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PREFERENCE INTENSITY WITHOUT CARDINALITY

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Abstract. Preference intensity is a relevant concept, more general than cardinal representable preference, and an according axiomatic definition is introduced. Uniform spaces are applied as a tool to study preference intensity. "Uniqueness" and the existence of cardinal representation are thus considered.

1. Introduction

Suppose that an individual prefers a particular car (say a Mercedes) to a particular handkerchief, and a rehearsal of Beethoven 7th Symphony to a rehearsal of Beethoven 8th Symphony. If she compares the two preferences, most likely she will find that the first one is stronger that the second one, i.e., for her the difference in preference in going from the handkerchief to the Mercedes is greater than the difference in preference in going from the 8th to the 7th Symphony.

In another context, two voters, A and B, facing a general election may agree that they prefer pro-life to pro-choice and high inheritance tax to low inheritance tax. But perhaps for A the first issue is more relevant than the second one when the time comes to decide his vote, whereas for B it is the other way round. In short, their strengths of preferences are opposite as for these two issues. Any politician knows that the strength of preference of voters is very important in an election. Of course, reality can be formalized in manifold ways (e.g., considering preferences between bundles of alternatives, one component of the bundle for each issue), and the bounded rationality of decision makers is to be considered here, but at any rate the strength of preference is a relevant fact in the decisions of economic and political agents. Is the strength of preference susceptible of formal analysis? Perhaps we may say, even recognizing that the strength of preference exists in reality, that "quae res in se neque consilium neque modum habet ullum, eam consilio regere non potes" ("something that has in itself neither reason nor measure at all, you cannot manage it according to reason"), as Terence put it [12]. Perhaps, in a more sanguine vein, we may try to deal formally with the strength of preference, as many before have done.

Traditionally the analysis of the strength of preference (or "preference intensity") has been seen as a way to obtain cardinal utilities. In the wake of the classical [9], the preference intensity is formalized through a binary relation between pairs of alternatives satisfying three sorts of axioms: (1) basic assumptions (such as being a weak order); (2) in the infinite case, technical solvability assumptions (entailing a "rich enough" set of alternatives and a "discriminating enough" decision maker); (3) an Archimedean assumption, necessary in order to build a cardinal scale. In this paper we intend to reflect the preference intensity without attempting to build a cardinal utility, and thus we dispense with (3) in our definition of "preference intensity". Our stronger "cardinal preference intensity" refers in the literature to terms like "strength of preference", "preference intensity" or "algebraic difference structucture".

Order topologies provide a suitable tool to study preferences (in the infinite case). The preference intensity involves further structure, and we introduce here a further tool: a certain uniformity, generating the order topology of the preference. Uniform spaces are, in a sense, between metric and topological spaces.

The "cardinality" of a preference intensity is linked to the semimetrizability of the corresponding uniformity.

While the question of the "uniqueness" of preferences involves the hypothesis of connectedness [6], a "uniqueness" result for preference intensities leads naturally to the hypothesis of compactness. Through the uniformity corresponding to the preference intensity, compactness can be characterized, in particular in the cardinal case.

2. Preliminaries

A binary relation $\precsim$ on a nonempty set X is said to be a *weak order* if it is transitive and total (*total* means that $x \precsim y$ or $y \precsim x$ for any two elements x and y), and a weak order is said to be a linear order if it is antisymmetric (*antisymmetric* means that if $x \precsim y$ and $y \precsim x$ then $x = y$). Weak orders are the standard way to model the wishes of consumers in economics.

If R is a binary relation on X, its conjugate $R^\diamond$ is defined by $xR^\diamond y$ iff not $yR^\diamond x$, for all $x, y \in X$. Obviously $(R^\diamond)^\diamond = R$. If we use the symbol $\precsim$ for the binary relation, we denote its conjugate by $\prec$, i.e. $x \prec y$ iff not $x \succsim y$, where $\succsim$ has the obvious meaning (the inverse relation). It is immediate that $\precsim$ is a weak order if and only if $\prec$ is asymmetric and negatively transitive.

We may associate to a weak order $\precsim$ an equivalence relation $\sim$ defined by $x \sim y$ iff $x \precsim y$ and $y \precsim x$. The quotient set by the indifference $\sim$ will be denoted by $X/\sim$, and $\precsim$ induces naturally a linear order on $X/\sim$. The order $\precsim$ is linear precisely when $\sim$ is trivial (the equality). In practice, we shall refer to $\precsim$ as *preference*, to $\prec$ as *strict preference*, and to $\sim$ as *indifference*.

The *order topology* associated to a weak order $\precsim$ on X, denoted by $\tau_{or}(\prec)$ (or simply τ_{or} if no confusion is possible), has as base the set of all the intervals of the kind $]a, b[:= \{x \in X : a \prec x \prec b\}$ with $a, b \in X \cup \{\pm\infty\}$ (equivalently, generated by all the subsets $]-\infty, x[$ and $]x, \infty[$ with $x \in X$). Thus, if $\precsim'$ is the linear order naturally induced on $X/\sim$, and we consider the topology $\tau_{or}(\prec)$ on X, then $\tau_{or}(\prec')$ coincides with the quotient topology on $X/\sim$.

If $\precsim$ is a linear order, then X (with the order topology) is T_4 and hereditarily normal.

In order to generalize from linear orders to weak orders the properties of order topologies, we consider saturated identifications [2]. Let (X, τ) and (X', τ') be topological spaces. We say that a quotient map $f : X \longrightarrow X'$ is a *saturated identification* with the topologies τ, τ' if, for each $A \in \tau$, $A = f^{-1}f(A)$. Topological properties can often be translated between the dominion and the range through saturated identifications, as it is stated in the next proposition.

Proposition 2.1. [1]*Let $f : (X, \tau) \longrightarrow (X', \tau')$ be a saturated identification. Then the following properties are satisfied by (X, τ) if and only if they are satisfied by (X', τ'): first countability, second countability, separability, compactness, countable* *compactness, local compactness, Lindelöff, connectedness, local connectedness, normality, hereditary normality, regularity. Moreover, if* (X', τ') *is completely regular, then the same holds for* (X, τ).

Let (X, τ) be a topological space. We say that a binary relation R *saturates* τ when: $x \in A \in \tau$ and yRx imply $y \in A$. If R is an equivalence relation on X, then the quotient map $p : (X, \tau) \longrightarrow (X/R, \tau_{quot})$ is a saturated identification if and only if R saturates τ. Besides, if $\precsim$ is a weak order, then $\sim$ saturates $\tau_{or}(\prec)$. It follows that if $\precsim$ is a weak order on X and $\precsim'$ is the induced linear order on $X/\sim$, then the quotient map $p : X \longrightarrow X/\sim$ is a saturated identification with the order topologies. By considering this, many results on linear orders can be easily adapted to weak orders. In particular, If $\precsim$ is a weak order, then X (with the order topology) is completely regular and hereditarily normal.

A function $f : (X, R) \longrightarrow (X', R')$ between spaces with binary relations is called *isotone* when: xRy if and only if $f(x)R'f(y)$ for any $x, y \in X$. If R' is a weak order, then R is a weak order. If R is a linear order, then f is injective. If R' is a linear order and f is injective, then R is a linear order. An injective isotone function is called an (order) embedding, and then (X, R) is said to be *(order) embeddable* in (X', R').

Let $(X, \precsim)$ and $(X', \precsim')$ be spaces with weak orders. If $f : (X, \precsim) \longrightarrow (X', \precsim')$ is isotone, then $x \sim y$ if and only if $f(x) \sim' f(y)$, and an embedding naturally associated to f can be defined between the linearly ordered spaces $X/\sim$ and $X'/\sim'$. Conversely, an embedding between the linearly ordered spaces $X/\sim$ and $X'/\sim'$ leads (using the axiom of choice if $\sim'$ is not trivial) to an isotone function between $(X, \precsim)$ and $(X', \precsim')$. An isotone function u from a space X with a weak order $\precsim$ to the real line $\mathbb{R}$ (with its usual linear order) is called a *utility representation* of $\precsim$. Thus the problem of the existence of utility representations reduces to that of embeddability of linearly ordered spaces in the real line.

The following theorem is based on ideas of Cantor going back to 1895 (cf. [5]).

Theorem 2.2. *Let* $(X, \precsim)$ *be a linearly ordered space with more than one element. Then:*

[4],[10] $(X, \precsim)$ *can be embedded in the real line (with its usual linear order) if and only if* X *(with the order topology) is second countable.*

A weak order $\precsim$ and a topology τ on the same set X are called *compatible* if $\tau_{or}(\prec) \subseteq \tau$. A result on "uniqueness" of compatible orders is available.

Theorem 2.3. *Let* (X, τ) *be a connected topological space. Then:*

[6] *Two linear orders compatible with* τ *are either identical or inverse to each other.*

3. Preference intensity

Let $A \neq \emptyset$ be the set of alternatives available for a decision maker. In the wake of [9], the assumptions for preference intensity, including an Arquimedean assumption, differ in detail, but not greatly in content. We follow those of [8], but dispensing with the Arquimedean assumption.

Definition 3.1. *Suppose that* A *is a nonempty set and* $\precsim$ *a binary relation on* $A \times A$. *Then* $\precsim$ *is a preference intensity on* A *iff, for all* $a, b, c, d, a', b', c' \in A$, *the following five assumptions are satisfied:*

(A.1) $\precsim$ *is a weak order on* $A \times A$.
(A.2) If $(a,b) \precsim (c,d)$, *then* $(d,c) \precsim (b,a)$.
(A.3) If $(a,b) \precsim (a',b')$ *and* $(b,c) \precsim (b',c')$, *then* $(a,c) \precsim (a',c')$.
(A.4) Given a,b, *there exists* $d \in A$ *such that* $(a,d) \sim (d,b)$.
(A.5) Given a,b,c, *there exists* $d \in A$ *such that* $(c,d) \sim (a,b)$.

Preference intensity is a stronger structure than preference, and thus every preference intensity determines a preference (see below). $(a,b) \precsim (c,d)$ may be interpreted as "the difference in preference in going from d to c is greater (or equal) than the difference in preference in going from b to a". "Negative differences in preference" are part of the model (cf. (A.2)). (A.4) and (A.5) are solvability axioms, entailing a "rich enough" set of alternatives and a "discriminating enough" decision maker.

Some easy consequences result from the definition.

Proposition 3.1. *Let* $\precsim$ *be a preference intensity on* A. *Then, for all* $a,b,c,a',b',c' \in A$,
(i) $(a,a) \sim (b,b)$.
(ii) If $(a,b) \precsim (a',b')$ *and* $(b,c) \prec (b',c')$, *then* $(a,c) \prec (a',c')$.
(iii) If $(a,b) \prec (c,d)$, *then* $(d,c) \prec (b,a)$.
(iv) If $(a,b) \prec (a,c)$, *then* $(b,c) \succ (b,b)$.

Proof. (i) follows from $\precsim$ being total and (A.2). To prove (ii), suppose that $(a,b) \precsim (a',b')$ and $(b,c) \prec (b',c')$, but $(a,c) \succsim (a',c')$. Then, by (A.2), $(c,a) \precsim (c',a')$, and thus, by (A.3), $(c,b) \precsim (c',b')$. From (A.2), $(b,c) \succsim (b',c')$, and a contradiction results. In order to verify (iii), suppose that $(a,b) \prec (c,d)$, but $(b,a) \precsim (d,c)$. Now, by (ii), $(b,b) \prec (d,d)$, a contradiction. Finally, (iv) follows from (ii), considering that $(b,a) \precsim (b,a)$. ∎

In Proposition 3.1, (i) means that all the pairs of the form (x,x) are mutually indifferent. Given $a,b \in A$, we write $(a,b) \succsim 0$ (resp. $(a,b) \succ 0$) (resp. $(a,b) \sim 0$) iff $(a,b) \succsim (x,x)$ (resp. $(a,b) \succ (x,x)$) (resp. $(a,b) \sim (x,x)$) for some (and thus for all) (x,x). It is easy to check that in order to determine the preference intensity $\precsim$ it is sufficient to provide $A^+ := \{(x,y) \in A \times A : (x,y) \succ 0\}$ and the restriction of $\precsim$ to A^+.

Definition 3.2. *Let* $\precsim$ *be a preference intensity on* A. *The binary relation* $\precsim^*$ *induced on* A *by* $\precsim$ *is defined by* $a \precsim^* b$ *iff* $(b,a) \succsim 0$, *for all* $a,b \in A$.

It is immediate that $\precsim^*$ is a weak order on A. The notations $\prec^*$ and $\sim^*$ have the natural meaning. Given $a,b \in A$, obviously $a \prec^* b$ iff $(b,a) \succ 0$, and $a \sim^* b$ iff $(b,a) \sim 0$.

We now intend to make correspond a uniform space, generating the order topology of the induced preference, to every preference intensity.

Definition 3.3. *Let* $\precsim$ *be a preference intensity on* A. *If* $(b,a) \succ 0$, *we define* $U_{ba} := \{(x,y) \in A \times A : (x,y) \prec (b,a)$ *and* $(y,x) \prec (b,a)\}$.

Given $(b,a) \succ 0$ and $(d,c) \succ 0$, it is immediate that

$$(b,a) \precsim (d,c) \text{ if and only if } U_{ba} \subseteq U_{dc}$$

If $V \subseteq A \times A$ and $x \in A$, we denote $V(x) := \{y \in A : (x,y) \in V\}$. If $V \subseteq A \times A$ and $W \subseteq A \times A$, we write $W \circ V := \{(x,y) \in A \times A : (\exists z \in A : (x,z) \in V \text{ and } (z,y) \in W)\}$.

Proposition 3.2. *Let $\precsim$ be a preference intensity on A. The collection of sets $\mathcal{B} := \{U_{ba} : (b,a) \succ 0\} \cup \{A \times A\}$ is a base for a uniformity $\mathcal{U}$ on A.*

Proof. Firstly, it is immediate that all the members of $\mathcal{B}$ include the diagonal $\Delta := \{(x,x) : x \in A\}$ of $A \times A$. Now we see that $\mathcal{B}$ is a filterbase. Obviously $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$. Let $(b,a) \succ 0$ and $(d,c) \succ 0$. There are two possibilities: either $(b,a) \precsim (d,c)$ or $(b,a) \succ (d,c)$; in the first case $U_{ba} \subseteq U_{dc} \cap U_{ba}$, and in the second case $U_{dc} \subseteq U_{ba} \cap U_{dc}$. Thus we have shown that $\mathcal{B}$ is a filterbase. Moreover, $U_{ba} = U_{ba}^{-1} := \{(y,x) : (x,y) \in U_{ba}\}$, and all members of $\mathcal{B}$ are symmetric. Finally, given U_{ba}, we see that there exists $V \in \mathcal{B}$ such that $V \circ V \subseteq U_{ba}$. By (A.4), there exists $d \in A$ such that $(b,d) \sim (d,a)$. Now $(b,d) \succ 0$ (otherwise $(b,d) \precsim (b,b)$ and $(d,a) \precsim (b,b)$, thus $(b,a) \precsim (b,b)$, a contradiction). We take $V = U_{bd}$. If $(x,y) \in U_{bd} \circ U_{bd}$, then there is $z \in A$ such $(x,z) \in U_{bd}$ and $(z,y) \in U_{bd}$. Hence $(x,z) \prec (b,d), (z,x) \prec (b,d) \sim (d,a)$ and $(z,y) \prec (b,d) \sim (d,a), (y,z) \prec (b,d)$. Applying (ii) in Proposition 3.1, $(x,y) \prec (b,a)$ and $(y,x) \prec (b,a)$; thus $(x,y) \in U_{ba}$. We conclude that $U_{bd} \circ U_{bd} \subseteq U_{ba}$. ∎

Theorem 3.3. *Let $\precsim$ be a preference intensity on A. Then the uniformity $\mathcal{U}$ generates the order topology $\tau_{or}(\prec^*)$ on A.*

Proof. Let $\mathcal{T}$ be the topology generated by $\mathcal{U}$. Firstly we prove that $\tau_{or}(\prec^*) \subseteq \mathcal{T}$. Given $c \in]a,b[$, with $a,b,c \in A$, we shall show that there is $V \in \mathcal{B}$ such that $V(c) \subseteq]a,b[$. There are two possible cases: (i) $(c,a) \precsim (b,c)$; (ii) $(c,a) \succ (b,c)$. In case (i), we take $V = U_{ca}$. If $z \in U_{ca}(c)$, then $(c,z) \prec (c,a)$ and $(z,c) \prec (c,a) \precsim (b,c)$. From (iv) in Proposition 3.1, $(z,a) \succ 0$, and $z \succ^* a$. Also, $(c,b) \prec (c,z)$; again from (iv) in Proposition 3.1, $(b,z) \succ 0$, and $z \prec^* b$. We obtain that $U_{ca}(c) \subseteq]a,b[$. In case (ii), the proof that $U_{bc}(c) \subseteq]a,b[$ is similar. We conclude that $\tau_{or}(\prec^*) \subseteq \mathcal{T}$.

Now we prove that $\mathcal{T} \subseteq \tau_{or}(\prec^*)$. Given $U_{ba}(c)$, with $a,b,c \in A$ and $(b,a) \succ 0$, we shall show that there is $]h,k[$ such that $c \in]h,k[\subseteq U_{ba}(c)$. By (A.5), there are $h,k \in A$ such that $(c,h) \sim (b,a)$ and $(c,k) \sim (a,b)$. Thus $(c,h) \succ 0$ and $(k,c) \succ 0$, i.e. $h \prec^* c \prec^* k$. Let $z \in]h,k[$. Now, on the one hand, $(c,h) \sim (b,a)$ and $(h,z) \prec (a,a)$. From (ii) in Proposition 3.1, $(c,z) \prec (b,a)$. On the other hand, $(z,k) \prec (b,b)$ and $(k,c) \sim (b,a)$. From an obvious variation of (ii) in Proposition 3.1, $(z,c) \prec (b,a)$. We conclude that $z \in U_{ba}(c)$. ∎

Let $\precsim_1$ and $\precsim_2$ be two preference intensities on A inducing the same weak order $\precsim^*$ on A. Then, given $c,d \in A$, we have that $(c,d) \succ_1 0$ if and only if $(c,d) \succ_2 0$.

It is immediate that $\precsim_2$ includes $\precsim_1$ when

$$\forall (c,d) \succ_2 0,\ (a,b) \prec_1 (c,d) \Rightarrow (a,b) \prec_2 (c,d)$$

A weaker condition is (3.1) below.

Proposition 3.4. *Let $\precsim_1$ and $\precsim_2$ be two preference intensities on A inducing the same weak order $\precsim^*$ on A , with respective uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$. Then $\mathcal{U}_1 \supseteq \mathcal{U}_2$ if and only if*

$$\forall (c,d) \succ_2 0\ \exists (u,v) \succ_1 0 \text{ such that } (a,b) \prec_1 (u,v) \Rightarrow (a,b) \prec_2 (c,d) \tag{3.1}$$

Proof. Firstly we show that if $\mathcal{U}_1 \supseteq \mathcal{U}_2$ then (3.1) holds. Let $(c,d) \succ_2 0$. Since $U^2_{cd} := \{(x,y) \in A \times A : (x,y) \prec_2 (c,d) \text{ and } (y,x) \prec_2 (c,d)\} \in \mathcal{U}_2 \subseteq \mathcal{U}_1$, there is $(u,v) \succ_1 0$ such that $U^1_{uv} := \{(x,y) \in A \times A : (x,y) \prec_1 (u,v) \text{ and } (y,x) \prec_1 (u,v)\} \subseteq U^2_{cd}$. If $(a,b) \prec_1 (u,v)$ there are two possibilities: either $(a,b) \succ_1 0$ or $(a,b) \precsim_1 0$;

in the first case $(a,b) \in U^1_{uv} \subseteq U^2_{cd}$, and $(a,b) \prec_2 (c,d)$; in the second case $a \precsim^* b$, and thus $(a,b) \precsim_2 0$ and again $(a,b) \prec_2 (c,d)$. We infer (3.1).

Now we prove that (3.1) implies that $\mathcal{U}_1 \supseteq \mathcal{U}_2$. If $(c,d) \succ_2 0$ then $U^2_{cd} := \{(x,y) \in A \times A : (x,y) \prec_2 (c,d) \text{ and } (y,x) \prec_2 (c,d)\} \supseteq$
$U^1_{uv} := \{(x,y) \in A \times A : (x,y) \prec_1 (u,v) \text{ and } (y,x) \prec_1 (u,v)\}$, where $(u,v) \succ_1 0$ is as defined in (3.1). We conclude that $\mathcal{U}_1 \supseteq \mathcal{U}_2$. ∎

Given $\precsim_1$ and $\precsim_2$ as above, they are said u-equivalent if $\mathcal{U}_2 = \mathcal{U}_1$.

4. Utility representations

If $U : A \times A \longrightarrow \mathbb{R}$ is a utility representation of a preference intensity $\precsim$ on A, then utility representations of the weak order $\precsim^*$ induced on A by $\precsim$ are available:

Proposition 4.1. *Let $\precsim$ be a preference intensity on A. Let $\overline{y} \in A$. If $U : A \times A \longrightarrow \mathbb{R}$ is a utility representation of $\precsim$, then the function $u : A \longrightarrow \mathbb{R}$ defined by $u(x) := U(x, \overline{y})$, for all $x \in A$, is a utility representation of $\precsim^*$.*

Proof. Let $a, b \in A$. We shall prove that $a \precsim^* b$ if and only if $u(a) \leq u(b)$. Firstly, if $a \precsim^* b$, then $(a,b) \precsim (b,b)$. By (A.3), taking $(b,\overline{y}) \precsim (b,\overline{y})$, we have that $(a,\overline{y}) \precsim (b,\overline{y})$. It follows that $U(b,\overline{y}) \leq U(b,\overline{y})$, and $u(a) \leq u(b)$. Conversely, if $u(a) \leq u(b)$, then $(a,\overline{y}) \precsim (b,\overline{y})$. We apply again (A.3), taking now $(\overline{y}, b) \precsim (\overline{y}, b)$. Thus $(a,b) \precsim (b,b)$, and $a \precsim^* b$. ∎

Definition 4.1. *Let $\precsim$ be a preference intensity on A. Then $U : A \times A \longrightarrow \mathbb{R}$ is a value-difference utility representation of $\precsim$ iff U is a utility representation of $\precsim$ such that, for every $x, y \in A$,*

$$U(x,y) = u(x) - u(y)$$

where $u : A \longrightarrow \mathbb{R}$ is some utility representation of $\precsim^*$.

The following proposition is immediate from Proposition 4.1.

Proposition 4.2. *Let $\precsim$ be a preference intensity on A and $U : A \times A \longrightarrow \mathbb{R}$ a utility representation of $\precsim$. Then U is a value-difference utility representation of $\precsim$ if and only if*

$$U(x,y) = U(x,z) - U(y,z),\ \forall x,y,z \in A \tag{4.1}$$

Definition 4.2. *A cardinal preference intensity on A is a preference intensity on A admitting a value-difference utility representation.*

As for the ensuing uniqueness up to positive linear transformations of the utility representations of $\precsim^*$, cf. [9], [3]. In the literature terms like "preference intensity", "strength of preference" ([7]), "algebraic difference structure" ([9],[11]) or "value difference order" ([8]) refer essentially to our "cardinal preference intensity".

We consider the relation between the fact that a preference intensity is cardinal and the semimetrizability of its uniformity.

Proposition 4.3. *Let $\precsim$ be a cardinal preference intensity on A. Then the uniformity $\mathcal{U}$ is semimetrizable.*

Proof. Let $U : A \times A \longrightarrow \mathbb{R}$ be a value-difference utility representation of $\precsim$. We define $d : A \times A \longrightarrow \mathbb{R}$ by $d(x,y) := |U(x,y)|$, for all $x, y \in A$. By Proposition 4.2, we have that $U(x,x) = 0$, $U(y,x) = -U(x,y)$ and $U(x,y) = U(x,z) + U(z,y)$; thus d is a semimetric on A. We shall prove that $\mathcal{U}$ is equal to the uniformity $\mathcal{V}$ induced by the semimetric d. Recall that $\mathcal{B}$ is a base for $\mathcal{U}$, and $\mathcal{C} := \{V_\varepsilon : \varepsilon > 0\}$, with $V_\varepsilon := \{(x, y) \in A \times A : d(x, y) < \varepsilon\}$, is a base for $\mathcal{V}$. In order to check that $\mathcal{U} \subseteq \mathcal{V}$, let $(b, a) \succ 0$, and consider U_{ba}. We take $\varepsilon = U(b, a) > 0$. Now it is immediate that $V_\varepsilon \subseteq U_{ba}$. We conclude that $\mathcal{U} \subseteq \mathcal{V}$. On the other hand, suppose that $\mathcal{V} \nsubseteq \mathcal{U}$. Then there is $\varepsilon > 0$ such that, for every $(b, a) \succ 0$, $U_{ba} \nsubseteq V_\varepsilon$. It follows that $(b, a) \notin V_\varepsilon$, and thus $U(b, a) \geq \varepsilon$, for every $(b, a) \succ 0$. This a contradiction: by (A.4) and (4.1), for given $(x, y) \succ 0$ there is $(d, y) \succ 0$ such that $U(d, y) = (1/2)U(x, y)$. ∎

5. Compactness

Let $\precsim$ be a preference intensity on A. We consider the case when A is compact (with the order topology $\tau_{or}(\prec^*)$).

The following proposition is a "uniqueness" result. In contrast with Theorem 2.3, now compactness is assumed instead of connectedness, and the conclusion refers to preference intensities. This proposition is an immediate consequence of Theorem 3.3, recalling that a compact space has at most one uniformity which induces its topology.

Proposition 5.1. *Let $\precsim_1$ and $\precsim_2$ be two preference intensities on A inducing the same weak order $\precsim^*$ on A. If A is compact, then $\precsim_1$ and $\precsim_2$ are u-equivalent.*

If $\precsim$ is a preference intensity on A whose corresponding uniformity $\mathcal{U}$ is semimetrizable, then A (with the order topology $\tau_{or}(\prec^*)$) is paracompact (since it is semimetrizable). Let $\mathcal{N}$ be the set of all neighbourhoods of the diagonal Δ, in $A \times A$ (with the product topology). We recall that the paracompactness of A implies that $\mathcal{N}$ is a uniformity and generates the topology of A; moreover, $\mathcal{N}$ is the largests uniformity generating $\tau_{or}(\prec^*)$.

We can observe the uniformity $\mathcal{U}$ corresponding to a cardinal preference intensity $\precsim$ on A. If A is compact, then obviously $\mathcal{U} = \mathcal{N}$. Conversely, we may conclude the compactness of A from the fact that $\mathcal{U} = \mathcal{N}$, as it is stated in the next corollary.

Proposition 5.2. *Let $\precsim$ be a preference intensity on A whose corresponding uniformity $\mathcal{U}$ is semimetrizable. Then A is compact if and only if $\mathcal{U} = \mathcal{N}$.*

Proof. One of the implications is trivial. Conversely, suppose that the uniformity corresponding to $\precsim$ is $\mathcal{N}$; we shall prove that A is compact. By hypothesis, there is a semimetric d inducing the uniformity $\mathcal{N}$, and thus inducing the order topology $\tau_{or}(\prec^*)$ of A. We write $A' := A/\sim^*$. Let $\pi : A \longrightarrow A'$ be the quotient map defined by $\pi(x) := [x]$, for all $x \in A$, where $[x]$ is the equivalence class of x. Since A is regular and $\sim^*$ saturates $\tau_{or}(\prec^*)$, we have that $[x] = cl(\{x\})$. Therefore $[x] = \{y \in A : d(x, y) = 0\}$. Hence $D : A' \times A' \longrightarrow \mathbb{R}$ defined by $D([x], [y]) = d(x, y)$, for $x, y \in A$, is a metric on A' inducing the quotient topology. Let $\mathcal{N}'$ be the set of all neighbourhoods of the diagonal Δ' in $A' \times A'$, and $\mathcal{D}$ the uniformity on A' induced by the metric D. Now we shall show that $\mathcal{N}' \subseteq \mathcal{D}$. Let $\pi \times \pi : A \times A \longrightarrow A' \times A'$ be defined by $\pi \times \pi(x, y) := ([x], [y])$, for all $x, y \in A$. If $M' \in \mathcal{N}'$, then $(\pi \times \pi)^{-1}(M')$ is a neighbourhood of the diagonal Δ in $A \times A$, since $\pi \times \pi$ is continuous. As d induces the uniformity $\mathcal{N}$, there is $\varepsilon > 0$ such that $\{(x, y) \in A \times A : d(x, y) < \varepsilon\} \subseteq (\pi \times \pi)^{-1}(M')$. Therefore $\{([x], [y]) \in A' \times A' : D([x], [y]) < \varepsilon\} \subseteq (\pi \times \pi)(\{(x, y) \in A \times A : d(x, y) < \varepsilon\}) \subseteq M'$, and we obtain that $M' \in \mathcal{D}$. We have shown that $\mathcal{N}' \subseteq \mathcal{D}$. Now A' is paracompact (since it is metrizable), and thus $\mathcal{N}'$ is the largests uniformity generating its topology. It follows that $\mathcal{N}' = \mathcal{D}$.

We shall prove that A' has no isolated points. Suppose that there is an isolated point $[z]$ of A'. Then there is $][a],[b][$ such that $\{[z]\} \cap][a],[b][= \{[z]\}$, with $[a],[b] \in A' \cup \{\pm\infty\}$. There is no loss of generality in assuming that A' has more than one point (otherwise A is compact by Proposition 2.1), and thus, say, $[a] \in A'$. By (A.4), there exists $d \in A$ such that $(a,d) \sim (d,z)$. Now $(d,a) \succ 0$ (otherwise $(d,a) \precsim (a,a)$ and $(z,d) \precsim (a,a)$, thus $(z,a) \precsim (a,a)$, a contradiction, since $a \prec^* z$) and $(z,d) \succ 0$ (otherwise $(z,d) \precsim (a,a)$ and $(d,a) \precsim (a,a)$, thus $(z,a) \precsim (a,a)$, a contradiction again). Therefore $a \prec^* d \prec^* z$, a contradiction, because $][a],[z][= \emptyset$. If $[b] \in A'$ we are analogously led to a contradiction. We conclude that A' has no isolated points. To summarize, we have obtained that A' is a uniform space without isolated points such that $\mathcal{N}'$ (the set of all neighbourhoods of the diagonal) is a metrizable uniformity. By a known result (see e.g. Section 14.4 in [13]), A' is compact. The compactness of A follows from Proposition 2.1. ∎

The next corollary follows by considering Proposition 4.3.

Corollary 5.3. *Let $\precsim$ be a cardinal preference intensity on A. Then A is compact if and only if the uniformity corresponding to $\precsim$ is $\mathcal{N}$.*

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