A Comparative Analysis of Problem Solving Methods on the Basis of the Teacher's Knowledge and Student's Knowledge

Thesis of PhD dissertation

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2017
Szeged

1 Reasons behind the choice of the topic

The focus of my research work is to investigate the problem-solving ability seen from two points of view. As a practicing teacher I consider it is necessary a dual approach of mathematical problems: elementary mathematical knowledge (wich is available to pupils) and a superior mathematical knowledge, which is part of the university curriculum so the students do not possess it (but this type of knowledge is available to teachers). In this regard we can approach a problem by "pupil's tools" and "teacher's tools", respectively. So we can discuss a problem-solving activity by "pupil's method" and "teacher's method" respectively. In many cases the "teacher's solution" is more straightforward, but it requires some tools that are not available to pupils. So the teacher could not explain this solution to the pupils. In this case the teacher must elaborate the "pupil's solution", which often requires more creativity and is more ingenious than the "teacher's solution". The pupil can find the "pupil's solution" completely alone or with the aid of the teacher. In the teaching of mathematics, the teacher's advantage is the additional knowledge which can be used in various forms during the educational process.

Teacher's additional knowledge plays an important role not only in solving problems, but also in creating new exercises. The teacher chooses exercises from textbooks or exercise books during the teaching activity. However, in many cases teachers encounter difficulties in finding the exercises that fit perfectly the pupils' level of knowledge and the education curriculum. However the exercise books contain varied and differentiated problems, they can not satisfy the educational needs in every respect, because the each pupil's level of knowledge has particular features. Sometimes teacher has to improvise new exercises during a teaching activity. This happens, for example, when students' questions and ideas require the discussion of analogue problems. That is why the developement of problem posing skills during university studies becomes more important.

One way to create new problems is to start from a relatively simple exercise and then to generalize it using teachers' additional knowledge (this generalised exercise can also contain parameters). Giving concrete values to the parameters teacher can create the special cases of the generalized exercise. The exercises created in this way can be solved by "pupil's method" during the educational process.

During our work we have discussed many different methods in the following areas: solving word problems and extreme value problems; problem posing in some areas, such as mathematical induction, divisibility and number theory. In many cases we have given a sort of overview of the history of a particular problem and of the developmental aspects of its solution.

2 Aims of research

The research focuses on student's problem solving skills and teacher's problem posing opportunities in different fields of mathematics.
The aims of research are the following:

- To determine the theoretical background of problem solving activities taking into account teacher's knowledge and pupil's knowledge, respectively.
- To survey the pupils' problem solving skills and to investigate development opportunities in the following areas: word problems and extreme-value problems.
- To investigate the transition from arithmetic to algebra in lower secondary school education, to analyze problem solving methods and main error types.
- To investigate the tools of teachers' problem posing by the use of the teachers' additional knowledge.
- To compare the mathematical and language tools, respectively, regarding to the negation of sentences.

3 Process of research

The main components of the process of my research are the following:

- Comparing the teaching practice and the theoretical background of the transition from arithmetic to algebra in lower secondary school education.
- Studying the most adequate methods to solve extreme-value problems in the secondary school education.
- Summing up the possibilities of teachers' problem posing activities by the use of teachers' methods.
- Analyzing the linkage between the tools of Mathematical logic and Language, respectively.

I have reported the above mentioned research activities in various chapters of the dissertation, which I would summarize as follows.

3.1 Solving word problems in lower secondary school education

We have examined the pupils' problem solving skills related to word problems in the lower secondary school educational processes. We have discussed some word problems whose generalised algebraic structure is a system of equation with two or more unknowns. However these systems of equations are not available for the lower secondary school pupils, so teachers have to adopt some problem solving strategies that require elementary arithmetical and algebraic procedures, such as:

- drawing bubble charts and line segments,
- making a balance,
- making arithmetical computations,
- using the false position method,
- writing equations with one unknown.

Research in teaching and learning algebra has detected a number of serious cognitive difficulties and obstacles especially to novice students (see [3]). An important challenge in international research and thinking on Mathematics education curriculum is to consider ways in which the transition from arithmetic to algebra can be achieved more smooth (see [6], [13], [18], [19], [88],[97]). In my reasearch work, I tried to reveal the main difficulties concerning the transition from arithmetic to algebra and to compare my findings with the above mentioned works (see [20], [26], [28]). In particular, the "early algebra" movement has examined how to teach arithmetic in a way that prepares pupils for algebra, and which emphasises the thinking processes that underlie algebra. The main aim is not to introduce algebraic symbols at an earlier age, but to change the emphasis of arithmetic teaching. It is no longer appropriate to have an arithmetic curriculum which focuses exclusively on computation, so there is opportunity to include experiences of generalisation, mathematical structure and properties of operations that underpin algebra.

We have examined the transition from arithmetic thinking to algebraic thinking in case of grade 6 pupils in the Reformed Primary School in Veresegyház (in 2015-2016 school-year). I have to mention, that I am a Mathematics teacher in this school, and the entire project was carried out with my grade 6 pupils. For this purpose we have adopted a three-phase program that contained the followings.

Aritmetic calculations: At the begining, we have solved several word-problems with arithmetic methods, such as drawing bubble figures and line segments, using a balance, thinking backwards. We showed multiple solution methods for each problems and completed the pupils' problem-solving ideas if they could not find a correct solution themselves. At the end of this stage the pupils solved a list of six word problems to assess their problem solving skills by arithmetic methods (*Test 1.*).

Algebraic methods: Thereafter the pupils were initiated in the use of algebraic methods. Firstly, we solved equations thinking backwards and with the balance-principle. Thereafter, in our problem solving activities the most important strategy was to translate word problems into first-degree equations with one unknown. Finally, in order to delineate the procedures used to solve the exercises and to identify the most accepted methods (and to test especially the pupils' attitude towards arithmetic and algebraic methods, respectively), pupils were given a paper-and-pencil test with six word problems to solve in 45 minutes (*Test 2.*). We have to mention that the algebraic structure of the word problems in *Test 1.* and *Test 2.* was the same, but the context of the problems was quite different.
False-position method: Our experience is that lower secondary school pupils in word problem solving mainly prefer numerical checking strategies (we refered to these as *Groping*, according to Pólya), such as estimation/guess and check and trial-and-error (see [28]). In this way the importance of the false-position method increases in a considerable way, because it is a pattern, a model which the students can use in case they have to deal with word problems. The usage of the false-position method is convenient to the pupils who try to solve these problems by groping, because this is a deductive method in which the students should not follow severe arithmetic or algebraic rules.
During 2 Mathematics lessons we solved word problem by false position method. Thereafter pupils had to solve a list of six word problems (*Test 3.*). Problems on *Test 3.* were not quite simple, some of them were of moderate difficulty. We underlined that we agree with every learned method, they have to choose, according to their point of view the most suitable method to solve the given problem.
Most students used the false position method and gave the right answer. This is a good evidence that pupils in every-day word problems mainly prefer to use intuitive, non-algebraic methods. Pupils tend to use numerical procedures mainly because they are used to perform procedural computations rather than to represent in an arithmetic or algebraic way the relations involved in a given problem.

3.2 Solving extreme-value problems in secondary school education

We have presented some possible ways of solving extreme value problems by elementary methods with which the generally available method of differential calculus can be avoided. In the secondary school education the extreme value problems (or maximum and minimum problems, or problems concerned with the greatest and the least values), in some ways, are more attractive than other mathematical problems. The importance of the extremum problems is ignored in the regular curriculum; however they are in the main stream of competition problems - there-

fore they are useful tools in the selection and developement of talented students. The differential calculus provides a general method for solving problems on minima and maxima. The present day secondary school curriculum does not contain the elements of differencial and integral calculus (we do not mean the secondary schools with advanced mathematical teaching programme). But it does not mean that we cannot deal with extreme value problems in secondary school educational processes, since most of these problems can be solved by elementary methods. Therewith, by elementary methods we can solve many problems which need the partial differential of a function of several real variables (and this method is not included in any secondary school curriculum).
Problem-solving by elementary methods means the replacement of the methods based on differential calculus, which are quite stereotyped, by the elementary methods collected from different fields of Mathematics, such as elementary inequalities between geometric, arithmetic and square means, the codomain of the quadratic and trigonometric functions, the scalar product of the vectors, etc.
In case of elementary methods there are no generally valid rules to solve the problems, every exercise is a particular problem. However, having solved a problem with real insight and interest, the students aquire a precious possession: a pattern, a model, that they can imitate in solving similar problems. They develop this pattern if they try to follow it, if they reflect upon the analogy of problems solved, upon the relevant circumstances that make a problem accessible to this kind of solution, etc. Developing such a pattern, they may finally attain a real discovery and they have a chance to aquire some well ordered and readily available knowledge.
We have shown some patterns that students can imitate in solving similar problems. These patterns can also provide some ideas for Hungarian teachers on how to introduce this topic in their practice. We also have shown some useful ideas about how teachers can use their additional knowledge to create some novel extreme-value problems so we touched upon the teachers' problem posing activities.
We have discussed the results of a survey carried out with a group of 79 secondary school students in grades 10 and 11. The two schools selected were Boronkay György Technical High School (Vác town, Hungary) and Reformed High School from Gödöllő.
We looked for answers to the following questions:
• Which are the most agreed methods to solve extreme-value problems?
• What kind of misconceptions can occur in the extreme-value problem solving activities?
• Which are the main difficulties and deficiencies with regard to the maximum and minimum problems?
• How can students synthesize their knowledge collected from different fields of

Mathematics (such as geometry, trigonometry, quadratic functions, means, etc.) during the problem solving activities?

3.3 Teachers' problem posing

We examined how the teachers (by the use of their additional knowledge) can be able to create new mathematical problems in different areas of Mathematics, such as mathematical induction or problems concerning divisibility and number theory.

3.4 Mathematics in Language

We have touched upon the relationships between Language and Mathematics. Formulating sentences is wider and reacher in Language, this kind of problem in Mathematics is narrow but more precise. We have considered some statements and we have set up the problem of their precise negation. In general we can get various variants of answers, but Mathematics accepts only one precise answer (we called this the perfect negation of the statement). Our main objective have been to analyse three statements and their negation with the tools of Mathematics, such as conjunction, disjunction, implication, etc. We have also appealed to the tools of the Hungarian Language. The central question is how the Language methods fits the requirements of Mathematics related to the problem of the perfect negation. This general question contains many more special questions:

1. Which are the main competencies students posses while they have to deal with the negation of sentences?

2. Can the students find the perfect negation of a statement while they possess only Language and Grammar knowledge?

3. How can the grade 12 students, armed with the tools of Mathematical Logics, find the perfect negation of sentences?

Most of all, our attempt was to highlight the students' way of thinking related to the problem of the negation. We carried out a survey with a group of 817 students (grade 7-12). The four schools selected were Boronkay György Technical High School (Vác town), Reformed High School from Gödöllő, Reformed Primary School from Veresegyház and Fabriczius József Primary School (Veresegyház). The test-paper contained the following statements and the students had to choose the perfect negation of each statement from six variants of answer.

1. If Béla gets up early then he find gold.

2. It is snowing and Winnie the Pooh is cold.

3. If it snows then I do not go to cinema.

We have to mention that only one is considered the right answer, if we argue with the tools of Mathematics. The aim of the research was to find out how the students can handle this kind of problem by the use of their Language and Grammar knowledge, because the tools of Mathematics necessary to solve the problems are contained in the grade 12 curriculum.

4 Results

4.1 Solving word problems in lower secondary school education

We can collect the most information concerning the use of algebraic methods by examining *Test 2*. In this stage pupils have known both arithmetic and algebraic methods. The test-paper contained the following exercises.
Problem 1. *Paul paid a chocolate worth 960 HUF in coins of 20 and 50 HUF, respectively. In the aggregate he used 20 coins. How many coins did he use of each sort?*
Problem 2. *A rectangle has a perimeter of 136 cm and its length is 12 cm greater than its width. Find the dimensions of the rectangle!*
Problem 3. *Kukutyin has three times as many inhabitants as Nekeresd. The inhabitants of Kukutyin number 264 more than those of Nekeresd. How many inhabitants does Nekeresd have?*
Problem 4. *Bea spent 5450 HUF in three days. On the first day he spent three times as much she spent on the second day. On the third day she spent 40 HUF more than on second day. How much did she spend on the first day?*
Problem 5. *In a locality the number of inhabitants doubled, then 456 inhabitants moved from the locality. In this way the number of inhabitants became 1230. How many inhabitants were at the beginning?*
Problem 6. *On a farm there are geese, ducks and turkeys. A quarter of the birds are turkeys and a third are geese. The number of ducks is 65. How many birds are on the farm?*

We can summarize our experiences with the following tables.

Table 1 - The distribution by chosen methods

Chosen method	Pr. 1	Pr. 2	Pr. 3	Pr. 4	Pr. 5	Pr. 6
Algebraic method	5	26	21	40	21	20
Line segments	0	10	17	6	0	3
Working backwords (bubble chart)	0	0	0	0	17	0
False position method	22	0	0	0	0	0
Groping	17	0	0	1	0	0
Arithmetic operations	1	11	9	2	12	19

Table 2 - The efficiency of the chosen method

Chosen method	Pr. 1	Pr. 2	Pr. 3	Pr. 4	Pr. 5	Pr. 6
Algebraic method	20%	31%	38%	58%	62%	30%
Line segments	-	50%	82%	83%	-	33%
Working backwords (bubble chart)	-	-	-	-	88%	-
False position method	100%	-	-	-	-	-
Groping	76%	-	-	0%	-	-
Arithmetic operations	0%	55%	11%	50%	58%	47%

We will formulate our conclusions taking into account our experiences in the teaching process and the students' works on the test papers:

- In the aggregate the pupils were more successful with arithmetic methods then with algebraic tools.
- We were able to delineate some kind of problems, where most of the pupils were able to make up equations and solve them successfully.
- We have identified the prominent error types and major difficulties in the transition from arithmetic to algebra, such as the meaning of the unknowns, the order of the operations, parenthesis omited, the meaning of the equal sign, closure, etc.
- We have to mention that as pupils encounter difficulties in writing an equation or applying an arithmetical method they resort to the method of guess and check.

During 2 lessons we solved word problems exclusively by the false position method. After that, *Test 3.* followed, where students could use any of the learned methods. We would highlight two of these exercises.
I: *A bus on the first day went a distance four times than on the second day. On the first day it went 135 km more than on the second day. What distance did it go each day?*

Table 3

Chosen method	Right answer	Wrong answer
Algebraic method	0	1
False position method	16	6
Line segments	8	1
Arithmetic operations	1	6
Groping	6	0

16 pupils gave the right answer by the false position method. There are two conditions in the exercise, so we can divide the pupils' works in two main categories. We detail two pupils works.

Table 4 (First pupil's work)

	First day	Second day	difference	error
First position	80	20	60	75
Second position	84	21	63	72
Answer	180	45	135	0

This pupil used the condition "on the first day went a distance four times greater than on the second day" to make the suppositions, and he calculated the error from the condition "on the first day it went 135 km more than on the second day". He found out that if he increase the second day by one then the error will decrease by three. 10 students gave the right answer by this train of thought (with other numbers in the suppositions, of course).

Table 5 (Second pupil's work)

	First day	Second day	Four times	Error
First position	136	1	4	132
Second position	137	2	8	129
Answer	180	45	180	0

In this case the condition "on the first day it went 135 km more than on the second day" delivered the data to the suppositions. The pupil calculated the error by comparing the distance on the first day and four times the distance on the second day. 4 pupils gave the right answer working in the same way.

II. *There are books on two bookshelves. On the second shelf there are three times as many books as on the first shelf. We take 13 books from the second shelf and put 10 books on the first shelf. In this way on the second shelf there will be twice as many books as on the first shelf. How many books were on each shelf at the beginning?*

Table 6

Chosen method	Right answer	Wrong answer
Algebraic method	0	1
False position method	14	6
Groping	13	4
Line segments	0	1
Arithmetic operations	0	1

Table 6 shows that 14 pupils gave the right answer by false position method. Most of them worked in five coloumns as we can see in the following (one pupil's work).

Table 7

	1st shelf	2nd shelf	1st shelf +10	2nd shelf −13	error
First position	10	30	20	17	23
Second position	11	33	21	20	22
Answer	33	99	43	86	0

In *Test 3* most of the pupils gave the right answer by false position method. This is due to the fact that grade 6 pupils tend to use intuitive, non-algebraic metods to solve word problems. They mostly perform numerical procedures and make calculations instead of analyzing the relationships between unknown quantities by arithmetic or algebraic methods.

During lessons before the survey the pupils were fond of the false position method. Several students with arithmetic or algebraic methods have failed however they gave the right answer by false position method. Therefore the false position method plays an important role in the developement of affective aspects too.

We draw the conclusion that the false position method has its own right place in the lower secondary school education. This method can not replace arithmetic and algebraic methods, it appears as an effective complementary method. The method presents an efficient alternative in case of mathematical problems where arithmetic or algebraic approach is relatively complicated. The method is also useful in the process of differentiation. Those pupils who have difficulties in using algebraic tools can work with the false position method while the other pupils work with algebraic methods.

From my own educational experience I can state that, by consolidating the knowledge of algebraic methods, the false position method gradually gets neglected. For example the secondary school students can easily solve problems with algebraic methods so for them the false position method seems more laborious. In this way, it's nothing to worry about the false position method would completely suppress the algebraic approach to word problems.

We have tried to analyze our findings in parallel with the results of recent international research projects. Students errors in algebra can be ascribed to fundamental

differences between arithmetic and algebra. For instance, if students want to adopt an algebraic way of reasoning, they have to break away from the arithmetical conventions and need to learn to deal with algebraic symbolism. According to Filloy and Rojano (see [19]), in the transition from arithmetic to algebra it is necessary to break with certain arithmetical notions, hence the presence of a "didactic cut". This is especially noticeable when we make the transition from equations of the form $A \cdot x + B = C$ (arithmetical equations) to equations of the form $A \cdot x + B = C \cdot x + D$ (non-arithmetical equations). In arithemtical terms, the left side of an equation corresponds to a sequence of operations and the right side represent the consequence of having performed such operations. From this point of view, an equation of the form $A \cdot x + B = C$ can be solved in a pure arithmetic way, thinking backwards starting with the number C. However, we can not apply the arithmetical train of thought to an equation of the form $A \cdot x + B = C \cdot x + D$, its resolution involves operations drawn from outside the domain of arithmetic, namely operations on the unknowns. Solving such a non-arithmetical equation requires a different approach (for example, the way of thinking while we solve the equation by the balance-principle) and solving word problems with these equations is more difficult.
One of the most important steps is to understand the correct meaning of the equal sign. One of the main features of the transition is the pupil is aware that the sign of equality means equivalence and the left and right sides of equality can be interchanged [42], [58]. In many cases pupils interpret that the right hand side of equality must be the result of the operations on the left [84], [69].
Summarizing our experiences, we can conclude that grade 6 students encounter difficulties in solving problems by algebraic methods. We have identified those types of problems that students easily solved by algebraic methods (these are mainly problems that can be described by the arithmetical equation $A \cdot x + B = C$). Pupils in our study, when faced with other types of word problems approached them with arithmetic methods and numerical procedures (we mean guess-and-check, trial-and-error, false position method, etc.)

4.2 Solving extreme-value problems in secondary school education

We will analyse our survey concerning grade 10 and 11 students' extreme-value problem solving abilities. Every student received a test-paper with four exercises on it, and had to solve three exercises, every student could omit an exercise (our intention also was to investigate which are the most agreeable exercises).
Problem 1 (grade 10): *Being given $c = 10$, the length of the hypotenuse of a right-angled triangle, find the maximum of its area!*

Problem 1 (grade 11): *Being given* $b = 4$, *the length of the leg of an isosceles triangle, find the maximum of its area!*

Table 8 - The distribution of answers (Problem 1.)

	10th grade	11th grade
Right answer	34	10
Wrong answer	9	20
No response	3	3

Problem 2 (grade 10): *Find the maximum and minimum of the function* $f : [3, 7] \to \mathbb{R}$, $f(x) = \sqrt{x-3} + \sqrt{7-x}$ *!*
Problem 2 (grade 11): *Find the maximum of the function* $f : [-3, 2] \to \mathbb{R}$, $f(x) = \sqrt{x+3} + 2 \cdot \sqrt{2-x}$ *!*

Table 9 - The distribution of answers (Problem 2.)

	10th grade	11th grade
Right answer	8	6
Wrong answer	11	17
No response	27	10

Problem 3 (grades 10 and 11) *Find the maximum of the product* $a \cdot b$ *where* $a + 2 \cdot b = 4$ *and* $a, b \geq 0$ *!*

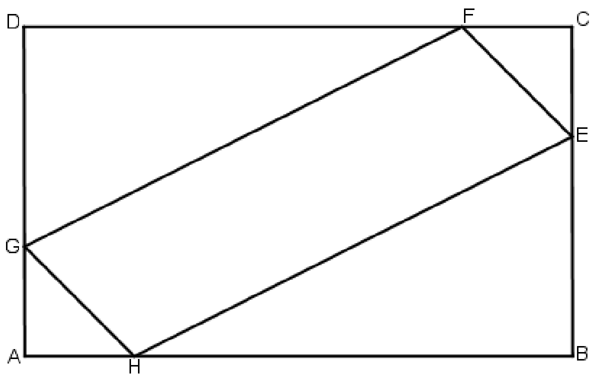
Table 10 - The distribution of answers (Problem 3.)

	10th grade	11th grade
Right answer	20	12
Wrong answer	20	17
No response	6	4

Problem 4 (grades 10 and 11) *The length of the sides of a rectangle* $ABCD$ *are* $AB = 10$ *cm and* $BC = 6$ *cm , respectively. On the sides of the rectangle we consider the points E, F, G and H such that* $CE = CF = AH = AG = x$ *(see Figure). Find the maximum of the area of the parallelogram EFGH!*

Table 11 - The distribution of answers (Problem 4.)

	10th grade	11th grade
Right answer	6	1
Wrong answer	32	16
No response	8	16



We have formulated our conclusions concerning the improvement of students' performance in solving these kind of problems. Based on the evaluation of the answers we consider that the maximum and minimum problems are difficult enough for secondary school students, this fact is seen in the large number of wrong answers. Also, we can see that grade 10 students' results were slightly better than the results of grade 11 ones. Besides efficiency, grade 10 students apply more adequate methods than the grade 11 ones to solve the extreme value problems. In my opinion, this is due to the fact that the curriculum for grade 11 students does not prescribe any kind of maximum-minimum problem solving activities, these kind of exercises were not practiced for over a year.
A large number of students think that an extreme value problem somehow involves the equality of two quantities, so they look for two quantities which must be equal and they often give erroneous answers. The main cause of these errors is the misinterpretation of the basic inequalities between arithmetic, geometric and square means. We can also state that the students textbooks mostly deal with extreme value problems where the equality between two or more quantities delivers the right answer (see [29] and [51]). To make an improvement in this sense we have shown some problems which we consider to be necessary to deal with. Many students gave the right answer after they calculated the value of a function or expression for several certain values of the variable. This is an eloquent proof that most of them did not find any way to solve the problem and they ultimately appealed to this lacunary method. Students face difficulties when they have to synthetize knowledge concerning functions, algebraic expressions or geometry. It is also difficult for them to find analogous problems to the actual discussed problem.
In our opinion a serious improvement is necessary in the extreme value problem solving activities. It is necessary to enlarge the number of methods which are ad-

equate to solve extreme value problem. It is also necessary to deal with a greater variety of problems, not only those that focuse on the inequalities between means. Futhermore, the extreme value problems have their rightful place in almost every chapter of the secondary school Mathematics education, not only in the grade 10 curriculum.

4.3 Teachers' problem posing

The teachers' additional knowledge is quite adequate in problem posing. We have shown, on some issues, how a teacher can create new exercises by generalization and by using some knowledge from university studies, such as modular arithmetic, Fermat's Little Theorem, Euler-Fermat Theorem, Lagrange-Theorem, integral calculus and de Moivre's formula. In my opinion, as a practising teacher, it is necessary to develop the teachers' problem posing skills. On this line, the above mentioned ideas could be useful.

4.4 Mathematics in Language

We will analyse the pupils' answers concerning the second statement on the test paper. The possible variants of answers are shown in the following.
2-nd statement: It is snowing and Winnie the Pooh is cold.
The variants of answers:
A. It is not snowing and Winnie the Pooh is not cold.
B. It is not snowing and Winnie the Pooh is cold.
C. It is not snowing or Winnie the Pooh is not cold.
D. It is snowing and Winnie the Pooh is not cold.
E. It is not snowing or Winnie the Pooh is cold.
F. It is snowing or Winnie the Pooh is not cold.

Table 12 - The distribution of pupils' answers

	Grade 7	Grade 8	Grade 9	Grade 10	Grade 11	Grade 12	Aggr.
A.	106	88	126	82	84	57	543
B.	6	7	5	9	3	6	36
C.	3	3	8	9	8	34	65
D.	18	34	35	18	28	19	152
E.	1	5	7	0	0	3	16
F.	2	0	2	0	0	1	5
Aggr.	136	137	183	118	123	120	817

The results of the survey show that a small part of the students gave the right answer, namely the perfect negation of the statements. Our conclusion is that the

Language and Grammar knowledge is not enough to find the perfect negation of a statement, it is necessary the students be acquainted with the tools of Mathematics, especially with the tools of Mathematical Logics. In our opinion the tools of Mathematical Logics have to be introduced at an earlier age, not only in the grade 12 curriculum.

5 Perspectives

The study concerning the transition from arithmetic to algebra is worth continuing. Exploring factors influencing the transition and possible sources of error can make the acquisition of algebraic knowledge in lower secondary school education more smooth.
The approach of extreme-value problems necessarily requires more emphasis and spread in the secondary school Mathematics curriculum.
The devepolement of teachers' problem posing skills needs a serious improvement. For this purpose each chapter of mathematics must be processed on the basis of "teacher's knowledge-student's knowledge" dichotomy. Those aspects of teacher's additional knowledge must be highlighted that can be useful tools in teacher's problem posing activity.
The teaching of Mathematical Logics could be enhanced. This could be effectively implemented by the teachers' cooperation in the field of Mathematics and Language.

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Test 1

Problem 1: *A hotel has 23 rooms with 52 beds. The rooms have two or three beds. How many double bed rooms are there in the hotel?*

Problem 2: *In a school there are 760 pupils. The girls' number is 168 more than the boys' number. How many girls and how many boys are in the school, respectively?*

Problem 3: *A farmer has ducks and geese. The total number of birds is 165. The number of ducks is four times the number of geeses. How many geese has the farmer?*

Problem 4: *There are books on three bookshelves. On the first shelf there are 5 books more than on the second. On the third shelf there are twice as many books as on the second. In the aggregate, there are 149 books. How many books are on each shelf?*

Problem 5: *Bea doubled her money, then she spent 368 HUF. Thereafter she has 432 HUF. How much money did she have at the beginning?*

Problem 6: *A half of the pupils went to Debrecen. A third of the pupils went to Székesfehérvár. The other 6 pupils went to Budapest. How many pupil were in the aggregate?*

Test 2

Problem 1: *Paul paid a chocolate worth 960 HUF in coins of 20HUF and 50HUF, respectively. In the aggregate he used 20 coins. How many coins did he use of each sort?*

Problem 2: *A rectangle has a perimeter of 136 cm and its length is 12 cm greater than its width. Find the dimensions of the rectangle!*

Problem 3: *Kukutyin has three times as many inhabitants as Nekeresd. The inhabitants of Kukutyin number 264 more than those of Nekeresd. How many inhabitants does Nekeresd have?*

Problem 4: *Bea spent 5450 HUF in three days. On the first day he spent three times as much she spent on the second day. On the third day she spent 40 HUF* *more than on second day. How much did she spend on the first day?*

Problem 5: *In a locality the number of inhabitants doubled, then 456 inhabitants moved from the locality. In this way the number of inhabitants became 1230. How many inhabitants were at the beginning?*

Problem 6: *On a farm there are geese, ducks and turkeys. A quarter of the birds are turkeys and a third of the birds are geese. The number of ducks is 65. How many birds are on the farm?*

Test 3.

Problem 1: *Hajni has collected spiders and swords. In the aggregate, there are 38 insects. She has counted 250 legs. A spider has 8 legs and a sword has 6 legs. What is the number of swords and spiders, respectively?*

Problem 2: *Bea bought a toy worth 2410 HUF. She paid in 20HUF and 50HUF coins, respectively. The number of 20HUF coins is five more than the number of 50HUF coins. How many coins did she use of each sort?*

Problem 3: *A bus on the first day went a distance four times than on the second day. On the first day it went 135 km more than on the second day. What distance did it go each day?*

Problem 4: *Peti and Zoli have together 168 stamps. Zoli has 12 stamps more than the double of Peti's stamps. How many stamps has each of them?*

Problem 5: *There are books on two bookshelves. On the second shelf there are three times as many books as on the first shelf. We take 13 books from the second shelf and put 10 books on the first shelf. In this way on the second shelf there will be twice as many books as on the first shelf. How many books were on each shelf at the beginning?*

Problem 6: *András has read one quater of a book and another 12 pages and there are two thirds of the book left. How many pages are the book?*

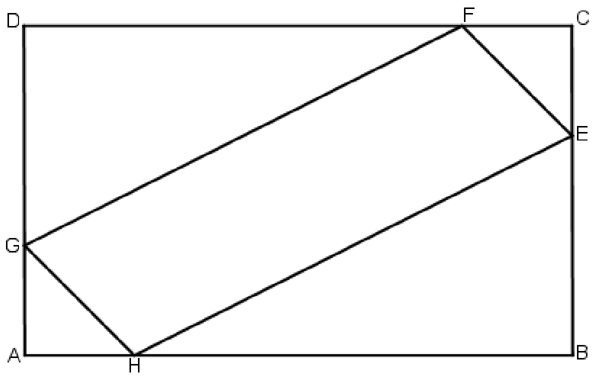
Test paper - Extreme-value problems (grade 10)

Problem 1: *Being given $c = 10$, the length of the hypotenuse of a right-angled triangle, find the maximum of its area!*

Problem 2: *Find the maximum and minimum of the function $f : [3, 7] \to \mathbb{R}$, $f(x) = \sqrt{x-3} + \sqrt{7-x}$!*

Problem 3: *Find the maximum of the product $a \cdot b$ where $a + 2 \cdot b = 4$ and $a, b \geq 0$!*

Problem 4: *The length of the sides of a rectangle ABCD are $AB = 10$ cm and $BC = 6$ cm , respectively. On the sides of the rectangle we consider the points E, F, G and H such that $CE = CF = AH = AG = x$ (see Figure). Find the maximum of the area of the parallelogram EFGH!*



Test paper - Extreme-value problems (grade 11)

Problem 1: *Being given $b = 4$, the length of the leg of an isosceles triangle, find the maximum of its area!*

Problem 2: *Find the maximum of the function $f : [-3, 2] \to \mathbb{R}$, $f(x) = \sqrt{x+3} + 2 \cdot \sqrt{2-x}$!*

Problem 3 and 4: *Problem 3 and 4, respectively, from the grade 10 students' test-paper*

Test paper - Mathematics and Language

1. If Béla gets up early then he find gold.

A. Béla do not get up early and he do not find any gold.

B. Béla do not get up early and he find gold.

C. Béla gets up early and he do not find any gold.

D. If Béla do not get up early then he do not find any gold.

E. If Béla do not get up early then he find gold.

F. If Béla gets up early then he do not find any gold.

2. It is snowing and Winnie the Pooh is cold.

A. It is not snowing and Winnie the Pooh is not cold.

B. It is not snowing and Winnie the Pooh is cold.

C. It is not snowing or Winnie the Pooh is not cold.

D. It is snowing and Winnie the Pooh is not cold.

E. It is not snowing or Winnie the Pooh is cold.

F. It is snowing or Winnie the Pooh is not cold.

3. If it snows then I do not go to cinema.

A. It does not snow and I go to cinema.

B. If it does not snow then I do not go to cinema.

C. It does not snow and I do not go to cinema.

D. If it snows then I go to cinema.

E. It snows and I go to cinema.

F. If it does not snow then I go to cinema.