The complexity of recognizing minimally tough graphs

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Abstract: Let t be a positive real number. A graph is called t-tough, if the removal of any cutset S leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t-tough. A graph is minimally t-tough, if the toughness of the graph is t and the deletion of any edge from the graph decreases the toughness. The complexity class DP is the set of all languages that can be expressed as the intersection of a language in NP and a language in coNP. We prove that recognizing minimally t-tough graphs is DP-complete for any positive integer t and for any positive rational number $t \leq 1/2$.

Keywords: 3–6 keywords toughness, complexity, DP-complete

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let $\omega(G)$ denote the number of components and $\alpha(G)$ denote the independence number. For a graph G and a vertex set $V \subseteq V(G)$, let $G[V]$ denote the subgraph of G induced by V.

The complexity class DP was introduced by C. H. Papadimitriou and M. Yannakakis [4].

Definition 1 *A language L is in the class* DP *if there exist two languages $L_1 \in NP$ and $L_2 \in coNP$ such that $L = L_1 \cap L_2$.*

We mention that DP $\neq$ NP $\cap$ coNP, if NP $\neq$ coNP. Moreover, NP $\cup$ coNP $\subseteq$ DP. A language is called DP-hard if all problems in DP can be reduced to it in polynomial time. A language is DP-complete if it is in DP and it is DP-hard.

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A critical-type DP-complete problem is CriticalClique [5], in our proofs we use an equivalent form of it, α-Critical.

CriticalClique
Instance: a graph G and a positive integer k.
Question: is it true that G has no clique of size k, but adding any missing edge e to G, the resulting graph $G+e$ has a clique of size k?

By taking the complement of the graph, we can obtain α-Critical from CriticalClique.

Definition 2 *A graph G is called* α-critical, *if $\alpha(G-e) > \alpha(G)$ for all $e \in E(G)$.*

α-Critical
Instance: a graph G and a positive integer k.
Question: is it true that $\alpha(G) < k$, but $\alpha(G-e) \geq k$ for any edge $e \in E(G)$?

Since a graph is clique-critical if and only if its complement is α-critical, α-Critical is also DP-complete.

Corollary 3 α-Critical *is DP-complete.*

The notion of toughness was introduced by Chvátal [2].

Definition 4 *Let t be a positive real number. A graph G is called* t-tough, *if*

$$\omega(G-S) \leq \frac{|S|}{t}$$

for any cutset S of G (i.e. for any S with $\omega(G-S) > 1$). The toughness *of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.*
We say that a cutset $S \subseteq V(G)$ is a tough set *if $\omega(G-S) = |S|/\tau(G)$.*

For all positive rational number t we can define a separate problem:

***t*-Tough**
Instance: a graph G,
Question: is it true that $\tau(G) \geq t$?

Bauer et al. proved the following.

Theorem 5 ([1]) *For any positive rational number t,* t-Tough *is coNP-complete.*

The critical form of this problem is minimally toughness.

Definition 6 *A graph G is* minimally t-tough, *if $\tau(G) = t$ and $\tau(G-e) < t$ for all $e \in E(G)$.*

Given t we define:

Min-*t*-Tough
Instance: a graph G,
Question: is it true that G is minimally t-tough?

Our main result is the following.

Theorem 7 Min-t-Tough *is DP-complete for any positive integer t and for any positive rational number $t \leq 1/2$.*

First we prove this theorem for $t = 1$, then we generalize that proof for positive integers, and finally we prove it for any positive rational number $t \leq 1/2$.

2 Preliminaries

In this section we prove some useful lemmas.

Proposition 8 *Let G be a connected noncomplete graph on n vertices. Then $\tau(G) \in \mathbb{Q}^+$, and if $\tau(G) = a/b$, where a, b are positive integers and $(a, b) = 1$, then $1 \le a, b \le n-1$.*

PROOF: By definition,

$$\tau(G) = \min_{\substack{S \subseteq V(G) \\ \text{cutset}}} \frac{|S|}{\omega(G-S)}$$

for a noncomplete graph G. Since G is connected and noncomplete, $1 \le |S| \le n-2$ and since S is a cutset, $2 \le \omega(G-S) \le n-1$. □

Corollary 9 *Let G and H be two connected noncomplete graphs on n vertices. If $\tau(G) \neq \tau(H)$, then*

$$|\tau(G) - \tau(H)| > \frac{1}{n^2}.$$

Claim 10 *For every positive rational number t, MIN-t-TOUGH $\in$ DP.*

PROOF: For any positive rational number t,

$$\begin{aligned} \text{MIN-}t\text{-TOUGH} &= \{G \text{ graph} \mid \tau(G) = t \text{ and } \tau(G-e) < t \text{ for all } e \in E(G)\} = \\ &= \{G \text{ graph} \mid \tau(G) \ge t\} \cap \{G \text{ graph} \mid \tau(G) \le t\} \cap \\ &\cap \{G \text{ graph} \mid \tau(G-e) < t \text{ for all } e \in E(G)\}. \end{aligned}$$

Let

$$L_{1,1} = \{G \text{ graph} \mid \tau(G-e) < t \text{ for all } e \in E(G)\},$$

$$L_{1,2} = \{G \text{ graph} \mid \tau(G) \le t\}$$

and

$$L_2 = \{G \text{ graph} \mid \tau(G) \ge t\}.$$

$L_2 \in$ coNP, a witness is a cutset $S \subseteq V(G)$ whose removal leaves more than $|S|/t$ components. $L_{1,1} \in$ NP, the witness is a set of cutsets: $S_e \subseteq V(G)$ for each edge e whose removal leaves more than $|S_e|/t$ components.

Now we show that $L_{1,2} \in$ NP, i.e. we can express $L_{1,2}$ in a form of

$$L_{1,2} = \{G \text{ graph} \mid \tau(G) < t + \varepsilon\},$$

which belongs to NP. Let a, b be positive integers such that $t = a/b$ and $(a, b) = 1$, and let G be an arbitrary graph on n vertices. If G is disconnected, then $\tau(G) = 0$, and if G is complete, then $\tau(G) = \infty$, so in both cases G is not minimally t-tough. By Proposition 8, if $1 \le a, b \le n-1$ does not hold, then G is also not minimally t-tough. So we can assume that $t = a/b$, where a, b are positive integers, $(a, b) = 1$ and $1 \le a, b \le n-1$. With this assumption

$$L_{1,2} = \{G \text{ graph} \mid \tau(G) \le t\} = \left\{ G \text{ graph} \;\middle|\; \tau(G) < t + \frac{1}{|V(G)|^2} \right\},$$

so $L_{1,2} \in$ NP.

Since $L_{1,1} \cap L_{1,2} \in$ NP, $L_2 \in$ coNP and MIN-t-TOUGH $= (L_{1,1} \cap L_{1,2}) \cap L_2$, we can conclude that MIN-t-TOUGH $\in$ DP. □

Claim 11 *Let t be a positive rational number and G a minimally t-tough graph. For every edge e of G,*

1. *the edge e is a bridge in G, or*
2. *there exists a vertex set $S = S(e) \subseteq V(G)$ with*

$$\omega(G - S) \leq \frac{|S|}{t} \quad \text{and} \quad \omega\big((G - e) - S\big) > \frac{|S|}{t},$$

   and the edge e is a bridge in $G - S$.

In the first case, we define $S = S(e) = \emptyset$.

Proof: Let e be an arbitrary edge of G, which is not a bridge. Since G is minimally t-tough, $\tau(G - e) < t$. So there exists a cutset $S = S(e) \subseteq V(G - e) = V(G)$ in $G - e$ satisfying $\omega\big((G - e) - S\big) > |S|/t$. On the other hand, $\tau(G) = t$, so $\omega(G - S) \leq |S|/t$. This is only possible if e connects two components of $(G - e) - S$. $\square$

Finally we cite a Lemma that our proof relies on.

Lemma 12 (Problem 14 of 8 in [3]) *If we replace a vertex of an α-critical graph with a clique, and connect every neighbor of the original vertex with every vertex in the clique, then the resulting graph is still α-critical.*

3 Recognizing minimally 1-tough graphs

To show that Min-1-Tough is DP-hard, we reduce α-Critical to it.

Theorem 13 Min-1-Tough *is DP-complete.*

Proof: In Claim 10 we have already proved that Min-1-Tough $\in$ DP.

Let G be an arbitrary connected graph on the vertices $v_1, \ldots, v_n$. Let G_α be defined as follows. It will be easy to see that it can be constructed from G in polynomial time. For all $i \in [n]$, let

$$V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,\alpha}\}$$

and place a clique on the vertices of V_i. For all $i, j \in [n]$, if $v_i v_j \in E(G)$, then place a complete bipartite graph on $(V_i; V_j)$. For all $i \in [n]$ and for all $j \in [\alpha]$ add the vertex $u_{i,j}$ to the graph and connect it to $v_{i,j}$. Let

$$V = \bigcup_{i=1}^{n} V_i$$

and

$$U = \{u_{i,j} \mid i \in [n], j \in [\alpha]\}.$$

Add the vertex set

$$W = \{w_1, \ldots, w_\alpha\}$$

to the graph and for all $j \in [\alpha]$ connect w_j to $v_{1,j}, \ldots, v_{n,j}$.

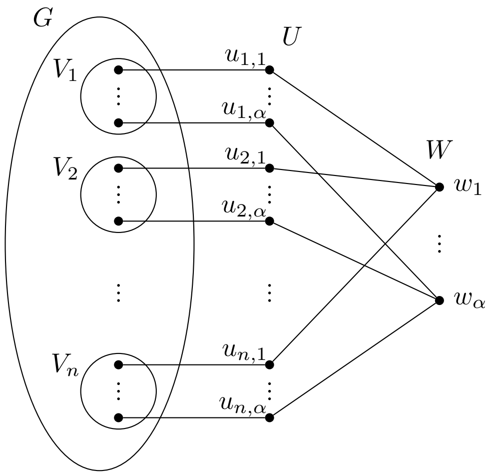

Figure 1: The graph G_α.

We need to prove that G is α-critical with $\alpha(G) = \alpha$ if and only if G_α is minimally 1-tough. First we prove the following lemma.

Lemma 14 *Let G be a graph with $\alpha(G) \leq \alpha$. Then G_α is 1-tough.*

Proof: Let $S \subseteq V(G_\alpha)$ be a cutset. We show that $\omega(G_\alpha - S) \leq |S|$.

Case 1: $W \subseteq S$. If a vertex of U has only one neighbor in $V(G_\alpha) \setminus S$, then we can assume that this vertex is not in S. Then there are two types of components in $G_\alpha - S$: isolated vertices from U and components containing at least one vertex from V. There are at most $\alpha(G)$ components of the second type and (exactly) $|V \cap S| = |S| - \alpha$ components of the first type. Thus $\omega(G_\alpha - S) \leq |S| - \alpha + \alpha(G) \leq |S|$.

Case 2: $W \nsubseteq S$. First, we make two convenient assumptions for S.

(1) $U \cap S = \emptyset$.

It is easy to see that if $u_{i,j} \in S$, then we can assume that $v_{i,j} \notin S$. Now there are two cases.

Case 2.1: $v_{i,j}$ is not isolated in $G_\alpha - S$. Then we can consider $S' = (S \setminus \{u_{i,j}\}) \cup \{v_{i,j}\}$ instead of S.

Case 2.2: $v_{i,j}$ is isolated in $G_\alpha - S$. Since there are no isolated vertices in G, there exists $k \in [n]$ such that $v_i v_k \in E(G)$. Then $v_{k,j} \in S$, so $u_{k,j} \notin S$, which means that w_j is not isolated in $G_\alpha - S$, so we can consider $S' = (S \setminus \{u_{i,j}\}) \cup \{w_j\}$ instead of S.

(2) For all $i \in [n]$, either $V_i \subseteq S$ or $V_i \cap S = \emptyset$.

After the assumption (1), assume that only a proper subset of V_i is contained in S. Let v be an element of this subset. We can consider the cutset $S \setminus \{v\}$ instead of S, since this decreases the number of components by at most one. So we can repeat this procedure until $V_i \cap S = \emptyset$.

So in $G_\alpha - S$ there are isolated vertices from U and one more component containing the remaining vertices of W and V. So there are less than $|V \cap S|$ isolated vertices, thus

$$\omega(G_\alpha - S) \leq |V \cap S| \leq |S|.$$

So G_α is 1-tough. $\square$

We show that G is α-critical with $\alpha(G) = \alpha$ if and only if G_α is minimally 1-tough.

Let us assume that G is α-critical with $\alpha(G) = \alpha$. So by Lemma 14 G_α is 1-tough. Let $e \in E(G_\alpha)$ be an arbitrary edge. If e has an endpoint in U, then this endpoint has degree 2, so $\tau(G_\alpha - e) < 1$. If e does not have an endpoint in U, then it connects two vertices of V. By Lemma 12 $G_\alpha[V]$ is α-critical, so in $G_\alpha[V] - e$ there exists an independent vertex set I of size $\alpha(G) + 1$. Let $S = (V \setminus I) \cup W$. Then $|S| = \big(|V| - \alpha(G) - 1\big) + \alpha = |V| - 1$ and $\omega\big((G_\alpha - e) - S\big) = |V|$, so $\tau(G_\alpha - e) < 1$.

Let us assume that G is not α-critical with $\alpha(G) = \alpha$.

Case 1: $\alpha(G) > \alpha$. Let I be an independent vertex set of size $\alpha(G)$ in $G_\alpha[V]$ and let $S = (V \setminus I) \cup W$. Then $|S| = \big(|V| - \alpha(G)\big) + \alpha < |V|$ and $\omega(G_\alpha - S) = |V|$, so $\tau(G_\alpha) < 1$, which means that G_α is not minimally 1-tough.

Case 2: $\alpha(G) \leq \alpha$. Since G is not α-critical there exists an edge $e \in E(G)$ such that $\alpha(G - e) \leq \alpha$. By Lemma 14 $(G - e)_\alpha$ is 1-tough, but we can obtain $(G - e)_\alpha$ from G_α by edge-deletion, which means that G_α is not minimally 1-tough. $\Box$

4 Further results

Theorem 15 *For every positive integer t,* MIN-t-TOUGH *is DP-complete.*

To prove this more general theorem, first we generalize the construction on Figure 1. We follow a similar argument to show that this construction has the required properties. However, due to the more complicated construction, the proof is harder.

The case when $t \leq 1/2$ is also covered in the paper.

Theorem 16 *For every positive rational number $t = a/b \leq 1/2$,* MIN-t-TOUGH *is DP-complete.*

It is shown that MIN-1-TOUGH can be reduced to this problem. The construction and the proof uses different ideas than the previous proofs.

We were not able to prove the DP-completeness for the remaining t values, but we make the following conjecture.

Conjecture 17 MIN-t-TOUGH *is DP-complete for any positive rational number t.*

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