Partitions of the set of nonnegative integers with the same representation functions

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Abstract

For a set of nonnegative integers S let $R_S(n)$ denote the number of unordered representations of the integer n as the sum of two different terms from S. In this paper we focus on that partitions of the set of natural numbers into two sets such that the corresponding representation functions are identical. We solve two recent problems of Lev and Chen.

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1 Introduction

Let S be a set of nonnegative integers and let $R_S(n)$ denote the number of solutions of the equation $s + s^{'} = n$, where s, $s^{'} \in S$ and $s < s^{'}$. The binary representation of an integer n is the representation of n in the number system with base 2. Let A be the set of those nonnegative integers which contains even number of 1 binary digits in its binary representation and let B be the complement of A. The set A is called Thue-Morse sequence. The investigation of the partitions of the set of nonnegative integers with identical representation functions was a popular topic in the last few decades [1], [2], [7], [8], [9]. By using the Thue - Morse sequences in 2002 Dombi [5] constructed two sets of nonnegative integers with infinite symmetric difference such that the corresponding representation functions are identical. Namely, he proved the following theorem.

Theorem 1. *(Dombi) The set of positive integers can be partitioned into two subsets C and D such that $R_C(n) = R_D(n)$ for all positive integer n.*

The complete description of the suitable partitions is the following.

Theorem 2. *Let C and D be sets of nonnegative integers such that $C \cup D = \mathbb{N}$, $C \cap D = \emptyset$ and $0 \in C$. Then $R_C = R_D$ if and only if $C = A$ and $D = B$.*

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As far as we know this theorem has never been formulated in this form, but the nontrivial part was proved by Dombi, therefore this theorem is only a little extension of Dombi's result. We give an alternative proof of the previous theorem.

A finite version of the above theorem is the following. Put $A_l = A \cap [0, 2^l - 1]$ and $B_l = B \cap [0, 2^l - 1]$.

Theorem 3. *Let C and D be sets of nonnegative integers such that $C \cup D = [0, m]$ and $C \cap D = \emptyset$, $0 \in C$. Then $R_C = R_D$ if and only if there exists an l natural number such that $C = A_l$ and $D = B_l$.*

If $C = A \cap [0, m]$ and $D = B \cap [0, m]$ then by Theorem 2 we have $R_C(n) = R_D(n)$ for $n \le m$, therefore Theorem 3. implies the following corollary.

Corollary 1. *If $C = A \cap [0, m]$ and $D = B \cap [0, m]$, where m is not of the form $2^l - 1$, then there exists an $m < n < 2m$ such that $R_C(n) \ne R_D(n)$.*

In Dombi's example the union of the set C and D is the set of nonnegative integers, and they are disjoint sets. Tang and Yu [10] proved that if the union of the sets C and D is the set of nonnegative integers and the representation functions are identical from a certain point on, then at least one cannot have the intersection of the two sets is the non-negative integers divisible by 4 i.e.,

Theorem 4. *(Tang and Yu, 2012) If $C \cup D = \mathbb{N}$ and $C \cap D = 4\mathbb{N}$, then $R_C(n) \ne R_D(n)$ for infinitely many n.*

Moreover, they conjectured that under the same assumptions the intersection cannot be the union of infinite arithmetic progressions.

Conjecture 1. *(Tang and Yu, 2012) Let $m \in \mathbb{N}$ and $R \subset \{0, 1, \dots, m-1\}$. If $C \cup D = \mathbb{N}$ and $C \cap D = \{r + km : k \in \mathbb{N}, r \in R\}$, then $R_C(n) = R_D(n)$ cannot hold for all sufficiently large n.*

Recently Chen and Lev [2] disproved this conjecture by constructing a family of partitions of the set of natural numbers such that all the corresponding representation functions are the same and the intersection of the two sets is an infinite arithmetic progression properly contained in the set of natural numbers.

Theorem 5. *(Chen and Lev, 2016) Let l be a positive integer. There exist sets C and D such that $C \cup D = \mathbb{N}$, $C \cap D = (2^{2l} - 1) + (2^{2l+1} - 1)\mathbb{N}$ and $R_C = R_D$.*

Their construction is based on the following lemma:

Lemma 1. *If there exist sets C_0 and D_0 such that $C_0 \cup D_0 = [0, m - 1]$, $C_0 \cap D_0 = \{r\}$ and $R_{C_0} = R_{D_0}$ then there exist sets C and D such that $C \cup D = \mathbb{N}$, $C \cap D = r + m\mathbb{N}$ and $R_C = R_D$.*

Chen and Lev [2] posed the following problem (we use different notations then they applied).

Problem 1. *Given $R_C = R_D$, $C \cup D = [0, m - 1]$, and $C \cap D = \{r\}$ with integer $r \ge 0$ and $m \ge 2$, must there exist an integer $l \ge 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$, $C = A_{2l} \cup (2^{2l} - 1 + B_{2l})$ and $D = B_{2l} \cup (2^{2l} - 1 + A_{2l})$?*

In this paper we solve this problem affirmatively.

Theorem 6. *Let C and D be sets of nonnegative integers such that $C \cup D = [0, m-1]$ and $C \cap D = \{r\}$, $0 \in C$. Then $R_C = R_D$ if and only if there exists an l natural number such that $C = A_{2l} \cup (2^{2l} - 1 + B_{2l})$ and $D = B_{2l} \cup (2^{2l} - 1 + A_{2l})$.*

The previous theorem suggests that there are no other counterexample for Tang and Yu's conjecture.

Problem 2. *Given $R_C = R_D$, $C \cup D = \mathbb{N}$, and $C \cap D = r + m\mathbb{N}$ with integer $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$?*

The following theorem extends Tang and Yu's theorem.

Theorem 7. *Let $m \geq 2$ be an even positive integer and let A and B be sets of nonnegative integers such that $A \cup B = \mathbb{N}$ and $A \cap B = m\mathbb{N}$. Then there exist infinitely many positive integer n such that $R_A(n) \neq R_B(n)$.*

Similar questions were investigated for unordered representation functions in [], [], [].

Thoughout this paper the characteristic function of the set A is denoted by $\chi_A(n)$, i.e.,

$$\chi_A(n) = \begin{cases} 1, \text{ if } n \in A \\ 0, \text{ if } n \notin A \end{cases}.$$

2 Proof of Theorem 2. and 3.

First we prove that if there exists a natural number l such that $C = A_l$ and $D = B_l$, then $R_C = R_D$.

We prove by induction on l. For $l = 1$, $A_1 = \{0\}$ and $B_1 = \{1\}$ thus $R_{A_1}(n) = R_{B_1}(n) = 0$. Assume the statement holds for any l and we prove it to $l+1$. By definition of A and B we have $A_{l+1} = A_l \cup (2^l + B_l)$ and $B_{l+1} = B_l \cup (2^l + A_l)$. Hence

$$R_{A_{l+1}}(n) = R_{A_l \cup (2^l + B_l)}(n) = |\{(a, a') : a < a', a, a' \in A_l, a + a' = n\}|$$
$$+|\{(a, a') : a \in A_l, a' \in 2^l + B_l, a + a' = n\}| + |\{(a, a') : a, a' \in 2^l + B_l, a + a' = n\}|$$
$$= R_{A_l}(n) + |\{(a, a') : a \in A_l, a' \in B_l, a + a' = n - 2^l\}| + R_{B_l}(n - 2^{l+1}).$$

On the other hand

$$R_{B_{l+1}}(n) = R_{B_l \cup (2^l + A_l)}(n) = |\{(a, a') : a < a', a, a' \in B_l, a + a' = n\}|$$
$$+|\{(a, a') : a \in B_l, a' \in 2^l + A_l, a + a' = n\}| + |\{(a, a') : a < a'a, a' \in 2^l + A_l, a + a' = n\}|$$
$$= R_{B_l}(n) + |\{(a, a') : a \in B_l, a' \in A_l, a + a' = n - 2^l\}| + R_{A_l}(n - 2^{l+1}),$$

thus we get the result.

Observe that if $k \leq 2^l - 1$, then $R_{A_l}(k) = R_A(k)$ and $R_{B_l}(k) = R_B(k)$. On the other hand $R_{A_l}(k) = R_{B_l}(k)$ thus we have $R_A(k) = R_B(k)$ for $k \leq 2^l - 1$. This equality holds for every l, therefore we have

$$R_A(k) = R_B(k) \qquad \text{for every } k. \tag{1}$$

To prove Theorem 2. and 3. we need the following three claims.

Claim 1. Let $0 < r_1 < \ldots < r_s \le m$ be integers. Then there exists at most one pair of sets (C, D) such that $C \cup D = [0, m]$, $0 \in C$, $C \cap D = \{r_1, \ldots, r_s\}$, $R_C(k) = R_D(k)$ for every $k \le m$.

Proof of Claim 1. We prove by contradiction. Assume that there exist at least two pairs of different sets (C_1, D_1) and (C_2, D_2) which satisfies the conditions of Claim 1. Let v denote the smallest positive integer such that $\chi_{C_1}(v) \neq \chi_{C_2}(v)$. It is clear that $R_{C_1}(v) = R_{D_1}(v)$ and $R_{C_2}(v) = R_{D_2}(v)$. We will prove that $R_{D_1}(v) = R_{D_2}(v)$ but $R_{C_1}(v) \neq R_{C_2}(v)$ which is a contradiction. Obviously

$$R_{D_1}(v) = |\{(d, d^{'}) : d < d^{'}, d, d^{'} \in D_1, d + d^{'} = v\}|.$$

As $0 \notin C \cap D$ and $0 \in C$, we have d, $d^{'} < v$. We prove that $D_1 \cap [0, v-1] = D_2 \cap [0, v-1]$, which implies that $R_{D_1}(v) = R_{D_2}(v)$. Clearly we have $C_1 \cap [0, v-1] = C_2 \cap [0, v-1]$ and $[0, v-1] = (C_1 \cap [0, v-1]) \cup (D_1 \cap [0, v-1])$ and $[0, v-1] = (C_2 \cap [0, v-1]) \cup (D_2 \cap [0, v-1])$. Let $(C_1 \cap [0, v-1]) \cap (D_1 \cap [0, v-1]) = \{r_1, \ldots, r_t\}$. Thus we have $D_1 \cap [0, v-1] = \Big([0, v-1] \setminus (C_1 \cap [0, v-1])\Big) \cup \{r_1, \ldots, r_t\}$. Similarly $D_2 \cap [0, v-1] = \Big([0, v-1] \setminus (C_2 \cap [0, v-1])\Big) \cup \{r_1, \ldots, r_t\}$, which implies $D_1 \cap [0, v-1] = D_2 \cap [0, v-1]$. On the other hand

$$R_{C_1}(v) = |\{(c, c^{'}) : c < c^{'} < v, c, c^{'} \in C_1, c + c^{'} = v\}| + \chi_{C_1}(v),$$

and

$$R_{C_2}(v) = |\{(c, c^{'}) : c < c^{'} < v, c, c^{'} \in C_1, c + c^{'} = v\}| + \chi_{C_2}(v),$$

thus $R_{C_1}(v) \neq R_{C_2}(v)$.

Claim 2. Let (C, D) be a pair of different sets, $C \cup D = [0, m]$, $C \cap D = \{r_1, \ldots, r_s\}$, and $R_C(n) = R_D(n)$ for every n nonnegative integer and if $C^{'} = m - C$ and $D^{'} = m - D$ then $C^{'} \cup D^{'} = [0, m]$, $C^{'} \cap D^{'} = \{m - r_s, \ldots, m - r_1\}$, and $R_{C^{'}} = R_{D^{'}}$.

Proof of Claim 2. Clearly,

$$R_C(k) = |\{(c, c^{'}) : c < c^{'}, c, c^{'} \in C^{'}, c+c^{'} = k\}| = |\{(c, c^{'}) : c < c^{'}, m-c, m-c^{'} \in C, c+c^{'} = k\}|$$

$$= |\{(m-c, m-c^{'}) : c < c^{'}, m-c, m-c^{'} \in C, 2m-(c+c^{'}) = 2m-k\}| = R_C(2m-k).$$

Similarly, $R_{D'}(k) = R_D(2m-k)$, which implies $R_{D'}(k) = R_D(2m-k) = R_C(2m-k) = R_{C'}(k)$, as desired.

Claim 3. If for some positive integer M, the integers $M-1, M-2, M-4, M-8, \ldots, M-2^u$, $u = \lfloor log_2 M \rfloor - 1$ are all contained in the set A (or B), then $M = 2^{u+1} - 1$.

Proof of Claim 3. Let us suppose that the integers $M-1, M-2, M-4, M-8, \ldots, M-2^u$, $u = \lfloor log_2 M \rfloor - 1$ are all contained in the set A. If M is even then $M-2$ is also an even and $M-1 = (M-2)+1$, therefore $\chi_A(M-1) \neq \chi_A(M-2)$, thus we may assume that M is an odd positive integer, and M is not of the form $2^k - 1$

Obviously, $\chi_B(M) \neq \chi_B(M-1)$. Let $M = \sum_{i=0}^{w} b_i 2^i$ be the representation of M in the number system based 2. Let x denote the largest index i such that $b_i = 0$. Then $x \le \lfloor \log_2 M \rfloor - 1$. Thus we have

$$M = \sum_{i=0}^{x-1} b_i 2^i + 2^{x+1} + \sum_{i=x+2}^{w} b_i 2^i, \quad b_i \in \{0, 1\}$$

thus

$$M - 2^x = \sum_{i=0}^{x-1} b_i 2^i + 2^x + \sum_{i=x+2}^{w} b_i 2^i = \sum_{i=0}^{w} b_i^{'} 2^i, \quad b_i' \in \{0,1\}$$

thus $\sum_{i=0}^{w} b_i = \sum_{i=0}^{w} b_i^{'}$. It follows that $\chi_A(M) = \chi_A(M - 2^x)$. On the other hand

$$\chi_A(M) \neq \chi_A(M-1) = \chi_A(M-2) = \chi_A(M-4) = \ldots = \chi_A(M - 2^x),$$

and

$$\chi_B(M) \neq \chi_B(M-1) = \ldots = \chi_B(M - 2^x),$$

which proves Claim 3.

Theorem 2. is a consequence of (1) and Claim 1. (for $s = 0$).

In the next step we prove that if the sets C and D satisfies $C \cup D = [0, m]$, $C \cap D = \emptyset$ and $R_C = R_D$, then there exists an l positive integer such that $C = A_l$ and $D = B_l$. Claim 1. and (1) imply that $C = A \cap [0, m]$ and $D = B \cap [0, m]$. Let $C^{'} = m - C$ and $D^{'} = m - D$. By Claim 1., Claim 2. and (1) we have $C^{'} = A \cap [0, m]$ or $C^{'} = B \cap [0, m]$. It follows that

$$\chi_{C'}(2^0) = \chi_{C'}(2^1) = \chi_{C'}(2^2) = \ldots = \chi_{C'}(2^u),$$

$u = \lfloor \log_2 M \rfloor - 1$ which implies that

$$\chi_C(m-1) = \chi_C(m-2) = \chi_C(m-4) = \ldots = \chi_C(m - 2^u).$$

By Claim 3. we get $m = 2^{u+1} - 1$. The proof of Theorem 3. is completed.

3 Proof of Theorem 6.

First, assume that there exists a positive integer l such that $C = A_{2l} \cup (2^{2l} - 1 + B_{2l})$, $D = B_{2l} \cup (2^{2l} - 1 + A_{2l})$. Obviously $C \cup D = [0, 2^{2l+1} - 2]$, $C \cap D = \{2^{2l} - 1\}$, $0 \in C$ and we will prove that $R_C = R_D$. It is easy to see that

$$\begin{gathered} R_C(n) = |\{(c, c^{'}) : c < c^{'}, c, c^{'} \in A_{2l}, c + c^{'} = n\}| + \\ |\{(c, c^{'}) : c \in A_{2l}, c' \in 2^{2l} - 1 + B_{2l}, c + c' = n\}| \\ + |\{(c, c^{'}) : c, c' \in 2^{2l} - 1 + B_{2l}, c + c' = n\}| \\ = R_{A_{2l}}(n) + |\{(c, c') : c \in A_{2l}, c' \in B_{2l}, c + c' = n - (2^{2l} - 1)\}| + R_{B_{2l}}(n - 2(2^{2l} - 1)). \end{gathered}$$

Moreover,

$$\begin{gathered} R_D(n) = |\{(d, d') : d < d', d, d' \in B_{2l}, d + d' = n\}| + \\ |\{(d, d') : d \in B_{2l}, d' \in 2^{2l} - 1 + A_{2l}, d + d' = n\}| \\ + |\{(d, d') : d, d' \in 2^{2l} - 1 + A_{2l}, d + d' = n\}| \\ = R_{B_{2l}}(n) + |\{(d, d') : d \in A_{2l}, d' \in B_{2l}, d + d' = n - (2^{2l} - 1)\}| + R_{A_{2l}}(n - 2(2^{2l} - 1)). \end{gathered}$$

and by Theorem 3. $R_{A_{2l}} = R_{B_{2l}}$ thus we get the result.

In the next part we prove that if $C \cup D = [0, m]$, $C \cap D = \{r\}$, $0 \in C$ and $R_C(n) = R_D(n)$, then there exists a positive integer l such that $C = A_{2l} \cup (2^{2l} - 1 + B_{2l})$, $D = B_{2l} \cup (2^{2l} - 1 + A_{2l})$ and $m = 2^{2l+1} - 2$, $r = 2^{2l} - 1$.

By Claim 2. if $C \cup D = [0,m]$, $C \cap D = \{r\}$ and $R_C = R_D$, then for $C' = m - C$, $D^{'} = m - D$ we have $C' \cup D^{'} = [0,m]$, $C' \cap D^{'} = [0, m-r]$ and by Claim 2. $R_{C'}(n) = R_{D'}(n)$, thus we may assume that $r \le m/2$.

Let

$$p_C(x) = \sum_{i=0}^{m} \chi_C(i)x^i, \tag{2}$$

$$p_D(x) = \sum_{i=0}^{m} \chi_D(i)x^i = \frac{1-x^{m+1}}{1-x} - p_C(x) + x^r. \tag{3}$$

Then we have

$$\sum_{n=0}^{\infty} R_C(n)x^n = \sum_{n=0}^{\infty} R_D(n)x^n,$$

thus

$$\frac{1}{2}p_C(x)^2 - \frac{1}{2}p_C(x^2) = \frac{1}{2}p_D(x))^2 - \frac{1}{2}p_D(x^2). \tag{4}$$

It follows from (3) and (4) that

$$(p_C(x))^2 - p_C(x^2) = \Big(\frac{1-x^{m+1}}{1-x} - p_C(x) + x^r\Big)^2 - \Big(\frac{1-x^{2m+2}}{1-x^2} - p_C(x^2) + x^{2r}\Big).$$

An easy calculation shows that

$$2p_C(x^2) = \frac{1-x^{2m+2}}{1-x^2} + 2p_C(x)\frac{1-x^{m+1}}{1-x} - \Big(\frac{1-x^{m+1}}{1-x}\Big)^2 - 2x^r\frac{1-x^{m+1}}{1-x} + 2x^r p_C(x). \tag{5}$$

We will prove that r must be odd. If r would be even and $r \le k \le 2r \le m$ is also even then from the coefficient of x^k in (5) we have

$$2\chi_C\Big(\frac{k}{2}\Big) = 1 + 2\sum_{i \le k} \chi_C(i) - (k+1) - 2 + 2\chi_C(k-r), \tag{6}$$

If $k+1 \le 2r \le m$, then from the coefficient of x^k in (5) we have

$$0 = 2\sum_{i \le k+1} \chi_C(i) - (k+2) - 2 + 2\chi_C(k+1-r). \tag{7}$$

By (6) - (7) and dividing by 2 we get that

$$\chi_C\Big(\frac{k}{2}\Big) = 1 - \chi_C(k+1) + \chi_C(k-r) - \chi_C(k+1-r).$$

As Claim 1., $k+1-r < r$, $k-r$ is even, $C \cap [0, r-1] = A \cap [0, r-1]$ and by definition of A we get $\chi_C(k+1-r) + \chi_C(k-r) = 1$ thus we have $\chi_C(k-r) - \chi_C(k+1-r) = \pm 1$. If $\chi_C(k-r) - \chi_C(k+1-r) = 1$ then we get that $\chi_C(k-r) = 1$, $\chi_C(k-1-r) = 0$, which yields $\chi_C(k+1) = 0$ and $\chi_C\Big(\frac{k}{2}\Big) = 1$.

On the other hand if $\chi_C(k-r) - \chi_C(k+1-r) = -1$ then we get that $\chi_C(k-r) = 0$, $\chi_C(k+1-r) = 1$, which yields $\chi_C(k+1) = 0$ and $\chi_C\Big(\frac{k}{2}\Big) = 0$.

This gives that $\chi_C\left(\frac{k}{2}\right) = \chi_C(k-r)$. Taking $k = 2r - 2^{i+1}$ where $i+1 \le \lfloor \log_2 r \rfloor$ we obtain $\chi_C(r-2^{i+1}) = \chi_C(r-2^i)$. It follows that

$$\chi_C(r-1) = \chi_C(r-2) = \chi_C(r-4) = \ldots = \chi_C(r-2^t).$$

for $t = \lfloor \log_2 r \rfloor - 1$. Claim 3. implies that $r = 2^l - 1$, which is a contradiction.

We know that r must be odd. If $r \le k < 2r \le m$ and k is even then from the coefficient of x^k in (5) we have we have

$$2\chi_C\left(\frac{k}{2}\right) = 1 + 2\sum_{i \le k} \chi_C(i) - (k+1) - 2 + 2\chi_C(k-r).$$

In this case $k-1$ is odd, and $k-1 \ge r$ therefore from the coefficient of x^{k-1} in (5) we have

$$0 = 2\sum_{i \le k-1} \chi_C(i) - k - 2 + 2\chi_C(k-1-r).$$

Subtracting the above equalities and dividing by 2 we get that

$$\chi_C\left(\frac{k}{2}\right) = \chi_C(k) + \chi_C(k-r) - \chi_C(k-1-r).$$

If r is odd, then $k-1-r$ is even, we know from Claim 1. that $C \cap [0, r-1] = A \cap [0, r-1]$ and by definition of A we get $\chi_C(k-1-r) + \chi_C(k-r) = 1$ thus we have $\chi_C(k-r) - \chi_C(k-1-r) = \pm 1$. If $\chi_C(k-r) - \chi_C(k-1-r) = 1$ then we get that $\chi_C(k-r) = 1$, $\chi_C(k-1-r) = 0$, which yields $\chi_C(k) = 0$ and $\chi_C\left(\frac{k}{2}\right) = 1$.

If $\chi_C(k-r) - \chi_C(k-1-r) = -1$ then we get that $\chi_C(k-r) = 0$, $\chi_C(k-1-r) = 1$, which yields $\chi_C(k) = 1$ and $\chi_C\left(\frac{k}{2}\right) = 0$. This gives that $\chi_C(k-r) = \chi_C\left(\frac{k}{2}\right)$ when $r \le k < 2r$ and k is even. Values $k = 2r - 2^{i+1}$, where $i+1 \le \lfloor \log_2 r \rfloor$ imply $\chi_C(r-2^{i+1}) = \chi_C(r-2^i)$. It follows that

$$\chi_C(r-1) = \chi_C(r-2) = \chi_C(r-4) = \ldots = \chi_C(r - 2^{\lfloor \log_2 r \rfloor - 1}),$$

which yields by Claim 3 that $r = 2^u - 1$. If $k = r$, then from the coefficient of x^k in (5) we have

$$0 = 2\sum_{i \le r} \chi_C(i) - (r+1) - 2 + 2. \tag{8}$$

On the other hand if $k = r-1$, then from the coefficient of x^{k-1} in (5) we have

$$2\chi_C\left(\frac{r-1}{2}\right) = 1 + 2\sum_{i \le r-1} \chi_C(i) - r. \tag{9}$$

By (8) - (9) we get that

$$-2\chi_C\left(\frac{r-1}{2}\right) = -2 + 2\chi_C(r) = 0,$$

we get that $0 = \chi_C\left(\frac{r-1}{2}\right) = \chi_A\left(\frac{r-1}{2}\right)$, thus $\frac{r-1}{2} = 2^{u-1} - 1$, where $u-1$ is odd, so that $r = 2^{2l} - 1$.

Because of $r \leq m/2$, Claim 1. and the first part of Theorem 3. we have $C \cap [0, 2r] = A_{2l} \cup (2^{2l} - 1 + B_{2l})$ and $D \cap [0, 2r] = B_{2l} \cup (2^{2l} - 1 + A_{2l})$. We will show that $m < 3 \cdot 2^{2l} - 2$. We prove by contradiction. Assume that $m > 3 \cdot 2^{2l} - 2$. We verify

$$C \cap [0, 3 \cdot 2^{2l} - 3] = A_{2l} \cup (2^{2l} - 1 + B_{2l}) \cup (2^{2l+1} - 1 + (B_{2l} \cap [0, 2^{2l} - 2]))$$

and

$$D \cap [0, 3 \cdot 2^{2l} - 3] = B_{2l} \cup (2^{2l} - 1 + A_{2l}) \cup (2^{2l+1} - 1 + (A_{2l} \cap [0, 2^{2l} - 2]))$$

If $2^{2l+1} - 2 < n \leq 3 \cdot 2^{2l} - 3$, then we have

$$R_C(n) = |\{(c, c') : c \in A_{2l}, c' \in 2^{2l} - 1 + B_{2l}, c + c' = n\}| +$$
$$|\{(c, c') : c \in A_{2l}, c' \in 2^{2l+1} - 1 + (B_{2l} \cap [0, 2^{2l} - 2]), c + c' = n\}| +$$
$$|\{(c, c') : c < c', c, c' \in 2^{2l} - 1 + B_{2l}, c + c' = n\}|$$
$$= |\{(c, c') : c \in A_{2l}, c' \in B_{2l}, c + c' = n - (2^{2l} - 1)\}|$$
$$+ |\{(c, c') : c \in A_{2l}, c' \in B_{2l}, c + c' = n - (2^{2l+1} - 1)\}| +$$
$$R_{B_{2l}}(n - 2(2^{2l} - 1))$$

and

$$R_D(n) = |\{(d, d') : d \in B_{2l}, d' \in 2^{2l} - 1 + A_{2l}, d + d' = n\}|$$
$$+ |\{(d, d') : d \in B_{2l}, d' \in 2^{2l+1} - 1 + (A_{2l} \cap [0, 2^{2l} - 2]), d + d' = n\}|$$
$$||\{(d, d') : d < d', d, d' \in 2^{2l} - 1 + A_{2l}, d + d' = n\}$$
$$= |\{(d, d') : d \in B_{2l}, d' \in A_{2l}, d + d' = n - (2^{2l} - 1)\}| +$$
$$|\{(d, d') : d \in B_{2l}, d' \in A_{2l}, d + d' = n - (2^{2l+1} - 1)\}| + R_{A_{2l}}(n - 2(2^{2l} - 1)),$$

which imply $R_C(n) = R_D(n)$ by Theorem 3., therefore by Claim 1. this is the only possible beginning of C and D. We will prove that $3 \cdot 2^{2l} - 2 \in C$. We prove by contradiction. Assume that $3 \cdot 2^{2l} - 2 \in D$, that is $C \cap [0, 3 \cdot 2^{2l} - 2] = A_{2l} \cup (2^{2l} - 1 + B_{2l}) \cup (2^{2l+1} - 1 + B_{2l})$ and $D \cap [0, 3 \cdot 2^{2l} - 2] = B_{2l} \cup (2^{2l} - 1 + A_{2l}) \cup (2^{2l+1} - 1 + A_{2l})$. We have a solution $3 \cdot 2^{2l} - 2 = (2^{2l+1} - 1) + (2^{2l} - 1)$ in set D, thus we have

$$R_D(3 \cdot 2^{2l} - 2) = 1 + |\{(d, d') : d \in B_{2l}, d' \in 2^{2l+1} - 1 + A_{2l}, d + d' = 3 \cdot 2^{2l} - 2\}|$$
$$|\{(d, d') : d < d', d, d' \in 2^{2l} - 1 + A_{2l}, d + d' = 3 \times 2^{2l} - 2\}|$$
$$= 1 + |\{(d, d') : d \in B_{2l}, d' \in A_{2l}, d + d' = 2^{2l} - 1)\}| + R_{A_{2l}}(2^{2l}).$$

On the other hand

$$R_C(3 \cdot 2^{2l} - 2) = |\{(c, c') : c \in A_{2l}, c' \in 2^{2l+1} - 1 + B_{2l}, c + c' = 3 \cdot 2^{2l} - 2\}| +$$
$$|\{(c, c') : c < c', c, c' \in 2^{2l} - 1 + B_{2l}, c + c' = 3 \times 2^{2l} - 2\}|$$
$$= |\{(c, c') : c \in A_{2l}, c' \in B_{2l}, c + c' = 2^{2l} - 1)\}| + R_{B_{2l}}(2^{2l}),$$

therefore by Theorem 3 we have $R_D(3 \cdot 2^{2l} - 2) > R_C(3 \cdot 2^{2l} - 2)$, which is a contradiction. We may assume that $3 \cdot 2^{2l} - 2 \in C$.

Using the fact $1 \notin C$ we have

$$R_C(3\cdot 2^{2l}-1) = |\{(c,c') : c \in A_{2l}, c' \in 2^{2l+1}-1+B_{2l}, c+c' = 3\cdot 2^{2l}-1\}|$$
$$|\{(c,c') : c < c', c, c' \in 2^{2l}-1+B_{2l}, c+c' = 3\times 2^{2l}-1\}| + \chi_C(3\cdot 2^{2l}-1)$$
$$= |\{(c,c') : c \in A_{2l}, c' \in B_{2l}, c+c' = 2^{2l}\}| + R_{B_{2l}}(2^{2l}+1)\chi_C(3\cdot 2^{2l}-1).$$

On the other hand using $1 \in B_{2l}$, $2^{2l}-1 \in A_{2l}$ and $2\cdot 2^{2l}-2 \in C$ we get

$$R_D(3\cdot 2^{2l}-1) = |\{(d,d') : d \in B_{2l}, d' \in 2^{2l+1}-1+A_{2l}\cap[0,2^{2l}-2], d+d' = 3\cdot 2^{2l}-1\}| +$$
$$|\{(d,d') : d < d', d, d' \in 2^{2l}-1+A_{2l}, d+d' = 3\times 2^{2l}-1\}|$$
$$= |\{(d,d') : d \in B_{2l}, d' \in A_{2l}, d+d' = 2^{2l}\}| + R_{A_{2l}}(2^{2l}+1) - 1,$$

therefore by Theorem 3 we have $R_C(3\cdot 2^{2l}-2) > R_D(3\cdot 2^{2l}-2)$, which is a contradiction, that is we have $m \le 3\cdot 2^{2l}-2$.

Assume that $m = 3\cdot 2^{2l}-2$. In this case $3\cdot 2^{2l}-3$, $3\cdot 2^{2l}-2 \in C$. Thus we have $R_C(6\cdot 2^{2l}-5) = 1$, but $R_D(6\cdot 2^{2l}-5) = 0$, which is absurd, therefore $m < 3\cdot 2^{2l}-2$. It follows that

$$C = A_{2l} \cup (2^{2l}-1+B_{2l}) \cup (2^{2l+1}-1+(B_{2l}\cap[0, m-(2^{2l+1}-1)]))$$

and

$$D = B_{2l} \cup (2^{2l}-1+A_{2l}) \cup (2^{2l+1}-1+(A_{2l}\cap[0, m-(2^{2l+1}-1)])).$$

We will prove that $m = 2^{2l+1}-2$. Assume that $m > 2^{2l+1}-2$. If $m-(2^{2l+1}-1) \ne 2^k-1$, then by Corollary there exists an $m-(2^{2l+1}-1) < u < 2(m-(2^{2l+1}-1))$ such that

$$R_{A\cap[0,m-(2^{2l+1}-1)]}(u) \ne R_{B\cap[0,m-(2^{2l+1}-1)]}(u).$$

Since $m + 2^{2l+1}-1 < u + 2(2^{2l+1}-1) < 2m$ we obtain

$$R_C(2(2^{2l+1}-1)+u) = R_{2^{2l+1}-1+B_{2l}\cap[0,m-(2^{2l+1}-1)]}(2(2^{2l+1}-1)+u) = R_{B_{2l}\cap[0,m-(2^{2l+1}-1)]}(u).$$

Similarly

$$R_D(2(2^{2l+1}-1)+u) = R_{A_{2l}\cap[0,m-(2^{2l+1}-1)]}(u),$$

which is a contradiction. Thus we may assume that $m-(2^{2l+1}-1) = 2^k-1$, where $k < 2l$. Hence

$$C = A_{2l} \cup (2^{2l}-1+B_{2l}) \cup (2^{2l+1}-1+B_k)$$

and

$$D = B_{2l} \cup (2^{2l}-1+A_{2l}) \cup (2^{2l+1}-1+A_k).$$

If $k = 0$, then $C, D \subset [0, 2^{2l+1}-1]$ and $2^{2l+1}-2, 2^{2l+1}-1 \in D$, therefore $R_C(2^{2l+2}-3) = 0$ and $R_D(2^{2l+2}-3) = 1$, a contradiction. Thus we may assume that $k > 0$. Then if $C' = 2^{2l+1}+2^k-2-C$ and $D' = 2^{2l+1}+2^k-2-D$ it follows that $C' \cup D' = [0, 2^{2l+1}+2^k-2]$, $C' \cap D' = \{2^{2l}+2^k-1\}$ and by Claim 2. $R_{C'} = R_{D'}$. Thus we have $C'\cap[0,2^{2l}+2^k-2] = A\cap[0,2^{2l}+2^k-2]$ or $C'\cap[0,2^{2l}+2^k-2] = B\cap[0,2^{2l}+2^k-2]$. We prove that $C'\cap[0,2^{2l}+2^k-2] = A\cap[2^{2l}+2^k-2]$. Assume that $C'\cap[0,2^{2l}+2^k-2] = B\cap[2^{2l}+2^k-2]$. Then $C'\cap[0,2^k-1] = B_k$ and $C'\cap[2^k,2^{k+1}-1] = 2^k+A_k$, which is a contradiction because $C'\cap[2^k,2^{k+1}-1] = 2^k+B_k$. We know $C'\cap[0,2^k-1] = A_k$, then $C'\cap[2^k,2^{k+1}-1] = 2^k+B_k$, and $C'\cap[2^{k+1},3\cdot 2^k-2] = 2^{k+1}+B_k\cap[0,2^k-2]$. On the other hand for $C'' = 2^{2l+1}-2-(A_{2l}\cup(2^{2l}-1+B_{2l}))$ we have $C''\cap[0,2^{2l}-2] = B\cap[0,2^{2l}-2]$, therefore $C''\cap[0,2^k-1] = B_k$ and $C''\cap[2^k,2^{k+1}-2] = 2^k+A_k\cap[0,2^k-2]$, which is a contradiction because $C' = A_k\cup(2^k+C'')$. The proof of Theorem 6. is completed.

4 Proof of Theorem 7.

We prove by contradiction. Assume that there exist sets of positive integers A and B such that $A \cup B = \mathbb{N}$, $A \cap B = m\mathbb{N}$ and $R_A(n) = R_B(n)$ for every large enough n. As we know for the generating function of A $f_A(x) = \sum_{n=0}^{\infty} \chi_A(n)x^n$ we have $\sum_{n=0}^{\infty} R_A(n)x^n = \frac{1}{2}f_A(x)^2 - \frac{1}{2}f_A(x)$, therefore

$$f_B^2(x) - f_B(x^2) - (f_A^2(x) - f_A(x^2)) = p(x), \tag{10}$$

where $p(x)$ is a polynomial. As $A \cup B = \mathbb{N}$ and $A \cap B = m\mathbb{N}$ it follows that

$$f_A(x) + f_B(x) = \frac{1}{1-x} + \frac{1}{1-x^m},$$

thus we have

$$f_B(x) = \frac{1}{1-x} + \frac{1}{1-x^m} - f_A(x),$$

which implies that

$$\Big(\frac{1}{1-x} + \frac{1}{1-x^m} - f_A(x)\Big)^2 - \Big(\frac{1}{1-x^2} + \frac{1}{1-x^{2m}} - f_A(x^2)\Big) - (f_A^2(x) - f_A(x^2)) = p(x),$$

i.e.,

$$\Big(\frac{1}{1-x}\Big)^2 + \Big(\frac{1}{1-x^m}\Big)^2 + \frac{2}{(1-x)(1-x^m)} - \frac{2f_A(x)}{1-x} - \frac{2f_A(x)}{1-x^m} - \frac{1}{1-x^2} - \frac{1}{1-x^{2m}}$$
$$+2f_A(x^2) = p(x).$$

Multiplying both sides by $(1-x^m)$ and after ordering we obtain that

$$\frac{1+x+\ldots+x^{m-1}}{1-x} + \frac{1}{1-x^m} + \frac{2}{1-x} - \frac{1-x^m}{1-x^2} - \frac{1}{1+x^m} =$$
$$2(2+x+\ldots+x^{m-1})f_A(x) - 2f_A(x^2)(1-x^m) + p(x)(1-x^m).$$

Substituting the definition of $f_A(x)$ we have

$$\frac{1+x+\ldots+x^{m-1}}{1-x} + \frac{1}{1-x^m} + \frac{2}{1-x} - \frac{1-x^m}{1-x^2} - \frac{1}{1+x^m} =$$
$$2(2+x+\cdots+x^{m-1})\Big(\sum_{n=0}^{\infty}\chi_A(n)x^n\Big) - 2\Big(\sum_{n=0}^{\infty}\chi_A(n)x^{2n}\Big)(1-x^m) + p(x)(1-x^m).$$

Let n be a large odd positive integer. In this case it is easy to see that the coefficient of x^n on the left hand side is $m+2$ and on the right hand side is $2(2\chi_A(n) + \chi_A(n-1) + \ldots + \chi_A(n-m+1))$, thus we have

$$\frac{m}{2} + 1 = 2\chi_A(n) + \chi_A(n-1) + \ldots + \chi_A(n-m+1). \tag{11}$$

If $m = 2$ we get that $2 = 2\chi_A(n) + \chi_A(n-1)$ which implies that $n \in A$, $n-1 \notin A$ which contradicts the fact that $A \cap B = 2\mathbb{N}$. Let us suppose that $m > 2$. Considering the coefficient of x^{n+2} we have

$$\frac{m}{2} + 1 = 2\chi_A(n+2) + \chi_A(n+1) + \chi_A(n) + \ldots + \chi_A(n-m+3). \qquad (12)$$

Substracting (12) from (11) we get that $2\chi_A(n+2) + \chi_A(n+1) = \chi_A(n) + \chi_A(n-m+2) + \chi_A(n-m+1)$. The values $\chi_A(n), \chi_A(n-1), \ldots, \chi_A(n-m+1)$ determine values $\chi_A(n+2)$ and $\chi_A(n+1)$:

$$\chi_A(n) + \chi_A(n-m+2) + \chi_A(n-m+1) = 0 \Rightarrow \chi_A(n+1) = 0, \chi_A(n+2) = 0$$

$$\chi_A(n) + \chi_A(n-m+2) + \chi_A(n-m+1) = 1 \Rightarrow \chi_A(n+1) = 1, \chi_A(n+2) = 0$$

$$\chi_A(n) + \chi_A(n-m+2) + \chi_A(n-m+1) = 2 \Rightarrow \chi_A(n+1) = 0, \chi_A(n+2) = 1$$

$$\chi_A(n) + \chi_A(n-m+2) + \chi_A(n-m+1) = 3 \Rightarrow \chi_A(n+1) = 1, \chi_A(n+2) = 1,$$

therefore if n is large enough the values $\chi_A(n), \chi_A(n-1), \ldots, \chi_A(n-m+1)$ determine $A \cap [n+1, \infty)$. We can do the same process for the set B. We know that a sequence defined by a linear recurrence on a finite set must be eventually periodic. This means that A and B are periodic from a certain point on. In other words there exist N, M positive integers, where $m \mid M$ and sets $F_A \subset [0, MN-1]$, $F_B \subset [0, MN-1]$, $F_A \cup F_B = [0, NM-1]$ and $F_A \cap F_B = \{0, m, 2m, \ldots, MN-m\}$, moreover there exist disjoint sets M_A, $M_B \subset [0, M-1]$ such that $M_A \cup M_B = [0, M-1]$ and $M_A \cap M_B = \{0, m, 2m, \ldots M-m\}$ for which $A = F_{\mathcal{A}} \cup \{kM + a : k \geq N, a \in M_A\}$ and $B = F_{\mathcal{B}} \cup \{kM + b : k \geq N, b \in M_B\}$ and hence

$$f_A(x) = F_A(x) + \frac{M_A(x)x^{NM}}{1-x^M},$$

$$f_B(x) = F_B(x) + \frac{M_B(x)x^{NM}}{1-x^M}.$$

Substituting in (1) we have

$$\Big(F_A(x) + \frac{M_A(x)x^{NM}}{1-x^M}\Big)^2 - \Big(F_A(x^2) + \frac{M_A(x^2)x^{2NM}}{1-x^{2M}}\Big)$$

$$-\Big(F_B(x) + \frac{M_B(x)x^{NM}}{1-x^M}\Big)^2 - \Big(F_B(x^2) + \frac{M_B(x^2)x^{2NM}}{1-x^{2M}}\Big) = p(x),$$

where $p(x)$ is a polynomial. As $F_A(x)$ and $F_B(x)$ are polynomials we have

$$(M_A^2(x) - M_B^2(x))\frac{x^{2NM}}{(1-x^M)^2} + \frac{2(M_A(x)F_A(x) - M_B(x)F_B(x))x^{NM}}{1-x^M} -$$

$$\frac{(M_A(x^2) - M_B(x^2))x^{2NM}}{1-x^{2M}} = P(x),$$

where $P(x)$ is a polynomial. It follows that

$$(1-x^M)^2(1+x^M) \mid (M_A^2(x) - M_B^2(x))x^{2NM}(1+x^M)$$

$$+2(M_AF_A(x)-M_BF_B)x^{NM}(1-x^{2M})- \tag{13}$$
$$(M_A(x^2)-M_B(x^2))x^{2NM}(1-x^M).$$

In the next step we describe the sets M_A and M_B. We will prove that if k is an arbitrary nonnegative integer and $M>2^k$ then $2^k\mid M$ and if $M_A^{(k)}\cap M_A=\{0,\dots,\frac{M}{2^k}-1\}$, $M_B^{(k)}\cap M_B=\{0,\dots,\frac{M}{2^k}-1\}$, then we have

$$M_A=\bigcup_{i=0}^{2^k-1}\Big(\frac{M}{2^k}i+M_A^{(k)}\Big),$$

and

$$M_B=\bigcup_{i=0}^{2^k-1}\Big(\frac{M}{2^k}i+M_B^{(k)}\Big).$$

We prove it by induction on k. When $k=0$, then obviously $2^0\mid M$ and $M_A^{(0)}=M_A$, $M_B^{(0)}=M_B$. Assume that we have already constructed the desired sets for k, in the next step we construct the sets for $k+1$. We know

$$M_A(x)=M_A^{(k)}(x)\sum_{t=0}^{2^k-1}x^{t\frac{M}{2^k}}=M_A^{(k)}(x)\prod_{j=1}^{k}(1+x^{\frac{M}{2^j}}).$$

$$M_B(x)=M_B^{(k)}(x)\sum_{t=0}^{2^k-1}x^{t\frac{M}{2^k}}=M_B^{(k)}(x)\prod_{j=1}^{k}(1+x^{\frac{M}{2^j}}).$$

Substituting in (13) we get

$$(1-x^M)^2(1+x^M)\mid(M_A^{(k)}(x)^2-M_B^{(k)}(x)^2)x^{2NM}(1+x^M)\left(\prod_{j=1}^{k}(1+x^{\frac{M}{2^j}})\right)^2$$

$$+2(M_A^{(k)}(x)F_A(x)-M_B^{(k)}(x)F_B(x))x^{NM}(1-x^{2M})\cdot\prod_{j=1}^{k}(1+x^{\frac{M}{2^j}})$$

$$-x^{2MN}(M_A^{(k)}(x^2)-M_B^{(k)}(x^2))(1-x^M)\prod_{j=1}^{k}(1+x^{2\frac{M}{2^j}}).$$

It is easy to see that

$$1+x^{\frac{M}{2^k}}\mid M_A^{(k)}(x^2)-M_B^{(k)}(x^2).$$

Write

$$M_A^{(k)}(x^2)-M_B^{(k)}(x^2)=\sum_{l=0}^{2\frac{M}{2^k}-2}c_l^{(k)}x^l$$

and $M_A(x^2) - M_B(x^2) = \sum_{l=0}^{2M-2} c_l x^l$, where $c_l = 0$ if l is odd and for even l

$$c_l = \begin{cases} 0, \text{ if } l/2 \in A \cap B \\ 1, \text{ if } l/2 \in A \setminus B \\ -1, \text{ if } l/2 \in B \setminus A \end{cases}.$$

Thus we have

$$M_A^{(k)}(x^2) - M_B^{(k)}(x^2) = q^{(k)}(x)(1 + x^{\frac{M}{2^k}}), \tag{14}$$

where $q^{(k)}(x)$ is a polynomial. Considering the degrees of the polynomials in both sides we get the equation

$$2\frac{M}{2^k} - 2 = deg\, q^{(k)}(x) + \frac{M}{2^k}.$$

It follows that

$$deg\, q^{(k)}(x) = \frac{M}{2^k} - 2.$$

Thus we have

$$q^{(k)}(x) = \sum_{l=0}^{\frac{M}{2^k}-2} q_l^{(k)} x^l.$$

On the other hand $c_l^{(k)} = c_l$ for $l \leq 2\frac{M}{2^k} - 2$ and therefore

$$M_A^{(k)}(x^2) - M_B^{(k)}(x^2) = \sum_{l=0}^{2\frac{M}{2^k}-2} c_l^{(k)} x^l = \sum_{l=0}^{2\frac{M}{2^k}-2} c_l x^l = q^{(k)}(x)(1 + x^{\frac{M}{2^k}}) =$$

$$\sum_{l=0}^{\frac{M}{2^k}-2} q_l^{(k)} x^l + \sum_{l=0}^{\frac{M}{2^k}-2} q_l^{(k)} x^{l+\frac{M}{2^k}}.$$

It follows that $q_l^{(k)} = c_l = c_l^{(k)}$. and $c_{l+\frac{M}{2^k}} = c_l$ for $l < \frac{M}{2^k}$. Since $1 \in (A \setminus B) \cup (B \setminus A)$, we get $0 \neq c_2 = c_{2+\frac{M}{2^k}}$. If $2 \nmid \frac{M}{2^k}$, then obviously $c_{2+\frac{M}{2^k}} = 0$, which is absurd, therefore $2^{k+1} \mid M$. Let us suppose that $l < \frac{M}{2^k}$ and l is even. Then

$$\frac{l}{2} \in A \cap B \Leftrightarrow c_l = 0 \Leftrightarrow c_{l+\frac{M}{2^k}} = 0 \Leftrightarrow \frac{l}{2} + \frac{M}{2^{k+1}} \in A \cap B$$

$$\frac{l}{2} \in A \setminus B \Leftrightarrow c_l = 1 \Leftrightarrow c_{l+\frac{M}{2^k}} = 1 \Leftrightarrow \frac{l}{2} + \frac{M}{2^{k+1}} \in A \setminus B$$

$$\frac{l}{2} \in B \setminus A \Leftrightarrow c_l = -1 \Leftrightarrow c_{l+\frac{M}{2^k}} = -1 \Leftrightarrow \frac{l}{2} + \frac{M}{2^{k+1}} \in B \setminus A$$

which implies that for $M_A^{(k+1)} = M_A^{(k)} \cap [0, \frac{M}{2^{k+1}} - 1]$ we have $M_A^{(k)} = M_A^{(k+1)} \cup (\frac{M}{2^{k+1}} + M_A^{(k+1)})$, therefore

$$M_A = \bigcup_{i=0}^{2^{k+1}-1} \Big(\frac{M}{2^{k+1}} i + M_A^{(k+1)}\Big).$$

A similar argument prove the statement for set B.

Now we are ready to prove Theorem 7. Assume that $M = 2^u v$, where $v \geq 3$ is odd and $m \mid M$. Then obviously $M > 2^{u+1}$, thus $2^{u+1} \mid M$ a contradiction.

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