

Efficient Computation Method for Strong Stability Area of Neutral Equations

Daniel Bachrathy*

* *Department of Applied Mechanics, BME
H-1111 Budapest, Műegyetem rkp. 5., Hungary
(e-mail: bachrathy@mm.bme.hu).*

Abstract: A reliable method for the computation of the strong stable area of neutral delay differential equations is presented. A special neutral system with commensurate delays is analysed. Phase parameters are used to determine all possible parameter points where a critical characteristic root may occur for a given time delay ratio. An extra condition is formulated to define the boundary curves of the robust stable area. The corresponding co-dimension 3 parameter problem in the 4 dimensional parameter space is solved efficiently by the Multi-Dimensional Bisection Method. Finally the so-called instability gradients are used to classify the distinct domains of the chart. It is shown, that the computational time of the proposed method to obtain the strong stable area is comparable to the CPU time needed for the computation of the stability chart for a given fixed delay ratio.

Keywords: Strong Stability, Neutral Equation, Instability Gradient, Multi-Dimensional Bisection Method (MDBM),

1. INTRODUCTION

The determination of the stability of dynamical systems described by neutral equations has a long history [Grammatikopoulos et al. (1986); Sficas and Stavroulakis (1987); Graef et al. (1991); Sipahi and Olgac (2006); Olgac et al. (2008); Michiels et al. (2009); Sipahi et al. (2010); Henrion and Vyhldal (2012); Cesari et al. (2014)]. Such systems can occur in different fields of engineering problems [Bellen et al. (1999); Murray et al. (1998); Niculescu and Brogliato (1999)], for which the determination of stability conditions is of high importance. For dynamical systems where only the states are delayed, the stability properties can be well defined by the approximated characteristic roots based on methods shown in [Stepan (1989); Insperger and Stepan (2011); Hill (1886); Nayfeh and Mook (1979); Khasawneh et al. (2010)]. However, if the derivative terms are delayed, too, the characteristic roots may be sensitive to arbitrarily small perturbations of the time delays [Melvin (1974); Logemann and Townley (1996); Hale and Lunel (2001a); Michiels et al. (2002, 2009)]. The parameter ranges for which the neutral system is stable for any time delay perturbation are so-called strong stable, also referred to as robust stable or delay independent stable. The computation of these parameters is complicated due to the infinite sensitivity of the characteristic roots, thus special methods have to be applied. For the stability computation of systems with free delay parameters the Cluster Treatment of Characteristic Roots (CTCR) method is an appropriate choice [Olgac and Sipahi (2004); Sipahi (2005); Sipahi and Olgac (2006); Olgac et al. (2006, 2008); Sipahi et al. (2010)] or algorithms presented in [Jarlebring (2007); Péics and Karsai (2002)] can be applied. In [Michiels et al. (2009)] the delay dependency structure is also considered. Neutral functional differential equations are analysed in [Rabah

et al. (2012)], while in [Bellen and Guglielmi (2009)] state-dependent delays are also described.

In this paper we consider a special neutral system with commensurate delays, for which all the delays are integer multiples of certain base delays. This model can describe mechanical systems with acceleration feedback. In paper [Insperger et al. (2010)], for instance, the tilt angle coordinate of a segway model is measured by an accelerometer. The angular velocity is approximated by a finite difference between some delayed values of the acceleration. This way, higher order finite difference schemes in the measured acceleration values can lead to neutral equations with delays which are strictly integer multiples of the sampling time. Another example is a digital acceleration feedback with distributed delay in which the integral along the range of the time delay is realized by a finite sum.

In this paper we focus on the stability of the corresponding difference equation of the neutral governing equation. An efficient method for the computation of the Strong Stable Area of the parameter space is presented.

2. NEUTRAL DELAYED SYSTEM

Based on the description above, a neutral system is considered to include multiple commensurate delays [Michiels et al. (2009)] (integer multipliers of a parameter).

$$\frac{d}{dt} \left(x(t) + a \sum_{k=1}^{N_a} \hat{a}_k x(t - k\tau_a) + b \sum_{l=1}^{N_b} \hat{b}_l x(t - l\tau_b) \right) + (1) \\ x(t) + c \sum_{k=1}^{N_c} \hat{c}_k x(t - k\tau_a) + d \sum_{l=1}^{N_d} \hat{d}_l x(t - l\tau_b) = 0.$$

To ensure the stability of the neutral equation, the necessary (but not sufficient) condition must be fulfilled: the equation formed by the derivative terms of Eq. (1) (see its first line) have to be stable [Cesari et al. (1976); Hale and Lunel (2001b)]. In the present paper we will focus on the stability of this difference equation only:

$$x(t) + a \sum_{k=1}^{N_a} \hat{a}_k x(t - k\tau_a) + b \sum_{l=1}^{N_b} \hat{b}_l x(t - l\tau_b) = 0. \quad (2)$$

In our test case, treated in this study, the stability computations were carried out for free control parameters a and b (parameters of the stability charts) and fixed coefficients $\hat{a} = [2, -1]$ and $\hat{b} = [1]$, which refer to a selected differential scheme of the control. Note, that in a general case \hat{a} and \hat{b} can be arbitrary vectors.

3. STABILITY LIMITS FOR FIXED DELAYS

In case, the exact values of the delays are known, the characteristic function $D(\lambda)$ of Eq. (2) can be found by substituting the trial solution $x(t) = e^{\lambda t}$ according to:

$$D(\lambda) := 1 + a \sum_{k=1}^{N_a} \hat{a}_k e^{-k\lambda\tau_a} + b \sum_{l=1}^{N_b} \hat{b}_l e^{-l\lambda\tau_b}. \quad (3)$$

The stability boundaries are determined for the critical values of the roots $\lambda = i\omega_c$:

$$D(\omega_c) = 1 + a \sum_{k=1}^{N_a} \hat{a}_k e^{-ik\omega_c\tau_a} + b \sum_{l=1}^{N_b} \hat{b}_l e^{-il\omega_c\tau_b}. \quad (4)$$

A co-dimension 2 problem is defined by the real and imaginary part of the characteristic equation:

$$\text{Re}(D(a, b, \omega_c)) = 0 \quad (5)$$

$$\text{Im}(D(a, b, \omega_c)) = 0, \quad (6)$$

in the three dimensional parameter space (a, b, ω_c) (\hat{a}_k and \hat{b}_l are considered to be constant). The so-called Multi-Dimensional Bisection Method (MDBM) [Bachrathy and Stepan (2012); Bachrathy (2012)] is a numerical computation algorithm designed to find the submanifolds of the roots of a system of non-linear equations. It can even be applied for higher parameter dimensions and co-dimensions. The roots of (5) and (6) are determined by the MDBM and are presented in Fig. 1 for different time delay ratios (τ_a/τ_b).

If the resulting numerator and denominator form of the delay ratio τ_a/τ_b contains only 'small' integer numbers (as is the case for the systems presented in the top row in Fig. 1), then the computation is not sensitive for the resolution of ω_c and a smaller range of ω_c is sufficient for the analysis. Consequently, in these cases the computation time is relatively small (for the final resolution of a and b 97x97 it is 2-5 seconds using Matlab 2014b, Intel Core i7-4710HQ CPU 2.70 GHz, 16 GB memory). Meanwhile, for $\tau_a/\tau_b = 10.01$ the CPU time is two orders of magnitude higher (134 s) and the resultant chart is still fragmented.

The results presented in Fig 1 show the (infinite) sensitivity to the delay ratio, which means that a small perturbation or the slightest uncertainty can change the stability chart completely. In case τ_a/τ_b is irrational, infinitely many bifurcation lines would occur and the range of ω_c during the computation would have to be infinity large.

4. INDEPENDENT PHASE PARAMETERS

If the periodicity of the exponential terms is considered in Eq. (4), then the phase parameters $\Phi_1 = \text{mod}(\omega_c\tau_a, 2\pi)$ and $\Phi_2 = \text{mod}(\omega_c\tau_b, 2\pi)$ can be introduced (similarly to [Michiels et al. (2009)]). During the numerical analysis it is sufficient to analyse the range $[0, 2\pi]$ for both parameters. If further symmetry properties are considered it might be enough to analyse the range $[0, \pi]$ for one of the variables.

This idea is applied in the CTCR method [Sipahi et al. (2010)], where the time delays τ_a and τ_b are the independent variables. In this study the idea is slightly different: even though Φ_1 and Φ_2 are connected through ω_c , they can be treated as independent variables due to the infinite sensitivity of the characteristic roots. In other words, if we consider an infinitesimal perturbation in τ_2 and take into account the infinite range of ω_c $[0, \infty]$, then Φ_2 can have any arbitrary value on the interval $[0, 2\pi]$, hence it is independent from Φ_1 .

The characteristic function (4) can be rewritten with the help of the phase parameters:

$$D(a, b, \Phi_1, \Phi_2) = 1 + a \sum_{k=1}^{N_a} \hat{a}_k e^{-ik\Phi_1} + b \sum_{l=1}^{N_b} \hat{b}_l e^{-il\Phi_2}. \quad (7)$$

For the 4 dimensional parameter space and co-dimension 2 problem, the resultant roots form a surface, which contains all possible parameter sets for (a, b) where critical characteristic roots ($\lambda = i\omega$) can occur for a given delay scenario. This surface can be computed by means of MDBM. It can be presented in an impressive chart (see Fig.4), however, even with MDBM it requires large computation effort and only its boundary in the (a, b) parameter plane contains relevant information.

5. EXTRA CONDITIONS

The boundary of the strong stable area is defined by the envelope of this surface. As presented in [Bachrathy (2015)], in the vicinity of these parameter points, the real part of the roots λ of the characteristic equation of (3) do not change as a function of the perturbation parameter Φ_2 , which is now considered as the perturbed parameter. This condition can be described as follows:

$$\text{Re} \left(\frac{\partial \lambda}{\partial \Phi_2} \right) = 0 \quad (8)$$

in which we focus on the critical value $\lambda = i\omega_c$, which is linearly depends on Φ_1 . The left hand side of Eq. (8) can be determined by the implicit derivation of (7) [Stepan (1989)], and after the rearrangement of the terms one ends up with:

$$\text{Re} \left(\frac{\partial \Phi_1}{\partial \Phi_2} \right) = \text{Re} \left(- \frac{\frac{\partial D}{\partial \Phi_2}}{\frac{\partial D}{\partial \Phi_1}} \right) = 0. \quad (9)$$

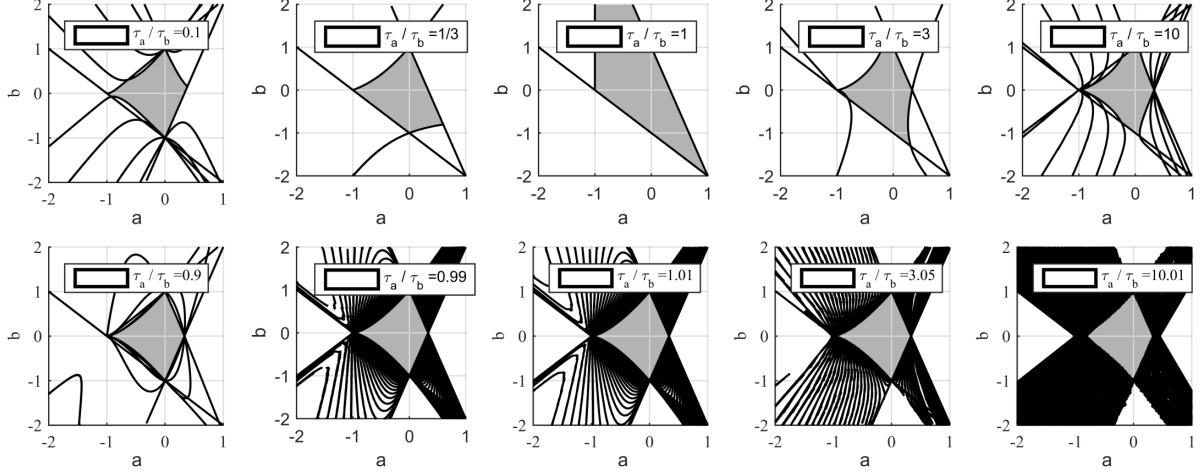


Fig. 1. Stability charts for fixed delay ratios ($\hat{a} = [2, -1]$, $\hat{b} = [1]$).

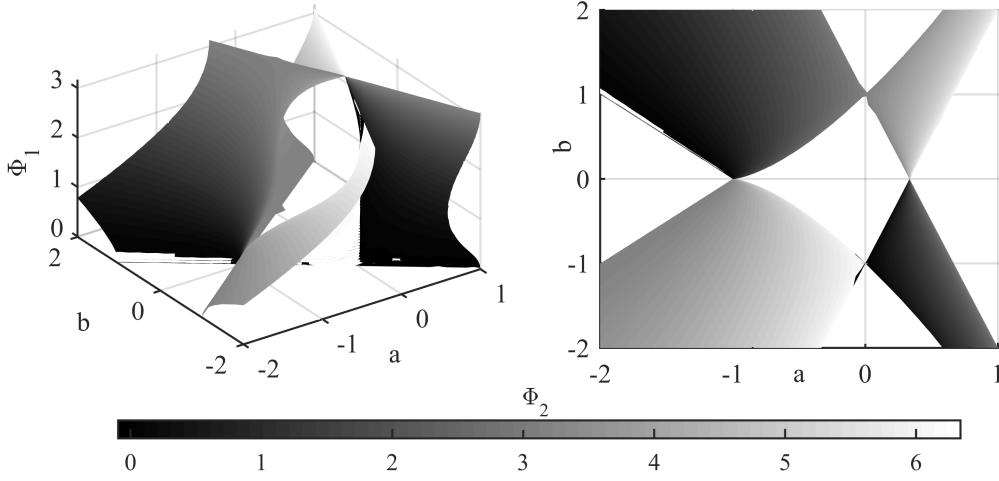


Fig. 2. Surface formed by roots of Eq. (7) in the 4 dimensional parameter space (a, b, Φ_1, Φ_2 and $\hat{a} = [2, -1]$, $\hat{b} = [1]$).

In order to eliminate the division in the numerical implementation, it is advised to use the rearranged form

$$\text{Im} \left(\frac{\partial \overline{D}}{\partial \Phi_1} \frac{\partial D}{\partial \Phi_2} \right) = 0. \quad (10)$$

Note, that in paper [Michiels et al. (2009)] the extra equations are presented in a different form. Apply this extra condition Eq. (10) to the characteristic function Eq. (7)

$$\text{Im} \left(\left(a \sum_{k=1}^{N_a} k \hat{a}_k e^{ik\Phi_1} \right) \left(b \sum_{l=1}^{N_b} l \hat{b}_l e^{-il\Phi_2} \right) \right) = 0. \quad (11)$$

With the extra independent parameter (Φ_2) and the extra condition Eq. (10) we can reduce the dimension of the entity to compute by one, thus the boundaries of the robust stability area are obtained in form of curves.

The boundaries are computed by MDBM and are presented in Fig. 3.

Now only the distinct separated areas have to be checked and classified into strongly stable (delay independent stable) areas, conditionally stable (delay dependent stable)

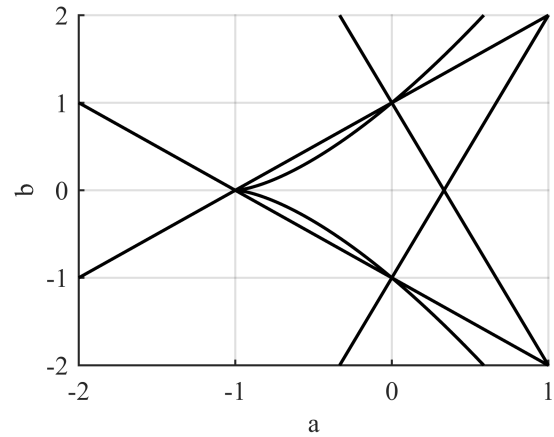


Fig. 3. Boundaries of the strong stability area ($\hat{a} = [2, -1]$, $\hat{b} = [1]$).

areas, or strongly unstable (delay independent unstable) areas. The conditionally stable regions can be treated as unsafe zones, where the system could be stable under special conditions.

7. CONCLUSION

In the present study the main steps of a strong stability computation method were presented for neutral differential equations. First, independent phase variables are introduced in order to connect the stability boundary curves of different time delay ratios, forming a surface. After that an extra condition is formulated which defines the envelope curves of this surface, defining the boundaries of the strong stable areas. Finally, the instability gradients are used to efficiently identify the strong stable area. These type of problems with 5 parameter dimension and 3 co-dimensions can efficiently be solved by means of MDBM.

The presented method can easily be extended for larger number of commensurate delay series. The topological dimension of the strong stable boundary will remain 1 for any arbitrary problem, because for each introduced extra phase parameter an additional condition is applied. Note, that the increase of the computational time in case of the MDBM for a 1 dimensional entity is significantly smaller compared to that of the brute force method. Note, that the MDBM is effective until 7 parameters, above this number of parameters different numerical techniques have to be applied, like continuation methods, which have their limitations with regards to automation and finding all the solutions.

ACKNOWLEDGEMENTS

This paper was supported by the Hungarian Scientific Research Fund - OTKA PD-112983 and the Janos Bolyai Research Scholarship of the Hungarian Academy of Sciences.

REFERENCES

- Bachrathy, D. (2012). Multi-dimensional bisection method (mdbm) - fast computation of all the roots of a non-linear function for higher parameter dimension and co-dimension.
- Bachrathy, D. (2015). Robust stability limit of delayed dynamical systems. *Periodica Polytechnica. Engineering. Mechanical Engineering*, 59(2), 74.
- Bachrathy, D. and Stepan, G. (2012). Bisection method in higher dimensions and the efficiency number. *Periodica polytechnica. Mechanical engineering*, 56(2), 81–86.
- Bachrathy, D. and Stepan, G. (2013). Improved prediction of stability lobes with extended multi frequency solution. *CIRP Annals - Manufacturing Technology*, -. doi:10.1016/j.cirp.2013.03.085.
- Bellen, A. and Guglielmi, N. (2009). Solving neutral delay differential equations with state-dependent delays. *Journal of computational and applied mathematics*, 229(2), 350–362.
- Bellen, A., Guglielmi, N., and Ruehli, A.E. (1999). Methods for linear systems of circuit delay differential equations of neutral type. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, 46(1), 212–215.
- Cesari, L., Hale, J., and LaSalle, J. (1976). *Dynamical Systems, An International Symposium*, volume 1. New York: Academic Press.
- Cesari, L., Hale, J.K., and LaSalle, J.P. (2014). *Dynamical systems: an international symposium*. Academic Press.

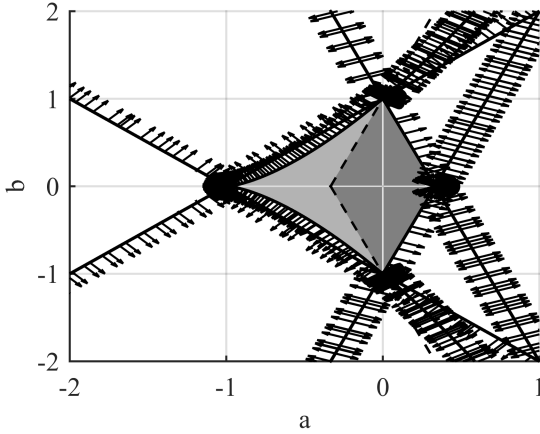


Fig. 4. Strong stability boundaries and the instability gradients. Strong stable area shaded by grey colors. The solutions of the conservative approximation Eq. (13) denoted by dashed lines and the corresponding approximated strong stable area shaded by dark grey color ($\hat{a} = [2, -1]$, $\hat{b} = [1]$).

6. INSTABILITY GRADIENTS

The classification of the areas can also be performed based on the so-called "instability gradients" [Bachrathy and Stepan (2013)], defined as:

$$\mathbf{n}_i = \text{Im} \left(\mathbf{grad}(D(a, b, \Phi_1, \Phi_2)) \frac{\partial \bar{D}(a, b, \Phi_1, \Phi_2)}{\partial \Phi_i} \right) = 0. \quad (12)$$

These instability gradients point out which side of the limit lines has more unstable characteristic roots. Thus, those areas, the corresponding instability gradients point to cannot be strong stable areas. In Fig. 4 these instability gradients are shown, too. Furthermore, if we take into account, that point (0,0) in the a - b plane is robustly stable (hence the delays in the governing equation disappear), it is straightforward, that the area around the origin is the only strong stable area (see the grey areas in Fig. 4).

The computation time of Fig. 3 is 12 seconds for the same resolution applied for Fig. 1, for which the CPU time was 2-150 seconds. Even though the dimension of the problem defined in Eq. (7) is higher by one, the computational time is comparable. The MDBM method provides the gradients automatically, hence the computation of the instability gradients only requires negligible CPU time.

Note, that the classical sufficient condition for the robust stability area [Cesari et al. (2014)] for Eq. (2) is

$$|a| \sum_{k=1}^{N_a} |\hat{a}_k| + |b| \sum_{l=1}^{N_b} |\hat{b}_l| < 1, \quad (13)$$

which results in the dark grey area in Fig. 4. Note, that this conservative approach delivers a poor approximation for the strong stable areas, because it considers all delays to be independent, but in fact only τ_a and τ_b are independent, their integer multiples are not.

- Graef, J.R., Grammatikopoulos, M.K., and Spikes, P.W. (1991). Asymptotic and oscillatory behavior of solutions of first order nonlinear neutral delay differential equations. *Journal of Mathematical Analysis and Applications*, 155(2), 562–571.
- Grammatikopoulos, M.K., Grove, E., and Ladas, G. (1986). Oscillations of first-order neutral delay differential equations. *Journal of mathematical analysis and applications*, 120(2), 510–520.
- Hale, J.K. and Lunel, S.M.V. (2001a). Effects of small delays on stability and control. In *Operator theory and analysis*, 275–301. Springer.
- Hale, J.K. and Lunel, S.M.V. (2001b). *Operator Theory and Analysis: The M.A. Kaashoek Anniversary Volume Workshop in Amsterdam, November 12–14, 1997*. Birkhäuser Basel.
- Henrion, D. and Vyhlídal, T. (2012). Positive trigonometric polynomials for strong stability of difference equations. *Automatica*, 48(9), 2207 – 2212. doi: <http://dx.doi.org/10.1016/j.automatica.2012.06.021>. URL <http://www.sciencedirect.com/science/article/pii/S0005109812002749>.
- Hill, G.W. (1886). On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon. *Acta Math*, 8, 1–36.
- Inspurger, T. and Stepan, G. (2011). *Semi-Discretization for Time-Delay Systems*. Applied Mathematical Sciences. Springer, 1, 1 edition.
- Inspurger, T., Stepan, G., and Turi, J. (2010). Delayed feedback of sampled higher derivatives. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 368(1911), 469–482.
- Jarlebring, E. (2007). On critical delays for linear neutral delay systems. In *Proc. Europ. Contr. Conf.*
- Khasawneh, F.A., Mann, B.P., and Butcher, E.A. (2010). Comparison between collocation methods and spectral element approach for the stability of periodic delay systems. In *9th IFAC Workshop on Time Delay Systems*, volume 9, 69–74.
- Logemann, H. and Townley, S. (1996). The effect of small delays in the feedback loop on the stability of neutral systems. *Systems & Control Letters*, 27(5), 267–274.
- Melvin, W.R. (1974). Stability properties of functional difference equations. *Journal of Mathematical Analysis and Applications*, 48(3), 749–763.
- Michiels, W., Engelborghs, K., Roose, D., and Dochain, D. (2002). Sensitivity to infinitesimal delays in neutral equations. *SIAM Journal on Control and Optimization*, 40(4), 1134–1158.
- Michiels, W., Vyhlídal, T., Zitek, P., Nijmeijer, H., and Henrion, D. (2009). Strong stability of neutral equations with an arbitrary delay dependency structure. *SIAM Journal on Control and Optimization*, 48(2), 763–786.
- Murray, R.M., Jacobson, C.A., Casas, R., Khibnik, A.I., Johnson Jr, C., Bitmead, R., Peracchio, A., and Proscia, W. (1998). System identification for limit cycling systems: a case study for combustion instabilities. In *American Control Conference, 1998. Proceedings of the 1998*, volume 4, 2004–2008. IEEE.
- Nayfeh, A.H. and Mook, D.T. (1979). *Nonlinear Oscillations*. John Wiley and Sons, New York.
- Niculescu, S.I. and Brogliato, B. (1999). Force measurement time-delays and contact instability phenomenon. *European Journal of Control*, 5(2), 279–289.
- Olgac, N. and Sipahi, R. (2004). A practical method for analyzing the stability of neutral type lti-time delayed systems. *Automatica*, 40(5), 847–853.
- Olgac, N., Vyhlídal, T., and Sipahi, R. (2006). Exact stability analysis of neutral systems with cross-talking delays. In *Time Delay Systems*, volume 6, 175–180.
- Olgac, N., Vyhlídal, T., and Sipahi, R. (2008). A new perspective in the stability assessment of neutral systems with multiple and cross-talking delays. *SIAM Journal on Control and Optimization*, 47(1), 327–344.
- Péics, H. and Karsai, J. (2002). Positive solutions of neutral delay differential equation. *Novi Sad J. Math*, 32(2), 95–108.
- Rabah, R., Sklyar, G.M., and Barkhayev, P.Y. (2012). Stability and stabilizability of mixed retarded-neutral type systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 18(03), 656–692.
- Sficas, Y. and Stavroulakis, I. (1987). Necessary and sufficient conditions for oscillations of neutral differential equations. *Journal of Mathematical Analysis and Applications*, 123(2), 494–507.
- Sipahi, R. (2005). Cluster treatment of characteristic roots, ctr, a unique methodology for the complete stability robustness analysis of linear time invariant multiple time delay systems against delay uncertainties. *Mechanical Engineering*, 97.
- Sipahi, R. and Olgac, N. (2006). Complete stability analysis of neutral-type first order two-time-delay systems with cross-talking delays. *SIAM journal on control and optimization*, 45(3), 957–971.
- Sipahi, R., Olgac, N., and Breda, D. (2010). A stability study on first-order neutral systems with three rationally independent time delays. *International Journal of Systems Science*, 41(12), 1445–1455.
- Stepan, G. (1989). *Retarded Dynamical Systems*. Longman, Harlow.