Tight contact structures on some small Seifert fibered 3–manifolds

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Abstract We classify tight contact structures on the small Seifert fibered 3–manifold $M(-1; r_1, r_2, r_3)$ with $r_i \in (0,1) \cap \mathbb{Q}$ and $r_1, r_2 \geq \frac{1}{2}$. The result is obtained by combining convex surface theory with computations of contact Ozsváth–Szabó invariants. We also show that some of the tight contact structures on the manifolds considered are nonfillable, justifying the use of Heegaard Floer theory.

1 Introduction

In this paper we call a Seifert fibered 3–manifold M *small* if M is closed, the base surface is S^2 and M has exactly three singular fibers. Using normalized Seifert invariants, a small Seifert fibered 3–manifold M can be described by the surgery diagram of Figure 1, where $e_0 \in \mathbb{Z}$ and $r_i \in (0,1) \cap \mathbb{Q}$, with $r_1 \geq r_2 \geq r_3$. Conversely, a 3–manifold given by Figure 1 carries a natural structure of a small Seifert fibered 3–manifold. We shall denote such a 3–manifold by $M(e_0; r_1, r_2, r_3)$. The classification of tight contact structures on
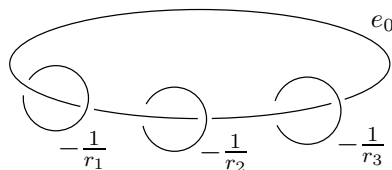

Figure 1:   Surgery diagram for the Seifert fibered 3–manifold $M(e_0; r_1, r_2, r_3)$

$M(e_0; r_1, r_2, r_3)$ has been given in [35] when $e_0 \neq 0, -1, -2$, and then extended in [8] to

the case $e_0 \geq 0$. In the present article we consider a further special case:

$$e_0 = -1, \quad r_1 \geq r_2 \geq \frac{1}{2}.$$

Let us assume that $-\frac{1}{r_i}$ has continued fraction expansion $[a_0^i, \dots, a_{k_i}^i]$, that is,

$$-\frac{1}{r_i} = -a_0^i - \cfrac{1}{-a_1^i - \cfrac{1}{\ddots - \cfrac{1}{-a_{k_i}^i}}}, \quad a_j^i \in \mathbb{N}, \ a_j^i \geq 2.$$

Define

$$h(r_1, r_2, r_3) := (a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1),$$

$$\varphi(r_1, r_2, r_3) := \begin{cases} \left(2(a_1^1 - 1)(a_1^2 - 1) + (a_0^3 - 1)(a_1^1 + a_1^2 - 2)\right) h(r_1, r_2, r_3) & \text{if } r_2 > \frac{1}{2}, \\ \left(2(a_1^1 - 1) + (a_0^3 - 1)\right)(a_1^3 - 1) \prod_{i \neq 2} \prod_{j \geq 2}(a_j^i - 1) & \text{if } r_1 > r_2 = \frac{1}{2}, \\ 2 \prod_{j \geq 1}(a_j^3 - 1) & \text{if } r_1 = r_2 = \frac{1}{2}, \end{cases}$$

and

$$\psi(r_1, r_2, r_3) := \begin{cases} (a_1^1 - 1)(a_1^2 - 1) a_1^3 \prod_{i=1}^{3} \prod_{j \geq 2}(a_j^i - 1) & \text{if } r_3 \neq \frac{1}{a_0^3}, \\ \prod_{i=1}^{2} \prod_{j \geq 1}(a_j^i - 1) & \text{if } r_3 = \frac{1}{a_0^3}, \end{cases}$$

where one should conventionally set $a_j^i = 2$ when $j > k_i$. Our main result is the following.

Theorem 1.1 *Let* $M = M(-1; r_1, r_2, r_3)$ *be a small Seifert fibered 3–manifold with* $r_1 \geq r_2 \geq \frac{1}{2}$. *Then,* M *supports exactly*

$$\varphi(r_1, r_2, r_3) + \psi(r_1, r_2, r_3)$$

tight contact strucures up to isotopy.

In order to put this result in perspective, recall that the classification of tight contact structures on small Seifert fibered 3–manifolds with $e_0 \neq -1, -2$ (as it is given in [8, 35]) required two steps. First, using convex surface theory, an upper bound on the number of tight contact structures was achieved. In the second step, using Legendrianizations of appropriate surgery diagrams for the 3–manifold at hand, a collection of Stein fillable contact structures was constructed. The isotopy classes of these Stein fillable (hence tight) structures were distingushed by the first Chern classes of their Stein fillings, resting on a result of [17]. For the class of 3–manifolds considered in this paper, this strategy needs to be modified, since — as it will be shown in Section 4, cf. Theorem 4.13 — some 3–manifolds addressed by Theorem 1.1 admit tight, nonfillable contact structures. Tightness of such nonfillable structures can be either shown by state traversal methods (which becomes extremally complicated when the underlying 3–manifold is atoroidal, as is the case of the 3–manifolds considered in Theorem 1.1), or by computing contact Ozsváth–Szabó invariants. By finding appropriate contact (± 1)–surgery diagrams for all the potential tight contact structures, here we describe a simple way to determine their contact Ozsváth–Szabó invariants.

Our results imply that the mod 2 reduced contact Ozsváth–Szabó invariant is a complete invariant for tight contact structures on the 3–manifolds $M(-1; r_1, r_2, r_3)$, $r_1, r_2 \geq \frac{1}{2}$. Moreover, two tight structures on these 3–manifolds are isotopic if and only if they induce the same spin^c structure.

Notice that the class of small Seifert fibered 3–manifolds considered in Theorem 1.1 is large enough to contain each small Seifert fibered 3–manifold with finite fundamental group (with one orientation) — with the unique exception of the Poincaré homology sphere $-\Sigma(2, 3, 5) = M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{5})$.

In Section 2 we derive upper bounds for the number of isotopy classes of tight contact structures on the Seifert fibered 3–manifolds listed in Theorem 1.1. The heart of these arguments is to establish isotopies between tight contact structures presented in different ways. Section 3 is devoted to a recollection of relevant results of Heegaard Floer theory related to the contact Ozsváth–Szabó invariants. Contact surgeries are also briefly discussed. In Section 4 the special case of $M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{p})$ is examined, and by using contact Ozsváth–Szabó invariants we achieve a complete classification for these Seifert fibered 3–manifolds. Stein fillability and nonfillability is also discussed. Finally in Section 5 we complete the proof of Theorem 1.1 by giving lower bounds for the number of tight contact structures on $M(-1; r_1, r_2, r_3)$ satisfying $r_1, r_2 \geq \frac{1}{2}$.

There is a final remark in place. Notice that in the definition of continued fractions we introduced negative signs, so that the coefficients a^i_j became positive. Although this convention might differ from many results established in the literature, for our purposes it seemed to be more convenient to work with positive rather than negative numbers.

2 Upper bounds from convex surface theory

In order to get an upper bound for the number of tight contact structures, we will follow the methods developed in [6, 10] and implemented [6, 9]. Suppose that (M, ξ) is a contact, Seifert fibered 3–manifold. Then, a Legendrian knot in M smoothly isotopic to a regular fiber admits two framings: one coming from the fibration and another one coming from the contact structure ξ. The difference between the contact framing and the fibration framing is the *twisting number* of the Legendrian curve. We say that ξ has *maximal twisting equal to zero* if there is a Legendrian knot L isotopic to a regular fiber such that L has twisting number zero. Applying [34, Theorem 1.5(1)] we have:

Proposition 2.1 *Every tight contact structure on $M(-1; r_1, r_2, r_3)$ with $r_1 \geq r_2 \geq \frac{1}{2}$ has maximal twisting equal to zero.*

Proof Using unnormalised Seifert invariants, we write $M(-1; r_1, r_2, r_3)$ as $M(0; r_1 - 1, r_2, r_3)$. In the notation of [34, Theorem 1.5] this means that $\frac{q_3}{p_3} = r_1 - 1$ and $\frac{q_2}{p_2} = r_2$.

Therefore $\frac{q_2}{p_2} + \frac{q_3}{p_3} = r_1 + r_2 - 1$, which is nonnegative by our assumptions. Consequently [34, Theorem 1.5(1)] applies, implying the statement. $\square$

Let F_i $(i = 1, 2, 3)$ be the three singular fibers of the Seifert fibration on M. First, in view of Proposition 2.1 we can isotope the Seifert fibration until there is a Legendrian regular fiber L with twisting number zero with respect to the fibration. Then, we can isotope each F_i further so that it becomes Legendrian.

Let V_i be a standard convex neighbourhood of F_i, $i = 1, 2, 3$. Then, $M \setminus (V_1 \cup V_2 \cup V_3)$ can be identified with $\Sigma \times S^1$ where Σ is a pair of pants. This diffeomorphism determines identifications of $-\partial(M \setminus V_i)$ with $\mathbb{R}^2/\mathbb{Z}^2$ so that $\binom{1}{0}$ is the direction of the section $\Sigma \times \{1\}$ and $\binom{0}{1}$ is the direction of the regular fibers. In order to fix one among the infinitely many product structures on $M \setminus (V_1 \cup V_2 \cup V_3)$ we also require the meridian of each V_i to have slope $-\frac{\beta_i}{\alpha_i}$ in $-\partial(M \setminus V_i)$, with

$$\frac{\beta_1}{\alpha_1} = r_1 - 1, \quad \frac{\beta_2}{\alpha_2} = r_2, \quad \frac{\beta_3}{\alpha_3} = r_3.$$

We also choose an identification between ∂V_i and $\mathbb{R}^2/\mathbb{Z}^2$ so that $\binom{1}{0}$ is the direction of the meridian of V_i and $\binom{1}{0}$ is the direction of a longitude. Notice that ∂V_i and $-\partial(M \setminus V_i)$ coincide as sets, but are identified with $\mathbb{R}^2/\mathbb{Z}^2$ in different ways. We can choose the longitude on V_i so that these two identifications are related by gluing matrices $A_i : \partial V_i \to -\partial(M \setminus V_i)$ given by

$$A_i = \begin{pmatrix} \alpha_i & \alpha_i' \\ -\beta_i & -\beta_i' \end{pmatrix}$$

with $\beta_i \alpha_i' - \alpha_i \beta_i' = 1$ and $0 < \alpha_i' < \alpha_i$.

Since V_i is a standard convex neighbourhood of a Legendrian curve, $-\partial(M \setminus V_i)$ is a standard torus with exactly 2 dividing curves. By flexibility of the Legendrian ruling [14, Corollary 3.6] we can modify the characteristic foliation of $-\partial(M \setminus V_i)$ so that its Legendrian ruling has infinite slope. Consider convex vertical annuli A_i with Legendrian boundary between a vertical Legendrian ruling curve of $-\partial(M \setminus V_i)$ and the Legendrian regular fiber L with twisting number zero. By the Imbalance Principle [14, Proposition 3.17] the annuli A_i give bypasses on $-\partial(M \setminus V_i)$. The Bypass Attachment Lemma [14, Lemmas 3.12 and 3.15] implies that the attachments of these bypasses reduce the denominator of the slope $-\partial(M \setminus V_i)$ in absolute value, therefore after a finite number of bypass attachments we obtain tubular neighbourhoods U_i of F_i containing V_i such that $-\partial(M \setminus U_i)$ has infinite slope for $i = 1, 2, 3$.

Lemma 2.2 *The convex torus ∂U_i has slope*

$$-\frac{\alpha_i}{\alpha_i'} = [a_{k_i}^i, \ldots, a_0^i], \quad i = 1, 2, 3.$$

Proof For $i = 2, 3$ the statement follows from [23, Lemma A4] because $r_i = \frac{\beta_i}{\alpha_i}$. When $i = 1$ we can still apply [23, Lemma A4] because $r_i = \frac{\beta_1 - \alpha_1}{\alpha_1}$ and we have

$$\alpha_1'(\beta_1 - \alpha_1) - \alpha_1(\beta_1' - \alpha_1') = 1.$$

$\square$

Since $-\frac{\alpha_i}{\alpha_i'} < -1$, by [14, Proposition 4.16] applied to $U_i \setminus V_i$ there exist tubular neighbourhoods $V_i' \subset U_i$ of the singular fibers such that $\partial V_i'$ is a standard convex torus with slope -1. Following [14, Section 4.4] we decompose $U_i \setminus V_i'$ into consecutive layers N_j^i diffeomorphic to toric annuli with convex boundary and boundary tori with slopes

$$[a_{k_i}^i, \ldots, a_1^i - 1] \quad \text{and} \quad [a_{k_i}^i, \ldots, a_0^i] \quad \text{if} \quad j = 0,$$

$$[a_{k_i}^i, \ldots, a_{j+1}^i - 1] \quad \text{and} \quad [a_{k_i}^i, \ldots, a_j^i - 1] \quad \text{if} \quad 0 < j < k_i,$$

$$-1 \quad \text{and} \quad [a_{k_i}^i - 1] \quad \text{if} \quad j = k_i,$$

where $[a_{k_i}^i, \ldots, a_1^i - 1]$ should be interpreted as -1 when $r_i = \frac{1}{a_0^i}$. If $a_j^i = 2$ then by definition N_j^i is an invariant neighbourhood of a convex torus, and if $a_j^i > 2$ by [14, Proposition 4.14] N_j^i is a continued fraction block.

We define q_j^i as the number of positive basic slices in N_j^i. We have

$$0 \le q_0^i \le a_0^i - 1 \quad \text{and} \quad 0 \le q_j^i \le a_j^i - 2 \qquad (j > 0).$$

For $j > k_i$ there are no more layers N_j^i and we define $q_j^i = \infty$.

Our assumption $r_1 \ge r_2 \ge \frac{1}{2}$ implies that $a_0^1 = a_0^2 = 2$, therefore $q_0^1, q_0^2 \in \{0, 1\}$ and N_0^1, N_0^2 are basic slices.

By the classification of tight contact structures on solid tori [14, Theorem 2.3] there are

$$a_0^i(a_1^i - 1) \ldots (a_{k_i}^i - 1)$$

distinct tight contact structures on each U_i. We will see that not every combination of tight contact structures on U_1, U_2, and U_3 is the restriction of a tight contact structure on M. But first we need to analyse the tight contact structures on the complement of U_1, U_2, and U_3. There are infinitely many distinct tight contact structures on $M \setminus (U_1 \cup U_2 \cup U_3)$ which can be the restriction of a tight contact structure on M. However, Lemma 2.4 below implies that the isotopy class of a tight contact structure ξ on M does not depend on its restriction to $M \setminus (U_1 \cup U_2 \cup U_3)$.

Definition 2.3 Let Σ be a pair of pants. A tight contact structure ξ on $\Sigma \times S^1$ is *appropriate* if there is no contact embedding

$$(T^2 \times I, \xi_\pi) \hookrightarrow (\Sigma \times S^1, \xi)$$

with $T^2 \times \{0\}$ isotopic to a boundary component, where ξ_π is a tight contact structure with convex boundary and twisting π (see [14, § 2.2.1] for the definition of twisting).

Lemma 2.4 *Let Σ be a pair of pants and let ξ be an appropriate contact structure on $\Sigma \times S^1$ with convex boundary $-\partial(\Sigma \times S^1) = T_1 \cup T_2 \cup T_3$, boundary slopes*

$$s(T_1) = -n, \quad s(T_2) = -1, \quad s(T_3) = \infty, \quad n \in \mathbb{N} \cup \{0\},$$

and $\#\Gamma_{T_i} = 2$ for $i = 1, 2, 3$. Then, there is a pair of pants Σ' contained in Σ and a factorization

$$\Sigma \times S^1 = (\Sigma' \times S^1) \cup B_1 \cup B_2$$

such that

(1) *the restriction of ξ to $\Sigma' \times S^1$ is appropriate and has convex boundary with infinite boundary slopes;*

(2) *the restrictions of ξ to B_1 and B_2 are basic slices;*

(3) *the isotopy class of ξ is determined by the signs of the restrictions of ξ to B_1 and B_2.*
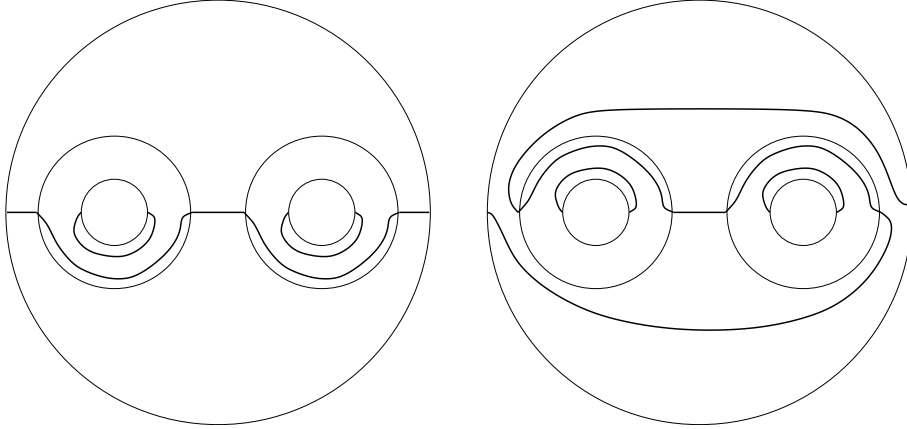

Figure 2: Nonisotopic dividing sets on a convex horizontal section of $\Sigma' \times S^1$ become isotopic when extended to a section of $\Sigma'' \times S^1$

Proof The proof of this lemma is similar to the proof of [35, Lemma 4.1], where different boundary slopes are considered. Here we give a sketch of the argument.

The existence of the factorization is stated in [15, Lemma 5.1(a)]. The restriction of ξ to $\Sigma' \times S^1$ is appropriate because the restriction of an appropriate contact structure is always appropriate. This proves (1). B_1 and B_2 are basic slices because they are appropriate (which, for thickened tori, is the same as being minimally twisting) and their boundary slopes are consecutive in the Farey Tessellation. This proves (2). Let Σ'_0 be a convex horizontal section of $\Sigma' \times S^1$ with Legendrian boundary consisting of horizontal Legendrian ruling curves of T_1, T_2, and T_3 so that the dividing set $\Gamma_{\Sigma'_0}$ has the minimum possible number of dividing curves in the isotopy class of Σ'_0. The dividing set of Σ'_0 cannot contain a boundary parallel dividing curve because ξ is appropriate, therefore it consists of three arcs, each joining distinct boundary components: see [6, Lemma 10]. By [15, Lemma 5.2] there is a unique extension of $(\Sigma \times S^1, \xi)$ to $(\Sigma'' \times S^1, \xi'')$ obtained by adding basic slices B_1'' and B_2'' so that the resulting contact structure ξ'' is tight and has infinite boundary slopes. Let Σ_0'' be a convex horizontal section of $\Sigma'' \times S^1$ with Legendrian boundary extending Σ'_0 so that the dividing set $\Gamma_{\Sigma_0''}$ has the minimum number of dividing curves in the isotopy class of Σ_0''. Then the dividing set $\Gamma_{\Sigma_0''}$ consists of three boundary parallel arcs, as in Figure 2. Therefore, the isotopy class of ξ'' is determined by the signs of the boundary parallel regions. These signs depend on the signs of the basic slices B_1'' and B_2'', which must be equal to the signs of the basic slices B_1 and B_2 by the Gluing Theorem [14, Theorem 4.25]. $\square$

Lemma 2.5 *Let $(T^2 \times [0,1], \eta)$ be a minimally twisting tight contact structure with boundary slopes s_0 and s_1. Let $T^2 \times [\frac{1}{2}, 1]$ be the last basic slice in the basic slice decomposition of $(T^2 \times [0,1], \eta)$. Then, the slope $s_{\frac{1}{2}}$ of $T^2 \times \{\frac{1}{2}\}$ is the vertex of the Farey Tessellation in the counterclockwise arc starting at $s_{\frac{1}{2}}$ and ending at s_1 which is the closest to s_0 among those connected to s_1 by an edge.*

Proof In [14, Section 4.4.3] $T^2 \times \{\frac{1}{2}\}$ is obtained from $T^2 \times \{1\}$ by attaching a bypass along a Legendrian ruling curve with slope s_1. The bypass attachment lemma [14, Lemma 3.15] concludes the proof. □

Let Z_i be the complement of the outermost basic slice in U_i. In particular, $U_1 \setminus Z_1$ and $U_2 \setminus Z_2$ coincide with N_0^1 and N_0^2 respectively.

Lemma 2.6 $-\partial(M \setminus Z_1)$ *has boundary slope 0, while $-\partial(M \setminus Z_2)$ and $-\partial(M \setminus Z_3)$ have both boundary slopes equal to -1.*

Proof Since

$$A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 - \alpha_1' \\ -\beta_1 + \beta_1' \end{pmatrix},$$

using the relation $\alpha_1' \beta_1 - \alpha_1 \beta_1' = 1$ and the fact that

$$\frac{\beta_1}{\alpha_1} = r_1 - 1 < 0 \quad \text{implies} \quad \beta_1 \leq -1,$$

we see that the slope of $-\partial(M \setminus V_i')$ is

$$\frac{-\beta_1 + \beta_1'}{\alpha_1 - \alpha_1'} = \frac{-\alpha_1 \beta_1 + \alpha_1 \beta_1'}{\alpha_1(\alpha_1 - \alpha_1')} = \frac{-\beta_1(\alpha_1 - \alpha_1') - 1}{\alpha_1(\alpha_1 - \alpha_1')} = -\frac{\beta_1}{\alpha_1} - \frac{1}{\alpha_1(\alpha_1 - \alpha_1')} \geq -\frac{\beta_1 + 1}{\alpha_1} \geq 0.$$

On the other hand,

$$\frac{-\beta_1 + \beta_1'}{\alpha_1 - \alpha_1'} = -\frac{\beta_1}{\alpha_1} - \frac{1}{\alpha_1(\alpha_1 - \alpha_1')} \leq -\frac{\beta_1}{\alpha_1} = 1 - r_1 < 1.$$

Applying Lemma 2.5 to $U_1 \setminus V_1'$ with its boundary slopes computed with respect to the basis of $-\partial(M \setminus U_1)$, we find that $-\partial(M \setminus Z_1)$ has boundary slope 0. In the same way we see that $-\partial(M \setminus Z_2)$ and $-\partial(M \setminus Z_3)$ have both boundary slope -1. □

It is easy to check that if ξ is a tight contact structure on M, its restriction to $M \setminus (Z_1 \cup Z_2 \cup U_3)$ is appropriate, therefore Lemma 2.4 implies that the isotopy class of a tight contact structure ξ on M depends only on the restriction of ξ to the solid tori U_i. This fact implies that the number of possible tight contact structures on M is at most

$$\prod_{i=1}^{3} a_0^i \prod_{j \geq 1} (a_j^i - 1).$$

We will sharpen this upper bound by excluding some overtwisted contact structures and eliminating the overcounting of different presentations of the same tight contact structure. Lemmas 2.7 and 2.8 are essentially more precise reformulations of a particular case of [9, Theorem 4.13].

Lemma 2.7 *Let Σ be a pair of pants and let ξ be a contact structure on $\Sigma \times S^1$ with convex boundary $-\partial(\Sigma \times S^1) = T_1 \cup T_2 \cup T_3$, boundary slopes*

$$s(T_1) = -n, \quad s(T_2) = -1, \quad s(T_3) = \infty, \quad n \in \mathbb{N} \cup \{0\},$$

and $\#\Gamma_{T_i} = 2$ for $i = 1, 2, 3$. Suppose that there exists a collar neighbourhood $B_3 \subset \Sigma \times S^1$ of T_3 such that

(1) *B_3 is a basic slice with boundary slopes ∞ and n;*

(2) *the restriction of ξ to $(\Sigma \times S^1) \setminus B_3 = \Sigma'' \times S^1$ coincides, up to isotopy, with the unique tight contact structure on $\Sigma'' \times S^1$ without vertical Legendrian curves with twisting number 0, and with boundary slopes $-n$, -1, and n.*

Then $(\Sigma \times S^1, \xi)$ is appropriate, and in the decomposition $\Sigma \times S^1 = B_1 \cup B_2 \cup (\Sigma' \times S^1)$ of Lemma 2.4, where B_1 and B_2 are basic slices with boundary slopes $-n$, ∞ and -1, ∞ respectively, the signs of B_1 and B_2 are both opposite to the sign of B_3.
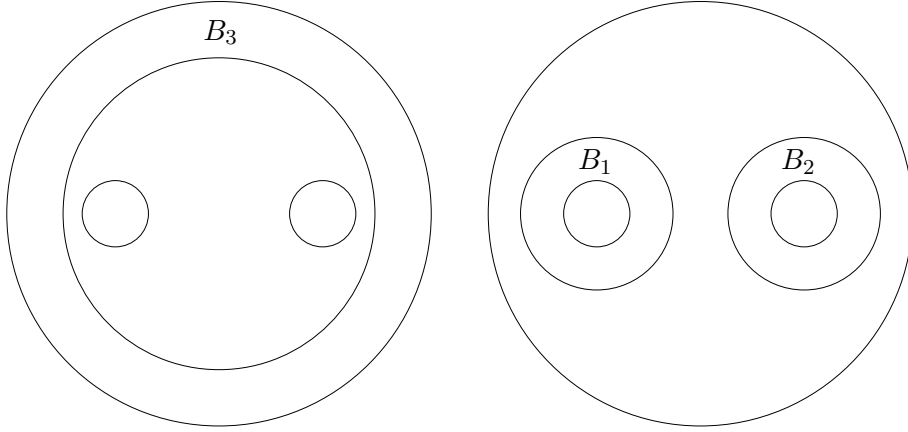

Figure 3: The two decompositions of $(\Sigma \times S^1, \xi)$.

Here we warn the reader that our convention for the computation of the boundary slopes differs from that of [15, Lemma 5.1] in the sense that our slopes are computed with respect to $-\partial(\Sigma \times S^1)$, as opposed to $\partial(\Sigma \times S^1)$.

Proof Define $-\partial B_i = T_i - T_i'$ for $i = 1, 2, 3$. $(\Sigma'' \times S^1, \xi|_{\Sigma'' \times S^1})$ is contactomorphic to the complement of a vertical Legendrian curve with twisting number -1 in a nonrotative thickened torus with boundary slopes $-n$ because there is a unique tight contact structure up to isotopy on $\Sigma'' \times S^1$ with these boundary slopes and without vertical Legendrian curves with twisting number 0, see [15, Lemma 5.1(4b)]. A collar of T_3' in B_3 is isomorphic to a nonrotative thickened torus, therefore we can identify $(\Sigma \times S^1, \xi)$ with the complement of a vertical Legendrian curve with twisting number -1 in B_3. Since B_3 is tight and minimally twisting we can conclude that $(\Sigma \times S^1, \xi)$ is appropriate.

To prove that the signs of the restrictions of ξ to B_1 and B_2 are the opposite of the sign of its restriction to B_3, we argue by evaluating the relative Euler class $e(\xi)$ of ξ on vertical annuli $A_1 \subset B_1$ and $A_2 \subset B_2$ with Legendrian boundary

$$\partial A_i = (A_i \cap T_i') - (A_i \cap T_i)$$

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such that $A_i \cap T_i$ is a Legendrian ruling curve of T_i, and $A_i \cap T_i'$ is a Legendrian divide of T_i'.

Let A_i', for $i = 1, 2$, be a vertical annulus in $\Sigma' \times S^1$ such that $\partial A_i' = (A_i' \cap T_3) - (A_i' \cap T_i')$ consists of Legendrian divides of T_i' and T_3, and $A_i' \cap T_i' = A_i \cap T_i'$. We consider also the vertical annuli $A_i'' = A_i \cup A_i'$ between T_3 and T_i. From the Thurston–Bennequin inequality it follows that

$$\langle e(\xi), [A_i'] \rangle = 0,$$

therefore

$$\langle e(\xi), [A_i] \rangle = \langle e(\xi), [A_i''] \rangle.$$

Take a vertical annulus $A_3 \subset B_3$ with Legendrian boundary $\partial A_3 = (A_3 \cap T_3') - (A_3 \cap T_3)$, and a vertical annulus $A_3' \subset \Sigma'' \times S^1$ with Legendrian boundary $\partial A_3' = (A_3 \cap T_i) - (A_3 \cap T_3')$ for either $i = 1, 2$. Assume moreover that $A_3 \cap T_3' = A_3' \cap T_3'$. The dividing set of A_3' can contain no boundary parallel dividing arc because $\Sigma'' \times S^1$ contains no vertical Legendrian curve with twisting number 0, therefore [14, Proposition 4.5] implies that $\langle e(\xi), [A_3'] \rangle = 0$. If we call $A_3'' = A_3 \cup A_3'$, then we have

$$\langle e(\xi), [A_3] \rangle = \langle e(\xi), [A_3''] \rangle = -\langle e(\xi), [A_i''] \rangle.$$

The change of sign in the evaluation of the relative Euler class is due to the different orientations of A_3'' on one hand, and A_i'' with $i = 1, 2$ on the other hand. This implies that

$$\langle e(\xi), [A_3] \rangle = -\langle e(\xi), [A_i] \rangle$$

for $i = 1, 2$. □

Combining Lemma 2.4 and Lemma 2.7 we obtain the following corollary, which contains the basic move which allows us to go from a sign configuration of the basic slice decompositions of U_1, U_2, and U_3 to a different one without affecting the isotopy class of ξ.

Lemma 2.8 *Let Σ be a pair of pants and let ξ be an appropriate contact structure on $\Sigma \times S^1$ with convex boundary $-\partial(\Sigma \times S^1) = T_1 \cup T_2 \cup T_3$, boundary slopes*

$$s(T_1) = -n, \quad s(T_2) = -1, \quad s(T_3) = \infty, \quad n \in \mathbb{N} \cup \{0\},$$

and $\#\Gamma_{T_i} = 2$ for $i = 1, 2, 3$. If the basic slices B_1 and B_2 of Lemma 2.4 have the same sign, then there exists a collar neighbourhood B_3 of T_3 such that B_3 is a basic slice with boundary slopes ∞ and n having sign opposite to the sign of B_1 and B_2 and the restriction of ξ to $\Sigma'' \times S^1 = (\Sigma \times S^1) \setminus B_3$ coincides, up to isotopy, with the unique tight contact structure on $\Sigma'' \times S^1$ without vertical Legendrian curves with twisting number 0, and with boundary slopes $-n$, -1, and n.

Proof Let Σ'' be a pair of pants, and let η' denote the the unique tight contact structure on $\Sigma'' \times S^1$ without vertical Legendrian curves with twisting number 0, and with boundary slopes $-n$, -1, and n. Consider the contact structure η obtained by gluing to the boundary component of $(\Sigma'' \times S^1, \eta')$ with slope n a basic slice B_3 with boundary slopes ∞ and n, and with sign opposite to the sign of $(B_1, \xi|_{B_1})$ and $(B_2, \xi|_{B_2})$. Then, by Lemma 2.7, in the decomposition of Lemma 2.4 for η the signs of the basic slices B_1 and

B_2 are both opposite to the sign of B_3, and therefore are equal to the sign of $(B_1, \xi|_{B_1})$ and $(B_2, \xi|_{B_2})$. By Lemma 2.4 the contact structures η and ξ are isotopic, so the statement follows. $\square$

We will see that the nontrivial behaviour of the tight contact structures on M comes from the first two outermost layers N_0^i and N_1^i in the decomposition of U_i ($i = 1, 2, 3$). This fact justifies the introduction of the following notation: to a potentially tight contact structure on M we associate the matrix

$$\begin{pmatrix} q_0^1 & q_0^2 & q_0^3 \\ q_1^1 & q_1^2 & q_1^3 \end{pmatrix}$$

whose entries q_j^i are the number of positive basic slices in the basic slice decompositions of N_j^i for $i = 1, 2, 3$ and $j = 0, 1$. (Recall that we defined $q_j^i = \infty$ if $j > k_i$.) We will call this matrix *the matrix of signs* of the contact structure.

We will study two separate cases. Suppose first that $q_0^1 \neq q_0^2$.

Proposition 2.9 *A tight contact structure on $M(-1; r_1, r_2, r_3)$ with matrix of signs*

$$\begin{pmatrix} 1 & 0 & q_0^3 \\ q_1^1 & q_1^2 & q_1^3 \end{pmatrix}$$

is isotopic to either a tight contact structure with matrix of signs

$$\begin{pmatrix} 0 & 1 & q_0^3 + 1 \\ q_1^1 + 1 & q_1^2 & q_1^3 \end{pmatrix}$$

or to a tight contact structure with matrix of signs

$$\begin{pmatrix} 0 & 1 & q_0^3 - 1 \\ q_1^1 & q_1^2 - 1 & q_1^3 \end{pmatrix}$$

provided that the expressions are defined, and all the further basic slice decompositions are the same. Here we use the convention $\infty \pm 1 = \infty$.

Proof We start with the proof of the first isotopy. If $q_0^3 < a_0^3 - 1$ then there is a negative basic slice in the decomposition of N_0^3, therefore we can arrange the basic slice decomposition of U_3 so that the outermost basic slice is negative. We recall that $M \setminus (U_1 \cup Z_2 \cup Z_3)$ has boundary slopes ∞, -1, and -1. Applying Lemma 2.8 to $M \setminus (U_1 \cup Z_2 \cup Z_3)$ we obtain a positive basic slice B_1 such that $U_1'' = U_1 \cup B_1$ is a tubular neighbourhood of F_1 and $-\partial(M \setminus U_1'')$ has slope $s(-\partial(M \setminus U_1'')) = 1$. Now we divide the proof in two cases.

Case 1. If $r_1 = \frac{1}{2}$ then

$$A_1^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

therefore $s(\partial U_1'') = \infty$. It follows that U_1'' is the standard neighbourhood of a destabilization of F_1 with twisting number 0. Stabilize F_1 again, and remove a standard neighbourhood U_1' of the stabilized curve. We can choose the sign of the stabilization so that $U_1'' \setminus U_1'$ is a negative basic slice. The Gluing Theorem [14, Theorem 4.25] implies that $U_1' \setminus V_1' = N_0^1$ is also a negative basic slice, because it glues to $U_1'' \setminus U_1'$, which is a

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negative basic slice, to give a tight contact structure on $U_1'' \setminus V_1'$ with boundary slopes -1 and ∞. For this reason q_0^1 changes from $q_0^1 = 1$ to $q_0^1 = 0$. Notice that, in this case, $V_1' = Z_1$. Using Lemma 2.4 applied to $M \setminus (U_1' \cup Z_2 \cup Z_3)$ we obtain basic slices B_2' and B_3' with boundary slopes -1 and ∞. The basic slices B_2' and B_3' are positive by Lemma 2.7, therefore q_0^2 changes from $q_0^2 = 0$ to $q_0^2 = 1$, and q_0^3 changes to $q_0^3 + 1$. This changes the first row of the matrix of signs from $\begin{pmatrix} 1 & 0 & q_0^3 \end{pmatrix}$ to $\begin{pmatrix} 0 & 1 & q_0^3 + 1 \end{pmatrix}$.

Case 2. Suppose now that $r_1 > \frac{1}{2}$. Observe that, in view of Lemma 2.6, $B_1 \cup N_0^1$ has boundary slopes 0 and 1. Since B_1 and N_0^1 are both positive basic slices, by [14, Theorem 4.25] $B_1 \cup N_0^1$ is a positive basic slice as well. Since

$$A_1^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\beta_1' - \alpha_1' \\ \beta_1 + \alpha_1 \end{pmatrix},$$

it follows that B_1 has boundary slopes

$$-\frac{\alpha_1}{\alpha_1'} \quad \text{and} \quad -\frac{\beta_1 + \alpha_1}{\beta_1' + \alpha_1'}$$

in the basis of $-\partial(M \setminus U_1)$. Moreover,

$$-\frac{\alpha_1}{\beta_1 + \alpha_1} = -\frac{1}{r_1} = [a_0^1, \dots, a_{k_1}^1]$$

implies by Lemma 2.2 that

$$-\frac{\alpha_1}{\alpha_1'} = [a_{k_1}^1, \dots, a_0^1],$$

and by an inductive argument over k_1 as in [23, Lemma A4] that

$$-\frac{\beta_1 + \alpha_1}{\beta_1' + \alpha_1'} = [a_{k_1}^1, \dots, a_1^1].$$

This implies that $B_1 \cup N_0^1 \cup N_1^1$ is a continued fraction block, therefore the signs of the basic slices in $B_1 \cup N_0^1 \cup N_1^1$ can be shuffled. If we shuffle the sign of the positive basic slice $B_1 \cup N_0^1$ with the sign of a negative basic slice in N_1^1 we obtain the claimed isotopy.

The proof of the second isotopy is analogous: if $q_0^3 > 0$, we can arrange the basic slice decomposition of U_3 so that the outermost basic slice is positive. Applying Lemma 2.8 to $M \setminus (Z_1 \cup U_2 \cup Z_3)$ and proceeding as above we obtain the second isotopy. $\qquad \square$

Corollary 2.10 *The number of distinct tight contact structures on $M(-1; r_1, r_2, r_3)$ with $q_0^1 \neq q_0^2$ is bounded above by*

$$\left(2(a_1^1 - 1)(a_1^2 - 1) + (a_0^3 - 1)(a_1^1 + a_1^2 - 2) \right) (a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1)$$

when $r_2 > \frac{1}{2}$, by

$$\left(2(a_1^1 - 1) + (a_0^3 - 1) \right) (a_1^3 - 1) \prod_{i \neq 2} \prod_{j \geq 2} (a_j^i - 1)$$

when $r_1 > r_2 = \frac{1}{2}$, and by

$$2 \prod_{j \geq 1} (a_j^3 - 1)$$

when $r_1 = r_2 = \frac{1}{2}$. In the above formulae $a_j^i = 2$ by convention if $j > k_i$.

Proof By Proposition 2.9 we can always assume $(q_0^1, q_0^2) = (1, 0)$ unless $(q_0^1, q_0^2) = (0, 1)$ and one of the following cases occurs:

(1) $q_0^3 = 0$ and $q_1^2 = a_1^2 - 2$,

(2) $q_0^3 = a_0^3 - 1$ and $q_1^1 = 0$,

(3) $q_1^1 = 0$ and $q_1^2 = a_1^2 - 2$.

There are at most

$$(a_1^1 - 1)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1)$$

isotopy classes of tight contact structures in Case (1),

$$(a_1^2 - 1)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1),$$

in Case (2), and

$$a_0^3(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1)$$

in Case (3). But we counted twice the configurations of signs belonging to the groups (1) and (3) or (2) and (3) simultaneously, so we have to subtract

$$2(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1)$$

from the sum of the above expressions. This shows that the maximum number of tight contact structures when $(q_0^1, q_0^2) = (0, 1)$ and $r_2 > \frac{1}{2}$ is

$$(a_0^3 + a_1^1 + a_1^2 - 4)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1). \tag{2.1}$$

If $r_1 > r_2 = \frac{1}{2}$, only Case (2) above is possible, so the upper bound when $(q_0^1, q_0^2) = (0, 1)$ is

$$(a_1^3 - 1) \prod_{i \neq 2} \prod_{j \geq 2} (a_j^i - 1). \tag{2.2}$$

If $r_1 = r_2 = \frac{1}{2}$, then none of the above cases can occur, and we may always assume $(q_0^1, q_0^2) = (1, 0)$.

Now we consider the case when $(q_0^1, q_0^2) = (1, 0)$. By Proposition 2.9 the contact structure with matrix of signs

$$\begin{pmatrix} 1 & 0 & q_0^3 \\ q_1^1 & q_1^2 & q_1^3 \end{pmatrix}$$

is isotopic to one with matrix

$$\begin{pmatrix} 1 & 0 & q_0^3 \pm 2 \\ q_1^1 \pm 1 & q_1^2 \pm 1 & q_1^3 \end{pmatrix}$$

where the same sign must be chosen in each entry, assuming that all the expressions are defined. Therefore, we can always assume that one of the following cases holds:

(4) $q_0^3 = 0$,

(5) $q_0^3 = 1$,

(6) $q_0^3 \neq 0, 1$ and $q_1^1 = 0$,

(7) $q_0^3 \neq 0, 1$ and $q_1^2 = 0$.

Each one of Cases (4) and (5) allows the existence of at most

$$(a_1^1 - 1)(a_1^2 - 1)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1)$$

distinct isotopy classes of contact structures, Case (6) allows

$$(a_0^3 - 2)(a_1^2 - 1)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1),$$

and Case (7) allows

$$(a_0^3 - 2)(a_1^1 - 1)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1).$$

However, we have counted twice the contact structures with $(q_1^1, q_1^2) = (0, 0)$ belonging to both Cases (6) and (7). Therefore, we must subtract

$$(a_0^3 - 2)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1).$$

Thus, when $r_2 > \frac{1}{2}$ the total number of potential tight contact structures with $(q_0^1, q_0^2) = (1, 0)$ is

$$\left(2(a_1^1 - 1)(a_1^2 - 1) + (a_0^3 - 2)(a_1^1 + a_1^2 - 3)\right)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1). \qquad (2.3)$$

Adding up (2.1) and (2.3) we obtain the stated formula in the case $r_2 > \frac{1}{2}$.

If $r_1 > r_2 = \frac{1}{2}$ then Case (7) cannot occur because $q_1^2 = \infty$, therefore the total number of potential tight contact structures with $(q_0^1, q_0^2) = (1, 0)$ is

$$\left(2(a_1^1 - 1) + (a_0^3 - 2)\right)(a_1^3 - 1) \prod_{i \neq 2} \prod_{j \geq 2} (a_j^i - 1). \qquad (2.4)$$

Adding up (2.2) and (2.4) gives the statement in this case.

When $r_1 = r_2 = \frac{1}{2}$ only Cases (4) and (5) can occur, giving the upper bound

$$2 \prod_{j \geq 1} (a_j^3 - 1),$$

which coincides with the stated formula. $\qquad \square$

Next we turn to the second possibility, when $q_0^1 = q_0^2$.

Proposition 2.11 *Let ξ is a tight contact structure on $M = M(-1; r_1, r_2, r_3)$ such that $q_0^1 = q_0^2$. If $(q_0^1, q_0^2) = (1, 1)$ then $q_0^3 = 0$ and if $(q_0^1, q_0^2) = (0, 0)$ then $q_0^3 = a_0^3 - 1$.*

Proof Recall that $M \setminus (Z_1 \cup Z_2 \cup U_3)$ has boundary slopes 0, -1, and ∞. Applying Lemma 2.8 to $M \setminus (Z_1 \cup Z_2 \cup U_3)$ we get a basic slice B_3 with the sign opposite to the sign of N_0^1 and N_0^2. Then $U_3'' = B_3 \cup U_3$ is a tubular neighbourhood of F_3 so that $-\partial(M \setminus U_3'')$ has slope 0. Since

$$A_3^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\beta_3' \\ \beta_3 \end{pmatrix},$$

U_3'' has boundary slope $-\frac{\beta_3}{\beta_3'}$. Now suppose that $r_3 \neq \frac{1}{a_0^3}$. Then, by induction on k_3 as in [23, Lemma A4], we have

$$-\frac{\beta_3}{\beta_3'} = [a_{k_3}^3, \ldots, a_1^3].$$

Since in the basis of $-\partial(M \setminus V_3)$ the toric annulus $B_3 \cup N_0^3$ has boundary slopes

$$[a_{k_3}^3, \ldots, a_1^3 - 1] \quad \text{and} \quad [a_{k_3}^3, \ldots, a_1^3],$$

which are joined by an edge in the Farey Tessellation, by [14, Theorem 4.25] $B_3 \cup N_0^3$ is a basic slice, and it is tight if and only if B_3 and all the basic slices in the basic slice decomposition of N_0^3 have the same sign. This happens if and only if

$$(q_0^1, q_0^2, q_0^3) = (1, 1, 0) \quad \text{or} \quad (q_0^1, q_0^2, q_0^3) = (0, 0, a_0^3 - 1).$$

Now suppose that $r_3 = \frac{1}{a_0^3}$. Since

$$A_3^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0^3 & a_0^3 - 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$\partial U_3''$ has slope 1. Therefore, in this case $B_3 \cup N_0^3$ has boundary slopes -1 (as observed after Lemma 2.2) and 1. Thus, we can argue as in the case $r_3 \neq \frac{1}{a_0^3}$ and draw the same conclusion. $\square$

Proposition 2.12 *Two tight contact structures on $M(-1; r_1, r_2, r_3)$ with matrices of signs*

$$\begin{pmatrix} 1 & 1 & 0 \\ q_1^1 & q_1^2 & q_1^3 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & a_0^3 - 1 \\ q_1^1 & q_1^2 & q_1^3 + 1 \end{pmatrix}$$

and with identical basic slice decomposition in all the further continued fraction blocks are isotopic, whenever the symbols are defined. Here we use the convention $\infty \pm 1 = \infty$.

Proof Suppose that $q_0^1 = q_0^2$. Applying Lemma 2.8 to $M \setminus (Z_1 \cup Z_2 \cup U_3)$ we obtain a basic slice B_3 such that $U_3'' = B_3 \cup U_3$ is a tubular neighbourhood of F_3, and $-\partial(M \setminus U_3'')$ has slope $s(-\partial(M \setminus U_3'')) = 0$. Now we divide the proof into two cases.

Case 1. Suppose first that $r_3 = \frac{1}{a_0^3}$. As in the proof of Proposition 2.11, we see that U_3'' has slope 1, and therefore it is the standard neighbourhood of a destabilization of F_3 with twisting number 1. Stabilize F_3 again, and remove a standard neighbourhood U_3' of the stabilized curve. We can choose the sign of the stabilization so that $U_3'' \setminus U_3'$ is a negative

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basic slice. Proceeding as in Case 1 of the proof of Proposition 2.9, we can change the first row of the matrix of signs from $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 0 & a_0^3 - 1 \end{pmatrix}$.

Case 2. If $r_3 \neq \frac{1}{a_0^3}$ then $B_3 \cup N_0^3$ is a basic slice with boundary slopes $[a_{k_3}^3, \ldots, a_1^3 - 1]$ and $[a_{k_3}^3, \ldots, a_1^3]$, and B_3 and all the basic slices in N_0^3 have the same sign (cf. the proof of Proposition 2.11). $B_3 \cup N_0^3 \cup N_1^3$ has boundary slopes $[a_{k_3}^3, \ldots, a_2^3 - 1]$ and $-\frac{\beta_3}{\beta_3'} = [a_{k_3}^3, \ldots, a_1^3]$ computed in the basis of $-\partial(M \setminus V_3)$. This implies that $B_3 \cup N_0^3 \cup N_1^3$ is a continued fraction block. According to [14, Lemma 4.14] we can swap the signs of the basic slice $B_3 \cup N_0^3$ and the sign of a basic slice of N_1^3. This gives the stated isotopy. □

Corollary 2.13 *The number of isotopy classes of tight contact structures with $q_0^1 = q_0^2$ carried by $M(-1; r_1, r_2, r_3)$ is bounded above by*

$$ (a_1^1 - 1)(a_1^2 - 1)a_1^3 \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1) $$

if $r_3 \neq \frac{1}{a_0^3}$, and by

$$ \prod_{i=1}^{2} \prod_{j \geq 1} (a_j^i - 1) $$

if $r_3 = \frac{1}{a_0^3}$. In the above formulae $a_j^i = 2$ by convention if $j > k_i$.

Proof By Proposition 2.11 there are two possibilities for the first row of the matrix of signs (q_0^1, q_0^2, q_0^3) defined by the number of positive basic slices in the outermost continued fraction blocks N_0^i: either $(q_0^1, q_0^2, q_0^3) = (1, 1, 0)$ or $(q_0^1, q_0^2, q_0^3) = (0, 0, a_0^3 - 1)$. By Proposition 2.12 each potentially tight contact structure with $(0, 0, a_0^3 - 1)$ as first row of the matrix of signs is isotopic to one with $(1, 1, 0)$ unless $q_1^3 = 0$. If $r_3 = \frac{1}{a_0^3}$ we have $q_1^3 = \infty$, therefore in this case we get the bound

$$ (a_1^1 - 1)(a_1^2 - 1) \prod_{i=1}^{2} \prod_{j \geq 2} (a_j^i - 1) = \prod_{i=1}^{2} \prod_{j \geq 1} (a_j^i - 1). $$

When $r_3 \neq \frac{1}{a_0^3}$ we can consider two cases:

(1) $(q_0^1, q_0^2, q_0^3) = (1, 1, 0)$, or
(2) $(q_0^1, q_0^2, q_0^3) = (0, 0, a_0^3 - 1)$ and $q_1^3 = 0$.

Case (1) gives the upper bound

$$ (a_1^1 - 1)(a_1^2 - 1)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1) \tag{2.5} $$

and Case (2) gives

$$ (a_1^1 - 1)(a_1^2 - 1) \prod_{i=1}^{3} \prod_{j \geq 2} (a_j^i - 1). \tag{2.6} $$

Adding up (2.5) and (2.6) gives the statement when $r_3 \neq \frac{1}{a_0^3}$. □

3 Contact Ozsváth–Szabó invariants

Ozsváth–Szabó homology groups

In a remarkable series of papers [24, 25, 26, 30] Ozsváth and Szabó defined new invariants of many low–dimensional objects — including contact structures on closed 3–manifolds. Heegaard Floer theory associates to a closed, oriented spin^c 3–manifold $(Y, \mathbf{t})$ the abelian groups $\widehat{HF}(Y, \mathbf{t})$, $HF^\infty(Y, \mathbf{t})$, $HF^-(Y, \mathbf{t})$ and $HF^+(Y, \mathbf{t})$, called the *Ozsváth–Szabó homology groups*. If $(W, \mathbf{s})$ is an oriented spin^c cobordism between two spin^c 3–manifolds $(Y_1, \mathbf{t}_1)$ and $(Y_2, \mathbf{t}_2)$ and $HF^\bullet(Y, \mathbf{t}_i)$, $i = 1, 2$ is any of the groups above, there is a homomorphism

$$F^\bullet_{W, \mathbf{s}} \colon HF^\bullet(Y_1, \mathbf{t}_1) \to HF^\bullet(Y_2, \mathbf{t}_2).$$

In this paper we shall be mainly concerned with the groups $\widehat{HF}(Y, \mathbf{t})$, which are always finitely generated. The symbol $\widehat{HF}(Y)$ will denote the direct sum

$$\widehat{HF}(Y) := \oplus_\mathbf{t} \widehat{HF}(Y, \mathbf{t})$$

over all spin^c structures $\mathbf{t}$ on Y. Since there are only finitely many spin^c structures with nonvanishing $\widehat{HF}$–group, $\widehat{HF}(Y)$ is still finitely generated. A rational homology 3–sphere Y is called an *L–space* if $\widehat{HF}(Y, \mathbf{t}) \cong \mathbb{Z}$ for all $\mathbf{t} \in Spin^c(Y)$.

Let Y be a closed, oriented 3–manifold and let $K \subset Y$ be a framed knot with framing f. Let $Y(K)$ denote the 3–manifold given by surgery along $K \subset Y$ with respect to the framing f and $Y'(K)$ the 3-manifold we get by performing surgery along K with framing $f + \mu$, where μ denotes the meridian of K. The surgeries determine cobordisms X_1 from Y to $Y(K)$, X_2 from $Y(K)$ to $Y'(K)$ and X_3 from $Y'(K)$ back to Y. The following result can be deduced (cf. the discussion at the beginning of [27, Section 3] and [19, page 934]) from [25, Theorem 9.16] and [26, Subsection 4.1].

Theorem 3.1 (Surgery exact triangle) *The Ozsváth–Szabó homology groups of Y, $Y(K)$ and $Y'(K)$ fit into an exact triangle*

$$\widehat{HF}(Y) \xrightarrow{\;\;F_{X_1}\;\;} \widehat{HF}(Y(K))$$
$$F_{X_3} \nwarrow \qquad \swarrow F_{X_2}$$
$$\widehat{HF}(Y'(K))$$

where

$$F_{X_i} = \sum_{\mathbf{s} \in Spin^c(X_i)} \pm F_{X_i, \mathbf{s}}, \quad i = 1, 2, 3.$$

$\square$

It was proved in [24, 27] that for each spin^c structure $\mathbf{t}$ the Ozsváth–Szabó homology group $\widehat{HF}(Y, \mathbf{t})$ comes with a natural relative $\mathbb{Z}/div(\mathbf{t})\mathbb{Z}$–grading, where $div(\mathbf{t})$ is the divisibility of $c_1(\mathbf{t})$ in $H^2(Y; \mathbb{Z})$. If $\mathbf{t} \in Spin^c(Y)$ is torsion, that is, $c_1(\mathbf{t}) \in H^2(Y; \mathbb{Z})$ is a torsion element, then $div(\mathbf{t}) = 0$, and therefore $\widehat{HF}(Y, \mathbf{t})$ has a natural relative $\mathbb{Z}$–grading. This relative $\mathbb{Z}$–grading admits a natural lift to an absolute $\mathbb{Q}$–grading. In

conclusion, for a torsion spin^c structure $\mathbf{t}$ the Ozsváth–Szabó homology group $\widehat{HF}(Y, \mathbf{t})$ splits as

$$\widehat{HF}(Y, \mathbf{t}) = \oplus_{n \in \mathbb{Z}} \widehat{HF}_{d_0 + n}(Y, \mathbf{t}),$$

where the degree $d_0 \in \mathbb{Q}$ is determined mod 1 by $\mathbf{t}$. Moreover, when $\mathbf{t} \in \mathrm{Spin}^c(Y)$ has torsion first Chern class, there is an isomorphism between the homology groups $\widehat{HF}_d(Y, \mathbf{t})$ and $\widehat{HF}_{-d}(-Y, \mathbf{t})$.

Let $(W, \mathbf{s})$ be a spin^c cobordism between two spin^c manifolds $(Y_1, \mathbf{t}_1)$ and $(Y_2, \mathbf{t}_2)$. If the spin^c structures $\mathbf{t}_i$ are both torsion and $x \in \widehat{HF}(Y_1, \mathbf{t}_1)$ is a homogeneous element of degree $d(x)$, then $F_{W, \mathbf{s}}(x) \in \widehat{HF}(Y_2, \mathbf{t}_2)$ is also homogeneous of degree

$$d(x) + \frac{1}{4}(c_1^2(\mathbf{s}) - 3\sigma(W) - 2\chi(W)). \tag{3.1}$$

We need one more piece of information. Recall that the set of spin^c structures comes equipped with a natural involution, usually denoted by $\mathbf{t} \mapsto \overline{\mathbf{t}}$. The spin^c structure $\overline{\mathbf{t}}$, called the *conjugate* of $\mathbf{t}$, is defined as follows: If one thinks of a spin^c structure as a suitable equivalence class of nowhere zero vector fields (cf. [24]) then the above involution is the map induced by multiplying a representative vector field by (-1).

Theorem 3.2 ([25], Theorem 2.4) *There is a natural isomorphism*

$$\mathcal{J}_Y \colon \widehat{HF}(Y, \mathbf{t}) \to \widehat{HF}(Y, \overline{\mathbf{t}})$$

$\square$

A spin^c structure $\mathbf{t} \in Spin^c(Y)$ is induced by a *spin* structure exactly when $c_1(\mathbf{t}) = 0$, or equivalently when $\mathbf{t} = \overline{\mathbf{t}}$. According to [26, Theorem 3.6], given a spin^c cobordism $(W, \mathbf{s})$ we have

$$\mathcal{J}_{Y'} \circ F_{W, \mathbf{s}} = F_{W, \overline{\mathbf{s}}} \circ \mathcal{J}_Y, \tag{3.2}$$

where $\overline{\mathbf{s}}$ is the spin^c structure on the 4–manifold W conjugate to $\mathbf{s}$. This means that, if one thinks of $\mathbf{s} \in Spin^c(W)$ as a suitable equivalence class of almost–complex structures defined on $W \setminus \{\text{finitely many points}\}$, then $\overline{\mathbf{s}}$ is represented by the negative $-J$ of any almost–complex structure J representing $\mathbf{s}$.

Contact (± 1)–surgery

Suppose that $L \subset (Y, \xi)$ is a Legendrian knot in a contact 3–manifold. Let $Y_L^{\pm}$ denote the 3–manifold obtained by doing (± 1)–surgery along L, where the surgery coefficient is measured with respect to the contact framing of L. According to the classification of tight contact structures on a solid torus [14], the contact structure $\xi|_{Y - \nu L}$ extends uniquely (up to isotopy) to the surgered manifolds Y_L^+ and Y_L^- as a tight structure on the glued–up torus. Therefore, the knot L with a $(+1)$ or (-1) on it uniquely specifies a contact 3–manifold (Y_L^+, ξ_L^+) or (Y_L^-, ξ_L^-). (For more about contact surgery see [1, 2, 3].) In particular, a Legendrian link $\mathbb{L} \subset (S^3, \xi_{st})$ in the standard contact 3–sphere (which can be represented by its front projection) defines a contact structure once the surgery coefficients $(+1)$ and (-1) are specified on its components.

Contact Ozsváth–Szabó invariants

In [30] Ozsváth and Szabó define an invariant

$$c(Y, \xi) \in \widehat{HF}(-Y, \mathbf{t}_\xi)/\{\pm 1\}$$

assigned to a positive, cooriented contact structure ξ on Y. In fact, ξ (as an oriented 2–plane field) determines an element $(d(\xi), \mathbf{t}_\xi) \in \mathcal{H}$ and according to [30] the contact invariant $c(Y, \xi)$ is an element of $\widehat{HF}_{-d(\xi)}(-Y, \mathbf{t}_\xi)/\{\pm 1\}$. Moreover, if $c_1(\xi) \in H^2(Y; \mathbb{Z})$ is torsion then

$$d(\xi) = \frac{1}{4}(c_1^2(X, J) - 3\sigma(X) - 2\chi(X) + 2), \tag{3.3}$$

where (X, J) is a compact almost–complex 4–manifold with $\partial X = Y$, and ξ is homotopic to the distribution of complex tangencies on ∂X.

The main properties of the contact Ozsváth–Szabó invariant are summarized in the following two theorems.

Theorem 3.3 ([30]) *If (Y, ξ) is overtwisted, then $c(Y, \xi) = 0$. If (Y, ξ) is Stein fillable then $c(Y, \xi) \neq 0$. In particular, for the standard contact structure (S^3, ξ_{st}) the invariant $c(S^3, \xi_{st})$ is nonzero.* $\qquad\square$

Given a spin^c cobordism $(W, \mathbf{s})$ between spin^c 3–manifolds $(Y_1, \mathbf{t}_1)$ and $(Y_2, \mathbf{t}_2)$, the homomorphism $F_{W, \mathbf{s}}$ clearly induces a map between the sets $\widehat{HF}(Y_i, \mathbf{t}_i)/\{\pm 1\}$, $i = 1, 2$. Likewise, for any spin^c 3–manifold $(Y, \mathbf{t})$ the isomorphism $\mathcal{J}_Y \colon \widehat{HF}(Y, \mathbf{t}) \to \widehat{HF}(Y, \mathbf{t})$ induces a map

$$\widehat{HF}(Y, \mathbf{t})/\{\pm\} \to \widehat{HF}(Y, \mathbf{t})/\{\pm\}.$$

Abusing notation, throughout the paper we shall keep denoting such maps by $F_{W, \mathbf{s}}$ and $\mathcal{J}_Y$.

Theorem 3.4 ([18, 30]) *Suppose that (Y_2, ξ_2) is obtained from (Y_1, ξ_1) by a contact $(+1)$–surgery, and let $-W$ be the cobordism induced by the surgery with reversed orientation. Then,*

$$F_{-W}(c(Y_1, \xi_1)) = c(Y_2, \xi_2).$$

In particular, if $c(Y_2, \xi_2) \neq 0$ then (Y_1, ξ_1) is tight. $\qquad\square$

Since by [1, Proposition 8] contact $(+1)$–surgery along a Legendrian knot L is cancelled by a contact (-1)–surgery along a Legendrian push–off of L, Theorem 3.4 immediately implies:

Corollary 3.5 *If (Y_2, ξ_2) is obtained by Legendrian surgery along a Legendrian knot in (Y_1, ξ_1) and $c(Y_1, \xi_1) \neq 0$, then $c(Y_2, \xi_2) \neq 0$. In particular, (Y_2, ξ_2) is tight.* $\qquad\square$

An easy application of the surgery exact triangle together with Theorem 3.4 gives

Lemma 3.6 ([18], Lemma 2.5) *The contact structure η_1 on $S^1 \times S^2$ given as $(+1)$–surgery on a Legendrian unknot with Thurston–Bennequin number -1 in (S^3, ξ_{st}) has nonvanishing contact Ozsváth–Szabó invariant.* $\qquad\square$

4   Tight contact structures on $M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{p})$

In this section we define three contact structures ξ_1, ξ_2 and Ξ on the 3–manifold $M_p = M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{p})$ for each $p \geq 2$, we prove that they are distinct up to homotopy, that ξ_1 and ξ_2 are Stein fillable and that Ξ has nonzero Ozsváth–Szabó contact invariant. Combined with the results of Section 2, this gives the complete classification of tight contact structures on M_p for every p. In the last subsection we show that for $p > 2$ the contact 3–manifold (M_p, Ξ) is not Stein fillable, and for $p \not\equiv 2 \bmod 8$ is not symplectically fillable.

Heegaard Floer groups of M_p

The oriented manifold $-M_p$ is represented by the third surgery diagram of Figure 8. The three dotted circles denoted by μ_a, μ_b and μ_c represent (up to sign) elements of $H_1(-M_p; \mathbb{Z})$. It is easy to check that

$$H_1(M_p; \mathbb{Z}) = \begin{cases} \langle \mu_b \mid 4\mu_b = 0 \rangle \cong \mathbb{Z}/4\mathbb{Z} & \text{for } p \text{ odd,} \\ \langle \mu_b, \mu_c \mid 2\mu_b = 2\mu_c = 0 \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{for } p \text{ even,} \end{cases}$$

In particular, M_p has four spin^c structures for every p.

The sequence of Kirby calculus moves going from the third to the seventh diagram of Figure 8 shows that $-M_p$ is the boundary of P, the plumbing of spheres given by Figure 4. This amounts to saying that $-M_p$ is the link of the singularity D_{p-2}.
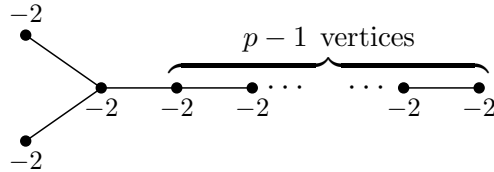

Figure 4: The 4–dimensional plumbing P with boundary $-M_p$

Consider the four 2–cohomology classes K_i, $i = 1, \ldots, 4$, on P whose values on the standard homology generators are given by Figure 5 (each number in parentheses indicates the value of K_i on the homology generator corresponding to the nearby vertex, and by convention no number is present if such value is zero).
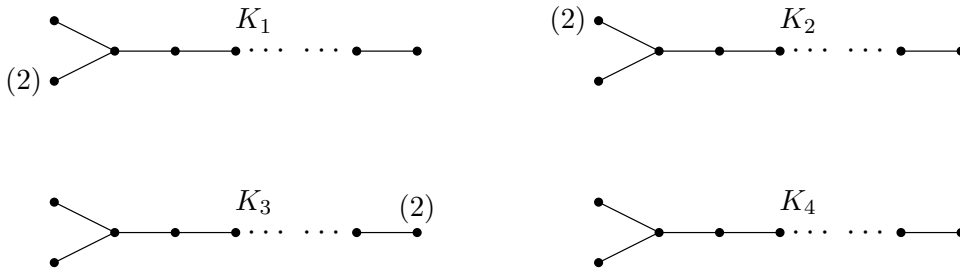

Figure 5: Initial vectors

Definition 4.1 Define $\mathbf{t}_i$, for $i = 1, \ldots, 4$, to be the spin^c structure on $-M_p$ which is the restriction of the spin^c structure on P specified by the characteristic element K_i.

An easy calculation shows that

$$\mathbf{t}_1 = \mathbf{t}_4 + \mu_b, \quad \mathbf{t}_2 = \mathbf{t}_4 + \mu_a + \mu_b = \mathbf{t}_4 + \mu_c \quad \text{and} \quad \mathbf{t}_3 = \mathbf{t}_4 + \mu_a,$$

where μ_a, μ_b and μ_c are the homology classes defined in Figure 8. This implies that $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\}$ is the whole set of spin^c structures on $-M_p$. When p is even, M_p has 4 spin structures, therefore in this case each of the spin^c structures $\mathbf{t}_i$ is induced by a spin structure. When p is odd, M_p carries 2 spin structures, which induce $\mathbf{t}_3$ and $\mathbf{t}_4$. In fact, observe that $\mathbf{t}_4$ is always induced by a spin structure because $K_4 = 0$ and therefore $c_1(\mathbf{t}_4) = 0$. On the other hand, we always have $\mu_a = \mu_b - \mu_c$, while when p is odd $\mu_c = 3\mu_b$ and therefore

$$c_1(\mathbf{t}_3) = c_1(\mathbf{t}_4) + 2\mu_a = -4\mu_b = 0.$$

Proposition 4.2 *We have*

$$\widehat{HF}(-M_p, \mathbf{t}_1) \cong \widehat{HF}(-M_p, \mathbf{t}_2) \cong \mathbb{Z}_{(0)}, \quad \widehat{HF}(-M_p, \mathbf{t}_3) \cong \mathbb{Z}_{(\frac{p-2}{4})}, \quad \widehat{HF}(-M_p, \mathbf{t}_4) \cong \mathbb{Z}_{(\frac{p+2}{4})}.$$

Proof Since $-M_p$ is the boundary of the plumbing P, we can apply the algorithm of [29] to determine its Ozsváth–Szabó homology groups. It is easy to check that the four cohomology classes K_i, $i = 1, \ldots, 4$, of Figure 5 provide initial characteristic vectors in the sense of [29]. An easy computation shows that

$$K_1^2 = K_2^2 = -p - 2, \quad K_3^2 = -4 \quad \text{and} \quad K_4^2 = 0.$$

The 3–manifold $-M_p$ has elliptic geometry and therefore it is an L–space by [31, Proposition 2.3]. Since by [29]

$$d(-M_p, \mathbf{t}_i) = \frac{K_i^2 + p + 2}{4}, \quad i = 1, \ldots, 4,$$

this immediately implies the statement. $\qquad\square$

Contact structures on M_p

Definition 4.3 Let ξ_1 and ξ_2 be the contact structures defined respectively by the contact surgery diagrams of Figure 6(a) and 6(b).

Definition 4.4 Let Ξ be the contact structure defined by the contact surgery diagram of Figure 7.

Lemma 4.5 *We have*

$$\{\mathbf{t}_{\xi_1}, \mathbf{t}_{\xi_2}\} = \{\mathbf{t}_1, \mathbf{t}_2\} \quad \text{and} \quad \mathbf{t}_\Xi = \mathbf{t}_3.$$
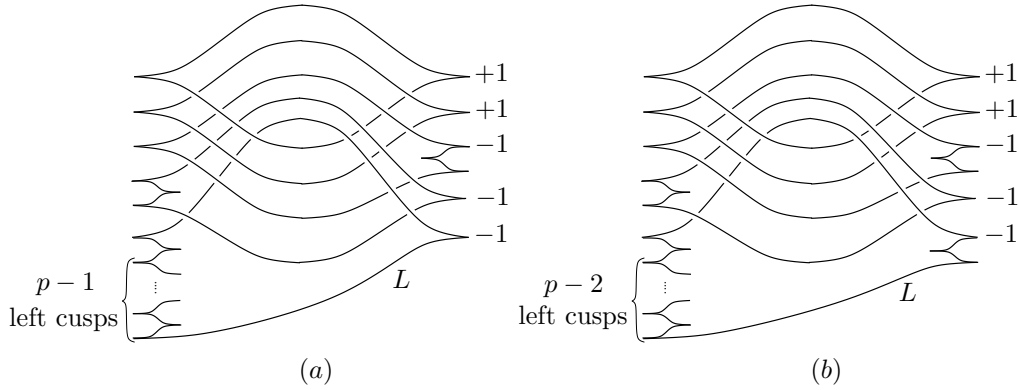
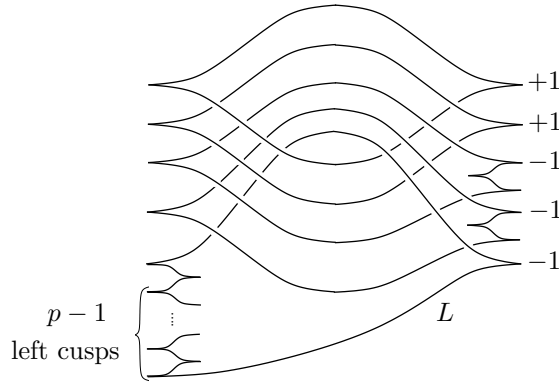Figure 6:   The contact structures ξ_1 and ξ_2 on M_p

Figure 7: The contact structure Ξ on M_p

Proof The contact surgery presentation of each contact structure $\xi \in \{\xi_1, \xi_2, \Xi\}$ can be interpreted as a simply connected 4–manifold with boundary, endowed with a characteristic 2–cohomology class K_ξ. The class K_ξ is uniquely determined by requiring that it evaluates on a 2–homology generator corresponding to a given Legendrian knot $\mathcal{L}$ in the surgery presentation as the rotation number of $\mathcal{L}$ (once an orientation for $\mathcal{L}$ is chosen). Moreover, by [3] the spin^c structure determined by K_ξ restricts to the spin^c structure associated to ξ on the boundary.

On the other hand, the given surgery presentation can be viewed smoothly as the first diagram of Figure 8. By carrying along the class K_ξ during the Kirby moves of Figure 8 (and observing that blowups and blowdowns do not change the spin^c structure on the boundary) one can check that the spin^c structures $\mathbf{t}_{\xi_1}$, $\mathbf{t}_{\xi_2}$ and $\mathbf{t}_\Xi$ are the restrictions to the boundary of the spin^c structures on the 4–dimensional plumbing determined by, respectively, the characteristic classes C_1, C_2 and C_3 given in Figure 9. Since when p is even $2\mu_b = 0$, while when p is odd $\mu_c = 3\mu_b$, we have

$$\mathbf{t}_{\xi_1} = \mathbf{t}_1 - 2\mu_b = \begin{cases} \mathbf{t}_1 & \text{for } p \text{ even,} \\ \mathbf{t}_1 - \mu_c + \mu_b = \mathbf{t}_2 & \text{for } p \text{ odd} \end{cases}$$
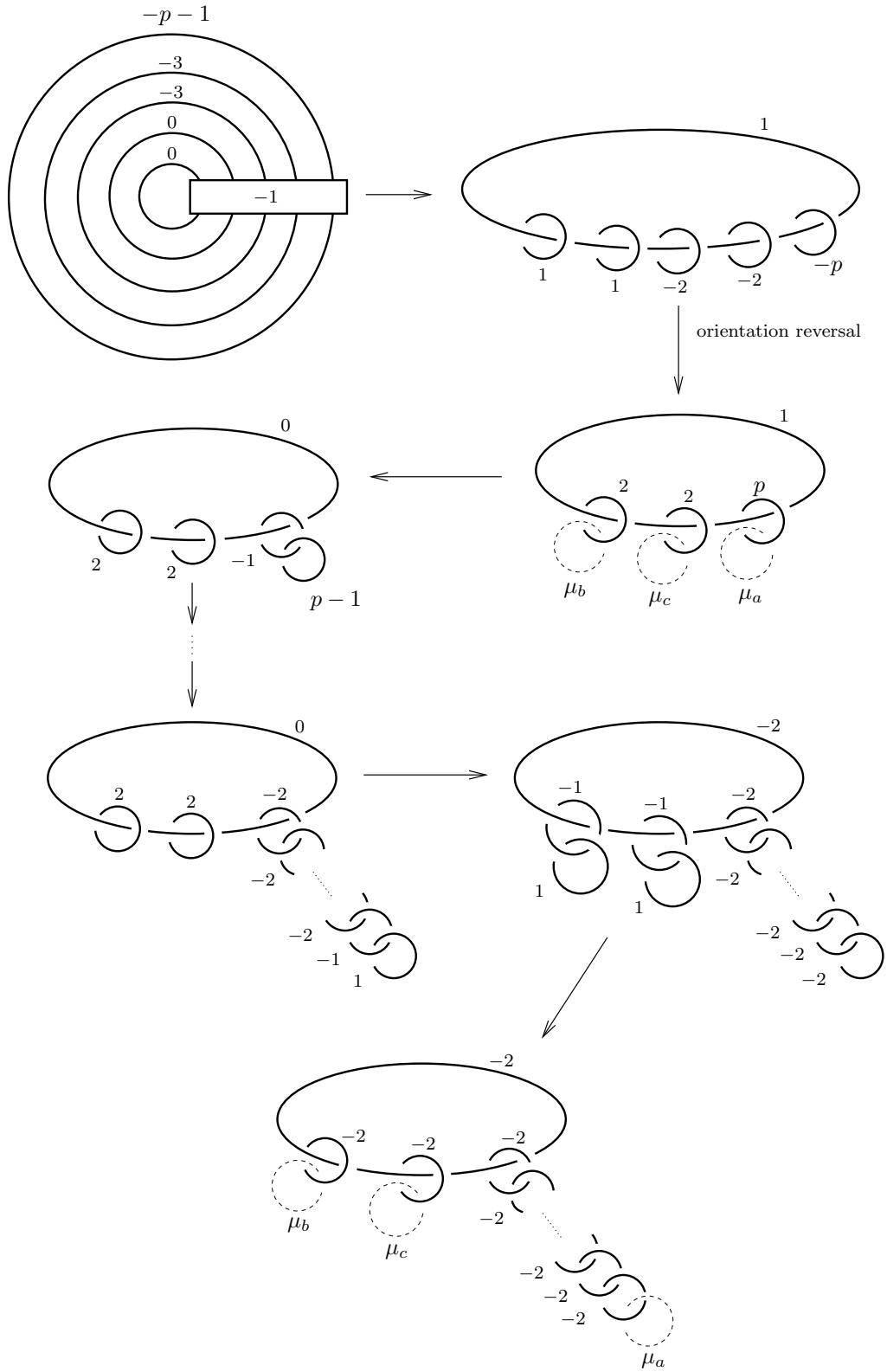
21

$-p-1$

-3

-3

0

0

-1

1

1

1

-2

-2

$-p$

orientation reversal

0

2

2

-1

$p-1$

1

2

2

p

μ_b

μ_c

μ_a

0

2

2

-2

-2

-2

-1

1

-2

-1

1

-1

-2

1

1

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-2

μ_b

μ_c

-2

-2

-2

-2

μ_a

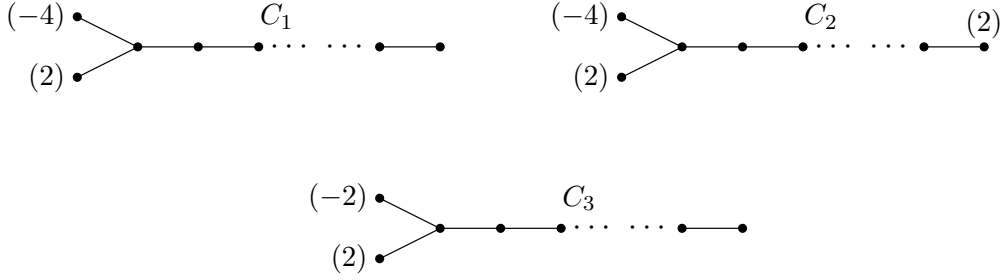Figure 8: Kirby diagrams for $\pm M_p$

Figure 9: The characteristic classes determining $\mathbf{t}_{\xi_1}$, $\mathbf{t}_{\xi_2}$ and $\mathbf{t}_\Xi$

Since $\mu_a = \mu_c - \mu_b = \mu_b - \mu_c$, we have

$$\mathbf{t}_{\xi_2} = \mathbf{t}_{\xi_1} + \mu_a = \mathbf{t}_{\xi_1} + \mu_b - \mu_c = \begin{cases} \mathbf{t}_1 + \mu_b - \mu_c = \mathbf{t}_2 & \text{for } p \text{ even,} \\ \mathbf{t}_1 & \text{for } p \text{ odd} \end{cases}$$

and

$$\mathbf{t}_\Xi = \mathbf{t}_1 - \mu_b = \mathbf{t}_1 + \mu_a - \mu_c = \mathbf{t}_3.$$

$\square$

We have shown that for $p \geq 2$ the contact structures ξ_1, ξ_2 and Ξ are distinct up to homotopy. Next, we are going to prove that they are tight for every $p \geq 2$.

Definition 4.6 Let η be the contact structure defined by the diagram obtained from any of the diagrams of Figure 6 by omitting the Legendrian knot L.

A simple Kirby calculus computation shows that η is a contact structure on $S^1 \times S^2$.

Proposition 4.7 *The contact Ozsváth–Szabó invariant of η is nonzero.*

Proof Consider the contact structure ζ given by the surgery diagram obtained from Figure 6(a) by erasing both L and one of the $(+1)$–framed Legendrian unknots. According to Corollary 3.5 and Lemma 3.6, the contact Ozsváth–Szabó invariant of the resulting structure ζ is nontrivial. It is easy to see that the 3–manifold underlying ζ is the lens space $L(4,1)$. Let X denote the cobordism from $L(4,1)$ to $S^1 \times S^2$ obtained by the handle attachment defined by the remaining contact $(+1)$–surgery. According to Theorem 3.4 we have

$$F_{-X}(c(L(4,1),\zeta)) = c(S^1 \times S^2, \eta),$$

where $F_{-X} = \sum_{\mathbf{s} \in Spin^c(X)} \pm F_{-X,\mathbf{s}}$. The cobordism $-X$ induces an exact triangle

The Ozsváth–Szabó homology groups in this triangle are well–known (see [27]):

- $\widehat{HF}(-L(4,1)) \cong \mathbb{Z}^2_{(0)} \oplus \mathbb{Z}_{(\frac{3}{4})} \oplus \mathbb{Z}_{(-\frac{1}{4})}$

- $\widehat{HF}(S^1 \times S^2) \cong \mathbb{Z}_{(\frac{1}{2})} \oplus \mathbb{Z}_{(-\frac{1}{2})}$

- $\widehat{HF}(\mathbb{RP}^3 \# \mathbb{RP}^3) \cong \mathbb{Z}^2_{(0)} \oplus \mathbb{Z}_{(\frac{1}{2})} \oplus \mathbb{Z}_{(-\frac{1}{2})}$

A simple computation shows that the cobordisms $-X$ and U have zero signature. More-over, since the 4–manifolds X and U are obtained by attaching a 2–handle to $S^1 \times S^2$, the restriction maps $H^2(X; \mathbb{Z}) \to H^2(\partial X; \mathbb{Z})$ and $H^2(U; \mathbb{Z}) \to H^2(\partial U; \mathbb{Z})$ are injective. Since the group $\widehat{HF}(S^1 \times S^2)$ is concentrated at the torsion spin^c structure, this implies that each possibly nontrivial component of the maps F_{-X} and F_U is induced by a torsion spin^c structure. Therefore, by the degree–shift formula (3.1), both F_{-X} and F_U shift degrees by $-\frac{1}{2}$.

Exactness of the triangle immediately shows that the kernel of F_{-X} is 3–dimensional and, since F_{-X} shifts degree by $-\frac{1}{2}$, we see that

$$\mathbb{Z}_{(\frac{3}{4})} \oplus \mathbb{Z}_{(-\frac{1}{4})} \subseteq \ker F_{-X} \quad \text{and} \quad F_{-X}(\mathbb{Z}^2_{(0)}) = \mathbb{Z}_{(-\frac{1}{2})}.$$

Since the $\mathcal{J}$–map preserves degree and fixes spin structures, the summands $\mathbb{Z}_{(\frac{3}{4})}$ and $\mathbb{Z}_{(-\frac{1}{4})}$ inside $\widehat{HF}(-L(4,1))$ correspond to the two spin structures on $-L(4,1)$. Therefore, since $c(\zeta) \neq 0$,

$$\langle c(\zeta), \mathcal{J}_{-L(4,1)} c(\zeta) \rangle = \mathbb{Z}^2_{(0)} \subseteq \widehat{HF}(-L(4,1)).$$

The group $\widehat{HF}(S^1 \times S^2)$ is pointwise fixed (up to sign) by the $\mathcal{J}$–action because it is concentrated at the only spin^c structure induced by a spin structure, and has rank at most one in each degree. Hence,

$$F_{-X}(\mathcal{J}_{-L(4,1)} c(\zeta)) = \mathcal{J}_{S^1 \times S^2} F_{-X}(c(\zeta)) = F_{-X}(c(\zeta)).$$

Since $F_{-X}(\mathbb{Z}^2_{(0)}) \neq \{0\}$, this implies $c(\eta) = F_{-X}(c(\zeta)) \neq 0$. $\qquad\square$

Corollary 4.8 *The contact structures ξ_1 and ξ_2 are Stein fillable, hence tight.*

Proof The contact structure η has nonzero contact Ozsváth–Szabó invariant, and there-fore it is tight. It is well–known that $S^1 \times S^2$ carries a unique tight contact structure up to isotopy, and this contact structure is Stein fillable. Therefore, η is Stein fillable. Since contact (-1)–surgery preserves Stein fillability, the statement follows. $\qquad\square$

Theorem 4.9 *The Ozsváth–Szabó invariant of the contact structure Ξ is nonzero.*

Proof Denote by (Y, β) the contact 3–manifold whose contact surgery presentation is obtained from Figure 7 by erasing one of the unknots with contact surgery coefficient equal to $+1$. Let $-X$ be the cobordism from Y to M_p determined by the missing contact $(+1)$–surgery, with orientation reversed. Denote by $c^+(\beta)$ the image of $c(\beta)$ under the map induced by the natural homomorphism [24, 25]

$$\widehat{HF}(-Y, \mathbf{t}_\beta) \to HF^+(-Y, \mathbf{t}_\beta),$$

and define $c^+(\Xi)$ in the analogous way. Clearly, it is enough to show that $c^+(\Xi) \neq 0$. By [7, Lemma 2.11], there is a spin^c structure $\mathbf{s}$ on $-X$ such that $F^+_{-X,\mathbf{s}}(c^+(\beta)) = c^+(\Xi)$ and

$$-d_3(\beta) + \delta(\mathbf{s}) = -d_3(\Xi),$$

where

$$\delta(\mathbf{s}) := \frac{1}{4}(c_1^2(\mathbf{s}) - 3\sigma(-X) - 2\chi(-X)).$$

The 3–manifold $-Y$ is given by the surgery presentation obtained from the third diagram of Figure 8 by changing the framing of the (1)–framed unknot to 0. Then, Kirby moves similar to those of Figure 8 show that $-Y$ is the boundary of a plumbing whose graph is obtained from the graph of Figure 4 by changing the framing of the central vertex to -3. By [28, Theorem 7.1] it follows that $-Y$ is an L–space. By Lemma 3.6 and Corollary 3.5 we have $c(\beta) \neq 0$, therefore $-d_3(\beta) = d(-Y, \mathbf{t}_\beta)$. This immediately implies $c^+(\beta) \neq 0$. Moreover, using [3, Corollary 3.6] (where $b_2(X)$ should be plugged into the formula instead of the Euler characteristic $\chi(X)$ because the 3–dimensional invariant used in Heegaard Floer theory is shifted by $\frac{1}{2}$) a simple calculation gives

$$d_3(\Xi) = \frac{2-p}{4}.$$

Therefore, by Proposition 4.2 and Lemma 4.5 we have $-d_3(\Xi) = d(-M_p, \mathbf{t}_\Xi)$.

Another simple calculation shows that $b_2^-(-X) = 1$. Since $\mathbf{t}_\beta$ and $\mathbf{t}_\Xi$ are torsion and $HF^\infty(-Y, \mathbf{t}_\eta) \cong \mathbb{Z}[U, U^{-1}]$, by [27, Proposition 9.4] the induced map

$$F^\infty_{-X,\mathbf{s}} \colon HF^\infty(-Y, \mathbf{t}_\beta) \to HF^\infty(-M_p, \mathbf{t}_\Xi)$$

is an isomorphism. Since the group $HF^-_d(-Y, \mathbf{t}_\beta)$ vanishes if the absolute degree d is sufficiently large, the commutative diagram (see [27, Section 2])

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & HF^-(-Y, \mathbf{t}_\beta) & \longrightarrow & HF^\infty(-Y, \mathbf{t}_\beta) & \longrightarrow & HF^+(-Y, \mathbf{t}_\beta) & \longrightarrow \cdots \\
& \downarrow {\scriptstyle F^-_{-X,\mathbf{s}}} & & \downarrow {\scriptstyle F^\infty_{-X,\mathbf{s}}} & & \downarrow {\scriptstyle F^+_{-X,\mathbf{s}}} & \\
\cdots \longrightarrow & HF^-(-M_p, \mathbf{t}_\Xi) & \longrightarrow & HF^\infty(-M_p, \mathbf{t}_\Xi) & \longrightarrow & HF^+(-M_p, \mathbf{t}_\Xi) & \longrightarrow \cdots
\end{array}
$$

implies that the map

$$F^+_{-X,\mathbf{s}} \colon HF^+_d(-Y, \mathbf{t}_\beta) \to HF^+_{d+\delta(\mathbf{s})}(-M_p, \mathbf{t}_\Xi)$$

is also an isomorphism when d is large enough. Since $F^+_{-X,\mathbf{s}}$ is a homomorphism of $\mathbb{Z}[U]$–modules [26], this immediately implies that $F^+_{-X,\mathbf{s}}$ restricted to $HF^+_{-d_3(\beta)}(-Y, \mathbf{t}_\beta)$ is an isomorphism if and only if $-d_3(\Xi) = d(-M_p, \mathbf{t}_\Xi)$, and the conclusion follows. $\qquad \square$

Remark 4.10 (1) The proof of Corollary 4.8 applies to show that the result of any Legendrian surgery on $(S^1 \times S^2, \eta)$ is Stein fillable. In particular, for any choice of zig–zag distribution for the Legendrian knot L of Figure 6, the resulting contact structures ξ_i $(i = 1, 2, \ldots, p)$ are Stein fillable. In addition, by reversing the stabilizations on the other two contact (-1)–framed knots of Figure 6, another collection of tight contact surgery diagrams — denoted by ξ'_i $(i = 1, 2, \ldots, p)$ — can be given. Although these diagrams give

isotopic structures to ξ_1 or ξ_2 on M_p, they will play an important role in the classification results discussed in the next section.

(2) Let Ξ' be the contact structure on M_p with surgery presentation obtained from Figure 7 by applying a $180°$ rotation around an axis perpendicular to the plane of the picture. Then, $\mathbf{t}_{\Xi'} = \mathbf{t}_{\Xi}$ and $d_3(\Xi') = d_3(\Xi)$. Since the auxiliary 3–manifold Y used in the proof of Theorem 4.9 is the same for Ξ', the same proof also shows that Ξ' has nonzero Ozsváth–Szabó invariant.

(3) For a more general form of Theorem 4.9 see [21].

The results above lead to

Corollary 4.11 *For every $p \geq 2$, the 3–manifold $M_p = M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{p})$ admits exactly three tight contact structures (up to isotopy).*

Proof Corollaries 2.10 and 2.13 imply that M_p admits at most three tight contact structures, while to combination of Lemma 4.5, Corollary 4.8 and Theorem 4.9 verifies that M_p admits at least three distinct tight contact structures, concluding the proof. Notice that this argument shows, for example, that Ξ and Ξ' are isotopic on M_p. $\square$

Nonfillability of (M_p, Ξ)

In this subsection we give simple proofs of the facts that the tight contact 3–manifold (M_p, Ξ) is not Stein fillable for $p > 2$, and not symplectically fillable for $p \not\equiv 2 \bmod 8$, justifying our use of contact Ozsváth–Szabó invariants in the proof of their tightness. First we need the following general observation:

Lemma 4.12 *Let (Y, ξ) be a contact 3–manifold such that $b_1(Y) = 0$, and suppose that for every symplectic filling (X, ω) of (Y, ξ) we have $b_2^+(X) = 0$. Then, for every symplectic filling (X, ω) of (Y, ξ) we also have $b_1(X) = 0$.*

Proof Let n be the order of the finite group $H_1(Y; \mathbb{Z})$. By contradiction, suppose that (X, ω) is a symplectic filling of (Y, ξ) with $b_1(X) > 0$. Then X admits a connected $(n+1)$–fold cover $\tilde{X}$ which is necessarily trivial over $\partial X = Y$. By [4, 5] we can cap off n of the boundary components of $\tilde{X}$ with symplectic caps having $b_2^+ > 0$, obtaining a symplectic filling of (Y, ξ) with $b_2^+ > 0$, which is against our assumptions. $\square$

Theorem 4.13 *The tight contact 3–manifold (M_p, Ξ) is not Stein fillable for every $p > 2$, and is not symplectically fillable for $p \not\equiv 2 \bmod 8$.*

Proof By [16, Theorem 2.2] if (X, ω) is any symplectic filling of (M_p, Ξ), then the intersection form Q_X is a standard diagonal negative definite form. Therefore, Lemma 4.12 implies $b_1(X) = 0$.

Now let (X, J) be a Stein filling of (M_p, Ξ) with complex structure J inducing the contact structure Ξ. Then, the complex structure $-J$ gives another Stein structure on X inducing

a contact structure $\overline{\overline{\Xi}}$ on M_p. Since the associated spin^c structure satisfies $\mathbf{s}_{-J} = \overline{\mathbf{s}_J}$, we have

$$\mathbf{t}_{\overline{\overline{\Xi}}} = \overline{\mathbf{t}_{\overline{\Xi}}} = \overline{\mathbf{t}_3} = \mathbf{t}_3$$

because, as we observed previously, $\mathbf{t}_3$ is induced by a spin structure. Therefore, by Corollary 4.11 $\overline{\overline{\Xi}}$ is isotopic to Ξ. But then, by [17, Theorem 1.2] we have $\overline{\mathbf{s}_J} = \mathbf{s}_J$, which implies $c_1(J) = 0$. Since Q_X is standard diagonal and $b_1(X) = 0$, this implies $\sigma(X) = 0$ and $\chi(X) = 1$. In view of Formula (3.3) we have $d_3(\Xi) = 0$, and by Proposition 4.2 this is possible only if $p = 2$.

When (X, ω) is a general symplectic filling of (M_p, Ξ) we are unable to conclude that $c_1(X, \omega) = 0$, but we still know that Q_X is negative definite, diagonal and $b_1(X) = 0$, therefore Formula (3.3) and Proposition 4.2 give

$$\frac{2-p}{4} = \frac{1}{4}\left(-\sum_{i=1}^{b_2(X)} (2n_i + 1)^2 + b_2(X)\right)$$

for some $n_i \geq 0$. Since $(2n_i + 1)^2 - 1 = 4n_i(n_i + 1)$ is divisible by 8, the equation provides the desired contradiction once $(p - 2)$ is not divisible by 8. $\qquad\square$

It is natural to expect that (M_p, Ξ) is not symplectically fillable for every $p > 2$. On the other hand, from the obvious $\mathbb{Z}/3\mathbb{Z}$–symmetry of the surgery diagram of $M_2 = M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ it is not hard to see that for $p = 2$ the structures ξ_1, ξ_2 and Ξ are contactomorphic (although not isotopic), hence for $p = 2$ the structure Ξ is Stein fillable.

5 Lower bounds and the proof of Theorem 1.1

Now we are ready to prove the general lower bounds for the number of tight contact structures on the manifolds under consideration. We will proceed by first constructing a set of contact structures which are — due to our previous computations — all tight, and then determining how many distinct structures are in that set. Consider the surgery presentations of the contact structures ξ_i, ξ_i' ($i = 1, 2, \ldots, p$), Ξ and Ξ' on M_p (cf. Figure 6, Figure 7 and Remarks 4.10(1) and (2)). According to the continued fraction expansions of the surgery coefficients $-\frac{1}{r_i}$, attach chains of Legendrian unknots K_j^i ($i = 1, 2, 3; j = 1, \ldots, k_i$) stabilized $(a_j^i - 2)$–times to the contact (-1)–framed knots of Figures 6 and 7. Notice that there are many choices for the required stabilizations, hence this procedure gives rise to a number of contact structures.

Define $A(\xi)$ as the set of contact structures on $M(-1; r_1, r_2, r_3)$ obtained by Legendrian surgery on the knots K_j^i on either diagram of Figure 6 or its modifications ξ_i, ξ_i' ($i = 1, 2, \ldots, p$) described in Remark 4.10(1). In a similar manner, $A(\Xi)$ denotes the set of contact structures obtained either from the diagram of Figure 7 giving Ξ or its symmetric giving Ξ'.

Lemma 5.1 *The set $A(\xi) \cup A(\Xi)$ consists of tight contact structures having nonzero Ozsváth–Szabó invariant.*

Proof Any element of $A(\xi) \cup A(\Xi)$ is constructed by Legendrian surgery on a contact structure with nonzero contact Ozsváth–Szabó invariant, therefore the statement follows immediately from Corollary 3.5. □

Proposition 5.2 *If $\zeta_1 \in A(\xi)$ and $\zeta_2 \in A(\Xi)$ then $c(M, \zeta_1) \neq c(M, \zeta_2)$. In particular, ζ_1 is not isotopic to ζ_2.*

Proof Denote by (Y_p, γ_i), $i = 1, 2$ the contact 3–manifolds obtained by contact $(+1)$–surgeries in (M, ζ_i) along a push–off of K_1^i for every i. The 3–manifold Y_p is a connected sum $M_p \# L$, where L is the connected sum of at most three lens spaces. Correspondingly, the contact structure γ_i can be written as

$$\gamma_i = \gamma_i^{M_p} \# \gamma_i^L,$$

where $\gamma_1^{M_p}$ is equal to ξ_i or ξ_i' $(i = 1, 2, \ldots, p)$ and $\gamma_2^{M_p}$ to Ξ or Ξ'. By Corollary 4.8 and Theorem 4.9 $c(M_p, \gamma_1^{M_p})$ and $c(M_p, \gamma_2^{M_p})$ are both nonzero. Since by Lemma 4.5 $c(M_p, \gamma_1^{M_p})$ and $c(M_p, \gamma_2^{M_p})$ live in groups corresponding to different spin^c structures, we have $c(\gamma_1^M) \neq c(\gamma_2^M)$. Thus, the map corresponding to the cobordism induced by the prescribed contact $(+1)$–surgeries sends $c(M, \zeta_1)$ and $c(M, \zeta_2)$ to distinct elements, and the statement follows. □

As a consequence of Proposition 5.2, in order to get a lower bound for the number of nonisotopic contact structures on M, we can examine the sets $A(\xi)$ and $A(\Xi)$ separately.

Lower bound on $|A(\xi)|$

Suppose first that $k_1 = k_2 = 1$ and $k_3 = 0$, that is, there are two circles on the first and second legs and there is a single one on the third. Let the corresponding 3–manifold be denoted by $M_{p,k,l}$. A surgery presentation for this 3–manifold is given by Figure 10(a). Notice that if $r_1 > r_2 = \frac{1}{2}$ then $k_2 = 0$. Therefore, to cover this case we shall also consider the 3–manifold $M_{p,k}$ defined as in Figure 10(a) but omitting the $(-l)$–framed knot. (The case $r_1 = r_2 = \frac{1}{2}$ leads to the manifold M_p— we have already dealt with this manifold in Section 4.)
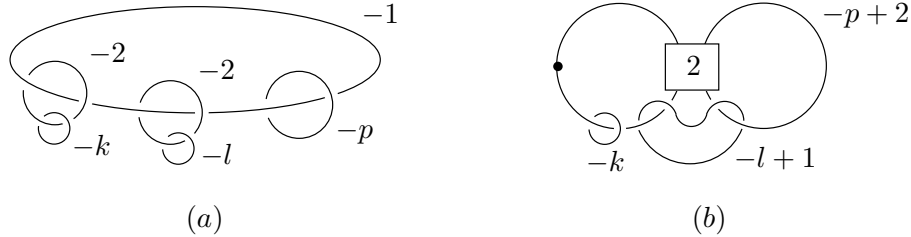

Figure 10:   Surgery diagrams for (a) $M_{p,k,l}$ and (b) $X_{p,k,l}$

Proposition 5.3 *On the 3–manifold $M_{p,k,l}$ there are at least*

$$(2(k-1)(l-1) + p - 1)(k + l - 2)$$

isotopy classes of tight contact structures belonging to $A(\xi)$, while on $M_{p,k}$ there are at least

$$2(k-1) + p - 1$$

such isotopy classes.

Proof Let us start with the case of $M_{p,k,l}$, i.e. when $r_2 > \frac{1}{2}$. We will apply [17, Theorem 1.2], which implies that if two Stein structures on a 4–manifold X have distinct first Chern classes, then the induced contact structures on ∂X are nonisotopic. As proved by Plamenevskaya [33], tight contact structures distinguished in this way have different contact Ozsváth–Szabó invariants. Notice, however, that the contact surgery diagrams giving the elements of $A(\xi)$ do not provide Stein fillings. A simple surgery operation, however, can turn the 4–manifold $W_{p,k,l}$ given by each surgery diagram into a Stein domain. Namely, let us consider the codimension–0 submanifold $Z \subset W_{p,k,l}$ defined by the union of the two $(+1)$–framed Legendrian unknots together with the two once stabilized (-1)–knots. By Proposition 4.7, the corresponding contact structure η is the unique tight (and hence Stein fillable) contact structure on $S^1 \times S^2$.

Replacing Z with a 4–dimensional 1–handle H we obtain a 4–manifold

$$X_{p,k,l} = (W_{p,k,l} \setminus Z) \cup H$$

with a decomposition involving a 1–handle and three 2–handles. Then, $X_{p,k,l}$ can be thought of as obtained by attaching three Stein 2–handles to a Stein 1–handle, and therefore carries a Stein structure. The Legendrian attaching circles are L from Figure 6 plus two Legendrian meridional unknots M_1 and M_2 linking the once stabilized unknots in the same picture. Smoothly, a handlebody decomposition for $X_{p,k,l}$ is given by Figure 10(b), where the $(2-p)$–framed knot corresponds to L, and M_1, M_2 correspond, respectively, to the $(-k)$– and the $(1-l)$–framed knots. Suppose now that (for some choice of the orientations) the rotation numbers of the once stabilized Legendrian unknots in Figure 6 are A and $-A$ with $A \in \{\pm 1\}$, while the rotation numbers of M_1, M_2 and L are given, respectively, by x, y and z. These rotation numbers satisfy the constraints

$$
\begin{aligned}
x &\in \{-k + 2i_1, \ i_1 = 1, \ldots, k-1\}, \\
y &\in \{-l + 2i_2, \ i_2 = 1, \ldots, l-1\}, \\
z &\in \{-p-1 + 2i_3, \ i_3 = 1, \ldots, p\}.
\end{aligned}
\tag{5.1}
$$

(Note the special behaviour of z, which is the rotation number of L, linking the $(+1)$–surgery curves in Figure 6.) Denote by a, b and c the homology classes in $H_2(W_{p,k,l}; \mathbb{Z})$ defined by M_1, M_2 and L respectively, and observe that there are homology classes $\alpha, \beta \in H_2(W_{p,k,l} \setminus Z; \mathbb{Z})$ which map to $c-a-b$ and $a-b$, respectively, and such that their images $\overline{\alpha}$ and $\overline{\beta}$ under the map induced by the inclusion $(W_{p,k,l} \setminus Z) \subset X_{p,k,l}$ generate the group $H_2(X_{p,k,l}; \mathbb{Z}) \cong \mathbb{Z}^2$. It is not hard to see that $X_{p,k,l}$ is simply connected, and the values of the first Chern class of its Stein structure on $\overline{\alpha}$ and $\overline{\beta}$ are, respectively, $z - x - y + 2$ and $x - y - A + 2$. Therefore, to apply [17, Theorem 1.2] we need to count the number of elements of the set $S(p, k, l) \subset \mathbb{Z}^2$ of pairs $(x - y - A, z - x - y)$ such that $A \in \{\pm 1\}$ and x, y, z satisfy the constraints of (5.1).

In order to do this, we first consider the set $T(k, l)$ consisting of pairs $(x - y, -x - y)$, where x and y satisfy the constraints given by (5.1). Setting $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$, we

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have

$$S(p,k,l) = \pm e_1 + \bigcup_z \left(T(k,l) + z e_2 \right),$$

where z satifies (5.1). Let $\varphi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ be the injective map given by $\varphi(x,y) = (x - y, -x - y)$. Clearly $T(k,l) = \varphi(Q(k,l))$, where $Q(k,l) \subset \mathbb{Z}^2$ is the set of pairs (x,y) which satisfy the constraints given by (5.1). Clearly, $Q(k,l)$ has the shape of a rectangle and contains $(k-1)(l-1)$ elements. Since

$$\varphi(\tfrac{1}{2}, -\tfrac{1}{2}) = e_1, \quad \varphi(-\tfrac{1}{2}, -\tfrac{1}{2}) = e_2,$$

the set $S(p,k,l)$ has the same number of elements as the set

$$T(p,k,l) := \pm \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} + \bigcup_z \left(Q(k,l) + z \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix} \right) \subset \mathbb{Z}^2.$$

To count the number of elements in this set, observe that the $(k-1)(l-1)$ elements of $Q(k,l)$ form a rectangle in the plane, and they are at two integral units of distance from each other. In Figure 11 the points of the set $Q(5,4)$ are represented by a's. The set
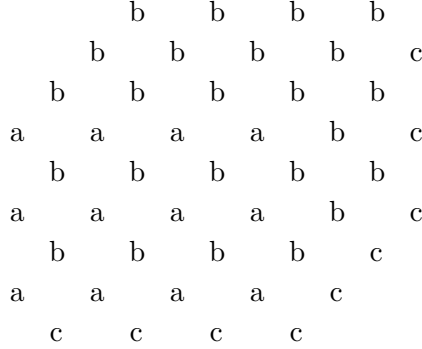
```
          b     b     b     b
       b     b     b     b     c
       b     b     b     b     b
    a     a     a     a     b     c
       b     b     b     b     b
    a     a     a     a     b     c
       b     b     b     b     c
    a     a     a     a     c
       c     c     c     c
```

Figure 11: Counting the number of elements of $T(p,k,l)$

$$R(p,k,l) := \bigcup_z \left(Q(k,l) + z \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix} \right)$$

is obtained as the union of p shifts of $Q(k,l)$ by integral units in the North–East direction. If we assume $p = 4$, for instance, we see in Figure 11 how the results of these shifts create the elements denoted by b's in the picture. It is easy to compute that the number of elements increases by $(k-1)(l-1) + (k+l-2)(p-2)$. Finally, $T(p,k,l)$ is obtained as the union of 2 shifts of $R(p,k,l)$, one integral unit apart from each other in the South–East direction. In Figure 11 the resulting new elements have been denoted by c's. It is easy to see that the number of such elements is $(k-1) + (l-1) + (p-2)$. Therefore, the cardinality of $T(p,k,l)$ is obtained by adding the number of a's, b's and c's:

$$(k-1)(l-1) + (k-1)(l-1) + (k+l-3)(p-2) + (k-1) + (l-1) + (p-2) =$$
$$2(k-1)(l-1) + (p-1)(k+l-2).$$

When $r_1 > r_2 = \tfrac{1}{2}$, i.e. in the case of $M_{p,k}$, there is no meridian M_2, no variable y nor homology class b, and one can work with 4–manifolds $W_{p,k}$ and $X_{p,k}$ by analogy to what

we did before. There is a class $\alpha \in H_2(W_{p,k} \setminus Z; \mathbb{Z})$ which is sent to $c - 2a \in H_2(W_{p,k}; \mathbb{Z})$ by the map induced by inclusion, and whose image $\overline{\alpha} \in H_2(X_{p,k}; \mathbb{Z}) \cong \mathbb{Z}$ is a generator. $X_{p,k}$ is still simply connected, and the possible values of the first Chern classes of $X_{p,k}$ on $\overline{\alpha}$ are of the form $z + A - 2x$ while z, A and x range as in (5.1). An easy count yields the stated formula. $\qquad\square$

Corollary 5.4 *The number of isotopy classes of tight contact structures on the 3–manifold $M(-1; r_1, r_2, r_3)$ belonging to $A(\xi)$ are at least*

$$\left(2(a_1^1 - 1)(a_1^2 - 1) + (a_0^3 - 1)(a_1^1 + a_1^2 - 2)\right)(a_1^3 - 1) \prod_{i=1}^{3} \prod_{j \geq 2}(a_j^i - 1)$$

if $r_2 > \frac{1}{2}$,

$$\left(2(a_1^1 - 1) + (a_0^3 - 1)\right)(a_1^3 - 1) \prod_{i \neq 2} \prod_{j \geq 2}(a_j^i - 1)$$

if $r_1 > r_2 = \frac{1}{2}$, and

$$2 \prod_{j \geq 1}(a_j^3 - 1)$$

if $r_1 = r_2 = \frac{1}{2}$. In the above formulae $a_j^i = 2$ by convention if $j > k_i$.

Proof Let ζ be an element of the set $A(\xi)$. Perform contact $(+1)$–surgeries along the Legendrian push–offs of K_1^1 (if it exists), K_1^2 (if it exists) and K_0^3. By Proposition 4.7, the resulting contact 3–manifold is the tight contact $S^1 \times S^2$ connected sum with at most three tight contact lens spaces. But for such contact lens spaces it is known that the zig–zag distribution is determined by the contact invariant. Therefore, applying Theorem 3.4 we see that different zig–zag distributions in the diagram for ζ after the second circle on the first two legs and after the first circle on the third leg yield nonisotopic structures. If $r_2 > \frac{1}{2}$ or $r_1 > r_2 = \frac{1}{2}$ the statement follows by Proposition 5.3 and a simple computation. If $r_1 = r_2 = \frac{1}{2}$, by Lemma 4.5 and Corollary 4.8, the set $A(\xi)$ contains at least 2 elements in the case of the 3–manifold M_p. Thus, the stated formula follows immediately. $\qquad\square$

Lower bound on $|A(\Xi)|$ and the proof of Theorem 1.1

If $k_1 = k_2 = k_3 = 1$, the same idea used to study the set $A(\xi)$ suggests the existence of an appropriate function $g(p, k, l, m)$ such that there are at least

$$g(a_0^3, a_1^1, a_1^2, a_1^3) \cdot \Pi_{i=1}^{3} \Pi_{j \geq 2}(a_j^i - 1)$$

different elements in $A(\Xi)$: just perform contact $(+1)$–surgeries along the push–offs of the Legendrian curves K_1^i $(i = 1, 2, 3)$. (It turns out that it is not useful to do surgery along the push–off of the first circle of the third leg, because in the present case the resulting contact structure on $S^1 \times S^2$ would be overtwisted.) So our aim will be to find a lower bound for the number of distinct structures in $A(\Xi)$ on the 3–manifolds $M_{p,k,l,m}$ defined by Figure 12. To cover the cases when $k_i = 0$ for some $i \in \{1, 2, 3\}$, we shall consider also analogously defined manifolds $M_{p,k,m}$, $M_{p,k,l}$ and $M_{p,k}$. Let $K_1, \ldots, K_5$ be the components of the contact surgery diagram in Figure 7 defining M_p with one of the two
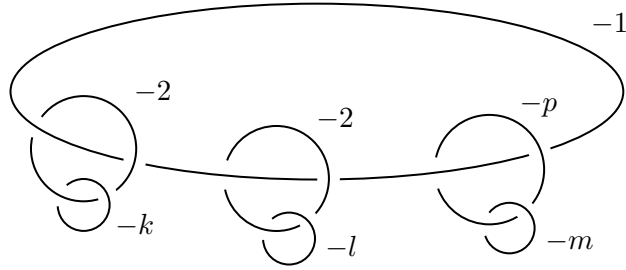
Figure 12:   Kirby diagram for the 3–manifold $M_{p,k,l,m}$

tight contact structures Ξ or Ξ'. Let K_6, K_7, K_8 be the three extra knots linked once to K_3, K_4, K_5 respectively as shown in Figure 13, which gives a contact surgery presentation of $(M_{p,k,l,m}, \zeta)$ with $\zeta \in A(\Xi)$. Since the contact surgery coefficient of K_6, K_7 and
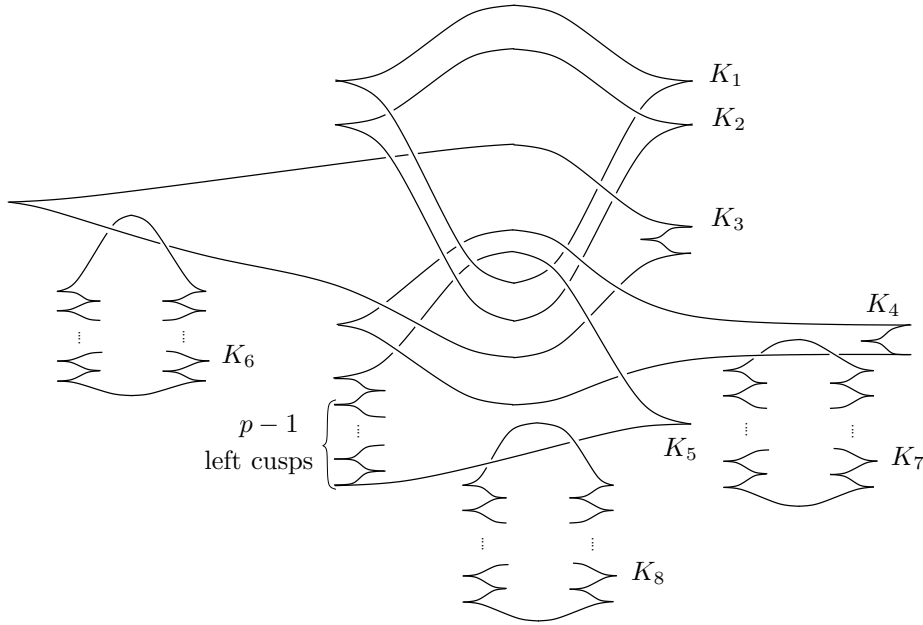

Figure 13:   Contact surgery diagram for ζ on $M_{p,k,l,m}$

K_8 is -1, they determine a Stein cobordism $W_{p,k,l,m}$ between (M_p, Ξ) or (M_p, Ξ') and $(M_{p,k,l,m}, \zeta)$. Denote by $\mathbf{t}$ the spin^c structure induced on $W_{p,k,l,m}$ by the Stein structure. The contact surgery diagram of Figure 13 determines also a 4–manifold $X_{p,k,l,m}$ bounded by $M_{p,k,l,m}$ and a spin^c structure $\mathbf{s}$ on $X_{p,k,l,m}$. Let X_p be the 4–manifold bounding M_p obtained by surgery on the link in Figure 7. Since this link is a sublink of the link in Figure 13, X_p is a submanifold of $X_{p,k,l,m}$ and $W_{p,k,l,m} = X_{p,k,l,m} \setminus X_p$ is the above mentioned cobordism between M_p and $M_{p,k,l,m}$. Moreover,

$$\mathbf{s}|_{W_{p,k,l,m}} = \mathbf{t}.$$

The above discussion remains essentially unchanged if the knot K_7, the knot K_8 or both

the knots K_7 and K_8 are omitted from Figure 13. In fact, it suffices to replace the triple

$$(M_{p,k,l,m}, W_{p,k,l,m}, X_{p,k,l,m})$$

by, respectively, the triples

$$(M_{p,k,m}, W_{p,k,m}, X_{p,k,m}), \qquad (M_{p,k,l}, W_{p,k,l}, X_{p,k,l}) \qquad \text{and} \qquad (M_{p,k}, W_{p,k}, X_{p,k}).$$

Lemma 5.5 *Consider two contact surgery diagrams as in Figure 13, where in both diagrams K_7, K_8 or both might be missing. Denote by ζ_1 and ζ_2 the tight contact structures induced on $M_{p,k,l,m}$, $M_{p,k,m}$, $M_{p,k,l}$ or $M_{p,k}$, and by $\mathbf{t}_1$ and $\mathbf{t}_2$ the corresponding spin^c structures induced respectively on $W_{p,k,l,m}$, $W_{p,k,m}$, $W_{p,k,l}$ or $W_{p,k}$. If ζ_1 is isotopic to ζ_2 then $\mathbf{t}_1$ is isomorphic to $\mathbf{t}_2$.*

Proof Consider the case of $M_{p,k,l,m}$. If ζ_1 is isotopic to ζ_2 then $c(\zeta_1) = c(\zeta_2)$. By [7, Lemma 2.11] $F_{\overline{W}_{p,k,l,m},\mathbf{s}}(c(\zeta_i)) = c(\Xi) \neq 0$ if $\mathbf{s} = \mathbf{t}_i$ and $F_{\overline{W}_{p,k,l,m},\mathbf{s}}(c(\zeta_i)) = 0$ for any other spin^c structure $\mathbf{s}$ on $W_{p,k,l,m}$, where $\overline{W}_{p,k,l,m}$ denotes the cobordism $W_{p,k,l,m}$ viewed upside down. The same argument applies to $W_{p,k,m}$, $W_{p,k,l}$ and $W_{p,k}$. This immediately implies the statement. $\qquad\square$

Lemma 5.6 *Let ζ_1, $\zeta_2 \in A(\Xi)$ be two contact structures on $M_{p,k,l,m}$, $M_{p,k,m}$, $M_{p,k,l}$ or $M_{p,k}$ given by contact surgery diagrams as in Figure 13, and let $\mathbf{t}_1$ and $\mathbf{t}_2$ be the spin^c structures on the cobordism $W_{p,k,l,m}$, $W_{p,k,m}$, $W_{p,k,l}$ or $W_{p,k}$ induced by the contact surgery diagrams. Denote by x_i, y_i, z_i, $i = 1, 2$, respectively, the rotation numbers (for some choice of orientations) of the Legendrian knots K_6, K_7 and K_8 for ζ_1 and ζ_2. Then, $\mathbf{t}_1$ is isomorphic to $\mathbf{t}_2$ if and only if one of the following conditions hold:*

(1) ζ_1 *and* ζ_2 *are both built by Legendrian surgery on Ξ or both on Ξ', and*

$$(x_1, y_1, z_1) = (x_2, y_2, z_2) \quad \text{for} \quad M_{p,k,l,m},$$

$$(x_1, z_1) = (x_2, z_2) \quad \text{for} \quad M_{p,k,m},$$

$$(x_1, y_1) = (x_2, y_2) \quad \text{for} \quad M_{p,k,l},$$

$$x_1 = x_2 \quad \text{for} \quad M_{p,k}$$

(2) ζ_1 *is built by Legendrian surgery on Ξ and ζ_2 is built by Legendrian surgery on Ξ', and*

$$(x_1, y_1, z_1) = (x_2, y_2, z_2 - 2) \quad \text{for} \quad M_{p,k,l,m},$$

$$(x_1, z_1) = (x_2, z_2 - 2) \quad \text{for} \quad M_{p,k,m},$$

$$(x_1, y_1) = (x_2, y_2) \quad \text{for} \quad M_{p,k,l},$$

$$x_1 = x_2 \quad \text{for} \quad M_{p,k}$$

Proof We consider first the case of $M_{p,k,l,m}$. Associated to any knot K_i in the contact surgery diagram in Figure 13 there is a surface $\Sigma_i \subset X_{p,k,l,m}$ obtained by capping off a Seifert surface of K_i with the core of the 2–handle attached along K_i. The homology classes represented by the surfaces Σ_i freely generate $H_2(X_{p,k,l,m}; \mathbb{Z})$. Denote by

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$[\Sigma_1]^*, \ldots, [\Sigma_8]^*$ the dual basis of $H^2(X_{p,k,l,m}; \mathbb{Z})$. The meridional discs N_i of K_i represent relative homology classes which freely generate $H_2(X_{p,k,l,m}, M_{p,k,l,m}; \mathbb{Z})$. Denote by $[N_1]^*, \ldots, [N_8]^*$ the dual basis of $H^2(X_{p,k,l,m}, M_{p,k,l,m}; \mathbb{Z})$. The cohomology exact sequence for the pair $(X_{p,k,l,m}, W_{p,k,l,m})$ together with the excision isomorphism

$$H^2(X_{p,k,l,m}, W_{p,k,l,m}; \mathbb{Z}) \cong H^2(X_p, M_p; \mathbb{Z})$$

gives the short exact sequence

$$0 \longrightarrow H^2(X_p, M_p; \mathbb{Z}) \xrightarrow{\varphi^*} H^2(X_{p,k,l,m}; \mathbb{Z}) \longrightarrow H^2(W_{p,k,l,m}; \mathbb{Z}) \longrightarrow 0$$

where the map φ^* is defined for $i = 1, \ldots, 5$ as

$$\varphi^*([N_i]^*) = \sum_{j=0}^{8} \mathtt{lk}(K_i, K_j)[\Sigma_j]^*$$

where $\mathtt{lk}(K_i, K_j)$ denotes the linking number between K_i and K_j if $i \neq j$, and the smooth surgery coefficient of K_i if $i = j$. In terms of the dual bases chosen above, the map φ^* is given by the matrix

$$\Phi^* = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & -3 & -1 & -1 \\ -1 & -1 & -1 & -3 & -1 \\ -1 & -1 & -1 & -1 & -p-1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Let $\mathbf{s}$ be a spin^c structure on $X_{p,k,l,m}$ defined by contact surgery on the Legendrian link $K_1 \cup \ldots \cup K_8$ describing ζ. By [12, Proposition 2.3] and [3, Proposition 3.4] the first Chern class of $\mathbf{s}$ is given by the formula

$$c_1(\mathbf{s}) = \sum_{i=0}^{8} \mathtt{rot}(K_i)[\Sigma_i]^*,$$

where $\mathtt{rot}(K_i)$ denotes the rotation number of the Legendrian knot K_i. Since K_1 and K_2 are Legendrian unknots with Thurston–Bennequin invariant $\mathtt{tb}(K_1) = \mathtt{tb}(K_2) = -1$, their rotation numbers are $\mathtt{rot}(K_1) = \mathtt{rot}(K_2) = 0$. If $K_1 \cup \ldots \cup K_5$ is a contact surgery diagram for Ξ, then (for a suitable choice of orientations) $\mathtt{rot}(K_3) = \mathtt{rot}(K_4) = +1$ and $\mathtt{rot}(K_6) = -(p-1)$. If it is a contact surgery diagram for Ξ' then $\mathtt{rot}(K_3) = \mathtt{rot}(K_4) = -1$, and $\mathtt{rot}(K_6) = (p-1)$.

Consider two contact surgery diagrams describing tight contact structures ζ_1, $\zeta_2 \in A(\Xi)$ on $M_{p,k,l,m}$ and inducing spin^c structures $\mathbf{s}_1$ and $\mathbf{s}_2$ on $X_{p,k,l,m}$. Since $X_{p,k,l,m}$ is simply connected, the restrictions $\mathbf{t}_1$ and $\mathbf{t}_2$ of $\mathbf{s}_1$ and $\mathbf{s}_2$ to $W_{p,k,l,m}$ are isomorphic if and only if

$$\frac{1}{2}(c_1(\mathbf{s}_1) - c_1(\mathbf{s}_2))|_{W_{p,k,l,m}} = 0.$$

If ζ_1 and ζ_2 are both built from Ξ or from Ξ', then

$$\frac{1}{2}(c_1(\mathbf{s}_1) - c_1(\mathbf{s}_2)) = \frac{1}{2}((x_1 - x_2)[\Sigma_6]^* + (y_1 - y_2)[\Sigma_7]^* + (z_1 - z_2)[\Sigma_8]^*), \qquad (5.2)$$

while if ζ_1 is built from Ξ and ζ_2 is built from Ξ' then

$$\frac{1}{2}(c_1(\mathbf{s}_1) - c_1(\mathbf{s}_2)) = [\Sigma_3]^* + [\Sigma_4]^* - (p-1)[\Sigma_5]^* + \frac{1}{2}((x_1 - x_2)[\Sigma_6]^* + (y_1 - y_2)[\Sigma_7]^*$$
$$+ (z_1 - z_2)[\Sigma_8]^*).$$
$$(5.3)$$

The matrix formed by the top five rows of Φ^* is invertible over $\mathbb{Q}$, therefore $[\Sigma_6]^*|_{W_{p,k,l,m}}$, $[\Sigma_7]^*|_{W_{p,k,l,m}}$ and $[\Sigma_8]^*|_{W_{p,k,l,m}}$ are linearly independent in $H^2(X_{p,k,l,m}; \mathbb{Z})$. This implies that

$$\frac{1}{2}((x_1 - x_2)[\Sigma_6]^* + (y_1 - y_2)[\Sigma_7]^* + (z_1 - z_2)[\Sigma_8]^*)$$

belongs to the image of φ^* if and only if $x_1 - x_2 = 0$, $y_1 - y_2 = 0$, and $z_1 - z_2 = 0$. Thus if ζ_1 and ζ_2 are both built from Ξ or from Ξ' then their surgery presentations induce isomorphic spin^c structures on $W_{p,k,l,m}$ if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

Let c_i denote the i-th column of Φ^*. The class

$$[\Sigma_3]^* + [\Sigma_4]^* - (p-1)[\Sigma_5]^* - [\Sigma_8]^*$$

can be expressed as $c_5 - c_1 - c_2$, and therefore its restriction in $H^2(W_{p,k,l,m}; \mathbb{Z})$ vanishes. Using this we see that Equation (5.3) implies

$$\frac{1}{2}(c_1(\mathbf{s}_1) - c_1(\mathbf{s}_2))|_{W_{p,k,l,m}} = \frac{1}{2}((x_1 - x_2)[\Sigma_6]^* + (y_1 - y_2)[\Sigma_7]^* + (z_1 - z_2 + 2)[\Sigma_8]^*)|_{W_{p,k,l,m}}.$$
$$(5.4)$$

Thus, by Equation (5.4) if ζ_1 is obtained by Legendrian surgery on Ξ and ζ_2 is obtained by Legendrian surgery on Ξ', then the surgery presentations of ζ_1 and ζ_2 induce isomorphic spin^c structures on $W_{p,k,l,m}$ if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2 - 2$.

The same argument given above works in the case of $M_{p,k,m}$. One just needs to omit the knot K_7 from Figure 13 and work with the analogously defined manifolds $W_{p,k,m}$ and $X_{p,k,m}$. The new matrix Φ^* is obtained from the original matrix Φ^* by simply dropping the seventh row. The remaining computations are essentially the same, except one does not have terms involving y_1, y_2 nor $[\Sigma_7]$. Similar considerations hold for the cases of $M_{p,k,l}$ and $M_{p,k}$. $\qquad\square$

Proposition 5.7 *The number of isotopy classes of tight contact structures in $A(\Xi)$ is at least*

$$(k-1)(l-1)m \quad \text{on} \quad M_{p,k,l,m},$$

$$(k-1)m \quad \text{on} \quad M_{p,k,m},$$

$$(k-1)(l-1) \quad \text{on} \quad M_{p,k,l},$$

and

$$(k-1) \quad \text{on} \quad M_{p,k}.$$

Proof In view of Lemma 5.5, the number of different spin^c structures induced on $W_{p,k,l,m}$ by the contact surgery diagrams of Figure 13 gives a lower bound for the number of isotopy classes of tight contact structures in $A(\Xi)$. Notice that $A(\Xi)$ can be decomposed as $A \cup A'$, where A contains the elements obtained by doing surgery on Ξ, while A' contains the ones

obtained from Ξ'. By Lemma 5.6(1) both A and A' contain $(k-1)(l-1)(m-1)$ elements distinguished by the induced spin^c structures on $W_{p,k,l,m}$. However, some elements may be contained both in A and in A'. In fact, by Lemma 5.6(2) for any contact structure ζ in A' there is a contact structure in A inducing an isomorphic spin^c structure on $W_{p,k,l,m}$ unless $\mathtt{rot}(K_8) = -m$ in the surgery diagram for ζ. Since the number of contact surgery diagrams on $M_{p,k,l,m}$ with $\mathtt{rot}(K_8) = -m$ giving tight contact structures belonging to A' is $(k-1)(l-1)$, there are at least

$$(k-1)(l-1)(m-1) + (k-1)(l-1) = (k-1)(l-1)m$$

nonisotopic tight contact structures in $A(\Xi)$.

In the case of $M_{p,k,m}$, a similar argument gives the lower bound $(k-1)m$. In the cases of $M_{p,k,l}$ and $M_{p,k}$, since there is no knot K_8 every spin^c structure induced by an element of A' is also induced by an element of A. Therefore, as a lower bound we just get the number of elements of A, that is $(k-1)(l-1)$ in the case of $M_{p,k,l}$ and $(k-1)$ in the case of $M_{p,k}$. $\square$

Corollary 5.8 *The number of isotopy classes of tight contact structures on the 3–manifold $M(-1; r_1, r_2, r_3)$ belonging to $A(\Xi)$ is at most*

$$(a_1^1 - 1)(a_1^2 - 1)a_1^3 \prod_{i=1}^{3} \prod_{j \geq 2}(a_j^i - 1)$$

if $r_3 \neq \frac{1}{a_0^3}$, and

$$\prod_{i=1}^{2} \prod_{j \geq 1}(a_j^i - 1)$$

if $r_3 = \frac{1}{a_0^3}$. In the above formulae $a_j^i = 2$ by convention if $j > k_i$.

Proof The statement follows immediately from Proposition 5.7 together with Theorem 4.9. $\square$

Proof of Theorem 1.1 The statement follows immediately combining Corollaries 2.10, 2.13, 5.4 and 5.8. $\square$

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