-making problems whereby both benefit and cost criteria are presented as subsystems. Thus the decision maker evaluates the performance of each alternative for decision process and further observes the performance for both benefit and cost criteria. The aggregation sub-stage in a fuzzy system maps the fuzzy membership functions for all rules to an aggregated fuzzy membership function representing the overall output for the rules. This approach improves significantly the transparency of the TOPSIS methods, while ensuring high effectiveness in comparison to established approaches. To ensure practicality and effectiveness, the proposed method is further tested on portfolio selection problems. The ranking produced by the method is comparatively validated using Spearman rho rank correlation. The results show that the proposed method outperforms the existing TOPSIS approaches in term of ranking performance.

Keywords: TOPSIS; Fuzzy networks; Selection alternatives; Fuzzy sets; Interval type 2 fuzzy sets; Z-numbers; Stock selection; Spearman rho; Type 1 fuzzy sets

1. Introduction

Multi-criteria decision making (MCDM) problems are often observed in reality, and decision makers are faced with the challenge of making decisions in the presence of multiple criteria. The focus is on identifying the best performing solution among feasible alternatives assessed by a group of decision maker and evaluated through multiple criteria [1]. There have been important advances in this field since the start of the modern multiple-criteria decision-making discipline in the early 1960s. Various MCDM techniques have been developed with the overall objective to assist decision makers solve complex decision problems in a systematic, consistent and more productive way.

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TOPSIS is an MCDM technique for ranking and selection of alternatives. The TOPSIS analysis considers two reference points – a positive ideal solution (PIS) and a negative ideal solution (NIS) – as well as the distances to both PIS and NIS. The preference order is ranked according to the closeness of PIS and NIS, and according to a combination of the two distance measures [2].

Fuzzy systems are vital within the armoury of fuzzy tools and applicable to real-life decision-making environment. There are three type of fuzzy systems introduced in the literature – systems with a single rule base, systems with multiple rule bases, and systems with networked rule bases. Systems with a single rule base are characterised with a black box nature, where the inputs are mapped directly to the output without considering any internal connection. Systems with multiple rule bases are characterises with a white box nature, where the inputs are mapped to the outputs through interval variables as connections. This type of systems is also termed chained fuzzy systems or hierarchical fuzzy systems. The third types of fuzzy systems incorporate networked rule bases, and are termed fuzzy networks (FN). Fuzzy networks are introduced as a theoretical concept in [3], and are characterised with a white box nature where the inputs are mapped to the outputs through intermediate variables.

According to [4], the accuracy of single rule base is moderate but the level of transparency is low, while multiple rule bases are regarded as having low accuracy in dealing with complex processes management. While in most decision making studies, single rule bases and multiple rule bases are common approaches [5], in this research we focus on fuzzy networks as they are highly transparent and moderately accurate. A node represents each subsystem in a FN and the interactions among subsystems are the connections between nodes. Therefore FNs consider the interaction between subsystems. This ability brings considerable benefits to modelling complex processes, and although FNs have been introduced recently, a cohort of researchers are dedicated to the theoretical development and applications of FNs [3,4,6–9].

The reliability of decision knowledge and the experience of experts are still in need of better incorporation into modelling complex decision-making processes. For instance, how assured in their choices are shareholders as decision makers, and how much experience experts as financial analysts have in relevant asset classes and markets. Besides, existing TOPSIS methods have a very low transparency level, and consequently are not able to track the performance of benefit and cost criteria [10]. In decision-making processes, it is essential that decision makers are aware of how the numerous criteria are performing.

The inadequacies described above bring the motivation of this study. This paper introduces novel application of fuzzy networks with aggregation of rule bases for decision-making problem solving. This approach is different from other similar approaches such as merging of rule bases [11]. In this case, aggregation is the process of combining a set of fuzzy rules into a single fuzzy rule. It also includes the defuzzification for each output of the system. This process is important because decisions are made based on considering all rules in the system. In order to make better decision, the rules must be combined. Moreover, the proposed methodology helps to improve significantly the transparency of TOPSIS methods, while ensuring high effectiveness in comparison to established approaches. Also the methodology can help experts to trace the performances of criteria and afterwards make better decisions.

The paper is structured as follows: Section 2 briefly reviews the concepts fuzzy systems, and the operation of fuzzy networks. The novel methodology of TOPSIS using fuzzy networks with aggregation of rule bases FN-TOPSIS is formulated in Section 3. Section 4 illustrates the application of FN-TOPSIS to the problem of ranking stock. Further discussion and analysis of the FN-TOPSIS ranking performance are provided in Section 5. The main conclusions are summarised in Section 6.

2. Preliminaries

2.1. Fuzzy systems

A fuzzy system consists of a single rule base where inputs are processed simultaneously without taking into account the connections and the structure of the system. This is shown in Fig. 1, where $\{p_1, \ldots, p_n\}$ is the set of inputs and q is the output of the system. For this type of system, the rules are derived based on expert knowledge about the process. The results are normally quite accurate but the poor transparency of the system can be an obstacle to understanding complex processes.

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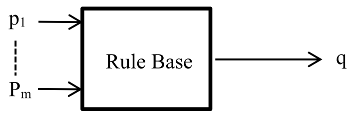

Fig. 1. Fuzzy system.

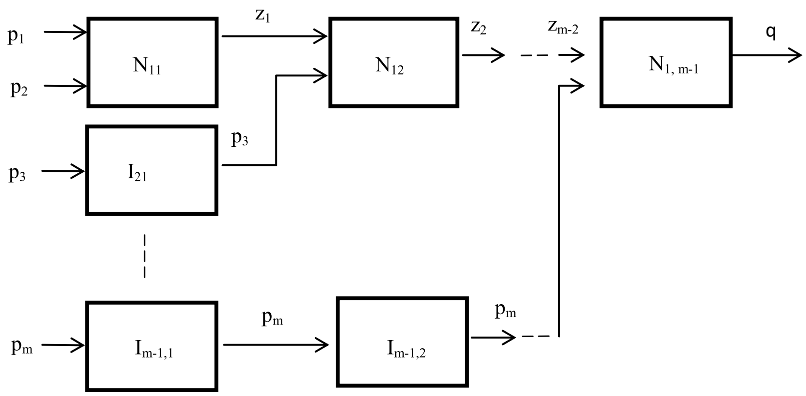

Fig. 2. Fuzzy network.

A fuzzy system with r rules, m inputs $p_1, \ldots, p_m$ taking linguistic terms from the sets $\{S_{11}, \ldots, S_{1r}\}, \ldots,$ $\{S_{m1}, \ldots, S_{mr}\}$, and n outputs $q_1, \ldots, q_n$ taking linguistic terms from the output sets $\{T_{11}, \ldots, T_{1r}\}, \ldots, \{T_{n1}, \ldots,$ $T_{nr}\}$, can be described by the following rule base in Eq. (1):

$$\begin{array}{l} \text{Rule 1: If } p_1 \text{ is } S_{11} \text{ and } \ldots \text{ and } p_m \text{ is } S_{m1} \text{ then } q_1 \text{ is } T_{11} \text{ and } \ldots \text{ and } q_l \text{ is } T_{n1} \\ \vdots \vdots \vdots \\ \text{Rule } r\text{: If } p_1 \text{ is } S_{1r} \text{ and } \ldots \text{ and } p_m \text{ is } S_{mr} \text{ then } q_1 \text{ is } T_{1r} \text{ and } \ldots \text{ and } q_{nl} \text{ is } T_{nr} \end{array} \quad (1)$$

2.2. Fuzzy networks

A fuzzy network is a new type of fuzzy system, which consists of networked rule bases (nodes) and deals with inputs sequentially, while taking into account the connections and structure of the system. The rules for both fuzzy systems and fuzzy networks are derived from expert knowledge. A type-1 fuzzy set A is defined on a universe X, The membership $\mu_A(x)$ describes the degree of belongingness of $x \in X$ in A. Throughout this study, type-1, type 2 fuzzy numbers and Z-numbers are presented in the form of trapezoidal fuzzy numbers [12]. The good coverage of trapezoidal fuzzy numbers is a good compromise between efficiency and effectiveness. A type-2 fuzzy set $\tilde{A}$ in the universe of discourse X is represented by a type-2 membership function $\mu_{\tilde{A}}$ as follows: $\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | \forall u \in J_X \subseteq [0, 1], 0 \leq \mu_{\tilde{A}}(x, u) \leq 1\}$,where J_X denotes an interval in [0, 1]. Z-number is an ordered pair of type-1 fuzzy numbers denoted as $Z = (\tilde{A}, \tilde{B})$. The first component $\tilde{A}$, a restriction on the values is a real-valued uncertain variable. The second component $\tilde{B}$ is a measure of reliability for the first component. Throughout this study, type-1, type 2 fuzzy set and Z-numbers are presented in the form of trapezoidal fuzzy numbers.

A networked fuzzy system is transparent and fairly accurate at the same time due to its hybrid nature, which facilitates the understanding and management of complex processes. As shown in Fig. 2, $\{p_1, p_2, \ldots, p_m\}$ is the set of inputs and $\{z_1, z_2, \ldots, z_{m-2}\}$ is the set of connections, while the set of network nodes is $\{N_{11}, N_{12}, \ldots, N_{1,m-1}\}$ and $\{I_{11}, I_{31}, I_{1,m-1}, \ldots, I_{m-1,m-2}\}$ are identity nodes. Here q represents the output of the system. A rule base is incorporated as a node within the fuzzy network.

There are four formal models for fuzzy networks characterised in [1], namely: (i) if-then rule and integer tables, (ii) block schemes and topological expressions, (iii) incidence and adjacency matrices, and (iv) Boolean matrices and binary relations. Here we employ if-then rules and Boolean matrices and binary relation, in order to represent the fuzzy rules. Hence the properties of such models will be reviewed briefly. The choice is justified by the ability of these formal models to work with any number of nodes and to handle dynamics in FNs.

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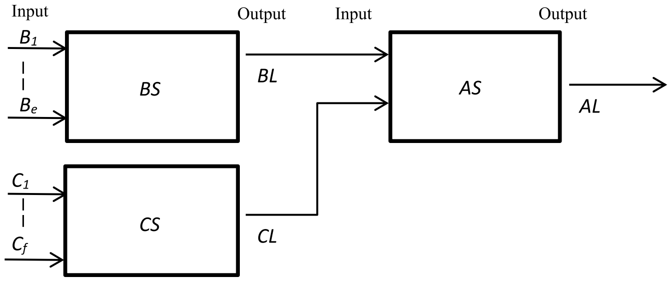

Fig. 3. Fuzzy network model for TOPSIS.

Table 1
Linguistic variables for the importance weight of each criterion.

Linguistic variables		Trapezoidal Fuzzy number
Very Low (VL)	1	(0.00, 0.00, 0.00, 0.10)
Low (L)	2	(0.00, 0.10, 0.10, 0.25)
Medium Low (ML)	3	(0.15, 0.30, 0.30, 0.45)
Medium (M)	4	(0.35, 0.50, 0.50, 0.65)
Medium High (MH)	5	(0.55, 0.70,0.70, 0.85)
High (H)	6	(0.80, 0.90, 0.90, 1.00)
Very High (VH)	7	(0.90, 1.00, 1.00, 1.00)

3. Method formulation

In this approach, the decision makers' opinions are evaluated independently, since they may have different influence degrees, depending on their experience in the area. Furthermore, criteria are categorised into benefit criteria or cost criteria. Each category will generate correspondingly benefit fuzzy systems or cost fuzzy systems, where the outputs of the systems are Benefit Levels (BL) or Cost Levels (CL), representing the performance of each category. Fig. 3 illustrates the proposed Generalised Fuzzy Network Model for TOPSIS, where Benefit Systems (BS), Cost Systems (CS) and Alternatives Systems (AS) are incorporated in the form of fuzzy network nodes. The inputs are the benefit criteria $B_1, \ldots, B_e$ and the cost criteria $C_1, \ldots, C_f$. At the end of the process, Alternatives Level (AL) are determined.

The next subsections 3.1, 3.2 and 3.3 are illustrated systematically the implementation of Type-1, Type-2 and Z-fuzzy sets to FN-TOPSIS respectively. In particular, subsection 3.1, describes the implementation of type 1 fuzzy set, when this type of fuzzy set is used. The implementation of type 2 and Z fuzzy set are explained in detail in subsections 3.2 and 3.3, respectively. In this case, Type 2 implementation is different from type 1 and Z fuzzy sets in terms of the foot of uncertainty that is used to represent the expert knowledge. Also, Z implementation is different from Type 1 and 2 fuzzy sets in terms of using a higher level of generalised number to represent expert knowledge.

However, the type of fuzzy set used is determined based on the linguistic terms and it is not used not simultaneously but sequentially. Also, decision makers decide whether to use type 1, type 2 or Z implementation of the proposed method to solve the problem.

3.1. Type-1 fuzzy set implementation

The following Table 1 and Table 2 are used by decision makers to evaluate the rating of alternatives and the importance of criteria, and Table 3 is used to determine the alternative level as the output, in generating fuzzy rule bases.

The following are the procedures involved in implementing a fuzzy network with merging rule bases to TOPSIS, based on Type 1-fuzzy set. Steps 1–6 are adopted from [16] and [13], while steps 7–9 are introduced as part of the proposed method in this paper.

Step 1: *Construct decision matrices where each decision maker opinion is evaluated independently, and categorise into two Criteria Categories as Benefit Criteria and Cost Criteria defined through a Benefit System and a Cost System.*

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Table 2
Linguistic variables for the rating of each alternative.

Linguistic variables		Trapezoidal Fuzzy number
Very Poor (VP)	1	(0, 0, 0, 1)
Poor (P)	2	(0, 1, 1,3)
Medium Poor (MP)	3	(1, 3, 3, 5)
Fair (F)	4	(3, 5, 5, 7)
Medium Good (MG)	5	(5, 7, 7, 9)
Good (G)	6	(7, 9, 9, 10)
Very Good (VG)	7	(9, 10, 10, 10)

Table 3
Linguistic variables for the level of alternatives.

Linguistic variables		Trapezoidal Fuzzy number
Very Bad (VB)	1	(0.00, 0.00, 0.00,0.25)
Bad (B)	2	(0.00, 0.25, 0.25, 0.50)
Regular (R)	3	(0.25, 0.50, 0.50, 0.75)
Good (G)	4	(0.50, 0.75, 0.75, 1.00)
Very Good (VG)	5	(0.75, 1.00, 1.00, 1.00)

In the decision matrices D_k^B, D_k^C and weight matrices W_k^B, W_k^C $(k = 1, \dots, K)$, it is assumed that e is the number of benefit criteria and f is the number of cost criteria, as shown in Eq. (2):

$$D_k^B = \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_e \end{array} \begin{bmatrix} x_{11,k} & x_{12,k} & \cdots & x_{1m,k} \\ x_{21,k} & x_{22,k} & \cdots & x_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{e1,k} & x_{e2,k} & \cdots & x_{em,k} \end{bmatrix} \quad \text{and} \quad D_k^C = \begin{array}{c} C_1 \\ C_2 \\ \vdots \\ C_f \end{array} \begin{bmatrix} y_{11,k} & y_{12,k} & \cdots & y_{1m,k} \\ y_{21,k} & y_{22,k} & \cdots & y_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{f1,k} & x_{f2,k} & \cdots & x_{fm,k} \end{bmatrix}; \tag{2}$$

$$W_k^B = [g_{1,k} \quad q_{2,k} \quad \cdots \quad q_{e,k}] \quad \text{and} \quad W_k^C = [h_{1,k} \quad h_{2,k} \quad \cdots \quad h_{f,k}], \quad \text{for } k = 1, \dots, K,$$

where $x_{ij,k}$ are Type-1 fuzzy sets representing the rating of alternatives A $(j = 1, \dots, m)$ with respect to benefit criteria B_i $(i = 1, \dots, e)$ according to the kth decision maker, and $q_{i,k}$ are Type-1 fuzzy sets representing the weights of benefit criteria $B_{i,k}$ $(i = 1, \dots, e)$ according to the kth decision maker, where $k = 1, \dots, K$. Also, $y_{ij,k}$ are Type-1 fuzzy sets describing the rating of alternatives A_j $(j = 1, \dots, m)$ with respect to cost criteria C_i $(i = 1, \dots, f)$ according to the kth decision maker, and $h_{i,k}$ are Type-1 fuzzy sets describing the weights of cost criteria C_i $(i = 1, \dots, f)$ according to the kth decision maker, where $K = 1, \dots, K$.

Step 2: *Construct weighted and normalized decision matrices.*

The fuzzy rating and weight of each criterion are variables described with Type-1 trapezoidal fuzzy numbers. The ratings of alternatives A_j $(j = 1, \dots, m)$ are described with the Type-1 trapezoidal fuzzy numbers $x_{ij,k} = (a^x_{ij,k}, b^x_{ij,k}, c^x_{ij,k}, d^x_{ij,k},)$ and $y_{ij,k} = (a^y_{ij,k}, b^y_{ij,k}, c^y_{ij,k}, d^y_{ij,k},)$, while the importance of benefit criteria B $(i = 1, \dots, e)$ and cost criteria C_i $(i = 1, \dots, f)$ are respectively represented by $g_{i,k} = (a^g_{i,k}, b^g_{i,k}, c^g_{i,k}, d^g_{i,k})$ and $h_{i,k} = (a^h_{i,k}, b^h_{i,k}, c^h_{i,k}, d^h_{i,k})$, for $k = 1, \dots, K$. The normalized fuzzy decision matrices R_k and weight normalized fuzzy decision matrices V_k are calculated as shown in Eq. (3):

$$R_k = [r_{ij,k}]_{(e+j)\times m}, \tag{3}$$

where

$$r_{ij,k} = \begin{cases} r^B_{ij,k} = \left(\frac{a^x_{ij,k}}{d^{x*}_{i,k}}, \frac{b^x_{ij,k}}{d^{x*}_{i,k}}, \frac{c^x_{ij,k}}{d^{x*}_{i,k}}, \frac{d^x_{ij,k}}{d^{x*}_{i,k}}\right), & \text{for } B_i \in B \\ r^C_{ij,k} = \left(\frac{a^{y*}_{ij,k}}{a^y_{i,k}}, \frac{a^{y*}_{ij,k}}{c^y_{i,k}}, \frac{a^{y*}_{ij,k}}{b^y_{i,k}}, \frac{a^{y*}_{ij,k}}{d^y_{i,k}}\right), & \text{for } C_i \in C \end{cases}$$

$$d_{i,k}^{x*} = \max_j d_{ij,k}^{x} \quad (i = 1, \dots, e)\ (j = 1, \dots, m)$$

$$a_{i,k}^{y*} = \min_j a_{ij,k}^{y} \quad (i = 1, \dots, f)\ (j = 1, \dots, m)$$

B and C are the sets of benefit criteria and cost criteria respectively; $V_k = [v_{ij,k}]_{(e+f)\times m}$, where

$$v_{ij,k} = \begin{cases} v_{ij,k}^{B} = r_{ij,k}(\cdot)g_{i,k}, & \text{for } B_i \in B \\ v_{ij,k}^{C} = r_{ij,k}(\cdot)h_{i,k}, & \text{for } C_i \in C \end{cases}$$

and

$$v_{ij,k} = \left(a_{ij,k}^{v}, b_{ij,k}^{v}, c_{ij,k}^{v}, d_{ij,k}^{v}\right)$$

are Type-1 fuzzy sets; for $k = 1, \dots, K$.

Step 3: *Find the Fuzzy Positive Ideal Solution (FPIS) and Fuzzy Negative Ideal Solution (FNIS) for each alternative, and the distance between each alternative to FPIS and FNIS.*

The FPIS and FNIS solutions are correspondingly $A_k^+ = (v_{1,k}^+, v_{2,k}^+, \dots, v_{(e+f),k}^+)$ and $A_k^- = (v_{1,k}^-, v_{2,k}^-, \dots, v_{(e+f),k}^-)$, where $v_{ij,k}^+ = (1\ 1\ 1\ 1)$ and $v_{ij,k}^- = (0\ 0\ 0\ 0)$ are Type-1 fuzzy sets, for $K = 1, \dots, K$. The distance for benefit criteria of each alternative j from A_k^+ is $\Delta_{j,k}^{B+}$. The latter is the distance between two or more fuzzy numbers and it is used as a defuzzification method to convert fuzzy numbers to crisp numbers. In this case, the distance value is calculated as shown in Eq. (4):

$$\Delta_{j,k}^{B+} = \sum_{i=1}^{e} \Delta_k^B\left(v_{ij,k}^B, v_{i,k}^+\right), \quad \text{where} \tag{4}$$

$$\Delta_k^B\left(v_{ij,k}^B, v_{i,k}^+\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij,k}^{v,B} - 1\right)^2 + \left(b_{ij,k}^{v,B} - 1\right)^2 + \left(c_{ij,k}^{v,B} - 1\right)^2 + \left(d_{ij,k}^{v,B} - 1\right)^2\right]},$$
$$\text{for } j = 1, \dots, m, \text{ and } B_i \in B, \text{ and } k = 1, \dots, K.$$

The distance for benefit criteria of each alternative from $A_{j,k}^-$ is $\Delta_{j,k}^{B-}$, calculated as shown in Eq. (5):

$$\Delta_{j,k}^{B-} = \sum_{i=1}^{e} \Delta_k^B\left(v_{ij,k}^B, v_{i,k}^-\right), \quad \text{where} \tag{5}$$

$$\Delta_k^B\left(v_{ij,k}^B, v_{i,k}^-\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij,k}^{v,B} - 0\right)^2 + \left(b_{ij,k}^{v,B} - 0\right)^2 + \left(c_{ij,k}^{v,B} - 0\right)^2 + \left(d_{ij,k}^{v,B} - 0\right)^2\right]},$$
$$\text{for } j = 1, \dots, m, \text{ and } B_i \in B, \text{ and } k = 1, \dots, K.$$

The distance for cost criteria of each alternative from A_k^+ is $\Delta_{j,k}^{C+}$, calculated as shown in Eq. (6):

$$\Delta_{j,k}^{C+} = \sum_{i=1}^{f} \Delta_k^C\left(v_{ij,k}^C, v_{i,k}^+\right), \quad \text{where} \tag{6}$$

$$\Delta_k^C\left(v_{ij,k}^C, v_{i,k}^+\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij,k}^{v,C} - 1\right)^2 + \left(b_{ij,k}^{v,C} - 1\right)^2 + \left(c_{ij,k}^{v,C} - 1\right)^2 + \left(d_{ij,k}^{v,C} - 1\right)^2\right]},$$
$$\text{for } j = 1, \dots, m, \text{ and } C_i \in C, \text{ and } k = 1, \dots, K.$$

Finally, the distance for cost criteria of each alternative from A_k^- is $\Delta_{j,k}^{C-}$, calculated as shown in Eq. (7):

$$\Delta_{j,k}^{C-} = \sum_{i=1}^{f} \Delta_k^C\left(v_{ij,k}^C, v_{i,k}^-\right), \quad \text{where} \tag{7}$$

$$\Delta_k^C\left(v_{ij,k}^C, v_{i,k}^-\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij,k}^{v,C} - 0\right)^2 + \left(b_{ij,k}^{v,C} - 0\right)^2 + \left(c_{ij,k}^{v,C} - 0\right)^2 + \left(d_{ij,k}^{v,C} - 0\right)^2\right]},$$
$$\text{for } j = 1, \dots, m, \text{ and } C_i \in C, \text{ and } k = 1, \dots, K.$$

Step 4: *Find the closeness coefficients for both the benefit and cost systems.*

The closeness coefficients $CC^B_{j,k}$ for the benefit systems, and the closeness coefficients $CC^C_{j,k}$ for the cost systems, are calculated in Eq. (8):

$$CC^B_{j,k} = \frac{\Delta^{B-}_{j,k}}{\Delta^{B+}_{j,k} + \Delta^{B-}_{j,k}}, \qquad CC^C_{j,k} = \frac{\Delta^{C-}_{j,k}}{\Delta^{C+}_{j,k} + \Delta^{C-}_{j,k}} \quad \text{for } j = 1, \dots, m \text{ and } k = 1, \dots, K. \tag{8}$$

Step 5: *Derive the Influenced Closeness Coefficients (ICC) by applying the influence degree of each decision maker. Then find the normalised ICC (NICC), dividing the ICC by the maximum value of ICC.*

Let θ_k denotes the influence degree, between 0 (uninfluential) and 10 (very influential), of decision maker k, where $k = 1, \dots, K$. Next, let σ_k stands for the normalized influence degree of the kth decision maker, $k = 1, \dots, K$, as evaluated with Eq. (9):

$$\sigma_k \frac{\theta_k}{\sum_{l=1}^{K} \theta_l}, \quad \text{for } k = 1, \dots, K. \tag{9}$$

Eq. (10) evaluates the influence closeness coefficients $ICC^B_{j,k}$ and $ICC^C_{j,k}$ for each DM k, respectively along the benefit and cost criteria.

$$ICC^B_{j,k} = \sigma_k^* CC^B_{j,k} \quad \text{and} \quad ICC^C_{j,k} = \sigma_k^* CC^C_{j,k}, \quad \text{for } j = 1, \dots, m \text{ and } k = 1, \dots, K, \tag{10}$$

It is further necessary to normalize the coefficients, in order to ensure that their values vary between 0 to 1. Eq. (11) evaluates the normalised coefficients, where $NICC^B_{j,k}$ and $NICC^C_{j,k}$ are respectively the normalized influence closeness coefficients for the benefit and cost systems, as related to the kth decision maker.

$$NICC^B_{j,k} = \frac{ICC^B_{j,k}}{\max_j ICC^B_{j,k}} \quad \text{and} \quad NICC^C_{j,k} = \frac{ICC^C_{j,k}}{\max_j ICC^C_{j,k}}, \quad \text{for } j = 1, \dots, m \text{ and } k = 1, \dots, K. \tag{11}$$

Both $NICC^B_{j,k}$ and $NICC^C_{j,k}$ will take linguistic variables from Table 3 for the level of alternatives performance.

Step 6: *Construct the antecedent matrices and the consequent matrices for the BS and CS systems, based on DMs opinions and the values of the NICC coefficients.*

Having the opinions D^B_k and D^C_k of all DMs ($k = 1, \dots, K$) on each alternative j ($i = 1, \dots, m$) in respect to each benefit criterion i ($i = 1, \dots, e$) and each cost criterion i ($i = 1, \dots, f$), we can define the BS antecedent matrix X_k and the CS antecedent matrix Y_k for each DM k, as introduced with Eq. (12):

$$X_k = \begin{bmatrix} x_{11,k} & x_{12,k} & \cdots & x_{1m,k} \\ x_{21,k} & x_{22,k} & \cdots & x_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{e1,k} & x_{e2,k} & \cdots & x_{em,k} \end{bmatrix} \quad \text{and} \quad Y_k = \begin{bmatrix} y_{11,k} & y_{12,k} & \cdots & y_{1m,k} \\ y_{21,k} & y_{22,k} & \cdots & y_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{f1,k} & y_{f2,k} & \cdots & y_{fm,k} \end{bmatrix} \quad \text{for } k = 1, \dots, K, \tag{12}$$

where $x_{ij,k}$ and $y_{ij,k}$ are linguistic variables describing decision makers' opinions.

Having determined the $NICC^{B,k}_j$ and $NICC^{C,k}_j$ coefficients for all decision makers ($k = 1, \dots, K$), next the benefit consequent matrix Λ_k and the cost consequent matrix Ψ_k are defined as shown in Eq. (13):

$$\Lambda_k = [\lambda_{1,k} \quad \lambda_{2,k} \quad \cdots \quad \lambda_{m,k}] \quad \text{and} \quad \Psi_k[\psi_{1,k} \quad \psi_{2,k} \quad \cdots \quad \psi_{m,k}] \quad \text{for } k = 1, \dots, K, \tag{13}$$

where $\lambda_{i,k}$ and $\psi_{i,k}$ are linguistic variables representing the output of the BS and CS systems, based respectively on the values of $NICC^B_{j,k}$ and $NICC^C_{j,k}$.

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The benefit system consists of K matrix decision rules presented in Eq. (14):

$$\text{If} \begin{bmatrix} x_{11,k} & x_{12,k} & \cdots & x_{1m,k} \\ x_{21,k} & x_{22,k} & \cdots & x_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{e1,k} & x_{e2,k} & \cdots & x_{em,k} \end{bmatrix} \text{ then } \Lambda_k = [\lambda_{1,k} \quad \lambda_{2,k} \quad \cdots \quad \lambda_{m,k}] \quad \text{for } k = 1, \dots, K; \tag{14}$$

and can be described with the rule bases in Eq. (15):

$$\begin{array}{l} \text{Rule 1: If } B_1 \text{ is } x_{11,k} \text{ and } \cdots \text{ and } B_e \text{ is } x_{e1,k} \text{ then } BL \text{ is } \lambda_{1,k} \\ \qquad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \text{Rule } m\text{: If } B_1 \text{ is } x_{1m,k} \text{ and } \cdots \text{ and } B_e \text{ is } x_{em,k} \text{ then } BLis\lambda_{m,k}, \end{array} \quad \text{for } k = 1, \dots, K; \tag{15}$$

where BL is the benefit level of alternatives, for $j = 1, \dots, m$.

The cost system consists of K matrix decision rules presented in Eq. (16):

$$\begin{bmatrix} y_{11,k} & y_{12,k} & \cdots & y_{1m,k} \\ y_{21,k} & y_{22,k} & \cdots & y_{2m,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{f1,k} & x_{f2,k} & \cdots & x_{fm,k} \end{bmatrix} \text{ then } \Psi_k = [\psi_{1,k} \quad \psi_{2,k} \quad \cdots \quad \psi_{m,k}] \quad \text{for } k = 1, \dots, K; \tag{16}$$

and can be described with the rule bases in Eq. (17):

$$\begin{array}{l} \text{Rule 1: If } C_1 \text{ is } y_{11,k} \text{ and } \cdots \text{ and } C_f \text{ is } y_{f1,k} \text{ then } CL_1 \text{ is} \psi_{1,k} \\ \qquad\qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \text{Rule } m\text{: If } C_1 \text{ is } y_{f1,k} \text{ and } \cdots \text{ and } C_f \text{ is } y_{fm,k} \text{ then } CL_m \text{ is } \psi_{m,k}, \end{array} \quad \text{for } k = 1, \dots, K; \tag{17}$$

where CL is the cost level of alternatives, for $j = 1, \dots, m$.

Step 7: *Construct the antecedent matrices and consequent matrices for the Alternatives System (AS).*

The AS antecedent matrices M_k are based on the Benefit Levels Λ_k and Cost Levels Ψ_k, which are the outputs of the BS and CS systems correspondingly. The antecedent matrix of a system with two inputs, i.e. BL and CL, each taking m possible values, will be usually of size $2 \times (m \cdot m)$, as presented in Eq. (18).

$$M_k = \begin{array}{c} BL \\ CL \end{array} \begin{bmatrix} \lambda_{1,k} & \cdots & \lambda_{1,k} & \cdots & \lambda_{m,k} & \cdots & \lambda_{m,k} \\ \psi_{1,k} & \cdots & \psi_{m,k} & \cdots & \psi_{1,k} & \cdots & \psi_{m,k} \end{bmatrix} \quad \text{for } k = 1, \dots, K. \tag{18}$$

However, in this case each tuple of inputs $(\lambda_{j,k}, \psi_{j,k})$ stands for the assessed levels of the same alternative j through two types of criteria – benefits and costs. Therefore, the AS antecedent matrices M_k are of size $2 \times m$, as constructed in Eq. (19):

$$M_k = \begin{array}{c} BL \\ CL \end{array} \begin{bmatrix} \lambda_{1,k} & \lambda_{2,k} & \lambda_{3,k} & \cdots & \lambda_{m,k} \\ \psi_{1,k} & \psi_{2,k} & \psi_{3,k} & \cdots & \psi_{m,k} \end{bmatrix} \quad \text{for } k = 1, \dots, K. \tag{19}$$

The AS consequent matrices are derived as follows:

(i) Calculate the aggregation $\xi_{j,k}$ of weighted $NICC^B_{j,k}$ and $NICC^C_{j,k}$, as shown in Eq. (20):

$$\xi_{j,k} = \frac{NICC^B_{j,k} \times (\frac{e}{e+f}) + NICC^C_{j,k} \times (\frac{f}{e+f})}{2} \quad \text{for } j = 1, \dots, m \text{ and } k = 1, \dots, K. \tag{20}$$

(ii) Normalize the values of $\xi_{j,k}$ to ensure they lie within [0, 1], as calculated in Eq. (21):

$$N\xi_{j,k} = \frac{\xi_{j,k}}{\max_j \xi_{j,k}} \quad \text{for } j = 1, \dots, m \text{ and } k = 1, \dots, K. \tag{21}$$

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(iii) For $N\xi_{j,k}$, take linguistic variables from Table 3 for the alternatives levels. Then the K AS consequent matrices, in this case of size $1 \times m$, rather than $1 \times m \cdot m$, are described in Eq. (22):

$$N_k = AL[N\xi_{1,k} \quad N\xi_{2,k} \quad \cdots \quad N\xi_{m,k}], \quad \text{for } k = 1, \dots, K \tag{22}$$

where AL is, the level of alternatives.

Therefore, the alternatives system is presented with K matrix decision rules, as constructed in Eq. (23):

$$\text{If } M_k = \begin{array}{c} BL \\ CL \end{array} \begin{bmatrix} \lambda_{1,k} & \lambda_{1,k} & \cdots & \lambda_{m,k} \\ \psi_{1,k} & \psi_{2,k} & \cdots & \psi_{m,k} \end{bmatrix} \quad \text{then } N_k = AL[N\xi_{1,k} \quad N\xi_{2,k} \quad \cdots \quad N\xi_{m,k}],$$
$$\text{for } k = 1, \dots, K; \tag{23}$$

and can be described with the rule bases in Eq. (24):

$$\begin{array}{l} \text{Rule 1: If } BL \text{ is } \lambda_{1,k} \text{ and } CL \text{ is } \psi_{1,k} \text{ then } AL \text{ is } N\xi_{1,k} \\ \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \text{Rule } m\text{: If } BL \text{ is } \lambda_{m,k} \text{ and } CL \text{ is } \psi_{m,k} \text{ then } AL \text{ is } N\xi_{m,k} \end{array} \quad \text{for } k = 1, \dots, K; \tag{24}$$

where BL is the level of benefits, CL is the level of costs, and AL is the level of alternatives.

Step 8: Derive the rules for the alternatives based on the generalised matrix from Eq. (24), as shown in Eq. (25) for $j = 1, \dots, m$:

$$\begin{array}{l} \text{Rule 1: If } B_1 \text{ is } x_{1j,1} \text{ and } \cdots \text{ and } B_e \text{ is } x_{ej,1} \text{ and } C_1 \text{ is } y_{1j,1} \text{ and } \cdots \text{ and } C_f \text{ is } y_{fj,1} \text{ then } AL \text{ is } N\xi_{j,1} \\ \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \text{Rule } n_j\text{: If } B_1 \text{ is } x_{1j,K} \text{ and } \cdots \text{ and } B_e \text{ is } x_{ej,K} \text{ and } C_1 \text{ is } y_{1j,K} \text{ and } \cdots \text{ and } C_f \text{ is } y_{fj,K} \text{ then } AL \text{ is } N\xi_{j,K} \end{array} \tag{25}$$

Step 9: *Derive a final score for each alternative.* In order to produce a final score Γ_j for each alternative j, take the average aggregate membership value of the consequent part of the n_j rules in Eq. (25). Then multiply with the influence multiplier based on the K DMs average influence degree for alternative j. This is shown in Eq. (26):

$$\Gamma_j = \frac{\sum_{Rule=1}^{n} N\xi_{j,k}}{n} \cdot \frac{\sum_{k=1}^{K}(NICC_{j,k}^{B} + NICC_{j,k}^{C})}{K}, \quad \text{for } j = 1, \dots, m. \tag{26}$$

Thus the ranking order of all alternatives can be determined: the better alternatives j have higher values of Γ_j. The alternatives we have developed the above ranking approach for are stock exchange traded equities. We have considered application to a developing financial market, and are currently extending the application to comparison of performance in developing and developed financial markets.

3.2. Interval type-2 fuzzy sets implementation

In this implementation of FN-TOPSIS, we use Interval Type-2 fuzzy sets, as detailed in Table 4, Table 5 and Table 6, for rating of alternatives and weighting the importance of criteria. All linguistics variables are written in the form of trapezoidal Type-2-fuzzy numbers.

In terms of steps involved in the implementation of Type-2 fuzzy sets in FN-TOPSIS, the concept of ranking trapezoidal interval Type-2 fuzzy sets is relevant to step 3 prior to finding the distance of alternatives from positive ideal solutions and negative ideal solutions. The other steps are the same as type-l fuzzy sets implementation discussed in subsection 3.1.

Step 3: *Find the Fuzzy Positive Ideal Solution (FPIS) and Fuzzy Negative Ideal Solution (FNIS) for each alternative, and the distance between each alternative to FPIS and FNIS.*

In order to construct the ranking weighted decision matrices, for $j = 1, \dots, m$ and $k = 1, \dots, m$, we need to calculate the ranking value of each Type-2 fuzzy set $v_{ij,k}$, i.e. $Rank(v_{ij,k})$. The maximum number n of edges in the upper membership function $v_{ij,k}^{U}$ and the lower membership function $v_{ij,k}^{L}$ are first defined, where $i = 1, \dots, e + f$

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Table 4
Linguistic terms for the importance weight of each criterion.

Linguistic		Trapezoidal Type 2 fuzzy number
Very Low (VL)	1	(0.00, 0.00, 0.00, 0.10, 1, 1)(0.00, 0.00, 0.00, 0.10, 1,1)
Low (L)	2	(0.00, 0.10, 0.10, 0.25, 1, 1)(0.00, 0.10, 0.10, 0.25, 1,1)
Medium Low (ML)	3	(0.15, 0.30, 0.30, 0.45, 1, 1)(0.15, 0.30, 0.30, 0.45, 1,1)
Medium (M)	4	(0.35, 0.50, 0.50, 0.65, 1, 1)(0.35, 0.50, 0.50, 0.65, 1,1)
Medium High (MH)	5	(0.55, 0.70, 0.70, 0.85, 1, 1)(0.55, 0.70, 0.70, 0.85, 1,1)
High (H)	6	(0.80, 0.90, 0.90, 1.00, 1, 1)(0.80, 0.90, 0.90, 1.00, 1,1)
Very High (VH)	7	(0.90, 1.00, 1.00, 1.00, 1, 1)(0.90, 1.00, 1.00, 1.00, 1,1)

Table 5
Linguistic terms for rating of all alternatives.

Linguistic		Trapezoidal Type 2 fuzzy number
Very Poor (VP)	1	(0, 0, 0, 1, 1) (0, 0, 0, 1, 1)
Poor (P)	2	(0, 1, 1, 3, 1, 1) (0, 1, 1, 3, 1,1)
Medium Poor (MP)	3	(1, 3, 3, 5, 1, 1) (1, 3, 3, 5, 1,1)
Fair (F)	4	(3, 5, 5, 7, 1, 1) (3, 5, 5, 7, 1,1)
Medium Good (MG)	5	(5, 7, 7, 9, 1, 1) (5, 7, 7, 9, 1,1)
Good (G)	6	(7, 9, 9, 10, 1, 1) (7, 9, 9, 10,1,1)
Very Good (VG)	7	(9, 10, 10, 10, 1, 1) (9, 10, 10,10,1,1)

Table 6
Linguistic variable for alternatives level.

Linguistic		Trapezoidal Type 2 fuzzy number
Very Bad(VB)	1	(0.00, 0.00, 0.00, 0.25, 1, 1) (0.00, 0.00, 0.00, 0.25, 1, 1)
Bad (B)	2	(0.00, 0.25, 0.25, 0.50, 1, 1) (0.00, 0.25, 0.25, 0.50, 1, 1)
Regular (R)	3	(0.25, 0.50, 0.50, 0.75, 1, 1) (0.25, 0.50, 0.50, 0.75, 1, 1)
Good (G)	4	(0.50, 0.75, 0.75, 1, 1, 1) (0.50, 0.75, 0.75, 1, 1, 1)
Very Good (VG)	5	(0.75, 1.00, 1.00, 1.00, 1, 1) (0.75, 1.00, 1.00, 1.00, 1, 1)

and $j = 1, \dots, m$. If n is an odd number and $n \geq 3$, then $r = n + 1$. If n is an even number and $n \geq 4$, then $r = n$. The $Rank(v_{ij,k})$ of an interval Type- 2 fuzzy set is presented in Eq. (27):

$$\begin{aligned} Rank(v_{ij,k}) = &\sum_{l \in \{U,L\}} M_1\big(v^l_{ij,k}\big) + \sum_{l \in \{U,L\}} M_2\big(v^l_{ij,k}\big) + \dots + \sum_{l \in \{U,L\}} M_{r-1}\big(v^l_{ij,k}\big) \\ &- \frac{1}{r}\Big(\sum_{l \in \{U,L\}} S_1\big(v^l_{ij,k}\big) + \sum_{l \in \{U,L\}} s_2\big(v^l_{ij,k}\big) + \dots + \sum_{l \in \{U,L\}} s_r\big(v^l_{ij,k}\big)\Big) \\ &+ \sum_{l \in \{U,L\}} H_1\big(v^l_{ij,k}\big) + \sum_{l \in \{U,L\}} H_2\big(v^l_{ij,k}\big) + \dots + \sum_{l \in \{U,L\}} H_{r-2}\big(v^l_{ij,k}\big) \end{aligned} \tag{27}$$

Here $M_p(v^l_{ij,k})$ denotes the average of the elements $a^{v,l}_{ij,k,p}$ and $a^{v,l}_{ij,k,(p+1)}$, i.e. $M_p(v^l_{ij,k}) = \frac{(a^{v,l}_{ij,k,p} + a^{v,l}_{ij,k,(p+1)}))}{2}$, for $p = 1, \dots, r-1$. Also $S_p(v^l_{ij,k})$ denotes the standard deviation of elements $a^{v,l}_{ij,k,1}, a^{v,l}_{ij,k,2}, \dots, a^{v,l}_{ij,k,p}$, i.e. $S_p(v^l_{ij,k}) = \sqrt{\frac{1}{p}\sum_{t=1}^{p}(a^{v,j}_{ij,k,t} - \frac{1}{p}\sum_{t=1}^{p} a^{v,j}_{ij,k,t})^2}$, for $p = 1, \dots, r$.

The standard deviation between two or more fuzzy numbers is used as defuzzifucation method to convert fuzzy numbers to crisp numbers. In this case the standard deviation value calculated using as $S_p(v^l_{ij,k})$. Finally, $H_p(v^l_{ij,k})$ denotes the membership value of the element $a^{v,l}_{ij,k,(p+1)}$ for $p = 1, \dots, r-2$, where $l \in \{U, L\}$ and r is an even number.

The fuzzy positive ideal solution $A^+_k = (v^+_{1,k}, v^+_{2,k}, \dots, v^+_{(e+f),k})$ and the fuzzy negative ideal solution $A^-_k = (v^-_{1,k}, v^-_{2,k}, \dots, v^-_{(e+f),k})$ are defined in Eq. (28):

Table 7
Linguistic variable for expert's reliability.

Linguistic variables	Trapezoidal Fuzzy number
Strongly Unlikely (SU)	(0.00, 0.00, 0.00, 0.10)
Unlikely (U)	(0.00, 0.10, 0.10, 0.25)
Somewhat Unlikely (SWU)	(0.15, 0.30, 0.30, 0.45)
Neutral (N)	(0.35, 0.50, 0.50, 0.65)
Somewhat Likely (SWL)	(0.55, 0.70,0.70, 0.85)
Likely (L)	(0.80, 0.90, 0.90, 1.00)
Strongly Likely (SL)	(0.90, 1.00, 1.00, 1.00)

$$A_k^+ = \left(v_{1,k}^+, v_{2,k}^+, \ldots, v_{(e+f),k}^+\right) \quad \text{and} \quad A_k^- = \left(v_{1,k}^-, v_{2,k}^-, \ldots, v_{(e+f),k}^-\right), \tag{28}$$

where

$$v_{i,k}^+ = \begin{cases} \max_{1\le j\le e+j}\{Rank(v_{ij,k}^B)\}, & B_i \in B \\ \min_{1\le j\le e+f}\{Rank(v_{ij,k}^C)\}, & C_i \in C \end{cases}$$

and

$$v_{i,k}^- = \begin{cases} \min_{1\le j\le e+f}\{Rank(v_{ij,k}^B)\}, & B_i \in B \\ \max_{1\le j\le e+f}\{Rank(v_{ij,k}^C)\}, & C_i \in C \end{cases}$$

Here, B denotes the set of benefit criteria, C denotes the set of cost criteria, and $i = 1, \ldots, m$. The distance $\Delta_{j,k}^+$ between each alternative $A_{j,k}$ and the fuzzy positive ideal solution A_k^+ is calculated with Eq. (29):

$$\Delta_{j,k}^+ = \sqrt{\sum_{i-1}^{e+f}\left(Rank(v_{ij,k}) - v_{i,k}^+\right)^2} \quad \text{for } j = 1, \ldots, m \text{ and } k = 1, \ldots, K \tag{29}$$

The distance $\Delta_{j,k}^-$ between each alternative $A_{j,k}$ and the fuzzy negative ideal solution A_k^- is calculated is calculated with Eq. (30):

$$\Delta_{j,k}^- = \sqrt{\sum_{i-1}^{e+f}\left(Rank(v_{ij,k}) - v_{i,k}^-\right)^2} \quad \text{for } j = 1, \ldots, m \text{ and } k = 1, \ldots, K \tag{30}$$

3.3. Z-numbers implementation

For the Z-numbers implementation of TOPSIS-FN, the Table 1, Table 2 and Table 3 from subsection 3.1 are used, with an additional Table 7 for the linguistic variable representing decision maker reliability.

Here, the reliability of experts is taken into consideration during the decision making process. The experts are advised to use the linguistic variables in Table 7 to evaluate the confidence in their decision. This applies at the start of step 1 of the algorithm described in Type-l fuzzy sets implementation of FN-TOPSIS. The other steps are the same as the implementation discussed in subsection 3.1.

Step 1: *Use the information from Table 7 to derive the second component B of the Z-numbers, and then convert the Z-numbers to Type-l fuzzy numbers.*

Let $Z = (A, B)$ is a Z-number, where $\{A = (x, \mu_A)|x \in [0, 1]\}$, $\{B = (x, \mu_B)|x \in [0, 1]\}$, and μ_A and μ_B are trapezoidal membership functions. The second part (reliability) needs to convert a fuzzy number into a crisp number using fuzzy expectation and whereby the centroid value between two or more fuzzy numbers is used as defuzzification method [2]. In this case the centroid value calculated using as shown in Eq. (31).

$$\alpha = \frac{\int x\mu_{\tilde{B}}dx}{\int \mu_{\tilde{B}}dx}, \tag{31}$$

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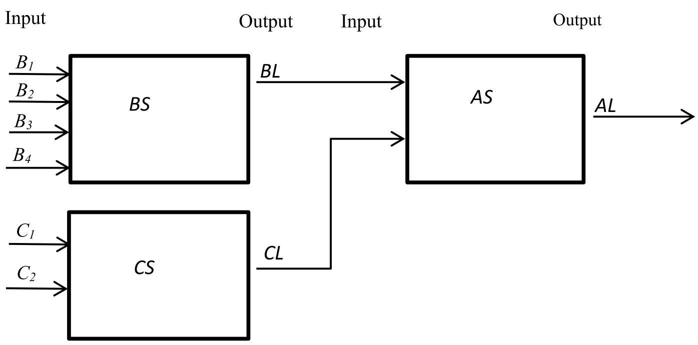

Fig. 4. Fuzzy network for the FN-TOPSIS application to ranking stock.

where $\int$ denotes an algebraic integration. Then add the weight of the second part (reliability) to the first part (restriction). Weighted Z-numbers can be denoted as:

$$\tilde{Z}^{\alpha} = \left\{(x, \mu_{\tilde{A}^{\alpha}}) \middle| \mu_{\tilde{A}^{\alpha}}(x) = \alpha \mu_{\tilde{A}}(x), x \in [0, 1]\right\}$$

These can be represented with Type-l fuzzy numbers as:

$$\tilde{Z}' = \left\{\langle x, \mu_{\bar{Z}^{\alpha}}(x)\rangle \middle| \mu_{\bar{Z}^{\alpha}}(x) = \mu_{\bar{A}}\left(\frac{x}{\sqrt{\alpha}}\right),\ x \in [0, 1]\right\}$$

It is proven in [14] that $\tilde{Z}'$ has the same Fuzzy Expectation as $\tilde{Z}^{\alpha}$. The remaining steps of the algorithm are the same as for the Type-l fuzzy sets implementation. The next section is illustrating systematically the application of Type-l fuzzy sets the proposed FN-TOPSIS method to solve the problem of se1ectio n/ranking of traded equity.

4. Simulation results

The proposed method is applied to a case study on stock selection where the main aim is to rank preferences of stock to invest. We study the problem of ranking traded equity in developing financial markets, in order to illustrate the applicability and validity of the proposed FN methodology in a realistic scenario. Decision makers with different levels of experience evaluate 25 company listed on the Main Board of the Kuala Lumpur Stock Exchange (KLSE). A set of financial ratios for the equities are considered towards the benefits and cost criteria in the FN-TOPSIS algorithm. These include: Market Value of Firm (B1), defined as market value of firm-to-earnings before amortization, interest and taxes. This is one of the critical financial indicators, and the lower the ratio the better the equity [15]; Return on Equity (B2), which evaluates how much the company earns on the investment of its shareholders. C2 is measured as net income divided by stockholder funds. Portfolio managers examine C2 when deciding whether to trade (buy or sell) equities. The higher values of the ratio indicate healthier companies. Debt-to-Equity ratio (C1), belonging to long-term solvency ratios that are intended to address the firm's long run ability to meet its obligations. It is considered by DMs that the lower the ratio the better [16]. Current Ratio (B3), which measures liquidity of companies, and explains the ability of a business to meet its current obligations when fall due. The higher the ratio, the more liquid is the company, and therefore in a better position. [17]. Market Value-to-Net Sales (B4) is market value ratios of particular interest to investors. The lower the ratio the better the equity [18]. Price/earnings ratios (C2) measure the ratio of market price of each share of common stock to the earnings per share. The lower this ratio is better the equity.

In this study, the process of ranking equities follows the proposed methods in Section 3. Fig. 4 illustrates the fuzzy network model for the problem of and includes 4 benefit criteria and 2 cost criteria.

Step 1: Based on the information provided by experts in Tables 8–11 and using Eq. (2), the decision matrices for the benefit and cost systems can be constructed. The linguistic terms in Tables 8–11 can be converted by using the fuzzy numbers in Tables 1–3, respectively. The rating of each criterion for each equity and the importance of criteria are based on decision makers' opinions.

Step 2: Considering the benefit system, the normalized decision matrix R_k^B and the weight normalized decision matrix V_k^B can be constructed for each k, using equations Eq. (3), correspondingly. For example, the calculations for E1 using the opinion of DM1 is as follows:

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Table 8
Importance of benefit and cost criteria based on DMs opinions.

Criteria	DM1	DM2	DM3
B1	VH	VH	MH
B2	MH	MH	MH
B3	M	ML	M
B4	H	H	VH
C1	MH	M	H
C2	ML	M	ML
Influnce of DM	8	10	7

Table 9
Rating of each criteria for each stock based on DM1 opinion.

Stock	Criteria					
	B1	B2	B3	B4	C1	C2
S1	VG	VG	VG	G	G	F
S2	MG	VG	M	G	F	G
S3	VG	MG	MG	G	VG	MG
S4	F	MP	MG	MP	F	G
S5	P	P	F	P	F	F
S6	G	G	F	MG	G	F
S7	MG	F	MP	G	F	F
S8	MG	F	F	P	MP	G
S9	VP	VP	P	VP	F	F
S10	F	G	MP	P	F	F
S11	P	G	F	P	F	G
S12	MG	G	MP	P	F	F
S13	P	G	MG	P	F	F
S14	F	F	VG	MG	MG	VG
S15	MG	G	G	F	MG	MG
S16	G	VG	G	MG	G	F
S17	P	G	F	P	VP	P
S18	P	VG	F	P	G	VG
S19	MG	F	G	F	F	F
S20	MG	G	F	G	F	MG
S21	G	VG	F	P	G	F
S22	F	F	P	F	F	P
S23	MG	MG	G	MG	F	VG
S24	MG	G	F	G	P	F
S25	MG	F	MP	G	F	G

$$g_{1,1} = (0.9, 1, 1, 1)$$

$$x_{11,1} = (9, 10, 10, 10)$$

$$d_{1,1}^{x^*} = 10$$

$$r_{111}^{B} = (9/10, 10/10, 10/10, 10/10) = (0.9, 1, 1, 1)$$

$$v_{1L1}^{B} = (0.9 \times 0.9, 1 \times 1, 1 \times 1, 1 \times 1) = (0.81, 1, 1, 1)$$

This step is repeated then for the cost system, in order to calculate the normalized decision matrix R_k^C and the weight normalized decision matrix V_k^c.

Step 3: The Fuzzy Positive Ideal Solution (FPIS) and the Fuzzy Negative Ideal Solution (FNIS) for each equity based on both systems, and the distances between the rating of criteria for each equity and the FPIS and FNIS, can be evaluated as follows.

Table 10
Rating of each criteria for each stock based on DM2 opinion.

Stock	Criteria				
	B1	B2	B3	B4	C1	C2
S1	VG	G	MG	MG	F	F
S2	G	VG	MG	G	G	VG
S3	VG	G	MG	G	VG	VG
S4	MP	F	F	P	MG	F
S5	P	F	G	VP	MG	MG
S6	G	MG	G	MG	G	MG
S7	MG	MG	MP	G	F	F
S8	F	MG	F	P	G	MP
S9	P	VP	F	F	VG	F
S10	F	MG	MG	F	F	G
S11	F	G	VG	F	VG	F
S12	MP	G	P	F	F	MG
S13	P	G	G	VP	MG	MG
S14	G	F	G	F	VG	MG
S15	MG	G	VG	F	G	G
S16	G	VG	G	VG	G	G
S17	F	G	G	F	VP	G
S18	F	VG	F	VP	G	F
S19	MG	P	G	G	VG	F
S20	VP	G	F	G	P	F
S21	MG	G	G	MG	VG	MG
S22	P	P	F	F	F	MP
S23	VG	G	VG	G	MG	MG
S24	G	MG	G	MG	F	F
S25	F	P	G	MG	F	G

Table 11
Rating of each criteria for each equity based on DM3 opinion.

Stock	Criteria					
	B1	B2	B3	B4	C1	C2
S1	G	MG	VG	G	G	F
S2	MG	VG	MG	VG	MG	G
S3	VG	MG	G	VG	VG	G
S4	F	MG	MP	P	F	F
S5	P	P	P	P	MG	G
S6	MG	VG	P	MG	G	G
S7	MG	MG	P	MP	F	P
S8	F	MG	MP	P	G	P
S9	VP	VP	P	VP	VG	VP
S10	F	MG	MG	F	P	G
S11	P	VG	G	P	G	G
S12	G	MG	P	P	F	MG
S13	P	G	MP	VP	MP	G
S14	G	F	F	MG	G	F
S15	F	G	MG	G	VG	MG
S16	G	G	MG	MG	VG	MG
S17	G	MG	P	P	VP	G
S18	P	VG	VP	VP	G	P
S19	F	F	MG	G	G	F
S20	G	P	P	F	F	P
S21	VG	G	MG	G	MG	MG
S22	F	F	P	F	F	P
S23	G	MG	VG	G	G	MG
S24	F	VG	MG	MG	G	G
S25	F	F	P	F	F	MG

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FPIS and FNIS are determined as:

$$A_k^+ = \left[(1,1,1,1)_{1,k}, (1,1,1,1)_{2,k}, \ldots, (1,1,1,1)_{25,k}\right]$$
$$A_k^- = \left[(0,0,0,0)_{1,k}, (0,0,0,0)_{2,k}, \ldots, (0,0,0,0)_{25,k}\right]$$

The distances $\Delta_{j,k}^{B+}$ or $\Delta_{j,k}^{B+}$, between the rating according to DM k of benefit criteria $i = 1, \ldots, 4$ for each equity j ($j = 1, \ldots, 25$) and the FPIS A_k^+ or FNIS A_k^- are calculated. For example, the distance between the first equity E1 according to DM1 and the FPIS A_1^+ is calculated for $j = 1$ and $k = 1$, as follows:

$$\Delta_k^B\left(v_{ij,k}, v_{i,k}^+\right) = \Delta_1^B\left(v_{1\text{L}1}, v_{1,1}^+\right) = \sqrt{\frac{1}{3}\left[(0.81-1)^2 + \cdots + (1-1)^2\right]} = 0.11$$

and similarly:

$$\Delta_k^B\left(v_{ij,k}, v_{i,k}^+\right) = \Delta_1^B\left(v_{211}, v_{2,1}^+\right) = 0.409$$
$$\Delta_1^B\left(v_{31,1}, v_{3,1}^+\right) = 0.668$$
$$\Delta_1^B\left(v_{31,1}, v_{3,1}^+\right) = 0.298$$

to produce overall:

$$\Delta_{j,k}^{B+} = \sum_{i=1}^{i} \Delta_k^B\left(v_{1j,k}, v_{1,k}^+\right) = \Delta_{1,1}^{B+} = \sum_{i=1}^{4} \Delta_1^B\left(v_{i1,1}, v_{i,1}^+\right) = 0.11 + 0.409 + 0.668 + 0.298 = 1.4841$$

Next, using Eq. (4) for $j = 1$ and $k = 1$, the distance between E1 according to DM1 and the FPIS A_1^- is calculated as:

$$\Delta_k^B\left(v_{ij,k}, v_{i,k}^-\right) = \Delta_1^B\left(v_{11,1}, v_{1,1}^-\right) = \sqrt{\frac{1}{3}\left[(0.81-0)^2 + \cdots + (1-0)^2\right]} = 1.373$$

and similarly

$$\Delta_k^B\left(v_{ij,k}, v_{i,k}^-\right) = \Delta_1^B\left(v_{21,1}, v_{2,1}^-\right) = 1.063$$
$$\Delta_1^B\left(v_{3\text{L}1}, v_{3,1}^-\right) = 0.789$$
$$\Delta_1^B\left(v_{31,1}, v_{3,1}^-\right) = 1.242$$

producing overall:

$$\Delta_{j,k}^{B-} = \sum_{i=1}^{i} \Delta_k^B\left(v_{1j,k}, v_{1,k}^-\right) = \Delta_{1,1}^{B-} = \sum_{i=1}^{4} \Delta_1^B\left(v_{i1,1}, v_{i,1}^-\right) = 1.373 + 1.063 + 0.789 + 1.242 = 4.4671$$

Now, the distances $\Delta_{j,k}^{C+}$ and $\Delta_{j,k}^{C+}$, between the rating according to DM k of cost criteria $i = 1, \ldots, 2$ for each equity j ($j = 1, \ldots, 25$) and the FPIS A_k^+ or FNIS A_k^- are calculated. For example, the distance between the first equity E1 according to DM1 and the FPIS A_1^+ is calculated using Eq. (6) for $j = 1$ and $k = 1$, as follows:

$$\Delta_k^c\left(v_{ij,k}, v_{i,k}^+\right) = \Delta_1^c\left(v_{11,1}, v_{1,1}^+\right) = \sqrt{\frac{1}{3}\left[(0.39-1)^2 + \cdots + (0.85-1)^2\right]} = 0.49$$

and similarly:

$$\Delta_k^c\left(v_{ij,k}, v_{i,k}^+\right) = \Delta_1^c\left(v_{21,1}, v_{2,1}^+\right) = 1.12$$

to produce overall:

$$\Delta_{j,k}^{c+} = \sum_{i=1}^{i} \Delta_k^c\left(v_{1j,k}, v_{1,k}^+\right) = \Delta_{1,1}^{c+} = \sum_{i=1}^{2} \Delta_1^C\left(v_{i1,1}, v_{i,1}^+\right) = 0.49 + 1.12 = 1.61$$

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Next, using Eq. (7) for $j = 1$ and $k = 1$, the distance between E1 according to DM1 and the FPIS A_1^- is calculated as:

$$\Delta_k^C\left(v_{ij,k}, v_{i,k}^-\right) = \Delta_1^C\left(v_{11,1}, v_{1,1}^-\right) = \sqrt{\frac{1}{3}\left[(0.39-0)^2 + \cdots + (0.85-0)^2\right]} = 1.017$$

and similarly

$$\Delta_k^c\left(v_{ij,k}, v_{i,k}^-\right) = \Delta_1^c\left(v_{21,1}, v_{2,1}^-\right) = 0.339$$

producing overall:

$$\Delta_{j,k}^{c-} = \sum_{i=1}^{i} \Delta_k^c\left(v_{1j,k}, v_{1,k}^-\right) = \Delta_{1,1}^{c-} = \sum_{i=1}^{2} \Delta_1^c\left(v_{i1,1}, v_{i,1}^-\right) = 1.017 + 0.339 = 1.358$$

Step 4: Find the closeness coefficients for the benefit system $CC_{j,k}^B$ and for the cost system $CC_{j,k}^C$, using Eq. (19) for each equity Ej, $j = 1, \ldots, 25$.

For example, the closeness coefficient for E1 in the benefit system under the first decision maker $k = 1$ is calculated using Eq. (8) as follows:

$$CC_{j,k}^B = \frac{\Delta_{j,k}^{B-}}{\Delta_{j,k}^{B+} + \Delta_{j,k}^{B-}} = CC_{1,1}^B = \frac{\Delta_{1,1}^{B-}}{\Delta_{1,1}^{B+} + \Delta_{1,1}^{B-}} = \frac{4.4671}{1.4841 + 4.4671} = 0.751$$

and the closeness coefficient in the cost system

$$CC_{j,k}^C = \frac{\Delta_{j,k}^{C-}}{\Delta_{j,k}^{c+} + \Delta_{j,k}^{c-}} = CC_{1,1}^C = \frac{\Delta_{1,1}^{C-}}{\Delta_{1,1}^{c+} + \Delta_{1,1}^{c-}} = \frac{1.358}{1.61 + 1.358} = 0.457$$

Step 5: The Influenced Closeness Coefficients $ICC_{j,k}^B$ and $ICC_{j,k}^c$ for each DMk are derived by applying the influence degree θ_k of each decision maker, using Eq. (9) and Eq. (10). Then the normalized coefficients $NICC_{j,k}^B$ and $NICC_{j,k}^C$ are calculated with Eq. (11).

For example, the influence degree of DM1 is $\theta_1 = 8$, as detailed in Table 8, and using Eq. (9) his normalised expertise is:

$$\sigma_k = \frac{\theta_k}{\sum_{l=1}^{K} \theta_l} = \sigma_1 = \frac{\theta_1}{\sum_{l=1}^{3} \theta_l} = \frac{8}{8 + 10 + 7} = 0.32$$

Then the Influenced Closeness Coefficient $ICC_{1,1}^B$ for the benefit system for equity E1 according to DM1 is calculated with Eq. (10) as:

$$ICC_{j,k}^B = \sigma_k^* CC_{j,k}^B = ICC_{1,1}^B = \sigma_1^* CC_{1,1}^B = 0.32^* 0.751 = 0.2403,$$

and similarly the corresponding Influenced Closeness Coefficient for the cost system $ICC_{1,1}^C$ is produces as:

$$ICC_{j,k}^c = \sigma_k^* CC_{j,k}^c = ICC_{1,1}^c = \sigma_1^* CC_{1,1}^c = 0.32^* 0.457 = 0.1462.$$

Next, the influenced closeness coefficients have to be normalized prior to matching the coefficients to the linguistic variable in Table 3. Using Eq. (11), $NICC_{1,1}^B$ and $NICC_{1,1}^C$ are calculated as:

$$NICC_{j,k}^B = \frac{ICC_{j,k}^B}{\max_j} = ICC_{j,k}^B = NICC_{1,1}^B = \frac{ICC_{1,1}^B}{\max_j} ICC_{j,k}^B = \frac{0.2403}{0.2403}$$

and

$$NICC_{j,k}^C = \frac{ICC_{j,k}^C}{\max_j} = ICC_{j,k}^C = NICC_{1,1}^C = \frac{ICC_{1,1}^C}{\max_j} ICC_{1,1}^C = \frac{0.1462}{0.1659}$$

Finally, the normalised coefficients are matched to the variable in Table 3:

$$NICC^{B}_{1,1} = 1 \cong VG$$
$$NICC^{c}_{1,1} = 0.8812 \cong VG$$

Step 6: The antecedent matrices X_k for the benefit system are constructed using Eq. (23) for $k = 1, \dots, K$, based on DMk opinions detailed in Tables 8–11. Each decision maker has a separate benefit antecedent matrix. The consequent matrices Λ_k for the benefit system are constructed using Eq. (12) for $k = 1, \dots, K$, based on the values of $NICC^{B}_{j,k}$ calculated at Step 5 above and matched to the linguistic variable in Table 3. Each decision maker has a separate benefit antecedent matrix. Similarly, the antecedent matrices Y_k and the consequent matrices Ψ_k are produced for the cost system. Thus the antecedent and consequent matrices for the benefit and cost rule bases are generated in this step. For example using Eq. (11), and according to the first decision maker $k = 1$ as detailed in Tables 8 and 9, the antecedent matrix X_1 for the benefit system is:

$$X_k = \begin{array}{c} \\ B_1 \\ B_2 \\ \vdots \\ B_e \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \dots & E_m \end{array} \\ \begin{bmatrix} x_{11,k} & x_{12,k} & \cdots & x_{1m,k} \\ x_{21,k} & x_{22,k} & \cdots & x_{2m,k} \\ \vdots & \ddots & \vdots & \\ x_{e1,k} & x_{e2,k} & \cdots & x_{em,k} \end{bmatrix} \end{array} = X_1 = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} x_{1,1,1} & x_{1,2,1} & \cdots & x_{1,25,1} \\ x_{2,1,1} & x_{2,2,1} & \cdots & x_{2,25,1} \\ x_{3,1,1} & x_{3,2,1} & \ddots & x_{3,25,1} \\ x_{4,1,1} & x_{4,2,1} & \cdots & x_{4,25,1} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_3 \end{array} \\ \begin{bmatrix} VG & MG & \cdots & MG \\ VG & VG & \cdots & F \\ VG & M & \cdots & MP \\ G & G & \cdots & G \end{bmatrix} \end{array},$$

where B_i are the four benefit criteria. Then using Eq. (13), the consequent matrix Λ_1 is:

$$\Lambda_k = BL \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_m \end{array} \\ \begin{bmatrix} \lambda_{1,k} & \lambda_{2,k} & \cdots & \lambda_{m,k} \end{bmatrix} \end{array} = \Lambda_1 = BL \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} \lambda_{1,1} & \Lambda_{2,1} & \dots & \Lambda_{25,1} \end{bmatrix} \end{array} = BL \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & VG & \cdots & G \end{bmatrix} \end{array}$$

where BL is the benefit level.

Next using Eq. (12), and according to the first decision maker $k = 1$ as detailed in Tables 8 and 9, the antecedent matrix Y_1 for the cost system is:

$$Y_k = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_e \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \dots & E_m \end{array} \\ \begin{bmatrix} y_{11,k} & y_{12,k} & \cdots & y_{1m,k} \\ y_{21,k} & y_{22,k} & \cdots & y_{2m,k} \\ \vdots & \ddots & \vdots & \\ y_{e1,k} & y_{e2,k} & \cdots & y_{em,k} \end{bmatrix} \end{array} = Y_1 = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \dots & E_{25} \end{array} \\ \begin{bmatrix} y_{1,1,1} & y_{1,2,1} & \cdots & y_{1,25,1} \\ y_{2,1,1} & y_{2,2,1} & \cdots & y_{2,25,1} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \dots & E_{25} \end{array} \\ \begin{bmatrix} G & F & \cdots & F \\ F & G & \cdots & G \end{bmatrix} \end{array}.$$

Then using Eq. (13), the consequent matrix Ψ_1 is:

$$\Psi_k = CL \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_m \end{array} \\ \begin{bmatrix} \lambda_{1,k} & \lambda_{2,k} & \cdots & \lambda_{m,k} \end{bmatrix} \end{array} = \Psi_1 = CL \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} \psi_{1,1} & \psi_{2,1} & \dots & \psi_{25,1} \end{bmatrix} \end{array} = CK \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \end{array} \\ \begin{bmatrix} VG & G & \cdots & G \end{bmatrix} \end{array}$$

where CL is the benefit level.

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The rule base of the benefit system for DM1 is constructed using Eq. (14) and Eq. (15), as follows:

$$X_1 = \begin{array}{c|cccc} & E_1 & E_2 & \cdots & E_{25} \\ B_1 & VG & MG & \cdots & MG \\ B_2 & VG & VG & \cdots & F \\ B_3 & VG & M & \cdots & MP \\ B_4 & G & G & \cdots & G \end{array}, \quad \text{then } \Lambda_1 = BL \begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \\ VG & VG & \cdots & G \end{array};$$

Rule 1: If B_1 is VG and B_2 is VG and B_3 is VG and B_4 is G Then the output BL is VG,

Rule 2: If B_1 is MG and B_2 is VG and B_3 is M and B_4 is G Then the output BL is VG,

$\vdots \quad \vdots \quad \vdots$

Rule 25: If B_1 is MG and B_2 is F and B_3 is MP and B_4 is G Then the output BL is G.

By analogy, the rule base for the cost system is constructed.

Step 7: The Alternatives System (AS) in this application is the Equity System (ES), and the antecedent matrices M_k of each DM k for ES are constructed using Eq. (18) based on the Benefit Level (BL) and Cost Level (CL), which are the outputs of the benefit system BS and cost system CS, respectively. Each decision maker has a separate equity antecedent matrix M_k. Next, the ES consequent matrices N_k are derived using Eq. (20)–(24), while calculating the aggregations $\xi_{j,k}$ of weighted coefficients $NICC^B_{j,k}$ and $NICC^c_{j,k}$ for each equity j ($j = 1, \ldots, 25$), then producing the normalised aggregations $N\xi_{j,k}$, and constructing the ES consequent matrices N_k based on $N\xi_{j,k}$. Each decision maker k has a separate equity consequent matrix N_k.

For example, based on the benefit and cost levels BL and CL evaluated in Step 6 above and using Eq. (20), the ES antecedent matrix M_1 according to DM1 is evaluated as:

$$M_k = \begin{array}{c} \\ BL \\ Cl \end{array}\begin{array}{ccccc} E_1 & E_2 & E_3 & \cdots & E_m \\ \lambda_{1,k} & \lambda_{2,k} & \lambda_{3,k} & \cdots & \lambda_{m,k} \\ \psi_{1,k} & \psi_{2,k} & \psi_{3,k} & \cdots & \psi_{m,k} \end{array} = M_1 = \begin{array}{c} \\ BL \\ Cl \end{array}\begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \\ \lambda_{1,1} & \lambda_{2,1} & \cdots & \lambda_{25,1} \\ \psi_{1,1} & \psi_{2,1} & \cdots & \psi_{25,1} \end{array}$$

$$= \begin{array}{c} \\ BL \\ Cl \end{array}\begin{array}{cccc} E_1 & E_2 & \cdots & E_{25} \\ VG & VG & \cdots & G \\ VG & G & \cdots & G \end{array}$$

Next, the ES consequent matrix N_1 according to DM1 is derived through:

(i) calculating the aggregated closeness coefficient $\xi_{j,1}$ for each equity $j = 1, \ldots, 25$, with Eq. (20) and based on the normalised closeness coefficients $NICC^B_{j,1}$ and $NICC^C_{j,1}$ according to DM1; e.g. for $j = 1$:

$$\xi_{j,k} = \frac{NICC^B_{j,k} \times (\frac{e}{e+f}) + NICC^C_{j,k} \times (\frac{f}{e+f})}{2} = \xi_{1,1} = \frac{NICC^B_{1,1} \times (\frac{4}{4+2}) + NICC^C_{1,1} \times (\frac{1}{4+2})}{2}$$

$$= \xi_{1,1} = \frac{1.00 \times (\frac{2}{3}) + 0.8822(\frac{1}{3})}{2} = 0.480$$

(ii) calculating the normalised aggregated closeness coefficients $N\xi_{j,1}$ for each equity $j = 1, \ldots, 25$, with Eq. (21) and based on the values $\xi_{j,1}$ produced in Step 7(i) above; e.g. for $j = 1$:

$$N\xi_{j,k} = \frac{\xi_{j,k}}{\max_j xi_{j,k}} = N\xi_{1,1} = \frac{\xi_{1,1}}{\max_j xi_{j,1}} = \frac{0.48}{0.50} = 0.96$$

and the value of $N\xi_{1,1}$ is matched to the linguistic variable for equity levels in Table 3:

$$N\xi_{1,1} = 0.960 \cong VG$$

(iii) The ES consequent matrix N_1 for DM1 is constructed using Eq. (22) and based on the values $N\xi_{j,1}$ for each equity j produced in Step 7(ii) above; e.g. for $j = 1$:

$$N_4k = EL \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_m \end{array} \\ \begin{bmatrix} N\xi_{1,k} & N\xi_{2,k} & \cdots & N\xi_{m,k} \end{bmatrix} \end{array} = N_1 = EL \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_25 \end{array} \\ \begin{bmatrix} N\xi_{1,1} & N\xi_{2,1} & \cdots & N\xi_{25,1} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_25 \end{array} \\ \begin{bmatrix} VG & VG & \cdots & G \end{bmatrix} \end{array}$$

where EL is the equity level.

Therefore, the equity system rule base according to DM1 is evaluated using Eq. (18) and Eq. (19) as:

$$\text{If} \begin{array}{cc} & \begin{array}{cccc} E_1 & E_2 & \cdots & E_25 \end{array} \\ \begin{array}{c} BL \\ CL \end{array} & \begin{bmatrix} VG & VG & \cdots & G \\ VG & G & \cdots & G \end{bmatrix} \end{array} \text{ Then } N_1 = EL \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_25 \end{array} \\ \begin{bmatrix} VG & VG & \cdots & G \end{bmatrix} \end{array};$$

Rule 1: If BL is VG and CL is VG then EL is VG,

Rule 2: If BL is VG and CL is VG then EL is VG.

$\vdots \quad \vdots \quad \vdots$

Rule 25: If BL is G and CL is G then EL is G.

Step 8: The final score for each alternative $j = 1, \ldots, 25$ is derived with Eq. (26), by taking average of the aggregate membership value of the consequent part of all active rules in the overall system for equity j, and then multiplying with the influence multiplier based on the average influence degree across all K decision makers DMs for each equity j.

For example, there are 3 active rules for E1 generated from the Boolean matrix operation. Eq. (26) is used in order to obtained final score for E1, the average aggregate membership value for the output of the 3 rules is calculated, and then multiplied with the influence multiplier for E1 across all DMs.

$$\Gamma_j = \frac{\sum_{Rule=1}^{n} N\xi_{j,k}}{n} \cdot \frac{\sum_{k=1}^{K}(NICC_{j,k}^{B} + NICC_{j,k}^{C})}{K} = \Gamma_1 = \frac{\sum_{Rule=1}^{3} N\xi_{1,k}}{3} \cdot \frac{\sum_{k=1}^{3}(NICC_{j,k}^{B} + NICC_{j,k}^{C})}{3}$$
$$= \frac{0.9 + 0.9 + 0.09}{3} \cdot \frac{0.961 + 0.829 + 0.932}{3} = 0.8162$$

The final score and ranking positions for all 25 equities considered in this case study, and based on Type-l, Type 2 and Z fuzzy numbers implementation of the proposed FN-TOPSIS method are provided in Table 12 and Table 13.

5. Result analysis

For the validation of the proposed rule-based FN-TOPSIS, the authors consider established TOPSIS methods, as the non-fuzzy TOPSIS [19] and the non-rule based fuzzy TOPSIS approaches – TI-TOPSIS [20], T2-TOPSIS [21], Z-TOPSIS. All these methods are applied to evaluate the score and final ranking of the equities from the case study in Section 5, and compared with the performance of FN-TOPSIS. The actual monthly equity returns in November 2007, based on trading the shares of the 25 companies on the Kuala Lumpur Stock Exchange and holding for a month, are used for benchmarking. The rankings are compared using the Spearman rho correlation coefficient ρ, where ρmeasures the strength of association between two ranked variables. This comparison approach is intuitively interpretable, and less sensitive to bias due to the effect of outliers [22]. The Spearman's Rank coefficient is evaluated as shown in Eq. (32).

$$\rho = 1 - \frac{6\sum \partial_i^2}{n^3 - n}, \tag{32}$$

where ∂_i represents the difference between the ranks, and n is the number of considered alternatives.

The coefficient ρ takes values between $+1$ to -1. Perfect positive relationship of ranks is indicated with $\rho = 1$, and $\rho = -1$ indicates perfect negative association of ranks, while $\rho = 0$ shows no relationship.

Based on the analysis in Table 14, it is observed that the proposed three approaches – T1, T2 and Z-Fuzzy Network TOPSIS – outperform the established TOPSIS methods (EM).

Table 12
Ranking based on type-1, type-2 and Z fuzzy number implementation of proposed FN-TOPSIS methods.

Stock	Type 1 implementation		Type 2 implementation		Z implementation	
	Final score	Rank	Final score	Rank	Final score	Rank
S1	0.8162	4	0.7496	4	0.7317	3
S2	0.8196	3	0.7427	5	0.5966	10
S3	0.8763	1	0.8742	1	0.8194	1
S4	0.3288	20	0.2600	20	0.2588	20
S5	0.2352	24	0.1470	24	0.1869	24
S6	0.7781	7	0.7675	3	0.6215	9
S7	0.4339	15	0.4183	12	0.3257	17
S8	0.3830	18	0.2739	19	0.3070	18
S9	0.1399	25	0.1171	25	0.0956	25
S10	0.4159	16	0.3405	15	0.3417	15
S11	0.4356	14	0.3954	14	0.3865	13
S12	0.4429	13	0.2909	18	0.3727	14
S13	0.2710	21	0.2428	21	0.2447	22
S14	0.7332	8	0.6003	9	0.6261	8
S15	0.7813	6	0.7021	7	0.7186	4
S16	0.8493	2	0.8499	2	0.7727	2
S17	0.2588	23	0.1906	23	0.2453	21
S18	0.3807	19	0.3248	17	0.3052	19
S19	0.5875	11	0.5579	11	0.5031	11
S20	0.4096	17	0.3340	16	0.3326	16
S21	0.7167	9	0.6965	8	0.6950	5
S22	0.2608	22	0.2164	22	0.2064	23
S23	0.8099	5	0.7420	6	0.6271	7
S24	0.6094	10	0.5821	10	0.6757	6
S25	0.4791	12	0.4029	13	0.4907	12

Table 13
Ranking based on 4 Established methods (EM) and 3 proposed methods (PM).

Stock	Ranking							
	Actual	Non-Fuzzy (EM)	T1 (EM)	T2 (EM)	Z (EM)	T1 FRBN (PM)	T2 FRBN (PM)	Z FRBN (PM)
S1	2	2	4	3	7	4	4	3
S2	4	7	3	5	9	3	5	10
S3	1	1	1	1	1	1	1	1
S4	21	21	20	18	20	20	20	20
S5	19	24	24	23	24	24	24	24
S6	11	6	6	6	8	7	3	9
S7	17	11	12	12	17	15	12	17
S8	24	14	18	17	18	18	19	18
S9	23	25	25	24	25	25	25	25
S10	22	15	16	14	14	16	15	15
S11	8	20	14	15	13	14	14	13
S12	13	12	17	16	16	13	18	14
S13	25	23	22	22	22	21	22	22
S14	9	10	9	10	10	8	9	8
S15	3	8	8	8	3	6	7	4
S16	5	3	2	2	2	2	2	2
S17	18	18	21	21	21	23	23	21
S18	12	19	19	19	19	19	17	19
S19	15	13	11	11	11	11	11	11
S20	16	17	15	13	15	17	16	16
S21	7	4	7	7	4	9	8	5
S22	20	22	23	20	23	22	21	23
S23	6	5	5	4	5	5	6	7
S24	14	9	10	9	6	10	10	6
S25	10	16	13	25	12	12	13	12

Table 14
Spearman rank correlation coefficient based on ranking performance.

	Non-Fuzzy		Fuzzy non-rule based system approach						Fuzzy network approach (aggregation)					
	Non-Fuzzy (EM)		T1 (EM)		T2 (EM)		Z (EM)		T1-AFN (NM)		T2-AFN (NM)		Z-AFN (NM)	
	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2	∂_i	∂_i^2
S1	0	0	−2	4	−1	1	−5	25	−2	4	−2	4	−1	1
S2	−3	9	1	1	−1	1	−5	25	1	1	−1	1	−6	36
S3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S4	0	0	1	1	3	9	1	1	1	1	1	1	1	1
S5	−5	25	−5	25	−4	16	−5	25	−5	25	−5	25	−5	25
S6	5	25	5	25	5	25	3	9	4	16	8	64	2	4
S7	6	36	5	25	5	25	0	0	2	4	5	25	0	0
S8	10	100	6	36	7	49	6	36	6	36	5	25	6	36
S9	−2	4	−2	4	−1	1	−2	4	−2	4	−2	4	−2	4
S10	7	49	6	36	8	64	8	64	6	36	7	49	7	49
S11	−12	144	−6	36	−7	49	−5	25	−6	36	−6	36	−5	25
S12	1	1	−4	16	−3	9	−3	9	0	0	−5	25	−1	1
S13	2	4	3	9	3	9	3	9	4	16	3	9	3	9
S14	−1	1	0	0	−1	1	−1	1	1	1	0	0	1	1
S15	−5	25	−5	25	−5	25	0	0	−3	9	−4	16	−1	1
S16	2	4	3	9	3	9	3	9	3	9	3	9	3	9
S17	0	0	−3	9	−3	9	−3	9	−5	25	−5	25	−3	9
S18	−7	49	−7	49	−7	49	−7	49	−7	49	−5	25	−7	49
S19	2	4	4	16	4	16	4	16	4	16	4	16	4	16
S20	−1	1	1	1	3	9	1	1	−1	1	0	0	0	0
S21	3	9	0	0	0	0	3	9	−2	4	−1	1	2	4
S22	−2	4	−3	9	0	0	−3	9	−2	4	−1	1	−3	9
S23	1	1	1	1	2	4	1	1	1	1	0	0	−1	1
S24	5	25	4	16	5	25	8	64	4	16	4	16	8	64
S25	−6	36	−3	9	−15	225	−2	4	−2	4	−3	9	−2	4
		556		362		630	0	404		318		386		358
	ρ	0.786		0.86		0.75	0.84			0.878		0.852		0.862
				1		8	5							
Method ranking according performance		6		3		7	5			1		4		2

6. Conclusion

This paper presents a novel TOPSIS method – FN-TOPSIS – extending the abilities of rule-based fuzzy networks in multi-criteria decision-making analysis. FN-TOPSIS uses Type-l, Type-2 and Z fuzzy numbers, and integrates experts' knowledge or data into decision analysis as well as experts degree of experience and influence. At the same time, the approach improves transparency of decision analysis; specifically in the TOPSIS process, by explicitly taking into account each subsystems and interactions. FN-TOPSIS provides an effective way to process imperfect information in decision-making practice in a more flexible and intelligent manner, also presents expert knowledge more accurately. The performance of the proposed method is validated using a benchmark, and comparing against a set of competitive approaches. The results show that the proposed method outperforms the existing non-rule based TOPSIS methods in terms of ranking performance.

This research successfully applies fuzzy networks with rule bases aggregation to a decision-making problem. The next objective is to implement and compare the theory of hesitant and intuitionistic fuzzy TOPSIS with the proposed method, particularly using Z numbers approximation. This requires a more detailed study in hesitant and intuitionistic fuzzy sets. In this context, it would be interesting to see how these fuzzy sets that have been implemented only to type 1 and type 2 TOPSIS could be extended to Z-TOPSIS.

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